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THEORY OF TIME VARYING ATMOSPHERIC ELECTRIC FIELDS AND SOME APPLICATIONS TO FIELDS OF IONOSPHERIC ORIGIN

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Abstract. The propagation of slowly varying electric disturbances in the atmosphere is investigated theoretically. The approach used is to solve the continuity equation for the electric current assuming the electric field to be a time varying potential field. An earlier, quite generally quoted result that the temporal variation of the electric field at each point is governed by the local relaxation time is shown to be false, also for slow variations, except for very special geometries. Atmospheric fields of ionospheric origin are studied. A general analytical solution is presented and examples of numerically computed solutions are given. Models are considered where the electric field is a standing or travelling sinusoidal wave with frequencies in the range 0-0.5 Hertz, wavelengths $\geq 3\times 10^4$ m, and velocities 0-3500 m/s. Also considered are fields from moving ionospheric structures that are spatially confined with dimensions in the range 10-200 km. It is concluded that ionospheric field variations give rise to substantial disturbances in the atmospheric electric field, but at low altitudes they are hard to distinguish from disturbances of meteorological origin. It is shown that it is quite feasible to make balloon studies (altitudes 30-40 km) of slowly varying fields of ionospheric origin. However, it is difficult to study ionospheric field variations at frequencies higher than about 0.1 Hz or to resolve spatial structures smaller than about 100 km.
1. INTRODUCTION

The propagation of slowly varying electric disturbances in the atmosphere is investigated theoretically. Special consideration is given to fields originating from potential variations in the lower ionosphere. For static fields this problem has been considered earlier (Atkinson et al., 1971; Volland, 1972) but here the phenomena associated with time varying fields are studied. Disturbances of different frequencies will be damped and phase shifted differently. Thus a particular disturbance of arbitrary form that may be considered to be built up of a number of Fourier components will be damped and distorted when it propagates into the lower atmosphere from the ionospheric source.

Suggestions that the atmospheric electric field is influenced by extraterrestrial phenomena such as the aurora have repeatedly appeared in the literature. Examples are the early reports of Wijkander (1874, 1876) and Andree (1887, 1890) and the more recent work of Freier (1961), Olson (1971), and Mühleisen (1971, 1972), see also the report by Dolezalek (1964). The observations are far from being understood and the explanations proposed include influence on the atmospheric conductivity, direct penetration of charges deep into the atmosphere, and direct electric coupling. The large-scale horizontal electric fields that exist in the ionosphere during times of aurorae and magnetic disturbances will definitely affect the field in the lower atmosphere and may be detected at balloon altitudes and possibly even at the ground level (Boström, 1967; Kellogg and Weed, 1969; Atkinson et al., 1971). The ionosphere is certainly not an equipotential surface and potential differences of several tens of kilovolts may occur between different points of the ionosphere, especially at high latitudes. Mozer and Serlin (1969) and Mozer and Manka (1971) have performed measurements of horizontal electric fields from balloons and correlated these with other phenomena of ionospheric origin.
2. BASIC RELATIONS

2.1. Fundamental equations for the electric field

The basic equations for the electric field in the atmosphere and the associated magnetic field are the two Maxwell equations

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$  \hspace{1cm} (2.1)

$$\frac{1}{\mu_0} \text{curl } \vec{B} = \mathbf{i} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$  \hspace{1cm} (2.2)

We have used Ohm's law to express \( \mathbf{i} \) in terms of \( \vec{E} \) and the atmospheric conductivity \( \sigma \). The conductivity will be assumed to be spatially variable but isotropic and constant in time. As we are only interested in the electric field we want to eliminate \( \vec{B} \). This can be achieved by taking the curl of the first equation and the time derivative of the second and combining the two. This gives the "telegraph" equation

$$\text{curl curl } \vec{E} = -\mu_0 \sigma \frac{\partial \vec{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$  \hspace{1cm} (2.3)

where \( c = (\varepsilon_0 \mu_0)^{-1/2} \) is the velocity of light.

An alternative approach is to take the divergence of (2.2) which gives the continuity equation for the current. Expressed in terms of the electric field this reads

$$\sigma \text{ div } \vec{E} + \frac{\varepsilon_0}{c^2} \text{ grad } \sigma + \varepsilon_0^2 \frac{\partial^2 \vec{E}}{\partial t^2} \text{ div } \vec{E} = 0$$  \hspace{1cm} (2.4)

The latter equation is simpler to work with and will form the basis of our analysis. However, we will also discuss equation (2.3), since some of the earlier studies of atmospheric electric fields have been based on this.

Quite generally the electric field may be split up into two components \( \vec{E}_V \) and \( \vec{E}_A \) related to a scalar potential \( V \) and a vector potential \( \vec{A} \) respectively. We have

$$\vec{E} = \vec{E}_V + \vec{E}_A = \text{ grad } V - \frac{\partial \vec{A}}{\partial t}$$  \hspace{1cm} (2.5)

The vector potential \( \vec{A} \) is also related to \( \vec{B} \) by \( \text{curl } \vec{A} = \vec{B} \). For \( V \) and \( \vec{A} \) to be uniquely defined they should satisfy a further so-called gauge-condition, generally taken to be
\[
\text{div } A + \varepsilon_0 \mu_0 \frac{\partial V}{\partial t} = 0 \tag{2.6}
\]

For the problems with rather slow temporal variations considered in this paper the component of the electric field \( F_A \) is negligibly small compared to \( F_Y \), as will be verified later. Thus, to describe the electric field we only need to find the scalar potential \( \Phi \). It is this fact that makes it possible to work only with the scalar equation (2.4) rather than the vector equation (2.3). However, for the analysis of problems involving high frequency variations equation (2.4) would not suffice.

2.2. Discussion of the use of the "telegraph" equation

If we for a moment assume that the electric field is strictly a potential field, or if we make the somewhat less stringent assumption that curl curl \( \mathbf{E} = 0 \), we find from (2.3) a peculiar result. The only possible form of solution would then be that with a particular exponential time dependence

\[
\mathbf{E} = E_1(x,y,z) \exp \left[ -\sigma(x,y,z)t/\varepsilon_0 \right] + E_2(x,y,z) \tag{2.7}
\]

In this solution the temporal variation of the electric field is at each point in space determined by the local relaxation time \( \varepsilon_0/\sigma(x,y,z) \). For the electrostatic potential the temporal variation is somewhat more complex.

