Cosmoparticle Physics
and
String Theory

Stefan Sjörs

Doctoral Thesis in Theoretical Physics

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Abstract

This thesis deals with phenomenological and theoretical aspects of cosmoparticle physics and string theory. There are many open questions in these topics. In connection with cosmology we would like to understand the detailed properties of dark matter, dark energy, generation of primordial perturbations, etc., and in connection with particle physics we would like to understand the detailed properties of models that stabilize the electroweak scale, for instance supersymmetry. At the same time, we also need to understand these issues in a coherent theoretical framework. Such a framework is offered by string theory.

In this thesis, I analyze the interplay between Higgs and dark matter physics in an effective field theory extension of the minimally supersymmetric standard model. I study a theory of modified gravity, where the graviton has acquired a mass, and show the explicit implementation of the Vainshtein mechanism, which allows one to put severe constraints on the graviton mass. I address the question of Planck scale corrections to inflation in string theory, and show how such corrections can be tamed. I study perturbations of warped throat regions of IIB string theory compactifications and classify allowed boundary conditions. Using this analysis, I determine the potential felt by an anti-D3-brane in such compactifications, using the explicit harmonic data on the conifold. I also address questions of perturbative quantum corrections in string theory and calculate one-loop corrections to the moduli space metric of Calabi-Yau orientifolds.

Key words: Supersymmetry, dark matter, cosmology, inflation, modifications of gravity, string theory, flux compactifications, perturbative string theory.
Svensk sammanfattning

Denna avhandling behandlar fenomenologiska och teoretiska aspekter av kosmologi och partikelfysik. Det finns många öppna frågor inom dessa områden. Inom kosmologin vill man förstå de detaljerade egenskaperna hos mörk materia, mörk energi, uppkomsten av störningar i det tidiga universum, etc., och inom partikelfysiken vill man förstå de detaljerade egenskaperna hos modeller som kan stabilisera den elektrosvaga skalan, som till exempel supersymmetriska modeller. Samtidigt så behöver vi också förstå dessa frågor inom ett sammanhängande teoretiskt ramverk. Strängteorin erbjuder ett sådant ramverk.

List of Accompanying Papers


Acknowledgments

First and foremost, I would like to thank my supervisor Marcus Berg, whom I owe the deepest of gratitude for all the support and encouragement I have received over the past five years. At the same time, I also feel a great responsibility to cherish the knowledge and the many possibilities, that he has bestowed upon me, and I hope to nurture them well.

I am also very grateful to both Liam McAllister, who readily accepted me as one of his students for one year at Cornell University, and Michael Haack, who arranged for me to stay in Munich for weeks; with whom I have enjoyed great collaboration. I am also pleased to thank my other collaborators Igor Buchberger, Joakim Edsjo, Sohang Gandhi, Paolo Gondolo, Erik Lundstrom, Edvard Mortsell and Enrico Pajer, whose particular efforts have been invaluable in the undertaken projects.

I have also enjoyed journal clubs, study groups, and research related discussions with Ingemar Bengtsson, Hans Hansson, Ariel Goobar, Fawad Hassan, Stefan Hofmann, Bo Sundborg, and Paolo Di Vecchia.

The past five years would not have been the same, if it were not for my officemates Jonas Enander, Lars Rosenstrom, Mikael von Strauss and Angnis Schmidt-May, and my friends in the corridor Michael Gustafsson, Soren Holst, Joel Johansson, David Marsh, Rachel Rosen, Sara Rydbeck, Alexander Sellerholm and Emma Wikberg; with whom I have enjoyed endless discussions over lunches, dinners and coffee breaks, as well as movie nights and training sessions. I would also like to thank the rest of the people in the CoPS and KoF groups for making the corridor the fun place it is.

Finally, I would like to thank the people closest to me, they who have supported me from home. My parents Karin and Ingemar, my brother Ambjorn, and my life partner Camilla, all deserve the warmest of thanks.
Preface

In the past five years I have had the pleasure to work with phenomenological and theoretical aspects of some of the greatest challenges faced in cosmology and particle physics today. In this thesis I present phenomenological work done on dark matter, supersymmetry, cosmological inflation and modifications of Einstein’s gravity, as well as theoretical investigations of compactifications in type IIB string theory and perturbative string calculations in models of low supersymmetry.

Thesis plan

The thesis consists of four parts. Part I provides background to the topics dealt with in the papers. Chapters 1, 2 and 3 are more general in character and deal with the basic observational and theoretical aspects of dark matter, supersymmetry, and cosmological inflation, in that order. These chapters provide a context to Paper I and Paper II, which deal with supersymmetric dark matter and cosmological inflation in string theory, respectively. Chapter 4 is a little bit more technical in nature and gives an introduction to massive gravity, which is the topic of Paper III. Chapters 5 and 6 are necessarily of more technical character. Chapter 5 provides an introduction to flux compactifications in type IIB string theory and sets the stage for Paper IV and Paper V. Finally, chapter 6 gives an introduction to perturbative string theory and how it can be used to infer the vacuum structure of low energy string theory. Part II is a summary of the results obtained in the papers. Part III is an appendix on some technical aspects of harmonic analysis, used in regard to Paper V. Last, Part IV is the collection of papers discussed in the introductory chapters.
Contribution to papers

In Paper I I focused mostly on the theoretical aspects. I analyzed electroweak symmetry breaking, derived sum rules and Feynman rules, that were implemented in DarkSUSY. The scanning over the BMSSM parameter space was done by my collaborators. The inflationary model considered in Paper II was invented collectively and calculations were shared equally between the authors. In Paper III I was responsible for the analytical work, while the work with the observational data was performed by my coauthor. The work in Paper IV and Paper V was very much a joint effort and we took turn in both deriving and writing. However, I am responsible for the harmonic analysis presented in the appendix of Paper V. The results in Paper VI are very much the effort of all authors and we developed the calculations as a collective.

Stefan Sjörs
Stockholm, March 2012
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Part I

Introduction
Chapter 1

Dark Matter

This chapter gives a brief introduction to the concept of dark matter. More extensive treatments can be found in the reviews [1, 2, 3, 4], or textbooks such as [5].

1.1 Observations

One of the first encounters with the dark side of the universe was made by Zwicky [6] in 1933. He measured the mass of the Coma cluster and compared it to the amount of luminous matter, and noticed that the average mass-to-light ratio, in solar units, was greater than one hundred. Zwicky’s technique was to first measure the velocities of the galaxies, using the Doppler shift of galaxies moving in the line-of-sight, then calculating the average kinetic energy $\langle T \rangle$ and using the virial theorem, $\langle V \rangle = -2 \langle T \rangle$, to infer the gravitational potential energy $\langle V \rangle$, thus measuring the mass of the cluster. The conclusion was that the amount of gravitational matter in the cluster was much higher than the amount of visible matter.

The same conclusions were also reached for single galaxies, when galaxy rotation curves were measured. The prediction from Newtonian theory of gravity is that, well outside the galaxy core, where most of the luminous matter is gathered, the orbital speed of a test particle, with acceleration $v^2/r$ and moving in a gravitational force field per unit test mass $GM/r^2$, should be

$$v = \sqrt{\frac{GM}{r}},$$  \hspace{1cm} (1.1)

which falls off as $1/\sqrt{r}$. This is not what Rubin, Thonnard and Ford found [7]. They instead found that the orbital speed goes more like a
constant, i.e. $v = \text{const}$. This observation can be explained by assuming that the mass of the galaxy is not only given by the luminous matter of the galactic core, but also given by a dark matter halo with density of $\rho \propto 1/r^2$, so that $M = \int d^3x \rho \propto r$, leading to constant orbital speeds $v = \text{const}$.

But what if the above results can be explained by modifying the laws of gravity? After all, the deviations discussed here manifest themselves only through gravitational effects. Such attempts run into difficulties especially with one recent observation that favors dark matter as some new matter component.\footnote{Some proponents maintain that the difficulties may eventually be resolved, see e.g. [8]}

This is the observation of the so-called “Bullet Cluster”, a collision of two galaxy clusters [9]. In Fig. 1.1 one clearly sees the luminous matter component of the clusters collide, emitting X-rays, while the bulk part of the matter, the dark matter component, is located away from the gas, as observed by gravitational lensing. This shows a clear separation of the luminous and the dark matter components.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{bullet_cluster.png}
\caption{The cluster 1E 0657-56, also known as the “Bullet Cluster”, consists of two colliding galaxy clusters. The picture shows a clear separation between the dark matter, as manifest by lensing data (in blue), and the luminous matter, as manifest by X-ray data (in pink). X-ray data: NASA/CXC/CfA/ M.Markevitch et al. Lensing map: NASA/STScI; ESO WFI; Magellan/U.Arizona/ D.Clowe et al. Optical data: NASA/STScI; Magellan/U.Arizona/D.Clowe et al.}
\end{figure}
1.2 WIMP

In the above, it was argued for the introduction of a new matter species, a dark matter particle. To accommodate observations, such a particle should be massive, stable and feebly interacting. These are the characteristics we assign to a so-called “WIMP”, which is a Weakly Interacting Massive Particle. There are many WIMP candidates in the literature, for example, supersymmetric particles, Kaluza-Klein states and axions. For a review, see [10]. In this thesis, i.e. in Paper I, we focus on the relatively popular supersymmetric candidate: the neutralino, to be discussed later.

1.2.1 Relic density

To explain the required abundance of dark matter, used to accommodate the above-mentioned observations, we must calculate the relic density, i.e. the number density of dark matter resulting from the Big Bang. Consider a WIMP, denoted by $\chi$, in the early universe. The WIMP will interact with itself, its anti-particle $\bar{\chi}$ (possibly itself), and the other Standard Model (SM) particles around at the time, collectively denoted by $\{X_i\}$

$$\chi + \bar{\chi} \leftrightarrow \sum_i X_i,$$

(1.2)

The number density $n_\chi$ of WIMPs is governed by the Boltzmann equation, which tells us how the expansion of the universe dilutes the density and how interactions with other particles feed the density, i.e.

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma_A v \rangle \left( n_\chi^2 - (n_{\chi}^\text{eq})^2 \right),$$

(1.3)

where $\langle \ldots \rangle$ denotes thermal averaging, $\sigma_A$ is the annihilation cross section for dark matter pairs, $v$ denotes the relative speed, and where the expansion rate of the universe $H$ gives the dilution factor.

As the universe expands, the number density becomes increasingly diluted, and when the annihilation rate equals the expansion rate, the WIMPs will stop annihilating and “freeze out”. This determines the number density of WIMPs as measured today. The exact results for the relic density is obtained by solving the equations above numerically. However, an estimate is given by [4]

$$\Omega_{DM} h^2 \approx \frac{3 \times 10^{-27} \text{cm}^3 \text{s}^{-1}}{\langle \sigma_A v \rangle},$$

(1.4)
where \( \Omega_{\text{DM}} h^2 \) is the fractional number density of dark matter today. From the observed value \( \Omega_{\text{DM}} h^2 = 0.1131 \pm 0.0034 \) [11] we find that the annihilation cross section is approximately given by

\[
\langle \sigma_A v \rangle \approx 3 \times 10^{-26} \text{cm}^3 \text{s}^{-1}.
\] (1.5)

What kind of particles can be responsible for these numbers? Now, a particle with mass and couplings at the electroweak scale, \( m_\chi \sim 100 \text{ GeV} \) and \( \alpha \sim 0.01 \), would produce a cross section of this order of magnitude

\[
\langle \sigma_A v \rangle \sim \frac{\alpha^2}{m_\chi^2} \sim 4 \times 10^{-26} \text{cm}^3 \text{s}^{-1}.
\] (1.6)

This coincidence is referred to as the “WIMP miracle” and fuels the dreams of many particle physicists, the dreams of finding a dark matter particle at the Large Hadron Collider (LHC).
Chapter 2

Low energy supersymmetry

The question of how the electroweak scale is stabilized is perhaps the most important question in theoretical particle physics of today. In this chapter I give a brief introduction to supersymmetry, which provides one of the most elegant answers to this question. Only the necessity of material is introduced in this chapter, and further details are referred to the vast literature on the subject. Among textbooks and reviews, I especially enjoy the books [12, 13, 14, 15] and the beautiful review [16]. The standard reference on issues related to conventions is [17].

2.1 The hierarchy problem of the Standard Model

In the SM of particle physics all particles obtain masses through their couplings to the vacuum expectation value (VEV) of the Higgs field. The VEV of the Higgs field is determined by minimizing the Higgs potential

\[ V(H) = \frac{\mu^2}{2} H^2 + \frac{\lambda}{4} H^4. \]  

From Eq. (2.1), the Higgs field acquires a VEV given by \( \langle H \rangle = \sqrt{-\mu^2/\lambda} \), for negative \( \mu^2 \) and positive \( \lambda \). From the fact that \( \langle H \rangle \) determines the masses of all particles in the SM we know experimentally that \( \langle H \rangle \) is about 174 GeV. This means that, for order one self coupling \( \lambda \), the Higgs mass must be of the order \(-\mu^2 \sim (100 \text{ GeV})^2\). Now, the problem is that this
quantity receives huge radiative corrections. For example, from one-loop diagrams of the form

\[ \begin{array}{c}
H \\
\downarrow \\
\uparrow \\
H
\end{array} , \]

where \( f \) denotes any of the fermions of the SM, we find a contribution to the Higgs mass that is quadratic in the momentum cutoff \( \Lambda \) of the theory. This contribution is supposed to be cancelled by the bare Higgs mass \(-\mu_0^2\). Now, if we believe in the SM up to the LHC scale \( \Lambda \sim 10^4 \text{ GeV} \), then we need to fine-tune \(-\mu_0^2\) to four decimal places to cancel off the cutoff, while if we believe the SM up the Planck scale \( \Lambda \sim 10^{19} \text{ GeV} \), we need cancellation in 34 decimal places. Now, imposing this cancellation by hand amounts to enormous fine-tuning.

One might ponder the idea of a mechanism that ensures the cancellation of the divergence, without having to impose it by hand, order by order. In the words of Martin [16]: “The systematic cancellation of the dangerous contributions to the Higgs mass can only be brought about by the type of conspiracy that is better known to physicists as a symmetry”. This is what happens when we introduce supersymmetry (SUSY). When performing the calculation of quadratic divergences in the SUSY theories, one finds that bosonic degrees of freedom all contribute positively while all fermionic degrees of freedom contribute negatively, in such a way that this cancellation comes about. In SUSY theories all particles have their corresponding superpartners, called sparticles, and together with detailed relations between couplings, due to supersymmetry, the cancellation of the quadratic mass divergences is manifest.

2.2 MSSM

2.2.1 Field content

By introducing a SUSY partner to all the SM particles we obtain the Minimally Supersymmetric Standard Model (MSSM). Thus, the leptons are assigned partners known as sleptons, the quarks are assigned partners known as squarks, the gauge fields are assigned partners known as gaugeinos, and the Higgs field is assigned a partner known as the higgsino. By the detailed structure of SUSY, one is also forced to introduce
a secondary Higgs, and higgsino, such that one can give masses to both up and down quarks. The field content is summarized in Tabs. 2.1-2.2.

### The chiral multiplets of the MSSM

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Spin-0</th>
<th>Spin-1/2</th>
<th>SU(3)_C</th>
<th>SU(2)_C</th>
<th>U(1)_Y</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_i = (\nu\nu_i L) (\nu_i L) e_i L$</td>
<td>1 2</td>
<td>$-\frac{1}{2}$</td>
<td>(0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{e}<em>i = \tilde{e}</em>{iR} e_{iR}$</td>
<td>1 1</td>
<td>+1</td>
<td>+1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q_i = (u_i L) (d_i L) (\nu_i L) e_i L$</td>
<td>3 2</td>
<td>$+\frac{1}{6}$</td>
<td>$\left(\frac{2}{3}\right)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U^c_i = \tilde{u}<em>{iR} u</em>{iR}$</td>
<td>3 1</td>
<td>$-\frac{2}{3}$</td>
<td>$\left(-\frac{2}{3}\right)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D^c_i = \tilde{d}<em>{iR} d</em>{iR}$</td>
<td>3 1</td>
<td>$+\frac{1}{3}$</td>
<td>$\left(+\frac{1}{3}\right)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_u = (H_u^+ H_u^0) (\tilde{H}_u^+ \tilde{H}_u^0)$</td>
<td>1 2</td>
<td>$+\frac{1}{2}$</td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_d = (H_d^- H_d^0) (\tilde{H}_d^- \tilde{H}_d^0)$</td>
<td>1 2</td>
<td>$-\frac{1}{2}$</td>
<td>(0)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1. The chiral multiplets of the MSSM and their charges under the SM gauge group.

### The vector multiplets of the MSSM

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Spin-1/2</th>
<th>Spin-1</th>
<th>SU(3)_C</th>
<th>SU(2)_L</th>
<th>U(1)_Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>$\tilde{g}^{1,\ldots,8}$</td>
<td>$g^{1,\ldots,8}$</td>
<td>8</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$W$</td>
<td>$\tilde{W}^{1,2,3}$</td>
<td>$W^{1,2,3}$</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>$\tilde{B}$</td>
<td>$B$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2.2. The vector multiplets of the MSSM that contains the vector bosons of the SM gauge group.

### 2.2.2 Interactions

When writing down the interactions of the MSSM one encounters two serious issues related to phenomenology. First, the spectrum of the SM is not supersymmetric, so we need to break supersymmetry spontaneously.
Second, generic supersymmetric couplings break baryon and lepton number conservation, so we need to invent a symmetry that forbids such couplings.

The first issue introduces a certain amount of arbitrariness into the MSSM. Introducing all possible symmetry breaking terms one has to introduce about one hundred new parameters. In particular we need to introduce mass splittings of the order of a TeV.

The second issue is rather easy to circumvent. Baryon and lepton number conservation can be enforced by imposing a so-called “$R$-parity” (or “matter parity”). Assigning $R$-parity is simple: all particles are assigned $P_R = +1$, while all sparticles are assigned $P_R = -1$. By demanding that interactions preserve $R$-parity, we in fact find that all the dangerous baryon and lepton violating couplings vanish. The presence of the discrete symmetry also has another important phenomenological consequence: no mixing between particles and sparticles is allowed. Thus the lightest supersymmetric particle (the LSP) is absolutely stable. In MSSM models with a neutral LSP, we get a potential dark matter candidate, i.e. a long-lived particle with electroweak mass and couplings.

With the MSSM field content presented, much can be said about the resulting phenomenology of collider physics and cosmology. This is not within the scope of this chapter and I will only focus on the part of the field content important for dark matter searches.

### 2.2.3 Neutralino

Glancing through the field content of the MSSM one finds, after electroweak symmetry breaking, four new neutral supersymmetric particles. The two partners of the Higgs fields: the two higgsinos $\tilde{H}_u^0$ and $\tilde{H}_d^0$, and the two partners of the gauge bosons: the wino $\tilde{W}^3$ and bino $\tilde{B}$. They all mix into four neutral particles, the “neutralinos” $\tilde{\chi}_i$, $i = 1, 2, 3, 4$. In models where the lightest neutralino is the LSP we have a good dark matter candidate and in **Paper I** we focus on such parameter space regions of the MSSM.
Chapter 3

Cosmological inflation

This chapter introduces the concept of cosmological inflation. Much of the material presented here can be found in textbooks [5, 18, 19], or in the reviews [20, 21].

3.1 The concordance model of cosmology

When gazing into the night sky, we see the same stars as the ancient Greeks once looked upon. Observing with the naked eye, one might come to the conclusion that the universe is both static and empty; with mostly voids, except for static configurations of stars. This view changes if we broaden our perspective and see farther away. In the late 1920’s it was observed that distant galaxies seemed to be moving away from us with a speed proportional to the distance separating us and the galaxy. If \( r \) denotes the distance to the far galaxy, then the recession speed \( \dot{r} \) is given by \( \dot{r} = H_0 r \), for some constant of proportionality \( H_0 \). This relation is attributed to Hubble, who determined \( H_0 \) to some precision [22]. However, the relation was considered earlier by Lemaitre, who gave an interpretation in terms of expansion of the universe [23, 24]. In fact, if all initial distances \( x \) are blown up by a scale factor \( a \), then for \( r = ax \) we see that \( \dot{r} = \dot{a} x = H r \), where the “Hubble constant” \( H = \dot{a}/a \) is interpreted as the rate of expansion.

We also learn from large scale studies (of sizes of the order of the observable universe) that the universe is in fact both homogeneous and
isotropic. The metric describing an expanding, homogeneous and isotropic universe is the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right),$$

(3.1)

where $k$ is the curvature of the universe, with $k = +1, 0, -1$ for an open, flat or closed universe, respectively. The dynamics of the universe is determined by Einstein’s equations. In particular, the rate of expansion is determined by the energy content of the universe

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3M_{pl}^2},$$

(3.2)

where the total energy density $\rho$ is a sum of many components

$$\rho = \sum_i \rho_i,$$

(3.3)

ranging over baryons ($\rho_B$), radiation ($\rho_{rad}$), etc. The great surprise of 20th century cosmology is that the ordinary matter is not enough to explain the expansion, but in fact we have to invoke two new components, “dark energy” (DE) and “dark matter” (DM). In terms of normalized energy densities, $\Omega_i = \rho_i/3H^2$, observations tell us that [11]

$$\Omega_{DE} = 0.721 \pm 0.015, \quad \Omega_{DM} = 0.233 \pm 0.013, \quad \Omega_B = 0.0462 \pm 0.0015.$$  

(3.4)

Thus, baryons constitute only a small fraction of the total energy content of the universe, which is instead dominated by dark energy and dark matter.

### 3.2 The horizon problem

Having outlined the concordance model of cosmology, we immediately encounter a puzzle. On the grandest observable scales of the universe, the scales of the cosmic microwave background (CMB), the universe is almost perfectly homogeneous and isotropic. Indeed, the temperature variations in the CMB are as small as $\Delta T/T \sim 10^{-5}$, see Fig. 3.1. This raises a question: Why are the CMB anisotropies so small? The question becomes even more acute when one realizes that, in the standard Big
3.2. The horizon problem

Following the rough estimates of [18]: We know that the universe is homogeneous and isotropic at the present horizon scale \(l_0 \sim ct_0\), where \(t_0\) is the age of the universe. This homogenous patch was initially about \(a_i/a_0\) times smaller, i.e. \(l_i \sim l_0a_i/a_0\). The initial patch should be compared to the causal region \(l_c \sim ct_i\)

\[
\frac{l_i}{l_c} \sim \frac{t_0a_i}{t_ia_0} \sim \frac{\dot{a}_i}{\dot{a}_0},
\]  

(3.5)

where in the last step, it was used that \(a\) grows as a power of time, for a matter or radiation dominated universe, such that \(\dot{a} \sim a/t\). From Eq. (3.5) we see that if the expansion of the universe has always been decelerating (due to gravitational attraction), then the ratio of the right-hand side is always less than one, thus the size of the initially homogenous universe is always less than the causally connected patch. This leads to the so called Horizon problem. The horizon problem is solved if the universe underwent a period of accelerated expansion so-called “inflation”. Inflation is, by definition, a stage of accelerated expansion of the universe, with gravity acting as a repulsive force.

Historically, inflation was used to solve the horizon problem for the first time by Guth in [25]. However, his model was plagued with problems and soon, ”old inflation” was replaced by Linde’s “new inflation” [26, 27].

![Figure 3.1. The WMAP7 sky map of the anisotropies in cosmic microwave background. The overall temperature of the CMB is \(T = 2.725\), with fluctuations in the fifth decimal place \(\Delta T/T \sim 10^{-5}\) as indicated by the varying colors in the skymap. Credit: NASA/WMAP Science Team.](image)
Along with a solution to the horizon problem, inflation also offers solution to the so-called “flatness problem” and problems of “unwanted relics” such as magnetic monopoles and topological defects of grand unified theories, see [21].

### 3.3 Inflation

To generate a period of accelerated expansion we need the universe to have been dominated by a nearly constant vacuum energy. The most studied mechanism for generating such a vacuum energy is to consider a scalar field \( \phi \), a so-called “inflaton”, that slowly rolls down a big potential \( V(\phi) \), dominating the energy content of the universe. The nearly constant vacuum energy feeds into the scale factor, through Eq. (3.2), and leads to nearly exponential expansion, which results in a big homogeneous universe starting from an initially small homogeneous patch.

