Determining the pressure distribution on submerged 2D bodies using dissipative potential flow.

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Abstract
Globalization is reaching the furthest corners of the world and with globalizations comes a rising demand on transportation. In shipping; a significant cost both to the ship owners and the environment are the fossil fuels used for propulsion, even a small reduction in the wave resistance can bring considerable reductions both in operating costs and emissions for such ships. When designing a ship it is important to be able to make fast and accurate predictions of its resistance so that more efficient hull forms can be selected early in the design process.

A panel method based on potential flow is a fast scheme to determine the wave resistance and is therefore suitable to be used early on in the design process. Here it is shown that potential flow can be improved by including Rayleigh damping, added viscous effects that will make the flow dissipative.

Dissipative Green functions are employed in the proposed technique with the resulting velocity potential determined from a combination of a source distribution and a modified distribution of vortices on submerged 2D bodies. NACA hydrofoils, Joukowski hydrofoils and cylinders are used to test the model.

The pressure distribution is more in line with experimental results than previous numerical methods without added viscosity for the NACA hydrofoils. The surface profile has very good comparison with existing numerical results for a NACA hydrofoil in subcritical speeds. However the results are very poor for the Joukowski hydrofoil.

There is therefore reason to develop this method further in both 2D and 3D.
NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Angle of attack</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Vortex strength</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Relative error</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Wave profile</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle between node and control point</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Panel angle of a cylinder</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Modified Rayleigh damping</td>
</tr>
<tr>
<td>$\mu'$</td>
<td>Rayleigh damping</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Reynolds viscosity coefficient</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Angle of circle section before mapped into a hydrofoil</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Fluid density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Source strength</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Domain specific parameter due to Cauchy’s theorem</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Perturbed velocity potential</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Velocity potential</td>
</tr>
<tr>
<td>$\phi_{in}$</td>
<td>Potential inside the body</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Complex velocity potential</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Stream line</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Perturbed stream function</td>
</tr>
</tbody>
</table>

$A_{ij}$ | Source influence matrix for control points |
$\tilde{A}_{ij}$ | Source influence matrix for surface points |
$a$ | Arbitrary example variable |
$a_{ij}$ | Source influence matrix |
$a_{i,N+1}$ | Vortex influence vector |
$B_{ij}$ | Vortex influence matrix for control points |
$\tilde{B}_{ij}$ | Vortex influence matrix for surface points |
$b$ | Arbitrary example variable |
$C_D$ | Drag coefficient |
$C_{DN}$ | Local drag coefficient |
$C_L$ | Lift coefficient |
$C_p$ | Pressure coefficient |
$C_{pa}$ | Analytical $C_p$ for a cylinder |
$C_{PN}$ | $C_p$ for panel $N$ |
$c$ | Cord length |
$d$ | Depth |
$F$ | Function |
$F_h$ | Froude depth number |
$Fn$ | Froude number |
$f$ | External force |
$f_s$ | Arbitrary source function |
$f_v$ | Arbitrary vortex function |
$\tilde{G}^{s,p}$ | Complex Green source function |
$G^{v,\mu}$ Complex Green vortex function

$G^{s,\mu} = G^s$ Real Green source function

$G^{v,\mu} = G^v$ Real Green vortex function

$g$ Gravitational constant

$g_s$ Arbitrary source function

$g_v$ Arbitrary vortex function

$h$ Depth of submerged body

$k$ Integration variable

$l_j$ Length of panel $j$

$M$ Summation variable to get an infinite limit

$N$ Number of panels

$n$ Summation variable

$n$ Outward normal vector

$n$ Complex conjugate of $n$

$p$ Pressure

$p_\infty$ Upstream pressure

$R$ Pressure Resistance

$r$ Arbitrary radius

$S$ Circumference of hydrofoil

$S$ Fluid domain boundary

$S_b$ Body surface

$S_F$ Free surface boundary

$S_s$ Boundary around singularity

$S_w$ Wake cut

$S_{\infty}$ Far away upstream, downstream and bottom boundary

$s$ Integration variable along the panel

$t$ Thickness of Joukowski hydrofoil

$t$ Tangential vector

$t$ Complex conjugate of the panel tangent

$U$ Free stream velocity

$v$ Fluid velocity

$x$ Cartesian coordinate or real coordinate in complex plane

$x_0$ Scaling factor when mapping a circle to a Joukowski foil

$\hat{x}$ Real coordinate of source or vortex

$y$ Cartesian coordinate or imaginary coordinate in complex plane

$y_0$ Pressure difference point

$\hat{y}$ Imaginary coordinate of source or vortex

$z$ Coordinate in complex plane

$\bar{z}$ Complex conjugate of $z$

$z_0$ Location of source or vortex

$\bar{z}_0$ Complex conjugate of $z_0$

$z_j$ Coordinate of first node of panel $j$

$z_{j+1}$ Coordinate of second node of panel $j$

$\hat{z}$ Coordinate of source or vortex/control point

$\bar{\hat{z}}$ Complex conjugate of $\hat{z}$
INDEX

Abstract............................................................................................................................................................................i
Nomenclature ................................................................................................................................................................ii
Introduction ...................................................................................................................................................................1
  Mathematical modelling...........................................................................................................................................1
  Resistance modelling...........................................................................................................................................1
Fluid dynamic background...........................................................................................................................................1
  Basic assumptions .....................................................................................................................................................3
  Basic equations and surface conditions.................................................................................................................6
  The complex potential ............................................................................................................................................7
  Green's functions .................................................................................................................................................7
  Expressing the source and vortex potential with Green's functions ........................................................................8
  The potential given by an integral around the boundaries ....................................................................................10
  Inner potential and Greens function on the body ..................................................................................................16
  Discretisation ...........................................................................................................................................................18
  Surface profile .........................................................................................................................................................22
  Resistance .................................................................................................................................................................23
Results ...........................................................................................................................................................................23
  Number of panels ...................................................................................................................................................24
  Comparison with conformal mapping ...................................................................................................................26
  Comparison to deep water method .......................................................................................................................28
  Comparison to experimental results ....................................................................................................................28
    Joukowski hydrofoil ...........................................................................................................................................31
  Comparison with numerical results ....................................................................................................................33
  Surface profile .........................................................................................................................................................34
Conclusions ..................................................................................................................................................................37
References.....................................................................................................................................................................38
INTRODUCTION

With globalization reaching the furthest corners of the world and with the associated rising demand on transportation, the volume and size of the merchant fleet will also increase since the majority of goods are transported by sea. In shipping, a significant cost both to the ship owners and the environment are the fossil fuels used for propulsion. Oil is not an unlimited resource and with the rising price of oil and the growing environmental concern, the motivation to reduce oil consumption has never been higher.

The propulsion resistance of a ship is divided into wave resistance and friction resistance. For slower going ships, like bulk carriers, the friction resistance is the most important whereas for fast ships, such as container and passenger ships, wave resistance is of more importance. This means that even a small reduction in the wave resistance can bring considerable reductions both in operating costs and emissions for such ships. When designing a ship it is important to be able to make fast and accurate predictions of its resistance so that more efficient hull forms can be selected early in the design process. A Reynolds Averaged Navier-Stokes (RANS) based Computational Fluid Dynamics (CFD) software is still too time consuming to be adopted in the initial design process. Thus, a faster tool, still able to detect how changes in hull geometry affect the wave resistance, would be valuable. The key to developing tools for resistance predictions is to set up a mathematical model that describes the physical conditions.

MATHEMATICAL MODELLING

Since the dawn of civilization man has tried to foresee the future. In engineering and science this is best done by mathematical modelling. This means that mathematical expressions are used to capture physical phenomena. If done in the right way, these models can tell what is currently happening in a system and also what is going to happen.

Mathematical models are not built all at once but gradually, starting with a very simple model and then, step by step, introducing more complex phenomena. This methodology gives the possibility to check that the results are reasonable every step of the way.

RESISTANCE MODELLING

A good way to model flowing water is potential theory where the fluid is described as a potential field. The benefits of this method are that is linear and enables capturing of surface effects.

To build a good wave resistance model, first a simple model is constructed and more complex functions can be added later on. A hull is a complex geometrical shape, so a simpler form is chosen, namely the hydrofoil. Somewhere major simplifications have to be made and the geometry is a good starting point since it can be changed reasonably easily later on.

A hydrofoil is a fairly easy object to model because it can be modelled as a 2D object but it still has some complex and interesting geometrical features such as the trailing edge. The main advantage of the hydrofoil is that there is an extensive amount of data, both numerical and experimental, to compare this model to.

Therefore; this report aims at determining the pressure distribution on submerged 2D bodies using dissipative potential flow.

Although this type of hydrofoil analysis has been around for fifty years this report still aims to show something new. Viscous effects will be included in order to get an even better prediction of the wave resistance.

FLUID DYNAMIC BACKGROUND

The knowledge of shipbuilders has been based on the experience of skilled craftsmen throughout history but with the age of enlightenment, science became more important to the shipbuilding industry. Investigations in engineering and physical problems tend to be analytical, numerical or experimental. The early work in this field was purely experimental and as early as 1669 Christian Huygens concluded trough experiments that the ship resistance is proportional to the velocity squared (Huygens 1669). This was
followed by Mariotte (1684), who specified that the resistance depended on the product of the density and the velocity squared.

Newton undertook analytical calculations and concluded that the resistance was not just dependent on the product of the density (of the fluid it was in) and the velocity squared but also the shape of the object. However, he only considered the shape of the projected front area since he assumed water to reflect off an object in a given angle, as light does (Newton 1687).

The basic properties of a fluid, such as pressure, density, and velocity, were investigated by both Johann and Daniel Bernoulli. In 1738 Daniel published the Bernoulli equation (Bernoulli 1738). The idea of a field variable to describe the flow features in a domain was first presented by d’Alembert (1744). However, this theory did not include the velocity nor the pressure and was therefore inadequate.

This was to be changed by Euler who developed a field theory based on Newton’s laws: the Euler-Lagrange differential equations (Euler 1755a, 1755b, 1755c, 1756). The viscosity had up to then been ignored until Navier presented what is today known as the Navier-Stokes equations for incompressible fluids in 1821 and for viscous fluids in 1822 (Anderson 1997). Even though the equations were first presented by Navier they were first correctly derived by Saint-Venant (Anderson 1997).