Evidently, the solution (2.7) can only be applied under very special circumstances with sources and boundary conditions matching this particular solution. It can not be the general solution to a problem where the electric field source varies with time in an arbitrary way. For other forms of the time dependence the two terms on the right hand side of (2.3) can not balance and curl curl \( \mathbf{E} \neq 0 \). Thus we conclude that to solve a more general problem we must not assume that the left hand member of (2.3) is zero (which does not exclude that the fields are nearly potential fields). This is so because (2.3) represents three scalar equations of which - loosely speaking - only one can be satisfied by varying the scalar \( \Phi \) alone.
We can obtain an estimate of the magnitude of curl curl $\mathbf{F}$ compared to that of other second order derivatives, such as grad div $\mathbf{F}$, by studying the terms on the right hand side of (2.3). In the atmosphere $\sigma$ varies from about $10^{-13}$ S/m at ground level to roughly $10^{-6}$ S/m at an altitude of about 100 km. For the problems considered the characteristic time scale $T$ is in the range 1 - 1000 seconds. Using the largest term on the right hand side of (2.3) we find $|\text{curl curl } \mathbf{F}| = E/R^2 = E_\varphi/R^2$, where $R = \min \left( \sqrt{T/\mu_0 \sigma}, cT \right) = 10^6$ to $10^{11}$ meters. On the other hand, we know that the atmospheric field typically varies on a scale $L \lesssim 10^4$ meters. Since $|\text{curl curl } \mathbf{F}| = E_\varphi/L^2$ (because $E_\psi$ does not contribute), we then find that $E_\varphi/E_\psi = L^2/R^2 \lesssim 10^{-4}$ to $10^{-14}$. For the problems considered here curl curl $\mathbf{F}$ can be smaller than other second order derivatives, which are of the order $E_\psi/L^2$, by a factor of $10^{-4}$ to $10^{-14}$ and still be important in equation (2.3). However, it is obvious that when we consider equation (2.4) it will be legitimate to approximate $\mathbf{F}$ by $-\text{grad } V$.

Kasemir (1950), in a discussion of the atmospheric electric field, introduced this, in general false, assumption that curl curl $\mathbf{F} = 0$ or $(\partial/\partial t) \text{curl } \mathbf{F} = 0$ (in his notation $\partial \mathbf{F}_n/\partial t = 0$). The conclusion obtained then, that the temporal variation of the electric field at each point is governed by the local relaxation time as indicated by equation (2.7), has been widely quoted. As pointed out above it can only be used under very special circumstances.

For a strictly spherically symmetric problem, e.g. such as discussed by Benndorf (1925, 1927), the electric field can have only a radial component, and curl $\mathbf{F}$ must vanish identically. Hence a solution of the form (2.7) is the only one possible satisfying equations (2.1) and (2.2). Thus it might seem that it would not be possible for the potential between the ionosphere and ground to vary in any other way than the one corresponding to this solution. This is true only as long as we do not introduce in the problem any other means of charge transport between the ionosphere and ground than the ohmic conduction current. However, if some current or voltage generators are operating, the potentials and fields could certainly vary in other ways, even for the strictly spherically symmetric case.
2.3. The continuity equation

Equation (2.3) is difficult to work with, since we have seen that in general we must not neglect the curl curl $\mathbf{E}$ term originating from the left hand member of (2.2). Equation (2.4), where we have been able to eliminate mathematically the influence of this member, is easier to handle. In the study of this equation we may, for not too rapid variations, apply the potential field approximation. That is, we neglect $E_A$ and introduce

$$\mathbf{E} = - \nabla V \quad (2.8)$$

Note that the potential $V$ still may be time dependent giving rise to a non-negligible displacement current. This description of the electric field may be used for our application as long as $E_A \ll E_V$ everywhere. It follows from the previous section that this is fulfilled for the problems considered, where the characteristic time $T \geq 1$ second, even if $L$ is taken to be as large as the distance from the ground to the ionosphere, that is $10^5$ meters. However, for rapid time variations the approximation breaks down. Thus, it is certainly necessary to use non-potential fields if the characteristic time $T$ is short compared to the time it takes for an electromagnetic wave of speed $c$ to propagate a distance $L$, that is, if $T \leq 3 \times 10^{-4}$ seconds.

Introducing the form (2.8) for $\mathbf{E}$ in (2.4) we obtain

$$\sigma \nabla^2 V + \nabla \sigma \cdot \nabla V + \epsilon_0 \frac{\partial}{\partial t} \nabla^2 V = 0 \quad (2.9)$$

We will solve this equation for a few different boundary conditions. The boundary conditions used for the potential are prescribed functions of space and time at a certain altitude in the ionosphere and constant potential at ground level.
It is not obvious that an arbitrary solution to (2.4) or (2.9) really satisfies (2.1) and (2.2). As we are using the approximation that $\mathbf{E}$ is curl-free the class of solutions has not been widened by taking the divergence of (2.2), but (2.1) will not be exactly satisfied. However, once a particular solution has been found, we can use (2.2) to derive $\mathbf{H}$ and then we can show that \( \frac{\partial \mathbf{B}}{\partial t} \ll E/L \) so equation (2.1) is approximately fulfilled by an electric field which is curl-free. Although our solutions do not satisfy Maxwell's equations exactly, the degree of approximation is extremely good.

We may remark here that the exponential form of solution given by (2.7) does satisfy the continuity equation, provided \( \text{div } \sigma \mathbf{E}_2 = 0 \). However, it is not the most general solution, and it cannot be applied here since it does not match the boundary conditions of our problem.

2.4. The conductivity
The atmospheric conductivity is assumed to have an exponential variation with the altitude coordinate $z$

$$\sigma = \sigma_0 \exp(z/H)$$

(2.10)

with the conductivity at the ground $\sigma_0 = 9 \times 10^{-14}$ S/m and a scale height $H = 7$ km. This makes a reasonable fit for example to the data given by Bourdeau et al. (1959), Cole and Pierce (1965), and Sagalyn (1965). For a total potential drop between the earth and ionosphere of 275 kvolt this conductivity would give an air-earth current density of $3.5 \times 10^{-12}$ amp/m$^2$, which seems reasonable. At high altitudes the conductivity probably increases somewhat faster than this expression would indicate. Above 70 km the earth's magnetic field affects the conductivity which no longer will be isotropic, but reduced in a direction transverse to the magnetic field. At high latitudes, where the magnetic field lines are nearly vertical, the effect will be to improve the propagation downwards of the ionospheric fields. This may readily be taken into account by introducing a modified height coordinate (Farley, 1959). We will neglect this effect since the damping and phase shift of the disturbances
of ionospheric origin in any case are small in these uppermost layers. We also neglect the deviations from an exponential variation of the conductivity that occur in the very lowest layers of the atmosphere.

