An approach to stiffness analysis methodology for haptic devices

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Abstract—In this work a new methodology is proposed to model the static stiffness of a haptic device. This methodology can be used for other parallel, serial and hybrid manipulators. The stiffness model considers the stiffness of; actuation system; flexible links and passive joints. For the modeling of the passive joints a Hertzian contact model is introduced for both spherical and universal joints and a simply supported beam model for universal joints. For validation of the stiffness model a modified JP Merlet kinematic structure has been used as a test case. A parametric Ansys FEM model was developed for this test case and used to validate the resulting stiffness model. The findings in this paper can provide an additional index to use for multi-objective structural optimization to find an optimum compromise between a lightweight design and the stiffness performance for high precision motion within a larger workspace.

Keywords: Haptics; Parallel/Serial kinematic structures; stiffness methodology;

I. INTRODUCTION

A haptic device is a computer-controlled actuated mechanical device that provides a physical interface between human sense of touch and computer generated virtual or remote environment. Based on manipulation and interaction with objects within the virtual or remote environment, these devices provide feedback forces and torques to the user. Applications of haptic devices are increasing in many fields particularly in medicine, tele-robotics, engineering design, and entertainment [1]. The intended application of the device presented in this work is a medical/dental simulator [1] which will be used to achieve manipulation capabilities and force/torque feedback in six degrees of freedom (6-DoF) during simulation of surgical or dental procedures in hard tissues such as bone structures [1]. Use of a haptic device in the above described application leads to three important design requirements on the design of the device [2], [3]:

1) Stiffness
2) Transparency and stability
3) Haptic feedback of force and torque in 6-DoF.

In precise positioning accuracy and high payload capability stiffness is an essential performance measure criteria. Stiffness can be defined as the capacity of a mechanical system to sustain loads without excessive changes of its geometry [4], or in other words these produced changes on geometry, due to the applied forces, are known as deformations or compliant displacements. Compliant displacements in a robotic system produces negative effects on static and fatigue strength, wear resistance, efficiency (friction losses), accuracy, and dynamic stability (vibration) [4].

By transparency it is meant that motion in free space should feel free while motion in contact with a virtual or remote object should result in feedback forces and torques as close as possible to those appearing in the remote or virtual world.

Haptic feedback is required to reflect both forces and torques in the case of surgical procedures of hard tissues like bones, which involves removing bone by drilling or milling to process channel and cavities, hence required 6 DoF capability.

Current trends in mechanical design of robotic manipulators have targeted essential reduction of moving masses, in order to achieve high dynamic performances with relatively small actuators and low energy consumption [5]. This motivates using advanced kinematical architectures, high strength and light-weight materials, as well as minimization of cross-sections of all critical elements. The primary constraint for such minimization is the mechanical stiffness of the manipulator, which is directly related with the robot accuracy defined by the design specifications.

There are still open problems related with stiffness. It has not yet been completely solved, for example, the problem of improving the stiffness analysis in order to have a better match between theoretical and experimental results. This aspect would require the development of more accurate stiffness models by taking into account also phenomena that cannot be easily modeled such as friction and backlash. Manufacturing tolerances and other geometrical indeterminacies in the kinematic model of a multi-body robotic system should be also properly considered.

The aim of this paper is to present a methodology for generating a compact and accurate generalized model for stiffness analysis of an any-DoF parallel/serial/hybrid manipulator. The generated model e.g. could be used for structure optimization within the workspace. The paper is organized as follows: section II summarizes a literature review on stiffness analysis of 6-DoF manipulators, and section III presents the stiffness analysis methodology. Section IV validates the presented methodology with an illustrative example comparing the proposed generalized stiffness model and a fairly detailed
FEM model. In addition the calculated stiffness variation in workspace, based of the stiffness model, is presented. Finally section V concludes the presented work.