The Navier-Stokes equations have to be solved in the entire fluid domain. A way to simplify this is with the use of the field variable developed by Euler in the form by Lagrange (1781) and Laplace (Ball 1960). Lagrange developed Euler’s field variable so that the state of the fluid flow could be described using a single field variable, the potential. This potential could also describe the motion of the fluid flow if it was said to be irrotational. Lagrange proposed what is today know as streamlines, the orthogonal lines in the flow with make it possible to use Bernoulli’s equation to determine the pressure in the fluid. Laplace proved that any field variables that were to describing the movements must satisfy a differential equation that today is known as the Laplace equation (Ball 1960).

The wave profile for a gravity surface wave was described by Green (1828) and Airy (1841). The potential under a surface wave over the seabed was proposed by Stokes (1847). To determine the potential boundary conditions are needed. There are two main boundary conditions: the Newman (1877) boundary condition and the Dirichlet boundary condition (1850). The Robin boundary is a condition which is a combination of these two.

The problem of constructing a model using potential flow that satisfies the surface condition can be solved by modelling the potential using a Green’s function, a method which was first introduced by Green (1828).

The field of fluid dynamics using potential flow with a Green’s function has been well explored. Up until the 1970’s over 700 papers had been published on wave resistance (Baar 1986). In this project the classical work of Yeung and Bouger (1979) and Giesing and Smith (1967) will be used as a comparison case. Both because they use a similar but totally inviscid model but also, because their work is well known and often referred to and compared with by others, for example by Bal (1999). Some basic analytic work with viscous potential flow has been carried out by Havelock (1928, 1932).

The flow around a hydrofoil will be modelled in order to determine the resistance of said hydrofoil. It will be explained why this particular model is chosen and the theory behind it will be outlined below.

The problem will be limited to the fluid domain shown in Figure 1.
Using potential flow, the flow will be modelled as a field as explained earlier. Mathematically this will be described using sources placed on the body. The influence of these sources on the fluid will be described using a Green's function. The main benefit of this model is that a Green's function will satisfy the surface condition so that it only needs to be solved on the body. This means that the problem is harder to set up but easier to solve.

Even though none of the following theory is completely new it will be derived from first principles so that others can duplicate the work using only this report. However a basic knowledge of fluid dynamic is required from the reader. To do this the report is broken down into steps where the mathematics is set up. This is shown in Figure 2.

**Basic Assumptions**

Potential theory models the fluid as a field where the velocity field is the gradient of the velocity potential $\phi$. When the flow is modelled as a field a lot of information can be extracted from only one variable, such as velocity and stream lines. In potential flow theory the flow is considered to be inviscid, irrotational and incompressible. The fundamental equation of incompressible flow is the Navier-Stokes equation.
\[ \bar{v} + \nu \nabla \bar{v} + \frac{1}{\rho} \nabla p + \mu' \bar{v} - \nu \nabla^2 \bar{v} = f \]

where \( \bar{v} \) is the fluid velocity, \( \nabla \) the gradient, \( \nabla^2 \) the Laplacian, \( f \) the external force \((0,Mg)\), \( \rho \) the density of the fluid, \( \nu \) is Reynolds viscosity coefficient and \( \mu' \) the Rayleigh parameter that measures the strength of the energy dissipation, which is described by the linear friction force \(-\mu'v\) with \( \mu' \geq 0 \). In common potential flow theory the Rayleigh viscosity is neglected. Here, it is included as described by Chen and Hearn (2010). The fundamental approach when modelling the problem as a potential flow is that bodies submerged in the fluid (or boundaries such as channel walls or the sea bed) are modelled with sources, vortices and sinks. The contributions from each source, vortex and sink are linearly added together to build the flow field. Instead of a wall which will reflect the incoming flow, a distribution of sources is used, which will have the same effect on the flow as a reflecting wall. A schematic picture of a source, doublet and vortex are shown in Figure 3.

![Figure 3 Streamlines and potential of a source, a doublet and a vortex (Chen, 2010a)](image)

Ideally, there would be sources continuously over the body; however that is not numerically practical. For that reason a panel method is used. The object, here the hydrofoil, is divided into small panels. The panels have no curvature which means that a lot of panels are needed to create curved bodies. The panels are of different size so that areas of interest, usually where the gradient changes rapidly, are covered by more densely spaced panels; such areas on hydrofoils are the leading and trailing edge. On each panel a source and vortex is placed, they are considered to have constant strength over that panel as if they were continuous. However, they are mathematically said to be located at a control point in the middle of each panel. The strength of the sources differs between the panels but the vortex strengths are the same for the entire body. The role of the source is to ensure that the boundary conditions are satisfied. For this type of problem, the Neumann boundary condition is suitable. It states that there can be no flow through the submerged body, thus:

\[ \frac{\partial \Phi}{\partial n} = 0 \quad \text{on the body} \]

where \( n \) is the outwards normal vector and \( \Phi \) the perturbed potential. On the free surface both the kinematic and dynamic free surface conditions must be satisfied. The kinematic free surface condition states that no water particles can move through the surface. The dynamic free surface condition states that the pressure below and above the surface must be the same.
\[ \frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial y} \bigg|_{y=0} \]

\[ \frac{Dp}{Dt} = 0 \]

Where \( p \) is the pressure and \( \eta \) is elevation of the surface profile. The boundary conditions are satisfied in discrete points, the control (or collocation points). In order to solve this problem the potential \( \Phi \) must be determined, after which the velocity, \( v \), can then be determined which will yield the pressure. Therefore this model aims at determining \( \Phi \), under the condition that the boundary conditions are satisfied.

It is easier to determine the resistance for submarines and aeroplanes since they only move through one medium. What makes resistance determination for sea going vessels hard is the surface where water and air meet. This why, in hydrodynamics, wave resistance and viscous resistance are divided and determined separately. In order to make the determination of the wave resistance easier, the flow is usually considered to be inviscid.

However, viscosity does affect the wave resistance, so to get a more correct estimation of the wave resistance this model incorporates a viscosity term to capture some of the viscous effects (Chen and Hearn 2010). In reality waves decay with distance due to friction. The added viscosity therefore enables a better modelling of dissipative waves. A single NACA profile hydrofoil is modelled in deep water. The hydrofoil is close to the surface so surface effects are included. A Greens function is used to describe the strength of the sources and vortices. In the case of deep water as for the submarine, the sources are most easily modelled with the fundamental solution, i.e. a Rankin source the influence of which in 2D is described as ln(\( r \)). The fundamental solution does not satisfy the surface condition. Therefore a Greens function which will satisfy all of the boundary conditions is used. The geometry and nomenclature of the domain is shown in Figure 4.

\[ U \] is the free stream velocity, \( \alpha \) the angle of attack, \( n \) the normal to the hydrofoil surface and \( t \) the tangent. The surface is located at \( y=0 \). The boundaries for this model are denoted \( S \), where \( S_\infty \) is the faraway upstream and downstream boundaries but also the downwards boundary since the water is considered to be deep. \( S_F \) is the free surface boundary, \( S_b \) the body boundary, \( S_w \) the wake cut and \( S_\epsilon \) is the boundary around the singularities. Each singularity has its own boundary around it. The singularities are located on the hydrofoil but are here placed at an arbitrary place in the fluid domain for better readability. A source model is used but since the sources are located on the hydrofoil which also is the area of interest, there is going to be a problem with obtaining the potential in the fluid domain. This is due to the fact the source is a mathematical singularity which means that any information from the domain will be useless as long as it contains singularities. The easiest way to handle this is to cut the singularities out by putting a boundary around them \( S_\epsilon \).
BASIC EQUATIONS AND SURFACE CONDITIONS

Some Green’s functions have singularities which lead to numerical problems. This is eliminated by adding a viscosity term according to Chen and Hearn (2010), which also enables modelling of some of the viscous effects. However, the flow is still considered to be irrotational, incompressible and inviscid. Starting from first principles and starting in the real Cartesian plane. The basic governing fluid equations are the Navier-Stokes and the continuity equation. The continuity equation is given by

\[ \nabla \cdot \mathbf{v} = 0 \]  

5

Using the basic properties

\[ \nabla \Phi = \mathbf{v} \]  \[ \Phi = Ux + \phi \]  

6 7

The velocity potential is given by equation 6, with the perturbed velocity potential in equation 7. Here \( U \) is the uniform free stream velocity, \( \Phi \) the total velocity potential and \( \phi \) the perturbed velocity potential according to Chen and Hearn (2010).

The continuity equation, equation 5 then reduces to the Laplace equation:

\[ \nabla^2 \phi = 0 \]  

8

Without viscosity the disturbances will not decay with time or distance. It is the friction that makes waves further away from the source decrease in amplitude, i.e. be dissipative. Using the continuity equation and the Laplace equation, the Navier–Stokes equation is reduced to the dissipative Bernoulli equation

\[ \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{\rho} p + gy + \mu' \phi = \frac{1}{2} U^2 + \frac{1}{\rho} p_\infty (y_0) + gy_0 \]  

9

Here a linearization is made, the location of the surface, is assumed to always be at approximately \( y = 0 \). This is a simplification, since the x-axis is a straight line but the surface profile is oscillating. The reference pressure is taken far upstream at a depth \( y = y_0 \). \( \eta \) is the wave evaluation.

\[ y = \eta(x) \]  

10

The pressure is considered to be constant along any streamline. With \( y_0 = 0 \) equation 9 can be rewritten as

\[ 0 = \frac{1}{2} |\nabla \Phi|^2 + g\eta + \mu' \phi - \frac{1}{2} U^2 \]  

11

The waves are considered to be long which means that \( \eta \) and \( \phi \) are of the same small magnitude. This is in line with typical presumption for linear wave theory, which alters equation 11 so that

\[ 0 = U \phi_x (x, \eta) + g\eta + \mu' \phi (x, \eta) \Rightarrow \]  

12

\[ 0 = U \phi_x (x, 0) + g\eta + \mu' \phi (x, 0) \]  

13
This is the potential function linearized around the free surface so that \( \phi(x, \eta) = \phi(x, 0) \). On the surface \( y = 0 \) but since \( \eta(x) \neq 0 \) this will yield a discrepancy between this method and experiments and the surface profiles will not match. This means that equation 10 is not entirely accurate. However, this will not affect the result on the hydrofoil. The free surface is a streamline which can be described as:

\[
\Psi = Uy + \psi
\]

where \( \Psi \) is a stream function, which only exist in 2D and it is constant along the surface. \( \psi \) is the perturbed stream function. The stream function is then going to satisfy the surface condition.

\[
0 = \Psi \approx U\eta(x) + \psi(x, 0) \Rightarrow \eta = -\frac{\psi}{U}
\]

Using the relations above and

\[
\nu = \frac{g}{U^2}, \quad \mu = \frac{\mu'}{U}
\]

equation 13 reduces to

\[
0 = \varphi_x - \nu \psi + \mu \varphi
\]

where \( \varphi \) is the complex velocity potential.

