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A zero-one law for *l*-colourable structures with a vectorspace pregeometry

Ove Ahlman

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Department of Mathematics
Uppsala University

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1 Introduction

Model theory is the study of abstract mathematical structures, their formal language and their theories. Especially interesting among these are the finite models which, because of their finiteness, are easily identified in real life and applications. When studying finite models of different types, logicians asked themselves what happens to the truth in the finite models when we let the models grow towards infinity. Glebskii, Kogan, Liogonkii and Talanov [5] and Fagin [4] answered this independently of each other for sets of finite relational structures by giving a so called $0-1$ law for the uniform probability measure. If for each $n \in \mathbb{N}$, $\mathbf{K}_n$ is a set of L-structures of size n, we say that $\mathbf{K} = \cup_{n=1}^{\infty}\mathbf{K}_n$ has a $0-1$ law for the probability measure μ_n defined on formulas on $\mathbf{K}_n$, if for each $\varphi \in L$

$$\lim_{n\to\infty} \mu_n(\varphi) = 1 \quad \text{or} \quad \lim_{n\to\infty} \mu_n(\varphi) = 0.$$

The $0-1$ law which Fagin [4] and Glebskii et. al. [5] proved was very general, speaking about all finite structures over a certain relational language, so researchers asked themselves how we could restrict the sets of structures in different ways and still have a $0-1$ law. Kolaitis, Prömel and Rothschild [9] showed, as a part of their proof that K_{l+1}-free graphs ($l \geq 2$) has a $0-1$ law for the uniform probability measure, that a $0-1$ law holds for l-colourable graphs. The question may arise if such a $0-1$ law is possible to generalise to l-colourable structures in general. If relation symbols of higher arity than 2 are in the formal language then there are two natural ways of generalising l-colourings and l-colourability; the "strong" and the "weak" versions of l-colourings.

Koponen [10] showed that both strongly and weakly l-colourable structures have a $0-1$ law for both the uniform probability measure and for the dimension conditional measure (defined in [10]). This is true even if you decide that certain relation symbols always should be interpreted as irreflexive and symmetric relations. A consequence is that if you have sets of L-structures $K_n, n = 1, 2, 3, ...$ where each $\mathcal{M} \in \mathbf{K}_n$ has universe $\{1, ..., n\}$ and a) each l-colourable L-structure with universe $\{1, ..., n\}$ is in $\mathbf{K}_n$ and b) "almost all"$\mathcal{M} \in \mathbf{K}_n$ are $l-$colourable (for big n), then $\mathbf{K} = \bigcup_{n=1}^{\infty} \mathbf{K}_n$ has a $0-1$ law. In [11], Schacht and Person let $\mathbf{K}_n$ be the set of all 3 hypergraphs without Fano planes and node-set $\{1, ..., n\}$, and show that almost all such hypergraphs are 2-colourable. Since each 2-colourable 3-hypergraph is missing a Fano plane it follows that $\mathbf{K}$ in this case has a $0-1$ law. Another example regarding 3-hypergraphs can be seen in [2]. One of the most fundamental and important mathematical structures are vector spaces (as well as affine and projective spaces) which in turn induces so called pregeometries. Pregeometries also play an important part in model theory. It is therefore natural to study sets $\mathbf{K}_n, n = 1, 2, ...$ of L-structures (for some fixed language L) which has a underlying pregeometry, definable by L-formulas. Especially interesting to study are (strongly or weakly) l-colourable L-structures with a underlying pregeometry where the l-colouring respects the pregeometry.

In this thesis we study strongly and weakly colourable l-structures whose underlying pregeometry is a vector space (of finite dimension) over a fixed finite field. We will show that both strongly and weakly l-colourable L-structures have a $0-1$ law for the "dimension conditional" probability measure in Theorem 5.6, which generalises theorem 9.1 in [10]. The dimension conditional measure has a natural interpretation as a process where you first randomly choose a l-colouring c on each finite dimensional vector space, then randomly choose relations on the 1-dimensional subsets, then on the 2-dimensional subsets (among those possibilities for which c is still a l-colouring) etc. for each r such that some relation symbol has at least the arity $d \geq r$. Whether the main result of this thesis holds also for the uniform probability measure is a topic for further research.

The proofs are using a result from Koponen [10] which is applicable to l-coloured structures whose underlying pregeometry is defined by a finite dimensional vector space over a finite field. The proof idea is to define certain "extension axioms" and to show that each such almost surely is true in a l-colourable structure with big enough dimension. To do this we need a formula $\xi(x, y)$ such that with probability approaching one as the dimension tends to infinity, two elements a and b have the same colour if and only if $\xi(a, b)$ holds in the given structure. Moreover, it is essential that $\xi(x, y)$ does not explicitly mention the colours; it only speaks about the relations of the structure and the pregeometry. In the case of strong l-colourings, in Section 3 this will be done in an explicit way. While when we speak of weak l-colourings, in Section 4, the strong colouring method doesn't work. Instead we seek aid in a result from Ramsey theory and a theorem by Graham, Leeb and Rothschild [6] which is about colouring vector spaces over an arbitrary finite field. This result shows that a formula ξ as we said we needed above, exists but without exactly talking about what ξ looks like.

2 Preliminaries

First order logic starts with a vocabulary V containing constant, function and relation symbols. By using function symbols together with constant symbols and variables we can create terms, and by putting terms inside relations and putting logical connectives and quantifiers between these relations with terms we get formulas. All the terms and formulas together build a language L. A L-structure $\mathcal{M}$ is a tuple which contains a set M which is called the universe, and interpretations of all the symbols in the vocabulary V, to the universe. An embedding is a function between L-structures which keep the structure intact, and $\mathcal{N}$ is a substructure of $\mathcal{M}$ (written $\mathcal{N} \subseteq \mathcal{M}$) if the inclusion map from $\mathcal{N}$ to $\mathcal{M}$ is an embedding. We may create the structure which is generated by a subset, by taking the least possible substructure which contain that subset, and call that structure, the generated structure of that subset. The reduct of a structure is obtained by restricting the vocabulary of a structure, but without changing anything else. For more information regarding these basic model theoretic concepts and formal definitions read [8].

Notation 2.1. *When we use structures in this thesis we will always write them with caligraphic letters, like $\mathcal{M}$ or $\mathcal{A}$, while their respective universes will be written using normal letters, like M and A. The notation $\mathcal{M} \upharpoonright L$ means the reduct of the structure to the language L. In the case $S \subseteq M$ then we will use the notation $\mathcal{M} \upharpoonright S$ to mean the substructure of $\mathcal{M}$ which is generated by S.*

The following definition of a pregeometry is rather compact and the reader should look up [1] for a more detailed explanation.

Definition 2.2. We say that (A, cl_A), with $cl_A : \mathcal{P}(A) \to \mathcal{P}(A)$ is a **pregeometry** if it satisfies the following:

1. (Reflexivity) $X \subseteq cl_A(X)$.
2. (Monotonicity) $Y \subseteq cl_A(X) \Rightarrow cl_A(Y) \subseteq cl_A(X)$.
3. (Exchange property) If $a, b \in A$ then $a \in cl_A(X \cup \{b\}) - cl_A(X) \Rightarrow b \in cl_A(X \cup \{a\})$.
4. (Finite Character) $cl(X) = \bigcup\{cl_A(X_0) | X_0 \subseteq X$ and $|X_0|$ is finite$\}$.

If it is obvious which structure we are talking about we will exclude the A subscript, and simply write cl. Throughout this thesis we will only use the pregeometry induced by taking the linear span, $Span(X)$, of a subset X of a vector space $\mathcal{G}_n$ which forms the pregeometry $(\mathcal{G}_n, Span)$.

If $X, Y, Z \subseteq A$ then we say that X is **independent from** Y **over** Z if for every $a \in X$, $a \in cl(Y \cup Z) \Leftrightarrow a \in cl(Z)$ and in the case $Z = \emptyset$ we simply say that X is **independent from** Y. We say that a set X is **independent** if for each $x \in X$, we have that $\{x\}$ is independent from $X - \{x\}$. Notice that in the common case $x, y \in A$ with $x, y \notin cl(\emptyset)$ then x independent from y i.e. $x \notin cl(\{y\})$ is, by the exchange property, equivalent to $y \notin cl(\{x\})$. When speaking of the closure operator as the span in a vector space then independence is the same as linear independence. The **dimension** dim of a set $X \subseteq A$ is the same as the cardinality of the largest independent subset of X. We say that a set $X \subseteq A$ is **closed** if $cl(X) = X$.

We'll prove a lemma considering pregeometries which will be used much later. The following lemma, is fairly obvious if we think of the pregeometry as a vector space, but in order to get a nice result and showing how to use the pregeometry axioms, we do it in the general pregeometric context.

Lemma 2.3. *Let $\mathcal{A} = (A, cl)$ be a pregeometry. If $\{a, v_1, ..., v_m, w_1, ..., w_n\} \subseteq A$ is an independent set then $cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n) = cl(a)$*

Proof. By reflexivity $a \in cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n)$ and so by monotonicity $cl(a) \subseteq cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n)$.

For the opposite direction we first assume that $x \in cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n)$ and do induction over n in order to prove that $x \in cl(a)$.

- Basis: If $n = 0$ then $cl(a, w_1, ..., w_n) = cl(a)$ so by definition of x we are done.
- Induction Step: We are in the case of $x \in cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_{n+1})$, so we have two sub cases to consider:

$$x \in (cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_{n+1})) - (cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n))$$

$$\text{or } x \in cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n).$$

In the first case we get the consequence that $x \in cl(a, w_1, ..., w_{n+1}) - cl(a, w_1, ..., w_n)$ and hence by the exchange property we get that $w_{n+1} \in cl(a, w_1, ..., w_n, x)$. We already know that $x \in cl(a, v_1, ..., v_m)$ hence we get that

$$1+m+n \geq \dim(a, v_1, ..., v_m, w_1, ..., w_n) = \dim(a, v_1, ..., v_m, w_1, ..., w_n, x) =$$
$$\dim(a, v_1, ..., v_m, w_1, ..., w_n, w_{n+1}, x) = \dim(a, v_1, ..., v_m, w_1, ..., w_{n+1}) = 1+m+n+1.$$

The second equality is because of $w_{n+1} \in cl(a, w_1, ..., w_n, x)$ and the last equality is because of the independence of the elements. This leads to a contradiction, by $m+n+1 \geq m+n+2$. Hence the correct sub case must be the second which state that $x \in cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n)$, so $x \in cl(a, w_1, ..., w_n)$ and hence by the induction hypothesis we get that $x \in cl(a)$.

The induction now concludes the other direction $cl(a, v_1, ..., v_m) \cap cl(a, w_1, ..., w_n) \subseteq cl(a)$ which finishes the proof. □

We won't directly use pregeometries in this thesis, but rather structures that may simulate a pregeometry. The following definition shows what we mean by that.

Definition 2.4. We say that a L-**structure** $\mathcal{A}$ **is a pregeometry** if both of the following items are satisfied:

- We may define a closure operator $cl_{\mathcal{A}}$ on $\mathcal{A}$ such that $(A, cl_{\mathcal{A}})$ is a pregeometry.
- For each $n \in \mathbb{N}$ there is a formula $\theta_n(x_1, ..., x_{n+1})$ such that

$$x_{n+1} \in cl_{\mathcal{A}}(x_1, ..., x_n) \Leftrightarrow \mathcal{A} \models \theta_n(x_1, ..., x_{n+1})$$

  for all $x_1, ..., x_{n+1} \in A$.

In such a structure we may say $\dim_{\mathcal{A}}$ which will mean the same as the dimension with respect to $cl_{\mathcal{A}}$. If $\mathbf{K}$ is a set of L-structures we will say that $\mathbf{K}$ is a pregeometry if for each $\mathcal{A} \in \mathbf{K}$, $\mathcal{A}$ is a pregeometry, and for each $n \in \mathbb{N}$ there is an $L-$formula θ_n which such that for each $\mathcal{A} \in \mathbf{K}$ we have $\mathcal{A} \models \theta_n(x_1, ..., x_{n+1}) \Leftrightarrow x_{n+1} \in cl_{\mathcal{A}}(x_1, ..., x_n)$ for all $x_1, ..., x_{n+1} \in A$.

