

Abstract

The capital requirements for insurance companies in the Solvency I framework are based on the premium and claim expenditure. This approach does not take the individual risk of the insurer into consideration and give policy holder little assurance. Therefore a framework called *Solvency II* is under development by EU and its members. The capital requirements in Solvency II are based on risk management and is related to the specific risks of the insurer. Moreover, the insurer must make disclosures both to the supervising authority and to the market. This puts pressure on the insurance companies to use better risk and capital management, which gives the policy holders better assurance.

In this thesis we present a stochastic model that describes the development of assets and liabilities. We consider the following risks: *Stock market, bond market, interest rate and mortality intensity*. These risks are modeled by stochastic processes that are aggregated to describe the change in the insurers *Risk Bearing Capital*. The capital requirement, *Solvency Capital Requirement*, is calculated using *Conditional Value-at-Risk* at a 99% confidence level and *Monte Carlo simulation*. The results from this model is compared to the *Swiss Solvency Test* model for three different types of life insurance policies. We can conclude that for large portfolios, the model presented in this thesis gives a lower solvency capital requirement than the Swiss model for all three policies. For small portfolios, the capital requirement is larger due to the stochastic mortality risk which is not included in the Swiss model.

Key words: Solvency II, Solvency Capital Requirement, life insurance.

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CHAPTER 1

Introduction

The financial situation of an insurance company can be described by considering the market value of a portfolio of assets and the market value of an insurance portfolio consisting of liabilities corresponding to the insurance policies. Market values of assets can generally be taken from a liquid market, this is not the case for the insurance portfolio. To value a life insurance policy we consider discounting the expected future cash flow for the specific policy, a best estimate value. The difference between market value of assets and market value of liabilities is the available solvency capital, in this report called *Risk-Bearing Capital (RBC)*. Figure 1.1 explains the relation between the three quantities.

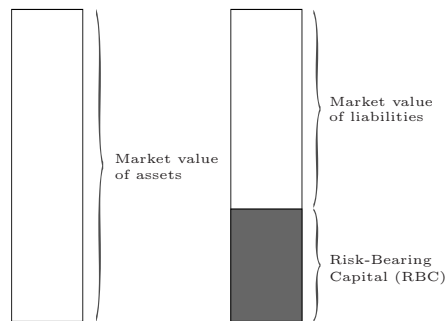


Figure 1.1: The relation between the market value of assets and liabilities and the Risk-Bearing Capital.

Obviously the RBC need to be greater than zero. However, there are regulations that controls how low the RBC is allowed to be. In the current EU solvency framework, *Solvency I*, the insurance company is obligated to hold 4% of the liabilities as solvency capital. This capital requirement only consider the insurance risk, excluding the capital and risk management of the insurer which gives prudent insurers a competitive disadvantage and policy holders little assurance. Therefore a new framework called Solvency II [2] is under development by CEIOPS¹. The main target is to erase the weak points of Solvency I. Following in the footsteps of the Basel II framework for the banking sector, Solvency II focus on risk management for capital

requirements and the individual risks of the insurer. The foundation of Solvency II is the following three pillars:

1. Capital adequacy requirement. The insurance company must cover the capital requirements from credit risk, market risk, insurance risk and operational risk.
2. Supervisory reviews. Each company must develop and use risk management techniques when monitoring and managing their risks.
3. Disclosure requirements. The insurer need to provide financial, and other, information to the market.

The part mainly considered in this thesis is Pillar 1, which sets out the capital requirements from market, insurance, credit and operational risks. Pillar 1 also implies that the assets and liabilities needs to valued on a market consistent basis. Pillar 2 sets out the internal risk and capital management and allows the supervisor to undertake an evaluation of the insurers risk-based capital requirement. Pillar 3 concerns the disclosures that an insurer is expected to make, this includes disclosure to the supervisor and disclosure to the public.

The capital requirement for an insurer consist of two levels. The first capital level is called *Solvency Capital Requirement* (SCR), which translate the individual risks of an insurer into a capital requirement. This capital level provides the insurer with a buffer that can absorb unforeseen losses and also give the policy holders a decent assurance. The second capital requirement level is the so-called *Minimal Capital Requirement* (MCR). This level constitute the lowest value of the RBC at which an insurer is allowed to continue its business. Should the RBC fall below this level, the supervisor can force the insurance company to run-off its business. Figure 1.2 displays the two capital levels, the black area representing the MCR, and the RBC. However, the MCR will not be discussed further in this thesis.

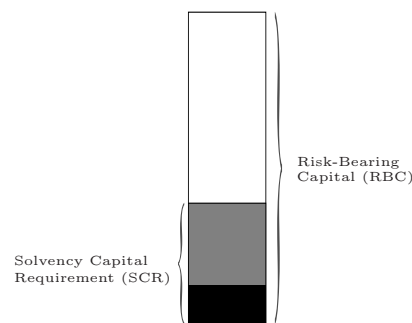


Figure 1.2: In the figure, Solvency capital requirement and minimal capital requirement (black area) is shown in relation to the Risk-Bearing capital. These relations, of course, are not fixed but are dependent on the individual risks of the insurer.

In the Solvency II framework, the capital requirements has to be calculated for a time horizon of one year. This requires that the change in RBC over time is modeled. What are the different risk factors that affect the change in RBC? We consider an insurer with a portfolio of life insurance policies that invests its capital only in bonds and stocks. The portfolio of assets depends on the fluctuations in both the stock market and the bond/interest rate market. If we consider the portfolio

¹Committee of European Insurance and Occupational Pensions Supervisors

of liabilities, we see that it also depends on the development of interest rate since the value of the portfolio is determined by discounting the cash flow using market interest rates. Moreover, the insurance portfolio depends on the uncertainty of the mortality intensity since the cash flow of the portfolio depend on the longevity of the individuals. Hence, we can say that the change in RBC depends on how the capital is invested, the characteristics of the insurance portfolio and how well diversified the two portfolios are. There are of course other risks like operational risk and credit risk, but in this thesis these risks will not be considered. For further information about Solvency II consider [1].

Within the Solvency II framework the SCR can be calculated either using a standard model, available for every insurance company, or the insurer can create an internal model, unique for the company. An example of a standard model consistent with the Solvency II framework is the *Swiss Solvency test* [17], under development by FOPI². This model is based on the assumption that the change in RBC due to market risks factors and insurance risk factors have a multivariate normal distribution. The standard approach includes standard models for market risk, insurance risk, credit risk and through aggregation of these models a distribution for the change in RBC is obtained. Insurers only need to estimate the sensitivity and volatility of each risk and the dependence between the risk factors, if these are not pre-defined by the supervisor. Another Solvency II consistent approach is discussed in [10]. This model is a factor-based model using a static co-efficient method with Excel as basis. This model however does not generate an actual distribution for the change in RBC as for the SST model.

An internal model can be created making a full or partial stochastic model, i.e all (or some) risks are modeled by stochastic processes giving scenarios for the possible outcomes of each risk factor. Through aggregation of these risks the distribution for the change in RBC over time can be obtained. The risk factors that we consider modeling in this thesis are the stock prices, bond prices, interest rate and the mortality intensity.

Maybe the most common approach for modeling the interest rate is the Cox-Ingersoll-Ross model (CIR) [6]. CIR is a type of short rate model describing the development of interest rate and has been used for valuation of interest rate derivatives. The main advantage of this model is that it is widely spread and is easy to implement, and thereby very practical. For instance, the CIR model has been used to model interest rate fluctuations in [11] and [14] to name a few. A more complex interest rate model growing more popular in recent years is the LIBOR market model (LMM), see [4] and [19]. The interest rate is modeled by a forward rate structure and the initial values are set from bond or swap prices. The LMM model is often used for pricing interest rate derivatives and is easily calibrated to current market data via the well know Black formula [5].