**THE COMPLEX POTENTIAL**

Since vector geometry is easier to describe in the complex plane, the analysis is therefore moved to the Argand plane. The complex potential is defined as

\[
\phi(z) = \phi(z) + i \psi(z) \quad \text{where} \quad z = x + iy
\]

This enables the use of the complex dissipative free surface condition. The dissipative free surface condition is derived using equation 19 and 20.

\[
\text{Im}(\varphi_z + i \nu \varphi + \mu \varphi) = 0
\]

The potential is a mathematical magnitude constructed to solve flow problems. To solve this problem the strength of each source and the vortex must be determined. The influence of the sources and vortex is modelled with a Green’s function. This will make it easy to obtain the potential \( \varphi \).

**Green’s functions**

The concept of Green’s functions was developed in the 1930s by the English mathematician George Green. It is usually used to solve inhomogeneous differential equations subject to boundary conditions. In physics it is used to solve different field problems.

The same way mechanical vibration and damping problems are best described with a differential equation; a Green’s function is a function that is well suited for describing field problems. A Green’s function is an
integral function that can be used to solve an inhomogeneous differential equation with boundary conditions. It is especially good in the complex plane and by solving a Green’s function on the contour one can obtain the solution inside the bounded area. The three main benefits of using a Green’s function to model a source distribution are: that it satisfies the Laplace equation, it satisfies the free surface condition and it is equal to the Dirac function in the fluid domain.

**Expressing the source and vortex potential with Green’s functions**

A source or vortex is located at \(z_0\) and has strength of \(2\pi\). The uniform flow \(U\) travels in positive \(x\)-direction and \(\varphi\) is the distributed complex potential; \(\varphi\) is holomorphic, which means that it is differentiable in a close proximity to a point and that it satisfies the Cauchy–Riemann equations. \(\varphi\) is a solution to the Laplace equation if \(y < 0\), i.e. in the fluid domain. The complex potential for a source and vortex, as seen in Figure 3, is generally expressed as

\[
\varphi(z) = \ln(z - z_0) + f_s(z)
\]

where \(z_0\) is the location of a source

\[
\varphi(z) = i[\ln(z - z_0) + f_v(z)]
\]

where \(z_0\) is the location of a vortex

where \(f_s\) and \(f_v\) are holomorphic in the fluid domain; they are general and arbitrary and will be eliminated later on. This resembles the fundamental solution and must therefore be complemented to satisfy the free surface condition. The complex dissipative free surface condition, equation 22, then becomes

\[
i(\varphi_z + (iv + \mu) \varphi) = i\left(\frac{1}{z - z_0} + (iv + \mu) \ln(z - z_0) + g_s(z)\right)
\]

\(z_0\) is the location of a source

\[
i(\varphi_z + (iv + \mu) \varphi) = i\left(\frac{i}{z - z_0} + i(v + \mu) \ln(z - z_0) + g_v(z)\right)
\]

\(z_0\) is the location of a vortex

where

\[
g_s = i(f_s'(z) + (iv + \mu)f_s(z))
\]

\[
g_v = i(f_v'(z) + (iv + \mu)f_v(z))
\]

In order to get a function which is zero at the surface (for any \(x\)) an image method is used which introduces a function above the surface, such that when it is added to the original function the new function yields zero on the surface. The free surface is a streamline so one source is put under the surface and another one is put above the surface, they are the mirror image of each other. They both produce a streamline at the surface which no flow can cross. This corresponds to the properties of both the free surface and a streamline as seen in Figure 5.
This leads to a new function (called $F$ in equation 29), so that $F(x,0)=0$ on the surface $y=0$. Thus

\[
F(x+iy) - F(x-iy) = 0 \quad \Rightarrow
\]

\[
i(\varphi_z + (i\nu + \mu) \varphi) = i \left( \frac{1}{z - z_0} + (i\nu + \mu) \ln(z - z_0) - \frac{1}{z - z_0} + (i\nu - \mu) \ln(z - z_0) \right)
\]

\[
i(\varphi_z + (i\nu + \mu) \varphi) = i \left[ \frac{i}{z - z_0} + i(i\nu + \mu) \ln(z - z_0) + \frac{i}{z - z_0} - i(i\nu - \mu) \ln(z - z_0) \right]
\]

In equation 30 $z_0$ is the location of a source and in equation 31 $\zeta$ is the location of a vortex. $\varphi$ is holomorphic in the whole conjugate, except on $\zeta_0$ (and $\zeta_0$). Observing the basic mathematical property

\[
\varphi_z + (i\nu + \mu) \varphi = e^{-(i\nu + \mu)z} \frac{d}{dz} \left( e^{(i\nu + \mu)z} \varphi \right)
\]

can be used to shorten these expressions. Integrating equation 30 and 31 using equation 32 gives

\[
\varphi = \ln(z - z_0) + \frac{i\nu - \mu}{i\nu + \mu} \ln(z - z_0) - \frac{2i\nu}{i\nu + \mu} \int_{t - z_0}^{z_0} e^{(i\nu + \mu)(t-z)} dt
\]

\[
\varphi = i \left[ \ln(z - z_0) + \frac{i\nu - \mu}{i\nu + \mu} \ln(z - z_0) - \frac{2i\nu}{i\nu + \mu} \int_{t - z_0}^{z_0} e^{(i\nu + \mu)(t-z)} dt \right]
\]

where equation 33 is for a source point and equation 34 for a vortex point. To make the integral easier a substitution is made, using

\[
i(\nu - i\mu)(t - z) = -ik(z - z_0)
\]

Which is the same as

\[
(\nu - i\mu)(t - z_0) = (\nu - i\mu - k)(z - z_0)
\]
Differentiating both sides of equation 35 gives, upon observing that \( k \) is depending on \( t \):

\[
(v - i \mu) = \frac{dk}{dt} \left( z - z_0 \right)
\]

Which is the same as

\[
\frac{dt}{t - z_0} = - \frac{dk}{v - i \mu - k}
\]

The complex potential can now be described as a Green’s function and will henceforth be denoted \( G \):

\[
\hat{G}_{s,\mu}^i(z, z_0) = \varphi_s(z, z_0) \quad \hat{G}_{v,\mu}^i(z, z_0) = \varphi_v(z, z_0)
\]

Using equation 35 and 38 in equation 33 and 34 will give the complex Green’s function

\[
\hat{G}_{s,\mu}^i(z, z_0) = \ln \left( z - z_0 \right) + i \frac{v - \mu}{iv + \mu} \ln \left( z - z_0 \right) - \frac{2iv}{iv + \mu} \int_0^\infty \frac{1}{k - v + i \mu} e^{-ik(z - z_0)} \, dk
\]

\[
\hat{G}_{v,\mu}^i(z, z_0) = i \left[ \ln \left( z - z_0 \right) - i \frac{v - \mu}{iv + \mu} \ln \left( z - z_0 \right) - \frac{2iv}{iv + \mu} \int_0^\infty \frac{1}{k - v + i \mu} e^{-ik(z - z_0)} \, dk \right]
\]

where index \( s \) is for the source and \( v \) is for the vortex. The real part of the above equation is given by

\[
G_{s,\mu}^i = \text{Re} \left\{ \hat{G}_{s,\mu}^i(z, z_0) \right\} \quad \text{or in brevity} \quad G_s^i
\]

\[
G_{v,\mu}^i = \text{Re} \left\{ \hat{G}_{v,\mu}^i(z, z_0) \right\} \quad \text{or in brevity} \quad G_v^i
\]

**The potential given by an integral around the boundaries**

The potential is given by integrating the derivative of the potential around the boundaries. The velocity potential that describes the moving hydrofoil is a combination of sources and vortices placed on the surface of the hydrofoil. \( \varphi \) is the distributed complex potential and \( \phi \) is the real distributed potential. The fluid domain has several boundaries where \( S_b \) is the body, \( S_f \) the free surface, \( S_w \) the wake cut, \( S_\infty \) the cut far up- or downstream and \( S_\epsilon \) a circle of small radius \( \epsilon \) centred at \( \hat{z} = \hat{x} + i\hat{y} \) which is the location of a singularity i.e. a source or vortex. This singularity can be placed anywhere in the fluid according to Figure 4; later on the specific case for which the singularities lie on the hydrofoil will be explained. The complex potential satisfies the Laplace equation. The potential has to be smooth which is the case only if a wake cut is made, since only the velocity \( \nabla \phi \) and not \( \phi \) is continuous over the wake. Using Green’s theorem, and in particular Green’s second identity gives

\[
0 = \oint_{\partial S} \phi(z) n \cdot \nabla G^i(\hat{z}, z) \, ds_z - \int_{\partial S} G^i(\hat{z}, z) n \cdot \nabla \phi(z) \, ds_z
\]

where region \( S = S_b \cup S_f \cup S_w \cup S_\epsilon \)

where the contours \( S \) are seen in Figure 4 and \( g(z) \) is the surface normal vector field pointing into the fluid. This might seem like a complicated integral but it will be shown that the integral around the
boundary $S$ can be reduced to the integral around the body and the singularity, which is much simpler. It is assumed that $\phi \to 0$ $\nabla \phi \to 0$ close to the far away boundary $S_\infty$. Hence, equation 45 is valid for

$$S = S_x \cup S_b \cup S_s \cup S_w$$

The easiest way to evaluate the integral in equation 45 is to do it separately for each subregion of $S$. Integration is carried out clockwise starting with the integral around the singularity:

$$\lim_{\varepsilon \to 0} \left( \int_{S_x} \phi(z) n \cdot \nabla G^*(\hat{z}, z) ds_z - \int_{S_s} G^*(\hat{z}, z) n \cdot \nabla \phi(z) ds_z \right)$$

All non-singular parts will be equal to zero according to Cauchy’s theorem or Green’s third identity. The only singularity in $\nabla G^*$ is $\nabla \ln |\hat{z} - z|$ so that equation 48 becomes

$$\lim_{\varepsilon \to 0} \int_{|\hat{z}| = \varepsilon} \phi(z) n \cdot \nabla \ln |\hat{z} - z| ds_z - \lim_{\varepsilon \to 0} \int_{|\hat{z}| = \varepsilon} \ln |\hat{z} - z| n \cdot \nabla \phi(z) ds_z$$

Changing the integration variable so that $ds = \varepsilon d\theta$ where $n \cdot \nabla \phi(z)$ is constant in this interval so it can be moved outside the integral

$$\lim_{\varepsilon \to 0} \int_{2\pi}^{0} \phi(z) n \cdot \varepsilon d\theta - \lim_{\varepsilon \to 0} \int_{2\pi}^{0} \ln \varepsilon \cdot \varepsilon d\theta =$$

$$\phi(\hat{z}) \int_{2\pi}^{0} d\theta - \lim_{\varepsilon \to 0} 2\pi \ln \varepsilon \cdot \varepsilon$$

As $\varepsilon$ approaches zero, $\ln \varepsilon$ slowly approaches minus infinity, with the result that $\varepsilon \ln \varepsilon$ approaches zero as $\varepsilon$ does. Equation 51 then yields $-2\pi \phi(\hat{z})$, which is consistent with Cauchy’s theorem.