II. RELATED WORK

Several methods exist for computation of the stiffness matrix: the virtual joint method (VJM), that is often called the lumped modeling [6]–[10], Finite Element Analysis (FEA) [10]–[12] and Matrix Structural Analysis (MSA) [13]–[16]. The first of them, the virtual joint method is based on the calculation of the Jacobian matrix that relate the joint displacement in joint space to the Tool Center Point (TCP) deflection in Cartesian space. In this method, only active joint stiffness is considered and links of mechanism are assumed strictly rigid. FEA is reliable for calculating the stiffness of components with arbitrary shape and complex contact interactions between components in a system. For example, the FEA model is adopted to characterize robot static rigidity and natural frequencies in [17] and it is used to validate an analytical model in [18]. However, FEA does not provide the analytical relationship between stiffness and structure dimensions of the mechanism. It does not fit the planned multi-objective optimization procedure in which the performance index such as stiffness index, most likely is to be represented as a function of design parameters. However this method is usually applied at the detail design stage because of the high computational expenses required for the repeated re-meshing of a complex structure.

The third of them, the MSA incorporates the main ideas of FEA, but operates with rather large elements and flexible beams to describe the manipulator structure [13]–[16]. Uchiyama [19] has derived an analytical model for the stiffness of a compact 6-DoF Haptic device, which he utilizes the design of a stiff platform for translational motion. The elastic elements were modeled as beam elements and considering the radial stiffness of the bearings. The model shows that the compliance matrix is a function of kinematic parameters as well as elastic parameters of each mechanical element. Also it was assumed that the base and traveling plates were rigid.

In literature, the compliance of joints is mostly modeled as a constant stiffness. Bonnemains et al. [20] has considered the stiffness of a spherical joints in the stiffness computation and identification of a kinematic machine tools.

III. STIFFNESS ANALYSIS METHODOLOGY

In this section we will describe a general methodology and the different steps required to develop a stiffness model of a robotic structure. This methodology can be applied on any robotic structure, but the discussions in this paper is focused on haptic devices based on parallel kinematic structures. The basic idea and assumption in this methodology is that the stiffness of the end-effector of a robotic structure can be obtained from the individual stiffness properties of its links using basic principles of the mechanical relation between force and displacement. The underlying idea is to resolve the force into individual link forces and then to compute the individual link deflection from the link and joint stiffness properties and finally to transform and add all these displacements to obtain the final stiffness matrix. The contact stiffness of passive joints like spherical and universal joints have been calculated as Hertzian springs [21]. The nominal point contact model was used for spherical joints and nominal line contact model was used for universal joints, while the bending stiffness of an universal joint is calculated based on the assumptions that the two axes act like a simply supported beam.

An example structure that we will use for formulating the different steps of the methodology is the Delta structure with 3 DoF [22], actuated by three rotational motors placed at the fixed base as shown in Fig.1.

![Fig. 1. ABB Flexible Automation’s IRB 340 FlexPicker.](image)

In a parallel kinematic structure we divide the total system into different kinematic chains. Each of these chains contain a number of components that need to be described in order to find the stiffness of one chain. The systems and components that need to be handled are:

A. Stiffness of actuation system
B. Stiffness of flexible links
C. Stiffness of passive joint

And as a final step the stiffness for the whole structure with \( N \) number of kinematic chains is formulated as

\[
K_{Total} = \sum_{i}^{N} K_{chain_{i}}
\]  

(1)

Each kinematic chain can be considered as a serial manipulator, so the same methodology is also applicable to serial and hybrid manipulators. The stiffness of each kinematic chain is given by equation (2):

\[
K_{chain}^{-1} = K_{act}^{-1} + K_{c}^{-1} + K_{sp}^{-1} + K_{U}^{-1}
\]  

(2)

Where \( K_{act}, K_{c}, K_{sp} \) and \( K_{U} \) are stiffness of actuation system, flexible links, spherical and universal joints, respectively.

A single kinematic chain can be modeled as shown in Fig.2, where the stiffness model of each compliant element is given in the coming sections.

A. Stiffness of actuation system

The stiffness of actuation system depends on stiffness of actuator and transmission system. The total stiffness of actuation system is given in equation (3):

\[
K_{act}^{-1} = K_{r}^{-1} + K_{Trans}^{-1}
\]  

(3)
where $K_r$ and $K_{trans}$ is the stiffness of actuator and transmission system.

The stiffness of the actuator was calculated from the torsional stiffness. The torsional stiffness of the actuator can be defined by simple expression given in equation (4):

$$K_{rotor} = \frac{J_{act}G}{L_r} \quad (4)$$

where $J_{act}$ is the polar moment of inertia, $G$ modulus of rigidity and $L_r$ is the length of motor shaft.