The mortality intensity is generally modeled using a deterministic model that does not consider the uncertainty of the mortality intensity. The alternative is to model the mortality rate as a stochastic process and thereby include the systematic mortality risk. One interesting approach for modeling the mortality risk as a stochastic process is suggested in [7]. The mortality intensity is modeled by applying the extended CIR model, previously considered as a model for interest rate, to a forward rate structure of the mortality intensity. Furthermore Dahl introduce a new type of contract called *mortality-linked-contracts* which provides the possibility to transfer the systematic mortality risk to the insured or to another company. This provides another possibility

²Federal Office of Private Insurance.

to hedge against mortality risk besides the tradable derivative *mortality swaps*. In [8] a third possibility of hedging against systematic mortality risk, namely the *survivor swap* is discussed. When buying a survivor swap we exchange a fixed number of survivors for the actual number of survivors. For further study of the effect of stochastic mortality rate and different hedging possibilities we recommend reading [7],[8] and [9].

The aim of this thesis is to create a general stochastic model that describes the change in RBC due to fluctuations in stock price, bond price, interest rate and mortality intensity. The development of each risk factor is described by a stochastic process.

The stock price is modeled by a geometric Brownian motion, used in the Black-Scholes-Merton framework for option pricing [16]. The bond price and interest rate is modeled mutually by the LMM model, since the Yield to Maturity of the bond can be taken from the yield curve modeled in the interest rate model. To model the mortality intensity we use the *Gompertz-Makeham* model [15] where the input parameters are regarded as independent stochastic processes. The distribution of the RBC is obtained by aggregating the risk factor models in the asset and liability portfolio. Values for the asset portfolio can be taken directly from a liquid market. The value of the liability portfolio is determined by calculating the present value of the cash flow using market interest rates and is considered to be a stochastic value.

The SCR is determined by using a risk measure that converts the risk in the change of the RBC to a numerical value. In this thesis we use *Conditional Value-at-Risk*, also known as *Expected Shortfall*, at a 99% confidence level. Briefly this is just the mean value of the 1% worst possible outcomes.

To generate the outcomes of the model we use Monte Carlo simulation, which is a method that uses random numbers to evaluate mathematical problems. It is especially attractive when evaluating integrals in high dimensions or when modeling phenomena with high uncertainty in inputs. For example when creating a stochastic model of an insurance company. Further applications of Monte Carlo methods can be found in [12].

Outline of this thesis

The outline of this thesis is the following:

- Chapter 2 presents the stochastic models used to describe the evolution of the different risks and the theory describing the lifetime of an individual. Moreover, we describe the characteristics of the cash flow for the insurance products.
- In Chapter 3 the model used in this thesis for estimating the SCR is presented. The characteristics of the insurance portfolio is described, starting with death process and the mortality intensity for the entire portfolio. Furthermore, the risk measure *Conditional Value at Risk* and the numerical evaluation technique *Monte Carlo method* is introduced.
- Chapter 4 presents the Swiss model [17] consistent with the Solvency II framework. This includes standard models for market and life insurance risk factors. We also consider extreme scenarios pre-defined in the SST model.
- In Chapter 5 the results from the stochastic model and SST model is presented and compared to the demand for solvency capital required by the cur-

rent swedish regulation. Three different life insurance policies are used, and all simulations has been made for different sizes of the portfolio.

- Chapter 6 presents the conclusions made from the numerical results.
- Finally in chapter 7 we present suggestions for future research and improvements of the model.

Modeling the risk factors

In this chapter we consider models for financial and insurance related risks that an insurance company is exposed to. We are interested in describing the development of the RBC (a quantity defined in the introduction) over time. This is done by modeling both assets and liabilities as stochastic processes. The assets consists of stocks and zero coupon bonds, so the value of the asset portfolio depends on the development of both stock price and bond price. The liability of the insurance portfolio depends on the development of mortality intensity, interest rate and the characteristics of the insurance product. Also, to model the insurance portfolio we need to define the lifetime of an individual and the mortality intensity connected to the expected lifetime. The following quantities need to be modeled:

- The lifetime of an individual
- Mortality intensity
- Insurance product
- Stock price
- Bond price
- Interest rate

2.1 Lifetime model

2.1.1 Theory

For a life insurance company payments to and from an insurance account depend on whether an individual is alive or not. Therefore it is natural to assume that the lifetime of an individual is a very important factor. In this section the probability theory necessary for valuation of insurance policies, and the parametric function used for estimating the mortality intensity are introduced.

In the following we use the notation in [3] and consider a population of individuals which is assumed to have independent lifetimes.

Definition 2.1 Let the lifetime, T , for an arbitrary individual be a non-negative stochastic variable with distribution function

$$F(x) = P(T \leq x), \quad x \geq 0, \quad (2.1)$$

and density function

$$f(x) = F'(x).$$

We now let $(x, x + dx)$ be a small interval and consider the probability that an arbitrary individual die in that interval under the assumption that the individual is alive at x . Moreover, if dx is small it is reasonable to assume that the probability is proportional to the length of the interval. Let μ_x be the proportionality factor, then we can write the probability as $\mu_x dx$ to get the approximative relation

$$\mu_x dx \cong \frac{F(x + dx) - F(x)}{1 - F(x)}$$

which can be written as

$$\mu_x \cong \frac{1}{dx} \frac{F(x + dx) - F(x)}{1 - F(x)}.$$

Let $dx \rightarrow 0$, then we get

$$\mu_x = \lim_{dx \rightarrow 0} \frac{1}{dx} \frac{F(x + dx) - F(x)}{1 - F(x)} = \frac{f(x)}{1 - F(x)}, \quad (2.2)$$

where μ_x is called the mortality intensity for an individual at age x .

The fact that the lifetime T is for a newborn individual makes it necessary to define a term for the remaining lifetime of an individual conditionally on the age of the individual.

Definition 2.2 Let T_x be a non-negative stochastic variable that represent the remaining lifetime for an arbitrary individual at age x with distribution function

$$F_x(t) = P(T_x \leq t), \quad t \geq 0.$$

Using conditional probability, the probability of an individual of age x surviving a time $t > x$ can be expressed as

$$P(T_x > t) = P(T > x + t | T > x) = \frac{P(T > x + t)}{P(T > x)}. \quad (2.3)$$

Let

$$l_x(t) = 1 - F_x(t) = P(T_x > t), \quad t > 0 \quad (2.4)$$

be the survival function for T_x . Then $l_x(t)$ is the probability that an individual at age x lives at least t years more. Moreover, note that $l_0(t)$ is the survival function for a newborn individual. $F_0(t), f_0(t), l_0(t)$ will be denoted as $F(t), f(t), l(t)$ to follow the notation in the definition of T . Furthermore, (2.3) can be written as

$$l_x(t) = \frac{l(x + t)}{l(x)}$$

This emphasize that the survival function for an individual at age x can be expressed using the survival function for a newborn individual.

To calculate the probability $P(T_x > t)$ we need to express the survival function in terms of the mortality intensity. By the fact that

$$l'(t) = \frac{d}{dt}(1 - F(t)) = -f(t)$$

and (2.2) the mortality intensity can be expressed as

$$\mu_x = \frac{f(x)}{1 - F(x)} = \frac{-l'(x)}{l(x)} = \frac{-d}{dt}(\ln l(x)).$$

Since $l(0) = 1$ the equation above can be written as

$$l(x) = \exp\left(-\int_0^x \mu_s ds\right). \quad (2.5)$$

Hence using (2.3) and (2.5) we see that

$$l_x(t) = \frac{l(x+t)}{l(x)} = \exp\left(-\int_x^{x+t} \mu_s ds\right). \quad (2.6)$$

The expected value of the remaining lifetime is defined as

$$E[T_x] = \int_0^\infty (1 - F_x(t))dt = \int_0^\infty l_x(t)dt. \quad (2.7)$$

Using (2.6) we get

$$e_x = E(T_x) = \int_0^\infty \frac{l(x+t)}{l(x)} dt. \quad (2.8)$$

The presented theory is the basis for the modeling of an individual's lifetime, the only missing piece is to specify the model for the mortality intensity. In this thesis we will use the so-called Gompertz-Makeham model [15] (henceforth referred to as the Makeham model) which is the most common model used in Sweden.