The free surface boundary is more complex so some mathematical simplifications of the integral are needed. This will enable the use of basic physical properties. On the free surface $S_F$, $\eta(x) = 0$. The downwards normal is $(\eta_x, -1)$ and the tangent to the free surface is $(dx, dy)$. The unit vector in the normal direction is therefore

$$n = \frac{\eta_x, -1}{\sqrt{\eta_x^2 + 1}}$$

so that the normal vector scalar products with the derivative of $G$ and $\phi$ become

$$n \cdot \nabla G^* = \frac{G_y \eta_x - G_x \eta_y}{\sqrt{\eta_x^2 + 1}}$$

$$n \cdot \nabla \phi = \frac{\phi_x \eta_x - \phi_y \eta_y}{\sqrt{\eta_x^2 + 1}}$$
By the use of the dynamic free surface boundary condition and the total velocity potential equation 7, and equation 54 give

\[
\frac{\partial \Phi}{\partial n} = 0 \iff n \cdot \nabla \Phi = 0 \Rightarrow n \cdot \nabla (Ux + \phi) = 0 \Rightarrow \frac{U \eta_x + \phi \eta_x - \phi_x}{\sqrt{\eta_x^2 + 1}} = 0 \Rightarrow
\]

\[
\eta \cdot \nabla \phi = -\frac{U \eta_x}{\sqrt{\eta_x^2 + 1}}
\]

This enables the evaluation of the integral in equation 45 over the region \( S_F \):

\[
\int_{S_F} (G' (\hat{z}, z) n \cdot \nabla \phi(z) - \phi(z) n \cdot \nabla G' (\hat{z}, z)) ds_z
\]

Inserting equation 53 and 56 in equation 57 yields

\[
\int_{S_F} \left( -\frac{U \eta_x G'}{\sqrt{\eta_x^2 + 1}} - \frac{\phi(z) (G' \eta_x - G'_x)}{\sqrt{\eta_x^2 + 1}} \right) ds_z
\]

However some more substitution is still needed to achieve a fairly simple integral:

\[
ds = \sqrt{dx^2 + dy^2} \Rightarrow ds = \frac{ds}{\sqrt{1 + \left( \frac{dy}{dx} \right)^2}}
\]

Using equations 59 and 10, the integral in equation 58 becomes

\[
\int_{S_F} \left( -U \eta_x G' - \phi(z) (G'_x G_x - G_x) \right) dx
\]

where both \( \phi \) and \( \eta \) are small. This is linearized around the free surface, by omitting \( \phi \cdot \eta \) and \( \eta^2 \) giving

\[
\int_{S_F} \left( -U \eta_x G' - \phi(z) (G'_x) \right) dx
\]

Integrating equation 61 by parts will give

\[
\int_{S_F} \left( -U \eta_x G' - \phi(z) (G_x) \right) dx = \left[ -U \eta G' \right]_0^\infty + \int_{S_F} \left( U \eta G'_x + \phi(z) (G_x) \right) dx
\]

Using equation 12, 17 and 18, the integral becomes

\[
\int_{S_F} \left( -\frac{\phi_x + \mu \phi}{\nu} G'_x + \phi(z) (G_x) \right) dx
\]

and integrating by part again yields
\[ \int_{s_x} \left( \frac{G^{s^*}_x + vG^{s^*}_y - \mu G^{s^*}_y}{\nu} \right) dx \]

since \( \phi \rightarrow 0 \) at \( \pm \infty \). Due to the radiation condition imposed on \( \phi \)
\[ G^{s^*}_x (\hat{z}, z) = -G^{s^*}_x (\hat{z}, z) \quad G^{s^*}_y (\hat{z}, z) = G^{s^*}_y (\hat{z}, z) \quad \text{at } y = 0 \]
where the later part of equation 65 is only valid for small values of \( \mu \). Equation 65 gives
\[ \int_{s_x} \left( \frac{ \phi \left( G^{s^*}_x + vG^{s^*}_y + \mu G^{s^*}_y \right) }{\nu} \right) dx \]

The complex function \( \tilde{G} \) is holomorphic which means it satisfies the Cauchy Riemann equations. This means that equation 66 can be written as
\[ \int_{s_x} \left( \frac{ \partial x \left( \text{Re} \tilde{G}^{s^*}_x - v \text{Im} G^{s^*} + \mu \text{Re} G^{s^*} \right) }{\nu} \right) dx \]

Using the properties of the complex function \( \{ \tilde{G}^s = \phi = \phi + i\psi \} \) leads to
\[ \int_{s_x} \left( \frac{ \phi \left( \partial_x (\phi - v\psi + \mu\phi) \right) }{\nu} \right) dx = 0 \]
which is only valid if \( \tilde{G}^s = \tilde{G}^{s^*\mu} \). The case when \( \tilde{G}^s = \tilde{G}^{s^*\mu} \) also has to be addressed. Equation 67 can be written as
\[ \int_{s_x} \left( \frac{ \partial x \left( \frac{ \text{Re} \tilde{G}^{s^*}_x - v \text{Im} G^{s^*} + \mu \text{Re} G^{s^*} - 2\mu \text{Re} G^s \right) }{\nu} \right) dx \]
where using equation 19, most of the terms vanish and the integral becomes very small:
\[ \int_{s_x} \left( \frac{ \partial x \left( -2\mu \text{Re} G^s \right) }{\nu} \right) dx \approx 0 \]

This leaves the integration over the body \( S_b \) and the wake cut \( S_w \); the wake cut is divided into an upper surface \( S_{+w} \) and a lower surface \( S_{-w} \).
\[ \int_{S_b \cup S_{+w} \cup S_{-w}} \left( \phi(z) n \cdot \nabla G^s (\hat{z}, z) - G^s (\hat{z}, z) n \cdot \nabla \phi(z) \right) ds \]
The Kutta conditions states that the pressure and normal velocity are continuous over the wake cut. This means that the normal velocity \( \mathbf{n} \cdot \nabla \phi \) has the same magnitude on the upper and lower surface and only differs in direction. Therefore there is no need to integrate the normal velocity over the wake cut.

\[
\Rightarrow \int_{S_u} \phi(z) n(z) \cdot \nabla G^* (\hat{z}, z) ds_z - \int_{S_l \cup S_u} G^* (\hat{z}, z) n(z) \cdot \nabla \phi(z) ds_z
\]

This integral is more complex than the previous ones and hence each scalar product will be described separately. As seen in the second term of equation 72, the expression \( \mathbf{n}(z) \cdot \nabla \phi(z) \) needs to be evaluated. However \( G^* (\hat{z}, z) \) is not holomorphic with respect to \( \hat{z} \) due to the fact that the conjugate of \( \hat{z} \), \( \overline{\hat{z}} \) is not holomorphic. \( n \) and \( t = in \) are the complex forms of the normal and tangential vectors \( \mathbf{g} \) and \( \xi \). Then,

\[
\mathbf{n} \cdot \nabla A = \overline{n} \frac{\partial A}{\partial \overline{\xi}} \quad \text{if } A \text{ contains } \overline{\xi}
\]

\[
\mathbf{n} \cdot \nabla A = n \frac{\partial A}{\partial \xi} \quad \text{if } A \text{ does not contain } \overline{\xi}
\]

These equations yield

\[
\mathbf{n}(z) \cdot \nabla G^{*\mu} (\hat{z}, z) = \frac{1}{2} \left( \mathbf{n}(z) \cdot \nabla \overline{G^*} (\hat{z}, z) + \mathbf{n}(z) \cdot \nabla G^* (\hat{z}, z) \right)
\]

which can be expanded in the Green’s function:

\[
\frac{1}{2} \left( \mathbf{n}(z) \cdot \nabla \ln (\hat{z} - z) + \mathbf{n}(z) \cdot \nabla \left( \frac{i\nu - \mu}{i\nu + \mu} \ln (\hat{z} - \overline{z}) + \frac{2i\nu}{i\nu + \mu} \int_0^\infty \frac{e^{-ik(\hat{z} - \tau)}}{k - \nu + i\mu} dk \right) \right)
\]

\[
+ \frac{1}{2} \left( \mathbf{n}(z) \cdot \nabla \ln (\overline{z} - \overline{\xi}) + \mathbf{n}(z) \cdot \nabla \left( \frac{-i\nu - \mu}{-i\nu + \mu} \ln (\overline{z} - \overline{z}) - \frac{2i\nu}{-i\nu + \mu} \int_0^\infty \frac{e^{ik(\overline{z} - \tau)}}{k - \nu - i\mu} dk \right) \right)
\]

