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AN EXTENSION OF THE BOLTZMANN RELATION TO COLLISIONLESS MAGNETIZED PLASMA

N. Brenning, C.-G. Fälthammar and M. Bohm

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Department of Plasma Physics
Royal Institute of Technology
S- 100 44 Stockholm, Sweden

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N. Brenning, C.-G. Fälthammar and M. Bohm

Department of Plasma Physics, the Royal Institute of Technology
S-100 40 Stockholm, Sweden

Abstract

The neutralization of positive space charge is studied for density perturbations of limited spatial extent in a collisionfree magnetized plasma. It is found that a local density maximum gets a positive potential which depends only on the ambient electron temperature T_e and the relative increase in density n_e/n_{e0}. For small density increases, below 5%, the resulting relation between potential and plasma density agrees closely with the Boltzmann relation, which applies in the presence of collisions. For larger density increases, the difference from the Boltzmann relation rapidly becomes large, *e.g.* a factor 2 for a 50% density increase, and a factor 3 for a 100% density increase. The result constitutes (1) a justification for using the Boltzmann relation also in collisionless magnetized plasma, provided that the density perturbations are small, and (2) a general relation which replaces the Boltzmann relation for larger-amplitude perturbations.

1. Introduction.

One of the basic properties of a plasma is that it maintains quasineutrality, *i.e.*, because of the high mobility of the electrons there is usually an almost equal concentration of positive and negative space charge. With the exception of special cases like boundary sheaths and double layers, it is therefore not practical to calculate the potential from Poissons equation. In the presence of collisions, the potential is instead obtained from the Boltzmann relation $n_e = n_{e0}\exp(eU/kT)$. Solving for U gives

$$U = \frac{kT}{e} \ln \left(\frac{n_e}{n_{e0}}\right), \tag{1}$$

i.e., U is a function only of the temperature and the density. The only condition of applicability of Eq. (1) is that the density varies slowly compared to the collision time of the electrons, so that they can maintain a thermal velocity distribution at all times. The cause of the density variation is irrelevant.

In the collisionfree case, Eq. (1) is still valid for density <u>decreases</u>. In response to an ion density decrease, quasineutrality can easily be restored by a negative plasma potential which reflects a corresponding part of the electrons. The electrons which have sufficient energy not to be reflected maintain a thermal velocity distribution, and Eq. (1) therefore remains valid. The situation is very different for density <u>increases</u>. Consider a limited region in a plasma where the ion density has increased for some unspecified reason, so that the region has momentarily obtained a positive potential. The electrons become separated into two groups, free and trapped. The free electrons enter the region from the outside, pass through, and escape on the other side. Because all the the free electrons increase their velocity when they enter the positive potential, their total density is reduced where the potential is positive. The trapped electrons, which have too small energy to escape from the positive potential, must therefore compensate two sources of excess positive space charge, (1) the increase in ion density and (2) the decrease in the density of free electrons. The density of trapped electrons depends on the time history of the system. Our aim here is to calculate the densities of free and trapped electrons under some general assumptions of that time history, and derive a relation between the density and the potential which can replace Eq. (1) in a collisionfree magnetized plasma.

2. Calculations

We consider a homogeneous magnetized plasma with density n_{e0} where the ion density has locally increased over a length L by an amount $\Delta n(x)$, and where the potential as a consequence has obtained a yet unknown value $U(x)$. The x axis aligned with the magnetic field.

We need to make two assumptions to calculate the density of free electrons: (1) that the potential $U(x)$ is nowhere negative, and (2) that the time scale of variation of $\Delta n(x)$ and $U(x)$ is slow compared to the time of transit of the free electrons over the distance L. For a thermal distribution of the electrons in the surrounding plasma, the incident velocity distribution along the magnetic field is

$$\mathrm{f}(V_{x0}) = \left(\frac{m_e}{2\,\pi k T_e}\right)^{1/2} \exp\left(-\frac{m_e V_{x0}^2}{2\,kT_e}\right). \tag{2}$$

Under the assumptions of positive and slowly varying potential, no electrons are reflected and their density is a function only of the potential. At the potential U, the electrons which started with velocity V_{x0} have a velocity

$$|V_x(U)| = \sqrt{V_{x0}^2 + \frac{2\,eU}{m_e}}\,. \tag{3}$$

Their density is therefore reduced by a factor $|V_{x0}/V_{x(U}|$. The total density of the free electrons at potential U is given by the integral over all velocities

$$n_{e,free} = n_{e0}\left(\frac{m_e}{2\pi kT_e}\right)^{1/2} \int_{-\infty}^{\infty} \frac{|V_{x0}|}{\left(V_{x0}^2 + \frac{2eU}{m_e}\right)^{1/2}} \exp\left(-\frac{m_e V_{x0}^2}{2kT_e}\right) \mathrm{d}V_{x0}. \tag{4}$$