The next definition gives us the characteristic formula of a structure. It is very important that we have a finite structure over a finite vocabulary when creating this formula, since else it will be a formula of infinite length.

Definition 2.5. Let $\mathcal{A}$ be a finite L-structure with universe $A = \{a_1, ..., a_\alpha\}$. We may then define **the characteristic formula** $\chi_{\mathcal{A}}(x_1, ..., x_\alpha)$ **of** $\mathcal{A}$ by letting $\chi_{\mathcal{A}}$ be a conjunction of the following:

- $(\bigwedge_{1 \leq i \leq j \leq \alpha} x_i \neq x_j)$.
- $R(x_{i_1}, ..., x_{i_n})$ iff $\mathcal{A} \models R(a_{i_1}, ..., a_{i_n})$ for some relation symbol $R \in V$ and $i_1, ..., i_n \in \{1, ..., \alpha\}$.
- $\neg R(x_{i_1}, ..., x_{i_n})$ iff $\mathcal{A} \models \neg R(a_{i_1}, ..., a_{i_n})$ for some relation symbol $R \in V$ and $i_1, ..., i_n \in \{1, ..., \alpha\}$.
- $f(x_{i_1}, ..., x_{i_n}) = x_j$ iff $\mathcal{M} \models f(x_{i_1}, ..., x_{i_n}) = x_j$ for some function symbol $f \in V$ and $i_1, ..., i_n, j \in \{1, ..., \alpha\}$.
- $c = x_j$ iff $\mathcal{M} \models c = x_j$ for some constant symbol $c \in V$ and $j \in \{1, ..., \alpha\}$

Notice that if $\mathcal{M} \models \chi_{\mathcal{A}}(x_1, ..., x_n)$ for some $x_1, ..., x_n \in M$ then we can create an embedding $f : \mathcal{A} \to \mathcal{M}$ which maps a_i to x_i. This induces a substructure $\mathcal{A}' \subseteq \mathcal{M}$ such that $\mathcal{A}' \cong \mathcal{A}$

Definition 2.6. Let $\mathbf{K}$ be a set of L-structures such that $\mathbf{K}$ is a pregeometry. Assume that the structures $\mathcal{A}, \mathcal{B}$ are isomorphic to some structures in $\mathbf{K}$ and that $\mathcal{A} \subseteq \mathcal{B}$ and with universes $A = \{a_1, ..., a_\alpha\}, B = \{a_1, ..., a_\alpha, b_{\alpha+1}, ..., b_\beta\}$ where $\alpha < \beta$.
Then we say that a L-structure $\mathcal{M}$ satisfies the **$\mathcal{B}/\mathcal{A}$-extension axiom** if:

> For each embedding $\tau : \mathcal{A} \to \mathcal{M}$ there exists an embedding $\pi : \mathcal{B} \to \mathcal{M}$ such that π extends τ i.e. $\forall a \in A,\ \tau(a) = \pi(a)$.

This may be expressed by the following first order formula

$$\forall x_1, ..., x_\alpha \exists x_{\alpha+1}, ..., x_\beta (\chi_{\mathcal{A}}(x_1, ..., x_\alpha) \to \chi_{\mathcal{B}}(x_1, ..., x_\alpha, x_{\alpha+1}, ..., x_\beta)).$$

In the case $\mathcal{A}$ has universe $A = \emptyset$ and if $\mathcal{A} \in \mathbf{K}$ then the $\mathcal{B}/\mathcal{A}$ extension axiom will be simply

$$\exists x_1, ..., x_\beta (\chi_{\mathcal{B}}(x_1, ..., x_\beta)).$$

In the case of $\dim_{\mathcal{B}}(B) \leq k + 1$ then we may call the $\mathcal{B}/\mathcal{A}$-extension axiom a k-extension axiom. We say that a structure $\mathcal{M}$ has the ***k*-extension property** if it satisfies all k-extension axioms.

We will now do some groundwork in order to define the dimension conditional measure which is the probability measure which we will use in this thesis when proving a zero-one law. The following definition of weak substructures is necessary for future definitions.

Definition 2.7. Left $\mathcal{M}$ be a L-structure. Then the L-structure $\mathcal{N}$ is a **weak substructure** of $\mathcal{M}$, written $\mathcal{N} \subseteq_w \mathcal{N}$ if $N \subseteq M$ and the following is satisfied:

- for each constant symbol c, $c^{\mathcal{N}} = c^{\mathcal{M}}$.
- for each function symbol f and each tuple $\bar{a} \in N$, $f^{\mathcal{M}}(a_1, ..., a_n) = b$ iff $f^{\mathcal{N}}(a_1, ..., a_n) = b$.
- for each relation symbol R, if $\bar{a} \in R^{\mathcal{N}}$ then $\bar{a} \in R^{\mathcal{M}}$.

If $\mathcal{M}$ is a structure with a pregeometry and $d \in \mathbb{N}$ then we define the **d-dimensional reduct** of $\mathcal{M}$, denoted $\mathcal{M} \upharpoonright d$, as the weak substructure of $\mathcal{M}$ satisfying the following:

1. $\mathcal{M} \upharpoonright d$ has the same universe as $\mathcal{M}$.
2. For each relation symbol R we have that
$$\bar{a} \in R^{M \upharpoonright d} \Leftrightarrow \dim_{\mathcal{M}}(\bar{a}) \leq d \text{ and } \bar{a} \in R^{\mathcal{M}}.$$
3. For each constant symbol c and each function symbol f, $c^{\mathcal{M} \upharpoonright d} = c^{\mathcal{M}}$ and $f^{\mathcal{M} \upharpoonright d} = f^{\mathcal{M}}$.

For a set of structures $\mathbf{K}$ we define $\mathbf{K} \upharpoonright d = \{\mathcal{M} \upharpoonright d : \mathcal{M} \in \mathbf{K}\}$.

Notice that if $\mathbf{K}$ is a set of L-structures and ρ is the highest arity among all relation symbols in V then for each $\mathcal{M} \in \mathbf{K}$, $\mathcal{M} \upharpoonright \rho = \mathcal{M}$ and hence $\mathbf{K} \upharpoonright \rho = \mathbf{K}$.

The **uniform probability measure** μ on a set of structures $\mathbf{K}_n$ is defined as $\mu(\mathcal{M}) = \frac{1}{|\mathbf{K}_n|}$, so each structure has the same probability. We will now recursively define the dimension conditional measure.

Definition 2.8. Assume that for each $n \in \mathbb{N}$, $\mathbf{K}_n$ is a set of L-structures with the same universe such that $\mathbf{K} = \cup_{n=1}^{\infty} \mathbf{K}_n$ is a pregeometry and all structures in $\mathbf{K}_n$ have dimension n. Also assume that ρ is the highest arity among all relation symbols in V. Define $\mathbb{P}_{n,0}$ as the uniform probability measure on $\mathbf{K}_n \restriction 0$. For each $1 \leq r \leq \rho$, $n \in \mathbb{N}$ and $\mathcal{M} \in \mathbf{K}_n \restriction r$ define

$$\mathbb{P}_{n,r}(\mathcal{M}) = \frac{1}{|\{\mathcal{M}' \in \mathbf{K}_n \restriction r : \mathcal{M}' \restriction r-1 = \mathcal{M} \restriction r-1\}|} \cdot \mathbb{P}_{n,r-1}(\mathcal{M} \restriction r-1).$$

We say that $\delta_n^{\mathbf{K}} = \mathbb{P}_{n,\rho}$ is the **dimension conditional measure on $\mathbf{K}_n$**. If $\mathbf{K}_n$ is obvious we will just write δ_n.

This definition needs an example, but in order to really put it in context we'll first define what a coloured structure is, since this is what we'll use mostly in this thesis. But first some notation in order to do this properly.

Assumption 2.9. *From now on in this thesis we assume the following*

- *For each $n \in \mathbb{N}$, all structures of size n have the universe $\{1, ..., n\}$, and if $\mathcal{M}$ is a countable infinite structure then we assume that $\mathcal{M}$ has the universe $\mathbb{N}$.*
- *F is a finite field and V_F is the vocabulary of a vector space over that finite field, which contain a function symbol for each element in F representing scalar multiplication with that element, a binary function symbol $'+'$ representing vector addition and a constant symbol 0 for the zero vector. L_F is the induced language from V_F.*
- *For each $n \in \mathbb{N}$, $\mathcal{G}_n$ is the n-dimensional vector space over F (viewed as an L_F structure) such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$. Let $\mathcal{G} = \cup_{i=0}^{\infty} \mathcal{G}_n$ and notice that $\mathcal{G}$ is a pregeometry with $cl_{\mathcal{G}}$ being the linear span.*
- *Let $V_{rel} \supseteq V_F$ be a vocabulary such that $V_{rel} - V_F$ is non-empty, finite and contains only relations of arity two or higher. Let the language for V_{rel} be L_{rel}.*
- *Fixate an integer $l \geq 2$ and define the vocabulary $V = V_{rel} \cup \{P_1, ..., P_l\}$ with corresponding language L.*
- *Whenever we have a L-, L_{rel}- or L_F-structure $\mathcal{M}$ we assume that $\mathcal{M} \restriction L_F = \mathcal{G}_n$ for some n, and hence since $\mathcal{G}_n$ is a pregeometry, $\mathcal{M}$ is also a pregeometry.*

Observe that for each $k \in \mathbb{N}$, there is an L_F-formula $\theta_k(x_1, ..., x_{k+1})$ such that for every n and all $a_1, ..., a_{k+1} \in G_n, a_{k+1}$ belongs to the linear span of $a_1, ..., a_k$ if and only if $\mathcal{G}_n \models \theta_k(a_1, ..., a_{k+1})$.

The following definition is a bit more general than the naive thought of colouring, so put into less generalisation it does make very much sense. In the specific case of an undirected graph with trivial pregeometry (that is $cl(X) = X$ for all X) this definition is exactly the same as that of a coloured map of the world, where you want neighbouring countries to have different colour.

Definition 2.10. We say that a L-structure $\mathcal{M}$ which is a pregeometry is **weakly l-coloured** if it satisfies the following:

1. $\mathcal{M} \models \forall x(P_1(x) \vee ... \vee P_l(x))$

2. For all $i, j \in \{1, ..., l\}$ such that $i \neq j$ and all $a, b \in M - cl(\emptyset)$ such that $a \in cl(b)$ we have that $\mathcal{M} \models \neg(P_i(a) \wedge P_j(b))$ i.e. linearly dependent elements have the same colour.

3. If $R \in V_{rel}$ has arity $m \geq 2$ and $\mathcal{M} \models R(a_1, ..., a_m)$ then there are $b, c \in cl(a_1, ..., a_m)$ such that for every $k \in \{1, ..., l\}$ we have $\mathcal{M} \models \neg(P_k(b) \wedge P_k(c))$.

If $\mathcal{M}$, instead of satisfying 3, satisfies the following axiom, then we call $\mathcal{M}$ **strongly $l-$coloured**

3′. If $R \in V_{rel}$ has arity $m \geq 2$ and $\mathcal{M} \models R(a_1, ..., a_m)$ then for all $b, c \in cl(a_1, ..., a_m)$ which are linearly independent ($b \notin cl(c)$) and for each $k \in \{1, ..., l\}$ we have $\mathcal{M} \models \neg(P_k(b) \wedge P_k(c))$. Also for each $R \in V_{rel}$ we have that if $x \in cl(\emptyset)$ then $\mathcal{M} \models \neg R(x, ..., x)$.

If it is obvious if we talk about a strongly or weakly l-coloured structure we might just say that $\mathcal{M}$ is a l-coloured structure. In section 3 and 4 it will be obvious which kind of coloured structure we are using and in section 5 it won't really matter, as we will explain later.

From the following example we see that the dimension conditional measure isn't the same as the uniform measure and the reason why we still use it becomes a bit more apparent. The dimension conditional measure is a measure which takes notice of how "easy" it is to generate a certain structure from scratch. That is, if you build your structure from a universe and up by first adding relations to 0 dimensional elements, then to 1 dimensional etc. it tells how probable it is that you end up with a certain structure.