The dynamics of inflation is governed by the action

\[
S = \int d^4x \sqrt{-g} \left( \frac{M_{\text{pl}}^2}{2} R + \frac{1}{2} \nabla \phi \cdot \nabla \phi - V(\phi) \right). \tag{3.6}
\]

For inflation to occur, and to occur for a sufficiently long time, we need the derivative, and the second derivative, of the potential to be small in Planck units. Defining the so-called “slow-roll parameters”

\[
\epsilon(\phi) = \frac{M_{\text{pl}}^2}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta(\phi) = M_{\text{pl}}^2 \frac{V''(\phi)}{V(\phi)}, \tag{3.7}
\]

then “slow-roll” amounts to \( \epsilon, \eta \ll 1 \), throughout the inflationary dynamics.

### 3.4 Observables

Inflation can also explain the observed features of the fluctuations in the CMB. Throughout the inflationary dynamics, small perturbations in the metric (and the inflaton) are generated quantum mechanically and stretched to macroscopic size by the expansion. Soon they reach the horizon, which is constant during inflation, and they freeze. After the exit of inflation, the horizon starts to grow and the fluctuations enter the horizon once again. Those fluctuations that enter during recombination will leave their imprint on the CMB, for us to observe today.
3.4. **Observables**

Due to the homogeneity and isotropy of hypersurfaces of constant time, it makes sense to Fourier expand the perturbations, as well as splitting them into “scalars”, “vectors” and “tensors”, under three-transformations. The split is convenient since perturbations at different scales $\vec{k}$, and of different tensorial structure, evolve independently of each other in first order perturbation theory. In what follows we will neglect vector modes from the discussion since, first, they are not generated during inflation, and second, they decay away rapidly with the expansion [18]. Thus we will only study scalar and tensor perturbations. There is one scalar mode, which can be described in many equivalent ways (due to gauge redundancies). One parametrization of the scalar mode is in terms of the so-called “comoving curvature perturbation” $R$. There are two tensor modes, which describe the two polarizations $h^+, h^\times$ of the gravitational wave.

A measure of the amplitude of scalar fluctuations is given by the variance, or “power spectrum”, of the perturbations

$$\langle R(\vec{k}) R(\vec{k}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') P_R(k), \quad \Delta_s^2 \equiv \Delta_R^2 = \frac{k^3}{2\pi^2} P_R(k).$$

(3.8)

The power spectrum gives the squared amplitude of fluctuations on scales $k^{-1}$. The deviation from scale invariance is encoded in the so-called “spectral index” $n_s$, defined by

$$n_s - 1 = \frac{d \ln \Delta_s^2}{d \ln k}. \quad (3.9)$$

Likewise, the power spectra of each of the two tensor modes are given by

$$\langle h(\vec{k}) h(\vec{k}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') P_h(k), \quad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k).$$

(3.10)

The power spectra of the two modes are combined into

$$\Delta_t^2 = \Delta_{h^+}^2 + \Delta_{h^\times}^2,$$

(3.11)

with spectral index

$$n_t = \frac{d \ln \Delta_t^2}{d \ln k}. \quad (3.12)$$

Starting from the action in Eq. (3.6), one finds the following predictions for the power spectra

$$\Delta_R^2(k) = \frac{H_*^2}{(2\pi)^2} \frac{H_*^2}{\dot{\phi}_*^2}, \quad \Delta_t^2 = \frac{2}{\pi^2} H_*^2,$$

(3.13)
where \( \ast \) denotes evaluation at horizon crossing \( a_\ast H_\ast = k \).

The inflationary predictions of a few models, together with the latest WMAP7 results for the scalar and tensor power spectra are presented in Fig. 3.2. The results are presented in terms of the so-called “tensor-to-scalar ratio” \( r \), defined by

\[
r = \frac{\Delta^2_l}{\Delta^2_s}. \tag{3.14}
\]

Observations of \( r \) would be a smoking gun signature for inflation. Using the COBE normalization for the scalar fluctuations \( \Delta^2_s \sim 10^{-9} \), and rewriting the potential in terms of the tensor-to-scalar ratio one finds

\[
V^{1/4} \approx \left( \frac{r}{0.01} \right)^{1/4} 10^{16} \text{ GeV}. \tag{3.15}
\]

Thus a measurement of \( r \) determines the scale of inflation.

### 3.5 The Lyth bound

The tensor-to-scalar ratio \( r \) can be related to the length of the field excursion of the inflaton. To see this, first notice that \( r \) can be related to
the number of $e$-folds of expansion of the scale factor, $N$, and the field excursion via

$$ r = 8 \left( \frac{d\phi}{dN} \right)^2. $$

(3.16)

Integrating this relation, one we finds that the total field excursion of $\phi$ during inflation is given by

$$ \Delta \phi = \int_{N_{\text{end}}}^{N_{\text{cmb}}} dN \sqrt{\frac{r}{8}}. $$

(3.17)

Now, $r$ is approximately constant throughout the inflationary dynamics so one finds [29]

$$ \frac{\Delta \phi}{M_{\text{pl}}} \approx \sqrt{\frac{r}{0.01}}. $$

(3.18)

For detectable $r \gtrsim 0.01$ the inflaton field has to take on super-Planckian values $\Delta \phi \gtrsim M_{\text{pl}}$ during inflation. This is the Lyth bound. Thus, a detection of $r$ would let us probe dynamics of Planck scale physics.

### 3.6 The $\eta$-problem

The $\eta$-problem illustrates the sensitivity of inflation to Planck scale corrections. Given an inflationary potential $V_0$, we generically expect Planck scale corrections of the form

$$ V_0 \to V_0 + \frac{\phi^2}{M_{\text{pl}}^2} V_0 + \ldots $$

(3.19)

Such a correction introduces a Hubble scale mass to the inflaton

$$ \delta m_\phi^2 \approx \frac{2V_0}{M_{\text{pl}}^2} \approx 6H^2, $$

(3.20)

which ruins inflation by generating a large $\eta$-parameter, $\delta \eta \approx 2$. It should be stressed that the above argument is completely general and highlights a problem of all inflationary models, low-scale as well as high-scale. There are only two ways around this problem. Either we fine-tune the coefficient of the correction in Eq. (3.19) or we invent a symmetry that forbids the generation of such a correction. Either way, understanding of the full
UV completion is needed before one can fine-tune couplings or answer questions about symmetries. In Paper II we analyze these issues in an inflationary model in string theory, which provides UV complete description, and we show how to generate high scale inflation with explicit control over Planck scale corrections.
Massive gravity

The history of massive gravity dates back to 1939 and the work of Fierz and Pauli [30] who wrote down a consistent theory of non-interacting massive spin-2 particles in flat space. At the level of the Lagrangian, there are two possible mass terms for a spin-2 field as described by a symmetric two-tensor $h_{\mu\nu}$

$$ -m_1^2 h_{\mu\nu}^2 - m_2^2 h^2, \quad (4.1) $$

where $h_{\mu\nu}^2 = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\mu\nu} h_{\alpha\beta}$ and $h^2 = (\eta^{\mu\nu} h_{\mu\nu})^2$. The work of Fierz and Pauli consisted of showing that only for $m_1^2 = -m_2^2 > 0$ does the above mass term describe a massive spin-2 particle. For $m_1^2 \neq -m_2^2$ the above mass term gives rise to an additional ghost-like degree of freedom. The ghost becomes infinitely heavy in the limit $m_1^2 = -m_2^2$ and is therefore frozen out.

After this seminal work, not much attention was given the Fierz-Pauli theory of linear (non-interacting) massive spin-2 particles until the observational consequences of massive gravity was worked out in 1970 by van Dam and Veltman [31], and Zakharov [32] (assuming that $h_{\mu\nu} = M_{\text{pl}} (g_{\mu\nu} - \eta_{\mu\nu})$ describes metric perturbations). They discovered a $3/4$ difference in the bending of light around the sun (as a prediction in terms of Newton’s constant) between the massive and massless theory, a difference that persisted even in the limit of vanishingly small graviton mass. Even by that time, such a big discrepancy was enough to rule out massive gravity, as one can read in the note added in proof of [31]. This phenomenon is referred to as the “vDVZ-discontinuity” and will explained in detail in §4.1.

If that was the end of the story, massive gravity would at that point have been ruled out observationally. But in 1972 Vainshtein [33] noticed...
that the Fierz-Pauli theory of massive gravity breaks down at a distance scale $r_V$, the so-called “Vainshtein radius”, and inside this scale perturbation theory is not applicable. The Vainshtein radius sets an intermediate scale between the very small scale of the source, the gravitational radius $r_S = 2G_NM$, and the presumably very large inverse mass scale of the graviton $\lambda_g = 1/m_g$. Vainshtein showed that outside the Vainshtein radius there exists a perturbation series in $r_V/r$, with Fierz-Pauli theory representing first order. However, at the Vainshtein radius massive gravity becomes strongly coupled and at smaller scales there exists instead a small $r/r_V$ limit such that massless gravity, i.e. general relativity, is recovered. The interpolation between the two regimes is necessarily non-analytic in the graviton mass and access to a non-linear theory of massive gravity would be needed to fully explore this phenomenon, referred to as the “Vainshtein mechanism”.

The generalization of the Fierz-Pauli theory to a full non-linear theory of massive gravity turned out to be a much more difficult task. The ghost-like mode that Fierz and Pauli proved to be absent in the linear theory for $m_1^2 = -m_2^2$ always seemed to reappear in the many examples considered. This lead Boulware and Deser [34] to conjecture in 1972 that there are in fact no consistent (ghost-free) non-linear completions of massive gravity. However, Boulware and Deser were proven wrong in a recent series of papers by a number of authors. First, there were some early arguments in [35] that sharpened the issue, which were then developed further in [36]. Then it was proven that, in the so-called “decoupling limit”, one can tune the non-linear interactions such that the ghost mode disappears [37, 38]. In [39], Gabadadze, de Rham and Tolley proposed a non-linear completion that reduced to the ghost-free theory in the decoupling limit. Then in 2011, Hassan and Rosen made further progress in a series of papers [40, 41, 42], where they proved the absence of ghosts in a generalized model of [39], using an ADM analysis. In the light of these findings, we decided in Paper III to revisit the phenomenology of massive gravity and especially the explicit implementation of the Vainshtein mechanism.

In the sections to follow, we explain the basic aspects of massive gravity and for further details the reader is referred to a recent review [43].

### 4.1 The van Dam-Veltman-Zakharov discontinuity

We wish to study the motion of a probe, e.g. a satellite or a light ray, in the gravitational field generated by a massive source, e.g. a star or
4.1. The van Dam-Veltman-Zakharov discontinuity

a galaxy. In first order perturbation theory, the gravitational field of a source, described by a stress-energy tensor $T_{\alpha\beta}$, is given by

$$\delta\langle h_{\mu\nu}(x) \rangle = \frac{i}{M_{\text{pl}}} \int d^4y \left\langle h_{\mu\nu}(x) h^{\alpha\beta}(y) \right\rangle T_{\alpha\beta}(y),$$

(4.2)

where $M_{\text{pl}}$ denotes the Planck mass.

The structure of the graviton propagator $\langle h_{\mu\nu}(x) h^{\alpha\beta}(y) \rangle$ depends on whether the graviton is massless or massive. In momentum space the propagator has a pole at the mass of the graviton

$$\left\langle h_{\mu\nu}(p) h^{\alpha\beta}(-p) \right\rangle \xrightarrow{p^2 \to -m_g^2} \frac{-iZ^{\alpha\beta}_{\mu\nu}(m_g^2)}{p^2 + m_g^2 - i\varepsilon},$$

(4.3)

where $-iZ^{\alpha\beta}_{\mu\nu}(m_g^2)$ is the residue of the pole. In what follows, we show that there is a finite difference in the value of the residue in the massive $Z^{\alpha\beta}_{\mu\nu}(m_g^2)$ and massless $Z^{\alpha\beta}_{\mu\nu}(0)$ theory. The residue $Z^{\alpha\beta}_{\mu\nu}(m_g^2)$ can be fixed using the optical theorem, which gives the imaginary part of a graviton exchange diagram in terms of the cross section for production of gravitons. For definiteness, consider a $2 \to 2$ process

$$2\text{Im} = \sum_{\text{polarizations}} \int d\Pi \left( \begin{array}{c} k_2 \\ k_1 \\ k_1 \\ k_2 \\ k_2 \end{array} \right) \left( \begin{array}{c} k_2 \\ k_1 \\ k_1 \end{array} \right).$$

(4.4)

Now, the amplitude for production of gravitons is proportional to the graviton wave functions, i.e. the polarizations (mode tensors) $\varepsilon_{\alpha\beta}$, thus from the optical theorem we get

$$Z^{\alpha\beta}_{\mu\nu}(m_g^2) = \sum_{\text{polarizations}} \varepsilon_{\mu\nu}(\varepsilon^{\alpha\beta})^*. $$

(4.5)

This is where the finite difference between massive and massless gravitons enters. For $m_g^2 \neq 0$, the polarization sum runs over the five spin states, $s_z = 0, \pm 1, \pm 2$, of a massive spin-2 particle, whereas for $m_g^2 = 0$, the polarization sum runs over the two helicity states, $\lambda = \pm 2$, of a massless helicity-2 particle.
Chapter 4. Massive gravity

For massive gravitons we can go to the rest-frame \( p^\mu = (m_g; 0) \). Here the five polarization tensors span the space of traceless and symmetric two-tensors in the \( xyz \)-directions, i.e. we get

\[
Z_{\mu\nu}^{\alpha\beta}(m_g^2) = \delta_{(\mu}^{\alpha} \delta_{\nu)}^{\beta} - \frac{1}{3} \eta_{\mu\nu} \eta^{\alpha\beta} + \text{momenta}, \tag{4.6}
\]

where the dependence on terms proportional to the graviton four-momentum is suppressed in the above. These terms in any case drop out when contracted with conserved sources.

For massless gravitons we cannot go to the rest-frame, but instead the best we can do is to go to a frame where, say, \( p^\mu = (|\vec{p}|; 0, 0, |\vec{p}|) \). The two polarization tensors now span the space of traceless symmetric two-tensors in the \( xy \)-directions only, i.e. we now get

\[
Z_{\mu\nu}^{\alpha\beta}(0) = \delta_{(\mu}^{\alpha} \delta_{\nu)}^{\beta} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} + \text{momenta}. \tag{4.7}
\]

This is the famous vDVZ discontinuity

\[
\lim_{m_g^2 \to 0} Z_{\mu\nu}^{\alpha\beta}(m_g^2) \neq Z_{\mu\nu}^{\alpha\beta}(0), \tag{4.8}
\]

whose consequences we explore below.

4.1.1 Newton’s force law and deflection of light

For definiteness we now consider the gravitational field generated by a stationary point source at the origin with \( T_{\alpha\beta}(t, \vec{x}) = M \delta_0^\alpha \delta_0^\beta \delta(3)(\vec{x}) \). Then from Eq. (4.3) we get the familiar Yukawa-type of potential

\[
M_{\text{pl}}^{-1} \delta \langle h_{\mu\nu}(x) \rangle = \frac{Z_{\mu\nu}^{00}(m_g^2) M}{4\pi M_{\text{pl}}^2} \frac{1}{r} e^{-m_g r}. \tag{4.9}
\]

We now wish to study the motion of a probe, described by a stress-energy tensor \( \tau_{\mu\nu} \), in this background. The interaction energy with the background is given by

\[
U = -\frac{1}{M_{\text{pl}}} \int d^4x \, \tau^{\mu\nu}(x) \delta \langle h_{\mu\nu}(x) \rangle = -\frac{i}{M_{\text{pl}}^2} \int d^4x \int d^4y \, \tau^{\mu\nu}(x) \langle h_{\mu\nu}(x) h_{\alpha\beta}^{\alpha\beta}(y) \rangle T_{\alpha\beta}(y). \tag{4.10}
\]
4.2. Helicity decomposition

For a probe that travels on a path $\vec{r} = \vec{r}(t)$, the stress-energy tensor takes the form $\tau_{\mu\nu}(t, \vec{x}) = \frac{p_{\nu} p_{\nu}}{p^\mu} \delta^{(3)}(\vec{x} - \vec{r}(t))$. Thus for a massive probe with momentum, say, $p^\mu = (m; 0, 0, 0)$ then

$$U = -\frac{Z^{00}_0(m_g^2) m M}{4\pi M_{\text{pl}}^2} \frac{1}{r} e^{-m_g r} = -G_N(m_g^2) \frac{m M}{r} e^{-m_g r},$$

where we have identified Newton’s constant $G_N(m_g^2)$, that an observer at distance scales $r \ll m_g^{-1}$ observe

$$G_N(m_g^2) = \frac{Z^{00}_0(m_g^2)}{4\pi M_{\text{pl}}^2}.$$  

(4.11)

For a massless probe with momentum, say, $p^\mu = (\omega; 0, 0, \omega)$ then

$$U = -\frac{Z^{00}_0(m_g^2) + Z^{00}_3(m_g^2) \omega M}{4\pi M_{\text{pl}}^2} \frac{1}{r} e^{-m_g r}.$$  

(4.12)

The above prefactors are explicitly given by

$$Z^{00}_0(0) = \frac{1}{2}, \quad Z^{00}_0(0) + Z^{00}_3(0) = 1,$$

(4.13)

$$Z^{00}_0(m_g^2) = \frac{2}{3}, \quad Z^{00}_0(m_g^2) + Z^{00}_3(m_g^2) = 1.$$

(4.14)

Notice that light is affected in the same way in both massless and massive gravity. This stems from the fact that the stress-energy tensor of a photon is traceless and the only difference between massive and massless gravity enters in the trace-part of the residues in Eqs. (4.6-4.7). However, the massive observer will experience a different Newton’s constant. This leads to a factor $3/4$ difference in the prediction for light deflection as stated in terms of Newton’s constant

$$\frac{G_N(0)}{G_N(m_g^2)} = \frac{Z^{00}_0(0)}{Z^{00}_0(m_g^2)} = \frac{3}{4},$$

(4.15)


4.2 Helicity decomposition

The presence of the vDVZ-discontinuity is perhaps no surprise, since the massless and massive theories are truly different in that they propagate
different numbers of degrees of freedom. This is to be contrasted with the theories of massless and massive photons where there is a discrete difference in the number of degrees of freedom but there is no discontinuity in the current-current correlator. The absence of a discontinuity stems from the fact that the third polarization state of a massive photon becomes longitudinal in the massless limit, or equivalently in the high energy limit, as is familiar from the Goldstone boson equivalence theorem, see e.g. the textbook [44]. In addition to the two $\lambda = \pm 1$ polarization states $\varepsilon_{\alpha}(\pm 1)$ of the massless photon, the massive photon also carries a $\lambda = 0$ polarization state $\varepsilon_{\alpha}(0)$. For a massive photon boosted in, say, the $z$-direction $k^\mu = (E; 0, 0, |\vec{k}|)$ this state is given by

$$\varepsilon^{\alpha}(0, \vec{k}) = \frac{1}{m}(|\vec{k}|; 0, 0, E). \quad (4.17)$$

When we take the mass to zero, or equivalently go to high energies, this polarization becomes longitudinal, i.e. proportional to the four-momentum

$$\varepsilon^{\alpha}(0, \vec{k}) \xrightarrow{m \to 0} \frac{1}{m} k^\alpha. \quad (4.18)$$

This means that for massive photons there is no discontinuity when we take the massless limit. The extra degree of freedom simply decouples.

This is not the case for gravity. A Clebsch-Gordan decomposition of the five spin-2 states into tensor products of spin-1 states gives

$$\varepsilon_{\alpha\beta}(\pm 2) = \varepsilon_{\alpha}(\pm 1)\varepsilon_{\beta}(\pm 1), \quad (4.19)$$

$$\varepsilon_{\alpha\beta}(\pm 1) = \frac{1}{\sqrt{2}}(\varepsilon_{\alpha}(\pm 1)\varepsilon_{\beta}(0) + \varepsilon_{\alpha}(0)\varepsilon_{\beta}(\pm 1)), \quad (4.20)$$

$$\varepsilon_{\alpha\beta}(0) = \frac{1}{\sqrt{6}}(\varepsilon_{\alpha}(+1)\varepsilon_{\beta}(-1) + 2\varepsilon_{\alpha}(0)\varepsilon_{\beta}(0) + \varepsilon_{\alpha}(-1)\varepsilon_{\beta}(+1)). \quad (4.21)$$

Thus in the limit $m_g \to 0$ we find

$$\varepsilon_{\alpha\beta}(\pm 2) \to \varepsilon_{\alpha}(\pm 1)\varepsilon_{\beta}(\pm 1), \quad (4.22)$$

$$\varepsilon_{\alpha\beta}(\pm 1) \to \frac{1}{\sqrt{2}}\left(\varepsilon_{\alpha}(\pm 1)\frac{k_\beta}{m_g} + \frac{k_\alpha}{m_g}\varepsilon_{\beta}(\pm 1)\right), \quad (4.23)$$

$$\varepsilon_{\alpha\beta}(0) \to \frac{1}{\sqrt{6}}\left(\varepsilon_{\alpha}(+1)\varepsilon_{\beta}(-1) + 2\frac{k_\alpha k_\beta}{m_g^2} + \varepsilon_{\alpha}(-1)\varepsilon_{\beta}(+1)\right). \quad (4.24)$$
From Eqs. (4.23-4.24) we see that the $\varepsilon_{\alpha\beta}(\pm 1)$ states become longitudinal and decouple while $\varepsilon_{\alpha\beta}(0)$ does not. Indeed, using the completeness relation for the helicity-one modes we see that

$$
\varepsilon_{\alpha\beta}(0) \rightarrow -\frac{1}{\sqrt{6}} \eta_{\alpha\beta} + \text{momenta}.
$$

Thus the $\varepsilon_{\alpha\beta}(0)$ mode does not decouple but instead propagates a massless scalar degree of freedom that couples to the trace of the stress-energy.

These considerations can also be seen at the level of the Lagrangian. Let us decompose the spin-2 field $h_{\mu\nu}$ according to Eqs. (4.19-4.21)

$$
h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{m_g} (\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{\sqrt{3}} \left( \eta_{\mu\nu} + 2 \frac{\partial_\mu \partial_\nu}{m_g^2} \right) \pi.
$$

Then it is straightforward to show that

$$
\lim_{m_g^2 \rightarrow 0} \left( -\frac{1}{2} h_{\mu\nu} \varepsilon^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} - \frac{1}{4} m_g^2 (h_{\mu\nu}^2 - h^2) + \frac{1}{2M_{pl}^2} h_{\mu\nu} T_{\mu\nu} \right)
= -\frac{1}{2} \tilde{h}_{\mu\nu} \varepsilon^{\alpha\beta}_{\mu\nu} \tilde{h}_{\alpha\beta} - \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \pi \partial^2 \pi + \frac{1}{2M_{pl}^2} (\tilde{h}_{\mu\nu} + \frac{1}{\sqrt{3}} \pi \eta_{\mu\nu}) T_{\mu\nu}.
$$

Indeed, the massless limit of the Fierz-Pauli action describes the propagation of a massless helicity-2 field $\tilde{h}_{\mu\nu}$, a massless helicity-1 field $A_\mu$ (which is completely decoupled), and a massless helicity-0 field $\pi$. It is the presence of the additional helicity-0 mode that is responsible for the vDVZ discontinuity $Z_{00}^{00}(m_g^2) - Z_{00}^{00}(0) = (1/\sqrt{3})^2$.

### 4.3 St"uckelberg treatment

The dynamics of massless gravity is only specified up to changes of gauge

$$
h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial_\mu \xi_\nu,
$$

corresponding to linearized general coordinate transformations. This is not the case in massive gravity where the Fierz-Pauli mass term clearly depends on the choice of gauge. The gauge redundancy can be restored
by the introduction of so-called St"uckelberg vector fields $\pi_\mu$ (generalizing scalar St"uckelberg fields \cite{45}) and the field redefinition

$$H_{\mu\nu} = h_{\mu\nu} + \frac{1}{m_g} 2 \partial_{(\mu} \pi_{\nu)} .$$

(4.29)

In this way we reintroduce a gauge redundancy into the system

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)} , \quad \pi_{\mu} \rightarrow \pi_{\mu} - m_g \xi_{\mu} ,$$

(4.30)

with $H_{\mu\nu}$ invariant. We can introduce a further gauge redundancy for the St"uckelberg fields $\pi_\mu$ by introducing an additional St"uckelberg scalar field $\pi$ and the field redefinition

$$\Pi_\mu = \pi_\mu + \frac{1}{m_g} \partial_\mu \pi ,$$

(4.31)

such that the following gauge transformation

$$\pi_\mu \rightarrow \pi_\mu + \partial_\mu \xi , \quad \pi \rightarrow \pi - m_g \xi ,$$

(4.32)

leaves $\Pi_\mu$ unchanged.