The solution is found by a series of substitutions (M. A. Raadu, private communication 1989): $a=m_e/(2kT_e)$, $b=(2eU/m_e)^{1/2}$, $t=(V^2+b^2)^{1/2}$, $s=t\,a^{1/2}$, which yield

$$n_{e,free} = n_{e0}\, \mathrm{e}^{\frac{eU}{kT_e}} \left(1 - \mathrm{erf}\sqrt{\frac{eU}{kT_e}}\,\right) \tag{5}$$

where erf is the error function,

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-s^2)\mathrm{d}s \tag{6}$$

To calculate the number of trapped electrons we need to assume that the conditions of positive and slowly growing potential have held during the whole time history of the process we study. Under these assumptions, consider the electrons that at any given time come from the ambient plasma into the region with positive potential. Only the slowest electrons, out of the central part of the Maxwell distribution, have so small energy that they become trapped by the small change in potential during their time of transit. If the potential has grown slowly during the whole trapping process, then all the trapped electrons that are found at a later time have originated in the central (low-energy) part of the ambient electron distribution. From Liouville's theorem follows that in the absence of collisions their velocity space density is unchanged, and therefore equal to the value for low velocities in the ambient plasma:

$$\mathrm{f}(V_x) = \left(\frac{m_e}{2\pi k T_e}\right)^{1/2} . \tag{7}$$

At a potential U it is only possible to have trapped electrons in the velocity range

$$-(2eU/m_e)^{1/2} < V_x < (2eU/m_e)^{1/2}.$$

Electrons with higher velocity would escape. If this velocity range is completely filled, and the velocity space density is given by Eq. (7), then the number of trapped electrons is

$$n_{e,trapped} = n_{e0}\left(\frac{4\,e\,U}{\pi\,kT_e}\right)^{1/2} . \tag{8}$$

This is, strictly, only an *upper limit* to the number of trapped electrons at a potential U. Depending on the time history during the trapping process there could for example be holes in the velocity distribution. However, we believe that Eq (8) gives the actual value for a slowly growing potential which has been positive during the whole trapping process, for the following reasons. Brenning *et al.*, (1989) have made an analytical calculation of the electron trapping out of a thermal plasma into a positive region where the potential grows in a prescribed way, which gave the same result as Eq. (8). This calculation also explained two rather surprising features in Eq. (8), namely that the number of trapped electrons is independent of the rate of change of the potential, and that it is independent of the spatial extent L of the region where the electrons are trapped; these quantities drop out of the result in a natural way. However, this analytical calculation has the drawback that the potential is prescribed, while it should in reality be self-consistently determined by the electron and ion dynamics. This process is better studied by computer simulations of the trapping process. Such simulations have been made by Bohm *et al.*, (1990), for density increases $\Delta n > n_{e0}$. The result agrees with trapping of electrons according to Eq. (8).

The total electron density is

$$n_e = n_{e,free} + n_{e,trapped},$$

where $n_{e,free}$ and $n_{e,trapped}$ are given by Eq:s 5 and 8. This gives the relative density increase

$$\frac{n_e}{n_{e0}} = \left(\frac{4eU}{\pi kT_e}\right)^{1/2} + e^{\frac{eU}{kT_e}}\left(1 - \mathrm{erf}\sqrt{\frac{eU}{kT_e}}\right). \tag{9}$$

Eq. (9) is the sought relation between the density and the potential. It is plotted in Fig. 1 together with the Boltzmann relation, $n_e = n_{e0}\exp(eU/kT_e)$. For density increases below 5% the two solutions closely follow each other, but for larger density increases they rapidly diverge: at 50% density increase, the value of Eq. (9) is twice that of the Boltzmann relation, and for 100% density increase the difference is a factor three. Some values from earlier computer simulations on the process (Bohm *et al.*, 1990) are included in the figure, and agree well with the result obtained here.

3. Summary

We have given a justification for using the Boltzmann relation also for density increases in collisionless magnetized plasma, provided that the density perturbations are small, below 5%. We have also derived a more general relation, Eq. (9), which shows a large deviation from the Boltzmann relation for larger-amplitude density increases. The conditions of applicability of these results is that (1) the density changes so slowly that it varies only little during the transit time of the electrons, and (2) that during the time of density increase, the potential has been positive all through the region. This latter condition excludes the possibility that electrons are reflected before they enter the trapping region; such reflection would increase the potential for a given density increase compared to the value of Eq. (9).

