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Detection of the Change Point and Optimal Stopping Time by Using Control Charts on Energy Derivatives

Master's Thesis in Financial Mathematics

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Master's thesis in Financial Mathematics, 15 ECTS credits

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Preface

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Finally, our special thanks go to our beloved families, for their encouragements, and whole hearted supports all the time.

Abstract

In this technical report, what we have concentrated on is the change-point problem in terms of energy futures contracts. X_t is a normally distributed random process, which represents the futures contracts prices, and it is presumed that this process is a first degree autoregressive process, $AR(1)$. The change-point is observed in the mean process which tends from constant to exponentially growth in our assumptions. Control chart methods which are Cumulative Sum (CUSUM), Shewhart and Shiryaev-Roberts control charts are used comprehensively with their corresponded alarm functions. In order to make a decision about effectiveness of each alarm function under our assumptions for $AR(1)$, we compared the correlation between the probability of false alarm, means ARL^0 measures, Expected Delay and ARL^1 performance measures for each of applied methods.

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Chapter 1

Introduction

The paper is devoted to quick detection and diagnosis of change-point in the mean process. The usage of Statistical Process Control (SPC) enables us to check performance during time at each sample. Energy derivatives are known as exchanged-traded securities which comprise of options, futures and swaps. Energy derivatives have substantial role on financial market. They are unstable due to the growing innovations or deteriorations of many factors such as policy, global climate changes, and civil wars which affect both producers and consumers throughout economic of the world. Thus managers and investors have to make provisions against unexpected fluctuations in statistical or economic sample. There are statistical methods for the precautions of undesirable cases, and as one of the most important practice tools of Statistical Process Control is Control Chart Methods. The Control Chart enables us to define and perform the required operations. It helps to detect the change point of the process, which occurs as an alarm function dependent on how the alarm function is defined. And then it presents the results of the performances in the light of samples, whether they are as expected, or not. In both theoretical and practical researches, the Control Chart methods provide the schemes for monitoring of the performance of process in the fields such as medicine, finance, economical and other sectors.

The objective of the considered methods is to detect changes of the mean dependent on the defined alarm functions. These methods are presented by Shewhart chart, Cumulative Sum (CUSUM) chart, Exponentially Weighted Moving-Average (EWMA) chart, Shiryaev-Roberts chart, and Likelihood Ratio Test. The interest to the Control Chart methods have currently enhanced, based on the needs of economical design, industry processes and so forth. The Shewhart chart was firstly handled by Shewhart [1] (1931)for

detection of the process without getting an information of previous observations. Thereby it can be pointed out that Shewhart chart is just an efficient method to present whether a process is stable, or not. The CUSUM control chart was firstly proposed as a performance control for monitoring the changes in the given process by Page [2] (1954). The CUSUM chart is used to determine which performance is suitable or favorable in the forward scheme. The EWMA control chart is firstly presented by Roberts [17], [18] (1959), and the chart was enhanced by Crowder [19] (1989), Ng and Case [5] (1989), Lucas and Saccucci [21] (1990). The Shiryaev - Roberts procedure was firstly introduced in 1961 (see Shiryaev [2]), to detect the changes in the drift of a Brownian motion. Shiryaev used the properties of Brownian motion to give a proof, which set a restriction on the rate of false alarms. This procedure is optimal for decreasing probability of the expected delay in detection of a change point.

In order to determine the needed number of observations for detection of a change point, Average Run Length (ARL) is used as a measure associated with the quality of Control Chart. ARL measure can be classified as ARL_0 and ARL_1 , in-control and out of control situations, respectively.

In this study, the observed data are recent oil, gasoline, natural gas, heating oil, propane and crude oil for future contract prices. The data include daily, monthly and yearly price changes. The changes in the mean from constant to exponential growth under the Stochastic Autoregressive Process are observed. Thereafter, we reformulate the alarm function of each control chart methods under our assumptions. The mathematical methods such as Autoregression Process ($AR(1)$), Likelihood Ratio Test, CUSUM, EWMA, Shewhart, Shiryaev-Roberts control methods are involved in Chapter 2. In the next chapter, the results are used and presented for control chart methods. In addition, comparative studies and empirical data analysis part exist there.

Chapter 2

Methods

2.1 First Order Autoregressive Process ($AR(1)$)

Autoregression process is used basically to estimate the forward values of random process in linear time series and is denoted as $AR(p)$. $AR(p)$ model is related with past p observations and can be formulated as a linear function of the observations as presented below

Assume that X_t is a random variable, dependent on time parameter t ,

$$X_t = \phi_0 + \sum_{i=1}^p \phi_i X_{t-i} + \sigma_\epsilon \epsilon_t,$$

where ϕ_1, \dots, ϕ_p are factors of the model, ϕ_0 is a constant, ϵ_t is the white noise which has zero mean and unit variance σ_ϵ^2 , $\epsilon_t \sim N(0,1)$.

First order Autoregression Process with $p = 1$, $AR(1)$ is given by

$$X_t = \phi_0 + \phi_1 X_{t-1} + \sigma_\epsilon \epsilon_t$$

with $-1 < \phi_1 < 1$ because of weak stationarity principle.

The properties of mean are presented in the following expressions,

$$\mu = E[X_t] = \phi_0 + \phi_1 E[X_{t-1}]$$

$$E[cX_t] = cE[X_t]$$

$$E[X_t + Y_t] = E[X_t] + E[Y_t]$$

$$E[\epsilon_t] = 0$$

The conditional density function for $AR(1)$ process is Gaussian and presented below for observations X_i and X_{i-1}

$$f(x_i | x_{i-1}) = \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}$$

In the unconditional case, the joint density function for $AR(1)$ process is given by

$$f(x_1, \dots, x_n) = \frac{1}{(\sigma_\epsilon \sqrt{2\pi})^n} e^{-\frac{\sum_{i=1}^n (x_i - \phi_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}$$

2.2 The Change-Point Model in Mean Process

The change point model can be interpreted in a state of in-control and out of control scheme for the considered random process. The diagnostic of the change point is used to facilitate crucial decisions i.e changes in quality performance.

In our study, we deal with change of mean value which tends from constant to exponentially growth when change point can appear at any time t . Thus we can present the expectation function as

$$\mu = b + ce^{a \max(t, \theta)}$$

where a , b , c are constants and θ is the change-point. In our observations, we presume that $b + ce^{a\theta}$ is a constant value, which can be entitled as d in the mean process until the change-point moment. After alteration of the process, it tends to exponentially growth in the mean. If we use the presented model on the AR(1) process

$$X_t = \phi_0 + \phi_1 X_{t-1} + \sigma_\epsilon \epsilon_t$$

we get

$$E[X_t] = E[\phi_0] + E[\phi_1 X_{t-1}] + \sigma_\epsilon E[\epsilon_t]$$

Because of expectation process properties, $\sigma_\epsilon E[\epsilon_t]$ is zero and therefore:

\Rightarrow For $t < \theta$ case

$$E[X_t] = \phi_0 + \phi_1 E[X_{t-1}]$$

$$d = \phi_0 + \phi_1 d$$

$$\phi_0^1 = d(\phi_1 - 1)$$

⇒ For $t = \theta$ case

$$E[X_t] = \phi_0 + \phi_1 E[X_{t-1}]$$

$$d = \phi_0 + \phi_1 d$$

$$\phi_0^1 = d(\phi_1 - 1)$$

⇒ For $t > \theta$ case

$$E[X_t] = \phi_0 + \phi_1 E[X_{t-1}]$$

$$b + ce^{at} = \phi_0 + \phi_1(b + ce^{a(t-1)})$$

$$b + ce^{at} = \phi_0 + \phi_1 b + \phi_1 ce^{a(t-1)}$$

$$\phi_0^2 = b(\phi_1 - 1) + ce^{at}(\phi_1 e^{-a} - 1)$$

The exact results for $AR(1)$ process which are presented below express the components of ϕ_0 variable of different cases for analysis of shift in the mean process.

2.3 The Change Point Problem

Change-point detection is the problem for exploring the time series, where properties of time series unexpectedly changes in some time point. This problem includes a broad range of real-world problems and has been widely discussed in many areas. It is necessary to detect the change point for taking preventive measures. Therefore, the change point should be found as quick as possible without spending time, and it is also needed to avoid from false alarms. (see *Järpe* [9] (2010))

2.4 Likelihood

In order to find the best parameter values, the likelihood function was presented by Fisher [16], Neyman and Pearson [20], as a measure of fit quality. In statistical quality control, the likelihood ratio test is a statistical test being used to compare one model to another model, one of which is modified case of the other model. The likelihood ratio test is based on the probability density ratio for these two models, that explains how much time more likely the inspected data are under control of the one model than the other. The likelihood ratio can be used to compute the critical value to decide whether to accept alternative model against the initial model.