The principle of virtual work can be used to find the stiffness of the actuation system for the case of parallel/serial manipulators \[23\], $K_s, K_p$, as given in equation (5) and (6).

$$K_r = K_p = J^T diag[k_i]J \quad (5)$$

$$K_r = K_s = J^{-T} diag[k_i]J^{-1} \quad (6)$$

Here $J$ jacobian matrix derived by Khan et. al. [3], and $k_i = K_{rotor}$ is the rotor stiffness of motor shaft.

The stiffness of transmission system depends on the intended application for the manipulator. Here we have chosen to discuss three different types of transmission systems;

1) Cable based linear transmission
2) Cable based rotational transmission
3) Timing belt transmission

1) Cable based linear transmission: The cable based transmission consists of cable based linear actuator driven by a DC motor shown in Fig.3. Both the cable and shaft of the DC motor can be considered as compliant. A cable with stiffness coefficient $K_{cable}$, length $l_i$, and pretension $\tau$, that statically balance the force $F_i$, then stiffness coefficient $K_{cable}$ of the cable can be found by approximation as given in Equation (7): EQ13

$$K_{trans} = K_{cable} = \frac{AE}{l_i} \left[ I \right]_{1 \times 1} \quad (7)$$

Where $A$ is the cross sectional area of the cable, $E$ is the young modulus of the cable and $i$ is the number of cables in the system.

2) Cable based rotational transmission: In the case of rotational transmission both the cables are acting in parallel, which is actuated by DC motor, so the effective stiffness coefficient of the rotational transmission system can be expressed in equation (8):

$$K_{trans} = K_{cable1} + K_{cable2} \quad (8)$$

3) Timing belt transmission: The stiffness coefficient for timing belt can be calculated based on the assumption that, stress is proportional to strain. Defining the stiffness of a unit length and a unit wide belt as specific stiffness $c_{sp}$, the stiffness coefficients of the belt on the tight and slack side, $k_1$ and $k_2$ are proposed by [24], and are expressed by equation (9):

$$k_1 = c_{sp} \frac{b}{L_1}, \quad k_2 = c_{sp} \frac{b}{L_2} \quad (9)$$

Where $L_1$ and $L_2$ are the un-stretched lengths of the tight and slack sides of the cable respectively, and $b$ is the belt width.

Since the tight and slack sides can be considered as springs acting in parallel, their stiffness add linearly to form a resultant stiffness constant $K_{trans}$ given in equation (10):

$$K_{trans} = k_1 + k_2 = bc_{sp} \frac{L_1 + L_2}{L_1L_2} \quad (10)$$

B. Stiffness of flexible links

To calculate the stiffness of flexible links i.e the base link and proximal link we have used the following methodology: we have transformed the force acting on the TCP of the platform into individual link forces, then we compute the individual link deflections from the link stiffness properties, and finally we transform and add all these displacements to obtain the final stiffness matrix. The stiffness of the base and proximal links can be calculated by transformation of forces and moments from frame C to B. The following steps will be used to calculate the stiffness of flexible links.

1) Force/moment transformation: Let consider a force/torque vector $(C F, C M)^T$ expressed in frame C. To express this in frame B we need to know how C is oriented with respect to B. This information is typically specified by a rotation matrix $B R_C \in R^{3\times3}$ where the first, second and third columns of $B R_C$ are unit vectors describing the orientation of x, y and z axes of frame C expressed in frame B. The required force transformations is given in equation (11):

$$\begin{bmatrix} B F \\ B M \end{bmatrix} = \begin{bmatrix} B R_C & 0 \\ 0 & B R_C \end{bmatrix} \begin{bmatrix} C F \\ C M \end{bmatrix} \quad (11)$$
We take the straightforward approach to compute $(U F_B, U M_B)^T$ by taking a force and moment balance of the proximal link, as shown in Fig. 4

![Image](image_url)

**Fig. 4.** Force/Moment Transformation from C to B.

The force balance equation results in equation (12):

$$U F_B = U F_C$$ \hspace{1cm} (12)

While taking moments about point C yields the moment balance equation (13):

$$U M_B = U M_C + (U R_B^B P_C) \times U F_C$$ \hspace{1cm} (13)