The density increase is here treated as due to a motion of the ions, to which the electrons react in such a way that quasineutrality is maintained. The result does not depend on the mechanism which changes the ion density. It could be for example the ion dynamics in solitons or ion acoustic waves, or density variations which are externally imposed by injected ion beams or plasma clouds. The relation (9) should be applicable where the Boltzmann relation is usually applied.

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References

Brenning, N., Bohm, M., and Fälthammar, C.-G., Dynamic Trapping of Electrons in Space Plasmas, TRITA-EPP-89-07, The Royal Institute of Technology, Department of Plasma Physics, S-100 44 Stockholm.

Bohm, M., Brenning, N., and Fälthammar, C.-G., Dynamic Trapping: Neutralization of Positive Space Charge in a Collisionless Magnetized Plasma, Phys, Rev. Lett. 65, 859-862.

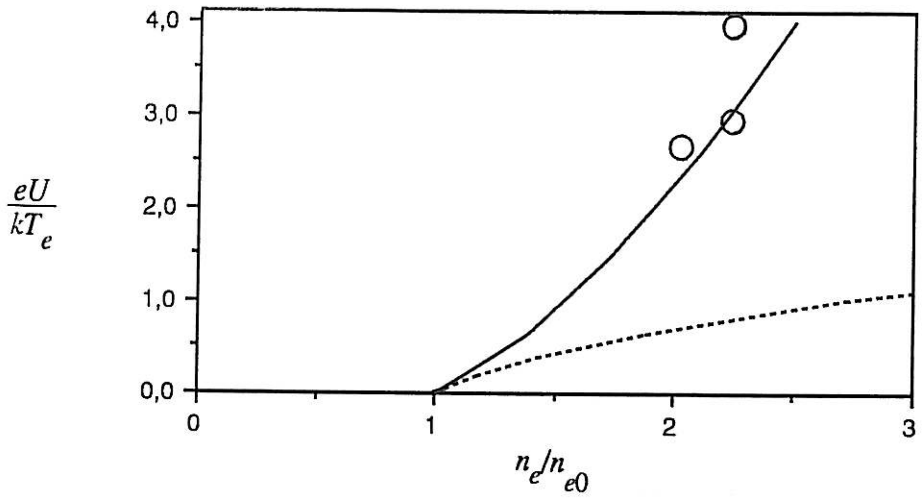

Fig.1a . The relation between the density and the potential according to Eq. 9 (the solid curve), and according to the Boltzmann relation (the dashed curve). The circles denote the computer simulations by Bohm. *et al.*, 1990.

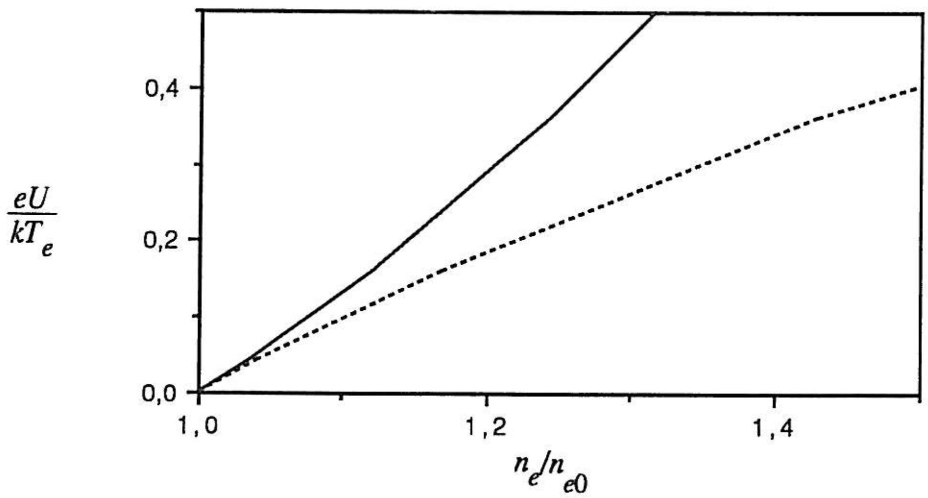

Fig. 1b. An enlargement of the small-amplitude part of Fig. 1a.

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Key words: Electron trapping, Boltzmann relation, Magnetic-field-aligned electric fields.