2.5. Conduction and displacement current
With the form (2.10) for the variation of $\sigma$ equation (2.7) gives

$$\sigma v^2 \nabla^2 v + \frac{\sigma}{H} \frac{\partial v}{\partial z} + \varepsilon_0 \frac{\partial}{\partial t} \nabla^2 v = 0 \quad (2.11)$$

Of the terms in (2.9) and (2.11) those proportional to $\sigma$ stem from the conduction current while those proportional to $\varepsilon_0$ stem from the displacement current. Depending on the angular frequency $\omega$ of a presumed sinusoidal temporal variation and the conductivity (or altitude) one or the other of these currents may dominate. For a period of oscillation $2\pi/\omega$ much longer than the relaxation time $\varepsilon_0/\sigma$ the conduction current dominates. Thus, for $\omega<10^{-2}$ s$^{-1}$ the conduction current dominates at all altitudes. The solutions will then be similar to the static solution, that is, the damping of the fields will be the same and no appreciable phase shifts occur between the waves at different altitudes. Note that the solution for this case depends only on the scale height $H$ and not on the absolute magnitude of $\sigma$. For somewhat higher frequencies the conduction current dominates at high altitudes and the displacement current at low altitudes. For $\omega>10^5$ s$^{-1}$ the displacement current dominates at all altitudes below the ionosphere and the atmosphere acts as a capacitive rather than a resistive medium. Before this limit is reached, however, our method of analysis is no longer valid since the electric field is no longer curl-free.

The equation (2.11) has been used also in studies of the electric field from thunderclouds, for example by Anderson and Freier (1969), Mann (1970), and Illingworth (1972) who investigated solutions with an exponential time dependence.
3. THE MODEL

3.1. Two fundamental types of solutions

We will assume that the ionospheric potential is given for a particular source altitude taken to be 112 km. This is the height where the horizontal conductivity in the ionosphere is at its maximum and the horizontal currents are most intense. Some physical mechanism in the ionosphere imposing potential variations is thus assumed. Two fundamental types of solutions will be investigated. In these the variations of the potential at this upper boundary is assumed to be given by a standing or travelling sinusoidal wave. Since equation (2.11) is linear in $V$ solutions for waves of other forms may be obtained by Fourier superposition as will be shown for a particular example. At all lower altitudes the potential field will then be a wave of the same frequency and wavelength, but with a different amplitude and phase. This disturbance of the atmospheric potential field should of course be superposed on the ordinary static field.

We will study solutions which are functions of only one horizontal coordinate $x$, the vertical coordinate $z$, and time $t$. Three dimensional fields may be analyzed in a completely analogous fashion. The two fundamental types of solutions to be investigated are (the superscript ' denotes differentiation with respect to $z$):

a) standing sinusoidal waves

$$V(x, z, t) = U(z) \cos kx \sin \omega t + W(z) \cos kx \cos \omega t =$$
$$= \sqrt{U'^2 + W'^2} \cdot \sin(\omega t + \arctan \frac{W'}{U'}) \cos kx$$

$$(3.1)$$

$$E_x = -\frac{\partial V}{\partial x} = k \sqrt{U'^2 + W'^2} \cdot \sin(\omega t + \arctan \frac{W'}{U'}) \sin kx$$

$$(3.2)$$

$$E_z = -\frac{\partial V}{\partial z} = -\sqrt{U'^2 + W'^2} \sin (\omega t + \arctan \frac{W'}{U'}) \cos kx$$

$$(3.3)$$
b) travelling sinusoidal waves

\[ V(x,z,t) = U(z) \cos(kx-\omega t) + W(z) \sin(kx-\omega t) = \sqrt{U^2 + W^2} \cdot \cos(kx-\omega t - \arctan \frac{W}{U}) \]  
\[ (3.4) \]

\[ E_x = -\frac{\partial V}{\partial x} = k \sqrt{U^2 + W^2} \cdot \sin(kx-\omega t - \arctan \frac{W}{U}) \]  
\[ (3.5) \]

\[ E_z = -\frac{\partial V}{\partial z} = -\sqrt{U^2 + W^2} \cos(kx-\omega t - \arctan \frac{W}{U}) \]  
\[ (3.6) \]

In the last case \( \omega \) and \( k \) are related by

\[ \omega = k \cdot v \]  
\[ (3.7) \]

where \( v \) is the velocity of the wave. The wavelength \( \lambda \) is related to \( k \) by \( \lambda = 2\pi/k \) and the period of oscillation \( T \) is related to \( \omega \) by \( T = 2\pi/\omega \)

Introducing (3.1) or (3.4) in (2.11) we obtain for both types of waves two linear coupled differential equations for \( U(z) \) and \( W(z) \)

\[ \sigma U'' + \frac{\sigma}{\mu} U' - \sigma k^2 U - \varepsilon_0 \omega (W'' - k^2 W) = 0 \]  
\[ (3.8) \]

\[ \sigma W'' + \frac{\sigma}{\mu} W' - \sigma k^2 W + \varepsilon_0 \omega (U'' - k^2 U) = 0 \]  
\[ (3.9) \]

3.2. Boundary conditions
The boundary conditions for the system of equations (3.8) and (3.9) are:

i) at the surface of the earth \( (z=0) \)

\[ U(0) = W(0) = 0 \]  
\[ (3.10) \]

This condition will make the potential \( V \) and the horizontal electric field \( E_x \) zero at the ground \( (z=0) \)
3. THE MODEL

3.1. Two fundamental types of solutions
We will assume that the ionospheric potential is given for a particular source altitude taken to be 112 km. This is the height where the horizontal conductivity in the ionosphere is at its maximum and the horizontal currents are most intense. Some physical mechanism in the ionosphere imposing potential variations is thus assumed. Two fundamental types of solutions will be investigated. In these the variations of the potential at this upper boundary is assumed to be given by a standing or travelling sinusoidal wave. Since equation (2.11) is linear in V solutions for waves of other forms may be obtained by Fourier superposition as will be shown for a particular example. At all lower altitudes the potential field will then be a wave of the same frequency and wavelength, but with a different amplitude and phase. This disturbance of the atmospheric potential field should of course be superposed on the ordinary static field.

We will study solutions which are functions of only one horizontal coordinate x, the vertical coordinate z, and time t. Three dimensional fields may be analyzed in a completely analogous fashion. The two fundamental types of solutions to be investigated are (the superscript ' denotes differentiation with respect to z):

a) standing sinusoidal waves

\[ V(x, z, t) = U(z) \cos kx \sin \omega t + W(z) \cos kx \cos \omega t = \]
\[ = \sqrt{U^2 + W^2} \cdot \sin(\omega t + \arctan \frac{W}{U}) \cos kx \]
\[ (3.1) \]

\[ E_x = -\frac{\partial V}{\partial x} = k \sqrt{U^2 + W^2} \cdot \sin(\omega t + \arctan \frac{W}{U}) \sin kx \]
\[ (3.2) \]

\[ E_z = -\frac{\partial V}{\partial z} = -\sqrt{U'^2 + W'^2} \sin (\omega t + \arctan \frac{W'}{U'}) \cos kx \]
\[ (3.3) \]
b) travelling sinusoidal waves

\[ V(x, z, t) = U(z) \cos(kx - \omega t) + W(z) \sin(kx - \omega t) = \]
\[ \frac{1}{\sqrt{U^2 + W^2}} \cdot \cos(kx - \omega t - \arctan \frac{W}{U}) \] (3.4)

\[ E_x = -\frac{3V}{dx} = k \sqrt{U^2 + W^2} \cdot \sin(kx - \omega t - \arctan \frac{W}{U}) \] (3.5)

\[ E_z = -\frac{3V}{dz} = -\sqrt{U'^2 + W'^2} \cos(kx - \omega t - \arctan \frac{W'}{U'}) \] (3.6)

In the last case \( \omega \) and \( k \) are related by

\[ \omega = k \cdot v \] (3.7)

where \( v \) is the velocity of the wave. The wavelength \( \lambda \) is related to \( k \) by \( \lambda = 2\pi/k \) and the period of oscillation \( T \) is related to \( \omega \) by \( T = 2\pi/\omega \).