Differentiating this function yields

\[
\left\{ \begin{array}{c}
\frac{1}{2} \frac{n(z) \partial \ln (\hat{z} - z)}{\partial \xi} + \overline{n}(z) \frac{\partial}{\partial \overline{\xi}} \left( \frac{i\nu - \mu}{i\nu + \mu} \ln (\hat{z} - \overline{z}) + \frac{2i\nu}{i\nu + \mu} \int_0^\infty \frac{e^{-ik(\hat{z} - \tau)}}{k - \nu + i\mu} dk \right) + \\
\frac{1}{2} \frac{\partial}{\partial \overline{\xi}} \left( \frac{\partial}{\partial \xi} \left( \frac{i\nu - \mu}{i\nu + \mu} \ln (\hat{z} - \overline{z}) + \frac{2i\nu}{i\nu + \mu} \int_0^\infty \frac{e^{-ik(\hat{z} - \tau)}}{k - \nu + i\mu} dk \right) \right)
\end{array} \right. +
\]

\[
\left\{ \begin{array}{c}
\frac{1}{2} \frac{\partial \ln (\overline{z} - \overline{\xi})}{\partial \xi} + \overline{n}(z) \frac{\partial}{\partial \overline{\xi}} \left( \frac{-i\nu - \mu}{-i\nu + \mu} \ln (\overline{z} - \overline{z}) - \frac{2i\nu}{-i\nu + \mu} \int_0^\infty \frac{e^{ik(\overline{z} - \tau)}}{k - \nu - i\mu} dk \right) + \\
\frac{1}{2} \frac{\partial}{\partial \overline{\xi}} \left( \frac{\partial}{\partial \xi} \left( \frac{-i\nu - \mu}{-i\nu + \mu} \ln (\overline{z} - \overline{\xi}) - \frac{2i\nu}{-i\nu + \mu} \int_0^\infty \frac{e^{ik(\overline{z} - \tau)}}{k - \nu - i\mu} dk \right) \right)
\end{array} \right. \]
Using the properties of the complex functions enables the function to be shortened using only the real part

\[
\text{Re} \left( n(z) \frac{\partial \ln(\hat{z} - z)}{\partial z} + \bar{n}(z) \right) - \frac{\partial}{\partial \bar{z}} \left( \frac{iv - \mu}{iv + \mu} \ln(\hat{z} - \bar{z}) + \frac{2iv}{iv + \mu} \int_{0}^{\infty} e^{-i(k - \nu)dk} \right)
\]

This can also be expressed as a function of the tangent instead of the normal:

\[
\text{Re} \left( it(z) \frac{\partial \ln(\hat{z} - z)}{\partial z} - i\bar{t}(z) \right) - \frac{\partial}{\partial \bar{z}} \left( \frac{iv - \mu}{iv + \mu} \ln(\hat{z} - \bar{z}) + \frac{2iv}{iv + \mu} \int_{0}^{\infty} e^{-i(k - \nu)dk} \right)
\]

With the same approach, applied to the vortex Green's function,

\[
\bar{t}(z) \cdot \nabla G^{v,\mu} =
\]

\[
\text{Re} \left( it(z) \frac{\partial \ln(\hat{z} - z)}{\partial z} - i\bar{t}(z) \right) - \frac{\partial}{\partial \bar{z}} \left( \frac{iv - \mu}{iv + \mu} \ln(\hat{z} - \bar{z}) + \frac{2iv}{iv + \mu} \int_{0}^{\infty} e^{-i(k - \nu)dk} \right)
\]

This is also true if \( \mu = -\mu \). Comparing equation 79 and 80 it is seen that

\[
n(z) \cdot \nabla G^{v,\mu} = \bar{t}(z) \cdot \nabla G^{v,\mu}
\]

so that equation 72 becomes

\[
\int_{S_{b}} G^{v} (\hat{z}, z) n \cdot \phi(z) \nabla dz - \int_{S_{b} \cup S_{w} \cup S_{w+v}} \phi(z) \bar{t}(z) \cdot \nabla G^{v} (\hat{z}, z) ds_{z}
\]

The second part of the equation above is integrated by parts from the infinite ending of \( S_{w} \) to \( S_{w+v} \), remembering that \( \phi \to 0 \) at \( S_{\infty} \):

\[
\int_{S_{w} \cup S_{b} \cup S_{w+v}} \phi(z) \bar{t}(z) \cdot \nabla G^{v} (\hat{z}, z) ds_{z} = - \int_{S_{w} \cup S_{b} \cup S_{w+v}} (\bar{t}(z) \cdot \nabla \phi(z)) G^{v} (\hat{z}, z) ds_{z}
\]

As before the Kutta condition states that the tangential velocity at the wake cut are continuous so that the integral only has to be carried out over the body. Hence, equation 82 becomes

\[
\int_{S_{b}} (n \cdot \nabla \phi(z)) \text{Re} \, G^{v} (\hat{z}, z) + (\bar{t}(z) \cdot \nabla \phi(z)) \bar{G}^{v} (\hat{z}, z) ds_{z}
\]
To conclude, this integration started with the integral in equation 45. The integral was then divided into sub integrals 51, 69 and 84, which combined result in

$$0 = \int_{S} \phi(z) n \cdot \nabla G^{i}(\hat{z}, z) \, ds_{z} - \int_{S} G^{i}(\hat{z}, z) n \cdot \nabla \phi(z) \, ds_{z} =$$

$$\phi(\hat{z}) \int_{0}^{2\pi} d\theta \lim_{\epsilon \to 0} 2\pi \ln \epsilon \cdot \epsilon + \int_{S_{p}} \phi \left( \frac{\partial x(-2\mu \text{Re} G^{i})}{\nu} \right) \, dx$$

$$+ \int_{S} (n \cdot \nabla \phi(z)) \text{Re} G^{i}(\hat{z}, z) + (t(z) \cdot \nabla \phi(z)) \hat{G}^{v}(\hat{z}, z) \, ds_{z} \Rightarrow$$

Inserting the result of these integrals gives

$$0 = -2\pi \phi(\hat{z}) + 0 + \int_{S} (n \cdot \nabla \phi(z)) \text{Re} G^{i}(\hat{z}, z) + (t(z) \cdot \nabla \phi(z)) \hat{G}^{v}(\hat{z}, z) \, ds_{z}$$

so that the body integral becomes

$$-2\pi \phi(\hat{z}) = \int_{S} (n \cdot \nabla \phi(z)) \text{Re} G^{i}(\hat{z}, z) + (t(z) \cdot \nabla \phi(z)) \hat{G}^{v}(\hat{z}, z) \, ds_{z}$$

If $\hat{z} \in S_{b}$ then the circle $S_{b}$ becomes a semi-circle which means that

$$\int_{S} (n \cdot \nabla \phi(z)) \text{Re} G^{i}(\hat{z}, z) + (t(z) \cdot \nabla \phi(z)) \hat{G}^{v}(\hat{z}, z) \, ds_{z} = \pi \phi(\hat{z})$$

$$\pi \text{ if } \hat{z} \in S_{b}$$

$$\tau = 2\pi \text{ if } \hat{z} \text{ is in the fluid domain}$$

$$0 \text{ if } \hat{z} \text{ is inside } S_{b}$$

**INNER POTENTIAL AND GREENS FUNCTION ON THE BODY**

There is also a flow potential inside the body $\phi_{m}$ that has to be considered. To get the source and vortex strength, the difference between the flow integral and the inner flow integral has to be determined. The inner flow integral is

$$\int_{S} (n \cdot \nabla \phi_{m}(z)) \text{Re} G^{i}(\hat{z}, z) + (t(z) \cdot \nabla \phi_{m}(z)) \hat{G}^{v}(\hat{z}, z) \, ds_{z} = 0$$

where source and vortex strengths are defined as

$$\pi \sigma = n \cdot \nabla \phi(z) - n \cdot \nabla \phi_{m}(z)$$

$$\pi \sigma = t \cdot \nabla \phi(z) - t \cdot \nabla \phi_{m}(z)$$

This gives the perturbed velocity potential in terms of a source and a vortex distribution on the hydrofoil surface as
\[
\int_{s_b} \sigma \text{Re} \hat{G}^i (\hat{z}, z) + \gamma \text{Re} \hat{G}^v (\hat{z}, z) \, ds_z = \phi (\hat{z})
\]

The singularities are located on the body, the normal vector \( \hat{n}(z) \) becomes \( \hat{n}(\hat{z}) \). Here, the Green’s function \( G^i (\hat{z}, z) \) is holomorphic with respect to \( \hat{z} \), \( n \) and \( t= in \) are the complex forms of the normal and tangential vectors \( g \) and \( t \):

\[
\hat{n}(\hat{z}) \cdot \nabla G^i (\hat{z}, z) = \frac{1}{2} \left( \hat{n}(\hat{z}) \cdot \nabla \hat{G}(\hat{z}, z) + \hat{n}(\hat{z}) \cdot \nabla \hat{G}(\hat{z}, z) \right) = \]

\[
\Re \left( \frac{\partial \hat{G}(\hat{z}, z)}{\partial \hat{z}} \right)
\]

In the same way as for the tangent and the Green’s function, we get the following expressions for a source point, normal vector and tangential vector, respectively:

\[
\hat{t}(\hat{z}) \cdot \nabla G^i (\hat{z}, z) = \Re \left( \frac{\partial \hat{G}^i(\hat{z}, z)}{\partial \hat{z}} \right)
\]

\[
\hat{n}(\hat{z}) \cdot \nabla G^v (\hat{z}, z) = \Re \left( \frac{\partial \hat{G}^v(\hat{z}, z)}{\partial \hat{z}} \right)
\]

\[
\hat{t}(\hat{z}) \cdot \nabla G^v (\hat{z}, z) = \Re \left( \frac{\partial \hat{G}^v(\hat{z}, z)}{\partial \hat{z}} \right)
\]

Using equation 94, 95, 96, 97 and 92 it is seen that when \( \hat{z} \in S_b \) the normal velocity expressions satisfies

\[
\hat{n}(\hat{z}) \cdot \nabla \phi(\hat{z}) = \Re \left( \hat{n}(\hat{z}) \int_{s_b} \sigma \frac{d\hat{G}^i(\hat{z}, z)}{d\hat{z}} + \gamma \frac{d\hat{G}^v(\hat{z}, z)}{d\hat{z}} \, ds_z \right) = 
\]

\[
\Re \left( i \hat{t}(\hat{z}) \int_{s_b} \sigma \frac{d\hat{G}^i(\hat{z}, z)}{d\hat{z}} + \gamma \frac{d\hat{G}^v(\hat{z}, z)}{d\hat{z}} \, ds_z \right)
\]

This yields the tangential velocity as

\[
\hat{t}(\hat{z}) \cdot \nabla \phi(\hat{z}) = \Re \left( \hat{t}(\hat{z}) \int_{s_b} \sigma \frac{d\hat{G}^i(\hat{z}, z)}{d\hat{z}} + \gamma \frac{d\hat{G}^v(\hat{z}, z)}{d\hat{z}} \, ds_z \right)
\]

This gives the free surface dependant integral equation, which is the solution to equation 45. In order to practically use this some numerical approximations are needed.
**DISCRETISATION**

In order to use numerical tools, discretisation of the boundary is necessary. One way of doing this is to describe the object as a series of panels i.e. using the panel method. Each panel has no curvature. The panels are identified by the nodes $z_j$ and the collocation points $\hat{z}_i$. A schematic picture of the discretisation is shown in Figure 6.