### 4.3.1 Goldstone boson limit and identification of ghost

The introduction of St"uckelberg fields allows us to identify the correct zero mass limit, or equivalently the high-energy limit, of massive gravity, as was first pointed out by Arkani-Hamed, Georgi and Schwartz in \cite{35}. We expect from the Goldstone boson equivalence theorem that the high-energy behavior should be dominated by the longitudinal modes, i.e. we expect $\pi$ to give the leading behavior in the limit $m_g \rightarrow 0$

$$M_{\text{pl}}^{-1} h_{\mu\nu} \rightarrow - \frac{2}{M_{\text{pl}} m_g^2} \partial_\mu \partial_\nu \pi \equiv - \frac{2}{\Lambda^3} \partial_\mu \partial_\nu \pi ,$$

(4.33)

where we in the last equality identified the scale $\Lambda^3 \equiv M_{\text{pl}} m_g^2$ where interactions grow strong. These considerations also allow us to identify the ghost mode present in a mass term with a non-Fierz-Paulian structure

$$\mathcal{L}|_{\text{mass-term}} = - \frac{1}{4} m_g^2 (h_{\mu\nu}^2 - ah^2) \to - \frac{(1-a)}{m_g^2} (\partial^2 \pi)^2 ,$$

(4.34)

up to total derivatives in the limit $m_g \rightarrow 0$. For $a = 1$, i.e. the Fierz-Pauli mass term, the above term vanishes. For $a \neq 1$ the higher derivatives signify the presence of a ghost.
4.3. St"uckelberg treatment

The presence of a ghost buried in higher derivatives is illustrated by the following toy model [36]

\[ \mathcal{L}_{\text{toy-model}} = -\frac{1}{2} (\partial \pi)^2 + \frac{\lambda}{2M^2} (\partial^2 \pi)^2. \]  

(4.35)

By introducing an additional field \( \chi \) the above model is equivalent to

\[ \mathcal{L}'_{\text{toy-model}} = -\frac{1}{2} (\partial \pi)^2 - \lambda \partial \chi \cdot \partial \pi - \frac{\lambda}{2} M^2 \chi^2. \]  

(4.36)

With the change of variables \( \pi' = \pi + \chi' \) and \( \chi' = \lambda \chi \) the Lagrangian takes the form

\[ \mathcal{L}' = -\frac{1}{2} (\partial \pi')^2 + \frac{1}{2} (\partial \chi')^2 - \frac{M^2}{2\lambda} (\chi')^2. \]  

(4.37)

The wrong-sign kinetic term of \( \chi' \) tells us that this field is a ghost. In the limit \( \lambda \to 0 \) the mass of the ghost \( M^2/\lambda \) goes to infinity and the ghost is frozen, leaving us with healthy theory of a free scalar field.

4.3.2 Non-linear considerations

The gauge transformations \( h_{\mu \nu} \to h_{\mu \nu} + 2 \partial (\mu \xi_\nu) \) correspond to linearized diffeomorphisms. We would now like to restore diffeomorphism invariance also at the non-linear level. First notice that to first order in the St"uckelberg fields we have

\[ g_{\mu \nu} = \eta_{ab} \partial_\mu (x^a - M_{\text{pl}}^{-1} m^{-1}_g \pi^a) \partial_\nu (x^b - M_{\text{pl}}^{-1} m^{-1}_g \pi^b) + M_{\text{pl}}^{-1} H_{\mu \nu} + \mathcal{O}(\pi^2) \]

\[ \equiv \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + M_{\text{pl}}^{-1} H_{\mu \nu} + \mathcal{O}(\pi^2). \]  

(4.38)

From these considerations we are led to introduce four St"uckelberg fields \( \phi^a = \phi^a(x) \), \( a = 1, 2, 3, 4 \), transforming as scalars under general coordinate transformations. In terms of these fields we write the perturbations of the metric \( M_{\text{pl}}^{-1} H_{\mu \nu} \) around a general background \( g_{\mu \nu}^{(0)} \) as

\[ g_{\mu \nu}(x) = g_{\mu \nu}^{(0)}(\phi(x)) \partial_\mu \phi^a(x) \partial_\nu \phi^b(x) + M_{\text{pl}}^{-1} H_{\mu \nu}. \]  

(4.39)

From the point of view of coordinates space, \( g_{ab}^{(0)} \) is just a collection of ten scalar fields, thus \( M_{\text{pl}}^{-1} H_{\mu \nu} \) transforms as a two-tensor, just like the metric. Unitary gauge corresponds to the choice \( \phi^a(x) = \delta^a_\alpha x^\alpha \)

\[ g_{\mu \nu}(x)|_{\text{unitary-gauge}} = g_{\mu \nu}^{(0)}(x) + M_{\text{pl}}^{-1} H_{\mu \nu}, \]  

(4.40)

which correctly describes metric fluctuations around the background \( g_{\mu \nu}^{(0)} \).
This is the starting point of [37, 38] where they consider the most general set of higher order potential interactions of $H_{\mu\nu}$

$$\mathcal{L} = M_{\text{pl}}^2 \sqrt{-g} R - \frac{m_g^2}{4} \sqrt{-g} \left( \mathcal{U}_2(g, H) + \mathcal{U}_3(g, H) + \ldots \right).$$  

Here $\mathcal{U}_n(g, H)$ denotes a general $n$th degree polynomial in $H_{\mu\nu}$

$$\mathcal{U}_2(g, H) = H_{\mu\nu}^2 - H^2,$$  

$$\mathcal{U}_3(g, H) = c_1 H_{\mu\nu}^3 + c_2 H H_{\mu\nu}^2 + c_3 H^3,$$  

$$\mathcal{U}_4(g, H) = d_1 H_{\mu\nu}^4 + d_2 H H_{\mu\nu}^3 + d_3 H_{\mu\nu}^2 H_{\alpha\beta}^2 + d_4 H_{\mu\nu}^2 H^2 + d_5 H^4,$$  

...  

The coefficients of the second-degree polynomial $\mathcal{U}_2$ are fixed by the Fierz-Pauli structure. The a priori arbitrary coefficients $c_i, d_i, \ldots$ of the polynomials $\mathcal{U}_3, \mathcal{U}_4, \ldots$ are fixed by demanding the absence of ghosts.

In [37, 38] the above action was analyzed in the so-called “decoupling limit”, which comprise a double-scaling limit

$$m_g \to 0, \quad M_{\text{pl}} \to \infty, \quad \Lambda^3 \equiv M_{\text{pl}} m_g^2 = \text{fixed}.$$  

This limit allows us to isolate the helicity-0 mode and study its dynamics. The amazing result of [38] is that, in this limit, they manage to tune the coefficients of the above expansion such that no ghosts appear in the spectrum. Moreover, in this limit all but the quadratic, cubic and quartic interactions vanish. Thus the infinite series in Eq. (4.41) is reduced to a finite polynomial with fixed coefficients apart from three overall coefficients $\alpha, \beta$ and $\gamma$ of the quadratic, cubic and quartic interactions respectively. The final result takes the form [46]

$$\mathcal{L} = -\frac{1}{2} h_{\mu\nu} \varepsilon_{\alpha\beta}^\mu h_{\alpha\beta} + h_{\mu\nu} \left( \alpha X_{\mu\nu}^{(1)} + \frac{\beta}{\Lambda^3} X_{\mu\nu}^{(2)} + \frac{\gamma}{\Lambda^6} X_{\mu\nu}^{(3)} \right),$$  

where the $X^{(n)}$ are certain $n$th powers of $\Pi_{\mu\nu} \equiv \partial_{\mu} \partial_{\nu} \pi$

$$X_{\mu\nu}^{(1)} = \varepsilon_{\mu}^{\alpha\rho\sigma} \varepsilon_{\nu}^{\beta} \Pi_{\alpha\beta},$$  

$$X_{\mu\nu}^{(2)} = \varepsilon_{\mu}^{\alpha\rho\sigma} \varepsilon_{\nu}^{\beta\gamma} \Pi_{\alpha\beta} \Pi_{\rho\gamma},$$  

$$X_{\mu\nu}^{(3)} = \varepsilon_{\mu}^{\alpha\rho\sigma} \varepsilon_{\nu}^{\beta\gamma\delta} \Pi_{\alpha\beta} \Pi_{\rho\gamma} \Pi_{\sigma\delta}.$$  

These terms are familiar from so-called “Galilean” theories, i.e. theories with the Galilean symmetry $\pi \to \pi + c + b \cdot x$. The action in Eq. (4.46) is the starting point for the investigations in Paper III.
Flux compactifications of type IIB string theory provide a promising framework for phenomenological and cosmological studies in string theory. The applications range from studies of inflationary and late-time cosmology, to low energy phenomenology such as dynamical supersymmetry breaking. Flux compactifications also provide a testing ground for generalizations of the AdS/CFT correspondence.

Perhaps the greatest challenge faced by string theory model builders is that of finding a stable vacuum describing the accelerated expansion of the universe [47, 48]. The vacuum structure of string theory is governed by the low energy effective action of various “moduli fields”, whose vacuum expectation values capture the geometry of the extra dimensions present in the compactification. The (often numerous) moduli fields must be stabilized in a vacuum with desirable properties such as a small positive cosmological constant and broken supersymmetry. A model with these characteristics was put forward by Kachru, Kallosh, Linde and Trivedi (KKLT) in 2003 [49] where they allowed for the possibility of having a long-lived metastable (as opposed to absolutely stable) de Sitter vacuum. The small and positive cosmological constant of KKLT is generated by an anti-D3-brane residing in a so-called “warped throat region” of the compactification, i.e. a region that is highly warped by a collection of D3-branes, as will be explained below. The presence of the anti-D3-brane, in the D3-brane background, lifts the negative supersymmetric vacuum energy of anti-de Sitter space to the positive supersymmetry breaking vacuum energy of de Sitter space. Furthermore, in KKLT the geometric moduli are stabilized, i.e. their expectation values are determined by minimizing the effective action. For example, the complex structure moduli
and dilaton are stabilized by fluxes and the Kähler moduli can be stabilized by non-perturbative effects. However, open string fields, associated with the position of the anti-D3-brane in the warped throat region, give rise to new moduli sectors whose stabilization is not ensured. It was noticed by Aharony, Antebi and Berkooz (AAB) in 2005 [50] that, although the background flux produces a potential for the moduli describing the position of the anti-D3-brane in the radial direction, the moduli describing the position in the angular directions correspond to massless Goldstone bosons, since to leading order the throat region without anti-D3-branes has continuous symmetries in the angular directions. AAB also characterized a restricted class of effects that could lift the angular moduli and found that, although non-zero, the masses were small.

However, the effective action for $D3$-branes in KKLT compactifications has been the subject of much more investigation [51, 52, 53, 54, 55], partly motivated by studies of brane-anti-brane inflation in warped throat regions. An example of such a construction is the inflationary model of [51] involving the attraction of a D3-brane towards an anti-D3-brane, with the brane-anti-brane potential energy driving inflation. Here contributions to the potential due to moduli stabilization have been incorporated in great detail [56, 53].

The success of these models rests partly on the presence of a warped throat region of the compactification. Such regions naturally occur when sources, such as D-branes, are included in the compactification. The tension and charge of the D-brane warp the surrounding space, yielding a “throat-like” region of the compactification. Initially, warped compactifications were envisaged as a way towards generating large hierarchies in string theory [57], which can for example be used to address naturalness problems. Further, warped throat regions are of great importance since they allow for full calculability and control over closed string moduli. The key property of such a region is that it can approximated by a finite segment of an infinite throat region for which the supergravity solution is completely known. The most well-studied example is the Klebanov-Strassler (KS) throat [58], which constitutes a non-compact and supersymmetric throat where the supergravity solution is known. However, realistic model building requires a finite throat region and the presence of supersymmetry breaking. Compactness is ensured by gluing the UV region of the throat (the far region) into a compact “bulk”. Supersymmetry breaking on the other hand can be taken into account by placing anti-D3-branes in the IR region of the throat (the near region). Thus, in
general the throat region is modified in both the UV and the IR. From the viewpoint of the supergravity fields in the throat, the deviations from the supersymmetric infinite throat approximation determine boundary conditions on a “UV-brane” and an “IR-brane”, as illustrated in Fig. 5.1. Then one could in principle pursue a solution for fields with the corresponding boundary conditions, in a perturbation expansion around the non-compact supersymmetric solution. This is the approach we take in Paper IV.

As an application the formalism developed in Paper IV, we determine the effective action of an anti-D3-brane residing at the tip of KS. As was noticed by AAB, the isometries of the throat (protecting the angular anti-D3-brane moduli) are necessarily broken when the throat is glued into a compact bulk. Thus, masses for the moduli are generated by compactification effects. In AAB the considerations were restricted to supersymmetry preserving and “no-scale” preserving effects, i.e. effects that do not lift the moduli for the D3-potential. The sizes of the various effects were estimated in the dual field theory (see below for a short discussion of what “dual” means here) by listing the several most relevant operators that could perturb the conformal field theory (CFT) Lagrangian. In Paper V we argue that perturbations that produce a force on a probe D3-brane introduces a new contribution to the anti-D3-brane potential that is parametrically larger than the leading contribution obtained in prior
work. The estimation of sizes of perturbations is performed by listing all
the most relevant possible perturbations of the supergravity solution.

There are also other complementary directions to KKLT, for example
the search for classically stabilized de Sitter vacua [59].

5.1 Warped compactifications in type IIB string
theory

In this section we give a general introduction to warped compactifications
in type IIB string theory.

5.1.1 Low energy action of type IIB string theory

In the conventions of [60] the bosonic part of the low energy action of
type IIB string theory is given by

\[
S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( R - \frac{\nabla \tau \cdot \nabla \tau}{2(\text{Im} \tau)^2} - \frac{G_3 \cdot \overline{G}_3}{12 \text{Im} \tau} - \frac{\tilde{F}_5^2}{4 \cdot 5!} \right) \\
+ \frac{1}{8i\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \overline{G}_3}{\text{Im} \tau} + S_{\text{loc}}. 
\]

(5.1)

In the above, the Ricci scalar, the covariant derivatives, and all contrac-
tions are constructed/performed using the full ten-dimensional metric.
The R-R axion $C_0$ and the NS-NS dilaton $\Phi$ are combined into a complex
scalar field

\[
\tau = C_0 + ie^{-\Phi},
\]

(5.2)

and the NS-NS three-form field strength $H_3 = dB_2$ and the R-R three-
form field strength $F_3 = dC_2$ are combined into a complex three-form

\[
G_3 = F_3 - \tau H_3,
\]

(5.3)

while the R-R five-form field strength $F_5 = dC_4$ enters in the combination

\[
\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3.
\]

(5.4)

The physical constraint $\ast \tilde{F}_5 = \tilde{F}_5$ should be imposed at the level of the
equations of motion.
5.1. Warped compactifications in type IIB string theory

The term $S_{\text{loc}}$ in Eq. (5.1) refers to the action of localized sources, e.g. D-branes and O-planes. For example, in the AdS/CFT correspondence, which we discuss below, the supergravity equations are sourced by a stack of $N$ D3-branes, with $S_{\text{loc}}$ given by Eq. (5.21) below. In the above formulation the action has a manifest $SL(2, \mathbb{R})$ symmetry

$$
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad G_3 \rightarrow \frac{G_3}{c\tau + d},
$$

(5.5)

with all other fields left unchanged.

5.1.2 Equations of motion and ansatz

Realistic models that approximate four-dimensional quantum field theories at low energies are obtained by compactifying the six extra dimensions. In [57], Giddings, Kachru and Polchinski describe a class of so-called “warped compactifications”, which is the most general class of compactifications consistent with four-dimensional Poincaré invariance. This amounts to compactifying the ten-dimensional theory down to a warped product of a four-dimensional Minkowski space $\mathbb{R}^{1,3}$ and a six-dimensional compact manifold $M_6$

$$
ds^2 = e^{2A(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-2A(y)} g_{mn}(y) dy^m dy^n.
$$

(5.6)

To preserve the symmetries of the four-dimensional metric $g_{\mu\nu}(x)$, the warp factor $e^{2A(y)}$ is only allowed to vary over the six-dimensional space $M_6$, which we equip with a metric $g_{mn}(y)$ in addition to the warp factor. We generalize the ansatz of [57] slightly by allowing for a possible four-dimensional cosmological constant $\Lambda$, i.e. we take $g_{\mu\nu}(x)$ to be the metric of maximally symmetric four-dimensional spacetime. For the present day value of $\Lambda$ this generalization has no effect on issues of the stability of the compactification but during inflation such a term is important and it contributes to the well-known $\eta$-problem of models of inflation in supergravity [55].

Consistent with these symmetries, the five-form flux must have components either orthogonal or proportional to the four-dimensional volume-form

$$
\hat{F}_5 = (1 + \star) d\alpha(y) \wedge \sqrt{-\det g_{\mu\nu}} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.
$$

(5.7)

The three-form flux $G_3$ can only have components in the extra dimensions, i.e. we consider only non-vanishing $G_{mnl}(y)$. Finally, we need the axidilaton $\tau = \tau(y)$ to be a function of the extra dimensions only.
If we define the quantities

\[ G_\pm = (\star_6 \pm i)G_3, \quad (5.8a) \]
\[ \Phi_\pm = e^{4A} \pm \alpha, \quad (5.8b) \]
\[ \Lambda = \Phi_+ G_- + \Phi_- G_+, \quad (5.8c) \]

then the equations of motion, that follows from the action in Eq. (5.1), and the Bianchi identities take the form

\[ \nabla^2 \Phi_\pm = \frac{(\Phi_+ + \Phi_-)^2}{96 \text{Im} \tau} |G_\pm|^2 + \mathcal{R}_4 + \frac{2}{\Phi_+ + \Phi_-} |\nabla \Phi_\pm|^2, \quad (5.9) \]
\[ d\Lambda + \frac{i}{2 \text{Im} \tau} d\tau \wedge (\Lambda + \bar{\Lambda}) = 0, \quad (5.10) \]
\[ d(G_3 + \tau H_3) = 0, \quad (5.11) \]
\[ \nabla^2 \tau = \frac{\nabla \tau \cdot \nabla \tau}{i \text{Im} (\tau)} + \frac{\Phi_+ + \Phi_-}{48i} G_+ \cdot G_-, \quad (5.12) \]
\[ R^6_{mn} = \frac{\nabla (m \nabla n)^\tau}{2 (\text{Im} \tau)^2} + \frac{2}{(\Phi_+ + \Phi_-)^2} \nabla (m \Phi_+ + n \Phi_-) - g_{mn} \frac{\mathcal{R}_4}{2 (\Phi_+ + \Phi_-)} - \frac{\Phi_+ + \Phi_-}{32 \text{Im} \tau} \left( G_+ (m \tilde{G}_{-n})_{pq} + G_- (m \tilde{G}_+ n)_{pq} \right), \quad (5.13) \]

where \( \mathcal{R}_4 \) is the four-dimensional Ricci scalar of \( g_{\mu \nu} \), and covariant derivatives \( \nabla_m \) and contractions are constructed and performed using \( g_{mn} \). The contribution from localized sources is suppressed in the above.

### 5.2 Warped compactifications on Calabi-Yau cones

We now turn to the discussion on some famous solutions to Eqs. (5.9-5.13). These solutions play an important role in the KKLT construction, as explained in the introduction of this chapter. They are also key ingredients in the \textit{AdS/CFT correspondence}, which we now discuss.

The original form of the AdS/CFT correspondence [61, 62, 63] is a duality between \( \mathcal{N} = 4 \) superconformal \( SU(N) \) Yang-Mills theory in four dimensions and type IIB superstring theory in ten-dimensional \( AdS_5 \times S^5 \). The field theory side gives a description of the world-volume theory living on a stack of \( N \) D3-branes while the gravity side gives a dual description in terms of supergravity solutions with \( F_5 \) flux carrying \( N \) units of D3-brane charge.
The most well-studied generalizations of the gauge/gravity correspondence involve placing D-branes at conical singularities. In the Klebanov-Witten (KW) theory [64] one places a stack of $N$ D3-branes at the singularity of the conifold $\mathcal{C}$. The conifold is a Calabi-Yau space and preserves $1/4$ of the supersymmetries, which leads to a duality between a certain $\mathcal{N} = 1$ superconformal $SU(N)$ gauge theory and strings in $AdS_5 \times T^{11}$, where $T^{11}$ is the base space of the conifold $\mathcal{C}$, which will be discussed in detail below. The breaking of conformality is described by the Klebanov-Tseytlin (KT) theory [65], which is a generalization of the KW theory by the inclusion of fractional D3-branes with magnetic $F_3$ flux that causes the gauge couplings to run logarithmically. In the Klebanov-Strassler theory (KS) [58], the conic singularity of the conifold is deformed by quantum effects and blown up to a finite size, which in the dual field theory corresponds to condensation of gluinos, i.e. condensation of the fermion superpartners of the gluon fields [58].

### 5.2.1 The AdS/CFT correspondence

We now wish to study a stack of $N$ D3-branes placed at a smooth point in the extra dimensions

\[ N \text{ D3-branes} \rightarrow \text{Smooth Calabi-Yau } \mathcal{M}_6. \]  

(5.14)

In the following we will give an approximate description of the setup by replacing the compact Calabi-Yau space $\mathcal{M}_6$ with a non-compact plane $\mathbb{R}^6$. This approximation is only valid close to the stack of branes and we expect corrections from the gluing of $\mathbb{R}^6$ into the compact bulk $\mathcal{M}_6$, as discussed in the introduction of this chapter.

The metric describing this setup is a warped product of four-dimensional Minkowski space and the six-dimensional plane

\[ ds^2 = h^{-1/2}(r)(-dt^2 + d\vec{x}^2) + h^{1/2}(r)(dr^2 + r^2 d\Omega_5^2), \]  

(5.15)

where the warp factor $h^{-1/2}(r) = e^{2A(r)}$ is given by

\[ h(r) = 1 + \frac{L^4}{r^4}, \]  

(5.16)

with $L^4 = 4\pi g_s N(\alpha')^2$. The metric in Eq. (5.15) is illustrated in Fig. 5.2.
Chapter 5. String compactifications

Figure 5.2. The placing of a stack of $N$ D3-branes on a smooth point of the plane $\mathbb{R}^6$ (the right figure) yields a throat-like space (the left figure) that is highly warped in the vicinity of the D3-branes. The plane $\mathbb{R}^6$ is described by the metric $ds^2 = dr^2 + r^2 ds_{S^5}^2$, with the five-spheres $S^5$ depicted as circles of constant radius in the above. The warped throat is described by the metric $ds^2 = h^{1/2}(r)(dr^2 + r^2 ds_{S^5}^2)$, with warp-factor $h(r) = 1 + L^4/r^4$. In the near horizon limit $r \to 0$ the warped metric takes the form $ds^2 \to dr^2/r^2 + L^2 ds_{S^5}^2$, i.e. the $S^5$ takes on a finite radius $L$ (the far most left in the throat).

As for the other fields, the axi-dilaton $\tau$ is constant, the three-form flux $G_\pm$ vanishes and only the five-form flux, sourced by the $N$ D3-branes, has a non-trivial profile

$$F_5 = (1 + *)F_5, \quad \mathcal{F}_5 = 16\pi(\alpha')^2 N \text{ vol } S^5,$$

where $\text{vol } S^5$ is the volume-form of the $S^5$. The D3-brane charge of the solution is obtained by integrating the field strength over an $S^5$ centered at the origin

$$Q_{\text{D3}} = \frac{1}{(4\pi^2\alpha')^2} \int_{S^5} *F_5 = N.$$

In this sense we can say that the solution describes a stack of $N$ D3-branes situated at the origin. However, as seen by taking the near horizon limit this is not quite true, but rather the whole throat region describes the D3-branes.
The near horizon limit consists of taking \( r \to 0 \)
\[
ds^2 \to \frac{L^2}{z^2} (-dt^2 + dx^2 + dz^2) + L^2 d\Omega_5^2,
\]
where we have defined \( z = L^2/r \). This is the metric of \( AdS_5 \times S^5 \), each factor with a radius \( L \). The above parametrization of \( AdS_5 \) is known as the Poincaré parametrization and it is well-known that it does not cover the whole of \( AdS_5 \). In fact the region \( z = \infty \) \( (r = 0) \) corresponds to the interior of \( AdS_5 \) and the metric in Eq. (5.19) can be continued past this region by going to the so-called global parametrization of \( AdS_5 \), which we will not need here. The region \( z = 0 \) \( (r = \infty) \) corresponds to the boundary of \( AdS_5 \). In the original metric Eq. (5.15) this region times the five-sphere is glued to ten-dimensional flat space \( r \to \infty \)
\[
ds^2 \to -dt^2 + dx^2 + dr^2 + r^2 d\Omega_5^2.
\]
In the dual description we describe the setup in terms of open strings living on the stack of D3-branes in ten-dimensional flat spacetime. Thus we arrive at a duality between strings in \( AdS_5 \times S^5 \) and the world-volume theory of the \( N \) D3-branes, for which the low-energy limit is a four-dimensional superconformal \( \mathcal{N} = 4 \) SU\((N) \) Yang-Mills theory.