Example 2.11. Choose as pregeometry the vector space pregeometry over $\mathbb{Z}_2$ and consider the case $\mathcal{G}_2 = \mathbb{Z}_2^2$. Let $\mathbf{K}_2$ be the set of all weakly 2-coloured structures (up to isomorphism) with vocabulary $V_F \cup \{P_1, P_2\} \cup \{R\}$ where R is a binary relation symbol and with vector space $\mathcal{G}_2$. There are exactly 114 different structures in $\mathbf{K}_2$, so if $\mathcal{M} \in \mathbf{K}_2$ is the structure which is mono-coloured with P_1 and having $R^{\mathcal{M}} = \emptyset$, then using the uniform probability measure we get $\mu(\mathcal{M}) = \frac{1}{114}$. If we want to calculate $\delta^{\mathbf{K}_2}(\mathcal{M})$ we first need to calculate $\mathbb{P}_{2,0}(\mathcal{M})$ which equals $\frac{1}{2}$. This is because we calculate $\mathbb{P}_{2,0}$ from the uniform measure on $\mathbf{K}_2 \upharpoonright 0$, and there are only two structures in there, one with colour P_1 and one with colour P_2 on the zero element. When we then continue on to $\mathbb{P}_{2,1}(\mathcal{M})$ we look at the structures in $\mathbf{K}_2 \upharpoonright 1$ i.e. with colours added and relations over sets of dimension 1. The number of structures in $\mathbf{K}_2 \upharpoonright 1$ is equal to 20, and the number of them who have the same colour (so $\mathcal{M} \upharpoonright 0 = \mathcal{N} \upharpoonright 0$ for those structures) on its zero element as $\mathcal{M}$ equals 10. Because of this, we can draw the following conclusion

$$\mathbb{P}_{2,1}(\mathcal{M}) = \frac{1}{|\{\mathcal{M}' \in \mathbf{K}_2 \upharpoonright 1 : \mathcal{M}' \upharpoonright 0 = \mathcal{M} \upharpoonright 0\}|} \cdot \mathbb{P}_{2,0}(\mathcal{M}) = \frac{1}{10} \cdot \frac{1}{2} = \frac{1}{20}.$$

The last step, to calculate $\delta^{K_2}(\mathcal{M}) = \mathbb{P}_{2,2}(\mathcal{M})$ is pretty easy, since the only structure in $\mathbf{K}_2 \upharpoonright 2 = \mathbf{K}_2$ which has the same colouring as $\mathcal{M}$ is $\mathcal{M}$ itself. Hence

$$\mathbb{P}_{2,2}(\mathcal{M}) = \frac{1}{|\{\mathcal{M}' \in \mathbf{K}_2 \upharpoonright 2 : \mathcal{M}' \upharpoonright 1 = \mathcal{M} \upharpoonright 1\}|} \cdot \mathbb{P}_{2,1}(\mathcal{M}) = \frac{1}{1} \cdot \frac{1}{20} = \frac{1}{20}.$$

We conclude this example by observing that $\delta^{K_2}(\mathcal{M}) = \frac{1}{20} \neq \frac{1}{114} = \mu(\mathcal{M})$.

The notion of being coloured may be abstracted out of a structure, in which case we get a colourable structure instead.

Definition 2.12. Let $\mathcal{A}$ be a L_{rel}-structure and let $\gamma : A \to \{1, ..., l\}$. We say that a tuple $\bar{a} = (a_1, ..., a_n) \in A^n$ is γ-**monochromatic** if $\gamma(a_1) = ... = \gamma(a_n)$. If $\bar{a}$ is not γ-monochromatic then it is γ-**multichromatic**, and in case $\gamma(a_i) \neq \gamma(a_j)$ for each $i \neq j$ such that $a_i \notin cl(a_j)$ then we call $\bar{a}$ **strongly** γ-**multichromatic**. If γ is obvious we just say monochromatic, multichromatic or strongly multichromatic. We say that γ is a (**strong**) l-**colouring** of $\mathcal{A}$ if it satisfies the following properties:

1. If $\bar{a} = cl(x)$ for some $x \in A$ then $\bar{a}$ is γ-monochromatic.
2. If $R \in V_{rel}$ and $\mathcal{A} \models R(\bar{a})$ then $cl(\bar{a})$ is (strongly) γ-multichromatic.

A structure is said to be (**strongly**) l-**colourable** if there exists a (strong) l-colouring of it. For notational simplicity, we may exclude writing strongly if it is obvious from the context, and we may write colourable without specifying the l if the l is obvious.

Remark 2.13. Each (strongly) l-coloured L-structure $\mathcal{M}$ is (strongly) l-colourable by defining a l-colouring $\gamma : M \to \{1, ..., l\}$ by $\gamma(a) = i$ if and only if $\mathcal{M} \models P_i(a)$. In the same way each l-colourable L_{rel}-structure can become l-coloured by expanding it to an L-structure by letting $P_i(a)$ be true if and only if $\gamma(a) = i$ for $i \in \{1, ..., l\}$ and $a \in M$. The big difference is that a coloured structure has the colouring in the definition of the structure, while a colourable structure just can be coloured with a "Meta colouring", which isn't necessarily known when inside the structure. This is a big difference, but what we are going to show in Sections 3 and 4 is that in a coloured structure, with sufficiently strong "extension properties", we can actually tell which elements have the same colours, without mentioning the colours. Hence we can also do this for colourable structures, since a colourable structure is the same as a coloured structure in which you can't mention the colours.

3 Strong l-colourings

In this section we'll let $\mathbf{K}_n$ be the set of all strongly l-coloured L-structures $\mathcal{M}$ such that $\mathcal{M} \upharpoonright L_F = \mathcal{G}_n$ and $\mathbf{K} = \bigcup_{i=1}^{\infty} \mathbf{K}_n$ is a pregeometry. The expression $\mathcal{N}$ is **represented with respect to K** will be used in this section, and means that $\mathcal{N} \cong \mathcal{M}$ for some $\mathcal{M} \in \mathbf{K}$, and if the set $\mathbf{K}$ is obvious, we may just say $\mathcal{N}$ is **represented**. Let $T(s) = \frac{|F|^s - 1}{|F| - 1}$ which is the number of 1-dimensional subspaces of F^s (an s-dimensional F-vector space) and put $t = \max\{m \in \mathbb{N} : T(m) \leq l\}$. That is we let t be the highest dimension we can have on a vector space and still have relations between any pair of elements in its basis in a strongly l-coloured structure. We'll assume that $t \geq 2$ that is $|F| + 1 \leq l$ which is motivated by the following example.

Example 3.1. Assume that $F = \mathbb{Z}_2$ and that $l = 2$, which gives us that $T(1) = 1$ and $T(2) = 3$ so $t = 1$. Also assume that $\mathcal{M}$ is a l-coloured L-structure such that $\mathcal{M} \upharpoonright L_F = \mathbb{Z}_2^2$. Then for any relation $R \in V_{rel}$ if $\mathcal{M} \models R(a_1, ..., a_r)$ for $a_1, ..., a_r \in M$, we know that the number of different one-dimensional subspaces in $cl(a_1, ..., a_r)$ has to be less than or equal to l, since each one-dimensional subspace has a different colour. But since $3 = T(2) > l = 2$ we get that $cl(a_1, ..., a_r)$ has to be one dimensional.

So we assume that $t \geq 2$ in order to not only get uninteresting structures where the relations do not interact at all with the colouring.

We'll be proving that for strong l-coloured structures (with the k-extension property), there is a L_{rel} formula $\xi(x,y)$ which describes the colouring. This will be done by creating a formula which induces relations between l independent elements and so by the definition of strong colourings, they will all need to have different colours. Using this reasoning twice, we'll prove in Lemma 3.3 that $\xi(a,b)$ forces the same colour on a and b. To prove that each pair (a,b) which have the same colour satisfies ξ we'll use the k-extension property, for a large enough k, to show in Lemma 3.4 that there exist elements with the required relations to satisfy ξ. These two results are then combined into Corollary 3.5, which say that ξ is true iff the two elements have the same colour.

To justify that we use cl in the following definition, remember from Assumption 2.9 that there is a formula $\theta_2(x_1,x_2) \in L_{rel} \subseteq L_F$ such that for each $M \in \mathbf{K}$ and $a,b \in \mathcal{M}$, $\mathcal{M} \models \theta_2(b,a) \Leftrightarrow a \in cl_{\mathcal{M}}(b)$. Let the relation symbols of V_{rel} be $R_1,...,R_\rho$ with arities $r_1,...,r_\rho \geq 2$. Assume that r_1 is the smallest among these arities. In the following definition we will use the notation $\underset{i=1}{\overset{l}{\exists}}\underset{j=1}{\overset{r_1}{\exists}} x_{i,j}$ which is the same as saying $\exists x_{1,1}...\exists x_{1,r}\exists x_{2,1}...\exists x_{2,r}...\exists x_{l,1}...\exists x_{l,r_1}$.

Definition 3.2. Define the L_{rel}-formula ξ as follows:

$$\xi(x,y) \quad \equiv \quad x \in cl(y) \ \vee \ y \in cl(x) \ \vee \exists y_2,...,y_l \underset{i=2}{\overset{l}{\exists}} \underset{j=1}{\overset{r_1-2}{\exists}} z_{(x,i,j)} \underset{i=2}{\overset{l}{\exists}} \underset{j=1}{\overset{r_1-2}{\exists}} z_{(y,i,j)} \underset{k=2}{\overset{l}{\exists}} \underset{i=2}{\overset{k-1}{\exists}} \underset{j=1}{\overset{r_1-2}{\exists}} z_{(k,i,j)}$$

$$\bigwedge_{i=2}^{l} \Big(R_1(x, y_i, z_{(x,i,1)}, ..., z_{(x,i,r_1-2)}) \wedge y_i \notin cl(x) \wedge R_1(y, y_i, z_{(y,i,1)}, ..., z_{(y,i,r_1-2)}) \wedge y_i \notin cl(y)$$

$$\bigwedge_{j=2}^{i-1} [R_1(y_i, y_j, z_{(i,j,1)}, ..., z_{(i,j,r_1-2)}) \wedge y_i \notin cl(y_j)] \wedge y_l \notin cl(\emptyset) \Big).$$

All the elements $z_{(k,i,j)}, z_{(x,i,j)}$ and $z_{(y,i,j)}$ will mostly be fillers to get the relation defined and not really important (yet necessary). In the case $r_1 = 2$ they won't even exist and ξ will look like this

$$\xi(x,y) \quad \equiv \quad x \in cl(y) \ \vee \ y \in cl(x) \ \vee \exists y_2,...,y_l \bigwedge_{i=2}^{l} \Big(R_1(x,y_i) \wedge y_i \notin cl(x) \wedge R_1(y,y_i) \wedge y_i \notin cl(y)$$

$$\bigwedge_{j=2}^{i-1} (R_1(y_i,y_j) \wedge y_i \notin cl(y_j)) \wedge y_l \notin cl(\emptyset) \Big).$$

Notice that all the independence clauses are needed, since in a strong l-colouring two elements can be in the same one-dimensional span but still related.

The following lemma will give us that each pair of elements which satisfies ξ has the same colour.

Lemma 3.3. *If $\mathcal{M} \in \boldsymbol{K}_n$, $a,b \in M - cl(\emptyset)$ and $\mathcal{M} \models \xi(a,b)$ then a and b has the same colour in $\mathcal{M}$, i.e. for some $i = 1,...,l$ we have $\mathcal{M} \models P_i(a) \wedge P_i(b)$.*

Proof. We assume that $\mathcal{M} \models \xi(a,b)$ and $a,b \notin cl(\emptyset)$. If $a \in cl(b)$ then we obviously are done by the definition of a colouring, hence assume that a and b are independent. Each y_i must have a different colour from a since they are independent, included in a tuple $(a, y_i, z_{(a,i,1),...,z_{(a,i,r_1-2)}}) \in R_1$ and we are using strong colourings. In the same way each y_i must have a different colour from b. From the second part of ξ we get, in the same way as for a and b, that y_i and y_j must have different colour in $\mathcal{M}$. Hence we can conclude that all the elements $a, y_2, ..., y_l$ have different colours and all the elements $b, y_2, ..., y_l$ have different colours. But since $\mathcal{M}$ is coloured by only l different colours this implies, by the pigeon hole principle, that a and b must have the same colour. $\square$

For the rest of this thesis let $k_0 = 2 + (t-1)(l-1)$. We'll now move on to try and prove the other direction for ξ, that is, for each structure with the k_0-extension property, elements with the same colour will satisfy ξ. This will be done by creating a structure $\mathcal{B}$ which has the same relations as described in ξ, and showing that this structure is strongly l-colourable. We will create $\mathcal{B}$ as big as possible (by using t), but still l-colourable, in order to get an as general proof as possible. Then using the extension property we'll show that if a and b have the same colour in a structure with the k_0-extension property, then they are included in a copy of $\mathcal{B}$ in such a way that, by construction of $\mathcal{B}$, $\xi(a, b)$ holds.