Introducing (3.1) or (3.4) in (2.11) we obtain for both types of waves two linear coupled differential equations for \( U(z) \) and \( W(z) \)

\[ \sigma U'' + \frac{q}{h} U' - \sigma k^2 U - \epsilon_0 \omega (W'' - k^2 W) = 0 \] (3.8)

\[ \sigma W'' + \frac{q}{h} W' - \sigma k^2 W + \epsilon_0 \omega (U'' - k^2 U) = 0 \] (3.9)

3.2. Boundary conditions

The boundary conditions for the system of equations (3.8) and (3.9) are:

i) at the surface of the earth \((z=0)\)

\[ U(0) = W(0) = 0 \] (3.10)

This condition will make the potential \( V \) and the horizontal electric field \( E_x \) zero at the ground \((z=0)\)
ii) at an altitude \( z = a \) (taken to be 112 km) in the ionosphere

\[
U(a) = 1 \tag{3.11}
\]

\[
W(a) = 0 \tag{3.12}
\]

That means that in the ionosphere disturbances are introduced:

a) \( V(x, a, t) = \cos kx \sin \omega t \tag{3.15} \)

or

b) \( V(x, a, t) = \cos (kx - \omega t) \tag{3.14} \)

As the equations are linear, a change in amplitude of the disturbance at the upper boundary by a certain factor will result in the same solutions as before multiplied by this factor. Thus the potential variations are normalized to unit amplitude at the upper boundary.

3.3. Field from a moving confined ionospheric source

We will also investigate a third kind of solution and that is the field from a moving ionospheric source of limited spatial extent. We will still consider only two dimensions so the source will be limited only in the \( x \) direction but not in the \( y \) direction.

Ionospheric electric field structures of this nature may be associated with field perturbations in the auroral zone caused by auroral arcs and auroral electrojets. These are known to be extended in the east-west direction but have a limited north-south extent. In the visual aurora the north-south dimensions may vary from less than 100 m for small scale structures to more than 100 km for systems of auroral arcs. In the model analyzed here a width of the structure of 10-200 km is studied, since this is the scale size that possibly could be resolved by measurements in the lower atmosphere. The visual auroral structures are observed to move in the north-south direction with velocities which typically are of the order of some hundred meters per second, but during auroral breakups they may be of the order a few kilometers per second.

In the source region \( (z = 112 \text{ km}, |x| < b/2) \) the horizontal ionospheric electric field is assumed to have a constant value.
and to be zero outside of this region, see Figure 1a. To make Fourier analysis easy the structure is continued periodically (Figure 1b) with a period L so long that the influence of the neighbouring structures is negligible. This means that in the solution the potential and electric field should in between the structures approach values obtained from a constant potential at the upper boundary.

A coordinate system moving with the velocity \( v \) is introduced. For the coordinate \( s \) the following relations hold

\[
\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial x^2}
\]

\[
\frac{\partial}{\partial t} = -v \frac{\partial}{\partial s}
\]

The periodical structure mentioned above can then be written as a Fourier sum

\[
E_x = E_s = E_0 \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin[(2n+1)\frac{b}{L}] \cos[(2n+1)\frac{2\pi}{L} s]
\]

This electric field corresponds to a potential distribution

\[
V = -E_0 \sum_{n=0}^{\infty} \frac{2L}{(2n+1)^2\pi^2} \sin[(2n+1)\frac{b}{L}]\sin[(2n+1)\frac{2\pi}{L} s]
\]

shown in Figure 1c.

With this potential as a boundary condition the procedure developed and discussed in the preceding sections can be used.

The constant \( E_0 \) is in all cases chosen to be \( 2/b \). With this choice the potential difference across each structure will be 2 volts. In the preceding section the peak-to-peak value of the potential was also 2 volts.
In the numerical evaluation the number of terms used in the
Fourier series has been chosen such that for the last term
the value of \( k \) is around \( 2 \times 10^{-4} \). The contribution from omitted
terms will be negligible because they are multiplied by a
smaller coefficient and, more important, since they are rapidly
damped as will be seen in Section 7.3.

The results of the numerical computations for all three kinds
of models are given in Section 7.

4. SOLUTIONS FOR SOME SPECIAL CASES

4.1. Time independent solutions with horizontal variations

For \( w = 0 \) but \( k > 0 \) the equations (3.8) and (3.9) reduce to

\[
U'' + H^{-1}U' - k^2 U = 0
\]  
(4.1)

\[
W'' + H^{-1}W' - k^2 W = 0
\]  
(4.2)

Taking the boundary conditions into account the solution is

\[
U = A(e^{r_1 z} - e^{-r_2 z})
\]  
(4.3)

\[
W = 0
\]  
(4.4)

where

\[
r_1 = \left(\sqrt{1 + 4k^2H^2} - 1\right)/2H
\]  
(4.5)

\[
r_2 = \left(\sqrt{1 + 4k^2H^2} + 1\right)/2H
\]  
(4.6)

A is determined by the value of \( U \) at the upper boundary, so
the final solution is
\[ V = \frac{e^{r_1 z} - e^{-r_2 z}}{e^{r_1 a} - e^{-r_2 a}} \cos kx = (e^{r_1 z} - e^{-r_2 z})e^{-r_1 a} \cos kx \] (4.7)

since \( r_2 a \gg 1 \).