The relation between the gauge coupling \( g_{YM} \) and the string coupling \( g_s \) can be read off from the Dirac-Born-Infeld and the Wess-Zumino terms of the low-energy action describing the fluctuations of the D3-branes
\[
S_{\text{loc}} = S_{\text{DBI}} + S_{WZ},
\]
where
\[
S_{\text{DBI}} = -T_3 \int_{D3} d^4x \text{tr} \left( e^{-\Phi} \left[ - \det \{ g_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} \} \right]^{1/2} \right),
\]
\[
S_{WZ} = iT_3 \int_{D3} \text{tr} \left( \exp \left[ 2\pi\alpha' F_2 + B_2 \right] \wedge \sum_q C_q \right).
\]
From these expressions we find
\[
S_{\text{DBI}} \ni -\frac{1}{4g_{YM}^2} \int d^4x F_{\mu\nu} F^{\mu\nu} \implies \frac{1}{g_{YM}^2} \sim \frac{1}{g_s},
\]
\[
S_{WZ} \ni -\frac{\theta_{YM}}{32\pi} \int d^4x F_{\mu\nu} \tilde{F}^{\mu\nu} \implies \theta_{YM} \sim iC_0.
\]
Now, the supergravity description is only valid for large curvature radius \((L/\sqrt{\alpha'})^4 = 4\pi g_s N\). Thus we require that the 't Hooft coupling
$g_{\text{YM}}^2 N \sim g_s N \gg 1$ is large, which corresponds to strong coupling in the field theory.

### 5.2.2 Singular Calabi-Yau cones

Next, we wish to generalize the above setting by placing the stack of $N$ D3-branes at a conic singularity in the extra dimensions

\[
N \text{ D3-branes} \quad \rightarrow \quad \text{Calabi-Yau } M_6 \text{ with conic singularities}
\]

That is, we consider a conical region of the Calabi-Yau $M_6$ approximated by a cone $C_6$ over some base space $B_5$. This will lead to new interesting warped solutions with near horizon geometry $AdS_5 \times B_5$, as illustrated Fig. 5.3. The most well-studied example is the so-called conifold $C$ with base $T^{11}$. We now give a digression on the conifold before coming back to the supergravity description. Also, in the appendix a summary is provided of some of the methods used in the following, but illustrated in the simpler cases of $S^2$ and $S^3$. For more explanations, see the classic review [66].

### 5.2.3 The conifold

The conifold $C$ is the set of points $z^a = (z^1, z^2, z^3, z^4)$ in $\mathbb{C}^4$ satisfying [67]

\[
C: \quad \sum_{a=1}^{4} (z^a)^2 = 0. \quad (5.27)
\]

As the name suggests, the conifold is indeed a cone since a rescaling $z^a \rightarrow \lambda z^a$, with $\lambda \in \mathbb{R}_+$, leaves Eq. (5.27) invariant. The apex of the cone, $z^a = 0$, is singular since there all the tangents to the surface vanish $d(\sum_{a=1}^{4} (z^a)^2)|_{z^a=0} = 0$. Therefore we also refer to the conifold as the singular conifold, as opposed to the resolved and deformed conifolds to be introduced below.

The base of the cone $C/\mathbb{R}_+$ defines the space $T^{11}$ (or more generally the class of $T^{pq}$ spaces to be defined below). The base space can be
obtained from the intersection of the conifold $C$ with unit seven-sphere $S^7$ inside $\mathbb{C}^4$, i.e.

\[
T^{11}: \sum_{a=1}^{4} (z^a)^2 = 0, \quad \sum_{a=1}^{4} |z^a|^2 = 1.
\] (5.28)

In terms of the real and imaginary parts $z^a = x^a + iy^a$ these equations become

\[
T^{11}: \quad x \cdot x = \frac{1}{2}, \quad y \cdot y = \frac{1}{2}, \quad x \cdot y = 0.
\] (5.29)

The first equation defines a three-sphere $S^3$ or radius $1/\sqrt{2}$. The second equation defines a two-sphere $S^2$ fibered over the three-sphere $S^3$ when taking into account the last equation. This is illustrated in Fig. 5.4.

We now solve the equations in Eq. (5.28) explicitly following [67]. First write

\[
Z = \sum_{a=1}^{4} z^a \sigma^a = \begin{pmatrix}
    z^3 + i z^0 & z^1 - i z^2 \\
    z^1 + i z^2 & -z^3 + i z^4
\end{pmatrix},
\] (5.30)

where $\sigma^a = (\bar{\sigma}, i1)$ and $\bar{\sigma}$ are the Pauli sigma matrices. Then the conditions in Eq. (5.28) translates into

\[
T^{11}: \quad \det Z = 0, \quad \text{tr} \, Z^\dagger Z = 1.
\] (5.31)
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Figure 5.4. The space $T^{11}$ as a fiber bundle $S^2$ over $S^3$. The inner (shadowed) sphere represents an $S^3_x$ defined by $x \cdot x = 1/2$, while the outer (transparent) sphere represents an $S^3_y$ defined by $y \cdot y = 1/2$. Given a point $x \in S^3_x$ the requirement $x \cdot y = 0$ forces $y \in S^3_y$ to lie in the space orthogonal to $x$, i.e. $y$ lies on the great circle, representing an $S^2_y$, that is obtained by intersecting $S^3_y$ with the plane orthogonal to $x$. E.g. for $x = \text{North Pole}$ then $S^2_y = \text{Equator}$. In this way, as we move across $S^3_x$ we pick out fibers $S^2_y \subset S^3_y$ and the fiber bundle of $S^2_y$ over $S^3_x$ defines $T^{11}$.

Now, given a particular solution to Eq. (5.31), say

$$Z_0 = \frac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(5.32)

then the general solution to Eq. (5.31) takes the form

$$Z = LZ_0 R^\dagger,$$

(5.33)

for a general pair of matrices $(L, R) \in SU_L(2) \times SU_R(2)$. However, this identification is not one-to-one since for

$$H = e^{i\sigma^3} = \begin{pmatrix} e^{ih} & 0 \\ 0 & e^{-ih} \end{pmatrix},$$

(5.34)

with $HZ_0H = Z_0$, we have that $(LH, RH^\dagger)$ defines the same point as $(L, R)$. Thus we are led to identify $(L, R) \cong (LH, RH^\dagger)$ and $T^{11}$ is given by

$$T^{11} = (SU_L(2) \times SU_R(2)) / U_H(1),$$

(5.35)

where the $U_H(1)$ is generated by $\frac{1}{2}\sigma^3_L - \frac{1}{2}\sigma^3_R$. 
More generally we define the $T^{pq}$ space as that obtained by identifying $(L, R) \simeq (L\Theta^q, R(\Theta^1)^p)$, for coprime $p, q \in \mathbb{Z}$. We will only be interested in $p = q = 1$, since for this choice the space preserves supersymmetry [68].

**Metric**

The conifold can be equipped with a cone metric [67]

$$ds_C^2 = dr^2 + r^2 ds_{T^{11}}^2,$$  \hspace{2cm} (5.36)

where $ds_{T^{11}}^2$ is the metric of $T^{11}$. The construction of a metric on a group space, or more generally a coset space, is straightforward. The details can be found in the appendix of Paper V, and the metric is

$$ds_{T^{11}}^2 = \frac{1}{9} (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2)^2$$

$$+ \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2),$$  \hspace{2cm} (5.37)

where $\psi/2 \in S^1$, $(\theta_1, \varphi_1) \in S^2$ and $(\theta_2, \varphi_2) \in S^2$.

**5.2.4 Resolving and deforming the conifold**

The singular behavior at the apex of the conifold can be lifted in two different ways: either by resolving or by deforming. The resolution of the conifold consists of replacing the original equation by

$$\det \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = 0 \xrightarrow{\text{Resolution}} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0,$$  \hspace{2cm} (5.38)

with $(\lambda_1, \lambda_2) \neq (0, 0)$. This replaces the singular apex with a finite size $S^2$. The deformation consists of performing the replacement

$$\sum_{a=1}^{2} (z^a)^2 = 0 \xrightarrow{\text{Deformation}} \sum_{a=1}^{2} (z^a)^2 = \epsilon^2,$$  \hspace{2cm} (5.39)

which instead yields a finite size $S^3$ at the former apex. This is illustrated in Fig. 5.5.
Figure 5.5. The singular conifold can be made smooth by either resolving Eq. (5.38) or deforming Eq. (5.2.6) the apex of the cone. In the case of the resolution we blow up an $S^2$ at the apex while in the case of the deformation we blow up an $S^3$, as illustrated in the figure.

5.2.5 The resolved conifold

The metric of the resolved conifold can be put on the form \[ \text{d}s^2_{\text{Res}} = \kappa^{-1}(\rho)\text{d}\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2\left(\text{d}\psi + \cos \theta_1 \text{d}\varphi_1 + \cos \theta_2 \text{d}\varphi_2\right)^2 \] \[ + \frac{1}{6}\rho^2 \left[ \text{d}\theta_1^2 + \sin^2 \theta_1 \text{d}\varphi_1^2 \right] + \frac{1}{6}(\rho^2 + 6a^2) \left[ \text{d}\theta_2^2 + \sin^2 \theta_2 \text{d}\varphi_2^2 \right], \] where $\kappa(\rho) = (\rho^2 + 9a^2)/(\rho^2 + 6a^2)$.

In the limit $\rho \to 0$ the resolved conifold takes the form of a two-sphere with radius $a$

\[ \text{d}s^2_{\text{Res}} = a^2 \left[ \text{d}\theta_2^2 + \sin^2 \theta_2 \text{d}\varphi_2^2 \right] + \mathcal{O}(\rho^2), \] (5.41)

thus confirming the picture in Fig. 5.5. In the limit $\rho \to \infty$ the resolved conifold takes the form

\[ \text{d}s^2_{\text{Res}} = \text{d}r^2 + r^2 \text{d}s^2_{T^{11}} + 3a^2 \delta(\text{d}s^2_{\text{Res}}) + \mathcal{O}(r^{-2}), \] (5.42)

where we have defined a new radial variable $r$ by $\rho = r - 3a^2/r$. To leading order, the resolved conifold reproduces the form of the singular conifold $\text{d}s^2_{\text{Sing}} = \text{d}r^2 + r^2 \text{d}s^2_{T^{11}}$. At order $\mathcal{O}(r^{-2})$, with respect to the leading term, the metric takes the form $3a^2 \delta(\text{d}s^2_{\text{Res}})$, where

\[ \delta(\text{d}s^2_{\text{Res}}) = \frac{1}{6} \left[ \text{d}\theta_1^2 + \sin^2 \theta_1 \text{d}\varphi_1^2 \right] - \frac{1}{6} \left[ \text{d}\theta_2^2 + \sin^2 \theta_2 \text{d}\varphi_2^2 \right]. \] (5.43)

In Paper V we show that the symmetric two-tensor $\delta(\text{d}s^2_{\text{Res}})_{\alpha\beta}$ is a traceless and divergence-free eigentensor of the Lichnerowicz operator, $\Delta_{T^{11}} \delta(\text{d}s^2_{\text{Res}})_{\alpha\beta} = \lambda \delta(\text{d}s^2_{\text{Res}})_{\alpha\beta}$, with eigenvalue $\lambda = 4$. In Paper IV we
show that the dimension $\Delta$ of the operator dual to a homogeneous solution to the supergravity equations with Lichnerowicz eigenvalue $\lambda$ is given by $\Delta = 2 + \sqrt{\lambda - 4}$, see Tab. 7.1. Thus, the dimension of $\delta(ds^2_{\text{Res}})$ is $\Delta = 2$, in agreement with the radial scaling in Eq. (5.42), and confirming that $\delta(ds^2_{\text{Res}})$ is a homogenous solution of the supergravity equations.

5.2.6 The deformed conifold

The metric of the deformed conifold takes the form [70]

$$ds^2_{\text{Def}} = e^{4/3} K(\tau) \left\{ \frac{\sinh^3 \tau}{3(\sinh 2\tau - 2\tau)} \left[ d\tau^2 + (d\psi + \cos \theta_1 d\varphi_1 + \cos \theta_2 d\varphi_2)^2 \right] + \frac{1}{4} \cosh \tau \left[ d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2 \right] + \frac{1}{2} \left[ \sin \psi (\sin \theta_2 d\theta_1 d\varphi_2 + \sin \theta_1 d\theta_2 d\varphi_1) + \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\varphi_1 d\varphi_2) \right] \right\}, \quad (5.44)$$

where $K(\tau) = (\sinh 2\tau - 2\tau)^{1/3}/(2^{1/3} \sinh \tau)$.

In the limit $\tau \to \infty$ the deformed conifold takes the form

$$ds^2_{\text{Def}} = dr^2 + r^2 ds^2_{T^{11}} + e^2 (3/2)^{3/2} (3r)^{-1} \delta(ds^2_{\text{Def}}) + \mathcal{O}(r^{-4}), \quad (5.45)$$

where we have defined a new radial variable $r^3 = e^2 (3/2)^{3/2} \cosh \tau$. To leading order the deformed conifold reproduces the form of the singular conifold $ds^2_{\text{Sing}} = dr^2 + r^2 ds^2_{T^{11}}$. At order $\mathcal{O}(r^{-3})$, with respect to the leading term, the metric takes the form $e^2 (3/2)^{3/2} (3r)^{-1} \delta(ds^2_{\text{Def}})$ where

$$\delta(ds^2_{\text{Def}}) = \sin \psi (\sin \theta_2 d\theta_1 d\varphi_2 + \sin \theta_1 d\theta_2 d\varphi_1) + \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\varphi_1 d\varphi_2). \quad (5.46)$$

In paper Paper V and Paper IV we show that the symmetric two-tensor $\delta(ds^2_{\text{Def}})_{\alpha\beta}$ is a traceless and divergence-free eigentensor of the Lichnerowicz operator with eigenvalue $\lambda = 5$ and dimension $\Delta = 3$, in agreement with the radial scaling in Eq. (5.42).

5.3 Conifold theories

We now study the generalizations of the AdS/CFT correspondence to theories on the conifold. This will lead to supergravity descriptions of
effects such as chiral symmetry breaking and condensation in the dual field theory. For the curious reader, there is a review on the subject [71] with one author whose name keeps reappearing in all the titles of the following subsections.

5.3.1 Klebanov-Witten theory

The supergravity solution and the dual field theory of a stack of $N$ D3 branes placed at the apex of the conifold was constructed in [64]. The metric takes the form of a warped product of flat four-dimensional space and the conifold

$$ds^2 = h^{-1/2}(r)(-dt^2 + dx^2) + h^{1/2}(r)(dr^2 + r^2 ds^2_{\mathbb{T}^{11}}), \quad (5.47)$$

where the warp factor is given by

$$h(r) = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N(\alpha')^2 \frac{27}{16}, \quad (5.48)$$

and the factor of $27/16$ comes from the volume $\text{vol} T^{11} = \frac{16}{27}\pi^3$ (compared to the volume $\text{vol} S^5 = \pi^3$).

The dual field theory is an $\mathcal{N} = 1$ superconformal $SU(N) \times SU(N)$ gauge theory with two chiral $SU(2)$ doublets $A_i, B_i, i = 1, 2$ in $(\mathbb{N}, \bar{\mathbb{N}})$, $(\bar{\mathbb{N}}, \mathbb{N})$ and superpotential

$$W = \epsilon_{ij} \epsilon^{kl} \text{tr} A_i B_k A_j B_l. \quad (5.49)$$

This superpotential is the most general marginal one consistent with the gauge symmetries and the $U_R(1)$ charge assignment $R_A = R_B = 1/2$, fixed by $R$-anomaly cancellation.

5.3.2 Klebanov-Tseytlin theory

The conformal invariance of the dual Klebanov-Witten field theory can be broken by the addition of ISD three-form flux $G_+$ on the supergravity side. In [65] the supergravity solution describing $M$ wrapped D5-branes on an $S^2$ of $T^{11}$ in addition to the $N$ D3-branes at the apex of the conifold was constructed. Now, a D5-brane couples to $C_6$, with field strength $F_7 = dC_6$ and magnetic dual $F_3 = *F_7$, so in addition to the $N$ units of
5.3. Conifold theories

$F_5$ flux sourced by the D3-branes, the solution also contains $M$ units of magnetic $F_3$ flux

$$\frac{1}{4\pi\alpha'} \int_{S^3} F_3 = M .$$

(5.50)

In [72] it is shown that the presence of $F_3$ flux turns on $H_3$ flux in the directions orthogonal to the $S^3$, i.e. $H_3 = g_s^{-1} \ast_6 F_3$ so that the solution carries ISD flux $G_3 = F_3 - \tau H_3$ with

$$\ast_6 G_3 = i G_3 .$$

(5.51)

Now, since $H_3$ is orthogonal to $S^3$, then $H_3 \propto \frac{dr}{r} \wedge \text{vol} S^2$, which implies that the two-form $B_2 \propto \log r \cdot \text{vol} S^2$ runs logarithmically. The flux also sources the metric through Einstein’s equations

$$R = \frac{1}{24} \left( H_3^2 + g_s^2 F_3^2 \right) ,$$

(5.52)

which causes the warp factor of the ansatz

$$d s^2 = h^{-1/2}(r) (-d t^2 + d \vec{x}^2) + h^{1/2}(r) (dr^2 + r^2 d s_{T_{11}}^2) ,$$

(5.53)

to run logarithmically

$$h(r) = \frac{27\pi (\alpha')^2}{4r^4} \left( g_s N + \frac{3(g_s M)^2}{2\pi} \left[ \ln r + \frac{1}{4} \right] \right) .$$

(5.54)

In the dual field theory, which is an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(N) \times SU(N + M)$, this causes the gauge couplings to run [72]

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s} = \text{const.} ,$$

(5.55)

$$\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} = \frac{1}{4\pi\alpha' g_s} \int_{S^2} B_2 = 6M \ln r + \text{const.} ,$$

(5.56)

with $\ln r = \ln \Lambda$ identified with the logarithm of the energy scale relative to some initial scale.
Let us now consider this directly in the dual field theory. The running
gauge coupling of a general $\mathcal{N} = 1$ theory is given by [73, 74, 75]

$$
\Lambda \frac{d}{d\Lambda} \left( \frac{8\pi^2}{g^2} \right) = 3 T(\text{Ad}) - \sum_i T(r_i)(1 - \gamma_i), \quad (5.57)
$$

where the Dynkin index is $T(\text{Ad}) = N$ for an $SU(N)$ gauge theory,
$T(r_i) = \frac{1}{2}$ for a field in the fundamental representation, and $\gamma_i$ denote
the anomalous dimensions. In our case

$$
\Lambda \frac{d}{d\Lambda} \left( \frac{8\pi^2}{g_1^2} \right) = 3 \cdot \left( \frac{N + M}{2} \right) - \frac{1}{2}(1 - \gamma) \cdot \frac{2}{SU(2)} \cdot \frac{2}{A,B} \cdot \frac{N}{SU(N)}, \quad (5.58)
$$

$$
\Lambda \frac{d}{d\Lambda} \left( \frac{8\pi^2}{g_2^2} \right) = 3 \cdot \left( \frac{N}{2} \right) - \frac{1}{2}(1 - \gamma) \cdot \frac{2}{SU(2)} \cdot \frac{2}{A,B} \cdot \left( \frac{N + M}{SU(N+M)} \right), \quad (5.59)
$$

which gives

$$
\Lambda \frac{d}{d\Lambda} \left( \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \right) = M(3 + 2(1 - \gamma)). \quad (5.60)
$$

Now, one can argue that $\gamma = -\frac{1}{2} + \mathcal{O} ((M/N)^{2n})$ [72], so

$$
\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M \ln \Lambda \quad (5.61)
$$

in exact agreement with the supergravity result.

### 5.3.3 Klebanov-Strassler theory

In [58] the setup is generalized to a warped product of flat space and the
deformed conifold

$$
\text{ds}^2 = h^{1/2}(\tau)(-dt^2 + dx^2) + h^{1/2}(\tau)\text{ds}^2_{\text{Def}}, \quad (5.62)
$$

where $\text{ds}^2_{\text{Def}}$ is given in Eq. (5.44). The full solution to the supergravity
equations is very involved and all details are referred to [58, 71]. However,
we would like to make the following remarks.
In the UV, $\tau \to \infty$, the warp factor reduces to that of the UV behavior of the near horizon limit of the Klebanov-Tseytlin solution
\[
h(\tau \to \infty) \to 2^{2/3}(g_s M \alpha')^2 \epsilon^{-8/3} \times 3 \cdot 2^{-1/3}(\tau - 1/4)e^{-4\tau/3}. \quad (5.63)
\]
However, the new feature of this solution is that in the IR, $\tau \to 0$, the warp factor takes on a constant value
\[
h(\tau \to 0) \to 2^{2/3}(g_s M \alpha')^2 \epsilon^{-8/3} \times a_0, \quad (5.64)
\]
with $a_0 \approx 0.71805$. This is to be contrasted to the warping of the solution of the singular conifold where the warp factor goes to infinity in the IR
\[
h(r \to 0) \to \frac{L^4}{r^4}. \quad (5.65)
\]
Thus the throat is no longer infinitely warped but rather smoothed off. This is illustrated in Fig. 5.6.

In the dual field theory we interpret this behavior as the presence of gluino condensation in the IR. As explained in §5.2.6, the deformation is a mode with $\Delta = 3$ and $|R| = 2$, which are the quantum numbers of a gluino condensation. The scale of the condensation is set by the scale of the tip [58]
\[
\langle \text{tr} \lambda \lambda \rangle \sim M\frac{\epsilon^2}{(\alpha')^3}. \quad (5.66)
\]
String perturbation theory

To relate string theory to phenomenology and cosmology we need to consider the low energy limit. The low energy limit consists of taking the energy $E$ of some process, say a string scattering experiment, normalized in terms of the string length $\ell_s$, to zero, i.e. $\ell_s E \rightarrow 0$. Indeed, at low energies the string behaves approximately like a point particle and the dynamics should be governed by quantum field theory. To determine the effective field theory that governs the low energy dynamics of strings we match $S$-matrix elements calculated in perturbative string theory to $S$-matrix elements calculated in known quantum field theories. In this way we find by trial and error the field theory that approximates the behavior of strings at low energies the best.

With the string effective action at hand we can ask questions about the vacuum structure of the theory. For example, we might ask whether the theory allows for a de Sitter vacuum and if any symmetries, such as supersymmetry, are spontaneously broken. In actual model constructions there will be no unique vacuum but rather a landscape of possible vacua. The shape of the mountains and valleys of this landscape is governed by the effective potential $V(\Phi)$ of the closed and open string moduli $\Phi$ describing the string compactification.