Lemma 3.4. *Let $\mathcal{M} \in \boldsymbol{K}_n$, $a, b \in M - cl(\emptyset)$, assume that $\mathcal{M}$ has the k_0-extension property and that $\mathcal{M} \models P_i(a) \wedge P_i(b)$ for some $i \in \{1, ..., l\}$. Then $\mathcal{M} \models \xi(a, b)$.*

Proof. Without loss of generality we may assume that $\mathcal{M} \models P_1(a) \wedge P_1(b)$. If $a \in cl(b)$ then $\mathcal{M} \models \xi(a, b)$ by definition, hence assume that $a \notin cl(b)$. Let $\mathcal{A} = \mathcal{M} \upharpoonright cl(a, b)$ and choose elements $v_{2,1}, ..., v_{2,t-1}, ..., v_{l,1}, ..., v_{l,t-1} \in \mathcal{M}$ such that $\{a, b, v_{2,1}, ..., v_{l,t-1}\}$ is an independent set. Let $\mathcal{B}_0$ be the L_F-structure (i.e. vector space pregeometry) which is spanned by $\{a, b, v_{2,1}, ..., v_{l,t}\}$, and so that $\mathcal{A} \upharpoonright L_F \subseteq \mathcal{B}_0$. Define $\mathcal{B}$ to be the structure which is created by expanding $\mathcal{B}_0$ to an L-structure in the following way. We know already that $\mathcal{A} \upharpoonright L_F \subseteq \mathcal{B} \upharpoonright L_F$, so for each $i \in \{1, ..., \rho\}$ every $R_i \in V - V_F$, and every $\bar{a} \in A$, assign $R_i^{\mathcal{B}}(\bar{a}) \Leftrightarrow R_i^{\mathcal{A}}(\bar{a})$ and for each $j \in \{1, ..., l\}$ let $P_j^{\mathcal{B}}(x) \Leftrightarrow P_j^{\mathcal{A}}(x)$ for each $x \in A$. In this way we obviously get that $\mathcal{A} \subseteq \mathcal{B}$ as L-structures, no matter how we define the rest of $\mathcal{B}$. For every $i \in \{2, ..., \rho\}$, relation symbol $R_i \in V_{rel} - \{R_1\}$ and $\bar{c} \in B^{r_i} - A^{r_i}$ let $\mathcal{B} \not\models R(\bar{c})$. For each $i \in \{1, ..., l\}$ and $i < j \leq l$ fix arbitrary elements $w_{(a,i,1)}, ..., w_{(a,i,r_1-2)} \in cl(a, v_{i,1}, ..., v_{i,t-1})$ and $w_{(b,i,1)}, ..., w_{(b,i,r_1-2)} \in cl(b, v_{i,1}, ..., v_{i,t-1})$ and $w_{(j,i,1)}, ..., w_{(j,i,r_1-2)} \in cl(v_{j,1}, v_{i,1}, ..., v_{i,t-1})$ to assign $R_1^{\mathcal{B}}$ such that

$$\mathcal{B} \models R_1(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)}) \wedge R_1(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)})$$

$$\bigwedge_{k=2}^{i} R_1(v_{k,1}, v_{i,1}, w_{(k,i,1)}, ..., w_{(k,i,r_1-2)}),$$

and such that $R_1^{\mathcal{B}}$ holds for no other tuples than those indicated above.

Let the set

$$S_i' = \{Q \subseteq cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r-2)}) : Q \text{ is a one dimensional closed subspace of } \mathcal{B}\},$$

then:

- If $i \leq |S_i'|$ then let $S_i = \{Q_p | Q_p \in S'\}_{1 \leq p \leq |S_i'|}$ be an enumeration of S_i' with $cl(a) = Q_1$ and $cl(v_{i,1}) = Q_i$. Assign colour to the subspaces in S_i by for each $p \in \{1, ..., |S_i'|\}$ and each $x \in Q_p - cl(\emptyset)$ let $\mathcal{B} \models P_p(x)$.
- If $i > |S_i'|$ then let $S_i = \{Q_p | Q_p \in S' - cl(v_{i,1})\}_{1 \leq p \leq |S_i'|-1}$ be an enumeration of $S_i' - (cl(v_{i,1}))$ with $cl(a) = Q_1$. Assign colour to the subspaces in S_i by for each $p \in \{1, ..., |S_i'| - 1\}$ and each $x \in Q_p - cl(\emptyset)$ let $\mathcal{B} \models P_p(x)$. Then for each $x \in cl(v_{i,1})$ let $\mathcal{B} \models P_i(x)$.

In the same way colour the span $cl(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)})$ but let $Q_1 = cl(b)$. Using the same method, assign colour to $cl(v_{k,1}, v_{i,1}, w_{(k,i,1)}, ..., w_{(k,i,r_1-2)})$ where the elements in $cl(v_{k,1})$ gets the colour P_k and the elements in $cl(v_{i,1})$ gets the colour P_i. The reason why we assign colours to $\mathcal{B}$ in this way is to get well coloured relations, w.r.t. strong colourings, and giving the spans $cl(v_{i,1})$,$cl(a)$ and $cl(b)$ their correct colour. Assign to any element $x \in B$, which yet has no colour, the colour $P_1(x)$, notice that these elements aren't in any relations nor in $\mathcal{A}$ and hence their colours don't matter.

Claim. The L-structure $\mathcal{B}$ is a strongly l-coloured structure.

Proof of claim. By the last part of the definition of the colouring we know that each element has attained at least one colour, so colouring condition one is satisfied. If we use the intersections

$$cl(v_{k,1}, v_{i,1}, w_{(k,i,1)}, ..., w_{(k,i,r_1-2)}) \cap cl(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)})$$
$$cl(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)}) \cap cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)})$$
$$cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)}) \cap cl(a, v_{j,1}, w_{(a,j,1)}, ..., w_{(a,j,r_1-2)})$$
$$cl(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)}) \cap cl(b, v_{j,1}, w_{(b,j,1)}, ..., w_{(b,j,r_1-2)})$$
$$cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)}) \cap cl(v_{k,1}, v_{i,1}, w_{(k,i,1)}, ..., w_{(k,i,r_1-2)}) \text{ and}$$
$$cl(v_{k,1}, v_{i,1}, w_{(k,i,1)}, ..., w_{(k,i,r_1-2)}) \cap cl(v_{j,1}, v_{i,1}, w_{(j,i,1)}, ..., w_{(j,i,r_1-2)})$$

in Lemma 2.3 we get that they actually consist of $cl(v_{i,1})$, $cl(v_{i,1})$, $cl(a)$, $cl(b)$, $cl(v_{i,1})$ and $cl(v_{i,1})$. But these closures have by definition specified unique colours and hence didn't get multiple colours defined for them. The colours on $\mathcal{A} \subseteq \mathcal{B}$ do, since $\mathcal{A}$ is a coloured structure, satisfy the required properties. All other elements are only spoken of once in the definition of the colouring on $\mathcal{B}$ and hence can only have one colour. If $(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)}) \in R_1^{\mathcal{B}}$ then by the definition of our colouring, all the one-dimensional linear spans in $cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)})$ have different colours, hence if $x, y \in cl(a, v_{i,1}, w_{(a,i,1)}, ..., w_{(a,i,r_1-2)})$ and are independent, then x and y have different colour. The same reasoning is true for $cl(v_{j,1}, v_{i,1}, w_{(j,i,1)}, ..., w_{(j,i,r_1-2)})$ and $cl(b, v_{i,1}, w_{(b,i,1)}, ..., w_{(b,i,r_1-2)})$ too and these indicated r_1-tuples are the only ones in $R_1^{\mathcal{B}}$. In the case $\bar{a} \in A^{r_i}$ and $\mathcal{B} \models R_i(\bar{a})$ for some $i \in \{1, ..., \rho\}$ and $R_i \in V_{rel}$ we already know that $\mathcal{A}$ is a l-coloured structure, so the strong colouring conditions are satisfied. Since we know that $\mathcal{B} \not\models R_i(\bar{x})$ for each $i \in \{2, ..., \rho\}$ and $\bar{x} \in B - A$, we have now checked all conditions and hence $\mathcal{B}$ is strongly l-coloured. □

Continuing the proof of Lemma 3.4. By the claim, $\mathcal{B}$ is a closed strongly l-coloured L-structure such that $\mathcal{A} \subseteq \mathcal{B}$ and since $B = cl(a, b, v_{2,1}, ..., v_{l,t})$ we know that $\dim(B) = 2 + (t-1)(l-1) = k_0$. Hence since $\mathcal{M}$ has the k_0-extension property and $\dim(B) \leq k_0$, we get that there is $\mathcal{B}' \cong \mathcal{B}$, $\mathcal{B}' \subseteq \mathcal{M}$ and isomorphism $f : \mathcal{B}' \to \mathcal{B}$ with which extends the identity function on $\mathcal{A}$. So since we know that $\mathcal{B} \models \xi(a, b)$, by the definition of $\mathcal{B}$, we get that $\mathcal{M} \models \xi(a, b)$. □

Using Lemmas 3.3 and 3.4 we directly get the following important corollary

Corollary 3.5. *If $\mathcal{M}$ is a strongly l-coloured structure with the k_0-extension property and $a, b \in M - cl(\emptyset)$ then*

$$\mathcal{M} \models \xi(a, b) \qquad \Leftrightarrow \qquad \mathcal{M} \models P_i(a) \wedge P_i(b) \textit{ for some } i \in \{1, ..., l\}.$$

4 Weak l-colourings

In this section we'll let $\mathbf{K}_n$ be the set of all weakly l-coloured L-structures $\mathcal{M}$ such that $\mathcal{M} \upharpoonright L_F = \mathcal{G}_n$ and $\mathbf{K} = \bigcup_{i=1}^{\infty} \mathbf{K}_n$ is a pregeometry. We will prove that there is a L_{rel}-formula $\xi(x,y)$ which is true in a weakly coloured structure $\mathcal{M}$, with the k-extension property for some k, if and only if x and y has the same colour. To prove this we'll need to use a result of Ramsey theory, that is, we need a theorem which says that if we have a "big enough" coloured vector space, then there will be a subspace of that vector space which is only coloured in one single colour. This theorem was first conjectured by Rota [7], as a more specific version of Ramseys classical colouring theorem, and was later proved by Graham, Leeb and Rothschild [6]. To the "big enough" vector space we'll then add relations while adding colours to it as strict as possible, in order to really fix how the vector space has to be coloured. Then let $\xi_0(x,y)$ be the formula, to be defined, such that it expresses that $\mathcal{M}$ has this vector space in it with x and y in a subspace which is coloured by only one colour, hence x and y must have the same colour. This is proved in Lemma 4.3. Using the k-extension property, Lemma 4.4 shows that if x and y have the same colour, then we can find a structure $\mathcal{D} \subseteq \mathcal{M}$ which has two mono coloured subspaces of the same colour which intersect in a point u and has x in one subspace and y in the other. This will conclude that ξ_0 is satisfied by both (x,u) and (y,u). Now the formula $\xi(x,y)$ which we are looking for can be created by taking the conjunction of these instances of ξ_0 (and finally existentially quantifying over u), so all three x, u and y have the same colour, which will be concluded in Corollary 4.8.