The mortality intensity is defined as

$$\mu_x = \alpha + \beta e^{\gamma(x-f)}, \quad x \geq 0. \quad (2.9)$$

where f depends on the gender and $\alpha + \beta > 0$, $\beta > 0$ and $\gamma \geq 0$. We assume that we only consider male individuals so $f = 0$ henceforth. Using (2.9) in (2.6) and (2.8) we get

$$\begin{aligned} e_x &= \int_0^\infty \frac{l(x+t)}{l(x)} dt = \int_0^\infty \frac{e^{-\alpha(x+t) - \frac{\beta}{\gamma}(e^{\gamma(x+t)} - 1)}}{e^{-\alpha x - \frac{\beta}{\gamma}(e^{\gamma x} - 1)}} dt \\ &= \int_0^\infty e^{-\alpha t - \frac{\beta}{\gamma} e^{\gamma x} (e^{\gamma t} - 1)} dt \end{aligned}$$

2.1.2 Stochastic mortality intensity

Usually life insurance companies performs their actuarial calculations using a deterministic mortality intensity which only depends on the age of the policy holder. In this thesis the mortality intensity will be treated as a stochastic process and therefore both time dependence and the uncertainty of future development will be included.

This will quantify the risk for insurance companies associated with the underlying mortality risk.

To model the mortality intensity we consider the parameters α, β and γ in (2.9) as stochastic processes. We consider a discrete set of modeling points $0 = t_0, \dots, t_i$. We get

$$\begin{aligned}\alpha(t_i) &= \alpha(t_{i-1}) + \sigma_\alpha W_1(t_i - t_{i-1}) \\ \beta(t_i) &= \beta(t_{i-1}) + \sigma_\beta W_2(t_i - t_{i-1}) \\ \gamma(t_i) &= \gamma(t_{i-1}) + \sigma_\gamma W_3(t_i - t_{i-1})\end{aligned}\tag{2.10}$$

where $\{W_i\}$ are independent standard Brownian motions. This means that for a future modeling point t_i the mortality intensity is dependent on the development of the parameters α, β, γ . We write the mortality intensity as $\mu_{x+T_i}(t_j)$ to emphasise that it depends on the modeling point. Further models of the mortality intensity can be found in [7] and [9].

2.2 Insurance product model

In this section the characteristics of the insurance product is specified. In the Solvency II framework we only consider the guaranteed part of the insurance product, the financial bonus will not be included. Considering discrete time steps we define period i as $[T_i, T_{i+1}]$. We assume that premiums are paid at the beginning of each period while benefits are paid at the end of the period. Administrative costs are assumed to be included in the premium. Moreover, we assume that the policy holder retire at an age τ and that the probability of surrender is zero.

The guaranteed part of the insurance product is defined by the following characteristics:

- Premium: $\Pi_1(t), \dots, \Pi_i(t)$ denotes the payments of an insurance holder at a time t for the beginning of period i , if he is still alive at that time.
- Death benefits: $D_1(t), \dots, D_i(t)$ is the guaranteed payments to an insurance holder at a time t for period i if he dies in that period.
- Survival benefits: $S_1(t), \dots, S_i(t)$, where $S_i(t)$ denotes the guaranteed payments to the policy holder at a time t for period i , if he survives period i .

We also define T_d and T_s to be the maturity of death benefits and survival benefits respectively. Given the benefits characteristics of an insurance product, the premiums are calculated such that the present value of death and survival benefits is equal to the present value of premiums at the start of the insurance.

In this thesis the present values are calculated using a fixed technical interest rate r_g that constitutes an interest rate guarantee. This means that the guaranteed part of the insurance policy increase with r_g . We define

$$a_i = \frac{1}{(1 + r_g/m)^i}$$

as the discount factor for period i . The guaranteed part $G_i(t)$ can be described as

$$G_i(t) = \frac{1}{a_1} (G_{i-1}(t) + \Pi_i(t)) - S_i(t) - D_i(t),$$

where $G_0(t)$ is the guaranteed value at the beginning of period $i = 0$ for a time t . Given the value of $G_n(t)$ at death or retirement the annuity paid monthly is calculated using the present value approach with the technical interest rate. Notice that if we, as before, consider discrete modeling points we replace $t \rightarrow t_j$. A more complete asset-liability model can be found in [11].

2.3 Asset model

In the asset model it is assumed that the insurance company invests its capital either in zero coupon bonds or stocks. The stock prices are modeled by a stochastic process and the initial prices can be taken directly from the stock market. The bond prices are modeled by the interest rate model, this because the Yield To Maturity (YTM) for the bonds are taken from the yield curve.

2.3.1 Stock market model

The foundation of the stock model is that the price of the underlying asset follows a geometric Brownian motion process described by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad (2.11)$$

where σ , μ are constants representing the volatility of the asset and the expected rate of return. $W(t)$ is a standard Brownian motion. The volatility can be set from historical data and should be seen as measure of the rate and magnitude of price fluctuations.

Under the assumption that σ and μ is known the stochastic differential equation in (2.11) can be solved using Ito's lemma, resulting in the Markov chain

$$S(t_i) = S(t_{i-1}) \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) (t_i - t_{i-1}) + \sigma \sqrt{t_i - t_{i-1}} Z_i \right\}. \quad (2.12)$$

Where $S(t)$ is the stock price at time t and $\{Z_i\}$ are independent $N(0, 1)$ distributed random variables. Note that this is in discrete time and that the time steps are arbitrary.

The simulation model for a single asset is easily extended to d assets, where $S(t)$, σ , μ and W are d -dimensional vectors. The correlation between assets is described by their covariance matrix Σ . Then the movement of these assets are described by the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu dt + \Sigma dW(t). \quad (2.13)$$

which can be solved for every time step

$$S(t_i) = S(t_{i-1}) \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) (t_i - t_{i-1}) + \Sigma \sqrt{t_i - t_{i-1}} Z_i \right\}. \quad (2.14)$$

For effective simulation one can set $AA^T = \Sigma$ where A is the cholesky matrix and thereby reduce the number of multiplications. The Markov chain is then expressed as

$$S^k(t_i) = S^k(t_{i-1}) \exp \left\{ \left(\mu_k - \frac{\sigma_k^2}{2} \right) (t_i - t_{i-1}) + \sqrt{t_i - t_{i-1}} \sum_{j=k}^d A_{k,j} Z_{i,j} \right\} \quad (2.15)$$

where the subindex k stands for the stock type and $\{Z_{i,\cdot}\}$ are independent d -dimensional $N(0, I)$ random vectors.

2.3.2 Bond model

A zero-coupon bond is a security making a single payment at maturity time T . Consider a bond price at $t = 0$ of one unit, i.e $B(0, T) = 1$. The value of the bond at maturity date is then given by

$$B(T, T) = 1 \cdot e^{Y(0,T) \cdot T}$$

where $Y(0, T)$ is the YTM, the growth rate of the bond. For a future time t we see that the value of the bond is

$$B(t, T) = e^{Y(0,T)T} \cdot e^{-Y(t,T)(T-t)}, \quad (2.16)$$

which is the payment of the bond at maturity date discounted to t with $Y(t, T)$. With m compounding intervals in (2.16) we get

$$B(t, T) = \frac{\left(1 + \frac{Y(0,T)}{m}\right)^{T \cdot m}}{\left(1 + \frac{Y(t,T)}{m}\right)^{(T-t) \cdot m}}. \quad (2.17)$$

Here $Y(0, T)$ is set from the initial yield curve structure and $Y(t, T)$ is set from the future spot rates simulated in the interest rate model presented below. Notice that in the next section the bond price is defined as $B_n(t)$ which corresponds to $B(t, T_n)$ in this section.

2.4 Interest rate model

The modeling of the interest rate development is assumed to be described by the LIBOR market model. The LIBOR model simulates forward rates over given time-steps and gives a forecast of the yield curve for each step. It is often used to price interest rate derivatives, caplets¹, and the main advantage of this model is that it can easily be calibrated to market values through the Black formula or historical data. For further reading we recommend [4] and [19].