For supersymmetric vacua the effective field theory is very much constrained (compared to the space of all effective theories) and it is generally technically easier to match the low energy string theory to a known field theory. Indeed, four-dimensional $\mathcal{N} = 1$ supergravity at the two-derivative level is completely specified in terms of the three functions:
the Kähler potential $K(\Phi, \bar{\Phi})$, the superpotential $W(\Phi)$ and the gauge kinetic function $f(\Phi)$. The bosonic part of the action takes the form [60]

$$\mathcal{L}_{\text{bos}} = \frac{1}{2\kappa^2} R - K_{ij} D_\mu \bar{\Phi}^i D^\mu \Phi^j - \frac{1}{4} \text{Re} (f_{ab}(\Phi))(F^a)_{\mu\nu}(F^b)^{\mu\nu}$$

$$- \frac{1}{8} \text{Im} (f_{ab}(\Phi)) \epsilon^{\mu\nu\rho\sigma} (F^a)_{\mu\nu}(F^b)_{\rho\sigma} - V(\Phi, \bar{\Phi}),$$  \hspace{1cm} (6.1)$$

where the potential is given by

$$V(\Phi, \bar{\Phi}) = e^{\kappa^2 K} (K^{ij} D_i \bar{W} D_j W - 3\kappa^2 |W|^2) + \frac{1}{2} f_{ab} D^a D^b.$$  \hspace{1cm} (6.2)$$

Here $K_{ij}$ denotes the Kähler metric and $K^{ij}$ its inverse

$$K_{ij} = \frac{\partial}{\partial \Phi^i} \frac{\partial}{\partial \Phi^j} K(\Phi, \bar{\Phi}), \quad K^{ij} = (K_{ij})^{-1},$$  \hspace{1cm} (6.3)$$

and $D_i$ denotes the Kähler covariant derivative

$$D_i W = W_i + \kappa^2 K_i W,$$  \hspace{1cm} (6.4)$$

where $W_i = \frac{\partial}{\partial \Phi^i} W$, $K_i = \frac{\partial}{\partial \Phi^i} K$. Furthermore

$$\text{Re} (f_{ab}(\Phi)) D^b = -2\xi_a - K_i t^{a}_{ij} \Phi^j.$$  \hspace{1cm} (6.5)$$

In a free theory, the Kähler potential is quadratic $K = \delta_{ij} \bar{\Phi}^i \Phi^j$ and the Kähler metric takes the form $K_{ij} = \delta_{ij}$. But in a general theory the Kähler metric $K_{ij}(\Phi, \bar{\Phi})$ will be a function of the moduli. This is important since the Kähler metric also feeds into the potential and affects the vacuum structure. That the kinetic terms are determined by a single function is not the only striking fact of supersymmetry but even more strikingly, the holomorphicity of the superpotential $W(\Phi)$ protects it from quantum corrections to all orders in perturbation theory [76].

The low energy matching procedure consists of matching amplitudes from string perturbation theory in the limit $\ell_s E \to 0$ to amplitudes calculated in the above theory, thus fixing $f, W, K$. There are two types of corrections to the classical supergravity field theory above. The first type is corrections in $\ell_s E$. Matching amplitudes to higher order requires the addition of higher derivative terms in Eq. (6.1). For example we expect that, in addition to the curvature $R$, there should be terms of order $O(\ell_s^2 R^2)$. Terms of this kind are important when analyzing effects of strong gravity, where the curvature is big, such as for example close to the singularity of a black hole. Indeed, corrections of this kind play a crucial role when identifying the entropy of microscopical black holes. The second type of correction to the above picture is perturbative string corrections in $g_s$ which is the focus of this chapter.
6.1 Applications

We now discuss two applications of $\ell_s$ corrections and perturbative string $g_s$ corrections, with importance for string model building.

6.1.1 KKLT

In KKLT, the complex structure moduli and the axi-dilaton $\tau$ are stabilized by fluxes at the classical level. However, the stabilization of the Kähler modulus $\rho$, describing the volume of the extra dimensions, is ensured by non-perturbative effects. The $\mathcal{N} = 1$ effective field theory of $\rho$ is specified by the Kähler potential

$$K = -3 \ln \left[ -i (\rho - \bar{\rho}) \right],$$  

(6.6)

and superpotential

$$W = W_0 + Ae^{iap},$$  

(6.7)

where $W_0$ is the tree-level superpotential [77] and $Ae^{iap}$ is a non-perturbatively generated superpotential specified by two parameters $A$ and $a$. For example, gluino condensation on D7-branes gives rise to a potential (see for example [15])

$$W_{n.p.} = \mu^3 \exp \left[ -\frac{8\pi^2}{N_c} \left( \frac{1}{g_{YM}^2} + i \frac{\theta_{YM}}{8\pi^2} \right) \right] = \mu^3 \exp \left[ -\frac{8\pi^2}{N_c} f \right],$$  

(6.8)

where $\mu$ is some renormalization scale, and $g_{YM}$ and $\theta_{YM}$ denote the gauge coupling and theta-angle of the $SU(N_c)$ super-Yang-Mills theory, which we combine into the gauge kinetic function $f$ of Eq. (6.1). Now, the Yang-Mills coupling is set by the volume $\frac{8\pi^2}{g_{YM}^2} = 2\pi \text{Im} \rho$, thus gluino condensation generates a non-perturbative potential for $\rho$, with $A = \mu^3$ and $a = \frac{2\pi}{N_c}$.

However, perturbative string corrections alter this result by introducing additional moduli dependence. To see this, we analyze quantum corrections to the gauge coupling

$$\frac{8\pi^2}{g_{YM}^2(\Phi)} = \left( \frac{8\pi^2}{g_{YM}^2} \right)_{\text{tree}} + \left( \frac{8\pi^2}{g_{YM}^2(\Phi)} \right)_{\text{one-loop}} + \ldots,$$  

(6.9)

where we expect the one-loop result to introduce both running and moduli dependent threshold-corrections

$$\left( \frac{8\pi^2}{g_{YM}^2(\Phi)} \right)_{\text{one-loop}} = \frac{b}{2} \ln \frac{M_s^2}{\mu^2} + \Delta(\Phi).$$  

(6.10)
Here, \( b \) is the coefficient of the \( \beta \)-function, \( \beta = bg_{YM}^3/16\pi^2 \), and \( \Delta(\Phi) \) denotes a threshold correction that will depend on some generic moduli \( \Phi \). The threshold corrections in a simple toroidal \( \mathcal{N} = 1 \) type IIB orientifold model were calculated in [78] with a result of the form

\[
\Delta(\Phi) = -\frac{1}{2} \ln \left| \frac{\vartheta_1(\phi/2\pi|U)}{\eta(U)} \right|^2 + \frac{(\text{Im}\, \phi)^2}{4\pi \text{Im} U},
\]

where \( \phi \) is the modulus describing the position of the D3-brane in a space-time two-torus, with modulus \( U \), that is orthogonal to the stack of D7-branes, and where \( \vartheta_1 \) is a Jacobi theta function and \( \eta \) is the Dedekind eta function.

When rephrasing the gauge coupling correction in terms of a correction to the gauge kinetic function, one finds that the one-loop result introduces additional moduli dependence in the non-perturbative superpotential

\[
W_{n.p.} = \mu^3 \exp \left[ -\frac{8\pi^2}{N_c} (f_{\text{tree}} + f_{\text{one-loop}} + \ldots) \right]
\]

\[
= \mu^3 \left[ \frac{\vartheta_1(\phi/2\pi|U)}{\eta(U)} \right]^{\frac{1}{N_c}} \exp \left[ \frac{8\pi^2}{N_c} (f_{\text{tree}} + \ldots) \right].
\]

Thus, the prefactor \( A \) acquires a moduli dependence \( A = A(\Phi) \). For small \( \phi \), we have that \( \vartheta_1(\phi/2\pi|U) \sim \phi \), so the potential vanishes when the D3-brane is placed on top of the D7-branes [79]

\[
W_{n.p.} \sim \phi^{1/N_c}.
\]

Furthermore, there is a monodromy from the \( 1/N_c \) power, when the D3-brane moves around the D7-branes placed at the origin of the transverse space.

6.1.2 LVS

In KKLT, the presence of non-perturbative effects plays an important role when we stabilize the Kähler moduli. In the the so-called “large volume scenario” (LVS), also corrections in \( \alpha' \) play a crucial role, in addition to the non-perturbative effects. At large volume, the most important \( \alpha' \) correction to the Kähler potential \( K = -2 \ln \mathcal{V} \), takes the schematic form [80]

\[
\Delta K_{\alpha'} = \frac{\xi}{\mathcal{V}} + \ldots
\]

Naively, it would seem to be inconsistent to keep only one term in an \( \alpha' \) expansion. Indeed, if the first order term is important, then the second
term ought to be important as well and we get to the familiar “truncation problem”. That it in fact exists a consistent truncation was shown in [81, 82, 83]. The surprising result is that for large volume (linear size $\gtrsim 10^3$ in string units) there is a consistent minimum to the potential.

One might ask if the consistent truncation was a pure accident and wonder if, for example, perturbative string corrections could alter this conclusion? In [84] it was argued that perturbative string corrections can enter at the same power in the volume

$$\Delta K_{gs} = \Xi + \ldots$$  \hspace{1cm} (6.15)

However, miraculous cancellations do occur when one computes the full potential $V$ and the conclusion is that the original LVS is indeed intact. However the cancellations are not fully understood in detail and it seems that $\Delta K_{gs}$ could make a difference [85].

### 6.2 Loop expansion

The action of the bosonic string whose propagation in spacetime is described by the coordinate $X^\mu = X^\mu(\sigma^1, \sigma^2)$, as a function of the worldsheet (WS) space and time $\sigma^\alpha = (\sigma^1, \sigma^2)$, is given by [86]

$$S = S_G + S_B + S_\Phi , \hspace{1cm} (6.16)$$

where

$$S_G = \frac{1}{4\pi \alpha'} \int_{\text{WS}} d^2\sigma \sqrt{g} g^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu , \hspace{1cm} (6.17)$$

$$S_B = \frac{1}{4\pi \alpha'} \int_{\text{WS}} d^2\sigma \varepsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu , \hspace{1cm} (6.18)$$

$$S_\Phi = \frac{1}{4\pi} \int_{\text{WS}} d^2\sigma \sqrt{g} R \Phi(X) . \hspace{1cm} (6.19)$$

Here $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ denote a background metric, NS-NS two-form field and dilaton field, respectively, while $R$ denotes the Ricci scalar of the two-dimensional worldsheet of the string.
The importance of the term $S_{\Phi}$ is realized when we consider the path integral of the string. For a dilaton $\Phi(X)$ with a constant vacuum expectation value $\langle \Phi \rangle = \text{const.}$ (either everywhere or at infinity) the path of the string is weighted by the factor

$$\langle e^{-S} \rangle = \exp \left( -\frac{1}{4\pi} \int_{\mathcal{W}} d^2\sigma \sqrt{g} R \langle \Phi \rangle \right). \quad (6.20)$$

Now the value of the integral in the exponent is a pure topological property of the worldsheet and determines the Euler characteristic $\chi$ of the worldsheet

$$\chi = \frac{1}{4\pi} \int_{\mathcal{W}} d^2\sigma \sqrt{g} R. \quad (6.21)$$

Thus the loop expansion of string theory is given in terms of worldsheets and every worldsheet is associated with a factor

$$\exp (-\chi \langle \Phi \rangle) = (g_s)^{-\chi}, \quad (6.22)$$

where we have identified the string coupling constant

$$g_s = e^{\langle \Phi \rangle}. \quad (6.23)$$

The Euler characteristic $\chi$ of a two-dimensional surface is determined by

$$\chi = 2 - 2g - b - c, \quad (6.24)$$

where

$$g = \text{number of handles}, \quad (6.25)$$

$$b = \text{number of boundaries}, \quad (6.26)$$

$$c = \text{number of cross-caps}, \quad (6.27)$$

where a “cross-cap” means to cut a hole in the surface and identify diametrically opposite points. The tree-level surface has $\chi = 2$ and is given by the sphere with $(g, b, c) = (0, 0, 0)$. The one-loop surfaces have $\chi = 0$ and are given by the torus with $(g, b, c) = (1, 0, 0)$, the cylinder (or annulus) with $(g, b, c) = (0, 2, 0)$, the Möbius strip with $(g, b, c) = (0, 1, 1)$, and the Klein bottle with $(g, b, c) = (0, 0, 2)$. These surfaces are illustrated in Fig. 6.1.

Whether all four one-loop surfaces of Fig. 6.1 are present in the loop expansion or not depends on if we allow for boundaries and cross-caps.
A boundary of a worldsheet, as for the cylinder, requires the presence of objects where open strings can end. These objects define spacetime hypersurfaces, so-called D-branes, where Dirichlet boundary conditions are imposed at the ends of the string

\[ X(\sigma^1, \sigma^2)|_{\text{worldsheet boundary}} = \text{fixed}. \quad (6.28) \]

The presence of a cross-cap on the worldsheet, as for the Möbius strip and the Klein bottle, tells us that the worldsheet is non-orientable. Non-orientable strings can be obtained by so-called orientifolding, i.e. by identifying worldsheet coordinates under worldsheet parity

\[ \Omega: (\sigma^1, \sigma^2) \rightarrow (-\sigma^1, \sigma^2). \quad (6.29) \]

The action of \( \Omega \) changes the orientation of the string and by identifying strings of opposite orientation we end up with non-oriented worldsheets.

### 6.3 The one-loop surfaces

We now give the mathematical construction of the one-loop surfaces, depicted in Fig. 6.1. We first give the construction of the torus and then later the other surfaces. Indeed, there is a powerful general strategy that allows us to obtain all one-loop surfaces from the torus by identifying
points under appropriate involutions. The torus $T$ itself can be defined by identifying points of the worldsheet $\nu = (\sigma^1 + i\sigma^2)/2\pi$, in the complex plane $\mathbb{C}$, under the translations $\nu \to \nu + 1$ and $\nu \to \nu + \tau$, for a complex number $\tau = \tau_1 + i\tau_2$ in the upper half-plane $\tau_2 > 0$. Repeated use of the translations defines a lattice $\Gamma = \mathbb{Z} \times \tau \mathbb{Z}$ and the torus is defined as $T = \mathbb{C}/\Gamma$. The construction of the torus is illustrated in Fig. 6.2.

The torus modulus $\tau \in H$, where $H$ is the upper half-plane, describes the shape of the torus. However, not all $\tau$ describe different shapes. If $\tau$ and $\tau'$ determines the same lattice then $\tau$ and $\tau'$ describe the same torus and ought to be identified. For example $\tau$ and $\tau' = \tau + 1$ obviously defines the same lattice. The same is true for $\tau$ and $\tau' = -1/\tau$. The transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ generate the group of Möbius transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad (6.30)$$

with integer coefficients $a, b, c, d \in \mathbb{Z}$, the so-called modular group $PSL(2, \mathbb{Z})$. Thus the space $\mathcal{F}$ of inequivalent tori is given by $\mathcal{F} = H/PSL(2, \mathbb{Z})$.

After having constructed the torus, we obtain the annulus $\mathcal{A}$, Möbius strip $\mathcal{M}$ and Klein bottle $\mathcal{K}$ by dividing out by further transformations.
6.4. Method of images

The advantage of formulating all one-loop surfaces in terms of involutions of a torus is that one can then use the well-known results that exists on the torus using the method of images. The method of images is familiar from electrostatics and provides a powerful method for generating new solutions from known ones.

Consider the following familiar example from electrostatics. The electrostatic potential at $\vec{x} = (x, y, z)$ induced by a point charge $Q$ at $\vec{x}_0 = \frac{1}{\epsilon} \frac{Q}{|\vec{x}_0 - \vec{x}|}$ is given by:

$$\phi(\vec{x}) = \frac{1}{\epsilon} \frac{Q}{|\vec{x}_0 - \vec{x}|}$$

where $\epsilon$ is the permittivity of the medium. The potential is a solution to the Poisson equation:

$$\nabla^2 \phi(\vec{x}) = - \frac{Q}{\epsilon} \delta(\vec{x})$$

The method of images involves the introduction of virtual images of the charge $Q$ located at $-\vec{x}_0$ and $2\vec{x}_0 - \vec{x}$ to satisfy the boundary conditions.

For the annulus $A$ we consider a straight torus with purely imaginary modulus $\tau = it/2$ and identify under the involution $\nu \rightarrow I_A(\nu) = 1 - \bar{\nu}$. The lines $\nu = 0$ and $\nu = 1/2$ are fixed lines of the involution and correspond to the boundaries of the annulus. For the Möbius strip $M$ we consider a tilted torus with modulus $\tau = 1/2 + it/2$ and identify under the involution $I_M(\nu) = 1 - \bar{\nu}$. The lines $\nu = 0$ and $\nu = 1$ are fixed lines of the involution and correspond to a single boundary and a cross-cap. For the Klein bottle $K$ we consider a straight torus with modulus $\tau = 2it$ and identify under the involution $I_K(\nu) = 1 - \bar{\nu} + \tau/2$. The Klein bottle has no boundaries but two cross-caps. The construction of the annulus, Möbius strip and Klein-bottle out of the torus is illustrated in Fig. 6.3.
Chapter 6. String perturbation theory

$(x_0, y_0, z_0)$ is given by $G(\vec{x}, \vec{x}_0) = \frac{Q}{4\pi\epsilon_0 |\vec{x} - \vec{x}_0|}$. The introduction of perfect conductor in the half-space $z < 0$ yields the same result as introducing a point source of opposite charge $-Q$ placed at the point $\vec{x}_1 = (x_0, y_0, -z_0)$, mirror to $\vec{x}_0$ in the plane $z = 0$. Introducing the mirror (or involution) map $\vec{I}: (x, y, z) \rightarrow \vec{I}(x, y, z) = (x, y, -z)$ the solution to the problem of the conducting half-space is obtained by anti-symmetrizing under the action of $\vec{I}$, i.e.

$$G_I(\vec{x}, \vec{x}_0) = \frac{1}{2} \left( G(\vec{x}, \vec{x}_0) - G(\vec{I}(\vec{x}), \vec{x}_0) - G(\vec{x}, \vec{I}(\vec{x}_0)) + G(\vec{I}(\vec{x}), \vec{I}(\vec{x}_0)) \right)$$

$$= G(\vec{x}, \vec{x}_0) - G(\vec{x}, \vec{x}_1), \quad (6.31)$$

where $\vec{I}(\vec{x}_0) = \vec{x}_1$.

The same methods can be used to construct the worldsheet Green’s functions $G_\sigma(\nu_1, \nu_2) = \langle X(\nu_1)X(\nu_2) \rangle_\sigma$, $\sigma = \mathcal{A}, \mathcal{M}, \mathcal{K}$, of the annulus $\mathcal{A}$, Möbius strip $\mathcal{M}$, and Klein bottle $\mathcal{K}$, by (anti-)symmetrizing the torus amplitude $G_T(\nu_1, \nu_2) = \langle X(\nu_1, \nu_2) \rangle_T$ under the involution $I_\sigma$

$$G_\sigma(\nu_1, \nu_2) = \frac{1}{2} \left( G_T(\nu_1, \nu_2) \pm G_T(I_\sigma(\nu_1), \nu_2) \right.$$  

$$\pm G_T(\nu_1, I_\sigma(\nu_2)) + G_T(I_\sigma(\nu_1), I_\sigma(\nu_2)) \right)$$

$$= G_T(\nu_1, \nu_2) \pm G_T(I_\sigma(\nu_1), \nu_2), \quad \sigma = \mathcal{A}, \mathcal{M}, \mathcal{K}, \quad (6.32)$$

where the choice $+$ corresponds to Neumann boundary conditions and the choice $-$ corresponds to Dirichlet boundary conditions at the boundary of the surface.

The method of images can also be used to determine correlation functions of more complicated fields, such as spinor and tensor fields [87, 88, 89, 90, 91]. The analysis is complicated a bit by the fact the involution (mirror map) change handedness of the surface, so for example left- and right-handed spinors are exchanged. The analogous analysis for spinors is carried out in the appendix of Paper VI.

6.5 S-matrix

The S-matrix element of a scattering process in string theory is obtained by attaching external legs, corresponding to asymptotic states, to the worldsheets of the loop expansion. This is achieved by inserting local operators on the worldsheet corresponding to the external particle. That this in fact works is due to the state-operator correspondence of the worldsheet
conformal field theory which states that there is a one-to-one correspondence between quantum states and local operators.

To identify the vertex operator $\mathcal{V}$, of say an external graviton, we will use the background field method, see for example [86]. To this end, consider first the partition function of the string

$$Z = \sum_{\text{compact worldsheets}} \int \frac{\mathcal{D}[X,g]}{\text{vol (diff} \times \text{Weyl)}} e^{-S[X,g]}.$$

(6.33)

Consider now a plane wave graviton with momentum $k^\mu$ and polarization $\epsilon_{\mu\nu}(k)$, described by a wave function $\epsilon_{\mu\nu}(k)e^{ik \cdot X}$, propagating in flat space. The metric then takes the form

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_s \epsilon_{\mu\nu}(k)e^{ik \cdot X},$$

(6.34)

where we choose to associate a factor $-4\pi g_s$ with the insertion of a graviton. Then by expanding in fluctuations we get (with $z = e^{-2\pi i \nu}$)

$$Z[\mathcal{V}] = Z[0] \left( 1 + \int d^2z \sqrt{g(z,\bar{z})} \langle \mathcal{V}(z,\bar{z}) \rangle + \ldots \right).$$

(6.35)

This allows us to identify the vertex operator associated with the insertion of a graviton

$$\mathcal{V}(z,\bar{z}) = \frac{g_s}{\alpha'} \epsilon_{\mu\nu}(k)g^{\alpha\beta}(z,\bar{z}) \partial_\alpha X^\mu(z,\bar{z}) \partial_\beta X^\nu(z,\bar{z}) e^{ik \cdot X(z,\bar{z})}.$$  

(6.36)

The S-matrix element of $n$ gravitons is given by

$$S_{j_1\ldots j_n}(k_1, \ldots, k_n) = \sum_{\text{compact worldsheets}} \int \frac{\mathcal{D}[X,g]}{\text{vol (diff} \times \text{Weyl)}}$$

$$\times \prod_{i=1}^n \int d^2z_i \sqrt{g(z_i,\bar{z}_i)} \mathcal{V}_{j_i}(z_i,\bar{z}_i; k_i).$$

(6.37)

For example, when we consider the torus-worldsheet, the path-integral over the worldsheet metric $g_{\alpha\beta}$ is reduced to an integral over tori of different shapes $\tau$

$$\int_{\mathcal{F}} \frac{d^2\tau}{2\tau_2},$$

(6.38)
where $d^2\tau\tau_2$ is modular invariant. Then the torus contribution to the S-matrix element takes the form

$$S_{j_1\ldots j_n}^{\text{Torus}}(k_1,\ldots,k_n) = \int \frac{d^2\tau}{2\tau_2} Z[0] \left\langle \prod_{i=1}^n \int d^2z_i \mathcal{V}_{j_i}(z_i,\bar{z}_i;k_i) \right\rangle_{\text{Torus}},$$

(6.39)

where $Z[0]$ and $\langle \ldots \rangle_{\text{Torus}}$ implicitly depend on $\tau$. As we see, the typical one-loop calculation in string theory consists of first calculating correlators of worldsheet operators, then integrating over the positions of the operators and finally integrating over the different shapes of the worldsheet.

In Paper VI we calculate the one-loop correction to the two-point amplitude of internal gravitons in a Calabi-Yau orientifold.

### 6.6 Orbifold compactifications

A simple way of reducing the number of spacetime supersymmetries is by imposing a discrete symmetry on the compactification manifold. By performing so-called orbifoldings we reach $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry. For simplicity we consider the compact manifold to be a product of three two-tori

$$T^6 = T^2 \times T^2 \times T^2,$$

(6.40)

with complex coordinatization

$$Z^1 = X^4 + iX^5, \quad Z^2 = X^6 + iX^7, \quad Z^3 = X^8 + iX^9.$$

(6.41)

The orbifolding of the tori consists of identifying points under a $\mathbb{Z}_N$ transformation

$$Z^i \rightarrow \Theta Z^i = e^{2\pi iv^i}Z^i, \quad i = 1, 2, 3,$$

(6.42)

with $\Theta^N = 1$. The vector $v^i = (v^1, v^2, v^3)$ is known as the twist vector. In Fig. 6.4 we give an example of a $\mathbb{Z}_2$ orbifolding of a square two-torus.