If c is a l-colouring of a vector space $\mathcal{V}$ then a subspace $\mathcal{W} \subseteq \mathcal{V}$ will be called c-**monochromatic** if all vectors (except possibly the zero vector) are assigned the same colour by c. If $\mathcal{W} \subseteq \mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{U}$ is also a c-monochromatic vector space implies that $\mathcal{U} = \mathcal{W}$ then we will call $\mathcal{W}$ **maximal** c-**monochromatic**. That is, $\mathcal{W}$ is maximal c-monochromatic if it isn't contained in any bigger c-mono-chromatic subspace of $\mathcal{V}$. The following theorem is a consequence of a theorem by Graham, Leeb and Rothschild [6] and is also depending on which field we are using, but since it is fixed for this thesis we exclude that part.

Theorem 4.1. *For each $d, l \in \mathbb{N}$ there is a number $N(d,l) \in \mathbb{N}$ such that if $n \geq N(d,l)$, $\mathcal{V}$ is an n-dimensional F-vector space and c is a l-colouring of all 1-dimensional vector spaces of $\mathcal{V}$ then there exists at least one subspace of $\mathcal{V}$ with dimension at least d which is monochromatic.*

Let $n = N(2,l)$ for $N(d,l)$ in the above theorem, let $\mathcal{V} = \mathcal{G}_n$, where G_n is an n-dimensional F-vector space as assumed in Assumption 2.9, and let c be an l-colouring of $\mathcal{V}$. By our choice of n and Theorem 4.1 there exists at least one c-monochromatic subspace of $\mathcal{V}$ of dimension at least two and hence there must also exists at least one maximal c-monochromatic subspace of $\mathcal{V}$. Let $\mathcal{W}_1^c, ..., \mathcal{W}_{t(c)}^c$ be all the maximal c-monochromatic subspaces of $\mathcal{V}$ of dimension at least two, where $t(c)$ is a number depending on which colouring c we choose. Define a set $C = \{c : V \to \{1, ..., l\} : c \text{ is a } l\text{-colouring of } \mathcal{V}\}$. For each l-colouring c choose a basis $\{d_1, ..., d_{e_c}\}$ for the set $\bigcup_{i=1}^{t(c)} \mathcal{W}_i$ which has a colouring specific dimension e_c, then let $e = \min\{e_c | c \in C\}$. Choose $c_0 \in C$ such that $e_{c_0} = e$ and for each other l-colouring $c \in C$ with $e_c = e$ we have that $t(c) \leq t(c_0)$.

For this colouring c_0 let $m = t(c_0)$ and call $\mathcal{W}_1 = \mathcal{W}_1^{c_0}, ..., \mathcal{W}_m = \mathcal{W}_{t(c)}^{c_0}$. Assume that the relation symbol $R \in V_{rel}$ have the least arity r in V_{rel}, so $r \geq 2$. Let $\mathcal{B}$ be the expansion of $\mathcal{V}$ to the language L_{rel} defined by, for each relation symbol $Q \in V_{rel} - \{R\}$, assigning $Q^{\mathcal{B}} = \emptyset$ and defining $R^{\mathcal{B}}$ in the following way:

- If $v_1, v_2, ..., v_r \in W_i$ for some $i = 1, ..., m$ or if $v_2, ..., v_r \in cl(v_1)$ then $\mathcal{B} \models \neg R(v_1, ..., v_r)$.

- If $\{v_1, ..., v_r\} \nsubseteq W_i$ for each $i = 1, ..., m$ and there is some $j = 2, ..., r$ so that $v_j \notin cl(v_1)$ then $\mathcal{B} \models R(v_1, ..., v_r)$.

Notice that the second case requirement is the opposite of the first case requirement, so $\mathcal{B}$ is unambiguously defined.
Let $a, b \in \mathcal{B} \restriction W_1 \subseteq \mathcal{B}$ be independent (notice that they exist because of Theorem 4.1) and create the structure $\mathcal{A} = \mathcal{B} \restriction cl(\{a, b\})$. Remember that the characteristic formula $\chi_{\mathcal{B}}(x_1, ..., x_\beta)$ is the formula which classifies the isomorphism-class of the finite structure $\mathcal{B}$, choose it so that $\mathcal{B} \models \chi_{\mathcal{B}}(a, b, x_3, ..., x_\beta)$. Then we define the formula ξ_0 as

$$\xi_0(v, w) \qquad \Leftrightarrow \qquad \exists x_3, ..., x_\beta \chi_{\mathcal{B}}(v, w, x_3, ..., x_\beta).$$

This formula ξ_0 isn't the real colour classification formula ξ which we are looking for, but it is going to be a part of it. The following lemma, which says that $\mathcal{B}$ may be weakly l-coloured, will be very important when using $\mathcal{B}$, especially in Lemma 4.5. The proof is pretty straightforward, just confirming that everything in Definition 2.10 is satisfied.

Lemma 4.2. *The L_{rel}-structure $\mathcal{B}$ may be coloured as a structure by the colouring c_0. That is, there exists a structure $\mathcal{B}_0 \in \boldsymbol{K}$ with such that $\mathcal{B}_0 \restriction L_{rel} \cong \mathcal{B}$ and which is coloured as c_0 describes.*

Proof. Define a L-structure $\mathcal{B}_0$ by putting colour on $\mathcal{B}$ through the c_0-colouring. If $x \in B$ and $c_0(x) = i$ then let $\mathcal{B}_0 \models P_i(x) \bigwedge_{j \neq i} \neg P_j(x)$. This gives us that $\mathcal{B}_0 \restriction L_{rel} \cong \mathcal{B}$ so we just need to prove the following claim to prove the whole lemma.

Claim. $\mathcal{B}_0$ is represented with respect to $\mathbf{K}$.

All we need to check is if the colour of $\mathcal{B}_0$ is according to Definition 2.10 since we already know $\mathcal{B}_0$ a well defined L-structure. By the definition of a l-colouring it follows that all elements have a colour in $\mathcal{B}_0$ and all elements (except possibly zero) in a one dimensional span have the same colour. So the only thing which is left to check is if the colour of $\mathcal{B}_0$ goes together with the defined relations. Let R, as defined above, be a relation symbol in V_{rel} with arity r. For all other relation symbols we already know the colouring is correct, since the relations are empty so we only need to check R. Let $x_1, ..., x_r \in B_0$. We have multiple cases:

- If $x_2, ..., x_r \in cl(x)$ then by definition of $\mathcal{B}$ we have $\mathcal{B}_0 \models \neg R(x_1, ..., x_r)$. But through our definition of c_0 we must have that $x_2, ..., x_r$ have the same colour. Which is what we expect for a structure in $\mathbf{K}$.

- If $x_1, ..., x_r \in W_i$ for some $i \in \{1, ..., m\}$ then c_0 will colour the elements in the same colour. But since we have $\mathcal{B}_0 \models \neg R(x_1, ..., x_r)$ this works well with the definition of weak coloured structures.

- If $\{x_1, ..., x_r\} \nsubseteq W_i$ for all $i = 1, ..., m$ and there is some $j = 2, ..., r$ so that $x_j \notin cl(x_1)$ then we will have $\mathcal{B} \models R(x_1, ..., x_r)$. But we will also have that c_0 will colour some element $y \in cl(x_1, ..., x_r)$ in some other colour than x_1 since otherwise $\{x_1, ..., x_r\}$ would be included in a maximal c_0-monochromatic subspace different from all of $\mathcal{W}_1, ..., \mathcal{W}_m$, contradicting our assumption about $\mathcal{W}_1, ..., \mathcal{W}_m$. This is in line with the weak coloured structures definition.

These are all the possible cases for r-tuples in $\mathcal{B}_0$ so $1 - 3$ in Definition 2.10 holds and hence $\mathcal{B}_0$ is represented in $\mathbf{K}$. $\square$

The structure $\mathcal{B}_0$ from the previous lemma will be used further on. In the next lemma we show that two elements satisfying ξ_0 implies that these two elements have the same colour. The proof will use the definition of $\mathcal{B}$ and Theorem 4.1 to show that the colours are forced to be fixed when ξ_0 is satisfied, and then satisfying ξ_0 is the same as being in one of the monochromatic subspaces of $\mathcal{B}$.

Lemma 4.3. *If the L-structure $\mathcal{M} \in \boldsymbol{K}$, $v, w \in M - cl(\emptyset)$ and $\mathcal{M} \models \xi_0(v, w)$ then v and w have the same colour, i.e. $\mathcal{M} \models P_i(v) \wedge P_i(w)$ for some $i \in \{1, ..., l\}$.*

In order to prove this lemma we'll use the following claim.

Claim. Any isomorphism f (if such exists) from the L_{rel}-structure $\mathcal{B}$ a the L_{rel}-structure $\mathcal{B}' \subseteq \mathcal{M} \upharpoonright L_{rel}$ induces a bijection between the maximal monochromatic subspaces of $\mathcal{B}$ and of those in $\mathcal{B}'$.

Proof of the claim. Assume that c is a l-colouring of $\mathcal{B}'$ such that for each $x \in B'$if $\mathcal{M} \models P_i(x)$ then $c(x) = i$, so c mimics the original colouring of $\mathcal{B}'$. Let $\mathcal{W}_1, ..., \mathcal{W}_m$ be the maximal monochromatic subspaces of $\mathcal{B}$ with the l-colouring c_0 and $\mathcal{W}'_1, ..., \mathcal{W}'_p$ all the maximal monochromatic subspaces of $\mathcal{B}'$ of dimension at least 2 with the l-colouring c. By Theorem 4.1 this sequence is non-empty.

Suppose the $f : \mathcal{B} \to \mathcal{B}'$ is an isomorphism. Choose non-zero vector elements $f(x_1), ..., f(x_r) \in \mathcal{W}'_i$ for some $i \in \{1, ..., p\}$ such that $f(x_j) \notin cl(f(x_1))$ for some $j \in \{2, ..., r\}$. We know that $\mathcal{W}'_i$ is monochromatic, hence we must have that $\mathcal{B}' \models \neg R(f(x_1), ..., f(x_r))$. But since f is an isomorphism we have that $\mathcal{B} \models \neg R(x_1, ..., x_r)$ and $x_j \notin cl(x_1)$ which by our definition of $\mathcal{B}$ implies that $x_1, ..., x_r \in \mathcal{W}_{j_i}$ for some $j_i \in \{1, ..., m\}$. Hence for each $i \in \{1, ..., p\}$ there is $j_i \in \{1, ..., m\}$ so that $\mathcal{W}'_i \subseteq f(\mathcal{W}_{j_i})$. By minimality of the dimension of $\bigcup_{i=1}^{m} \mathcal{W}_i$ we get $\dim \bigcup_{i=1}^{m} \mathcal{W}_i = \dim \bigcup_{i=1}^{p} \mathcal{W}'_i$. Let c_f be the colouring of $\mathcal{B}$ which is induced from c through the isomorphism f i.e. $c_f(x) = i \Leftrightarrow c(f(x)) = i$. By choice of the colouring c_0 of $\mathcal{B}$ (and $\mathcal{V}$) we have $p = t(c_f) \leq t(c_0) = m$. But since we just proved that for every $i \in \{1, ..., p\}$ there is j_i s.t. $f^{-1}(\mathcal{W}'_i) \subseteq \mathcal{W}_{j_i}$, we must have $p = m$ and hence every $\mathcal{W}_i$ must be mapped onto some $\mathcal{W}'_j$. □

Proof of the lemma. Assume that $\mathcal{M} \models \xi_0(v, w)$. Then there is a $\mathcal{B}' \subseteq \mathcal{M}$ with $B' = \{v, w, b'_3, ..., b'_\beta\}$ and $\mathcal{M} \models \chi_{\mathcal{B}}(v, w, b'_3, ..., b'_\beta)$. So because of $\chi_{\mathcal{B}}$ there is an isomorphism $f : \mathcal{B} \to \mathcal{B}'$ such that $f(a) = v$ and $f(b) = w$, where $a, b \in \mathcal{W}_1$. By our previous claim we have that $f(\mathcal{W}_1)$ is a monochromatic subspace of $\mathcal{B}'$ and since $v, w \in f(\mathcal{W}_1))$ we have that v and w must have the same colour. □

Remember that $\mathcal{A} = \mathcal{B} \upharpoonright cl(\{a, b\})$. Put $\alpha = \dim \mathcal{B} + 1$ and let $k_1 = \dim \mathcal{B} + \dim \mathcal{A} - 1 = \alpha = \dim \mathcal{G}_\alpha$ for the rest of this section, and notice that $\dim(B) < k_1$. The reason that we choose this particular k_1 is in order to be able to prove Lemma 4.5. The following lemma is a part of proving the second direction of the formula ξ, that is, any two elements having the same colour will satisfy ξ (if the structure has the k_1-extension property).