2.4.1 The LIBOR market model

LIBOR rates are based on simple interest. If L denotes the rate for an increment of length $\delta = T_{n+1} - T_n$ with time measured in years, then the interest earned on the accrual period is δL . A forward LIBOR rate $L_n(t)$ is the rate set at time t for the accrual period $[T_n, T_{n+1}]$. Let $B_n(t)$ denote a zero-coupon bond issued at time t and maturing at T_n . Consider a person entering a contract at time t to borrow 1 unit at time T_n and repay it with interest at T_{n+1} , then the total payment will be $1 + L_n(t)\delta$. At the same time an investor could, at time t , buy a zero-coupon bond maturing at T_n and financing it by issuing zero-coupon bonds with maturity T_{n+1} . If the number k of bonds issued satisfies

$$kB_{n+1}(t) = B_n(t)$$

¹A caplet is an interest rate derivate providing protection against an increase in an interest rate.

we have zero cash flow at t . To avoid arbitrage the amount paid at T_{n+1} in this transaction must be the same as the amount paid in the forward rate transaction. Thus

$$\begin{aligned} k &= 1 + L_n(t)\delta \\ k &= \frac{B_n(t)}{B_{n+1}(t)} \end{aligned}$$

which gives

$$L_n(t) = \frac{1}{\delta} \left(\frac{B_n(t) - B_{n+1}(t)}{B_{n+1}(t)} \right), \quad 0 \leq t \leq T_n, \quad n = 0, \dots, M, \quad (2.18)$$

where we consider a fixed set of maturities $0 = T_0 < \dots < T_{M+1}$, with M forward rates, and the increment $\delta = T_{n+1} - T_n$ is constant. In this calculation it is assumed that the bonds always make their payment, ignoring the fact that the issuers may indeed default. The figure below shows the forward rates, bond prices and spot rates to help the reader understand the relation between the quantities.

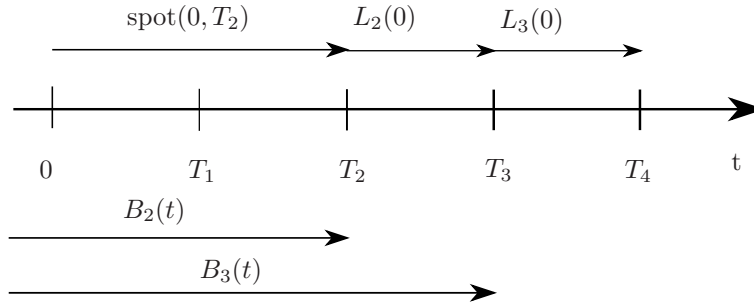


Figure 2.1: Shows the relation between spot rates, forward LIBOR rates and bond prices.

From (2.18) we see that bond prices determine the forward rates, this can be inverted to

$$B_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, \quad n = i + 1, \dots, M + 1 \quad (2.19)$$

Moreover, we notice that LIBOR rates does not determine bond prices for periods shorter then the accrual periods.

2.4.2 Forward rate dynamics

The forward rate dynamics for the Libor Market Model is described by the m-dimensional diffusion equation

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)^T dW(t), \quad 0 \leq t \leq T_n, \quad n = 1, \dots, M \quad (2.20)$$

with W as a m-dimensional standard Brownian motion and $\mu_n(t)$ can depend on both time and the current vector of forward rates. The volatility structure $\sigma_n(t)$ is chosen such that it is piecewise constant and only depends on the difference $T_n - t$.

I.e the volatility structure is stationary and can be written as $\sigma_n(t) = \sigma(T_n - t)$. For a set of M forward rates the volatility matrix becomes

$$\Sigma = \begin{pmatrix} \sigma(1) & & & & \\ \sigma(2) & \sigma(1) & & & \\ \vdots & \vdots & & & \\ \sigma(M) & \sigma(M-1) & \cdots & \sigma(1) & \end{pmatrix}.$$

The volatilities are usually calibrated to market values via the Black formula [12] for caplet pricing. It can also be set by using historical data or a combination of the alternatives, see [19].

2.4.3 The drift function

To determine the drift function $\mu_n(t)$ we start by rewriting (2.18) as

$$L_n(t)B_{n+1}(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta}. \quad (2.21)$$

$L_n(t)B_{n+1}(t)$ is a tradable asset since it can be replicated by buying and selling two bonds. Then it can be divided with the numeraire asset $B_{n+1}(t)$, and hence $L_n(t)$ is a martingale under the Q^{n+1} measure. Notice that Q^{n+1} is the martingale measure connected with the chosen numeraire asset B_{n+1} . Furthermore, if $L_n(t)$ is a martingale the diffusion process can be written as

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t)dW^{n+1}(t),$$

where $dW^{n+1}(t)$ is a Brownian motion under Q^{n+1} . Applying the reasoning above to the forward LIBOR rate $L_j(t)$, $j \neq n$ one notice that $L_j(t)B_{n+1}(t)$ is not a tradable asset and therefore under the measure Q^{n+1} only $L_n(t)$ is a martingale. To evolve all forward rates under the same measure we need a diffusion process for all forward rates under that measure. From the motivation above we get that under a given measure only one forward rate is martingale, i.e has a driftless diffusion equation, while all other forward rates have a non-zero drift term. In order to determine the drift term under the Q^{n+1} for the Q^n martingale $L_{n-1}(t)$ we consider the change of measure $dQ^n \rightarrow dQ^{n+1}$, see appendix A. The Radon-Nikodym derivative $p(t)$ is

$$\frac{dQ^n}{dQ^{n+1}} = p(t) = \frac{B_n(t)/B_n(0)}{B_{n+1}(t)/B_{n+1}(0)} = \frac{B_{n+1}(0)}{B_n(0)} (1 + \delta L_n(t)). \quad (2.22)$$

Using Girsanov's theorem, see Appendix A, we see that we need to find a process $k(t)$ such that

$$p(t) = \exp \left(\int_0^t k(s)dW^{n+1}(s) - \frac{1}{2} \int_0^t k^2(s)ds \right). \quad (2.23)$$

An application of Itô's lemma on (2.23) yields

$$dp(t) = p(t)k(t)dW^{n+1}(t), \quad (2.24)$$

thus $k(t)$ can be seen as the volatility of Radon-Nikodym derivative $p(t)$. So applying Itô's lemma on (2.22) and remembering the driftless diffusion process of $L_n(t)$ under Q^{n+1} we get

$$dp(t) = \frac{\delta \sigma_n(t)L_n(t)}{1 + \delta L_n(t)} p(t)dW^{n+1}(t). \quad (2.25)$$

Comparing (2.24) and (2.25) we identify that $k(t) = \frac{\delta\sigma_n(t)L_n(t)}{1+\delta L_n(t)}$. Hence, a change of measure from Q^n to Q^{n+1} affects the Q^n Brownian motion $dW^n(t)$ giving

$$dW^n(t) = dW^{n+1}(t) - \frac{\delta\sigma_n(t)L_n(t)}{1+\delta L_n(t)}dt.$$

The diffusion equation for $L_{n-1}(t)$ under the Q^{n+1} measure can be written as

$$\frac{dL_{n-1}(t)}{L_{n-1}(t)} = -\frac{\delta\sigma_n(t)L_n(t)}{1+\delta L_n(t)}\sigma_{n-1}(t) + \sigma_{n-1}(t)dW^{n+1}(t).$$

Following the argument above it is a straight forward calculation to show that under Q^{j+1} the drift term of $L_n(t)$ is

$$\mu_n(L(t), t) = -\sigma_n(t) \sum_{k=n+1}^j \frac{\delta\sigma_k(t)L_k(t)}{1+\delta L_k(t)}, \quad n < j$$

and for $n > j$ we get

$$\mu_n(L(t), t) = \sigma_n(t) \sum_{k=j+1}^n \frac{\delta\sigma_k(t)L_k(t)}{1+\delta L_k(t)}, \quad n > j.$$

Notice that the drift term depends on both time and, at a future time t , stochastic forward rates. Under the spot measure the SDE in (2.20) becomes

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t) \sum_{k=i+1}^n \frac{\delta\sigma_k(t)L_k(t)}{1+\delta L_k(t)}dt + \sigma_n(t)dW(t), \quad n = 1, \dots, M. \quad (2.26)$$

Simulating $L_n(t)$ under the spot measure results in a small discretization error described by Glasserman and Zhao [13].