#### 6.6.1 Supersymmetry

To determine the amount of supersymmetry in four dimensions we analyze how the ten-dimensional spinor supercharge $Q$ decomposes under the...
6.6. Orbifold compactifications

Figure 6.4. A $\mathbb{Z}_2$-orbifold obtained from a square torus, with modulus $U = i$, by identifying points under the involution $\mathbb{Z}_2: Z \to e^{\pi i} Z$. The resulting orbifold is obtained by 'folding' the colored strip along the dotted line and by identifying sides as indicated by the arrows. The resulting space resembles a 'pillow'. The white (and black) dots are fixed points of the $\mathbb{Z}_2$-action and become conic singularities of the orbifold. The deficit angle of each such cone is $180^\circ$. This can be seen by parallel transporting a vector around the apex of the cone, as illustrated by the series of black arrows that has been rotated $180^\circ$ after forming a closed loop.

decomposition $SO(9, 1) \to SO(3, 1) \times SO(6)$. The ten-dimensional Dirac algebra

$$\{\Gamma^M, \Gamma^M\} = 2\eta^{MN},$$

(6.43)
decomposes into an algebra in four dimensions $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ and an algebra in six dimensions $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$. By defining, in analogy to Eq. (6.41),

$$\Gamma^1_\pm = \frac{1}{2} (\Gamma^4 \pm i\Gamma^5), \quad \Gamma^2_\pm = \frac{1}{2} (\Gamma^6 \pm i\Gamma^7), \quad \Gamma^3_\pm = \frac{1}{2} (\Gamma^8 \pm i\Gamma^9),$$

(6.44)
the Dirac algebra in six dimensions takes the form of a set of three algebras of raising and lowering operators

$$\{\Gamma^a_+, \Gamma^b_+\} = \delta^{ab}, \quad \{\Gamma^a_+, \Gamma^b_-\} = \{\Gamma^a_-, \Gamma^b_-\} = 0, \quad a, b = 1, 2, 3.$$ 

(6.45)
Given a ground state $|0\rangle$, with $\Gamma^a |0\rangle = 0$, then a general state is given by

$$|s_1, s_2, s_3\rangle = (\Gamma^1_+)^{s_1+\frac{1}{2}} (\Gamma^2_+)^{s_2+\frac{1}{2}} (\Gamma^3_+)^{s_3+\frac{1}{2}} |0\rangle.$$ 

Now the orbifolding acts on this state as

$$|s_1, s_2, s_3\rangle \to \Theta |s_1, s_2, s_3\rangle = e^{2\pi is\vec{v}} |s_1, s_2, s_3\rangle.$$ 

(6.46)
Thus the number of states $|s_1, s_2, s_3\rangle$, and therefore the number of supercharges, invariant under $\Theta$ depends on the choice of twist vector.

Orbifolds with twist vector

$$v^1 + v^2 + v^3 = 0, \quad (6.47)$$

leave only the states with $s_1 = s_2 = s_3$ invariant and leads to $\mathcal{N} = 1$ supersymmetry in four dimensions. An example is the $\mathbb{Z}_3$ orbifold $\mathbb{T}^6/\mathbb{Z}_3$ with $v^i = (1/3, 1/3, -2/3)$.

Orbifolds with special case twist vector

$$v^1 + v^2 = 0, \quad v^3 = 0, \quad (6.48)$$

leave only the states with $s_1 = s_2$, $s_3$ unspecified, invariant and leads to $\mathcal{N} = 2$ supersymmetry in four dimensions. An example is the $\mathbb{Z}_3$ orbifold $\mathbb{T}^4/\mathbb{Z}_3 \times \mathbb{T}^2$ with $v^i = (1/3, -1/3, 0)$. 
Part II

Summary and outlook
Chapter 7

Summary of results

In Paper I we study the dark matter phenomenology of an effective field theory extension of the MSSM called “Beyond the MSSM” (BMSSM). This extension was put forward by Dine, Seiberg and Thomas in [92] as a solution to the little hierarchy problem. As is well-known, the MSSM tree-level Higgs mass satisfies a sum rule that makes it smaller than the Z-boson mass. To escape the Higgs bound from LEP it is required to tune loop-corrections such that the Higgs-boson becomes heavy. In the BMSSM one adds to the Higgs sector two higher dimensional operators, one supersymmetric and one supersymmetry breaking, whose effects are parametrized by two small parameters $\epsilon_1$ and $\epsilon_2$. Without resorting to fine-tuning of loop-corrections, these operators can raise the Higgs mass already at tree-level, and models that were once ruled out are now ruled in. The effects of these new operators compared to the one- and two-loop results is illustrated in Fig. 7.1.

Due to the nature of supersymmetry, modifications in the Higgs boson sector also find their way into the Higgsino sector. In those models where the Higgsino constitutes a dark matter candidate, these corrections can play an important role. We reanalyze electro-weak symmetry breaking and sum rules. We implement the Feynman rules of the BMSSM in the code DarkSUSY [96]. Using this code, we perform scans over the extended parameter space. At every point in parameter space we perform dark matter relic density calculations and compare with WMAP data, and check that the model passes some reasonable accelerator constraints. The new operators open up new areas in parameter space, see Fig. 7.2. As expected, the region in parameter space that is affected the most is that of light Higgsino-like dark matter. These regions involve interesting

\footnote{For earlier work in specific models that identified similar operators, see [93, 94, 95]}
interplay between the WMAP dark matter bounds and the LEP chargino bound. Furthermore we also find changes for gaugino dark matter, partly due to annihilation through a Higgs resonance, and partly due to coannihilation with light stops.

In Paper II we present a simple two-field model of high-scale inflationary cosmology based on the axion monodromy scenario [97, 98] in string theory. The model has two virtues: The first one, which is familiar from the bottom-up phenomenological model known as 'natural inflation', is the shift symmetry which is only broken by non-perturbative effects, leaving us with a protected small mass, thus evading the \( \eta \)-problem. The second one, which is a novel feature of the model, is that the effective inflaton field excursion can be super-Planckian even though the region in field-space of the two individual fields is parametrically sub-Planckian in diameter, due to the monodromy, see Fig. 7.3. This leads to control over Planck scale corrections at the same time as the model shows the
characteristic features of high-scale inflation, i.e. phenomenologically the predictions are equivalent to those of chaotic inflation, and in particular include observably large tensor modes. The whole high-scale large-field inflationary dynamics takes place within a region of field space that is parametrically subplanckian in diameter, hence improving our ability to control quantum corrections and achieve slow-roll inflation.

In Paper III we derive analytical solutions to the field equations of massive gravity corresponding to spherically symmetric objects. Massive gravity has attracted much attention since it was recently shown to be ghost-free [41, 40, 42]. Using the full solutions we can answer the question whether massive gravity allows for a Vainshtein mechanism or not. We find that a Vainshtein mechanism is not always present and we give the parameter ranges for this to happen. In Fig. 7.4 we show how as we vary the model parameters we go from a situation with a Vainshtein
Figure 7.3. The field space of the two-field model of axion monodromy inflation. The dynamics of inflation is as follows: Before inflation starts, the effective inflaton field $\phi_{\text{eff}}$ quickly rolls down into a ridge, in the $\tilde{r}$-direction, due to the steep walls of the potential. Well down the ridge, inflation occurs when the effective inflaton field $\phi_{\text{eff}}$ slowly rolls along the bottom of a ridge, in the $\tilde{\theta}$-direction, as denoted by the yellow line in the above. Due to the periodicity in the fields, the field excursion of the effective inflaton field can be shifted into a region of field space which is parametrically sub-Planckian in size, as denoted by the dotted box in the above.

mechanism to a pathological behavior. The analytical solutions also allow us to make predictions for the gravitational lensing and velocity dispersion measurements of galaxies. The absence of any deviations from general relativity allows us to put tight constraints on the mass of the graviton. We find that $m_g/H_0 \lesssim 0.01 - 0.02$ at 95% confidence level, i.e. the inverse mass scale of the graviton is necessarily pushed beyond scales of the order of a hundredth of the Hubble scale today.

Perhaps the greatest challenge of model building in string theory is that of finding a de Sitter vacuum describing the accelerated expansion of the universe. One prominent idea put forward by Kachru-Kallosh-Linde-Trivedi (KKLT) in [49], is to allow for a meta-stable de Sitter vacuum. Here a positive vacuum energy is generated using non-perturbative effects and supersymmetry breaking anti-branes residing in the extra dimensions. No exact solution describing this setup exists, so in Paper IV we decided to instead take a perturbative approach to the problem. In Paper IV we develop an expansion scheme for perturbations around any non-compact, supersymmetric, warped Calabi-Yau cone with imaginary self-dual (ISD) flux. Supersymmetry breaking and ISD violating effects in the bulk of the compactification effectively impose boundary conditions on the supergravity fields in the ultraviolet region of the cone. The behavior of the
Figure 7.4. The deviation in the gravitational force law $\varepsilon$ as a function of radius $r$ in units of the Vainshtein radius $r_V$. At short distances $\varepsilon$ goes to zero due to the Vainshtein mechanism that recovers general relativity at short distances, while at large distances $\varepsilon$ goes to $1/3$, due presence of the helicity-0 mode of the massive graviton which acts as an additional force carrier. Whether the theory allows for an interpolation between the two regions depends on the model parameters. As we vary $B$, with $C$ fixed, we go from a situation with a Vainshtein mechanism (yellow solid line) to a situation with a pathological behavior around the Vainshtein radius (pink and blue dashed lines).

Figure 7.5. The cosmological constraints on the inverse graviton mass scale $\lambda_g = m_g^{-1}$ coming from combined strong lensing and velocity dispersion data from galaxies. In the above we use the particular model with $B = -1$ and $C = 1$ but the constraints are rather insensitive to the specific choice of parameters and we typically find that $\lambda_g/r_H \gtrsim 0.01 - 0.02$ at 95% confidence level.
perturbations in the infrared are then determined by the running of the supergravity fields implied by the equations of motion.

When expanding around the supersymmetric ISD solution, the equations of motion simplify dramatically and we find a triangular structure allowing for all-order solutions. We present an algorithm that yields an explicit Green’s function solution for all the supergravity fields, to any desired order, for given boundary data and harmonic data on the base of the cone. In Tab. 7.1 we give the running of the homogeneous modes in terms of the harmonic data on the cone. In Paper IV we also give the running of the inhomogeneous modes at first and second order in perturbation theory using the explicit Green’s function solutions.

In Paper V we use the results of Paper IV to determine the potential felt by an anti-D3-brane residing at the tip of the warped deformed conifold. Even though the background produces a steep potential for the field describing the position of the anti-brane in the radial direction of the throat, the fields describing the angular motion around the tip correspond to massless fields, due to the isometries present in the infinite throat limit. However, compactification effects will produce a potential for these moduli. To make use of the results in Paper IV, the harmonic data of the base of the cone has to be known. The base space of the conifold is given by $T^{11} = SU(2) \times SU(2)/U(1)$ and the scalar, spinor, vector, and two-form harmonics of $T^{11}$ were derived by Ceresole, Dall’Agata, D’Auria in [99, 100]. In the appendix of Paper V we determine the spectrum of the Lichnerowicz operator on $T^{11}$, thus extending the results of [99, 100] to include also the harmonics of the symmetric two-tensor. The results for the smallest eigenvalues of the Lichnerowicz operator is summarized in Tab. 7.2. With all the harmonic data available we show that the dominant contribution to the anti-D3-brane potential arises from nonperturbative effects stabilizing the Kähler moduli. Incorporating these flux perturbations as sources for perturbations of the metric, we compute the leading contribution to the masses of the angular moduli. Even though the resulting masses are parametrically larger than those obtained in prior work, the angular moduli cannot be stabilized by compactification effects alone, without without resorting severe fine tuning.

In Paper VI we address the question of perturbative string loop corrections in string compactifications. We calculate the closed string one-loop corrections to the Kähler potential in an $\mathcal{N} = 1$ orientifold compactification with D-branes and O-planes. Novel to these calculations is the inclusion of truly $\mathcal{N} = 1$ twisted sectors. This is of utmost importance
### Homogeneous Scalings of the Non-Normalizable Modes

<table>
<thead>
<tr>
<th>Field</th>
<th>Scaling</th>
<th>Dimension</th>
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<tbody>
<tr>
<td>$r^{-4} \Phi^H_-$</td>
<td>$r^{\Delta(\Phi^{-})-4}$</td>
<td>$\Delta(\Phi^{-}) = \Delta(I_s) - 4, \lambda^{I_s} \neq 0$</td>
</tr>
<tr>
<td>$G^H_-$</td>
<td>$r^{\Delta(G^{-})-4}(1 + \ln r)$</td>
<td>$\Delta(G^{-}) = \begin{cases} -1 + \Delta(I_s) \ -2 + \Delta(I_s), \lambda^{I_s} \neq 0 \ -3 + \Delta(I_s), \lambda^{I_s} \neq 0 \end{cases}$</td>
</tr>
<tr>
<td>$\tau^H$</td>
<td>$r^{\Delta(\tau)-4}$</td>
<td>$\Delta(\tau) = \Delta(I_s), \lambda^{I_s} \neq 0$</td>
</tr>
<tr>
<td>$r^{-2} g^H_{ij}$</td>
<td>$r^{\Delta(g)-4}$</td>
<td>$\Delta(g) = \Delta(I_t)$</td>
</tr>
<tr>
<td>$G^H_+$</td>
<td>$r^{\Delta(G^{+})-4}(1 + \ln r)$</td>
<td>$\Delta(G^{+}) = \Delta(G^{-}), 4 + \Delta(G^{-})$</td>
</tr>
<tr>
<td>$r^A(\Phi^{+1})^H_-$</td>
<td>$r^{\Delta(\Phi^{+1})-4}$</td>
<td>$\Delta(\Phi^{+1}) = \Delta(I_s) + 4$</td>
</tr>
</tbody>
</table>

### Homogeneous Scalings of the Normalizable Modes

<table>
<thead>
<tr>
<th>Field</th>
<th>Scaling</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^{-4} \Phi^H_-$</td>
<td>$r^{-\Delta(\Phi^{-})}$</td>
<td>$\Delta(\Phi^{-}) = \Delta(I_s) + 4$</td>
</tr>
<tr>
<td>$G^H_-$</td>
<td>$r^{-\Delta(G^{-})(1 + \ln r)}$</td>
<td>$\Delta(G^{-}) = \begin{cases} -1 + \Delta(I_s) \ -2 + \Delta(I_s), \lambda^{I_s} \neq 0 \ -3 + \Delta(I_s), \lambda^{I_s} \neq 0 \end{cases}$</td>
</tr>
<tr>
<td>$\tau^H$</td>
<td>$r^{-\Delta(\tau)}$</td>
<td>$\Delta(\tau) = \Delta(I_s)$</td>
</tr>
<tr>
<td>$r^{-2} g^H_{ij}$</td>
<td>$r^{-\Delta(g)}$</td>
<td>$\Delta(g) = \Delta(I_t)$</td>
</tr>
<tr>
<td>$G^H_+$</td>
<td>$r^{-\Delta(G^{+})(1 + \ln r)}$</td>
<td>$\Delta(G^{+}) = \Delta(G^{-}), 4 + \Delta(G^{-})$</td>
</tr>
<tr>
<td>$r^A(\Phi^{+1})^H_-$</td>
<td>$r^{-\Delta(\Phi^{+1})}$</td>
<td>$\Delta(\Phi^{+1}) = \Delta(I_s) + 4, \lambda^{I_s} \neq 0$</td>
</tr>
</tbody>
</table>

**Table 7.1.** The radial scalings of the homogeneous modes of the supergravity fields. Here $\Delta(I_s) = 2 + \sqrt{4 + \lambda^{I_s}}$, where the $\lambda^{I_s}$ are the eigenvalues of the angular scalar Laplacian. Furthermore, $\Delta(I_t) = 2 + \sqrt{\lambda^{I_t} - 4}$, where the $\lambda^{I_t}$ are the eigenvalues of the angular Lichnerowicz operator.

The radial scalings are crucial for determining the string effective action including moduli fields. An interesting and phenomenologically relevant next step, building on [101], would be to include fluxes in a similar open string calculation.
### Eigenvalues of the Lichnerowicz operator

| $j_1$ | $j_2$ | $|R|$ | $\lambda_{\text{Longitudinal/Trace}}$ | $\lambda_{\text{Transverse-Traceless}}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 4, 20 |
| 0 | 0 | 2 | — | 5 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 8.25, 8.25, 11.25, 19.25 | 24.25, 34.25 |
| $\frac{3}{2}$ | $\frac{1}{2}$ | 3 | — | 16.25 |
| 1 | 0 | 0 | 12, 12, 24 | 20, 40 |
| 1 | 0 | 2 | 15 | 29 |
| 1 | 1 | 0 | 17.42†, 24, 24, 24, 24, 38.58* | 14.83§*, 32, 40, 40, 57.17†† |
| 1 | 1 | 2 | 21, 21, 27, 35 | 22.42**, 29, 43.58§§, 53 |
| 1 | 1 | 4 | — | 32 |

$\dag 28 - 4\sqrt{7}$, $\ast 28 + 4\sqrt{7}$, $\ddagger 36 - 8\sqrt{7}$, $\ddagger\ddagger 36 + 8\sqrt{7}$, $\ast\ast 33 - 4\sqrt{7}$, $\ddagger\ddagger\ddagger 33 + 4\sqrt{7}$

Table 7.2. The eigenvalues of the Lichnerowicz operator for the modes with the lowest quantum numbers. The two smallest eigenvalues $\lambda = 4, 5$ correspond to modes with $j_1 = j_2 = 0$ with $R = 0, 2$. The next-to-next smallest eigenvalue is $\lambda = 36 - 8\sqrt{7}$ and corresponds to a mode with $j_1 = j_2 = 1$ and $R = 0$. 
Part III

Appendix
Appendix A

Harmonics on $S^2$ and $S^3$

In Paper V we derived and made use of the harmonic data on the coset space

$$T^{11} = SU(2) \times SU(2)/U(1).$$  \hspace{1cm} (A.1)

To set up the conventions and develop some intuition about group spaces, and more generally coset spaces, we first worked out the harmonics of the more familiar spaces: the two-sphere and the three-sphere. The three-sphere can be regarded as a group space $S^3 = SU(2)$, while the two-sphere can be regarded as a coset space $S^2 = SO(3)/SO(2) \cong SU(2)/U(1)$. Thus $T^{11}$ picks out a little bit of this from $S^3$ and a little bit of that from $S^2$.

A.1 The three-sphere $S^3$

A.1.1 Identification $S^3 = SU(2)$

We start by identifying the three-sphere $S^3$ as the group space $SU(2)$. A general matrix $D$ of $SU(2)$ can be parametrized in terms of two complex parameters $a, b \in \mathbb{C}$, the so-called Cayley-Klein parameters,

$$D = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$  \hspace{1cm} (A.2)

with $\det D = |a|^2 + |b|^2 = 1$. In terms of the real and imaginary parts $a = a_1 + ia_2$ and $b = b_1 + ib_2$ the condition $|a|^2 + |b|^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$ defines an embedding of $S^3$ in $\mathbb{R}^4$. Thus we can identify $S^3 = SU(2)$. 

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The condition $|a|^2 + |b|^2 = 1$ can be solved for in terms of three angles $\alpha, \beta, \gamma$

$$a = \cos(\beta/2)e^{-i(\alpha+\gamma)/2}, \quad b = -\sin(\beta/2)e^{-i(\alpha-\gamma)/2}, \quad (A.3)$$

where $\alpha \in [0, 4\pi], \beta \in [0, \pi]$ and $\gamma \in [0, 2\pi]$. In terms of this parametrization, $D$ takes the form

$$D(\alpha, \beta, \gamma) = \begin{pmatrix}
\cos(\beta/2)e^{-i(\alpha+\gamma)/2} & -\sin(\beta/2)e^{-i(\alpha-\gamma)/2} \\
\sin(\beta/2)e^{i(\alpha-\gamma)/2} & \cos(\beta/2)e^{i(\alpha+\gamma)/2}
\end{pmatrix}. \quad (A.4)$$

This is the familiar parametrization of an $SU(2)$ rotation in terms of Euler angles $\alpha, \beta, \gamma$. Indeed, we have that

$$D(\alpha, \beta, \gamma) = e^{-iaJ_3}e^{-i\beta J_2}e^{-i\gamma J_3}, \quad (A.5)$$

where $J_i = \sigma_i^2/2$ denote the $SU(2)$ generators with $[J_i, J_j] = i\epsilon_{ij}^k J_k$ and $\sigma_i$ denote the Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (A.6)$$

with $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k$.

We have found that any group element of $SU(2)$, or equivalently any point on the three-sphere $S^3$, can be parametrized in terms of Euler angles $\alpha, \beta, \gamma$ in the following way

$$D(\alpha, \beta, \gamma) = e^{\alpha T_3}e^{\beta T_2}e^{\gamma T_3}, \quad (A.7)$$

where we have put the $-i$ together with generators $J_i$ and defined anti-Hermitian generators $T_i \equiv -iJ_i$, with $[T_i, T_j] = \epsilon_{ij}^k T_k$.

### A.1.2 Metric

When multiplying $D(\alpha, \beta, \gamma)$ from either the left or the right with some $SU(2)$ elements $g_L$ or $g_R$ we get a new element of $SU(2)$, which we can parametrize in terms of some new Euler angles, i.e.

$$g_L D(\alpha, \beta, \gamma) = D(\alpha', \beta', \gamma'), \quad D(\alpha, \beta, \gamma) g_R^{-1} = D(\alpha'', \beta'', \gamma''). \quad (A.8)$$

Thus left- and right-multiplication in $SU(2)$ induces a transformation of the three-sphere. Since matrix multiplication is associative left- and right-action commute. We now wish to construct a metric which is invariant under both of these transformations. Such a metric will have the right isometry group $SU(2)_L \times SU(2)_R$, appropriate for a three-sphere with isometry group $SO(4) \cong SU(2)_L \times SU(2)_R$. 

Maurer-Cartan forms

From the group elements $D(x)$, with $x^\alpha = (\alpha, \beta, \gamma)$, we now construct differentials $e^A(x) = dx^\alpha e_\alpha^A(x)$ by

$$D^{-1}dD = e^A T_A. \tag{A.9}$$

Writing out the terms $D^{-1}dD = d\alpha D^{-1}T_3 D + d\beta e^{-\gamma T_3}T_2 e^{\gamma T_3} + d\gamma T_3$ and using that $g^{-1}T_A g = (Ad g)^{}^B T_B$ defines the adjoint representation with $(Ad T_A)^{}^C = -\epsilon^C_{AB}$, i.e.

$$\begin{align*}
\text{Ad } T_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & \text{Ad } e^{\alpha T_1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \tag{A.10} \\
\text{Ad } T_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & \text{Ad } e^{\beta T_2} &= \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}, \tag{A.11} \\
\text{Ad } T_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \text{Ad } e^{\gamma T_3} &= \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.12}
\end{align*}$$

one explicitly finds for the differentials

$$\begin{align*}
e^1 &= -\sin \beta \cos \gamma \, d\alpha + \sin \gamma \, d\beta, \tag{A.13} \\
e^2 &= \sin \beta \sin \gamma \, d\alpha + \cos \gamma \, d\beta, \tag{A.14} \\
e^3 &= \cos \beta \, d\alpha + d\gamma. \tag{A.15}
\end{align*}$$

Regarding the differentials $e^A$ as one-forms they automatically satisfy the Maurer-Cartan equation $d(D^{-1} \wedge dD) = dD^{-1} \wedge dD = (D^{-1} D) \wedge D^{-1} dD = -D^{-1} dD \wedge D^{-1} dD$, or in components

$$de^A + \frac{1}{2} \epsilon^C_{BC} e^B \wedge e^C = 0. \tag{A.16}$$

Left- and right-translation \quad By construction, $e^A$ is left-invariant. Indeed, under a left-translation $g_L D(x) = D(x')$, for constant $g_L$, we have that

$$D^{-1}(x')dD(x') = D^{-1}(x)g_L^{-1}d(g_L D(x)) = D^{-1}(x)dD(x). \tag{A.17}$$

However, under a right-translation $D(x)g_R^{-1} = D(x')$ we have that

$$D^{-1}(x')dD(x') = g_R D^{-1}(x)dD(x)g_R^{-1} = e^A(x)g_R T_A g_R^{-1}, \tag{A.18}$$

so $e^A(x)$ transforms as $e^B(x') = e^A(x)(Ad g_R^{-1})^A_B$. 

Appendix A. Harmonics on $S^2$ and $S^3$

**Line element**

Using the differentials $e = e^A T_A$ we can construct a line element that is manifestly left- and right-invariant

$$ds^2 = -\frac{1}{2} \text{Tr} (\text{Ad} e \ Ad e) = \delta_{AB} e^A e^B = g_{\alpha \beta} dx^\alpha dx^\beta,$$  \hspace{1cm} (A.19)

where we have identified the metric $g_{\alpha \beta} = \delta_{AB} e^A e^B$, i.e. the $e^A_\alpha$ can be thought of as the vielbeins. Explicitly we have

$$ds^2 = d\beta^2 + \sin^2 \beta d\alpha^2 + (\cos \beta d\alpha + d\gamma)^2.$$  \hspace{1cm} (A.20)

Thus we have constructed a metric on $SU(2)$ ($= S^3$) with the isometry group $SU(2)_L \times SU(2)_R$ ($\cong O(4)$).