Lemma 4.4. *Assume that $\mathcal{M} \in \boldsymbol{K}$ has the k_1-extension property, $v, w \in M$ s.t. $v \notin cl(w)$, $w \notin cl(v)$ and $\mathcal{A}'$ is a substructure of $\mathcal{M}$ with universe $cl_{\mathcal{M}}(v, w)$. If all vectors in $\mathcal{A}'$ has the same colour (except possibly the zero vector) and there is an isomorphism $f_0 : \mathcal{A}' \upharpoonright L_{rel} \to \mathcal{A}$ such that $f_0(v) = a$ and $f_0(w) = b$ then $\mathcal{M} \models \xi_0(v, w)$.*

Proof. We know that f_0 is an isomorphism and $\mathcal{A} \subseteq \mathcal{B}$ with $\dim(B) < k_1$, hence since $\mathcal{M}$ satisfies the k_1-extension property there is and embedding $f : \mathcal{B} \to \mathcal{M} \upharpoonright L_{rel}$ which extends f_0^{-1}. Let $\mathcal{B}' = \mathcal{M} \upharpoonright \mathrm{im}(f)$ so $\mathcal{B} \cong \mathcal{B}'$ and since f extends f_0^{-1} we have that $v, w \in B'$. Let $B' = \{v, w, b'_3, ..., b'_\beta\}$, then since f is an isomorphism such that $f(v) = a$ and $f(w) = b$ we get that $\mathcal{M} \models \chi_{\mathcal{B}}(v, w, b'_3, ..., b'_\beta)$. Hence $\mathcal{M} \models \xi_0(v, w)$. □

The upcoming lemma is another part in proving that elements, inside a structure with the k_1-extension property, coloured the same will satisfy ξ.

Lemma 4.5. *Assume that $\mathcal{M} \in \boldsymbol{K}$ has the k_1-extension property. If $v, w \in \mathcal{M}$ are independent, and have the same colour, then $\exists u \in M - cl_{\mathcal{M}}(v, w)$ such that:*

> *Let $A_{v,u} = \mathcal{M} \restriction cl(v, u)$ and let $A_{w,u} = \mathcal{M} \restriction cl(w, u)$. Then there exist isomorphisms $f_{v,u} : A_{v,u} \restriction L_{rel} \to \mathcal{A}$ and $f_{w,u} : \mathcal{A}_{w,u} \restriction L_{rel} \to \mathcal{A}$ such that $f_{v,u}(v) = a$, $f_{v,u}(u) = b$, $f_{w,u}(w) = a$ and $f_{w,u}(u) = b$. Also $A_{v,u}$ and $A_{w,u}$ are mono-coloured.*

Proof. This proof will be in two parts. First we will create a L-structure $\mathcal{D}$ which satisfies everything that is described above. The structure $\mathcal{D}$ won't be a substructure of $\mathcal{M}$ so in the second part of this proof we'll use the extension property to show that we may embed $\mathcal{D}$ inside of $\mathcal{M}$ and hence there will be substructures of $\mathcal{M}$ which satisfy the needed conditions.

Part 1: Creating $\mathcal{D}$

We'll start from the L_F-structure $\mathcal{G}_{\dim \mathcal{B} + \dim A - 1} = \mathcal{G}_\alpha$ and expand it further and further into $\mathcal{D}$ until we have the structure that we want. We'll need to use the structure $\mathcal{B}_0$ from Lemma 4.2 and we assume, by permuting the colours of $\mathcal{B}_0$, that the colour of w and v in $\mathcal{M}$ is the same as a and b in $\mathcal{B}_0$. Let $g_1, ..., g_\alpha$ be basis vectors of $\mathcal{G}_\alpha$ and choose embeddings $f_1 : \mathcal{G}_\alpha \restriction cl(\{g_1, g_\alpha\}) \to \mathcal{M} \restriction L_F$, $f_2 : \mathcal{G}_\alpha \restriction cl(\{g_1, ..., g_{\alpha-1}\}) \to \mathcal{B}_0 \restriction L_F$ and $f_3 : \mathcal{G}_\alpha \restriction cl(\{g_2, ..., g_\alpha\}) \to \mathcal{B}_0 \restriction L_F$ so that $f_1(g_1) = v$, $f_1(g_\alpha) = w$, $f_2(g_1) = f_3(g_\alpha) = a$, $f_3(g_2) = f_2(g_2) = b$ and $\forall x \in \mathcal{G}_\alpha \restriction cl(\{g_2, ..., g_{\alpha-1}\})$ we have $f_2(x) = f_3(x)$. Notice that f_2 and f_3 are isomorphisms such that they agree on where elements are mapped for each element which exists in both domains. Now create the L-structure $\mathcal{D}$ by "removing the reducts from the embeddings", that is let the universe $D = G_\alpha$ and define structure on $\mathcal{D}$ as follows:

1. $\mathcal{D} \restriction L_F = \mathcal{G}_\alpha$.
2. If $Q \in V$ is a relation symbol with arity r_Q and $x_1, ..., x_{r_Q} \in D$, then $\mathcal{D} \models Q(x_1, ..., x_{r_Q})$ iff
   $x_1, ..., x_{r_Q} \in dom(f_1)$ and $\mathcal{M} \models Q(f_1(x_1), ..., f_1(x_{r_Q}))$ or
   $x_1, ..., x_{r_Q} \in dom(f_2)$ and $\mathcal{B}_0 \models Q(f_2(x_1), ..., f_2(x_{r_Q}))$ or
   $x_1, ..., x_{r_Q} \in dom(f_3)$ and $\mathcal{B}_0 \models Q(f_3(x_1), ..., f_3(x_{r_Q}))$.
3. If $x \in D$ does not have a colour after the previous item has been applied then let $\mathcal{D} \models P_1(x)$.

Notice that we are both defining most of the colours $P_1, ..., P_l$ and the other relations in the second item, the third item is to make up for any colours we missed.

Claim. The L-structure $\mathcal{D}$ is weakly l-coloured in accordance with Definition 2.10.

Proof of the claim. By the third item in the definition of $\mathcal{D}$ each element has a colour. We now check that colours have been assigned to elements of $\mathcal{D}$ in an unambiguous way. If $x \in cl(y) \subseteq D$ then there are a couple of cases. Case one is if $x, y \notin dom(f_1) \cup dom(f_2) \cup dom(f_3)$, in which case x and y must both have the colour P_1 and no other colour. In case two we see that if $x, y \in dom(f_2) \cap dom(f_3)$ then $f_2(x) = f_3(x)$ and $f_2(y) = f_3(y)$, so since $\mathcal{B}_0$ is a l-coloured structure and the colours of x and y in this case depends on the colouring of $\mathcal{B}_0$ we get that x and y have the same colour and no other colour. In the case $x, y \in dom(f_1) \cap dom(f_2)$, which is parallel with the case $x, y \in dom(f_1) \cap dom(f_3)$, we have assumed from before that $\mathcal{B}_0$ and $\mathcal{M}$ have the same colour on a,v and w and since $\mathcal{B}_0$ and $\mathcal{M}$ are l-coloured structure, this colour will be the only colour of $cl(y)$ by the definition of $\mathcal{D}$. The only case now not considered is if x, y is only in a domain of a single function, but that case follows fast from the knowledge that the colour of x and y is inherited from $\mathcal{B}_0$ or $\mathcal{M}$ who are l-coloured structures.

The last thing we need to check is what happens with the elements who are in relations. Recall from Assumption 2.9 that the arity of each relation symbol in V_rel is at least 2. If $R \in V_{rel}$, $x_1, ..., x_r \in D$ and $\mathcal{D} \models R(x_1, ..., x_r)$ then by the definition of $\mathcal{D}$ we must have that either $\mathcal{M} \models R(f_1(x_1), ..., f_1(x_r))$ or $\mathcal{B}_0 \models R(f_2(x_1), ..., f_2(x_r))$ or $\mathcal{B}_0 \models R(f_3(x_1), ..., f_3(x_r))$. In each case there must exist elements $x, y \in cl(f_i(x_1), ..., f_i(x_r))$, for $i = 1, 2$ or 3, of different colours (by definitions of l-colourable structures), and hence the colour of $f_i^{-1}(x), f_i^{-1}(y) \in cl(x_1, ..., x_r)$ must be different. $\square$

Part 2: Embedding $\mathcal{D}$
Let h_1, h_2, h_3 be f_1, f_2, f_3 extended into the language L, and hence they are defined on $\mathcal{D}$. These are obviously embeddings because of how we defined $\mathcal{D}$. Now we know that $\mathcal{D}$ is a l-coloured structure, $\dim(\mathcal{D}) = k_1$, h_1 is an isomorphism between $\mathcal{D} \restriction cl(\{g_1, g_\alpha\})$ and $\mathcal{M} \restriction cl(\{v, w\})$ s.t. $h_1(g_1) = v$ and $h_1(g_\alpha) = w$ and $\mathcal{M}$ satisfies the k_1-extension property. Hence there is an embedding $h : \mathcal{D} \to \mathcal{M}$ which extends h_1. Since h_2 and h_3 are both embeddings into $\mathcal{B}_0$ with $h_2(g_2) = h_3(g_2) = b$ and $h_2(g_1) = h_3(g_\alpha) = a$ we see that $\mathcal{D} \restriction cl(\{g_1, g_2\}) \cong \mathcal{D} \restriction cl(\{g_2, g_\alpha\}) \cong \mathcal{B}_0 \restriction cl(\{a, b\}) = \mathcal{A}_0$ where $\mathcal{A}_0 \restriction L_{rel} = \mathcal{A}$. Let $u = h(h_2^{-1}(b))$ and let $A_{v,u}$ and $A_{w,u}$ be defined through this as in the statement of the lemma. Since $h(h_2^{-1})$ is an embedding of $\mathcal{A}_0$, where $\mathcal{A}_0$ is mono-coloured, we get that $A_{v,u}$ is mono-coloured, and in the same way through $h(h_3^{-1})$ we get that $A_{w,u}$ is monocoloured. Then we may define $f_{v,u} = h_2(h^{-1})$ restricted to $\mathcal{M} \restriction cl(\{v, u\})$. In the same way we can define $f_{w,u} = h_3(h^{-1})$ restricted to $\mathcal{M} \restriction cl(\{w, u\})$. Since h, h_2, h_3 are all L_{rel}-isomorphisms, $f_{v,u}$ and $f_{w,u}$ are L_{rel}-isomorphisms. Also h, h_2, h_3 satisfy $h(g_1) = v, h(g_\alpha) = w, h(g_2) = u$ and $h_2^{-1}(a) = g_1, h_2^{-1}(b) = g_2$ and $h_3^{-1}(a) = g_\alpha, h_3^{-1}(b) = g_2$ so we can conclude that $f_{v,u}(v) = a$, $f_{v,u}(u) = b$, $f_{w,u}(w) = a$ and $f_{w,u}(u) = b$. $\square$

Now to finish it of, we put the previous two lemmas together to finally show that if elements have the same colour then we can create a formula which the elements satisfy.