2.4.4 Discretising the forward equation

When simulating forward Libor rates under the spot measure we can apply an Euler scheme to discretizes the SDE in (2.26), see [12]. We choose a time grid $0 = t_0 < t_1 < \dots < t_{M+1}$, where $t_i = T_i$ so that the simulation evolves from one maturity date to the next. The solution is

$$L_n(t_{i+1}) = L_n(t_i) + \mu_n(L(t_i), t_i)L_n(t_i)[t_{i+1} - t_i] + L_n(t_i)\sqrt{t_{i+1} - t_i}\sigma_n(t_i)Z_{i+1} \quad (2.27)$$

with

$$\mu_n(L(t_i), t_i) = \sigma_i(t) \sum_{k=i+1}^n \frac{\delta\sigma_k(t)L_k(t_i)}{1+\delta L_k(t_i)}.$$

and Z_1, Z_2, \dots, Z_M are independent $N(0, I)$ random vectors in \mathbb{R}^d . However, (2.27) can produce negative rates, so for numerical stability we apply, as in [12], the Euler scheme to $\log(L_n(t_i))$ instead. We get

$$L_n(t_{i+1}) = L_n(t_i) \exp \left(\left[\mu_n(L(t_i), t_i) - \frac{1}{2}\sigma_n(t_i)^2 \right] [t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\sigma_n(t_i)Z_{i+1} \right). \quad (2.28)$$

Which is especially useful when $\sigma_n(t)$ is deterministic because $\log L_n$ is close to lognormal.

2.4.5 Simulating forward rates

The method used to simulate the forward rates is the short step method, i.e exactly one forward rate will come to its reset time at every step. For example, in a 6-month forward rate structure 6-months long steps will be taken. The procedure follows the algorithm below.

1. Choose $B_1(0)$ as numeraire asset. Set $j = 0$
2. Evolve the forward rates according to the Euler scheme from t_j to t_{j+1} with the drift corresponding to the $j < i$ case.
3. If t_{j+1} is not the final destination and choose $L_{j+1}(t_{j+2})$ as numeraire asset.
4. Set $j = j + 1$, go to step 2.

This algorithm produces the following scheme, where each set of forward rates begin with a spot rate. To initialize the simulation we need to set the values for

$spot(t_0, T_1)$	$L_1(t_0)$	$L_1(t_0)$	\cdots	$L_{M-1}(t_0)$	$L_M(t_0)$
	$spot(t_1, T_2)$	$L_2(t_1)$	\cdots	$L_{M-1}(t_1)$	$L_M(t_1)$
		$spot(t_2, T_3)$	\cdots	$L_{M-1}(t_2)$	$L_M(t_2)$
			\ddots	\vdots	\vdots
				$spot(t_{M-1}, T_M)$	$L_M(t_{M-1})$
					$spot(t_M, T_{M+1})$

Table 2.1: Scheme from the simulation of forward rates in the LIBOR model.

$L_1(0), \dots, L_M(0)$. Assuming that the initial bond prices $B_1(0), \dots, B_{M+1}(0)$ exists the forward LIBOR rates are given by (2.18). The forward rates can also be set from a yield curve.

CHAPTER 3

The model

In this chapter we present the model for calculating the SCR for a portfolio of life insurance policies. We also describe the dependency on the risk factors for the asset and liability portfolios. For the insurance portfolio we specify the mortality intensity and the payment process for the entire portfolio. Moreover, we define the value process of assets and liabilities and the change in RBC over time as $\Delta_{RBC}(t)$. To calculate the SCR we introduce a risk measure that converts the risk reflected in the distribution of $\Delta_{RBC}(t)$ into a numerical value. In the last section we present the numerical evaluation method used in this thesis.

Considering $m = 12$ to be the number of periods in one year the formula for the SCR becomes

$$\text{SCR} = \text{CVaR}_\alpha [\Delta_{RBC}(t_j)] = \text{CVaR}_\alpha \left[\frac{\mathbf{A}(t_j) - \mathbf{V}(t_j)}{(1 + r_j(0)/m)^j} - (\mathbf{A}(0) - \mathbf{V}(0)) \right]. \quad (3.1)$$

In this thesis we consider a time horizon of one year and periods of one month, i.e. $T_i - T_{i-1} = 1/12$. Hence, the SCR in (3.1) is obtained by choosing $j=12$. In the following we have considered discrete modeling points such that $t_i = T_i$.

3.1 The model for the insurance portfolio

In the insurance portfolio we consider n individuals at the same age x . The individual remaining lifetimes of the insurer at t_0 is assumed to be described by the sequence T_x^1, \dots, T_x^n of identically distributed non-negative random variables. The lifetimes in a portfolio are mutually independent conditionally on the mortality intensity. The probability of a single individual surviving to time T_i is given by (2.6). Moreover, we consider the counting process $N_i(t_j)$ that describes the number of deaths in the portfolio until T_i for model point t_j as

$$N_i(t_j) = \sum_{j=0}^n I_{T_x^j \leq T_i}. \quad (3.2)$$

Where we remember that $T_i - T_{i-1}$ stands for the period i . It follows that the counting process $N_i(t_j)$ is a Markov process and that the stochastic intensity $\lambda_i(t_j)$

related to $N_i(t_j)$ is defined by

$$\lambda_i(t_j) = (n - N_{i-1}(t_j)) \mu_{x+T_i}(t_j).$$

The intensity of the portfolio for period i and at a modeling point t_j is just the number of survivors at period $i - 1$ multiplied with the mortality intensity for a single individual.

To value the liability of the insurance portfolio we need to define a payment process that describes the cash flow of the portfolio. The benefits less premiums is described by the payment process $C_i(t_j)$. Thus, $C_i(t_j)$ is the net payment to the policy holders for modeling point t_j in the period i . Using the quantities specified for the insurance product model we get

$$\begin{aligned} C_i(t_j) = & -n\Pi_0(t_j) - \Pi_i(t_j)I_{0 < T_i < \tau} (n - N_i(t_j)) \\ & + S_i(t_j)I_{\tau \leq T_i < T_s} (n - N_i(t_j)) + D_i(t_j) (N_i(t_j) - N_{i-1}(t_j)) I_{0 < T_i < T_d}. \end{aligned} \quad (3.3)$$

The first term, $-n\Pi_0(t_j)$, is the single premium paid at $i = 0$ by all policyholders, $\Pi_i(t_j)I_{0 < T_i < \tau} (n - N_i(t_j))$ is the premiums paid by the surviving policyholders until retirement age τ . In

$$S_i(t_j)I_{\tau \leq T_i < T_s} (n - N_i(t_j)) + D_i(t_j) (N_i(t_j) - N_{i-1}(t_j)) I_{0 < T_i < T_d},$$

the first term is the benefits to surviving policyholders after retirement age τ for period i . The last term is the death benefits to policy holders in period i for a time t_j .

3.2 Valuation

In this section we define the market value process for the asset and liability portfolios. The market value of assets is fixed at t_0 since the values of assets and bonds can be taken from a liquid market. The market value of liabilities however is stochastic both at t_0 and t_1 since we in this model does not consider a deterministic cash flow for the portfolios.

First we define the yield curve consisting of a set of spot rates as $r_i(t_j)$, where $r_i(t_j)$ is the spot rate for period i and modeling point t_j . Given a set of spot rates $r_i(t_j)$ the forward LIBOR rates are calculated using

$$L_i(t_j) = m \left[\frac{(1 + r_{i+1}(t_j)/m)^{i+1}}{(1 + r_i(t_j)/m)^i} \right] - m \quad (3.4)$$

which also can be used to determine spot rates from a set of forward rates.