4.2. Time dependent solutions with no horizontal variations

For \( k = 0 \) but \( \omega > 0 \) the equations (3.8) and (3.9) reduce to

\[ \sigma U'' + \frac{\sigma}{R} U' - \varepsilon_\omega \omega W'' = 0 \] (4.8)
\[ \sigma W'' + \frac{\sigma}{R} W' + \varepsilon_\omega \omega U'' = 0 \] (4.9)

The solution is

\[ U = A \arctan \frac{\sigma}{\varepsilon_\omega} + B \ln \frac{\sigma}{\sqrt{\sigma^2 + \varepsilon_\omega^2}} + C \] (4.10)
\[ W = B \arctan \frac{\sigma}{\varepsilon_\omega} - A \ln \frac{\sigma}{\sqrt{\sigma^2 + \varepsilon_\omega^2}} + D \] (4.11)

with \( \sigma \) given by (2.10)

\( A, B, C, \) and \( D \) are determined by the boundary conditions, giving

\[ V = \sqrt{(\arctan \frac{\sigma}{\varepsilon_\omega} - \arctan \frac{\sigma}{\varepsilon_\omega})^2 + (\frac{z + \frac{1}{2} \ln \frac{\sigma^2 + \varepsilon_\omega^2}{\varepsilon_\omega^2}}{\sqrt{\sigma^2 + \varepsilon_\omega^2}})^2 \sin (\omega t + \phi)} \] (4.12)

with

\[ \phi = \arctan \frac{a + \frac{1}{2} \ln \frac{\sigma^2 + \varepsilon_\omega^2}{\varepsilon_\omega^2}}{\arctan \frac{\sigma}{\varepsilon_\omega} - \arctan \frac{\sigma}{\varepsilon_\omega}} - \arctan \frac{\frac{z + \frac{1}{2} \ln \frac{\sigma^2 + \varepsilon_\omega^2}{\varepsilon_\omega^2}}{\sqrt{\sigma^2 + \varepsilon_\omega^2}}}{\arctan \frac{\sigma}{\varepsilon_\omega} - \arctan \frac{\sigma}{\varepsilon_\omega}} \] (4.13)

\[ \sigma_a = \sigma_0 \exp (a/H) \] (4.14)
5. GENERAL ANALYTICAL SOLUTION

5.1. Introduction
A general solution to equation (2.11) may be derived by separation of variables, and the solution is expressible in terms of hypergeometric functions as shown by Mann (1970). The analytical solution is best expressed in terms of a complex variable $\tilde{V}$ instead of the real potential $V$. For our applications we assume a solution of the form

a) for standing waves

$$\tilde{V} = Y(z) \cos kx e^{-j\omega t} \quad (5.1)$$

b) for travelling waves

$$\tilde{V} = Y(z) e^{j(kx-\omega t)} \quad (5.2)$$

Here $z, x, \text{ and } t$ are real variables as before, $j$ is the imaginary unit, and $Y$ is a complex quantity. Inserted into equation (2.11) both expressions yield the same result ($\frac{d}{dz}$)

$$(\omega - j\omega_0)(Y'' - kY) + \frac{\sigma}{H} Y' = 0 \quad (5.3)$$

The method of solving this equation is given in Section 5.3.

5.2. Physical interpretation of the solution

a) travelling waves

The boundary conditions are in this case (cf. Section 3.2)

$$z = 0 \quad \tilde{V} = 0 \quad (5.4)$$
$$z = a \quad \tilde{V} = \cos (kx-\omega t) \quad (5.5)$$

When $\tilde{V}$ has been found the physical situation is studied by analysing the real part of $\tilde{V}$

$$V = \text{Re} \tilde{V} = \text{Re} Y(z) \cos(kx-\omega t) - \text{Im} Y(z) \sin(kx-\omega t) \quad (5.6)$$
It is seen from this expression that the boundary condition for equation (5.3) can be written
\[
\begin{align*}
    z & = 0 \quad Y = 0 \\
    z & = a \quad Y = 1
\end{align*}
\] (5.7) (5.8)

Expression (5.6) is analyzed in terms of amplitude and phase functions as is described in Section 3.2.

b) standing waves
These waves are analyzed in a manner completely analogous to the one described above.

5.3. The solution of equation (5.3)
A change of variable is performed in equation (5.3)
\[
    \xi = \frac{\omega c}{j_0} e^{-\frac{z}{H}} \quad \omega \neq 0
\] (5.9)
The new differential equation is
\[
    \xi^2 (1-\xi) \frac{d^2 Y}{d\xi^2} - \xi^2 \frac{dY}{d\xi} - (1-\xi) \beta^2 Y = 0
\] (5.10)
where \( \beta = k H \).

With the substitution
\[
    Y(\xi) = \xi^\alpha W(\xi)
\] (5.11)
equation (5.10) can be transformed to
\[
    \xi (1-\xi) \frac{d^2 W}{d\xi^2} + \left[ 2\alpha - (2\alpha + 1)\xi \right] \frac{dW}{d\xi} - \alpha W = 0
\] (5.12)
with \( \alpha \) given by
\[
    \alpha = 0.5 \left[ 1 - \sqrt{1 + 4\beta^2} \right]
\] (5.13)
The solutions of equation (5.12) are the hypergeometric functions
\( W_1(\xi) = F(a,b;c;\xi) \)

\( W_2(\xi) = \xi^{1-c}F(a-c+1, b-c+1; 2-c;\xi) \)

where \( a = \alpha + \beta, \ b = \alpha - \beta \) and \( c = 2a \)

The hypergeometric function is defined as a series

\[
F(a,b;c;\xi) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \xi^n
\]

(5.16)

where \((a)_n = 1, (a)_n = a(a+1) \ldots (a+n-1)\)

and \( c \neq -m, m = 0, 1, 2 \ldots \).

For \( c \neq m \)

\[
\lim_{r \to (c)} \frac{1}{\Gamma(c)} F(a,b;c;\xi) = \frac{(a)_{m+1}(b)_{m+1}}{(m+1)!} \xi^{m+1}F(a+m+1, b+m+1; m+2;\xi)
\]

(5.17)

The series (5.16) converges only for \(|\xi| < 1\). As the range of our variable is \(|\xi| < \infty\) another set of solutions is needed.

Such solutions are

\( W_1(\xi) = (1-\xi)^{-a}F(a,c-b;b;\xi) \)

(5.18)

\( W_2(\xi) = \xi^{1-c} (1-\xi)^{c-a-1}F(a-c+1,1-b; 2-c;\xi) \)

(5.19)

The set of solutions \( Y_1 \) and \( Y_2 \) are constructed from \( W_1, W_2 \) and relation (5.11). The complete solution of equation (5.3) is

\[
Y(z) = AY_1(z) + BY_2(z)
\]

(5.20)

where \( A \) and \( B \) are constants which are determined by the boundary conditions (5.7) and (5.8). The last step is to form the expression (5.1) or (5.2) for \( V \).

The electric field is obtained from

\[
E_y = -\nabla V
\]

(5.21)

with the help of the relation
\[
\frac{d}{dz} F(a,b;c;z) = \frac{a b}{c} F(a+1,b+1;c+1;z) \tag{5.22}
\]

5.4. Numerical calculations

The advantage of this analytical solution is less than might be expected because of the complexity of the hypergeometric functions. Computer calculations are needed to determine the constants A and B and reveal the behaviour of the solutions.

A large number of terms in the series for the hypergeometric function must be used to reach a reasonable accuracy, especially at low altitudes where as much as 1000 terms have been found insufficient.