Lemma 4.6. *Assume that $\mathcal{M} \in K$ has the k_1-extension property. If $v, w \in \mathcal{M}$ are independent and have the same colour then $\exists u \in M - cl(v, w)$ such that $\mathcal{M} \models \xi_0(v, u) \wedge \xi_0(w, u)$.*

Proof. By Lemma 4.5, $\exists u \in M - cl(v, w)$ and mono-coloured structures $\mathcal{A}_{v,u}, \mathcal{A}_{w,u} \subseteq \mathcal{M}$ with $A_{v,u} = cl(v, u)$ and $A_{w,u} = cl(w, u)$, isomorphisms $f_{v,u} : \mathcal{A}_{v,u} \restriction L_{rel} \to \mathcal{A}$ and $f_{w,u} : \mathcal{A}_{w,u} \restriction L_{rel} \to \mathcal{A}$ with $f_{v,u}(v) = f_{w,u}(w) = a$ and $f_{v,u}(u) = f_{w,u}(u) = b$. So by Lemma 4.4 we get through $f_{v,u}$ that $\mathcal{M} \models \xi_0(v, u)$ and then, still by Lemma 4.4, using $f_{w,u}$ get that $\mathcal{M} \models \xi_0(w, u)$. Hence $\mathcal{M} \models \xi_0(v, u) \wedge \xi_0(w, u)$. $\square$

Using this lemma we can finally define ξ which is the desired formula which we later in 4.8 show has the desired property of describing if elements have the same colour or not.

Definition 4.7. We define the L_{rel}-formula ξ using ξ_0 in the following way:

$$\xi(x, y) = x \in cl(y) \vee (x \notin cl(y) \wedge \exists z(\xi_0(x, z) \wedge \xi_0(y, z)).$$

Corollary 4.8. *Assume that $\mathcal{M} \in \boldsymbol{K}$ has the k_1-extension property, $v, w \in M$ and $w, v \notin cl(\emptyset)$. Then*

$$\mathcal{M} \models \xi(v, w) \quad \Leftrightarrow \quad v \text{ and } w \text{ has the same colour}$$

Proof. If v, w are dependent, then they have the same colour by the colour definition and by the definition of ξ, we have $\mathcal{M} \models \xi(x, y)$.
Assume that $w \notin cl(v)$ and v, w has the same colour. By Lemma 4.6, $\exists u \in M$ s.t. $\mathcal{M} \models \xi_0(v, u) \wedge \xi_0(w, u)$, so by the definition of ξ we get that $\mathcal{M} \models \xi(v, w)$.
Now for the opposite case, assume that $\mathcal{M} \models \xi(v, w)$ and that $v \notin cl(w)$. Then by ξ, $\exists u \in \mathcal{M}$ s.t. $\mathcal{M} \models \xi_0(v, u) \wedge \xi_0(w, u)$ so, by Lemma 4.4, v has the same colour as u and u has the same colour as w and hence v must have the same colour as w. □

5 The almost sure theory, and the zero-one law

This section will wrap up what has been done previously in this thesis, and in the end the zero-one law will be proved. Sections 3 and 4 both concluded with corollaries which contained formulas $\xi(a, b)$ which were equivalent to saying that a and b had the same colour, in structures with the k-extension property for k large enough. That such formula ξ exists will be the only thing that matters, and not if the colouring is weak or not, and hence the reasoning in this chapter will not care which kind of colouring is actually used. Using ξ we'll extend the extension axioms which were defined in 2.6 into colour compatible extension axioms, i.e. extension axioms which also care about which colouring we have of the structures we extend with. These new extension axioms will be proven true in all models with big enough k-extension property in Lemma 5.2 which in turn will make them true with probability approaching 1 when the size of the structures tends to infinity, in Lemma 5.3. The proof of the zero-one law then follows the classical path, by collecting the colour compatible extension axioms into one big theory and proving that the theory is complete in Lemma 5.5. Keep $\mathbf{K}_n$ and $\mathbf{K}$ as defined in section 3 or 4 and define $\mathbf{C}_n$ to be the set of all (strongly) l-colourable L_{rel}-structures $\mathcal{M}$ such that $\mathcal{M} \restriction L_F = \mathcal{G}_n$ and $\mathbf{C} = \bigcup_{i=1}^{\infty} \mathbf{C}_n$ is a pregeometry. Notice that $\mathbf{K}_n \restriction L_{rel} = \{\mathcal{M} \restriction L_{rel} | \mathcal{M} \in \mathbf{K}_n\} = \mathbf{C}_n$. We want to define the dimension conditional measure on $\mathbf{C}$ as depeding on the measure we previously defined on $\mathbf{K}$. For each set $X \subseteq \mathbf{C}_n$ let

$$\delta_n^{\mathbf{C}}(\mathcal{M}) = \delta_n^{\mathbf{K}}(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \restriction L_{rel} \in X\}).$$

This may also be extended to formulas by defining for each formula $\varphi \in L_{rel}$, $\delta_n^{\mathbf{C}}(\varphi) = \delta_n^{\mathbf{C}}(\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \varphi\})$.

In order to define the colour compatible extension axiom we first need a help formula θ. We define the θ uniquely for each l-colouring of a certain structure so that it classifies that l-colouring. So assume that $\gamma : \{1, ..., \alpha\} \to \{1, ..., l\}$ is a l-colouring of a structure $\mathcal{A}$ with universe $A = \{1, ..., \alpha\}$. In the definition of θ_γ we will use the formula ξ from Section 4 in case we are using weak l-colourings and ξ from Section 3 in case of strong l-colourings. That said all references to l-colourings throughout this section will refer to either strong coloured structures or weakly coloured, which one won't matter. Define the γ-colouring specific formula $\theta_\gamma(x_1, ..., x_\alpha)$ as follows:

$$\theta_\gamma(x_1, ..., x_\alpha) \quad \Longleftrightarrow \quad \bigwedge_{\{i,j:\gamma(i)=\gamma(j)\}} (\xi(x_i, x_j) \vee x_i \in cl(\emptyset) \vee x_j \in cl(\emptyset))$$
$$\bigwedge_{\{i,j:\gamma(i)\neq\gamma(j)\}} (\neg\xi(x_i, x_j) \vee x_i \in cl(\emptyset) \vee x_j \in cl(\emptyset))$$

In the case $\mathcal{A} \subseteq \mathcal{B}$ are both L_{rel}-structures represented w.r.t. $\mathbf{C}$ and γ' is a l-colouring of $\mathcal{B}$ which extends a l-colouring γ of $\mathcal{A}$, we define the following to be an **instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom $\eta_{\gamma\gamma'}$**:

$$\forall y_1, ..., y_\alpha \exists y_{\alpha+1}, ..., y_\beta (\chi_\mathcal{A}(y_1, ..., y_\alpha) \wedge \theta_\gamma(y_1, ..., y_\alpha) \longrightarrow \chi_\mathcal{B}(y_1, ..., y_\beta) \wedge \theta_{\gamma'}(y_1, ..., y_\beta)).$$

Where the characteristic formulas $\chi_\mathcal{A}$ and $\chi_\mathcal{B}$ are as defined in 2.5. There are only finitely many l-colourings there are only finitely many instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extensions axiom. We define the **l-colour compatible $\mathcal{B}/\mathcal{A}$ extension axiom** η as the conjunction of all the instances.

Remark 5.1. Notice that θ_γ and $\theta_{\gamma'}$ will be the same for two colourings of the same structure, if the colourings γ and γ' are just permuting the colours of each other. This is because ξ only looks at which elements have the same colour and not which colour they have. This will also transfer to extensions of the colourings, so if γ can be extended into γ_0 and γ' extended to γ'_0, colouring $\mathcal{B}$ s.t. $\forall x, y \notin cl(\emptyset)(\gamma_0(x) = \gamma_0(y)) \Leftrightarrow (\gamma'_0(x) = \gamma'_0(y))$, then $\theta_{\gamma_0} = \theta_{\gamma'_0}$ Since θ_γ depends only on the partition induced by the colouring γ, $\eta_{\gamma\gamma'}$ will depends only on the partition induced by the colouring of γ'. This will be used in the proof of Lemma 5.2 in order to assume that the colours are in the way we want them to in the colouring γ'.

Define the set $X_{n,k} = \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M}$ has the k-extension property$\}$. We'll now prove a lemma which makes sure that our new colour compatible extension axioms really are as good as we want them to be, and are satisfied by all the structures satisfying the regular extension axioms. This will be done by using what we have learned about ξ from the previous chapters, dissecting the formula for θ_γ and then just show that it is satisfied.

Lemma 5.2. *Assume that $\mathcal{B}$ is a l-colourable L_{rel}-structure and $\mathcal{A} \subseteq \mathcal{B}$ (hence $\mathcal{A}$ is also l-colourable). Let η denote the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. If $k = max(k_0, k_1, \dim(B))$, where k_0 and k_1 come from Sections 3 and 4 respectively, and $\mathcal{M} \in X_{n,k}$, then $\mathcal{M} \models \eta$.*

Proof. In order to prove this, it is enough to prove that every $\mathcal{M} \in X_{n,k}$ satisfies each instance $\eta_{\gamma\gamma'}$ of the l-colour compatible $\mathcal{A}/\mathcal{B}$-extension axiom since η is a conjunction of these. So, choose an arbitrary instance $\eta_{\gamma\gamma'}$, which uses a colouring $\gamma' : B \to \{1, ..., l\}$ and its restriction to a A, $\gamma : A \to \{1, ..., l\}$. Assume that $\mathcal{M} \in X_{n,k}$ and

$$\mathcal{M} \models \chi_\mathcal{A}(a_1, ..., a_\alpha) \wedge \theta_\gamma(a_1, ..., a_\alpha)$$

for some $a_1, ..., a_\alpha \in M$ and let $\mathcal{A}' = \mathcal{M} \upharpoonright \{a_1, ..., a_\alpha\}$. Then by the definition of $\chi_\mathcal{A}$ there is an isomorphism $f : \mathcal{A}' \upharpoonright L_{rel} \to \mathcal{A}$. For all $i \in \{1, ..., \alpha\}$ let

$$\gamma_0(a_i) = j \quad \Leftrightarrow \quad \mathcal{M} \models P_j(a_i),$$

so γ_0 is a l-colouring of $\mathcal{A}'$. Since $\mathcal{M}$ has the k-extension property and $\mathcal{M} \models \theta_\gamma(a_1, ..., a_\alpha)$ it follows that, for all $x, y \in \{a_1, ..., a_\alpha\} - cl(\emptyset)$,

$$\gamma_0(x) = \gamma_0(y) \quad \Leftrightarrow \quad \gamma(f(x)) = \gamma(f(y)).$$

From this it follows that (by permuting the colours of γ') we can find a l-colouring γ'_1 of $\mathcal{B}$ such that if γ_1 is the restriction of γ'_1 to A, then $\forall x \in \{a_1, ..., a_\alpha\} - cl(\emptyset)$ we have $\gamma_1(f(x)) = \gamma_0(x)$ and for all $x, y \in B - cl(\emptyset)$,

$$\gamma'_1(x) = \gamma'_1(y) \quad \Leftrightarrow \quad \gamma'(x) = \gamma'(y).$$

Now expand $\mathcal{B}$ into the L-structure $\mathcal{B}'$ by adding colours to it according to γ_0' that is if $x \in B$ and $\gamma_0'(x) = i$ then let $\mathcal{B}' \models P_i(x)$. By the definition of $\mathcal{B}'$ it follows that f is an L-isomorphism from $\mathcal{B}' \upharpoonright A$ onto $\mathcal{M} \upharpoonright \{a_1, ..., a_\alpha\}$. Since $\mathcal{M}$ satisfies the k-extension property, we may extend f into an embedding $g : \mathcal{B}' \to \mathcal{M}$, which has the image $\{b_1'', ..., b_\beta''\}$. We know that $\mathcal{B}'$ is isomorphic to $\mathcal{B}$ and that it has the same colours as the colouring γ prescribes, and hence we get from g that

$$\mathcal{M} \models \chi_{\mathcal{B}}(b_1'', ..., b_\beta'') \wedge \theta_{\gamma'}(b_1'', ..., b_\beta'').$$

The chosen instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom is hence satisfied by $\mathcal{M}$, and since it was an arbitrary instance, $\mathcal{M}$ has to satisfy all of the instances. □

The next lemma is a consequence of Koponen's work [10] and important for the results here since the previous lemma only holds for $\mathcal{M} \in X_{n,k}$, that is, $\mathcal{M} \in \mathbf{K}_n$ which have the k-extension property with respect to $\mathbf{K}$.