The stocks in the portfolio evolve according to (2.14) with the initial values $S^1(0), \dots, S^d(0)$ for d assets. The value for the entire stock portfolio is

$$S(t_j) = \sum_{q=1}^d S^q(t_j).$$

For the bond portfolio we consider the development of a single bond in (2.17). We consider k bonds with initial values B_0^i and maturity dates T_B^i . The value of the bond portfolio is

$$B(t_j) = \sum_{q=0}^k B_0^q \cdot B(t_j, T_B^q).$$

Assuming that the correlation between the stock and interest rate market is zero we get that the value $A(t_j)$ of the asset portfolio is described by

$$A(t_j) = S(t_j) + B(t_j).$$

For the valuation of the liability of an insurance portfolio we consider the payment process $C_i(t_j)$ described in (3.3). The present value of this cash flow is calculated by discounting with the yield curve described by $r_i(t_j)$. We define

$$d_i(t_j) = (1 + r_i(t_j)/m)^{-i}$$

to be the discount factor for period i modeled at time t_j . Moreover we let $V(t_j)$ be the liability modeled at time t_j , the value at that time is

$$V(t_j) = \sum_{q=0}^N d_q(t_j) C_q(t_j) \quad (3.5)$$

where N is the period where the last payment is made for the portfolio. The definitions above induce that the value of the RBC can be described by

$$\text{RBC}(t_j) = A(t_j) - V(t_j). \quad (3.6)$$

and the change in RBC as

$$\Delta_{RBC}(t_j) = \frac{\text{RBC}(t_j)}{d_j(0)} - \text{RBC}(t_0). \quad (3.7)$$

3.3 Risk measure and definition of capital

The SCR is defined through a risk measure that converts the risk in the distribution of Δ_{RBC} into a numerical value. In the Solvency II framework the calculation of SCR is based on the so-called Value-at-Risk (VaR) measure at a 99.5 % confidence level, but a different measure may be used if it is approved by the supervising authority. In this thesis we will use Conditional Value-at-Risk (CVaR) at a 99 % confidence level, as suggested in [17]. In the following we refer to [20].

Consider a function $f(x, y)$ that generates values of Δ_{RBC} associated with the decision vector $\mathbf{x} \in \mathbb{R}^n$ and the random vector $\mathbf{y} \in \mathbb{R}^m$. The vector \mathbf{x} can be interpreted as the portfolio of assets and liabilities and the vector \mathbf{y} stands for the uncertainties in risk factor parameters that can affect Δ_{RBC} . Negative values of Δ_{RBC} is considered as losses. We assume that the underlying probability density for \mathbf{y} exists, and is denoted $p(\mathbf{y})$. Then, for every choice of \mathbf{x} we let $f(\mathbf{x}, \mathbf{y})$ be the distribution of Δ_{RBC} in \mathbb{R}^m induced by the vector \mathbf{y} . The probability of $f(\mathbf{x}, \mathbf{y})$ not exceeding a threshold β is given by

$$\Psi(\mathbf{x}, \beta) = \int_{f(\mathbf{x}, \mathbf{y}) \leq \beta} p(\mathbf{y}) d\mathbf{y}.$$

As a function of β for fixed \mathbf{x} , $\Psi(\mathbf{x}, \beta)$ is the cumulative distribution for Δ_{RBC} associated with \mathbf{x} . Moreover we let $\alpha \in (0, 1)$, for a 99 % confidence level we get that $\alpha = 1 - 0.99$, be the specified probability level for the $\alpha - VaR$ and $\alpha - CVaR$, denoted by $\delta_\alpha(\mathbf{x})$ and $\phi_\alpha(\mathbf{x})$ respectively. We get that

$$\delta_\alpha(\mathbf{x}) = \min \{ \beta \in \mathbb{R} : \Psi(\mathbf{x}, \beta) \geq \alpha \} \quad (3.8)$$

and

$$\phi_\alpha(\mathbf{x}) = (\alpha)^{-1} \int_{f(\mathbf{x}, \mathbf{y}) \leq \delta_\alpha(\mathbf{x})} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y}. \quad (3.9)$$

Thus, $\phi_\alpha(\mathbf{x})$ is the conditional expectation of Δ_{RBC} provided that the loss is $\delta_\alpha(\mathbf{x})$ or greater. Figure 3.1 shows the relation between VaR and CVaR.

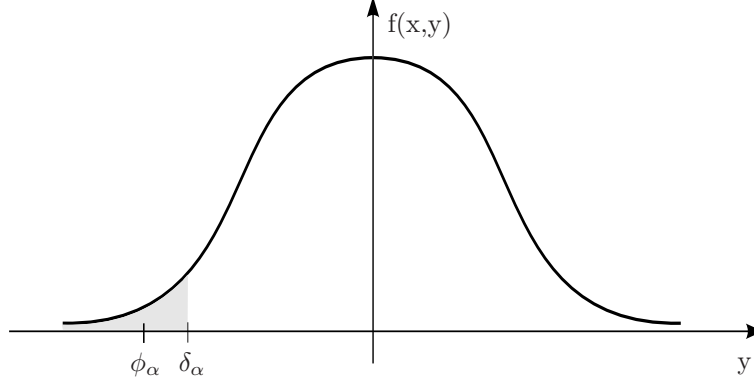


Figure 3.1: VaR is the right endpoint of the shaded area and CVaR is the mean value of all values exceeding VaR.

In our model the vector \mathbf{x} consists of stocks, bonds and insurance policies and the random vector \mathbf{y} is induced by the risk factors modeled in the previous chapter. The integral in (3.9) is solved by generating random samples from $p(\mathbf{y})$ with the use of Monte Carlo simulation.

3.4 Monte Carlo simulation

Monte Carlo simulation is a method that uses random numbers to find solutions to mathematical problems. It is especially useful when modeling phenomena with high uncertainty in inputs or creating stochastic models, for example when calculating the risk of an insurance company. In finance, Monte Carlo methods are often used to calculate the value of companies or evaluating financial derivatives.

The power of Monte Carlo lies in the generality of the method although the convergence rate is rather slow, only $O(n^{-1/2})$. This means that to half the error we need to increase the number of simulation four times. However, the convergence rate does not decrease when increasing the number of dimensions. Hence, Monte Carlo methods becomes attractive when we want to evaluate integrals in high dimensions. Further applications for the use of Monte Carlo methods in finance can be found in [12].

In this thesis Monte Carlo simulation is used to evaluate the integral in (3.9) by generating scenarios for the asset and insurance portfolio. This is done by creating scenarios for each risk in (2.10), (2.15), (2.27) and generating deaths in (3.2). We are only interested in the SCR at $t = 1$ so we consider simulating from $t = 0$ to $t = 1$ in a single step. The algorithm below describes the simulation process for this model.

1. Set $t_0 = 0$. Define the initial values for all parameters from market data, and set the number of scenarios to n .
2. For the insurance portfolio, generate the n -dimensional random vector $\mathbf{L}(0)$. The value $A(0)$ of the asset portfolio is constant at $t_0 = 0$.

3. Calculate the values for $\mathbf{RBC}(0)$.
4. Set $t_1 = 1$.
5. Simulate n random scenarios for the development of the yield curve using the interest rate model.
6. Generate n random values of the asset portfolio in the vector $\mathbf{A}(\mathbf{1})$.
7. Generate the random vector $\mathbf{L}(\mathbf{1})$ using stochastic mortality intensity and the scenarios for the yield curve.
8. Calculate the vector $\mathbf{RBC}(1)$.
9. Calculate $\Delta_{RBC}(1)$
10. Calculate the SCR using (3.1).