Another drawback is the difficulty to make a program that can handle all values of the parameters. The hypergeometric functions chosen here are not valid for certain values of the parameter c as seen in Section 5.3. It is difficult to keep the numerical error small when the parameter c is close to these values.

For these reasons it is easier to make a direct numerical integration of equations (3.8) and (3.9). This method is described in Section 6.

6. THE NUMERICAL SOLUTION

6.1. The method

The differential equations (3.8) and (3.9) can be written

\[
U'' = k^2 U - \frac{\sigma/H}{\sigma^2 + \varepsilon_0^2 \omega^2} (\sigma U' + \varepsilon_0 \omega W') \tag{6.1}
\]

\[
W'' = k^2 W - \frac{\sigma/H}{\sigma^2 + \varepsilon_0^2 \omega^2} (\sigma W' - \varepsilon_0 \omega U') \tag{6.2}
\]

By introducing two new variables

\[
R = U' \tag{6.3}
\]

\[
S = W' \tag{6.4}
\]

four first order differential equations, suitable for numerical integration, are obtained
\[ R' = k^2 U - \frac{\sigma/\mu}{\sigma^2 + \sigma_0^2} (\sigma R + \sigma_0 \omega S) \quad \text{(6.5)} \]

\[ U' = S \quad \text{(6.6)} \]

\[ S' = k^2 W - \frac{\sigma/\mu}{\sigma^2 + \sigma_0^2} (\sigma S - \sigma_0 \omega R) \quad \text{(6.7)} \]

\[ W' = S \quad \text{(6.8)} \]

For the numerical integration "initial" values for \( R \) and \( S \) (at \( z=0 \)) are desirable instead of values for \( U \) and \( W \) at the upper boundary (\( z=a \)). For that reason the problem is transformed into an initial value problem. The values of \( U \) and \( W \) at the upper boundary are linear combinations of the derivatives \( U' \) and \( W' \) at the ground.

\[ U(z=a) = \beta U'(z=0) + \gamma W'(z=0) \quad \text{(6.9)} \]

\[ W(z=a) = \gamma U'(z=0) + \delta W'(z=0) \quad \text{(6.10)} \]

One solution with initial values at \( z=0 \),

\[ U = 0; \quad W = 0; \quad U' = 1; \quad W' = 0 \]

gives

\[ \alpha = U(a)/U'(a) = U(a) \quad \text{(6.11)} \]

\[ \gamma = W(a) \quad \text{(6.12)} \]

Similarly a solution with initial values

\[ U = 0; \quad W = 0; \quad U' = 0; \quad W' = 1 \]

determines \( \beta \) and \( \delta \).

It can be seen from the differential equations that

\[ \beta = \gamma; \quad \delta = \alpha \quad \text{(6.13)} \]
When $\alpha, \beta, \gamma$, and $\delta$ are known, the appropriate initial values for $U'$ and $W'$ that give the desired condition at the upper boundary, are found from the system

\begin{align}
1 & = \alpha U' + \beta W' \\
0 & = \gamma U' + \delta W'
\end{align} \tag{6.14, 6.15}

Numerical integrations of (6.5) - (6.8) have been made using Hammings modified predictor-corrector method for the solution of general initial value problems.

7. RESULTS

7.1. The time independent case
Using the expressions (4.3) and (4.4) for $U$ and $W$, relations (3.4) to (3.6) show that the amplitude of $V, E_z$, and $E_x$ are represented by $U, U'$, and $kU$. These quantities are shown for some values of $k$ in Figures 2, 3, and 4. There it can be seen that small scale horizontal structures (large $k$) are damped very fast. The increase of $U'$ (that is $E_z$) at higher altitudes for $k>0$ is a consequence of current continuity, because $k>0$ means that a horizontal current ($-\sigma E_x$) is present. Furthermore this current increases with both $k$ and altitude as is seen in Figure 4, which shows $kU$ (that is $E_x$).

7.2. The case of no horizontal variation
The amplitudes of the potential and vertical field variations are shown for different $\omega$ in Figures 5 and 6. Figures 7 and 8 give corresponding phase shifts. The discussion of these is included in the discussion of the general case.

7.3. General solutions. Standing waves.
The discussion of the physical situation is in terms of amplitude and phase functions, as given in equations (3.1) - (3.3). For standing waves $k$ and $\omega$ vary independently.
Signal amplitude

Curves of signal amplitudes versus $z$ with $k$ and $\omega$ as parameters are shown in Figures 9-20. We can distinguish between two separate regions of simple behaviour.

a) At altitudes so high that the inequality

$$\sigma > \varepsilon_0 \omega$$  \hspace{1cm} (7.1)

is valid the complete equations are reduced to

$$U'' + H^{-1}U' - k^2 U = 0$$  \hspace{1cm} (7.2)
$$W'' + H^{-1}W' - k^2 W = 0$$  \hspace{1cm} (7.3)

These are the equations for the time independent case with the solutions for different $k$ in Figures 2, 3, and 4. The curves for $\omega \neq 0$ will therefore asymptotically approach these above a certain altitude which depends on the value of $\omega$. Figures 18-20 show this behaviour very clearly. The altitude above which this approximation is valid varies from about 50 km for $\omega = 0.03$ to about 80 km for $\omega = 3$. The potential is almost constant in the height interval under consideration if there is no horizontal variation ($k=0$) as is seen in Figures 5, 9, 12, and 15. $E_z$, shown in Figures 6, 10, 13, and 16, is then close to zero. However, if a horizontal structure is introduced ($k \neq 0$) a horizontal electric field, shown in Figures 11, 14, and 17, will exist and its magnitude will increase with height. A current will then flow in the $x$-direction. Because of current continuity there must be a vertical electric field, shown in Figures 10, 13, and 16, which increases with both altitude and the value of $k$.

b) For altitudes so low that (provided $\omega$ is not too small)

$$\sigma > \varepsilon_0 \omega$$  \hspace{1cm} (7.4)

the differential equations are reduced to

$$U'' = k^2 U$$  \hspace{1cm} (7.5)
$$W'' = k^2 W$$  \hspace{1cm} (7.6)
Because of the low conductivity in this height interval the displacement current is dominating while the conduction current is small. The faster the time variation of the potential and the fields the lower is the field intensity at these heights see Figures 18-20. For \( k = 0 \) one obtains the equations

\[
U'' = 0 \quad W'' = 0 \quad (7.7)
\]

which show that in this height interval the amplitude of \( E_z \) (Figure 10, 13, and 16) is constant and the amplitude of the potential (Figure 9, 12, and 15) is a linear function of \( z \). A variation in the x-direction \( (k \neq 0) \) gives according to the equations (7.5) and (7.6) an exponential variation of both potential and fields. Letting \( k \neq 0 \) is the same as introducing a horizontal displacement current at these altitudes. This current will increase with height. Because of current continuity the vertical field then also has to increase with height. At the upper level of this height interval the conduction current begins to influence the results.