Lemma 5.3. *For every $k \in \mathbb{N}$, $\lim_{n\to\infty} \delta_n^{\boldsymbol{K}}(X_{n,k}) = 1$*

Proof. Since we are using $\mathbf{K}_n$ with pregeometry $\mathcal{G}_n$ which is the vector space pregeometry, we know by Koponen [10], examples 7.9,7.22 and 7.23, that both in the case of strong and weak colourings, $\mathcal{G}_n$ is uniformly bounded, polynomially k-saturated and $\mathbf{K}$ accepts k-substitutions (all these terms are defined in [10]). Theorem 7.31 in [10] say that

> Let $k > 0$. Suppose that $\mathcal{G}_n : n \in \mathbb{N}$ is uniformly bounded and polynomially k-saturated for every $k \in \mathbb{N}$ and that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} K_n$ with pregeometry $\mathcal{G}$ accepts $k-$substitutions. Then for every $(k-1)$-extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$.

This theorem is applicable in the current case so we get that for every extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\varphi) = 1$. Since there are only a finite number of k-extension axioms, the lemma follows. □

Using the previous two lemmas we can prove that also the colour compatible extension axioms will be satisfied in almost all structures.

Corollary 5.4. *For every l-colour compatible extension axiom η, $\lim_{n\to\infty} \delta_n^{\boldsymbol{C}}(\eta) = 1$.*

Proof. Let η be an l-colour compatible extension axiom. Since $\eta \in L_{rel}$ we have

$$\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \eta\} = \{\mathcal{N} \upharpoonright L_{rel} : \mathcal{N} \in \mathbf{K}_n \text{ and } \mathcal{N} \models \eta\}$$

by definition of $\mathbf{C}_n$, $\mathbf{K}_n$ and the fact that μ is an L_{rel}-sentence. Hence by the definition of $\delta_n^{\mathbf{C}}$ and $\delta_n^{\mathbf{K}}$ we get $\delta_n^{\mathbf{C}}(\eta) = \delta_n^{\mathbf{K}}(\eta)$, for every $n \in \mathbb{N}$, for some l-colourable L_{rel}-structures $\mathcal{A} \subseteq \mathcal{B}$, with η as the l-colour compatible $\mathcal{B}/\mathcal{A}-$extension axiom. But then if we choose $k = \max(k_0, k_1, |B|)$ we get from Lemma 5.3 that $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(X_{n,k}) = 1$. Together with Lemma 5.2, which say that for every n and $\mathcal{M} \in X_{n.k}$ we have $\mathcal{M} \models \eta$, we get that $\delta_n^{\mathbf{C}}(\eta) = \delta_n^{\mathbf{K}}(\eta) \geq \delta_n^{\mathbf{K}}(X_{n,k})$, hence by Lemma 5.3 $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\eta) = 1$. □

Let T_{ext} be the set of all l-colour compatible extension axioms. Notice that the first part of the formula ψ_n is to ensure that the set $\{x_1, ..., x_{|F|}\}$ is closed. For each $n \in \mathbb{N}$ let $\mathcal{M}_{n,1}, ..., \mathcal{M}_{n,m_n} \in \mathbf{K}_n$

be the different isomorphism classes in $\mathbf{K}_n$ and put $\chi_i^n(x_1, ..., x_{|F|^n})$ equivalent to $\chi_{\mathcal{M}_{n,i}}(x_1, ..., x_{|F|^n})$ Define the sentence ψ_n for every $n \in \mathbb{N}$ by

$$\forall x_1, ..., x_{|F|^n}\Big(\forall x[x \in cl(x_1, ..., x_{|F|^n}) \to \bigvee_{i=1}^{|F|^n} x = x_i] \to \bigvee_{i=1}^{m_n} \bigvee_{\pi} \chi_i^n(x_{\pi(1)}, ..., x_{\pi(|F|^n)}))$$

where the second disjunction ranges over all permutations π of $\{1, ..., |F|^n\}$. When we collect all these formulas we get $T_{iso} = \bigcup_{n=0}^{\infty}\{\psi_n\}$. Remember from Definition 2.4 that we used a formula θ_n (not to be confused with θ_γ in the colour compatible extension axiom) which described the closure of n elements in our pregeometry $\mathcal{G}_m$. All properties of a general pregeometry can be described using θ_n except for the finiteness property. Let T_{pre} be the set of all sentences which express the properties of the vector space pregeometry over F except the finiteness property. Now we will define the theory $T_{\mathbf{C}} = T_{ext} \cup T_{iso} \cup T_{pre}$. Notice that $T_{\mathbf{C}}$ contains only L_{rel}-formulas and is consistent by Corollary 5.4 and compactness.

Lemma 5.5. *$T_{\mathbf{C}}$ is countably categorical.*

Proof. Assume that $\mathcal{M}$ and $\mathcal{M}'$ are L_{rel}-structures such that $\mathcal{M} \models T_{\mathbf{C}}$, $\mathcal{M}' \models T_{\mathbf{C}}$ and $|\mathcal{M}| = |\mathcal{M}'| = \aleph_0$. By a back and forth argument we will build up partial isomorphisms between $\mathcal{M}$ and $\mathcal{M}'$ and extend them, which when unified will give that $\mathcal{M} \cong \mathcal{M}'$.

Claim. Let $\mathcal{A} \subseteq \mathcal{M}$, $A' \subseteq \mathcal{M}'$ such that both $\mathcal{A}$ and $\mathcal{A}'$ are finite and there is an isomorphism $f : \mathcal{A} \to \mathcal{A}'$ such that $\forall a, b \in A$ we have $\mathcal{M} \models \xi(a, b) \Leftrightarrow \mathcal{M}' \models \xi(f(a), f(b))$. Then $\forall c \in M - A$ there exists $\mathcal{B}' \supsetneq \mathcal{A}'$ and an isomorphism $g : \mathcal{M} \restriction cl_M(A \cup c) \to B'$ such that g extends f and for each $a, b \in \mathcal{M} \restriction cl_M(A \cup c)$ we have $\mathcal{M} \models \xi(a, b) \Leftrightarrow \mathcal{M}' \models \xi(g(a), g(b))$

Proof of claim. Take any $b \in M - A$ and let $\mathcal{B} = \mathcal{M} \restriction (A \cup \{b\})$ as always we assume that $A = \{a_1, ..., a_\alpha\}$,$A' = \{a'_1, ..., a'_\alpha\}$ and $B = \{b_1, ..., b_\beta\}$ with obviously $\beta > \alpha$. In order to get a bit nicer notation we also assume that the isomorphism f is such that $f(a_i) = a'_i$. We then have $\mathcal{B} \subseteq \mathcal{M}$ and since $\mathcal{M} \models T_{iso}$ we know that $\mathcal{B}$ is isomorphic to an l-colourable structure and so, $\mathcal{B}$ is l-colourable by a l-colouring γ'. Since $\mathcal{A} \subseteq \mathcal{B}$, any l-colouring of $\mathcal{B}$ is also (when restricted) a colouring of $\mathcal{A}$, let γ be the restriction of the colouring γ' to $\mathcal{A}$. Hence there is a $\mathcal{B}/\mathcal{A}$-colour compatible extension axiom $\eta_{\gamma\gamma'} \in T_{ext}$, but $\mathcal{M}' \models T_{ext} \Rightarrow \mathcal{M}' \models \eta_{\gamma\gamma'}$ which gives us that

$$\mathcal{M}' \models \forall y_1, ..., y_\alpha \exists y_{\alpha+1}, ..., y_\beta(\chi_{\mathcal{A}}(y_1, ..., y_\alpha) \wedge \theta_\gamma(y_1, ..., y_\alpha) \longrightarrow \chi_{\mathcal{B}}(y_1, ..., y_\beta) \wedge \theta_{\gamma'}(y_1, ..., y_\beta)).$$

From the assumption we know that $\mathcal{A} \cong \mathcal{A}'$, so $\mathcal{M}' \models \chi_{\mathcal{A}}(a'_1, ..., a'_\alpha)$. We know by the assumption $\mathcal{M} \models \xi(a_i, a_j) \Leftrightarrow \mathcal{M}' \models \xi(f(a_i), f(a_j))$ and since f is an isomorphism, $x \in cl(\emptyset) \Rightarrow f(x) \in cl(\emptyset)$ so $\mathcal{M}' \models \theta_\gamma(a'_1, ..., a'_\alpha)$. This together with $\mathcal{M}' \models \eta_{\gamma\gamma'}$ gives us that $\exists a'_{\alpha+1}, ..., a'_\beta \in M - A'$ such that

$$\mathcal{M}' \models \chi_{\mathcal{B}}(a'_1, ..., a'_\beta) \wedge \theta_{\gamma'}(a'_1, ..., a'_\beta).$$

That there is an isomorphism $g : \mathcal{B} \to \mathcal{B}'$ extending f follows from $\mathcal{M}' \models \chi_B(a'_1, ..., a'_\beta)$ where $a'_1, ..., a'_\beta$ extends the sequence $a'_1, ..., a'_\alpha$ (and since $f : \mathcal{A} \to \mathcal{A}'$ is an isomorphism). We get from $\mathcal{M} \models \theta_{\gamma'}(a'_1, ..., a'_\beta)$ that for each $a_i, a_j \in \mathcal{B}$ we have $\mathcal{M} \models \xi(a_i, a_j) \Leftrightarrow \mathcal{M}' \models \xi(g(a_i), g(a_j))$. □

Continuation of the lemma proof. What we have left now to prove is that the assumptions hold for the basis case $A = \{a_1\}, A' = \{a'_1\}$. In these structures with just the zero vector, we must have $R^{\mathcal{A}} = R^{\mathcal{A}'} = \emptyset$ for each $R \in V_{rel}$, since else they aren't l-colourable. Also notice that as L_F

structures, $A \restriction L_F$ and $\mathcal{A}' \restriction L_F$ are isomorphic. Hence the mapping $a_1 \mapsto a_1'$ is an isomorphism between the L_{rel} structures $\mathcal{A}$ and $\mathcal{A}'$, and trivially we have that $\mathcal{A} \models \xi(a_1, a_1)$ and $\mathcal{A}' \models \xi(a_1', a_1')$. So we can use this basis case together with the claim to back and forth build a partial isomorphism. First from $\mathcal{M}$ to $\mathcal{M}'$ and in the next step, we use the claim in the opposite case, and continue the partial isomorphism but build it from $\mathcal{M}'$ to $\mathcal{M}$. In this way if we take the union of all the partial isomorphisms which go from $\mathcal{M}$ to $\mathcal{M}'$ we get an isomorphism (which isn't partial). Hence $T_{\mathbf{C}}$ is countably categorical. $\square$

We do now finally get our sought $0-1$ law using the previous theorem together with some standard model theoretical theorems.

Theorem 5.6. *For any sentence $\varphi \in L_{rel}$, either $\lim_{n\to\infty} \delta_n^{\boldsymbol{C}}(\varphi) = 1$ or $\lim_{n\to\infty} \delta_n^{\boldsymbol{C}}(\varphi) = 0$.*

Proof. By Lemma 5.5 $T_{\mathbf{C}}$ is a countably categorical theory and it can't have any finite models, since each finite model of $T_{\mathbf{C}}$ has to be arbitrary finitely large in order to satisfy the extension axioms. Hence by Vaught's test, we know that T_C is a complete theory so for each $\varphi \in L$ either $T_{\mathbf{C}} \models \varphi$ or $T_{\mathbf{C}} \not\models \varphi$. If $T_{\mathbf{C}} \models \varphi$ we know by compactness that there are $\psi_1, ..., \psi_m \in T_{\mathbf{C}}$ s.t. $\psi_1, ..., \psi_m \models \varphi$ but by Lemma 5.4 we know that $\lim_{n\to\infty} \delta_n^{\mathbf{C}} \psi_i = 1$ and hence $\lim_{n\to\infty} \delta_n^{\mathbf{C}} \varphi = 1$. In the case $T_{\mathbf{C}} \not\models \varphi$ we get $T_{\mathbf{C}} \models \neg\varphi$ so $\lim_{n\to\infty} \delta_n^{\mathbf{C}} \neg\varphi = 1$ which implies that $\lim_{n\to\infty} \delta_n^{\mathbf{C}} \varphi = 0$. $\square$

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