The Likelihood Ratio, generally denoted by λ , is the ratio between the likelihood functions dependent on two different parameter sets in the numerator and denominator. So Likelihood ratio method is a statistical method to make a decision.

LikelihoodFunction : Suppose that X is a random variable with the density function $g(x, \theta)$ and \hat{X} is a set of observations of X . The likelihood function of X in case of θ , is the function LRT: $\theta \rightarrow \mathbb{R}^+$ can be denoted as

$$LR(\theta) = g(\hat{X}, \theta),$$

where the function $g(x, \theta)$ is the joint density function for the sample \hat{G} .

$$LRT(\theta_1, \theta_0; X) = \frac{LR(\theta_1; X)}{LR(\theta_0; X)}$$

is called the likelihood ratio, and $\ln LR(\theta_1, \theta_0; X)$ is the log likelihood ratio.

The likelihood ratio is efficient to distinguish the expressions $\theta = \theta_0$ and $\theta \in Q(\theta_0)$.

The set $Q(\theta_0)$,

$$Q(\theta_0) = \begin{cases} (-\infty, \theta_0), & \text{to check } \theta < \theta_0 \\ (\theta_0, \infty), & \text{to check } \theta > \theta_0 \\ \mathbb{R} \setminus \theta_0, & \text{to check } \theta \neq \theta_0. \end{cases}$$

2.5 The Change-Point Detection Methods

In order to solve the change point problem, for example; to detect a time point θ , a change point detection method of the form $\min (t \geq 1 : A_t \geq C)$ can be used. There are a lot of procedures which are developed through the history of change point theory. Some of these methods for controlling the statistical process are as follows

- Shewhart Method
- CUSUM(Cumulative Sum) Method
- Ewma Method
- Shiryaev-Roberts Control Chart

2.5.1 The Shewhart Method

The concept of control and prevention of defects was proposed by Shewhart [1] in 1924, and he devised a new statistical tool, which is known as control chart today. This control chart signals during observation of the change, and the estimated changes are reduced, by removing the disturbing causes.

A control chart is a graph to present repeated calculations of some characteristic under a process which is plotted in time order. In the graph, a horizontal line that passes through the center, represents the mean value of the characteristic. In this control chart, the barriers that are above and below of the horizontal line shows, when the plotted points are outside of these boundaries, the existence of change(s). The plots which are outside of these boundaries are presumed to signal for a special investigation of the process during the change point, to identify the disturbing cause.

Dr. Shewhart published a book about statistical quality control named "Economic Control of Quality of Manufactured Products" in 1931, where he mentioned two separated causes of change(s) such as chance causes where random change(s) occurred in the process, and special causes where one need to search for the disturbing cause.

The alarm function for Shewhart method was defined by Shewhart [1] (1931) as followi

$$A(t) = LR(t, t), \forall t \in \mathbb{N}$$

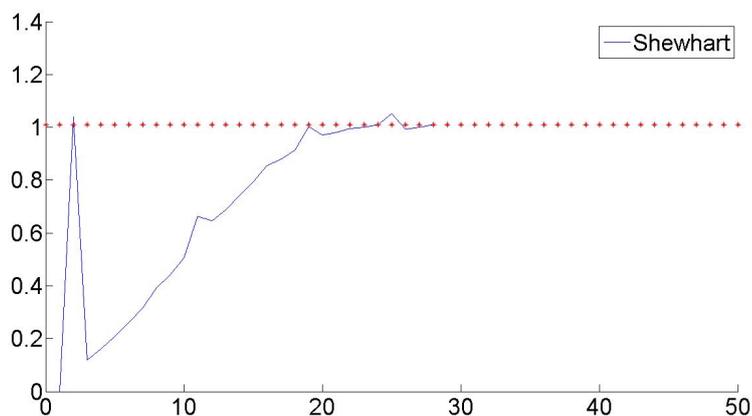


Figure 2.1: An example of Shewhart method

2.5.2 Cumulative Sum Method (CUSUM)

The Cumulative-Sum (CUSUM) control method was firstly laid out by Page [7] in 1954 as a mean to detect sequentially changes in distributions of discrete time random processes. The Cumulative Sum of the deviations from each sample value are plotted by CUSUM chart. The CUSUM method is good at controlling small shifts. Cusum is a sequential control procedure constructed from the previous observations to check whether the distribution is in control, or not. A minimum- maximum criterion for the change point problem was proposed by Lorden [12] in 1971. Moustakides [11] (1986) shown optimality for the identically and independently distributed case before and after the change point. Moustakides extended his work to a special class of processes with dependent observations.

The CUSUM is a surveillance method with alarm function

$$A(X_t) = \begin{cases} \log LR(1), & \text{when } t=1 \\ \max(0, \log LR(t) + A(X_{t-1})), & \text{when } t=2,3,4, \dots \end{cases}$$

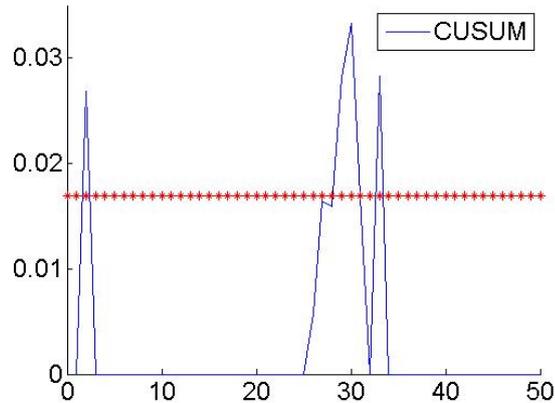


Figure 2.2: An example of Cusum method

2.5.3 Exponentially Weighted Moving Average (EWMA) Method

In recent years, the EWMA-based process mean estimator or adjustment has been studied by many researchers including Schlonea, Schmidb and Knothb [4] (1999), Roberts [17] (1959), Jang, Shu and Apley [23] (2008), Dasgupta and Wu [22] (2006), Yaschin [8] (1995), Crowder [19] (1989), and Koehler [3] (2001).

Exponentially Weighted Moving Average (EWMA) method is a control chart for the data which are quantitative and continuous in measurement in time order. EWMA chart plots exponentially weighted moving average values, and a weighting factor is found in order to estimate the influence of older data points on the average value compared to more recent one. EWMA chart uses all information that a given sample provides. EWMA control charts are used to monitor processes over time.

The EWMA method is defined by likelihood ratio as

$$\tau = \min(t \geq 1; \lambda^{t-n} LR(n, t) : n = 1, 2, 3, \dots, t \geq C)$$

where C is the value of the threshold and constant.

$\lambda \in (0, 1)$. If the $\lambda = 0$, then we get Cumulative Sum (CUSUM) method, or if the $\lambda = 1$, then we get Shewhart control chart.

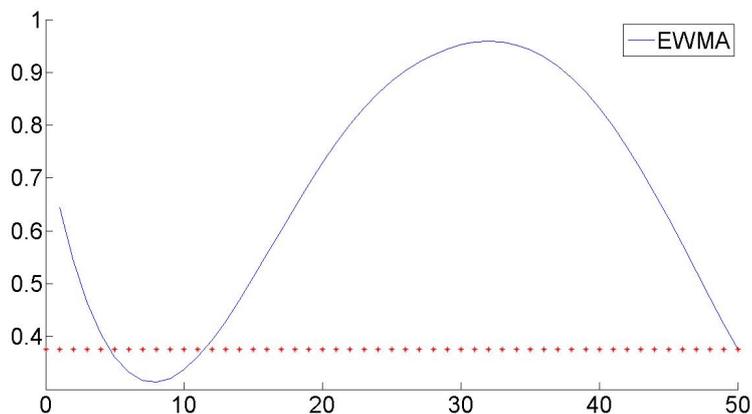


Figure 2.3: An example of Ewma method

2.5.4 Shiryaev-Roberts Method

Shiryaev presented a method to detect the change point in the drift of a Brownian motion which is referred to as Shiryaev-Roberts procedure nowadays ((Shiryaev [2] in 1963) and (Roberts [18] in 1966)). Shiryaev used the properties of Brownian motion to give a proof, for the case where the rate of false alarms are restricted. The procedure is optimal to decrease expected delay in case of detecting a change point. Shiryaev described this problem as "Quickest Detection of a Disorder in a Stationary Regime".