Where $B P_C$ is the position vector from $\{B\}$ to $\{C\}$ expressed in frame $\{B\}$. Equations (12) and (13) can be combined to a more compact matrix form of equation (14):

$$\begin{bmatrix} U F_B \\ U M_B \end{bmatrix} = \begin{bmatrix} I & \theta \end{bmatrix} \begin{bmatrix} U F_C \\ U M_C \end{bmatrix}$$ \hspace{1cm} (14)

we use the notation $[p \times] \in \mathbb{R}^{3 \times 3}$ to represent the linear transformation of a cross product with a vector $p \in \mathbb{R}^3$

$$[p \times] y = p \times y, \ y \in \mathbb{R}^3$$

where

$$[p \times] = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$ \hspace{1cm} (15)

$U J_{B,C,force} \in \mathbb{R}^{3 \times 3}$ as the force transformation Jacobian matrix which transforms the frame of application of the force vector from frame C to frame B. The forces are all expressed in the same frame $U$. $U J_{B,C,force}$ involves the relative orientation between frame B and frame U as well as the position vector describing the origin of frame C in frame B. Similarly, using Fig. 2 to derive the transformation involving displacements. With infinitesimal displacements, equations relating velocity transformations are equivalent to those for displacement transformations. Imagine frames B and C to be moving instantaneously with respect to frame U.

**2) Displacement transformation:** The linear and angular velocities $(U v_B, U \omega_B \in \mathbb{R}^3)$, respectively, of frame B may be computed from the velocities of frame C using:

$$U v_B = U v_C + (U R_B^C P_B) \times U \omega_C$$ \hspace{1cm} (16)

$$U \omega_B = U \omega_C$$ \hspace{1cm} (17)

The velocities in (16) and (17) may be replaced with the infinitesimal displacements $U d_B$ and $U \theta_B$, and combined to the following compact matrix form in equation (18):

$$\begin{bmatrix} U d_B \\ U \theta_B \end{bmatrix} = \begin{bmatrix} I & (U R_B^B P_C) \times \\ 0 & I \end{bmatrix} \begin{bmatrix} U d_C \\ U \theta_C \end{bmatrix}$$ \hspace{1cm} (18)

We refer to $U J_{B,C,disp} \in \mathbb{R}^{3 \times 3}$ as the displacement (or velocity) transformation Jacobian which transforms the frame that is in motion from frame B into frame C. Frame B and Frame C can be viewed as instantaneously attached to the same rigid body, and the transformation involves determining the motion of frame B due to the motion of frame C. All motions are expressed in the same frame $U$. $U J_{B,C,disp}$ involves the relative orientation between frames B and A as well as the position vector describing the origin of frame C in frame B. The force experience at frame B can be expressed in frame U using equation (19).

$$\begin{bmatrix} U F_B \\ U M_B \end{bmatrix} = U J_{B,C,force} \begin{bmatrix} U F_C \\ U M_C \end{bmatrix}$$ \hspace{1cm} (19)

Where $U J_{B,C,force} = \begin{bmatrix} I & 0 \\ \begin{bmatrix} (U R_B^B P_C) \times \end{bmatrix} & I \end{bmatrix}$

**3) Transformation of stiffness and compliance matrices:**

The stiffness matrix of a rotated beam in a task-based coordinate frame $U$ can be obtained from the local link stiffness matrix $L K$ and the knowledge of rotating matrix $U R_L \in \mathbb{R}^{3 \times 3}$ that describes the orientation of the limb with respect to the task frame. Since forces/moments and translational/rotational displacements are related by:

$$\begin{bmatrix} U F \\ U M \end{bmatrix} = U J_{L} \begin{bmatrix} L F \\ L M \end{bmatrix}$$

and

$$\begin{bmatrix} U d_L \\ U \theta_L \end{bmatrix} = U J_{L} \begin{bmatrix} L d \\ L \theta \end{bmatrix}$$

and from the definition of the stiffness matrices:

$$\begin{bmatrix} U F \\ U M \end{bmatrix} = U K \begin{bmatrix} U d \\ U \theta \end{bmatrix}$$

and

$$\begin{bmatrix} L F \\ L M \end{bmatrix} = L K \begin{bmatrix} L d \\ L \theta \end{bmatrix}$$