Swiss solvency test

The Swiss Solvency Test (SST) is the Swiss solvency framework, under development by the supervising authority FOPI (an abbreviation defined in the introduction), that is consistent with the Solvency II framework. As for the Solvency II framework the SST model requires that a capital requirement called Target Capital (TC) is calculated for the current year. This capital requirement consist of two different capitals. The first capital is the one-year risk capital that emanates from the change in RBC in one year and is calculated using the risk measure Expected Shortfall (ES) at a 99% confidence level. Notice that ES is also called CVaR. The second capital is the risk margin that is defined as the future regulatory capital needed to run-off the existing portfolio. In the *2004 field test* the risk margin was set to 6% of the best estimate of liabilities. The SST standard model contains models for market and insurance risk, the credit risk model is based on the Basel II standardized approach. Also, pre-defined extreme scenarios that have a negative impact on the RBC has to be evaluated and included in the model. In this thesis we only consider the market and life insurance model and two extreme scenarios. For further reading about the risk margin and the SST model in general we refer to [17] and [18].

4.1 Standard model for market risk

The standard model for market risk is based on the assumption that the change of RBC due to market risks can be described as a dependency on market risk factors. These risk factors includes interest rates over different terms, stock indices, currency exchange rates, real estate indices, bond spread, implicit volatilities etc. The market risk factors are assumed to have a multivariate normal distribution. For most factors, the volatility and correlation coefficients are given. Exceptions are made for private equity and different hedge funds since they can behave very differently. These parameters has to be determined by the insurer for its own portfolio. Moreover, each insurer has to estimate the sensitivity to the risk factors for its own portfolio, i.e the partial derivatives of the risk bearing capital with respect to market risk factors. In general they can be approximated by a difference quotient, here we will use the central-difference approach. Let $\Delta_{RBC}(t, h)$ be $\Delta_{RBC}(t)$ influenced by the risk factor

change h . Then the sensitivity is defined as

$$s_n = \frac{\Delta_{RBC}(t, h) - \Delta_{RBC}(t, -h)}{2h}.$$

Where s_n is the sensitivity to risk factor n . Given that the variances and covariances of the risk factors and the dependencies of assets and liabilities on the risk factors are known, the variance of the risk capital due to market risks are given by

$$\Sigma_{market} = (s_1\sigma_1, \dots, s_n\sigma_n) \cdot \begin{pmatrix} 1 & \rho_{1,2} & \dots & \dots & \rho_{1,n} \\ \rho_{2,1} & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & \rho_{n-1,n} \\ \rho_{n,1} & \dots & \dots & \rho_{n,n-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} s_1\sigma_1 \\ \vdots \\ s_n\sigma_n \end{pmatrix}.$$

Here, σ_i is the volatility of market risk factor i , $\rho_{i,j}$ is the correlation between risk factor i and j and s_i is the sensitivity of market risk factor i .

4.2 Standard model for life insurance

As in the market risk model it is assumed that the change in RBC due to the risk factors have a normal distribution. Notice that in the SST model the mortality intensity is defined as

$$\mu_{x,t} = \mu_{x,t_0} \cdot e^{-\lambda_x(t-t_0)}, \quad (4.1)$$

where λ_x describes the trend of the cohort. As before x is the age of the individuals in the portfolio. For the insurance risk the following parameters will be considered.

- Mortality, parameter μ_x with volatility $\sigma_m = 20\%$.
- Longevity, parameter λ_x with volatility $\sigma_l = 10\%$.

The insurance factors are assumed to be independent, i.e the total variance for market risk is given by

$$\Sigma_{insurance} = (s_m\sigma_m, s_l\sigma_l) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_m\sigma_m \\ s_l\sigma_l \end{pmatrix},$$

where s_m and s_l is the sensitivity for the mortality intensity and the longevity.

4.3 Aggregation of the risk factors

If X^j is the change in RBC due to risk factor j the market risk factors and insurance risk factors are aggregated through a correlation matrix R corresponding to the vectors

$$X(t) = (X^1(t), X^2(t), \dots, X^d(t)),$$

which can be seen as

$$X(t) = (X_A(t), X_B(t)).$$

Here $X_A(t)$ is the change with respect to market risk factors and $X_B(t)$ the change with respect to insurance risk factors. The correlation matrix has the following structure

$$R = \begin{bmatrix} A & C \\ C & B \end{bmatrix}.$$

A is the correlation matrix for vector X_A and B is the correlation matrix for vector X_B . For simplicity $C = 0$ so that market risk factors and insurance risk factors are uncorrelated. Hence the total variance is just

$$\Sigma = \Sigma_{market} + \Sigma_{insurance}.$$

Following the argument in [17], where it is assumed that the aggregated normal distribution has zero drift, we get that the TC is defined as,

$$TC = \frac{\sqrt{\Sigma}}{\alpha} \varphi(q_\alpha(Z)). \quad (4.2)$$

Where $q_\alpha(Z)$ is the α -quantile of a random variable Z having a standard normal distribution.

4.4 Extreme Scenarios

In this section we present the extreme scenarios considered in this thesis. There are numerous scenarios defined for the SST model, but we will only consider one financial distress scenario and one scenario for the longevity risk.

For the financial distress we consider that all shares lose 30 % of their value. In the longevity scenario it is assumed that the mortality decreases twice as quickly as assumed in (4.1). The mortality intensity becomes

$$\mu_{x,t} = \mu_{x,t_0} \cdot e^{-2\lambda_x(t-t_0)}.$$

The probability for a scenario to occur is set to $p_i = 0.5\%$ for both scenarios and $p_0 = 1 - p_1 - p_2$. Here p_1 refers to the financial distress scenario and p_2 refers to the longevity scenario. The sensitivities c_i for the scenarios is calculated by

$$c_i = E[RBC(t | \text{scenario occurs}) - RBC(t | \text{scenario does not occur})].$$

We let $F_0(z)$ be the distribution obtained from the standard model without scenarios for Δ_{RBC} . Then the new distribution function $F(z)$ is obtained by

$$F(z) = \sum_{j=0}^2 p_j \cdot F_0(z - c_j),$$

where $c_0 = 0$. The ES can then be calculated using a numerical method, for example Monte Carlo simulation. However, if $F_0(z)$ is a normal distribution we can calculate the ES using the following formula (shown in [17]).

$$ES = \frac{1}{\alpha} \sum_{j=0}^n p_j (-\sigma^2 \cdot \varphi_{c_j, \sigma}(q) + c_j \Phi_{c_j, \sigma}(q)). \quad (4.3)$$

Here q is the α -VaR value for $F(x)$ and $\Phi_{c_j, \sigma}$ is the cumulative normal distribution with mean c_j and standard deviation σ .

Numerical results

In this chapter we will present and analyze the results of our two capital requirement models for three different types of insurance portfolios. To compare the models we compute the SCR and put it in relation to the risk-bearing capital, i.e the SCR divided with the RBC. The market value of liabilities is calculated using the expected discounted cash flow for the portfolio, the so-called actuarial reserve (FTA). This because we consider two different types of mortality models in (2.9) and (4.1). This means that the market value of liabilities for the models will differ. Large values of the quota induce that the company need to hold a greater amount of its RBC to achieve the solvency requirement. Moreover, we compare the SCR from the two models to the current capital requirement, which is that the RBC must be greater then 4% of the FTA. In the simulation we consider time steps of one month, hence $m = 12$.

The insurance policies considered are the following:

- Policy A; Lifelong pension including survivor's protection with payments for 20 years or until the policy holder is at most $\tau+20$ years.
- Policy B; Temporary life annuity for as long as the policyholder is alive and at most for 10 years. Survivor's protection for 10 years or until the policyholder is $\tau + 10$ years.
- Policy C; Lifelong pension with fixed annuity of 20000 and survivor's protection for 10 years or until the policyholder is at most $\tau+10$ years.

In all simulations we assume that the policy holders age is 40 years and retire at an age of $\tau = 65$ years. Notice that the benefit characteristics of policy A and C indicate that the insured is entitled to hereditary gains after the retirement age. This due to the fact that if the insured dies within $\tau < T_i < T_d$ all of the guaranteed value $G_i(t)$ is not returned to the insured. The input data in the simulations is taken from an actual portfolio used by a swedish insurance company.