Now $ds^2$ is the metric of the three-sphere with radius two. To see this, we make connection to the standard parameterization of the three-sphere in terms of polar coordinates. First, write the complex numbers $a, b$ as $a = x_4 - i x_2$, $b = -(x_1 - i x_3)$, then $|a|^2 + |b|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and

$$\begin{align*}
x_1 &= \sin \frac{\beta}{2} \cos \frac{\alpha - \gamma}{2}, \hspace{1cm} (A.21) \\
x_2 &= \cos \frac{\beta}{2} \sin \frac{\alpha + \gamma}{2}, \hspace{1cm} (A.22) \\
x_3 &= \sin \frac{\beta}{2} \sin \frac{\alpha - \gamma}{2}, \hspace{1cm} (A.23) \\
x_4 &= \cos \frac{\beta}{2} \cos \frac{\alpha + \gamma}{2}. \hspace{1cm} (A.24)
\end{align*}$$

Then one finds explicitly

$$ds^2 = d\beta^2 + \sin^2 \beta d\alpha^2 + (\cos \beta d\alpha + d\gamma)^2,$$  \hspace{1cm} (A.25)

$$= 2^2 (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).$$  \hspace{1cm} (A.26)

In terms of the standard parameterization of $S^3$, with $\psi, \theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$, we have

$$\begin{align*}
x_1 &= \cos \psi, \hspace{1cm} (A.27) \\
x_2 &= \sin \psi \cos \theta, \hspace{1cm} (A.28) \\
x_3 &= \sin \psi \sin \theta \cos \varphi, \hspace{1cm} (A.29) \\
x_4 &= \sin \psi \sin \theta \sin \varphi. \hspace{1cm} (A.30)
\end{align*}$$
and

\[ ds^2_{S^3} = d\psi^2 + \sin^2 \psi \, d\theta^2 + \sin^2 \psi \sin^2 \theta \, d\varphi^2. \] (A.31)

We can now calculate the volume of the three-sphere

\[
\text{vol}(S^3) = \int_{S^3} d^3x \sqrt{g} \\
= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^\pi d\psi \, \sin^2 \psi \sin \theta \\
= \frac{1}{2^3} \int_0^{4\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^\pi d\beta \sin \beta \\
= 2\pi^2. \] (A.32)

### A.1.3 Killing vectors

The left- and right-actions are generated by six Killing fields \( L_A = L_A^\alpha \partial_\alpha \) and \( R_A = R_A^\alpha \partial_\alpha \) defined by their action on the group elements

\[
L_A D(\alpha, \beta, \gamma) \equiv -T_A D(\alpha, \beta, \gamma), \quad (A.33)
\]
\[
R_A D(\alpha, \beta, \gamma) \equiv D(\alpha, \beta, \gamma) T_A. \quad (A.34)
\]

It is easy to see that \( L_3 = -\partial/\partial \alpha \) and \( R_3 = \partial/\partial \gamma \). Using the definition of the vielbeins one can show that the Killing fields are given by

\[
L_A^\alpha = -(\text{Ad } D)_A^B \, e_B^\alpha, \quad R_A^\alpha = e_A^\alpha. \quad (A.35)
\]

Here \( e_A^\alpha \) denotes the vielbein inverse with \( e_A^\alpha e^B_\alpha = \delta^B_A \) and \( e_A^\alpha e_A^\beta = \delta^\beta_\alpha \). Notice that \( \langle R_A, e_B^\gamma \rangle = \delta_A^B \).

Explicitly we have for the left-Killing fields

\[
L_1 = \cos \alpha \cot \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} - \frac{\cos \alpha}{\sin \beta} \frac{\partial}{\partial \gamma}, \quad (A.36)
\]
\[
L_2 = \sin \alpha \cot \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} - \frac{\sin \alpha}{\sin \beta} \frac{\partial}{\partial \gamma}, \quad (A.37)
\]
\[
L_3 = -\frac{\partial}{\partial \alpha}. \quad (A.38)
\]
and for the right-Killing fields

\[ R_1 = -\cos \gamma \frac{\partial}{\partial \alpha} + \sin \gamma \frac{\partial}{\partial \beta} + \cos \gamma \cot \beta \frac{\partial}{\partial \gamma}, \quad (A.39) \]

\[ R_2 = \sin \gamma \frac{\partial}{\partial \alpha} + \cos \gamma \frac{\partial}{\partial \beta} - \sin \gamma \cot \beta \frac{\partial}{\partial \gamma}, \quad (A.40) \]

\[ R_3 = \frac{\partial}{\partial \gamma}. \quad (A.41) \]

From the definition of \( L_A \) and \( R_A \) follows that

\[ [L_A, L_B]D = -[T_A, T_B]D, \quad [R_A, R_B]D = D[T_A, T_B], \quad (A.42) \]

thus \( L_A \) and \( R_A \) generate two independent copies of \( SU(2) \)

\[ [L_A, L_B] = \epsilon_{AB}^C L_C, \quad [R_A, R_B] = \epsilon_{AB}^C R_C. \quad (A.43) \]

Notice that on the inverses, the roles of \( R_A \) and \( L_A \) are interchanged. Indeed, since \( dD^{-1} = -D^{-1}dDD^{-1} \) we have that

\[ L_A D^{-1}(\alpha, \beta, \gamma) = D^{-1}(\alpha, \beta, \gamma)T_A, \quad (A.44) \]

\[ R_A D^{-1}(\alpha, \beta, \gamma) = -T_A D^{-1}(\alpha, \beta, \gamma). \quad (A.45) \]

### A.1.4 Matrix representations

In an \( SU(2) \) representation \(|j, m\rangle\), with \( j = 0, 1/2, 1, \ldots \) and \(-j \leq m \leq j\), the action of the rotation matrix \( D(\alpha, \beta, \gamma) \) is represented by the Wigner \( D \)-matrices \( \mathcal{D}^{(j)}_{mn}(D(\alpha, \beta, \gamma)) \) defined by

\[
\mathcal{D}^{(j)}_{mn}(D(\alpha, \beta, \gamma)) \equiv \langle j, m|D(\alpha, \beta, \gamma)|j, n\rangle \\
= \langle j, m|e^{\alpha T_3}e^{\beta T_2}e^{\gamma T_3}|j, n\rangle. \quad (A.46)
\]

Now \( T_3 = -iJ_3 \), with \( J_3|j, m\rangle = m|j, m\rangle \), so in the “zyz-convention” we are employing here, we have

\[
\mathcal{D}^{(j)}_{mn}(D(\alpha, \beta, \gamma)) = e^{-ima}e^{-in\gamma}\langle j, m|e^{\beta T_2}|j, n\rangle \\
\equiv e^{-ima}e^{-in\gamma}d^{(j)}_{mn}(\beta), \quad (A.47)
\]

where \( d^{(j)}_{mn}(\beta) \) denotes the “small” Wigner \( d \)-matrix defined by

\[ d^{(j)}_{mn}(\beta) \equiv \langle j, m|e^{\beta T_2}|j, n\rangle. \quad (A.48)\]
Explicitly, Wigner’s $d$-matrix $d_{mn}^{(j)}(\beta)$ takes the form

$$
d_{mn}^{(j)}(\beta) = \sum_k (-1)^{k-n+m} \sqrt{(j+n)!(j-n)!(j+m)!(j-m)!}$$

$$
\times \left( \cos \frac{\beta}{2} \right)^{2j-2k+n-m} \left( \sin \frac{\beta}{2} \right)^{2k-n+m},
\quad (A.50)
$$

where we take the sum over $k$ whenever none of the arguments of the factorials in the denominator are negative, see the textbook [102].

**Left- and right-action**

We now evaluate the action of the left- and right-Killing fields on the Wigner $D, d$-matrices. First, for the square of the left- and right-Killing fields, $L^2 = \delta_{AB}L_AL_B$ and $R^2 = \delta_{AB}R_AR_B$, we have

$$
L^2 = R^2 = \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left( \sin \beta \frac{\partial}{\partial \beta} \right)
$$

$$
+ \frac{1}{\sin^2 \beta} \left( \frac{\partial^2}{\partial \alpha^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right),
\quad (A.51)
$$

or in a sometimes more useful form

$$
L^2 = R^2 = \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left( \sin \beta \frac{\partial}{\partial \beta} \right)
$$

$$
+ \frac{1}{\sin^2 \beta} \left( \frac{\partial}{\partial \alpha} - \cos \beta \frac{\partial}{\partial \gamma} \right)^2 + \frac{\partial^2}{\partial \gamma^2},
\quad (A.52)
$$

where we have separated out the $\partial^2/\partial \gamma^2 = R^2_3$ term. Now, by definition we have

$$
L^2 D = T^2 D, \quad R^2 D = DT^2,
\quad (A.53)
$$

thus on the Wigner $D$-matrices we get

$$
L^2 \mathcal{D}_{mn}^{(j)}(D) = \mathcal{D}_{mn}^{(j)}(T^2 D) = -j(j+1) \mathcal{D}_{mn}^{j}(D),
\quad (A.54)
$$

and

$$
R^2 \mathcal{D}_{mn}^{(j)}(D) = \mathcal{D}_{mn}^{(j)}(DT^2) = -j(j+1) \mathcal{D}_{mn}^{j}(D),
\quad (A.55)
$$

where we used that $T^2 = -J^2$, with $J^2|j,m\rangle = j(j+1)|j,m\rangle$ and $\langle j,m|J^2 = j(j+1)\langle j,m|$. 
Therefore we conclude that the Wigner $D$-matrices satisfy the following eigenvalue equations

$$
\left( \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left( \sin \beta \frac{\partial}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \left( \frac{\partial}{\partial \alpha} - \cos \beta \frac{\partial}{\partial \gamma} \right)^2 + \frac{\partial^2}{\partial \gamma^2} \right) \mathcal{D}^{(j)}_{mn}(\alpha, \beta, \gamma) = -j(j+1) \mathcal{D}^{(j)}_{mn}(\alpha, \beta, \gamma).
$$

(A.56)

From this equation we also read off the eigenvalue equations satisfied by the small Wigner $d$-matrices

$$
\left( \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left( \sin \beta \frac{\partial}{\partial \beta} \right) - \frac{1}{\sin^2 \beta} (m - n \cos \beta)^2 \right) d^{(j)}_{mn}(\beta) = -(j(j+1) - n^2) d^{(j)}_{mn}(\beta).
$$

(A.57)

**A.1.5 Curvature**

From the Maurer-Cartan equation we can read off the connection one-forms $\omega^{AB} = dx^\alpha (\omega^{AB})_\alpha$, defined by the torsion-free condition

$$
d e^A + \omega^A_B \wedge e^B = 0,
$$

(A.58)

see the textbook [103]. Explicitly we get

$$
\omega^A_B = \frac{1}{2} \epsilon_{CB}^A e^C.
$$

(A.59)

From the connection one-form we define the Riemann two-form

$$
R^A_B = d \omega^A_B + \omega^A_C \wedge \omega^C_B.
$$

(A.60)

The two-form is explicitly given by

$$
R^{AB} = \frac{1}{4} \delta^{A}_{[C} \delta^{B]} e^C \wedge e^D,
$$

(A.61)

and its components, which define the Riemann tensor $R^A_{B} = \frac{1}{2} R^A_{BCD} e^C \wedge e^D$, are given by

$$
R^{AB}_{CD} = \frac{1}{2} \delta^{A}_{[C} \delta^{B]}_{D]}.
$$

(A.62)

From here we can also construct the Ricci tensor $R^A_B = R^{CA}_{CD}$ with the result

$$
R^A_B = \frac{1}{2} \delta^A_B.
$$

(A.63)
\section{Covariant derivative}

The tangent-space covariant derivative \( \mathcal{D} = dx^\alpha \partial_\alpha \) is defined by

\[
\mathcal{D} = d + \frac{1}{2} \omega^{AB} \Sigma_{AB},
\]

where \( \Sigma_{AB} \) is the generator of tangent-space rotations. The covariant derivative is defined such that under a tangent-space rotation \( (e')^a = \Lambda^a_b e^b \), with a field \( \phi_M \) transforming in some representation \( (\phi')_M = L(\Lambda)_M^N \phi_N \), then

\[
((\mathcal{D}\phi')_M = L(\Lambda)_M^N (\mathcal{D}\phi)_N.
\]

For example, on a tangent-space vector \( V^A \) with \( (\Sigma_{AB})_{CD} = 2 \delta^C_A \delta^D_B \) we have

\[
\mathcal{D}V^A = dV^A + \frac{1}{2} \omega^{CD} (\Sigma_{CD})_{AB} V^B = dV^A + \omega_A^B V^B.
\]

When converting tangent-space indices to curved-space indices using the vielbein, e.g. \( V_\alpha = e_\alpha^A V_A \), the tangent-space covariant derivative \( \mathcal{D} \) is defined such that

\[
\nabla Y_\alpha = \nabla (e_\alpha^A V_A) = e_\alpha^A \partial_\alpha \]

where \( \nabla = dx^\alpha \nabla_\alpha \) denotes the curved-space covariant derivative. From this property we can read off the relation between the tangent-space connection \( (\omega^A_B)_\alpha \) and the curved-space Christoffel connection \( \Gamma^\alpha_{\beta\gamma} \)

\[
(\omega^A_B)_\alpha = e_\gamma^A \nabla_\alpha e_\gamma^B = e_\gamma^A (\partial_\alpha e_\gamma^B + \Gamma_{\alpha\beta}^{\gamma} e_\beta^B),
\]

or in form notation \( \omega^A_B = e_\gamma^A \nabla_B e_\gamma^B \). From this expression it is easy to show the torsion-free condition using the symmetry of the Christoffel symbol \( \Gamma_{\beta\gamma}^{\alpha} = \Gamma_{\gamma\beta}^{\alpha} \). Indeed

\[
\omega^A_B \wedge e^B = e_\gamma^A \nabla_B e_\gamma^B \wedge e^B = -\nabla e_\gamma^A e_\gamma^B \wedge e^B = -\nabla e_\gamma^A \wedge dx^\gamma = -d(e_\gamma^A dx^\gamma) = -de^A,
\]

that is \( de^A + \omega^A_B \wedge e^B = 0 \).
A.1.7 Harmonics

Scalar Harmonics

Since the collection of representation matrices \( D^{(j)}_{mn}(D(\alpha, \beta, \gamma)) \) of \( SU(2) \) are both orthogonal and complete (by the Peter-Weyl theorem) we can expand any function on the three-sphere in terms of them. For convenience we consider inverse representations and define three-sphere scalar harmonics as

\[
Y^{(j,m_R,m_L)}(\alpha, \beta, \gamma) = D^{(j)}_{m_Rm_L}(D^{-1}(\alpha, \beta, \gamma)).
\] (A.70)

This object has definite transformation properties under the isometry group \( SU(2)_L \times SU(2)_R \) of the three-sphere. Indeed, under a combined left- and right-translation \( g_L D(\alpha, \beta, \gamma) g^{-1}_R = D(\alpha', \beta', \gamma') \), we have that

\[
Y^{(j,m_R,m_L)}(\alpha', \beta', \gamma') = D^{(j)}_{m_Rm_L}(g_R) Y^{(j,m'_R,m'_L)}(\alpha, \beta, \gamma) D^{(j)}_{m'_Lm'_R}(g^{-1}_L),
\] (A.71)

that is \( Y^{(j,m_R,n_L)} \) transforms in the \( j_L = j \) and \( j_R = j \) representation of \( SU(2)_L \times SU(2)_R \). The fact that \( j_L = j_R \) for scalar harmonics comes from the equality of the two quadratic Casimirs \( R^2 = L^2 \).

Since \( D^{-1}(\alpha, \beta, \gamma) = e^{-\gamma T_3} e^{-\beta T_2} e^{-\alpha T_3} \) we get that the scalar harmonics take the form

\[
Y^{(j,m_R,m_L)}(\alpha, \beta, \gamma) = e^{im_R\gamma} e^{im_L\alpha} d^{(j)}_{m_Lm_R}(\beta),
\] (A.72)

where we used that \( d^{(j)}_{m_Rm_L}(-\beta) = d^{(j)}_{m_Lm_R}(\beta) \).

We now calculate the Laplacian \( D^2 = g^{AB} D_A D_B \) acting on the scalar harmonics. Since \( Y^{(j,m_R,m_L)} \) is a scalar we have that \( \Sigma_{AB} Y^{(j,m_R,m_L)} = 0 \), thus the covariant derivative acting on a scalar is simply

\[
\nabla Y^{(j,m_R,m_L)} = d Y^{(j,m_R,m_L)}. \] (A.73)

Now \( d = dx^\alpha \partial_\alpha = e^A R_A \) and the action of \( R_A = e^A_\alpha \partial_\alpha \) on the inverse representation matrices produces a factor \(-T_A\), i.e.

\[
d Y^{(j,m_R,m_L)} = -e^A T_A Y^{(j,m_R,m_L)},
\] (A.74)

or in components \( \nabla_A Y^{(j,m_R,m_L)} = -T_A Y^{(j,m_R,m_L)} \).
A.1. The three-sphere $S^3$

Now $\mathcal{D}_AY^{(j,m_R,m_L)}$ is a vector so

$$\mathcal{D}\mathcal{D}_AY^{(j,m_R,m_L)} = \mathcal{D}\mathcal{D}_AY^{(j,m_R,m_L)} + \frac{1}{2}\omega^{CD}(\Sigma_{CD})_A^B\mathcal{D}_BY^{(j,m_R,m_L)},$$

(A.75)

where the vector generator is given by $(\Sigma_{AB})^{CD} = 2\delta^C_A\delta^D_B$. Now using the expression for the connection one-form $\omega^A_B = \frac{1}{2}\epsilon^C_A e^C$ and using that $d$ acts as an operator on representation matrices we get that

$$\mathcal{D}_C\mathcal{D}_AY^{(j,m_R,m_L)} = T_AT_CY^{(j,m_R,m_L)} - \frac{1}{2}\epsilon^C_B A_TBY^{(j,m_R,m_L)}.$$  (A.76)

Thus squaring this equation we get for the Laplacian

$$\mathcal{D}^2Y^{(j,m_R,m_L)} = T^2Y^{(j,m_R,m_L)}.$$  (A.77)

Using that $T^2Y^{(j,m_R,m_L)} = -j(j+1)Y^{(j,m_R,m_L)}$ we get

$$\mathcal{D}^2Y^{(j,m_R,m_L)} = -j(j+1)Y^{(j,m_R,m_L)},$$  (A.78)

for $j = 0, 1/2, 1, 3/2, \ldots$

**Vector Harmonics**

To build vectors on $S^3$ we take combinations of $e^A_\alpha$ and $Y^{(j,m_R,m_L)}$. Now $e^A_\alpha$ is left-invariant so the object $e^A Y^{(j,m_R,m_L)}$ transforms in the definite representation $j_L = j$ of $SU(2)_L$ with magnetic quantum number $m_L$. But both $e^A_\alpha$ and $Y^{(j,m_R,m_L)}$ transform under $SU(2)_R$, so if we wish to construct definite $SU(2)_R$ representations we have to tensor them together using the appropriate Clebsch–Gordan coefficients.

Under a right transformation $D = D'g_R^{-1}$, the vielbeins transform as $(e')^A = eB(\text{Ad} g_R^{-1})_B^A$, while the harmonics transform as

$$Y^{(j,m_R,m_L)}(\alpha', \beta', \gamma') = \mathcal{D}^{(j)}_{m_R m_L'}(g_R)Y^{(j,m_R,m_L)}(\alpha, \beta, \gamma).$$  (A.79)

We now go to a basis $T_A \rightarrow T_m = U_mA_T$ and $e^A \rightarrow e^m = e^A(U^{-1})_A^m$, such that

$$(\text{Ad} g_R^{-1})_m^m = \mathcal{D}^{(j=1)}_{mm'}(g_R).$$  (A.80)

Then $e^m$ transforms in the definite $j_R = 1$ representation of $SU(2)_R$, i.e.

$$(e')^m = \mathcal{D}^{(j=1)}_{mm'}(g_R)e^{m'}.$$  (A.81)

Explicitly $T_{\pm 1} = \mp \frac{1}{\sqrt{2}}(T_1 \pm iT_2) = \mp \frac{1}{\sqrt{2}}T_{\pm}$ and $T_0 = T_3$, together with $e^{\pm 1} = \mp \frac{1}{\sqrt{2}}(e^1 \mp ie^2)$ and $e^0 = e^3$. 
We can now tensor $e^m$ together with $Y^{(j,m_R,m_L)}$, which results in three different $j_R = j - 1, j, j + 1$ representations

$$Y^{(j,m_R,m_L,j_R,m_R)} = \sum_m \langle j, m_R - m; 1, m | j_R, m_R \rangle e^m Y^{(j,m_R-m,m_L)}$$  \hspace{1cm} (A.82)

where $\langle j_1, m_1; j_2, m_2 | j_3, m_3 \rangle$ denotes the Clebsch-Gordan coefficient we get when tensoring $j_1$ and $j_2$ to get $j_3$. Before writing explicit expressions for the Clebsch-Gordan coefficients we determine the eigenvectors of the Lichnerowicz operator on $S^3$.

**Lichnerowicz operator**

The action of the Lichnerowicz operator on a vector is given by

$$\Delta_L V_A = -D^2 V_A + R_A^B V_B.$$  \hspace{1cm} (A.83)

Using that the covariant derivative acts on harmonics as

$$D_A = -T_A + \frac{1}{2} \omega^{AB} \Sigma_{AB},$$  \hspace{1cm} (A.84)

with $(\Sigma_{AB})^{CD} = \delta^C_A \delta^D_B - \delta^C_B \delta^D_A$ for vectors, the Laplacian takes the form

$$(D^2)^A_B = \delta^B_A T^2 - T^G (\omega^{CD}) G (\Sigma_{CD})_A^B + \frac{1}{4} (\omega^{CD}) G (\omega^{EF}) G (\Sigma_{CD} \Sigma_{EF})_A^B.$$  \hspace{1cm} (A.85)

In the complex basis $(+1, 0, -1)$, defined above, we get

$$(D^2)_A^B = \begin{pmatrix} T^2 - \frac{1}{2} - iT_3 & -\frac{i}{\sqrt{2}} T_+ & 0 \\ -\frac{i}{\sqrt{2}} T_- & T^2 - \frac{1}{2} - \frac{i}{\sqrt{2}} T_+ & 0 \\ 0 & -\frac{i}{\sqrt{2}} T_- & T^2 - \frac{1}{2} + iT_3 \end{pmatrix}.$$  \hspace{1cm} (A.86)

Since $R_A^B = \frac{1}{2} \delta_A^B$ we get for the Lichnerowicz operator

$$(\Delta_L)_A^B = \begin{pmatrix} -T^2 + iT_3 + 1 & \frac{i}{\sqrt{2}} T_+ & 0 \\ \frac{i}{\sqrt{2}} T_- & -T^2 + 1 & \frac{i}{\sqrt{2}} T_+ \\ 0 & \frac{i}{\sqrt{2}} T_- & -T^2 - iT_3 + 1 \end{pmatrix}.$$  \hspace{1cm} (A.87)

Now any vector $V^{(j,m_R,m_L)} = dx^\alpha V^{(j,m_R,m_L)}_\alpha$ on $S^3$ with a definite $SU(2)_L$ representation $j_L = j$ and magnetic quantum number $m_L$ of $SU(2)_L$ and $m_R$ of $SU(2)_R$ can be expanded as

$$V^{(j,m_R,m_L)}_\alpha = c_{+1} e_\alpha^{+1} Y^{(j,m_R,m_L)}_{+1} + c_0 e_\alpha^0 Y^{(j,m_R,m_L)}_0 + c_{-1} e_\alpha^{-1} Y^{(j,m_R,m_L)}_{-1}$$  \hspace{1cm} (A.88)
A.1. The three-sphere $S^3$

for some constants $c_{+1}, c_0, c_{-1}$ and where we have defined

$$\begin{pmatrix}
Y^{(j,m_R,m_L)}_{+1}
\ Y_0^{(j,m_R,m_L)}
\ Y^{(j,m_R+1,m_L)}_{-1}
\end{pmatrix} =
\begin{pmatrix}
Y^{(j,m_{R-1},m_L)}
\ Y^{(j,m_L,m_R)}
\ Y^{(j,m_{R+1},m_L)}
\end{pmatrix}.$$ (A.89)

In the tangent-space, the vector takes the form

$$V_A^{(j,m_R,m_L)} =
\begin{pmatrix}
c_{+1}Y^{(j,m_R,m_L)}_{+1}
\ c_0Y_0^{(j,m_R,m_L)}
\ c_{-1}Y^{(j,m_R,m_L)}_{-1}
\end{pmatrix}.$$ (A.90)

Now acting with $\Delta_L$ on this vector we should evaluate

$$\left(\Delta_L\right)^B_A V_B^{(j,m,n)} = (\Delta_L)^B_A Y_B^{(j,m_R,m_L)} = \sum_B (\Delta_L)^B_A Y_B^{(j,m_R,m_L)},$$ (A.91)

For fixed $A, B$, $(\Delta_L)^B_A$ is an operator taking $Y_B^{(j,m_R,m_L)}$ to $Y_A^{(j,m_R,m_L)}$, with some numerical coefficient $M^B_A$, i.e.

$$(\Delta_L)^B_A Y_B^{(j,m_R,m_L)} = M^B_A Y_A^{(j,m_R,m_L)}, \text{ no sum over } A, B. \quad (A.92)$$

We can now find the eigenvectors $c_A$ and the corresponding eigenvalues $\lambda$ of $M^B_A$, with $M^B_A c_B = \lambda c_A$. Then, the vector with coefficients $V_A^{(j,m_R,m_L)} = c_A Y_A^{(j,m_R,m_L)}$ will also have the eigenvalue $\lambda$. First we have

$$\Delta_L V_A^{(j,m_R,m_L)} = \Delta_L \left(c_A Y_A^{(j,m_R,m_L)}\right) = \sum_B (\Delta_L)^B_A \left(c_B Y_B^{(j,m_R,m_L)}\right)$$ (A.93)

then using that $(\Delta_L)^B_A Y_B^{(j,m_R,m_L)} = M^B_A Y_A^{(j,m_R,m_L)}$ we get

$$\Delta_L V_A^{(j,m_R,m_L)} = \sum_B M^B_A c_B Y_A^{(j,m_R,m_L)} = \lambda c_A Y_A^{(j,m_R,m_L)} = \lambda V_A^{(j,m_R,m_L)}.$$ (A.94)

Now, using that

$$iT^\pm Y^{(j,m_R,m_L)} = J^\pm (j, m_R) Y^{(j,m_R\mp 1,m_L)},$$ (A.95)

$$iT^3 Y^{(j,m_R,m_L)} = J_3 (j, m_R) Y^{(j,m_R,m_L)},$$ (A.96)
where \( J_{\pm}(j,m) = \sqrt{j(j+1) - m(m \pm 1)} \) and \( J_3(j,m) = m \), we explicitly get

\[
M_A^B = \begin{pmatrix}
\frac{j(j+1)+(m_R-1)+1}{\sqrt{2}} & \frac{1}{\sqrt{2}} J_{-}(j,m_R) & 0 \\
\frac{1}{\sqrt{2}} J_{+}(j,m_R-1) & j(j+1)+1 & \frac{1}{\sqrt{2}} J_{+}(j,m_R+1) \\
0 & \frac{1}{\sqrt{2}} J_{+}(j,m_R) & j(j+1)-(m_R+1)+1
\end{pmatrix}.
\]  \tag{A.97}

The matrix \( M_A^B \) has three eigenvalues

\[
\lambda_{(-1)} = j^2, \quad \lambda_{(0)} = j(j+1), \quad \lambda_{(+1)} = (j+1)^2,
\]  \tag{A.98}

with corresponding eigenvectors \( c_{A}^{(-1)} \), \( c_{A}^{(0)} \) and \( c_{A}^{(+1)} \). The coefficients \( c_{A}^{(B)} \) correspond to the Clebsch-Gordan coefficients arising when tensoring \( j_R = j \) with \( j_R = 1 \) getting \( j_R = j + B \). Thus \( c_{A}^{(-1)} \) corresponds to \( j_R = j - 1 \), \( c_{A}^{(0)} = j_R = j \), and \( c_{A}^{(+1)} \) to \( j_R = j + 1 \).