![Figure 6 Discretisation of the hydrofoil](image)

$N$ is the number of nodes used and the locations of the nodes are denoted as:

\[ z_j = x_j + iy_j \quad j = 1...N \]
\[ z_{j+1} = x_{j+1} + iy_{j+1} \eqno{101} \]

so that panel $j$ is between the nodes $j$ and $j+1$. The trailing edge point, $T_p$ in Figure 6 is $z_N = z_{N+1}$. The collocation point on panel $i$ is

\[ \hat{z}_i = \frac{z_{i+1} + z_i}{2} \quad i = 1...N \eqno{102} \]

The tangent along the $j$th panel at the collocation point $\hat{z}_i$ is defined as

\[ t_j = t(\hat{z}_i) = \frac{z_{j+1} + z_j}{l_j} \]
\[ l_j = |z_{j+1} - z_j| \]
\[ \hat{z}_i = \frac{z_{i+1} + z_i}{2} \quad i = 1...N \eqno{102} \]

Where $l_j$ is the length of panel $j$. Discretizing equation 98 yields

\[ n(\hat{z}_i) \cdot \nabla \phi(\hat{z}_i) = \text{Re} \sum_{j=1}^{N} \int_{\text{panel } j} \left( \sigma_j \frac{d\tilde{G}^{i \mu}(\hat{z}_i, z)}{d\hat{z}} + \gamma \frac{d\tilde{G}^{\nu \mu}(\hat{z}_i, z)}{d\hat{z}} \right) ds_z \quad i = 1...N \]
\[ \int_{\text{panel } j} \left( \sigma_j \frac{d\tilde{G}^{i \mu}(\hat{z}_i, z)}{d\hat{z}} + \gamma \frac{d\tilde{G}^{\nu \mu}(\hat{z}_i, z)}{d\hat{z}} \right) ds_z \quad i = 1...N \eqno{105} \]

which is the effect of all the sources and vortexes combined acting on one panel. For brevity this is denoted as
\[ A_j = \int_{\text{panel } j} \frac{dG^{x,y}(\hat{z}, z)}{d\hat{z}} \, dz \quad B_j = \int_{\text{panel } j} \frac{d\tilde{G}^{x,y}(\hat{z}, z)}{d\hat{z}} \, dz \]  

so that

\[ A_j = \int_{0}^{l_j} \frac{1}{\hat{z}_j - \hat{z}_j - st_j} \left( \frac{i\nu - \mu}{i\nu + \mu} \right) \left( \frac{2\nu}{i\nu + \mu} \right) \left( e^{-ik(\hat{z}_j - \hat{z}_j)} - e^{-ik(\hat{z}_j - \hat{z}_j)} \right) \, ds \]  

Integrating equation 108 using

\[ t_j \tilde{t}_j = 1 \quad z_j + l_j t_j = z_{j+1} \]  

gives

\[ A_j = -t_j \ln \left( \frac{\hat{z}_j - \hat{z}_{j+1}}{\hat{z}_j - \hat{z}_j} \right) - t_j \frac{i\nu - \mu}{i\nu + \mu} \ln \left( \frac{\hat{z}_j - \hat{z}_{j+1}}{\hat{z}_j - \hat{z}_j} \right) - i \frac{2\nu}{i\nu + \mu} \int_{0}^{\infty} \frac{e^{-ik(\hat{z}_j - \hat{z}_j)} - e^{-ik(\hat{z}_j - \hat{z}_j)}}{k - \nu + i\mu} \, dk \]

The integral in the last part of equation 110 can be replaced with a summation according to

\[ \int_{0}^{\infty} \frac{e^{-ik(\hat{z}_j - \hat{z}_j)} - e^{-ik(\hat{z}_j - \hat{z}_j)}}{k - \nu + i\mu} \, dk = \sum_{n=0}^{M} \int_{k_n}^{k_{n+1}} \frac{e^{-ik(\hat{z}_j - \hat{z}_j)} - e^{-ik(\hat{z}_j - \hat{z}_j)}}{k - \nu + i\mu} \, dk \]

\[ M \text{ does not have to be infinitely large, which is seen when looking at the exponential part of equation 111.} \]

\[ \text{Im} \left( \hat{z}_j \right) < 0 \quad \text{and} \quad \text{Im} \left( -\hat{z}_{j+1} \right) < 0 \]

So that

\[ i \text{ Im} \left( \hat{z}_j - \hat{z}_{j+1} \right) > 0 \]

which can be expressed as

\[ i \left( \hat{z}_j - \hat{z}_{j+1} \right) = i(a + bi) \Rightarrow e^{(a-b)k} = e^{iak} e^{-bk} \]

Finally letting \( k \) approach infinity

\[ \lim_{k \to \infty} e^{iak} e^{-bk} \Rightarrow e^{iak} \to 1 \quad e^{-bk} \to 0 \]

Hence, equation 111 will yield zero for large values of \( k \). It can be shown mathematically that after \( M=10 \) the equation is close to zero. The function
is smooth in this small interval and fairly constant with respect to \( k \) so that the following approximation is reasonable:

\[
\int_{k_{n+1}}^{k_{n}} \frac{e^{-ik(\hat{z}_i - \tau, \mu)} - e^{-ik(\hat{z}_i - \tau)}}{k - \nu + i\mu} \, dk = e^{-ik_n(\hat{z}_i - \tau)} \int_{k_{n+1}}^{k_{n}} \frac{dk}{k - \nu + i\mu}
\]

This gives

\[
A_{ij} = -t_j \ln \left( \frac{\hat{z}_j - z_{j+1}}{\hat{z}_j - z_j} \right) - t_j \frac{i\nu - \mu}{i\nu + \mu} \ln \left( \frac{\hat{z}_j - \bar{z}_{j+1}}{\hat{z}_j - \bar{z}_j} \right) - i \frac{2t_j \nu}{i\nu + \mu} \sum_{n=0}^{M} \frac{e^{-ik_n(z_i - \tau, \mu)} - e^{-ik_n(z_j - \tau)}}{k_n - \nu + i\mu} \ln \frac{k_{n+1} - \nu + i\mu}{k_n - \nu + i\mu}
\]

In the same way

\[
B_{ij} = i \left( -t_j \ln \left( \frac{\hat{z}_j - z_{j+1}}{\hat{z}_j - z_j} \right) + t_j \frac{i\nu - \mu}{i\nu + \mu} \ln \left( \frac{\hat{z}_j - \bar{z}_{j+1}}{\hat{z}_j - \bar{z}_j} \right) + i \frac{2t_j \nu}{i\nu + \mu} \sum_{n=0}^{M} \frac{e^{-ik_n(z_i - \tau, \mu)} - e^{-ik_n(z_j - \tau)}}{k_n - \nu + i\mu} \ln \frac{k_{n+1} - \nu + i\mu}{k_n - \nu + i\mu} \right)
\]

The complex natural logarithm is not uniquely defined. Here, a cut at the negative x-axis is made so that the function has a \( 2\pi i \) jump at \( x < 0 \). This due to the fact that

\[
\ln z = \ln re^{i\theta + 2\pi n} \bigg|_{\theta = 0} \quad n \in \mathbb{Z}
\]

The only part of \( A_{ij} \) and \( B_{ij} \) where this is going to be a problem is in the first logarithmic function. This is because the angular difference between \( \hat{z}_i \) and \( z_j \) is so large, since the conjugate of \( z \) is over the surface. The \( ln \) function is thus uniquely defined. The problem with the first logarithm function occurs when \( i=j \) i.e. when the influence on the panel is by the source on the same panel. The result differs if the panel is approached from inside or outside the boundary \( S_b \), as seen in Figure 7.
As

\[ z \to \text{panel} \quad \theta_1 \to 0 \text{ and } \theta_2 \to -\pi \]

so that

\[ \ln \left( \frac{\hat{z}_j - z_{i+1}}{\hat{z}_i - z_j} \right) = \pi i \]

However due to the ambiguity of the logarithm, the function does not always yield this true result. Therefore, when implementing this, the value for \(i=j\) has to be set. Using the total potential in equation 7 then

\[ n \cdot \nabla \Phi = n \cdot \nabla (Ux + \phi) = n \cdot U \begin{pmatrix} 1 \end{pmatrix} + n \cdot \phi \]

The boundary condition states that there is no flow through the surface of the body and therefore

\[ n \cdot \nabla \Phi = 0 \]

which means that

\[ n \cdot U \begin{pmatrix} 1 \end{pmatrix} + n \cdot \nabla \phi = 0 \]

so that

\[ n \cdot \nabla \phi = -U \Re n \left( \hat{z}_i \right) \]

The left hand side of equation 126 is equal to the expression in equation 105. So

\[ \Re \sum_{j=1}^{N} \alpha_i \left( \sigma_j A_{ij} + \gamma B_{ij} \right) = -U \Re n \left( \hat{z}_i \right) \quad i = 1\ldots N \]

For brevity
\[ a_{ij} = \text{Re} \left( it_j A_j \right) \quad a_{i,N+1} = \text{Re} \left( it_i \sum_{j=1}^{N} B_{ij} \right) \]

Using these definitions equation 127 becomes

\[ \sum_{j=1}^{N} \sigma_j a_{ij} + \gamma a_{i,N+1} = -U \text{Re} n(\hat{z}_i) \]

which gives \( N \) equations and \( N+1 \) unknowns, and hence one more equation is needed. Here the Kutta condition is useful; \( S_w \) is a streamline and according to the Kutta condition there is no jump in the velocity across the streamline, i.e. the tangential velocity for the first and last panel is the same.

\[ \hat{t}(\hat{z}_i) \cdot \nabla \phi(\hat{z}_i) + U \text{Re}(\hat{z}_i) = -\hat{t}(\hat{z}_N) \cdot \nabla \phi(\hat{z}_N) - U \text{Re}(\hat{z}_N) \]

The expression for the tangential velocity in equation 99 is used together with

\[ a_{N+1,j} = \text{Re} \left( t_1 A_{1,j} + t_N A_{N,j} \right) \quad a_{N+1,N+1} = \text{Re} \left( t_1 B_{1,j} + t_N B_{N,j} \right) \]

which gives

\[ \sum_{j=1}^{N} a_{N+1,j} \sigma_j + \gamma a_{N+1,N+1} = -U \text{Re}(\hat{z}_i) - U \text{Re}(\hat{z}_N) \]

The pressure coefficient is defined as:

\[ C_p(\hat{z}_i) = \frac{p - p_{\infty}}{\frac{1}{2} \rho U^2} \]

To evaluate the pressure coefficient \( C_p \) equation 9, the Bernoulli equation, is used. Here the upstream reference height \( y_0 \) is set to \( y \).