$U K$ is obtained from $L K$ using the same transformation matrix $U J_L$ and its transpose $U J_L^T = U J_L^{-1} U J_L$. Similarly, from the definition of the compliance matrices $U S$ and $L S$, i.e., being the inverse of the stiffness matrices:

$$\begin{bmatrix} U d \\ U \theta \end{bmatrix} = U S \begin{bmatrix} U F \\ U M \end{bmatrix}$$

and

$$\begin{bmatrix} L d \\ L \theta \end{bmatrix} = L S \begin{bmatrix} L F \\ L M \end{bmatrix}$$

So we have:

$$U S = U J_L L S U J_L^T$$ \hspace{1cm} (21)
C. Stiffness of passive joints

The contact stiffness of spherical and universal joint is calculated based on Hertzian contact model. The nominal point contact model was used for spherical joint and nominal line contact model was used for universal joint, while the bending stiffness of universal joint is calculated based on the assumptions that two axes act like a simply supported beam.

1) Nominal point contact model: The contact stiffness of the spherical joint can be found out by Hertzian theory, considering the nominal point contact between a sphere with radius $R$ and a plane loaded by force $F$ as shown in Fig.5, the radius $r$ of the circular contact area thus formed is given by the equation proposed by Hertz (Timoshenko and Goodier [25], Johnson [26]),

$$r = \left( \frac{3F}{E'} \right)^{1/3}$$  \hspace{1cm} (22)

Where $E'$ is the effective modulus of elasticity defined by:

$$\frac{1}{E'} = \frac{1-v_1^2}{2E_1} + \frac{1-v_2^2}{2E_2}$$  \hspace{1cm} (23)

And $R'$ the effective radius related to the individual components

$$\frac{1}{R'} = \frac{1}{R'_{x}} + \frac{1}{R'_{y}} = \frac{r_{1,x}}{1} + \frac{1}{r_{2,x}} \cdot \frac{1}{R'_{y}} = \frac{r_{1,y}}{1} + \frac{1}{r_{2,y}}$$  \hspace{1cm} (24)

The relation between contact radius and the indentation depth $\delta$ is given by Johnson [26]

$$r = (2R'\delta)^{1/3}$$  \hspace{1cm} (25)

Substituting equation (22) into equation (25), results in contact stiffness $K_{sp}$ of spherical joint

$$F = \left( \frac{8}{9}R' \right)^{1/2} E' \delta^{2}$$  \hspace{1cm} (26)

$$K_{sp} = \frac{dF}{d\delta} = E' (2R'\delta)^{1/3}$$  \hspace{1cm} (27)

2) Nominal line contact model: Stiffness of a universal joint is calculated from the Hertzian contact model for line contact, the line contact is shown in Fig.5, while the bending stiffness of universal joint is calculated based on the assumption that two axes act as a simply supported beam. The semi-axis $b$ of a nominal line contact is derived by Hertz theory

$$b = 2 \sqrt{\frac{2FR'}{\pi E'}}$$  \hspace{1cm} (28)

$$b = (2R'\delta)^{1/2}$$  \hspace{1cm} (29)

Where $E'$ and $R'$ are defined in equation (23) and (24), comparing both the equations (28) and (29), we get contact stiffness $K_{UC}$ of the universal joint

$$F = \frac{\pi l E' \delta}{4}, \quad K_{UC} = \frac{dF}{d\delta} = \frac{\pi l E'}{4}$$  \hspace{1cm} (30)

The general expression for the deflection of simply supported beam with length $L_u$ is given as

$$\delta_{u\text{joint}} = \frac{PL_u^{3}}{48EI}$$  \hspace{1cm} (31)

so the bending stiffness $K_{UB}$ of the universal joint is

$$K_{UB} = \frac{48EI}{L_u^{3}}$$  \hspace{1cm} (32)

so the total stiffness of the universal joint is

$$K_{U} = K_{UC} + K_{UB}$$  \hspace{1cm} (33)

Fig. 5. Contact model of spherical and universal joint.

IV. METHODOLOGY VALIDATION

In the following section, the methodology described above will be applied to a haptic device based on a parallel mechanism in the form of a modified version of the JP Merlet kinematic structure [3], [27]. The structure consists of a fixed base, a moving platform, and six identical legs connecting the platform to the base as shown in Fig.6. Each kinematic chain consists of an active actuator fixed to the base, a spherical joint, a constant length proximal link, and a universal joint. The joint attachment point pairs are symmetrically separated at 120° and lie on a circle, both on the base and platform. The platform attachment points are rotated 60° clockwise from the base attachment points. The 6-PSU (active prismatic, passive spherical and universal) joints configuration is used to get 6-DoF motion at TCP. Six independent DC motors are connected via cable transmissions to provide linear motion to each linear guideway. A parallel mechanism is a close loop mechanism consisting of a base plat, a traveling plate and elementary chains that connect the two plates. Its stiffness is determined by the stiffness of each kinematic chain. We assume the base and traveling plates are rigid, we begin with the analysis of elementary chains and drive a compliance matrix of the target chain.