The Shiryaev - Roberts method is a surveillance method with alarm function

$$\tau = \min\{t \geq 1 : A(x_0, \dots, x_t) > C\}$$

2.6 Performance Measures

In statistical process, there are some methods to estimate or measure the quality of statistical process monitoring such as Expected Delay (ED), Pre-

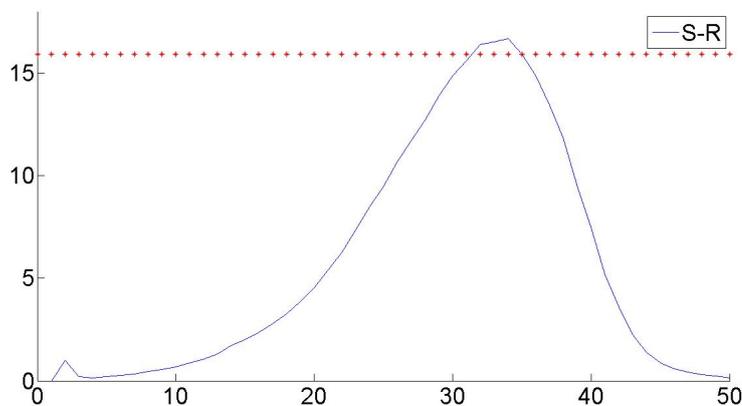


Figure 2.4: An example of Shiryaev - Roberts method

dictive Value (PV), Average Run Length (ARL), and Probability of Successful Detection (PSD). The main goal is to detect the change point time as quick as possible, as well as avoiding from the false alarms.

2.6.1 Expected Delay:

The expected delay of the alarm time was presented by Albert N. Shiryaev [2] in 1963. The Expected Delay method can also be used for measuring the performance which quality control charts present.

Let us assume that the change point time and the alarm time are θ and τ , respectively. Then the equation of Expected Delay can be expressed (see *Järpe* and *Wessman* [10] (2000)) as

$$ED(\theta) = E(\tau - \theta \mid \tau \geq \theta)$$

and equation of the Conditional Expected Delay at the change point time $\theta = t$ can be expressed as

$$CED(t) = E(\tau - \theta \mid \tau \geq \theta = t)$$

2.6.2 Predictive Value (PV):

The Predictive Value (PV) of the alarm (see Frisen [14]) was presented by Frisen in 1992. It is useful to have an information about the probability of the change point during the occurrence of a given alarm. The equation of the Predictive Value (PV) at the change point time θ can be expressed as

$$PV(t) = P(\theta \leq t \mid \tau = t)$$

2.6.3 Average Run Length (ARL)

In statistical quality control, the methods are generally composed on the basis of their run length (RL), that is, a number of observations are checked if they are at a certain quality level, before signal for a change point occurs. The Run Length (RL) dependent on whether the process is in control, or out of the control. Therefore, RL should be large if it is in control, and small if it is out of the control. Since the complexity of this problem, it is only discussed in terms of the expectation of the Run Length for most of the quality control charts. The expectation of the RL is called Average Run Length (ARL).

Chapter 3

Results

In this part, the alarm functions for the considered surveillance methods are obtained, dependent on the shift in mean for the 1st order Autoregressive Process, $AR(1)$. It will be introduced how the alarm functions are evaluated for the control chart models.

3.1 The Regenerated Form For CUSUM Control Chart

In statistical quality control, the Cumulative Sum (CUSUM) method is one of the popular sequential analysis techniques. The CUSUM method is typically used for monitoring and detection of the change point in the form,

$$A_0 = 0$$

$$A_t = \max_{1 \leq n \leq t} \log LR(n, t)$$

If A_t crosses the threshold value, then the control chart signals to give information about occurrence of the change point. The formula mentioned above only detects the change points in the positive direction. In order to find the change points in negative direction, *min* function would be used instead of *max* function.

Theorem :

The regenerated form of CUSUM Control Chart for $AR(1)$ process to detect the changes in the mean process is

$$\max \left(0, A_{t-1} + \frac{e^{-a}-1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_1 c e^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - c e^{ai} (\phi_1 + \phi_1 e^{-a} - 2)] \right)$$

Proof Part:

The likelihood ratio formula can be presented as follow

$$LR(n, t) = \prod_{i=1}^t \frac{f(x_i | x_{i-1}; \theta_1)}{f(x_i | x_{i-1}; \theta_0)}$$

or

$$LR(n, t) = \frac{f(x_t | \theta \leq t)}{f(x_t | \theta > t)}$$

where $1 \leq n \leq t$.

$$A_t = \max_{1 \leq n \leq t} \log LR(n, t)$$

We know that the alarm function A_t for CUSUM model but not its form for the considered model. We present the new regenerated version with incorporated density functions for considered process.

$$LR(n, t) = \frac{\prod_{i=n}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{\prod_{i=n}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}$$

$$\log LR(n, t) = \log \frac{\prod_{i=n}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{\prod_{i=n}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}$$

$$\begin{aligned}
 \log LR(n, t) &= \log \frac{e^{-\sum_{i=n}^t \frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{e^{-\sum_{i=n}^t \frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}} \\
 &= \log \exp \left(\sum_{i=n}^t -\frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2} + \frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2} \right) \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (x_i - \phi_0'' - \phi_1 x_{i-1})^2 - (x_i - \phi_0' - \phi_1 x_{i-1})^2 \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (x_i - \phi_0' - \phi_1 x_{i-1} - x_i + \phi_0'' + \phi_1 x_{i-1}) \times \\
 &\quad (x_i - \phi_0' - \phi_1 x_{i-1} + x_i - \phi_0'' - \phi_1 x_{i-1}) \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_0'' - \phi_0')(2x_i - 2\phi_1 x_{i-1} - \phi_0' - \phi_0'') \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [b(\phi_1 - 1) + ce^{ai}(\phi_1 e^{-a} - 1) - d(\phi_1 - 1)] \times \\
 &\quad [2x_i - 2\phi_1 x_{i-1} - d(\phi_1 - 1) - b(\phi_1 - 1) - ce^{ai}(\phi_1 e^{-a} - 1)] \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [(\phi_1 - 1)(b - d) + ce^{ai}(\phi_1 e^{-a} - 1)] \times \\
 &\quad [2x_i - 2\phi_1 x_{i-1} - (\phi_1 - 1)(d + b) - ce^{ai}(\phi_1 e^{-a} - 1)] \\
 &= \frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [(e^{-a} - 1)(\phi_1 ce^{ai})] [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]
 \end{aligned}$$

where $d = b + ce^{ai}$.

$$\log LR(n, t) = \frac{e^{-a} - 1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_1 ce^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]$$

Therefore, the alarm function for the CUSUM method can be expressed as

$$\max \left(0, A_{t-1} + \frac{e^{-a} - 1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_1 ce^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)] \right)$$

3.2 The Regenerated Form For Shewhart Control Chart

Shewhart control chart or process-behaviour chart in statistical process control, are helpful tools to be used to check whether the process is in a state of statistical control, or not.

If the analysis of the control chart shows the currently process is stable with the variates generated by the usual set of parameters for the process, then the result from the data can be used to forecast the future performance of the given process. If the process-behaviour chart shows that the process being monitored is out of the control, then the results of the charts could be helpful to identify the unusual variations, which can then easily be eliminated to get the process back in control state.

Shewhart control chart is a special sort of run chart which let fairly large change to be differentiated from the natural variability of the process.

Theorem :

The regenerated formula of Shewhart Control Chart for $AR(1)$ process to detect the change points in the mean process is

$$A_t = \exp\left(\left(\frac{e^{-a} - 1}{2\sigma_\epsilon^2}\right)(\phi_1 c e^{at}) [2(x_t - \phi_1 x_{t-1} - b(\phi_1 - 1)) - c e^{at}(\phi_1 + \phi_1 e^{-a} - 2)]\right)$$

Proof Part:

Alarm function for Shewhart Control Chart is expressed as following

$$A_t = LR(t, t)$$

$$LR(t, t) = \frac{\prod_{i=t}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{\prod_{i=t}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}$$

$$\begin{aligned}
 LR(t, t) &= \exp\left(\sum_{i=t}^t \frac{(x_i - \phi_0' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2} - \frac{(x_i - \phi_0'' - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}\right) \\
 &= \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t (x_i - \phi_0' - \phi_1 x_{i-1})^2 - (x_i - \phi_0'' - \phi_1 x_{i-1})^2\right) \\
 &= \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t (x_i - \phi_0' - \phi_1 x_{i-1} - x_i + \phi_0'' + \phi_1 x_{i-1})(x_i - \phi_0' - \right. \\
 &\quad \left. - \phi_1 x_{i-1} + x_i - \phi_0'' - \phi_1 x_{i-1})\right) \\
 &= \left(\exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t (\phi_0'' - \phi_0')(2x_i - 2\phi_1 x_{i-1} - \phi_0' - \phi_0'')\right)\right) \\
 &= \left(\exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t (\phi_0'' - \phi_0')(2x_i - 2\phi_1 x_{i-1} - \phi_0' - \phi_0'')\right)\right) \\
 &= \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t [b(\phi_1 - 1) + ce^{ai}(\phi_1 e^{-a} - 1) - d(\phi_1 - 1)] \times \right. \\
 &\quad \left. [2x_i - 2\phi_1 x_{i-1} - d(\phi_1 - 1) - b(\phi_1 - 1) - ce^{ai}(\phi_1 e^{-a} - 1)]\right) \\
 &= \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t [(\phi_1 - 1)(b - d) + ce^{ai}(\phi_1 e^{-a} - 1)] \times \right. \\
 &\quad \left. [2x_i - 2\phi_1 x_{i-1} - (\phi_1 - 1)(d + b) - ce^{ai}(\phi_1 e^{-a} - 1)]\right) \\
 &= \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=t}^t [(e^{-a} - 1)(\phi_1 ce^{ai})] [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]\right) \\
 &= \exp\left(\frac{e^{-a} - 1}{2\sigma_\epsilon^2} \sum_{i=t}^t (\phi_1 ce^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]\right)
 \end{aligned}$$

Therefore, the alarm function for Shewhart control chart is as following

$$A_t = \exp\left(\left(\frac{e^{-a} - 1}{2\sigma_\epsilon^2}\right)(\phi_1 ce^{at}) [2(x_t - \phi_1 x_{t-1} - b(\phi_1 - 1)) - ce^{at}(\phi_1 + \phi_1 e^{-a} - 2)]\right).$$

3.3 The Regenerated Form For Shiryaev-Roberts Control Chart

Shiryaev [2] (1963) and Roberts [18] (1966) presented a method, which is called the Shiryaev-Roberts method now. In Shiryaev-Roberts method an alarm is started at the first time t , when

$$\sum_{n=1}^t LR(n, t) > C$$

where C is a constant.

Theorem

The regenerated formula of Shiryaev-Roberts Control Chart for $AR(1)$ process to detect the change points in the mean process is

$$A_t = (1 + A_{t-1}) \left[\exp\left(\frac{e^{-a} - 1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_1 c e^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]\right) \right]$$

Proof Part :

Let us define moment τ as

$$\tau = \min\{t \geq 1 : A(x_0, \dots, x_t) > C\}$$

Alarm function can be presented in the form

$$A_t = \sum_{n=1}^t \prod_{i=n}^t \frac{f_{\theta_1}(x_i)}{f_{\theta_0}(x_i)}$$

where

$$\theta_t = \begin{cases} \theta_0, & \text{for } t < \theta \\ \theta_1, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} A_t &= \sum_{n=1}^t \frac{f_{\theta_0}(x_1) \dots f_{\theta_0}(x_{n-1}) f_{\theta_1}(x_n) f_{\theta_1}(x_{n+1}) \dots f_{\theta_1}(x_t)}{f_{\theta_0}(x_1) \dots f_{\theta_0}(x_{n-1}) f_{\theta_0}(x_n) f_{\theta_0}(x_{n+1}) \dots f_{\theta_0}(x_t)} \\ &= \sum_{n=1}^t \frac{f(x_1, \dots, x_{n-1} \mid \theta = \theta_0) f(x_n, \dots, x_t \mid \theta = \theta_1)}{f(x_1, \dots, x_t \mid \theta = \theta_0)} \\ &= \sum_{n=1}^t \frac{f(x_1, \dots, x_t \mid \theta = n)}{f(x_1, \dots, x_t \mid \theta > t)} \end{aligned}$$

where $0 < n \leq t$.

Therefore, we have the following expression

$$\begin{aligned} A_t &= \sum_{n=1}^t \frac{f(x_1, \dots, x_t \mid \theta = n \leq t)}{f(x_1, \dots, x_t \mid \theta > t)} \\ &= \sum_{n=1}^{t-1} \frac{f(x_1, \dots, x_t \mid x_0, \theta = n)}{f(x_1, \dots, x_t \mid x_0, \theta > t)} + \frac{f(x_1, \dots, x_t \mid x_0, \theta = t)}{f(x_1, \dots, x_t \mid x_0, \theta > t)} \end{aligned}$$

The extension of the density function can be substituted for $AR(1)$ process to detect the change points in the mean for obtaining the regenerated form of Shiryaev-Roberts method.

$$\begin{aligned} A_t &= \sum_{n=1}^{t-1} \frac{\prod_{i=1}^{n-1} \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi'_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}} \prod_{i=n}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi''_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{\prod_{i=1}^t \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi'_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}} \\ &= \sum_{n=1}^{t-1} \frac{\left(\frac{1}{\sigma_\epsilon \sqrt{2\pi}}\right)^{n-1} e^{-\frac{(x_i - \phi'_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}} \left(\frac{1}{\sigma_\epsilon \sqrt{2\pi}}\right)^{t-n+1} e^{-\frac{(x_i - \phi''_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}}{\left(\frac{1}{\sigma_\epsilon \sqrt{2\pi}}\right)^t e^{-\frac{(x_i - \phi'_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}} \end{aligned}$$

$$\begin{aligned}
&= 1 + \sum_{n=1}^t \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (x_i - \phi'_0 - \phi_1 x_{i-1})^2 - (x_i - \phi''_0 - \phi_1 x_{i-1})^2\right) \\
&= 1 + \sum_{n=1}^t \exp\left[\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (x_i - \phi'_0 - \phi_1 x_{i-1} - x_i + \phi''_0 + \phi_1 x_{i-1})(x_i - \phi'_0 - \phi_1 x_{i-1} + x_i - \phi''_0 - \phi_1 x_{i-1})\right] \\
&= 1 + \sum_{n=1}^t \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi''_0 - \phi'_0)(2x_i - 2\phi_1 x_{i-1} - \phi'_0 - \phi''_0)\right) \\
&= 1 + \sum_{n=1}^t \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [b(\phi_1 - 1) + ce^{ai}(\phi_1 e^{-a} - 1) - d(\phi_1 - 1)] \times \right. \\
&\quad \left. [2x_i - 2\phi_1 x_{i-1} - d(\phi_1 - 1) - b(\phi_1 - 1) - ce^{ai}(\phi_1 e^{-a} - 1)]\right) \\
&= 1 + \sum_{n=1}^t \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [(\phi_1 - 1)(b - d) + ce^{ai}(\phi_1 e^{-a} - 1)] \times \right. \\
&\quad \left. [2x_i - 2\phi_1 x_{i-1} - (\phi_1 - 1)(d + b) - ce^{ai}(\phi_1 e^{-a} - 1)]\right) \\
&= 1 + \sum_{n=1}^t \exp\left(\frac{1}{2\sigma_\epsilon^2} \sum_{i=n}^t [(e^{-a} - 1)(\phi_1 ce^{ai})] [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - \right. \\
&\quad \left. - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]\right) \\
&= 1 + \sum_{n=1}^t \exp\left(\frac{(e^{-a} - 1)}{2\sigma_\epsilon^2} \sum_{i=n}^t (\phi_1 ce^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - \right. \\
&\quad \left. - ce^{ai}(\phi_1 + \phi_1 e^{-a} - 2)]\right)
\end{aligned}$$

Therefore, the alarm function for Shiryaev-Roberts method can be defined

as

$$A_t = (1 + A_{t-1}) \left[\exp\left(\frac{e^{-a} - 1}{2\sigma_\epsilon^2}\right) \sum_{i=n}^t (\phi_1 c e^{ai}) [2(x_i - \phi_1 x_{i-1} - b(\phi_1 - 1)) - c e^{ai}(\phi_1 + \phi_1 e^{-a} - 2)] \right]$$

3.4 Simulation Results

The surveillance methods are applied to the detection of the change point in $AR(1)$ process with 1000000 processes with 1000 times series for 1000000 different θ in case of Geometrical distribution, with $\phi_1 = 10^{-3}$.

In the simulation part, it is supposed that $X(t)$ random variables appear one by one in time order. For the values of time $t = 1, 2, \dots, N$, the random variables for $AR(1)$ process are $X(1), X(2), \dots, X(N)$. Then by using the Likelihood ratio method, a table where the coordinates of the change point are investigated, has been created to find the most likely change point. In this process, it turned out that product of the Sequential Likelihood ratios, the rows and columns in the table represent the Likelihood ratios and change points, respectively. The purpose of choosing Sequential Likelihood ratio is, since the property of cumulatively prepared Likelihood ratio table, it is assumed there is no information about the time $t = 2$, when it is in the first moment where it's coordinates are mentioned in the first row. Therefore, what information we only have in the first column (i.e. first row) of the table at $t = 2$ is the change point has already occurred at $t = 1$. After it occurs at $t = 1$, the mean starts having an exponential growth as mentioned in our assumptions, and we see the cumulatively increasing products as the row number increases. By the way, what happens in second column of the Sequential Likelihood ratio table is as follows, the Sequential Likelihood ratio of first row is 1 as naturally, but second row behaves the change point has already occurred at that moment where time $t = 2$, and then exponential growth starts to happen as the row number increase in the table.