In the transition region between the two already considered height intervals the complete differential equations must be used and the behaviour of the solutions is more complex as can be seen from the figures.

**Signal phase**

The phase of potential and horizontal electric field variations (arctan \( W/U \)) and vertical electric field variations (arctan \( W'/U' \)) versus altitude are shown in Figures 21-26. As can be expected the phase lag of the potential and horizontal field increases with decreasing altitude. At high altitudes the phase lag is small because the conduction current is dominating. In the transition region the phase lag rapidly increases when the displacement current becomes comparable to the conduction current. At low altitudes the displacement current dominates and the result is an almost constant phase lag. As can be seen in the figures, the constant phase region extends approximately up to the altitude where \( \varepsilon_0 \omega = \sigma \), that is to 8, 24, and 40 km for \( \omega = 0.03, 0.3 \), and 3 respectively. The phase curves
for the vertical field $E_z$ (Figures 22, 24, and 26) are more complicated. For $k=0$ (no horizontal variation) it can be seen that the phase shift from the upper boundary to the ground is close to $\pi/2$. This is understandable since the nature of the current changes from being a conduction current high up to mainly a displacement current at low altitudes. When a horizontal structure is introduced the phase of $E_z$ becomes closely coupled to the phase of $V$ and $E_x$ at low and high altitudes. In the transition region the behaviour is once again complex.

7.4. General solutions. Travelling waves.

The solutions are easiest described in a coordinate system moving along with the waves. The new coordinate is

$$s = x - vt$$ (7.8)

At each altitude the solutions will vary with $s$ as $\cos ks$ and $\sin ks$. For given $\omega$ and $k$ the amplitude variations versus altitude are identical to those of the standing waves treated before. However, now $k$ and $\omega$ are related by (3.7), $\omega = kv$, so for given $v$ sets of curves like Figures 27-29 are obtained.

The corresponding phase curves are shown in Figures 30-31 and, for a higher value of $v$, in Figures 32-33. For small $k$ also $\omega$ is small and the phase shift tends to zero. When $k$ is large (small wave length) the coupling in the horizontal direction is good leading to increased phase shift. An overall view of the phase lag can easiest be obtained from the equipotential diagrams in Figures 34 and 35.

7.5. Moving confined structure.

By Fourier superposition of the preceding solutions as described in Section 3.3, the potential and fields associated with a confined moving ionospheric structure have been calculated. Results for a velocity of 350 m/s and various widths, $b$, of the structure are given in Figures 36-46. The potential drop across the structure is normalized to 2 volts.
It can be seen that the potential and fields lag the ionospheric structure considerably only at low altitudes. At altitudes below 50 km the width of the region with a considerable horizontal field is of the order of 100-200 km, practically independent of the width of the ionospheric structure if this is smaller than about 100 km. (Figures 37-40).

The behaviour of $E_z$ (Figures 41-46) is somewhat complex. The delay in the electric field change at low altitudes is very pronounced. At higher altitudes peaks occur in the vertical field before and after the structure. A special study has shown that similar peaks occur also at both sides of a stationary structure. The effect of the motion of the structure is to decrease the leading peak and increase the trailing one.

At high altitudes the vertical fields are generally weaker than the horizontal ones, at low altitudes the opposite is true. At low altitudes the vertical fields increase in strength as a consequence of the vertical currents flowing to the conducting earth through an atmosphere of low conductivity.

8. DISCUSSION
It has been demonstrated above that stationary or slowly varying ionospheric electric fields may penetrate deep into the atmosphere with small attenuation. This may be of importance for the physics of the atmosphere since it may be part of a mechanism of coupling from the ionosphere to the atmosphere. Whereas the ionospheric electric fields are generally small (typically 1-100 mV/m) compared to fields due to meteorological processes the potential differences due to large-scale ionospheric fields are often considerable and may well be of importance for atmospheric processes. Electric field measurements on satellites in polar orbits generally see large latitudinal potential variations (Cauffman and Burnett, 1972) and often a potential maximum above the auroral zone, sometimes of the order of 50 kV. Similarly,
the polar cap electric field is seen to vary much between different passes. The development of a 30 mV/m electric field across the entire polar cap, which has sometimes been seen, would for instance produce ionospheric potential variations of 50-100 kV. It is obvious that potential variations of this magnitude will influence the electric conditions in the atmosphere.

On the other hand, the variations of electric field strength at the earth's surface due to typical confined ionospheric disturbances are generally quite small. Even very large-scale ionospheric field variations will in general be unable to reverse the potential gradient at ground level. For this reason it seems impossible to explain the electric field reversals observed by Olson (1971) as due to direct downward propagation of auroral electric fields. This does of course not exclude the possible existence of some other coupling mechanism, e.g. due to energetic particles or X-rays penetrating into the atmosphere during auroral events.

The almost unattenuated propagation into the stratosphere of slowly varying, large-scale ionospheric electric fields is of practical importance since it offers a convenient method of measuring the ionospheric field with balloon-borne instruments (Kellogg and Weed, 1969; Mozer and Serlin, 1969; Mozer and Manka, 1971)

The attenuation of d.c. electric fields from various ionospheric structures has been investigated in a previous publication (Atkinson et al., 1971). The minimum balloon altitude for successful measurements of a large-scale ionospheric electric field was found to be of the order of 30 km, determined almost exclusively by the need of getting sufficiently far above disturbing potential sources of meteorological origin. A cut-off was found for fields from ionospheric structures of dimensions smaller than the approximate balloon-ionosphere distance. The present calculations give the same spatial cutoff also for
time-varying structures if the frequency is sufficiently low, as can be seen by comparing Figure 4 and Figure 11. However, for frequencies of the order of the atmospheric relaxation frequency $\sigma/\omega_o$ at the balloon altitude (about 0.2 Hz at 35 km), or higher, also fields from larger structures are more attenuated, see Figures 14 and 17.

Fields from a moving localized ionospheric disturbance have been found to map down to balloon altitudes with only a small phase-lag. When a localized disturbance passes above the balloon the horizontal field variation depends mainly upon the potential difference across the disturbance and little upon its detailed structure as long as this structure is not wider than the balloon-ionosphere distance as shown by Figures 37 to 40. Accurate information about ionospheric field structures from measurements of balloon-altitude vertical fields is even harder to get since strong vertical fields of atmospheric origin are present much more often than horizontal ones (Atkinson et al., 1971).

As a consequence of these results balloon-borne measurements of horizontal electric fields at altitudes of 30–40 km are very useful for a study of large-scale, time-averaged ionospheric fields. Quantitative measurements of fields from ionospheric structures smaller than about 100 km and time variations more rapid than about 0.1 Hz have to be done in other ways, however, e.g. by in situ measurements with rocket-borne double probes.
9. REFERENCES


Atkinson, W., S. Lundquist, and U. Fahlen, 1971: The electric field existing at stratospheric elevations as determined by tropospheric and ionospheric boundary conditions, Pure and Appl. Geophys., 84, 46.