Fig. 6. A schematic of 6 DoF haptic device.
A. Stiffness modeling

The kinematic chains consist of actuation system, linear guideways, spherical joint, constant length proximal link and universal joint. The following method has been used to calculate the stiffness of each kinematic chain.

B. Displacement due to linear guideway

From the local compliance matrix of the linear guideways, we transform the compliance matrix to the reference frame $U$ using (21):

$$ A_S_B = U J_A S_B U^T $$

(34)

$A_S_B$ is the stiffness matrix of the linear guideways. The displacement cause by the force acting on $B$ is then:

$$ d_B = \begin{bmatrix} U d_B \\ u \theta_B \end{bmatrix} = A_S_B \begin{bmatrix} U F_B \\ u M_B \end{bmatrix} = U J_A S_B U^T \begin{bmatrix} U F_B \\ u M_B \end{bmatrix} $$

(36)

The displacement $d_B$ is the contribution of the flexibility of the linear guideways. The effect of this displacement on $C$ is computed using (18), and transforming the local compliance matrix of the beam to the universal frame $U$ using (39):

$$ B J_U = \begin{bmatrix} BR_U & 0 \\ 0 & BR_U \end{bmatrix} $$

(39)

$BR_U$ is the unit vector describing the orientation of $x$, $y$ and $z$ axes of frame $B$ expressed in frame $U$. From the local frame $\{B\}$ of proximal link, the equations for rotation matrix $B J_U$ can be derived as

$$ Z_{pli} = \begin{bmatrix} L_{2y} \\ L_{2y} \\ 0 \end{bmatrix} $$

and

$$ X_{pli} = Z_{pli} \times X_{pli} $$

Finally, the rotation matrix for proximal link is given by equation (40)

$$ B R_U = \begin{bmatrix} X_{pli} & Y_{pli} & Z_{pli} \end{bmatrix} $$

(40)

C. Displacement due to proximal link

From the local flexibility matrix of the proximal link $B S_C$, we compute the displacement caused by $U F_C$ due to the flexibility of proximal link only:

$$ \begin{bmatrix} B d_C \\ B \theta_C \end{bmatrix} = B S_C \begin{bmatrix} B F_C \\ B M_C \end{bmatrix} = B S_C B J_U \begin{bmatrix} U F_C \\ u M_C \end{bmatrix} $$

(41)

Where $K_{prox} = B S_C$

The displacement solely due to the flexibility of the link 2 is expressed in the reference frame $U$ using (39):

$$ \begin{bmatrix} U d_C \\ u \theta_C \end{bmatrix} = U J_B \begin{bmatrix} B d_C \\ B \theta_C \end{bmatrix} $$

(42)

And substituting the (41) yields:

$$ \begin{bmatrix} U d_C \\ u \theta_C \end{bmatrix} = U J_B S_C B J_U \begin{bmatrix} U F_C \\ u M_C \end{bmatrix} $$

(43)
D. Total compliance due to linear guideways and proximal links

The total displacement of C due to the flexibility of linear guideways and proximal link is the sum of the individual displacement expressed in the reference frame U:

\[
\begin{bmatrix}
U_{dC}

U_{\theta C}
\end{bmatrix} = 
\begin{bmatrix}
U_{dC}

U_{\theta C}
\end{bmatrix}_{\text{link1}} +
\begin{bmatrix}
U_{dC}

U_{\theta C}
\end{bmatrix}_{\text{prox}}
\]

\[
= \left[ \begin{array}{c}
U_{c}J_{B,\text{disp}}(U_{c}J^{A}_{B}S_{B}^{U}J^{T}_{A}U_{c}J_{C,\text{force}} + U_{c}J^{B}_{B}S_{B}^{C}U_{c})\end{array} \right] U_{F_{C}}
\]

