**Singular-value decomposition of axial systems**

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Section, topic, and type of presentation

A method for reconstruction of 3D objects from 2D images, used e.g. in fluorescence microscopy, is taking several images along the axial direction. Singular-value decomposition is used to perform the reconstruction and determine its limits.

Introduction and theory

Singular-value decomposition [1, 2] is used to find the modes of a system. It is normally done for matrix operators, but can also be extended to linear analytical operators. We find the modes of the imaging system depicted in Fig. 1. The object \( f(x, z) \) is extended in two dimensions: one axial, where it is limited by \( z_1 \leq z \leq z_2 \), and one transverse where \( x \) ranges from \(-\infty\) to \(\infty\). \( M \) multiple images \( g_m(x_d) \) are taken, each at a different distance \( \zeta_m \). The images are also assumed to be of infinite extent in the transverse \( x_d \) direction. Although the examined model is two-dimensional, it is trivial to extend it to three-dimensional systems by adding a second transverse dimension. The imaging system is symmetric and telecentric, employing two lenses of focal length \( f \) each placed a distance \( f \) to either side of an aperture of size \( D \). The object distance, from the object to the first lens, is \( f - z \), and the image distance from the second lens to the image plane is \( f + z \). The telecentric system is used since its magnification doesn’t change with axial distance, which simplifies the form of the singular functions. All calculations are done in the geometrical optics approximation.

If the imaging operator is compact, we can expect the number of non-zero singular values to be finite. [2] In this case the convolution operator is used in the transverse direction, and its non-compactness implies there will be a continuous spectrum of singular values. A similar case, though with only one image plane, has been analysed by Barrett and Myers[1], chapter 7.2.10.

![Figure 1: The analyzed system.](image)

Given an object \( f(x, z) \), the \( m \)th image \( g_m(x_d) \) is found as

\[
g_m(x_d) = |\mathcal{H}f|_m(x_d) = \int_\infty^\infty dx \int_{z_1}^{z_2} dz \ h_m(x_d - x; z) f(x, z),
\]

(1)

where \( \mathcal{H} \) is the imaging operator and \( h_m \) the point-response function

\[
h_m(x - x_d; z) = \frac{1}{d_m(z)} \text{rect} \left( \frac{x - x_d}{d_m(z)} \right),
\]

(2)
for \( d_m(z) = \frac{D}{z - \zeta_m} \). The adjoint or back-projection operator is

\[
f(x, z) = [\mathcal{H}^\dagger g](x, z) = \sum_{m=1}^{M} \int_{-\infty}^{\infty} dx' h_m^*(x_d - x') g_m(x_d).
\]

Since the image space is discrete in the axial direction, we expect singular-value decomposition to be easier in this space. The Hermitian operator is formed as

\[
[\mathcal{H}^\dagger \mathcal{H}]_m(x_d) = \sum_{m' = 1}^{M} \int_{-\infty}^{\infty} dx' k_{mm'}(x_d - x') g_{m'}(x_d),
\]

where

\[
k_{mm'}(x_d - x') = \int_{-\infty}^{\infty} dx \int_{z_1}^{z_2} dz h_m(x_d - x; z) h_{m'}^*(x_d - x; z).
\]

The image-space singular functions/vectors \( \mathbf{v}_{\rho,j}(x_d) \) and singular values \( \sigma_{\rho,j} = (\mu_{\rho,j})^{1/2} \) are solutions of the eigenequation

\[
[\mathcal{H}^\dagger \mathcal{H}]_{\rho,j}(x_d) = \mu_{\rho,j} \mathbf{v}_{\rho,j}(x_d).
\]

Since the operator is a convolution in the transverse direction, the solution should be in the form

\[
\mathbf{v}_{\rho,j}(x_d) = \mathbf{V}_j(\rho) \exp(2\pi i \rho x_d)
\]

and insertion into Eq. (6) yields the vector \( \mathbf{V}_j(\rho) \) as a solution to the eigenequation

\[
K(\rho) \mathbf{V}_j(\rho) = \mu_{\rho,j} \mathbf{V}_j(\rho)
\]

which is solved for each value of \( \rho \). For \( M \leq 3 \) analytical solutions are readily obtained; for larger \( M \) the eigenvalue problem can be solved numerically. The matrix \( K(\rho) \) is formed from the elements

\[
K_{mm'}(\rho) = \int_{z_1}^{z_2} dz \frac{\sin[\pi \rho d_m(z)] \sin[\pi \rho d_{m'}(z)]}{\pi \rho d_m(z) \pi \rho d_{m'}(z)}
\]

which can be evaluated numerically. Once the image-space singular functions are known, the object-space singular functions \( u_{\rho,j} \) can be retrieved from a projection[1]

\[
\sigma_{\rho,j} u_{\rho,j}(x, z) = \exp(2\pi i \rho x) \sum_{m=1}^{M} |\mathbf{V}_j(\rho)|_m \int_{-\infty}^{\infty} dx_d h_m^*(x_d) \exp(2\pi i \rho x_d).
\]

This also shows that the object-space eigenfunctions may be written as \( u_{\rho,j}(x, z) = U_j(\rho, z) \exp(2\pi i \rho x) \).

Figure 2: The three object-space eigenfunctions (a) \( U_1(\rho, z) \), (b) \( U_2(\rho, z) \), and (c) \( U_3(\rho, z) \) resulting from three image planes.

**Numerical results and conclusions**

Numerical calculations were performed in Matlab. We assume \( z_1 = -20 \text{ mm}, z_2 = 20 \text{ mm}, \zeta_1 = -10 \text{ mm}, \zeta_2 = 0 \text{ mm}, \zeta_3 = 10 \text{ mm}, D = 20 \text{ mm}, \) and \( f = 100 \text{ mm} \). Fig. 2 shows the three resulting object-space eigenfunctions. Fig. 3 shows object, image and measurement component for a sample object. The measurement component is an expansion of the object in the singular functions, and represents the part of the object that is
transferred to the image plane, i.e., the part that can be measured. We note that the measurement component depends very much on position in the object; if an object part is more or less in focus in one of the images it is fairly well retrieved, but if it falls between two planes it is almost lost.

In conclusion, we have demonstrated singular-value decomposition for a through-focus imaging system. Since part of the imaging operator is a convolution in the transverse dimension, the analysis resembles Fourier analysis with the additional singular-value decomposition in the axial direction. The method can be used for projection or back-projection of object or image information to the other space, for object reconstruction in the shape of the measurement component, and for analyzing the limits of the optical system performance.

![Figure 3: (a) A sample object. (b) The three images at distances $\zeta_1 = -10$ mm (dotted), $\zeta_2 = 0$ mm (solid), and $\zeta_3 = 10$ mm (dashed). (c) The measurement component of the object.](image)

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References