In this manner, Sequential Likelihood ratio table with column numbers show the coordinate where change points occurred, are evaluated in time order. Finally, the most likely change point's coordinates are investigated by creating a vector. The maximum value of the vector gives then the most likely change point's coordinates.

The different types of surveillance methods characterizes the behavior dependent on the process' in or out of the control situations. If the process is

in control, then all alarms which has occurred are false alarms. It means that any alarm occurred before the change point is considered as false alarm. The distribution of the false alarms are usually expressed as in control Average Run Length, and denoted as $ARL^0 = E[\tau = \infty]$. One another measure is the probability of the false alarm,

$$P(\tau < \theta) = \sum_{t=1}^{\infty} P(\theta = t) P(\tau < \theta \mid \theta = t)$$

But, this case requires an assumption (In the considered model, it is Geometric distribution) for the distribution of the change point, θ . It means that the assumption is more suitable in case when the intensity of the shift is constant for the gradually increasing time. To compare the effectiveness of different methods, a comparison needs to be made between false alarms and expected delay times for optimal stopping.

In Figure 3.1, there are 3 plots for different methods, which are CUSUM, *Shiryayev – Roberts* ($S - R$) and Shewhart, respectively. We have 20 different threshold as control limits at Figure 3.2. If the alarm function passes boundaries comprised from control limits, then our process will stop as soon as Alarm function's local value is greater than threshold's local values. In the graph, as the False Alarms increase from the first control limit to last, it can be seen that CUSUM method start with lower Expected Delays. However, as our control limits start to get smaller numbers, the Expected Delays decrease and it cause the False Alarms to get bigger. If compare to CUSUM, *Shiryayev – Roberts* method started with a higher Expected Delay, and time by time as control limits decrease, the proportion of Expected Delays get lower and occurrence of false alarms emerge more. As a third method, the graph provided by Shewhart method had a sharp decrease depends on control limits values, but the numerical value of False Alarms did not change for changing Expected Delays, because they have the same proportional amount of false alarms. Depending on the control limits, when they get higher values, it is obvious that the number of false alarms decrease, however, if it gets smaller, then we have more false alarms. As a comparison of three methods in terms of False Alarms and Expected Delays, it can be mentioned that alarm functions of the CUSUM method works better than other surveillance methods used in the simulation.

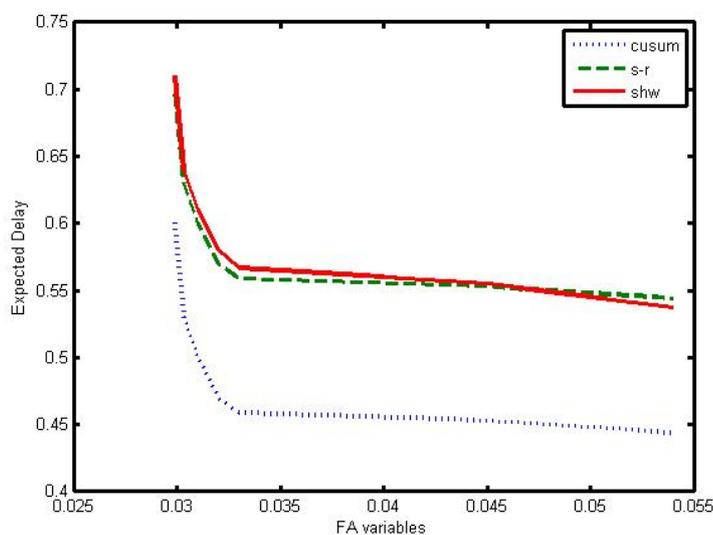


Figure 3.1: The graphic of "Expected Delay vs False Alarm"

The ARL^1 is one of the performance measures and it appears that the CUSUM method is faster than others dependent on increasing threshold values. The reason is CUSUM method is seemed more sensitive and quick for a forecast of the control situation.

$$ARL^1 = \frac{1}{10^6} \sum_{t=1}^{10^6} \tau_t$$

Figure 3.2 presents the ARL^1 values with respect to the threshold limits for each method.

3.5 Examples with Real Data

Energy derivatives markets consist of electricity, crude oil, natural gas and so on. Their price's fluctuation affect the investment schedules of traders dependent on financial and economical conditions. Energy derivatives are highly preferred by hedgers and speculators since a risk emerges from this

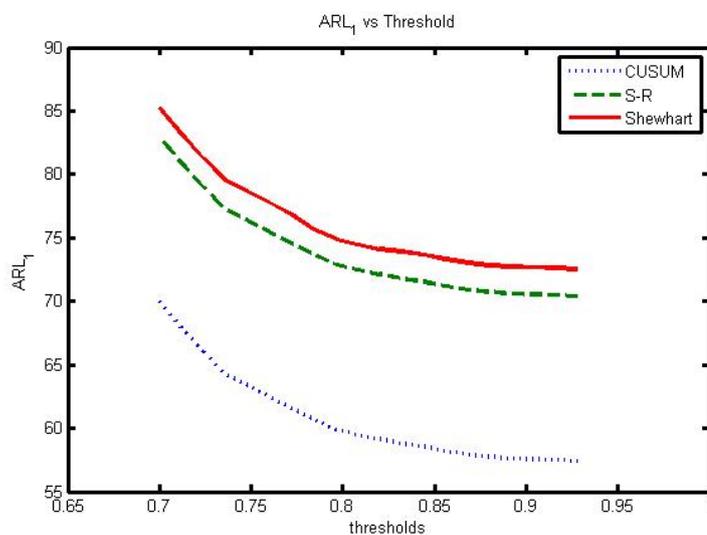


Figure 3.2: The graphic of "ARL¹ vs Threshold"

type of derivatives. A speculator desires capturing some profit opportunities for higher return which depends on energy price's fluctuations so the derivatives of interested commodity is used. A hedger requests hedging investment from differences among the energy prices so derivatives of interested commodity is used. Energy derivatives are composed of options, futures or swap compromises, whose values depend on buying and selling prices of interested commodity. Future contracts are the ones which include the pre-determined future price value of commodity and the delivery place at appointed time. They are standardized for both buyers and sellers who are participants of trading markets.

This technical report examined the futures contracts of an energy product based on Crude oil. The historical data of Crude oil futures contracts are provided by New York Mercantile Exchange's online electronic computer system, from July 21,1983 to August 30, 2011.

The logarithm of the prices have been used to detect the change point in returns and so the processes have been arranged as follows

$$P(t) = \text{Log} \left(\frac{X(t)}{X(t-1)} \right),$$

where $X(t-1)$ and $X(t)$ are daily and consecutive future price values.

According to the assumptions, the mean process is supposed to follow the formula

$$\mu = b + c e^{a \max(t, \theta)},$$

where the mean shifts from constant to an exponential growth, depending on presence of the change point. The way to proceed is to find the parameters a , b , and c as accurately as possible by using the least squares method. For the futures contracts daily exercise prices; that is, $X_{1, 2, 3, \dots, t-1, t} = X_1, X_2 \dots, X_{t-1}, X_t$, the value of the quadratic form will be calculated as follows

$$Q = \sum_{t=1}^{\theta-1} (P(t) - (b + c \exp(a \theta)))^2 + \sum_{t=\theta}^N (P(t) - (b + c \exp(a t)))^2,$$

where θ and N are the change point and the number of elements in the futures time series, respectively.

The value of the parameters a , b , c , and θ are the minimal values to get the minimal result from the least square method. So the value of the θ in this parameters will be the estimated change point, and by using other control charts methods, the optimal stopping time will be estimated later on.

The futures contracts' historical data has been divided into three samples which are assumed that these distinct data samples are independent of each other. The figures are shown at figure 3.3, figure 3.4 and figure 3.5 dependent on presumed mean function behaviour.