\[ C_p(\hat{z}_i) = \frac{1}{2} U^2 - \frac{1}{2} |\nabla \Phi|^2 = 1 - \frac{\left| U \text{Re} t(\hat{z}_i) + \hat{t}(\hat{z}_i) \cdot \nabla \phi(\hat{z}_i) \right|^2}{U} \]

**SURFACE PROFILE**

Equation 13 gives the surface profile, \( \eta \)

\[ \eta = -\frac{U \phi_x(x,0)}{g} - \frac{\mu \phi(x,0)}{g} \Rightarrow \]

\[ \eta \approx -\frac{U \phi_x(x,0)}{g} \]
This is solved using the same simplification as earlier: both $\mu'$ and $\phi(x,0)$ are small so their product can be omitted. However, the differential of the potential must be determined. With $\sigma_i$ and $\gamma$ given by equation 105 it is shown that the differential of $\phi$ is

$$\frac{\partial \phi}{\partial x} \bigg|_{z=z_i} = \text{Re} \sum_{j=1}^{N} \int_{\text{panel} j} \left( \sigma_j \frac{d\hat{G}^{n,\mu}(z_j, z)}{d\bar{z}} + \gamma \frac{d\hat{G}^{n,\mu}(\bar{z}_j, z)}{d\bar{z}} \right) ds$$

$$= \text{Re} \sum_{j=1}^{N} \left( \sigma_j \tilde{A}_{ij} + \gamma \tilde{B}_{ij} \right) \quad i = 1...N$$

where $\bar{z}_i$ are points located at the surface. $\tilde{A}_{ij}$ and $\tilde{B}_{ij}$ are integrated the same way as $A_{ij}$ and $B_{ij}$, where the infinite integral is replaced with a summation. This yields

$$\tilde{A}_{ij} = -\bar{t}_j \ln \left( \frac{\bar{z}_i - \bar{z}_{j+1}}{\bar{z}_i - \bar{z}_j} \right) - t_j \frac{i\nu - \mu}{i\nu + \mu} \ln \left( \frac{\bar{z}_i - \bar{z}_{j+1}}{\bar{z}_i - \bar{z}_j} \right) - i \frac{2t_j \nu}{i\nu + \mu} \sum_{n=0}^{M} e^{-ik_s(z_i - \tau_{j,n})} - e^{-ik_s(z_i - \tau_{j})} \ln \frac{k_{n+1} - \nu + i\mu}{k_n - \nu + i\mu}$$

$$\tilde{B}_{ij} = i \left( -\bar{t}_j \ln \left( \frac{\bar{z}_i - \bar{z}_{j+1}}{\bar{z}_i - \bar{z}_j} \right) + t_j \frac{i\nu - \mu}{i\nu + \mu} \ln \left( \frac{\bar{z}_i - \bar{z}_{j+1}}{\bar{z}_i - \bar{z}_j} \right) + i \frac{2t_j \nu}{i\nu + \mu} \sum_{n=0}^{M} e^{-ik_s(z_i - \tau_{j,n})} - e^{-ik_s(z_i - \tau_{j})} \ln \frac{k_{n+1} - \nu + i\mu}{k_n - \nu + i\mu} \right)$$

It can be seen from equation 140 and 141, that even though $\mu$ is not present in equation 138 it still affects the surface profile.

**Resistance**

The resistance can be determined through integration of the local pressure coefficient. The local drag coefficient, $C_{DN}$ is

$$C_{DN} = C_{PN} \begin{bmatrix} -1 & 0 \end{bmatrix}$$

The local drag coefficient is then integrate to give the total drag coefficient

$$C_D = \int_{S_b} C_{DN} ds$$

The total pressure resistance, $R$ is

$$R = C_D \frac{1}{2} \rho U_a^2 S$$

where $S$ is the length of $S_b$.

**Results**

The results will be compared and validated against experimental, analytical and numerical results. If the model is in agreement with conventional methods when there is no dissipative viscosity, i.e. $\mu=0$, the model can be assumed to be at least as good as current methods. The value of $\mu$ is then determined by comparison with experimental results. Both NACA and Joukowski hydrofoils will be used for comparison. First, the number of panels must be decided.
NUMBER OF PANELS

The pressure coefficient, $C_p$, of a cylinder is compared to the analytical solution

$$C_{pa} = 1 - 4 \sin^2 \theta$$

where $\theta$ is the angle defined in Figure 8.

The cylinder is considered to be in deep water, which in this numerical study is set to a depth of $10 \frac{b}{c}$. The depth $b$ is defined from $z=0$ to the location of the leading edge for the NACA hydrofoils and to the centroid for the Joukowski hydrofoils, $c$ is the cord length which is used to make the depth non-dimensional. It is well known that a panel method for a cylinder works best if the number of panels is even; the lowest number of panels that can be used to approximate a cylinder is 8. The error $\varepsilon$, representing the deviation from the analytical solution, is calculated using

$$\varepsilon = \frac{\int |C_{pa}| \, d\theta - \int |C_p| \, d\theta}{\int |C_{pa}| \, d\theta}$$

The relative errors of the integrals of the pressure coefficients are used because the relative error of the pressure coefficient is not valid in any point where $C_p$ is zero. The dissipative viscosity is set to zero and deep water is assumed. The error is seen in Table 1 and Figure 9 where it expressed as a function of the numbers of panels.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\varepsilon$</th>
<th>$N$</th>
<th>$\varepsilon$</th>
<th>$N$</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>-132 %</td>
<td>80</td>
<td>-3,4 %</td>
<td>152</td>
<td>-2,5 %</td>
</tr>
<tr>
<td>16</td>
<td>-35 %</td>
<td>88</td>
<td>-3,3 %</td>
<td>160</td>
<td>-2,5 %</td>
</tr>
<tr>
<td>24</td>
<td>-21 %</td>
<td>96</td>
<td>-3,3 %</td>
<td>168</td>
<td>-2,5 %</td>
</tr>
<tr>
<td>32</td>
<td>-10 %</td>
<td>104</td>
<td>-2,9 %</td>
<td>176</td>
<td>-2,4 %</td>
</tr>
<tr>
<td>40</td>
<td>-7,5 %</td>
<td>112</td>
<td>-2,8 %</td>
<td>184</td>
<td>-2,4 %</td>
</tr>
<tr>
<td>48</td>
<td>-6,8 %</td>
<td>120</td>
<td>-2,9 %</td>
<td>192</td>
<td>-2,4 %</td>
</tr>
<tr>
<td>56</td>
<td>-4,8 %</td>
<td>128</td>
<td>-2,6 %</td>
<td>200</td>
<td>-2,3 %</td>
</tr>
<tr>
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<td>-4,3 %</td>
<td>136</td>
<td>-2,6 %</td>
<td>208</td>
<td>-2,3 %</td>
</tr>
<tr>
<td>72</td>
<td>-4,2 %</td>
<td>144</td>
<td>-2,6 %</td>
<td>216</td>
<td>-2,3 %</td>
</tr>
</tbody>
</table>

Table 1 Error Relative an analytical solution
When deciding how many panels to use the computational time also has to be taken into account. The computational time is shown in Figure 10.
The CPU time is dependent on the capacity of the computer; however the quadratic increase seems to be machine-independent. The ultimate result would of course be an exact match with the analytical solution. However, when the geometry is approximated using panels, there is always going to be a discrepancy. The question is just how big it can be and still yield a valid result. When the CPU time and the error is considered it can be seen that less than 80 panels gives a relatively poor accuracy and that more than 100 panels gives a rapid increase in CPU time. It is therefore decided 88 panels will be used. This will give an accuracy of 3,3% according to Table 1.

**Comparison with conformal mapping**

Conformal mapping is a widely used technique to determine the forces on 2D shapes. It is most commonly applied to aerofoils or hydrofoils operating in deep water because it generally does not take the free surface into account. When the hydrofoil is deeply submerged, the surface effects will be negligible. There is no conventional method for deciding the precise depth for when the effects of the surface disappear. Therefore an iterative method is used to investigate when the submersion is big enough for the water to be considered deep.

The condition will be that when the method that takes free surface effects into account gives the same result as a method that does not take surface effect into account, the surface effect can be assumed to be small. For comparison $\mu$ is set to be zero in this study since there is no dissipation present in the used theoretical models. The results are shown for a NACA 4412 hydrofoil, which is a hydrofoil with camber as seen in Figure 11.

![NACA 4412 hydrofoil](image)

Figure 11 NACA 4412 hydrofoil

The result differs more from the conformal mapping the closer to the surface the foil is positioned as seen in Figure 12. The angle of attack is 6°.
As the hydrofoil is placed deeper and deeper the pressure coefficient will move towards the results from conformal mapping. However, at some stage the result is going to be within the error margin of the panel method itself, 3.3% as seen in Table 1. When this happens there is no point in increasing the depth. This happens when the depth is $6 \frac{h}{c}$ or greater as seen in Figure 13 and therefore can $6 \frac{h}{c}$ be assumed to be deep water.

![Figure 12 Pressure coefficient for different depths compared to conformal mapping](image)

![Figure 13 Cp for different depths compared to conformal mapping and the error margin](image)
COMPARISON TO DEEP WATER METHOD

The results are also compared to a previous panel method. This method follows the same discretisation as the presented method and is solved the same way. However the Green’s function is only taken to be the source $1/r$ and has no image part and hence does not take surface effects into account. For more detail see Chen (2010b). The hydrofoil is placed at $b/c = 6$ which has been shown to represent deep water and thus should produce equal results for both methods.

NACA 0012 is a symmetric hydrofoil; the profile of the hydrofoil is shown in Figure 15.

The conclusion of these results is that the proposed method gives a correct result when compared to existing methods when there is no dissipative viscosity and no free surface effects, however then main goal with this method is to determine if it is better than existing methods. The results must therefore be compared to experimental results since that is the only way to evaluate the dissipative properties of this method.