From the definition of the compliance matrix we therefore have:

\[
U_{S_{C}} = \left[ \begin{array}{c}
U_{c}J_{B,\text{disp}}(U_{c}J^{A}_{B}S_{B}^{U}J^{T}_{A}U_{c}J_{C,\text{force}} + U_{c}J^{B}_{B}S_{B}^{C}U_{c})\end{array} \right] U_{F_{M_{C}}}
\]

and the overall stiffness matrix of linear guideways and proximal link is then

\[
K_{C} = U_{K_{C}}
\]

\[
= \left[ \begin{array}{c}
U_{c}J_{B,\text{disp}}(U_{c}J^{A}_{B}S_{B}^{U}J^{T}_{A}U_{c}J_{C,\text{force}} + U_{c}J^{B}_{B}S_{B}^{C}U_{c})\end{array} \right] U_{F_{C}}
\]

(44) by considering all the compliant elements in series and using (2) as given in equation (45).

\[
K_{\text{chain}}^{-1} = K_{\text{acts}}^{-1} + K_{C}^{-1} + K_{sp}^{-1} + K_{U}^{-1}
\]

(45)

so the overall stiffness of stewart platform is

\[
K_{T_{\text{total}}} = \sum_{i=1}^{6} K_{\text{chain}_{i}}
\]

(46)

In order to validate the developed model experimentally in the future work, we have considered static analysis, so the stiffness of the actuation system is not considered in the simulation.

E. FEM-based validation

In order to validate the stiffness model developed in section B, a parametric model was developed with the Ansys Parametric Design Language (APDL). The aim was to validate the stiffness model at different configurations. In this parametric FEM model, the inverse kinematics developed in [3] was used to calculate the length of linear guideways at different configurations of the TCP. Beam, shell, and solid elements, as well as multi-point constraint (MPC) equations were used to model the linear guideways, the proximal, spherical and universal joints, and the platform. Table I shows the design parameters and material properties of the FEM model. Fig.7 shows prototype of a 6-DoF haptic device. Fig.8 shows the FEM mesh at the nominal configuration.

F. Comparison of generalized stiffness and FEM Models

To compare FEM analysis and stiffness modeling, the deflections are calculated at 9 points in the workspace of which 8 are shown in Fig.9. The first point (0) is located in the center of the cube. In this simulation a force of 50N was applied in each x,y and z-direction at the tool center point(TCP)of the platform, and then corresponding deflection in each direction was calculated. The resulting deflections and comparison between Analytical and FEM model is shown in Fig.10. The comparison shows that validity of the proposed methodology. The deflection in x, y directions at each configuration shows that when the TCP are at points 1, 2, 3 and 4, the deflection is high due to bending of the linear guideways.

G. Stiffness Variations in the workspace

To check the validity of the mathematical model developed, the TCP of the platform was traversed in the workspace and the deflection and stiffness was calculated in a number of grid points The stiffness variation is shown in Fig.11 and indicate that stiffness is isotropic in the workspace; the stiffness is high when the linear guideways are at the minimum actuation length, when the length of linear guideways is increasing due to bending phenomena the stiffness of the system decreases which is clear from the blue color.
Fig. 10. Linear deflections by applied force of 50 N in X,Y,Z-direction.

Fig. 11. Stiffness variation of TCP of platform.

V. CONCLUSIONS

This paper proposes a new methodology to compute a quasi-static stiffness model of parallel, serial, and hybrid manipulators. The proposed methodology is applied to an existing architecture of a 6-DoF haptic device based on a modified JP Merlet kinematic structure. The proposed model takes into account the stiffness of the actuation system, linear guideways, proximal links, and passive joints. In this methodology we have decomposed the force acting at the TCP of the platform into individual link forces, then computed the individual link deflections from the link stiffness properties, and finally transformed and added all these displacements to obtain the final global stiffness matrix. Another systems modeling approach covered by the proposed methodology is the introduction of the Hertzian contact model for both spherical and universal passive joints and a simply supported beam model for the universal joint.

The proposed model was validated with quasi-static simulations with a parametric Ansys FEM model. The comparative analysis between the proposed stiffness method and the fairly detailed FEA model shows that the stiffness results at the TCP are very close to each other.

This methodology also creates possibilities to develop a kinetostatic control algorithm, which will allow us to improve accuracy of the classical kinematic control and to compensate position errors caused by elastic deformation in links and joints due to external loading.

REFERENCES