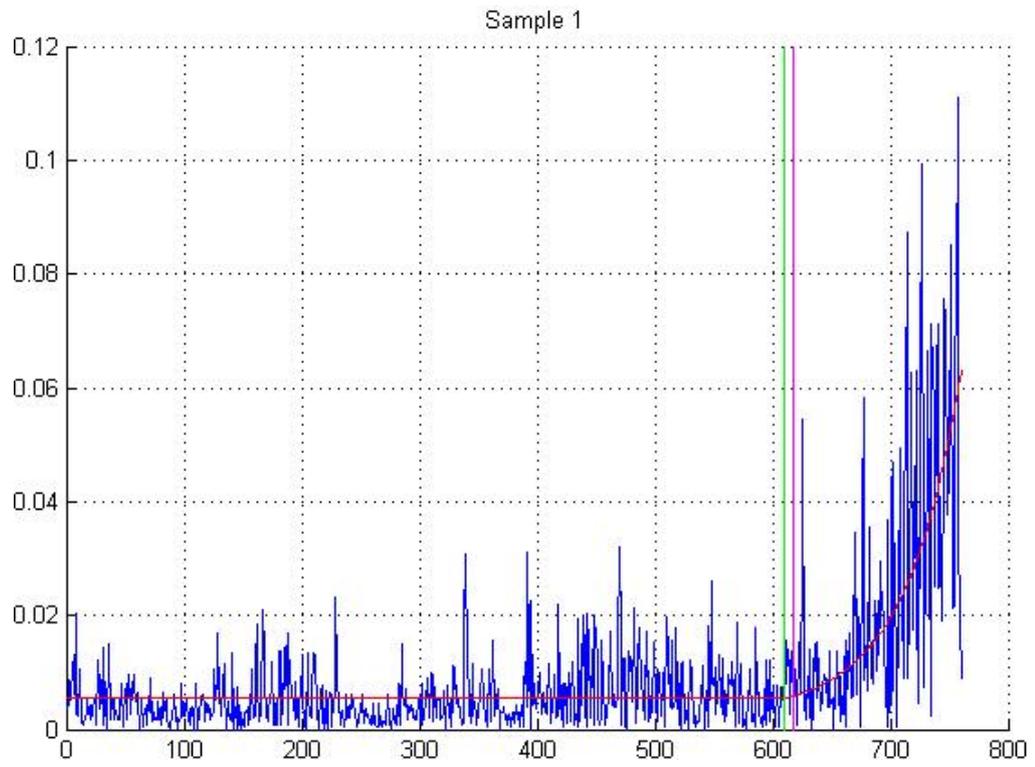


Figure 3.3: The graphic of monitored futures contracts' exercise prices data under the assumption for the data set "sample 1". As it is seen from the figure, the change point is at 610 on the x - coordinate. The optimal stopping time has been estimated about an 8 points delay with the alarm function of the CUSUM control chart. The colors green and purple represent the estimated change point and optimal stopping time, respectively.

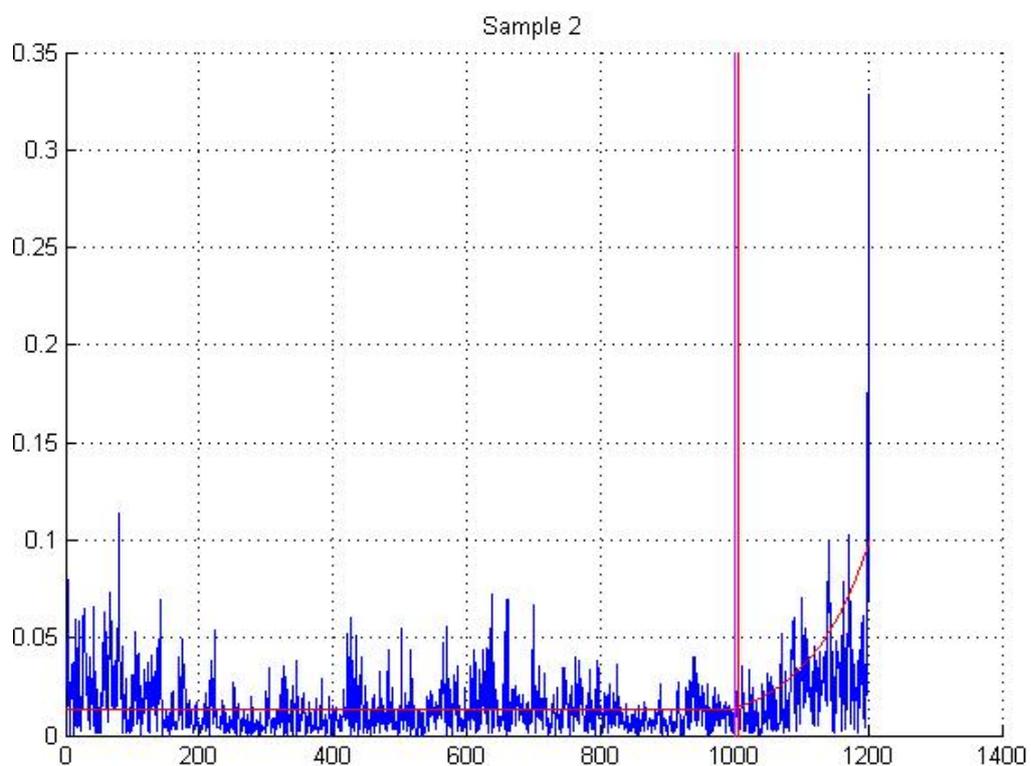


Figure 3.4: The graphic of monitored futures contracts' exercise prices data under the assumption for the data set "sample 2". As it is seen from the figure, the change point is at 1000 on the x - coordinate. The optimal stopping time has been estimated about a 5 points delay with the alarm function of the CUSUM control chart. The colors green and purple represent the estimated change point and optimal stopping time, respectively.

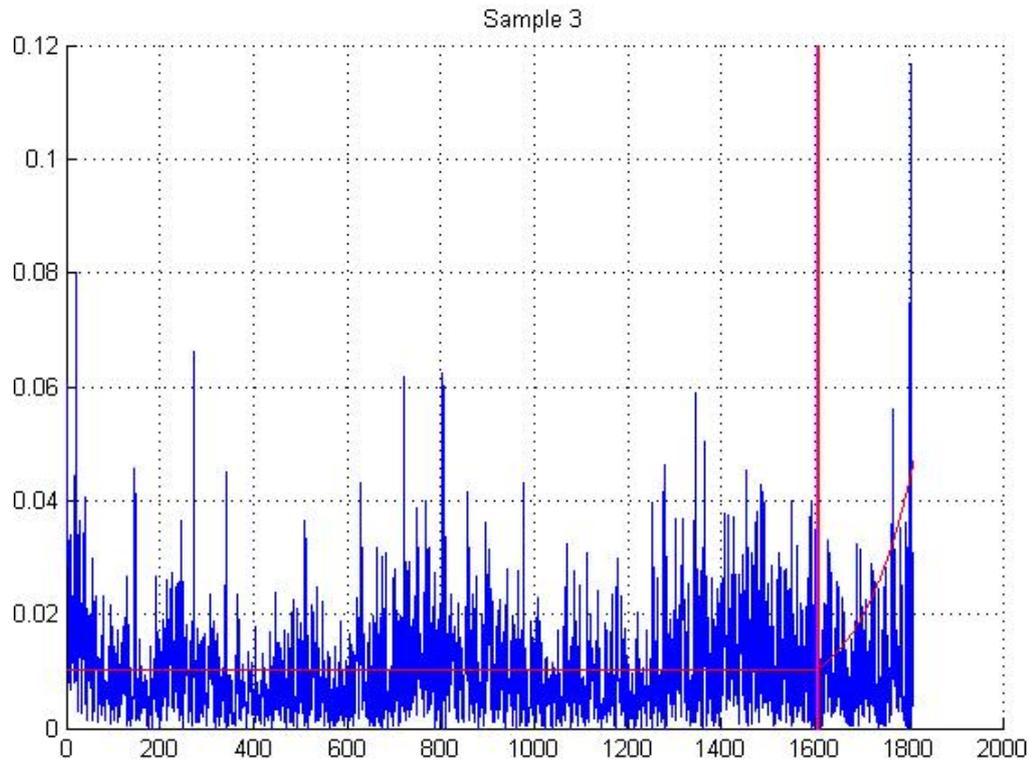


Figure 3.5: The graphic of monitored futures contracts' exercise prices data under the assumption for the data set "sample 3". As it is seen from the figure, the change point is at 1605 on the x - coordinate. The optimal stopping time has been estimated about a 4 points delay with the alarm function of the CUSUM control chart. The colors green and purple represent the estimated change point and optimal stopping time, respectively.