Explicitly we have for the \( j_R = j + 1 \) coefficients (see textbook [104])

\[
c_{\pm 1}^{(+1)} = \langle j, m_R \mp 1; 1, \pm 1|j, m_R \rangle \\
= \frac{\sqrt{(j \pm m_R)(j \pm m_R + 1)}}{(2j+1)(2j+2)}, \tag{A.99}
\]

\[
c_{0}^{(+1)} = \langle j, m_R; 1, 0|j, m_R \rangle \\
= \frac{\sqrt{(j - m_R + 1)(j + m_R + 1)}}{(2j+1)(j+1)}, \tag{A.100}
\]

the \( j_R = j \) coefficients

\[
c_{\pm 1}^{(0)} = \langle j, m_R \mp 1; 1, \pm 1|j, m_R \rangle \\
= \pm \frac{\sqrt{(j \mp m_R)(j \mp m_R + 1)}}{2j(j+1)}, \tag{A.101}
\]

\[
c_{0}^{(0)} = \langle j, m_R; 1, 0|j, m_R \rangle \\
= \frac{m_R}{\sqrt{j(j+1)}}, \tag{A.102}
\]
and the $j_R = j - 1$ coefficients
\[ c_{\pm 1}^{(-1)} = \langle j, m_R \mp 1; 1, \pm 1 | j - 1, m_R \rangle = \sqrt{\frac{(j \mp m_R)(j \mp m_R + 1)}{2j(2j + 1)}}, \tag{A.103} \]
\[ c_0^{(-1)} = \langle j, m_R; 1, 0 | j - 1, m_R \rangle = -\sqrt{\frac{(j - m_R)(j + m_R)}{j(2j + 1)}}. \tag{A.104} \]

Notice that for $j_R = j$ then
\[ \sqrt{j(j + 1)c_{\pm 1}^{(0)}} = \mp J_{j}(j, m_R) \text{ and } \sqrt{j(j + 1)c_0^{(0)}} = J_3(j, m_R), \]
thus the $j_R = j$ vector is a longitudinal vector
\[ \sqrt{j(j + 1)}Y^{(j, m_R; j = j, m_R; \pm 1, \pm 0)} = iDY/(j, m_R, m_L). \tag{A.105} \]

\section*{A.2 The Two-Sphere $S^2$}

\subsection*{A.2.1 Identification $S^2 = SO(3)/SO(2)$}

The standard way of constructing the two-sphere $S^2$ as a coset space $SO(3)/SO(2)$ is to use the fact that it is a symmetric space, i.e. starting from any point on the sphere, say the north pole $N = (0, 0, 1)$, any other point $(x, y, z)$ can be reached using an $SO(3)$ rotation. Indeed, using a rotation $D(\varphi, \theta) = e^{\varphi T_3}e^{\theta T_2}$ we get
\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_\varphi & -s_\varphi & 0 \\ s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s_\theta c_\varphi \\ s_\theta s_\varphi \end{pmatrix}, \tag{A.106} \]
which is the standard parametrization of the sphere in terms of angles. Now this correspondence is not one-to-one since $D$ takes you to the same point as $Dh$ for any $h = e^{\xi T_3}$. Thus we arrive at the conclusion that $S^2 = SO(3)/SO(2)_{3}$ where the $SO(2)_{3}$ subgroup of $SO(3)$ is generated by $T_3$. The coset elements of $SO(3)/SO(2)_{3}$ are given by those elements of $SO(3)$ with $\alpha = \varphi$, $\beta = \theta$ and $\gamma = 0$
\[ D(\varphi, \theta) = D(\alpha = \varphi, \beta = \theta, \gamma = 0) = e^{\varphi T_3}e^{\theta T_2}. \tag{A.107} \]
Appendix A. Harmonics on $S^2$ and $S^3$

A.2.2 Metric

To construct the metric we again define the Maurer-Cartan forms by

$$D^{-1} dD = e^A T_A = e^a T_a + e^3 T_3,$$  \hspace{1cm} (A.108)

where $a = 1, 2$. Explicitly we get

$$e^1 = - \sin \theta d \varphi,$$ \hspace{1cm} (A.109)
$$e^2 = d \theta,$$ \hspace{1cm} (A.110)
$$e^3 = \cos \theta d \varphi.$$ \hspace{1cm} (A.111)

They satisfy the Maurer-Cartan equation

$$d e^A + \frac{1}{2} \epsilon_{BC}^A e^B \wedge e^C = 0.$$ \hspace{1cm} (A.112)

However, this time the one-forms are no longer left-invariant.

**Left-translations** Consider multiplying $D(\varphi, \theta)$ by some element $g_L$ of $SO(3)$ from the left $g_L D(\varphi, \theta)$. Now $D(\varphi, \theta)$ takes the north pole to the point $(\theta, \varphi)$, while $g_L D$ will in general take the north pole to some other point $(\varphi', \theta')$ corresponding to the coset element $D(\varphi', \theta')$. Now modulo $SO(2)_3$ transformations, $D$ and $D'$ are the same, thus we have

$$g_L D(\varphi, \theta) = D(\varphi', \theta') h,$$ \hspace{1cm} (A.113)

for some $h = e^{\xi T_3} \in SO(2)_3$. In general $\xi$ will be a complicated function of $\varphi, \theta$ and $g_L$. Right-translations, apart from trivial $SO(2)_3$ translations, on the coset space $SO(3)/SO(2)_3$ are not well-defined.

Now under a left-translation the one-forms transform in the following way

$$D^{-1}(x') dD(x) = h^{-1} D^{-1}(x) g_L^{-1} d(g_L D(x) h)$$
$$= h^{-1} D^{-1}(x) dD(x) h + h^{-1} dh.$$ \hspace{1cm} (A.114)

Since $h^{-1} dh$ is proportional to $T_3$ and $(\text{Ad} T_3)_A^B = - \epsilon_{3A}^B$ we get that the $\{e^a\}$ transform amongst each other

$$e^a(x') = e^B(x) (\text{Ad} h)_B^a = e^b(x) (\text{Ad} h)_b^a,$$ \hspace{1cm} (A.115)

while $e^3$ instead gets shifted

$$e^3(x') = e^3(x) + d\xi.$$ \hspace{1cm} (A.116)
A.2. The Two-Sphere $S^2$

We now define the metric as

$$ds^2 = \delta_{ab} e^a e^b = d\theta^2 + \sin^2 \theta \, d\varphi^2,$$  \hfill (A.117)

which is the standard metric on the two-sphere. Since $\text{Ad} \, T_3$ is anti-symmetric, $\text{Ad} \, h$ is orthogonal, and the metric is indeed left-invariant

$$e^a(x') e_a(x') = e^b(x) (\text{Ad} \, h)_b^a (\text{Ad} \, h^T)_a^c e_c(x) = e^a(x) e_a(x).$$  \hfill (A.118)

This can also be easily seen in a complex coordinate system. Make a change of coordinates $T_m = U_m^a T_a$ and $e^m = e^a (U^{-1})_a^m$ such that

$$T_{\pm} = T_1 \pm i T_2, \quad e_{\pm} = \frac{1}{2} (e^1 \mp ie^2).$$  \hfill (A.119)

In this basis the metric becomes

$$g_{mn} = U_m^a U_n^b \delta_{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

so that $ds^2 = g_{mn} e^m e^n = 4 e^+ e^-$. Furthermore,

$$(\text{Ad} \, h)_{m}^n = (e^\xi \text{Ad} \, T_3)_{m}^n = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix},$$

thus $(e')_{\pm} = e^{\pm i\xi} e_{\pm}$. From these consideration it is clear that the metric is indeed left-invariant.

A.2.3 Curvature

We define the connection one-form $\omega^{ab} = dx^a \omega^a_{\ a}$ from the torsion free condition

$$de^a + \omega^a_{\ b} \wedge e^b = 0.$$  \hfill (A.122)

It is useful to split the connection into two pieces, one piece in the coset space $SO(3)/SO(2)_3$ and one piece outside in $SO(2)_3$

$$\omega^{ab} = e^3 \Omega_{3}^{ab} + e^c M_c^{ab}.$$  \hfill (A.123)

Comparing the equation for the connection one-form with the Maurer-Cartan equation we find

$$\omega^a_{\ b} = -e^a_{\ b} e^3,$$  \hfill (A.124)
where $\epsilon^a_b = -\epsilon^{3a}_b$. Thus $\omega^{ab}$ is purely in $SO(2)_3$

\[ \Omega^{ab}_3 = -\epsilon^{ab}, \quad M^{ab}_e = 0. \quad (A.125) \]

The curvature two-form $R^a_b$ is defined by

\[ R^a_b = d\omega^a_b + \omega^c_a \wedge \omega^e_b, \quad (A.126) \]

and we explicitly find

\[ R^{ab} = \frac{1}{2} \epsilon^{ab} \epsilon_{cd} e^c \wedge e^d. \quad (A.127) \]

From this expression we read off the components of the Riemann tensor $R^{ab} = \frac{1}{2} R^{ab}_{cd} e^c \wedge e^d$

\[ R^{ab}_{cd} = \epsilon^{ab} \epsilon_{cd} = 2 \delta^a_c \delta^b_d. \quad (A.128) \]

Then we get for the Ricci tensor and Ricci scalar $R^a_b = R^{ca}_{cb} = \delta^a_b, \quad R = R^a_a = 2. \quad (A.129)$

Starting from $ds^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$ and following the conventions of the textbook [105] with $R^{\alpha\beta\gamma\delta}_e = \partial_\beta \Gamma^e_\alpha_\gamma - \partial_\alpha \Gamma^e_\beta_\gamma + \Gamma^e_\epsilon_\gamma \Gamma^\epsilon_\beta - \Gamma^e_\epsilon_\beta \Gamma^\epsilon_\alpha$ one also finds $R^{\alpha\beta\gamma\delta}_e = 2 \delta^a_{[\gamma} \delta^b_{\delta]}$.

### A.2.4 Killing vectors

The Killing vectors corresponding to left-translations are given by

\[ L^A_\alpha = -(Ad \, D)_A^a e_a^\alpha. \quad (A.130) \]

Explicitly we have

\[ L_1 = \sin \varphi \frac{\partial}{\partial \theta} + \cos \varphi \cot \theta \frac{\partial}{\partial \varphi}, \quad (A.131) \]

\[ L_2 = -\cos \varphi \frac{\partial}{\partial \theta} + \sin \varphi \cot \theta \frac{\partial}{\partial \varphi}, \quad (A.132) \]

\[ L_3 = -\frac{\partial}{\partial \varphi}, \quad (A.133) \]

with $[L_A, L_B] = \epsilon_{AB}^C L_C$. Squaring the Killing vectors $L^2 = \delta^{AB} L_A L_B$ we get

\[ L^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (A.134) \]
A.2.5 Harmonics

We now define a set $Y_{(q)}^{(j,m)}$ of irreducible harmonics

$$Y_{(q)}^{(j,m)}(\theta, \varphi) = \mathcal{D}_{-q,m}^{(j)}(D^{-1}(\varphi, \theta)). \quad (A.135)$$

Under a translation $gD(\varphi, \theta) = D(\varphi', \theta')h$, with $h = e^{\xi T_3}$, we have that

$$Y_{(q)}^{(j,m)}(\theta', \varphi') = e^{iq\xi} Y_{(q)}^{(j,n)}(\theta, \varphi) \mathcal{D}_{mn}^{(j)}(g^{-1}). \quad (A.136)$$

Any tensor on the two-sphere can be expanded in terms of these harmonics.

Scalar harmonics

Now, $Y_{(q)}^{(j,m)}$ only carries the standard representation under $SU(2)$ for $q = 0$, so as spherical scalar harmonics we take $Y_{(0)}^{(j,m)}(\theta, \varphi) \equiv Y_{(q)}^{(j,m)}(\theta, \varphi)$

$$Y^{(j,m)}(\theta, \varphi) = \mathcal{D}_{0m}^{(j)}(D^{-1}(\varphi, \theta)) = e^{im\varphi} d_{0m}^{(j)}(-\theta) = e^{im\varphi} d_{m0}^{(j)}(\theta). \quad (A.137)$$

The scalar Laplacian on the sphere $ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is given by

$$\nabla^2 = g^{\alpha\beta} \left( \partial_\alpha \partial_\beta - \Gamma^\gamma_{\alpha\beta} \partial_\gamma \right) = \frac{1}{\sqrt{g}} \partial_\alpha \left( \sqrt{g}^{\alpha\beta} \partial_\beta \right) = L^2, \quad (A.138)$$

where we have identified the resulting expression with $L^2$ in Eq. (A.134). Then we get

$$\nabla^2 Y^{(j,m)}(\theta, \varphi) = e^{im\varphi} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} \right) d_{m0}^{(j)}(\theta), \quad (A.139)$$

and using the property of the small Wigner d-matrix, with $n = 0$, we get

$$\nabla^2 Y^{(j,m)}(\theta, \varphi) = -j(j+1)Y^{(j,m)}(\theta, \varphi). \quad (A.140)$$

Tensor harmonics

We can now build tensors on the two-sphere using $e^a$ and $Y_{(q)}^{(j,m)}$. We choose the charge $q$ of $Y_{(q)}^{(j,m)}$ so that the combined tensor has a well-defined $SU(2)$ representation.
Appendix A. Harmonics on $S^2$ and $S^3$

We can build two vector harmonics

$$e^+ Y^{(j,m)}_{(-1)}, \quad e^- Y^{(j,m)}_{(+1)},$$

(A.141)

four two-tensors harmonics; three symmetric

$$e^+ \otimes e^+ Y^{(j,m)}_{(-2)}, \quad \frac{1}{\sqrt{2}} (e^+ \otimes e^- + e^- \otimes e^+) Y^{(j,m)}_{(0)}, \quad e^- \otimes e^- Y^{(j,m)}_{(+2)}$$

(A.142)

and one anti-symmetric

$$e^+ \land e^- Y^{(j,m)}_{(0)}.$$  

(A.143)

Continuing like this allows us to construct a tensor of any rank.

A.2.6 Covariant derivative

The tangent-space covariant derivative $D = dx^\alpha D_\alpha$ is given by

$$D = d + \frac{1}{2} \omega^{ab} \Sigma_{ab},$$

(A.144)

where $d = dx^\alpha \partial_\alpha$ denotes the exterior derivative and $\omega^{ab} = dx^\alpha (\omega^{ab})_\alpha$ is the connection one-form. We now separate the connection into its two parts $\omega^{ab} = \Omega^{ab} + M^{ab}$ so that

$$D = d + \frac{1}{2} \Omega^{ab} \Sigma_{ab} + \frac{1}{2} M^{ab} \Sigma_{ab}.$$  

(A.145)

Defining the “$H$-covariant” derivative $D^H = d + \frac{1}{2} \Omega^{ab} \Sigma_{ab}$ we can write the above as

$$D = D^H + \frac{1}{2} M^{ab} \Sigma_{ab}.$$  

(A.146)

The split is useful since, as we now show, acting on harmonics we have

$$D^H = -e^a T_a.$$  

(A.147)

Now $\Omega^{ab}_3 = -\epsilon^{ab}_3$, thus

$$\frac{1}{2} \Omega^{ab} \Sigma_{ab} = -\frac{1}{2} \epsilon^a_3 \epsilon^{ab}_3 \Sigma_{ab}.$$  

(A.148)
But $(\text{Ad} T_3)^a_b = -\epsilon_a^b$ is exactly the generator of tangent-space rotations for vectors. Indeed, under a left-translation $gD = D'h$, with $h = e^{\xi T_3}$, then

\[(e^\prime)^a_b = e^b(e^{\xi T_3})^a_b; \quad (T_3)^b_a = (\text{Ad} T_3)^a_b. \quad (A.149)\]

Now $(T_3)^b_a \equiv \frac{1}{2}(T_3)^{cd}(\Sigma_{cd})_a^b$, so for any representation of the tangent-space rotation group we have

\[(T_3)_M^N = -\frac{1}{2}\epsilon_{3}^{ab}(\Sigma_{ab})_M^N \quad (A.150)\]

From this observation follows that

\[D^H = d + \frac{1}{2}\Omega^{ab}\Sigma_{ab} = d - \frac{1}{2}e^3\epsilon_{3}^{ab}\Sigma_{ab} = d + e^3T_3. \quad (A.151)\]

Furthermore, when acting on the harmonics we have that $d = -e^AT_A$ so

\[D^H = -e^AT_a. \quad (A.152)\]

Thus on harmonics we have that

\[D_a = -T_a + \frac{1}{2}M_a^{cd}\Sigma_{cd}, \quad (A.153)\]

and the Laplacian becomes

\[D^2_a = T_a^2 - g^{cd}M_c^{ab}\Sigma_{ab}T_d + \frac{1}{4}g^{ef}M_e^{ab}M_f^{cd}\Sigma_{ab}\Sigma_{cd}. \quad (A.154)\]

On the two-sphere $M_a^{ab} = 0$ and we get

\[D^2_a = T_a^2 = T_A^2 - T_3^2. \quad (A.155)\]

**Scalars**

On scalars $\Sigma_{ab} = 0$ and we get

\[D^2Y_{(0)}^{(j,m)} = (T_A^2 - T_3^2)Y_{(0)}^{(j,m)} = -j(j+1)Y_{(0)}^{(j,m)}. \quad (A.156)\]

As scalar harmonics we take $Y_{(0)}^{(j,m)}(\theta, \varphi)$. Notice that these are exactly the usual spherical harmonics (up to normalization)

\[Y_{(0)}^{(j,m)}(\theta, \varphi) = \mathcal{D}_{0m}^{(j)}(D^{-1}(\varphi, \theta)) = \mathcal{D}_{m0}^{(j)}(D(\varphi, \theta))^* \propto Y_{j}^{m}(\theta, \varphi), \quad (A.157)\]

see the textbook [102].
Vectors

The Lichnerowicz operator acting on vectors is given by

$$\Delta_L Y_a = -D^2 Y_a + R^b_a Y_b,$$  \hspace{1cm} (A.158)

where $D^2 = T^2_A - T^2_3$ and $R^b_a = \delta^b_a$. Now, a general vector can be written as

$$V^{(j,m)} = e^+ c_+ Y^{(j,m)}_{(-1)} + e^- c_- Y^{(j,m)}_{(+1)},$$ \hspace{1cm} (A.159)

that is in tangent-space notation

$$V^{(j,m)}_a = \begin{pmatrix} c_+ Y^{(j,m)}_{(-1)} \\ c_- Y^{(j,m)}_{(+1)} \end{pmatrix}.$$ \hspace{1cm} (A.160)

In this basis we readily check that

$$(T_3)_m^n = -\frac{1}{2} \epsilon^a_3 (\Sigma_{ab})_m^n = -(\Sigma_{12})_m^n = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix},$$ \hspace{1cm} (A.161)

which simply states that $T_3 Y^{(j,m)}_{(\pm 1)} = \pm i Y^{(j,m)}_{(\pm 1)}$, which is also apparent from the definition of $Y^{(j,m)}_{(\pm 1)}$.

The Lichnerowicz operator is given by

$$\Delta_L = \begin{pmatrix} -T^2_A + T^2_3 + 1 \\ 0 \\ -T^2_A + T^2_3 + 1 \end{pmatrix},$$ \hspace{1cm} (A.162)

and it has two eigenvalues which are the same $\lambda = j(j + 1)$.

Symmetric two-tensors

In the case of the symmetric two-tensor we introduce a complete and orthonomal basis $E^A_{ab}$, $A = +2, 0, -2$, of symmetric matrices

$$E^{+2}_{ab} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E^0_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad E^{-2}_{ab} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$ \hspace{1cm} (A.163)

with $E^A_{ab} E^B_{cd} = \delta^B_A$ and $E^A_{ab} E^c_d = \frac{1}{2} (\delta_a^c \delta_b^d - \delta_a^d \delta_b^c)$. In this basis

$$(T_3)_A^B = -\frac{1}{2} \epsilon^a_3 (\Sigma_{12})_A^B = -(\Sigma_{12})_A^B = \begin{pmatrix} 2i \\ 0 \\ 0 \end{pmatrix},$$ \hspace{1cm} (A.164)
and the Riemann tensor takes the form

\[ R^B_A = R^{c d}_{a b} E^a_A E^B_{c d} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(A.165)

The Lichnerowicz operator

\[ \Delta_L Y_{a b} = -D^2 Y_{a b} + 2 R^c_{a b} d Y_{c d} + 2 R^c_{(a} Y^d_{b) c} \]  

(A.166)

becomes

\[ \Delta_L = \begin{pmatrix} -T_A^2 + T_3^2 + 4 & 0 & 0 \\ 0 & -T_A^2 + T_3^2 & 0 \\ 0 & 0 & -T_A^2 + T_3^2 + 4 \end{pmatrix}. \]  

(A.167)

This matrix has three eigenvalues which are all the same \( \lambda = j(j + 1) \).

This concludes the discussion on tensor spherical harmonics on the two-sphere and the three-sphere.
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Part IV

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