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FIGURE CAPTIONS

Figure 1  Boundary conditions at the ionosphere to simulate the influence of a moving confined electric structure.
           a) Normalized horizontal electric field at the ionospheric boundary.
           b) Modified boundary condition to simplify analysis.
           c) Normalized electric potential corresponding to the field in b).

Figure 2  Normalized amplitude of electrostatic potential in the atmosphere due to a time-independent source at the ionospheric boundary, $V=\cos kx$.

Figure 3  Same as Figure 2 but for vertical electrical field.

Figure 4  Same as Figure 2 but for horizontal electric field.

Figure 5  Normalized amplitude of electrostatic potential variations in the atmosphere due to a spatially independent, time-varying potential at the ionospheric boundary, $V=\sin \omega t$.

Figure 6  Same as Figure 5 but for vertical electric field variations.

Figure 7  Phase of electrostatic potential variations in the atmosphere due to a spatially independent, time-varying potential at the ionospheric boundary, $V=\sin \omega t$.

Figure 8  Same as Figure 7 but for vertical electric field variations.

Figure 9  Normalized amplitude of electrostatic potential variations in the atmosphere due to a standing wave at the ionospheric boundary with potential $V=\cos kx \sin \omega t$.

Figure 10 Same as Figure 9 but for vertical electric field variations.

Figure 11 Same as Figure 9 but for horizontal electric field variations.

Figure 12 Same as Figure 9 but for a higher value of $\omega$.

Figure 13 Same as Figure 10 but for a higher value of $\omega$.

Figure 14 Same as Figure 11 but for a higher value of $\omega$.

Figure 15 Same as Figure 12 but for a still higher value of $\omega$. 
Figure 16  Same as Figure 13 but for a still higher value of $\omega$.

Figure 17  Same as Figure 14 but for a still higher value of $\omega$.

Figure 18  Same as Figure 9 but for constant $k$ and various values of $\omega$.

Figure 19  Same as Figure 10 but for constant $k$ and various values of $\omega$.

Figure 20  Same as Figure 19 but for a different value of $k$.

Figure 21  Phase of electrostatic potential and horizontal electric field variations in the atmosphere due to a standing wave at the ionospheric boundary with potential $V=\cos kx \sin \omega t$.

Figure 22  Same as Figure 21 but for vertical electric field variations.

Figure 23  Same as Figure 21 but for a higher value of $\omega$.

Figure 24  Same as Figure 22 but for a higher value of $\omega$.

Figure 25  Same as Figure 23 but for a still higher value of $\omega$.

Figure 26  Same as Figure 24 but for a still higher value of $\omega$.

Figure 27  Normalized amplitude of electrostatic potential variations in the atmosphere due to a travelling wave at the ionospheric boundary with potential $V=\cos k(x-\omega t)$.

Figure 28  Same as Figure 27 but for vertical electric field variations.

Figure 29  Same as Figure 27 but for horizontal electric field variations.

Figure 30  Phase of electrostatic potential and horizontal electric field variations in the atmosphere due to a travelling wave at the ionospheric boundary with potential $V=\cos k(x-\omega t)$.

Figure 31  Same as Figure 30 but for vertical electric field variations.

Figure 32  Same as Figure 30 but for a higher velocity.

Figure 33  Same as Figure 31 but for a higher velocity.

Figure 34  Normalized equipotential contours in the atmosphere due to a moving confined structure at the ionospheric boundary with potential $V=\cos k(x-\omega t) = \cos ks$.

Figure 35  Same as Figure 34 but for a higher velocity.
Figure 36  Normalized electrostatic potential at different altitudes due to a moving confined structure at the ionospheric boundary (See Figure 1). Width of structure $b = 94 \text{ km}$ and $s = x-vt$.

Figure 37  Same as Figure 36 but for horizontal electric field and $b = 11.7 \text{ km}$

Figure 38  Same as Figure 37 but for $b = 46.8 \text{ km}$

Figure 39  Same as Figure 37 but for $b = 94 \text{ km}$

Figure 40  Same as Figure 37 but for $b = 187 \text{ km}$

Figure 41  Same as Figure 36 but for vertical electric field and $b = 11.7 \text{ km}$

Figure 42  Same as Figure 41 but for higher altitudes

Figure 43  Same as Figure 41 but for $b = 94 \text{ km}$

Figure 44  Same as Figure 43 but for higher altitudes

Figure 45  Same as Figure 41 but for $b = 187 \text{ km}$

Figure 46  Same as Figure 45 but for higher altitudes
Fig. 1.
Fig. 12.

Fig. 13.

Fig. 14.
Fig. 15.

Fig. 16.

Fig. 17.
Fig. 23.

Fig. 24.
Fig. 25.

Fig. 26.
Fig. 30.

\[
\begin{align*}
\omega &= k \cdot v \\
v &= 350 \text{ m/s}
\end{align*}
\]

Phase, radians
(Pot., horizontal field)

Altitude, km

Fig. 31.

\[
\begin{align*}
\omega &= k \cdot v \\
v &= 350 \text{ m/s}
\end{align*}
\]

Phase, radians
(Vert., horizontal field)

Altitude, km

Fig. 32.

\[
\begin{align*}
\omega &= k \cdot v \\
v &= 3500 \text{ m/s}
\end{align*}
\]

Phase, radians
(Vert., horizontal field)

Altitude, km

Fig. 33.
Fig. 36.
The propagation of slowly varying electric disturbances in the atmosphere is investigated theoretically. The approach used is to solve the continuity equation for the electric current assuming the electric field to be a time varying potential field. An earlier, quite generally quoted result that the temporal variation of the electric field at each point is governed by the local relaxation time is shown to be false, also for slow variations, except for very special geometries. Atmospheric fields of ionospheric origin are studied. A general analytical solution is presented and examples of numerically computed solutions are given. Models are considered where the electric field is a standing or travelling sinusoidal wave with frequencies in the range 0-0.5 Hertz, wavelengths $\geq 3\times10^6$ m, and velocities 0-3500 m/s. Also considered are fields from moving ionospheric structures that are spatially confined with dimensions in the range 10-200 km. It is concluded that ionospheric field variations give rise to substantial disturbances in the atmospheric electric field, but at low altitudes they are hard to distinguish from disturbances of meteorological origin. It is shown that it is quite feasible to make balloon studies (altitudes 30-40 km) of slowly varying fields of ionospheric origin. However, it is difficult to study ionospheric field variations at frequencies higher than about 0.1 Hz or to resolve spatial structures smaller than about 100 km.

Keywords. Atmospheric electric fields, ionospheric electric fields.