The values of the parameters a , b , and c are estimated for each sample, as follows

Data	a	b	c
Sample1	0,003702883	0,0025	0,0051
Sample2	0,002884032	0,006	0,00119
Sample3	0,000175792	0,0342	0,072

As a result, the historical data which has been used to detect the change point was provided by NYMEX. It is essential for risk managers or investors to have information about the performance which the data has presented for $AR(1)$ process to make the right investment dependent on the result of the performance measures. The CUSUM control chart method is faster and more efficient than other methods for detection of small mean shifts for different values of changing threshold values as it has been experienced from the simulation. The performance measures are presented on the table as follows

Data	ED	ARL1
Sample1	150	464
Sample2	63	1167
Sample3	205	308

Chapter 4

Conclusions

To make investment is the case, when the assets have been offered to public trading, for investors to get optimal stopping rules to be proper and efficient. In this thesis, historical data from 20 different energy futures contracts have been investigated. What is assumed is that the price will move on a steady level until the change point, and then the price follows an exponential growth towards up. In order able to find the change point, it is assumed that the exercise price of the futures contract to be an ARL^1 process with a shift in the mean value.

The change point was assumed as normally distributed. The procedures which have been used to estimate the change point are as follows

- Shewhart Method
- CUSUM(Cumulative Sum) Method
- Ewma Method
- Shiryaev-Roberts Control Chart

All the statistical procedures mentioned above have been derived dependent on our assumptions. To help having efficient calculations, we have also found and used the recursive formulas. The Expected Delay for the change point which is assumed to be normally distributed, has been obtained by calibrated simulations. One can easily see that the higher expected delays

after the change point has been estimated, dependent on different statistical procedure from the results. Some statistical methods catches the change point quicker as its value gets smaller and some catches as it gets bigger. As we have experienced from our simulation, we have realised the CUSUM procedure gives more efficient results than others. And so, we have gone on to estimate the change point with CUSUM control chart on the data.

Notation

$\{X_t : t \in T\}$	A random process with time index set T .
$N(0, 1)$	Normal distributed random values.
$E(X_t)$	Expectation of random variable with respect to t time.
μ	An expectation function.
\hat{G}	The sample of X random variable.
$g(x, \theta)$	The joint density function of sample \hat{G} .
$A(t)$	Alarm function with respect to t time.
τ	Stopping time.
C	Threshold value.
θ	Change-point.
$LR(t)$	Likelihood Ratio with respect to t time.
ARL	Average Run Length.
ED	Expected Delay
CED	Conditional Expected Delay
PV	Predictive Value
LS	Least Square Method.

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Appendix

Proof of condition density function of standart normal distribution for AR(1) process

$$\begin{aligned} f(x_i | x_{i-1}) &= \frac{\partial}{\partial x_i} P(X_i \leq x_i | X_{i-1} = x_{i-1}) \\ &= \frac{\partial}{\partial x_i} P(\phi_0 + \phi_1 X_{i-1} + \sigma_\epsilon \epsilon_i \leq x_i | X_{i-1} = x_{i-1}) \\ &= \frac{\partial}{\partial x_i} P(\phi_0 + \phi_1 x_{i-1} + \sigma_\epsilon \epsilon_i \leq x_i | X_{i-1} = x_{i-1}) \\ &= \frac{\partial}{\partial x_i} P(\epsilon_i \leq \frac{x_i - \phi_0 - \phi_1 x_{i-1}}{\sigma_\epsilon}) \\ &= \frac{\partial}{\partial x_i} \Phi\left(\frac{x_i - \phi_0 - \phi_1 x_{i-1}}{\sigma_\epsilon}\right) \\ &= \frac{1}{\sigma_\epsilon} \phi\left(\frac{x_i - \phi_0 - \phi_1 x_{i-1}}{\sigma_\epsilon}\right) \\ &= \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{(x_i - \phi_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}} \end{aligned}$$

For joint density function, the same was has been followed to get the following expression for $AR(1)$ process

$$f(x_1, \dots, x_n) = \frac{1}{(\sigma_\epsilon \sqrt{2\pi})^n} e^{-\frac{\sum_{i=1}^n (x_i - \phi_0 - \phi_1 x_{i-1})^2}{2\sigma_\epsilon^2}}$$

Matlab programming language is "4th generation" language which is a high-level data construction that includes dictionary, data, and tuples, was firstly handled by Cleve Moler in order to facilitate the evaluations apart from Fortran, ensuing developments are came to this circumstance currently by MathWorks.

What Matlab enables users is controlling, sketching out the optimisation, analysing data, simulating operations and algorithmics like other compiled languages which are C++ and Python in industry, science areas and so on. The all programming languages portray the different codes, but they give constant algorithmic result when the program is run. The programmer who should have imaginative functions and math thinkings, just writes demanded scripts in accordance with choosen programming language structure, without using the memory evaluations . In order to decide which programming language is feasible for an algorithmic evaluation, the race time shows different values dependent on each programming language. For instance, Python is a language which is one of the high-level languages is almost the same as Matlab for the syntax it has.

The usage of Matlab is more simple and easier to get the algorithmic structure for the beginners. Furthermore, as the functions are constructed inside the program, it takes shorter time to compile than any other language. Matlab software system consists of matrix tasks, known as math formulas and data scheming.

The simulation results has been found with Matlab, and the following scripts have been used in our study.

```
clear all;
close all;
clc;
format longEng;
```

```

M=10^6;
  numb_obs=20;
observation_1=ones(M,12*numb_obs);

for obs_code=1:M;

N=10^6;
rnd_=rand(N,1);
normal_inverse=norminv(rnd_,0,1);
  %
%%%%%%%%%%
phi_1=0.001;    %
sigma_eps=0.005;    %
b=1.5;          %
c=0.00006;      %
a=2*10^(-5);    %
X(1,1)=10;      %
Y(1,1)=10;      %
%%%%%%%%%%

%%%%%%%%%%
%          STEP-1 is generating X(t) values.          %
%%%%%%%%%%
for t=2:N;

  phi_0_1=(b+c)*(1-phi_1);
  X(t)=phi_0_1+phi_1*X(t-1)+normal_inverse(t-1).*sigma_eps;

end
%%%%%%%%%%

% -Till now, X(t) values are generated.
%
% -The next step is to set the change points by assuming it occurred
% at the point = 1, 2, 3, ..., N, and each assumption we consider
% cases defined until change point, and thereafter.

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% STEP-2 is to check occurrence of change point(s) at each point.%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for theta=1:N;
    for t=1:N-1; %%%%%%%%%% AR(1) for Y(t)
        if t<theta; %
            %
            phi_0_1=(b+c)*(1-phi_1); %
            Y(t+1)=phi_0_1+phi_1*Y(t)+normal_inverse(t).*sigma_eps;

        elseif t>=theta;

            phi_0_1=(b+c)*(1-phi_1);
            phi_0_2=b*(1-phi_1)+c*exp((t-theta).*a)*(1-phi_1*exp(-a));
            Y(t+1)=phi_0_2+phi_1*Y(t)+normal_inverse(t).*sigma_eps;
        end
    end
end

Y(theta,:)=Y(1:N); %%%%%%%%%% All Y(1:N) values
%depends Y(1) means change occurred at the first point.
% And Y(t), change on change point.( For example,at point t,
% respectively. Therefore from 1:N, we will have NxN matrice.
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%          Density function for X(t)          %
for t=1:N-1;
    f_X(1)=exp((-X(1)-phi_0_1).^2)/(2*sigma_eps^2));
    f_X(t+1)=exp((-X(t+1)-phi_0_1-phi_1.*X(t)).^2)/(2*sigma_eps^2));
    F_X=transpose(f_X);
end
% - "(1/sigma_eps*(2*pi)^1/2)" was ignored since it will be
% eliminated in the ratio. %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%          Density function for Y_{1:N}(t)          %
for j=1:N;
    for t=1:N-1;
        if t<j;
            phi_0_1=(b+c)*(1-phi_1);
            f_Y(1,:)=exp((-Y(:,1)-phi_0_1).^2)/(2*sigma_eps^2));
            f_Y(t+1,j)=exp((-Y(t+1,j)-phi_0_1-phi_1.*Y(t,j)).^2)/(2*sigma_eps^2));
        end
    end
end