COMPARISON TO EXPERIMENTAL RESULTS

The novel nature of this method means that there is no established experience regarding the value of the Rayleigh damping. However, since conventional methods assume this damping to be negligible it can be assumed to be small. To investigate how small, results are compared to experimental results. As seen in Figure 16.
None of these results seems to give a very good accuracy at both the upper and lower side of the hydrofoil. To let $\mu$ be zero is not an option since the method gives very inaccurate result for no Rayleigh damping close to the surface. Any value that is less than 0.01 but larger than zero gives roughly the same result as $\mu = 0.01$. A small value of $\mu$ gives better accuracy on the lower side of the hydrofoil and a larger one gives a better accuracy on the upper side; however the largest value of $\mu$ has a discrepancy at the leading edge and therefore it is assumed that $\mu$ should lie between 0.01 and 1. This is shown in Figure 17.
It is widely known that numerical models assuming no viscosity (not to be confused with no Rayleigh damping) over predict the lift force. Since the fluid is inviscid, no boundary layer is formed. The formation of a boundary layer would slightly alter the velocity profile. This can be corrected by using a boundary layer displacement method where the velocity profile is offset; however this is not used in the current study. This means that at the same angle of attack, the current model will give too much lift compared to experiments. A closer match of the experimental conditions can be achieved by calculating the angle of attack corresponding to the correct lift coefficient $C_L$ (Gieseng and Smith 1967). The experimental results are therefore compared to numerical result not with the same angle of attack but with the same resulting $C_L$. For a NACA 4412 with an angle of attack of 5° $C_L=0.580$ according to Giesing and Smith (1967). For the current method $C_L$ is calculated according to

$$C_L = -\text{Im} \sum_{i=1}^{N} C_{p_i} l_i n_i$$

To gain the same lift, an angle of attack of 3.9° is needed; however this differs with the value of $\mu$ as seen in Figure 18.

![Figure 18 Comparison of calculated and experimental pressure distributions for different $\mu$ and corresponding $C_L$](image)

Experimental results by Ausman (1954)

This approach gives better results, especially at the leading edge, however the results are not a perfectly matched. A comparison with match $C_L$ and $\mu=0.5$ is shown in Figure 19b, where it is clearly seen that the numerical results are better when $C_L$ is matched, compared to when matching $\alpha$. 
Joukowski hydrofoil

The Joukowski hydrofoil is obtained by applying the Joukowski mapping

$$\zeta = z + \frac{1}{z}$$


where $\nu$ is the argument. Here a 12% thick hydrofoil is used; the thickness of a Joukowski hydrofoil was obtained by Wilson (2009)

$$\frac{t}{c} = \frac{3\sqrt{3}}{4} \frac{x_0 (1+2x_0)}{3x_0^2 + 3x_0 + 1}$$

Where it is assumed that the maximum thickness occurs when $\nu = \pi/3$. Equation 150 gives two roots where one of them will yield the desired thickness when inserted into equation 149. A Joukowski hydrofoil with 12% thickness is shown in Figure 20.
When the pressure is calculated around the Joukowski hydrofoil some singular behaviour is noticed around the trailing edge, which is therefore omitted from the graphs. $C_p$ will reach infinity at the trailing edge and scaling effects would otherwise make it impossible to see the value of $C_p$ at any point. The pressure coefficient $C_p$ for the upper surface is shown in Figure 21, where the results are compared to experimental results by Parkin et al (1955).

Figure 21 Comparison between numerical and experimental results. The depth is calculated to the trailing edge. Experimental results by Parkin et al (1955)

For the Joukowski hydrofoil the correspondence with the experimental results is very poor. Other methods have reached a better accuracy but it seems evident that they also suffer from lower accuracy when the Joukowski hydrofoil is used compared to a NACA hydrofoil (Chen & Hearn, 2010). This problem persists with both larger and negative values of $\mu$. The bad correspondence is also evident close to the surface as seen in Figure 22.

Figure 22 Comparison between numerical and experimental results, the depth is 0.2 h/c and calculated to the trailing edge. Experimental results by Parkin et al (1955)
Figure 21 and Figure 22 clearly show that the correspondence between this method and experimental results are very poor.

**COMPARISON WITH NUMERICAL RESULTS**

The aim has been to get as good a comparison with the experimental results as possible; however no numerical or theoretical method has total accuracy with experimental results. It is when comparing the correspondence to experimental results with other numerical methods, that it can be shown if the Rayleigh damping is an improvement or not. The present method is compared to the numerical results by Giesing and Smith (1967), as seen in Figure 23.

![Graph showing comparison with numerical results](image)

**Figure 23.** Comparison with numerical method with $F_n = 1.03$ and distance to centroid $h = 0.94$. The numerical reference values are by Giesing and Smith (1967) and the experiment by Ausman (1954). Results for the present method are shown for $\mu = 0.01$, $\mu = 0.3$, $\mu = 0.5$, $\mu = 1$.

A comparison match to $C_L$ instead of $\alpha$ is shown in Figure 24.

![Graph showing comparison with equal $C_L$](image)

**Figure 24.** Comparison with equal $C_L$. The reference has $C_L = 0.580$ and $h = 0.94$. The numerical reference values are by Giesing and Smith (1967) and the experiment by Ausman (1954). The results for the present method are shown for $\mu = 0.01$, $\mu = 0.3$, $\mu = 0.5$, $\mu = 1$.  

33
When comparing to numerical results for the Joukowski airfoil, the same features as for the comparison against experiments are present. The results around the trailing edge have to be omitted since they are infinite. At the leading edge is there a relatively good comparison. However, before the midcord $C_p$ unexpectedly drops as seen in Figure 25.

![Joukowsky thickness 0.12 $\alpha = 5$](image)

Figure 25. Comparison to numerical results (Giesing & Smith 1967) and experimental results (Parkin et al 1955) with $F_n = 0.95, h = 0.35$.

**SURFACE PROFILE**

The surface profile is compared to numerical results from Yeung and Bouger (1979) which do not include Rayleigh damping. A NACA 4412 hydrofoil as seen in Figure 11 with a $5^\circ$ angle of attack is used. Yeung and Bouger use a finite depth of $d=4c$ ($d$ is the depth of the fluid domain); however this is considered to be deep enough to be compared with the current method which assumes infinite depth. Yeung and Bouger (1979) use the Froude depth number, $F_h$

$$ F_h = \frac{U}{\sqrt{gd}} $$

where $d$ is the depth of the seabed, $g$ is gravity and $U$ the free stream velocity. The comparison is done so that that the current model has the same free stream velocity. As before $\mu=0$ yields an invalid result but there is a very good comparison for a small value of $\mu$ with the reference results for subcritical flow as seen in Figure 26. However the correspondence for supercritical flow is very poor.
As expected, the lowest value of $\mu$ gives the best comparison with a method without any damping. The best corresponding $\mu$ when considering experimental results for $C_p$ on the same hydrofoil (as seen in Figure 18 and Figure 24) is $\mu \approx 0.3$. It is therefore assumed that Figure 27, which uses $\mu = 0.31$, shows a more realistic wave profile, at least for subcritical flow.
The wave pattern following a submerged cylinder is also compared to Yeung and Bouger (1979). The submersion depth is calculated from the mean surface to the centre of the cylinder. The wave profile has lower amplitude than the reference case. When damping is added it is clearly shown that the wave quickly reduces to the undisturbed surface as shown in Figure 28.

![Figure 28. Comparison with numerical results for cylinder submerged at h/r = 2 with Fh=0.4. Numerical reference method by Yeung and Bouger (1979).](image)

For supercritical speeds the difference in amplitude between the current model and the reference case is even greater, as shown in Figure 29. It is likely to be because the influence of the bottom is greater on the cylinder than on the hydrofoil because a hydrofoil with the same cord length as the radius of the corresponding cylinder is further from the bottom, due to the fact that the hydrofoil has a much slimmer profile.
**CONCLUSIONS**

A method for numerical evaluation of the pressure coefficient and the wave profile has been presented. The results for a NACA profile correspond better to the experimental results by Ausman (1954) than the results by Giesing and Smith (1967) as seen in Figure 23 and Figure 24. However the results are very poor for the Joukowski hydrofoil as seen in Figure 25, where the numerical results by Giesing and Smith (1967) correspond well to the experimental results by Parkin et al (1955). The wave profile generated by a hydrofoil corresponds well to numerical results (Yeung and Bouger, 1979) for subcritical speeds. The wave profile generated by a cylinder is less accurate compared to numerical results by Yeung and Bouger (1979).

When the current model was developed each step was carefully evaluated against different numerical models to ensure that the results were reasonable. The work started as a simple panel method with a deeply submerged body. There was no Rayleigh damping and no consideration of the free surface condition. As the work progressed more and more complex features were introduced to the model, step by step increasing the complexity. At most of these steps, the results of a submerged NACA profile were used for comparison. This was done since it is the most popular shape for 2D calculations and there is much reference material available. This could have contributed to the fact that the method clearly shows a better result for the NACA hydrofoils.

This is a very mathematical project where the model is derived from first principles. Therefore, the highest possible standard of the accuracy on all of the derivations was ensured. However it must be pointed out that the later part of equation 65 is not strictly true; it is an approximation only completely true for \( \mu = 0 \). However, this does not seem to influence the final results so it might be that this is not as important practically as it is mathematically.

It is clear that the wave profiles shown in Figure 26 correspond well for lower speeds. This could be due to the fact that the bottom has a higher influence on the wave profile when the speed is supercritical. The
reason for the higher wave profile in the reference case (for supercritical speeds) could be due to an error in the present method since the pressure distribution by Yeung and Bouger (1979) is very similar to that of Giesing and Smith (1967), and by this, also similar to the present method, yet the present method yields a different wave profile for supercritical speeds. In Figure 28 and Figure 29 it is evident that the surface profile of a submerged cylinder corresponds poorly with the reference case. This could be due to the effect of the bottom not being negligible due to that the submerged cylinder is much closer to the bottom. The result shown previously for the surface profile is unfortunately not enough to evaluate the method; it would benefit from a future evaluation against experimental and numerical results with infinite depth.

Even though hydrofoil vessels are becoming more common as passenger vessels they are only a small part of global fleet since the lifting force from a hydrofoil is not enough to lift heavy cargo ships. This means that hydrofoil ships are likely to remain a small niche for fast passenger ships. This limits the applications of this method. Even though 2D is considered a generally accepted simplification for hydrofoil modelling, 2D will not be accurate enough when modelled together with the rest of the hull structure.

Therefore, the aim has been to use the current method as a stepping stone to a 3D model. This study shows that the introduction of viscous damping can improve the numerical results as seen in Figure 23 and Figure 24, where the current method corresponds better to experimental results (Ausman, 1954) than the numerical results by Giesing and Smith (1967). However, it still needs improvement to be considered an all-round model that can be used for any submerged shape (as seen in Figure 25.)

It is therefore argued that there are benefits to introducing Rayleigh damping into potential theory to get a more accurate estimation of the wave resistance. It is very hard to determine if this is valid for the 3D case. There is reason to investigate and improve the present 2D method further, but there are also sufficient results arguing that potential theory with damping is an improvement. The recommendation for further work is therefore to develop a 3D method based on potential flow with Rayleigh damping.

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