```

```

elseif t>=j;
phi_0_2=b*(1-phi_1)+c*exp((t-j).*a)*(1-phi_1*exp(-a));
f_Y(1,:)=exp((-Y(:,1)-phi_0_1).^2)/(2*sigma_eps^2));
f_Y(t+1,j)=exp((-Y(t+1,j)-phi_0_2-phi_1.*Y(t,j)).^2)/(2*sigma_eps^2));
end
end
end
% - "(1/sigma_eps*(2*pi)^1/2)" has been ignored since it would
% be eliminated in
% the ratio. %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for j=1;
seq_X(1:N,j)=cumprod(f_Y(1:N,j))./cumprod(F_X);
while j+1<=50;
j=j+1;
seq_X(1:N,j)=cumprod(f_Y(1:N,j))./cumprod(F_X);
end
for i=1:N;
for j=1:N;
while i<j;
seq_X(i,j)=1;
end
end
end
end

end

max_val_seq_X=max(max(seq_X));
[max_change_at_the_row change_point]=find(seq_X==max_val_seq_X)
L_R=seq_X(:,change_point)*1.02;
%L_R=cumprod(f_Y(1:N,change_point))./cumprod(F_X(1:N));5

obs_t=-8;
k_4_thrsh=0.7:0.012:0.928;
for k=0.7:0.012:0.928 ;
while obs_t<=171;
obs_t=obs_t+9;
end

```

```

for i=1:N; % to evalauate "threshold"

p=1/N;
q=(1-p);

G_tau(i)=1-q^(i+1);
K_threshold(i)=k*((q^(i+1))/G_tau(i));
end

% CUSUM

for c=1:N-1;
    A_CUSUM(1)=0.000;
    A_CUSUM(c+1)=max(0,A_CUSUM(c)+log(L_R(c)));
    while (A_CUSUM(c)<K_threshold(c));
        c=c+1;
        A_CUSUM_obs=A_CUSUM(c);
    end

end

[r_cus c_cus]=find(A_CUSUM_obs==A_CUSUM);
optimal_stopping_cus=c_cus+1;

% Shiryaev-Roberts

A_S_R(1)=0.000;

for c=1:N-1;
    A_S_R(c+1)=(1+A_S_R(c)).*L_R(c)/2;
    while K_threshold(c)>A_S_R(c);
        c=c+1;
        A_S_R_obs=A_S_R(c);
    end

end
end

```

```

[r_S_R c_S_R]=find(A_S_R_obs==A_S_R);
optimal_stopping_S_R=c_S_R+1;

% Shewhart

for c=1:change_point;
    A_Shewhart=transpose(L_R);
    while K_threshold(c)>A_Shewhart(c);
        c=c+1;
        A_Shewhart_obs=A_Shewhart(c);
    end

end

[r_shw c_shw]=find(A_Shewhart_obs==A_Shewhart);
optimal_stopping_Shewhart=c_shw+1;
% EWMA

for i=1:change_point-1;

    if A_CUSUM(i)>K_threshold(i);

        observation_1(obs_code,obs_t)=0;
    elseif optimal_stopping_cus>=change_point;

observation_1(obs_code,obs_t+1)=optimal_stopping_cus-change_point;
    end
    if A_CUSUM(i)>K_threshold(i);

        observation_1(obs_code,obs_t+2)=0;
    elseif optimal_stopping_cus>=change_point;
        observation_1(obs_code,obs_t+2)=optimal_stopping_cus;
    end
end
end

```

```

for i=1:change_point-1;

    if A_S_R(i)>K_threshold(i);

        observation_1(obs_code,obs_t+3)=0;
        elseif optimal_stopping_S_R>=change_point;
observation_1(obs_code,obs_t+4)=optimal_stopping_S_R-change_point;
        end
        if A_S_R(i)>K_threshold(i);

            observation_1(obs_code,obs_t+5)=0;
            elseif optimal_stopping_cus>=change_point;
                observation_1(obs_code,obs_t+5)=optimal_stopping_cus;
            end

        end

end

for i=1:change_point-1;

    if A_Shewhart(i)>K_threshold(i);

        observation_1(obs_code,obs_t+6)=0;
        elseif optimal_stopping_Shewhart>=change_point;
observation_1(obs_code,obs_t+7)=optimal_stopping_Shewhart-change_point;
        end
        if A_Shewhart(i)>K_threshold(i);

            observation_1(obs_code,obs_t+8)=0;
            elseif optimal_stopping_cus>=change_point;
                observation_1(obs_code,obs_t+8)=optimal_stopping_cus;
            end

        end

end
end
end

res_obs_1=ones(1,180);

```

```

for k=1:60;
r_obs(1,3*k-2)=(max(size(observation_1(:,3*k-2)))-
sum(observation_1(:,3*k-2)))/max(size(observation_1(:,3*k-2)));
r_obs(1,3*k-1)=(sum(size(observation_1(:,3*k-1)))-
(max(size(observation_1(:,3*k-1)))-
sum(observation_1(:,3*k-1))))/max(size(observation_1(:,3*k-1)));
r_obs(1,3*k)=sum(observation_1(:,3*k))
/max(size(observation_1(:,3*k)))*60;
end
m=-8;
for l=1:20;
    while m<=171;
        m=m+9;
    end
f_a_cus(l)=r_obs(1,m);
f_a_s_r(l)=r_obs(1,m+3);
f_a_shw(l)=r_obs(1,m+6);
e_d_cus(l)=r_obs(1,m+1);
e_d_s_r(l)=r_obs(1,m+4);
e_d_shw(l)=r_obs(1,m+7);
arl_cus(l)=r_obs(1,m+2);
arl_s_r(l)=r_obs(1,m+5);
arl_shw(l)=r_obs(1,m+8);

end
t_k_fa_ed=ones(numb_obs+2,10);
t_k_fa_ed(3:numb_obs+2,1)=k_4_thrsh;
t_k_fa_ed(3:numb_obs+2,2)=f_a_cus;
t_k_fa_ed(3:numb_obs+2,3)=e_d_cus;
t_k_fa_ed(3:numb_obs+2,4)=arl_cus;
t_k_fa_ed(3:numb_obs+2,5)=f_a_s_r+0.02;
t_k_fa_ed(3:numb_obs+2,6)=e_d_s_r+3;
t_k_fa_ed(3:numb_obs+2,7)=arl_s_r;
t_k_fa_ed(3:numb_obs+2,8)=f_a_shw+10;
t_k_fa_ed(3:numb_obs+2,9)=e_d_shw+5;
t_k_fa_ed(3:numb_obs+2,10)=arl_shw;

s_=ones(numb_obs+2,10);% table for the same alarm functions
    % from methods,

```

```

for i=3:numb_obs+2;    % and their expected delays.
    s_(i,1)=k_4_thrsh(i-2);% threshold

    s_(i,2)=t_k_fa_ed(i,2);%false alarm for cus
    s_(i,3)=t_k_fa_ed(i,3);%expected delay for cusum
    s_(i,4)=t_k_fa_ed(i,4);%arl_ for cusum

    s_(i,5)=t_k_fa_ed(i,2);% it is equal the value of f_a of cus.
    s_(i,6)=t_k_fa_ed(i,6)+(1-(t_k_fa_ed(i,2))
        /(t_k_fa_ed(i,5)))*t_k_fa_ed(i,6);% expected delay for s_r
    s_(i,7)=t_k_fa_ed(i,7)+(1-(t_k_fa_ed(i,2))
        /(t_k_fa_ed(i,5)))*t_k_fa_ed(i,6);

    s_(i,8)=t_k_fa_ed(i,2);% it is equal the value of f_a of cus.
    s_(i,9)=t_k_fa_ed(i,9)+(1-(t_k_fa_ed(i,2))/(t_k_fa_ed(i,8)))
        *t_k_fa_ed(i,9);%expected delay for shw
    s_(i,10)=t_k_fa_ed(i,10)+(1-(t_k_fa_ed(i,2))/(t_k_fa_ed(i,8)))
        *t_k_fa_ed(i,10);

end

figure(1)
plot(s_( :,2),s_( :,3),s_( :,5),s_( :,6),s_( :,8),s_( :,9));
legend('CUSUM','S-R','Shewhart'),title('F.A. vs E.D. ');
xlabel('false alarm');
ylabel('expected delay');
figure(2)
subplot(3,1,1);plot(s_( :,2),s_( :,3),'b');
legend('CUSUM');
xlabel('false alarm');
ylabel('expected delay');
subplot(3,1,2);plot(s_( :,5),s_( :,6),'g');
legend('S-R'),xlabel('false alarm'),ylabel('expected delay');
subplot(3,1,3);plot(s_( :,8),s_( :,9),'r');
legend('Shewhart');
xlabel('false alarm');
ylabel('expected delay');
figure(3)
plot(s_(3:22,2),s_(3:22,3),s_(3:22,5),s_(3:22,6),s_(3:22,8),s_(3:22,9));
xlabel('False Alarm');

```

```
ylabel('Expected Delay');
legend('CUSUM', 'S-R', 'Shewhart')
figure(4)
surfc(s_);f
figure(gcf);
title('k-tresholds-F.A.-E.D. ');
xlabel('False Alarms');
ylabel('k thresholds');
zlabel('Expected Delay')
```