5.1 Single policies

In this section we present the results from the market valuation of the three policies. Furthermore, we perform a first simulation of the three policies for one individual

using the stochastic model. The result is presented by displaying the distribution of the policies and comparing them to a normal distribution.

The input data for the insurance policies used in the calculation of the FTA and in the simulation is presented in the table below.

Policy	A	B	C
Premium	1.5'	5'	5'
Available capital	36'	105'	105'
Guaranteed value	28'	85'	85'

Table 5.1: Input data for the insurance policies. The available capital is the asset value disposed for the policy type. The guaranteed is the value of $G_0(0)$ for each policy. All values are in thousands.

The constants in the Makeham model is in all simulations set to $\alpha = 0.001$, $\beta = 0.000012$, $\gamma = 0.1013$ and the yield curve, without a deduction, used for market valuation can be found in appendix A.

In the table below the results for the calculation of the FTA in proportion to the available capital can be seen. Here we see that the FTA differ between the models due the different mortality models.

Policy	FTA	
	stoch.	SST
A	0.3202	0.3220
B	0.4260	0.4253
C	0.4042	0.4310

Table 5.2: The table shows the actuarial reserve (FTA) for the three policies in proportion to the available capital. Which is the discounted expected value of liabilities for the stochastic model and the SST model.

In the simulation we assume that the initial yield curve is as in Appendix A and that correlation between stocks and interest rates is zero, i.e we have the following correlation matrix for market risk

$$\rho_{market} = \begin{bmatrix} \rho_{st} & 0 \\ 0 & \rho_{i.r} \end{bmatrix}.$$

Where ρ_{st} is the correlation matrix for stocks and $\rho_{i.r}$ is the correlation matrix for interest rate. In the asset model the capital is allocated in the position vector $P = \{0.58, 0.42\}$ where the first entry is the part invested in the stock market and the second entry is the part of available capital invested in zero coupon bonds. In the stock model we consider six different stocks with drifts $\mu_{stock} = \{0.1, -0.05, 0.06, -0.02, 0.05, -0.1\}$ and volatilities $\sigma_{stock} = \{0.11, 0.15, 0.12, 0.14, 0.10, 0.13\}$. The capital divided between the assets is defined by the vector $s = \{0.20, 0.10, 0.25, 0.10, 0.25, 0.10\}$. The correlation between stocks are given by

$$\rho_{st} = \begin{pmatrix} 1 & 0.30 & 0.46 & 0.41 & 0.85 & 0.65 \\ 0.30 & 1 & 0.19 & 0.72 & 0.32 & 0.59 \\ 0.46 & 0.19 & 1 & 0.32 & 0.33 & 0.47 \\ 0.41 & 0.72 & 0.32 & 1 & 0.46 & 0.86 \\ 0.85 & 0.32 & 0.33 & 0.46 & 1 & 0.66 \\ 0.65 & 0.59 & 0.47 & 0.86 & 0.66 & 1 \end{pmatrix}.$$

For the zero coupon bonds three different maturity dates are considered, $T_{bonds} = (2, 5, 10)$ years, and the capital is divided by the vector $b = (0.80, 0.10, 0.10)$. The future zero coupon bond values can be calculated using the simulated forward rates in the LMM model and using (3.4) and (2.17). In the LMM model the initial forward rates are set from the yield curve in appendix A and the volatility structure can also be found in appendix A. The volatility for the stochastic mortality model in (2.10) is set to 1% of the parameter value. I.e in these simulations $\sigma_\alpha = 0.01\alpha$, $\sigma_\beta = 0.01\beta$, $\sigma_\gamma = 0.01\gamma$. The figure below shows the resulting distribution of Δ_{RBC} of one individual for the three policies.

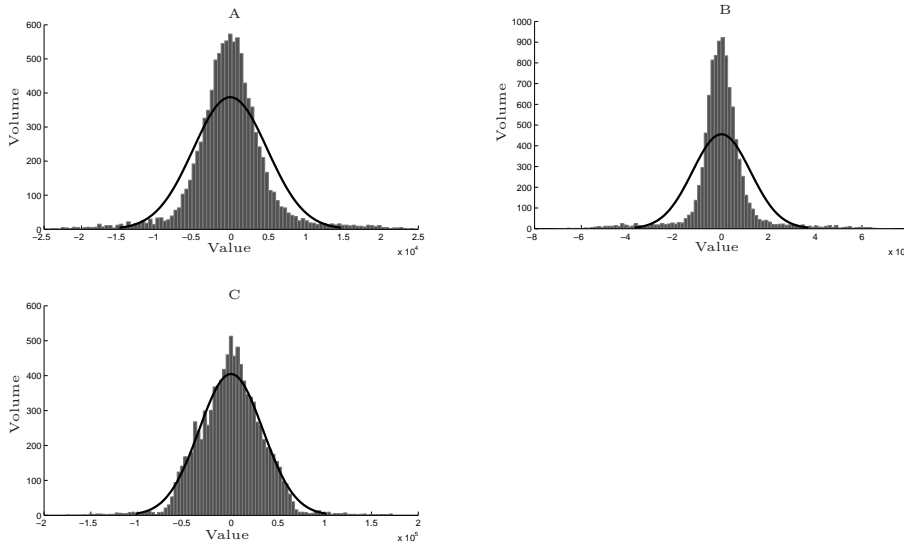


Figure 5.1: In the figure, the histogram shows the distribution of Δ_{RBC} for each policy. The simulated data is fitted to a normal distribution displayed by the black line. The simulation was done with 10 000 replications.

5.2 Portfolio of policies

In this section we present the results from the simulation of portfolios for our two models. We consider portfolios of $n = \{1, 10, 100, 1000, 10000\}$ individuals with same type of insurance policy. As in the previous section the SCR is in proportion to the RBC at $t = 0$. For the simulation of assets we use the same input parameters as before. The total available capital for the portfolio is just the available capital for one individual multiplied with the number of individuals in the portfolio. Moreover we present the distribution of the change in assets and liabilities for a portfolio with insurance policy B. In the SST model we consider the following risk factors:

- 6 different stocks with the same input as for the stochastic model.
- 7 different interest rates with maturity dates $T = (1/12, 1, 2, 5, 10, 20, 30)$ and volatility $\sigma_{i,r} = (0.2, 0.3, 1.1, 3.2, 5.3, 9.7, 14.3)\%$.
- Mortality risk with volatility $\sigma_{mortality} = 0.20$.
- Longevity risk with volatility $\sigma_{longevity} = 0.10$.

- The extreme scenarios defined in chapter 4.

The table below shows the numerical values from the simulation of the stochastic model and the SST model compared to the capital requirement for the Solvency I framework. All values are in proportion to the RBC at $t = 0$. We have assumed that the SST and Solvency I capital requirement only consider large portfolios. In table

Policy	n	Solvency II		Solvency I
		Stoch.	SST	
A	1	0.7480		
	10	0.2433		
	100	0.1922		
	1000	0.1897		
	10000	0.1859	0.2605	0.0188
B	1	0.8536		
	10	0.2780		
	100	0.2214		
	1000	0.2176		
	10000	0.2145	0.2995	0.0297
C	1	1.8062		
	10	0.5249		
	100	0.2636		
	1000	0.2230		
	10000	0.2164	0.4011	0.0271

Table 5.3: The capital requirements in proportion to the RBC at $t = 0$ for portfolios consisting of n individuals. The SCR for the SST and Solvency I model is valid for large portfolios. The simulation was made with 10 000 replications.

5.3 we see that the SCR for policies A and B is about 70% of the SCR in the SST model whereas it is about 50% for policy C.

In the first figure below we consider a portfolio of 10000 individuals for each policy and look at the distribution of Δ_{RBC} for these portfolios. The second figure shows the distribution of the change in value for the asset and liability portfolios, for policy B. The change in RBC to have normal distribution, as do the histograms in the last figure. The capital change dominates the change in liability.

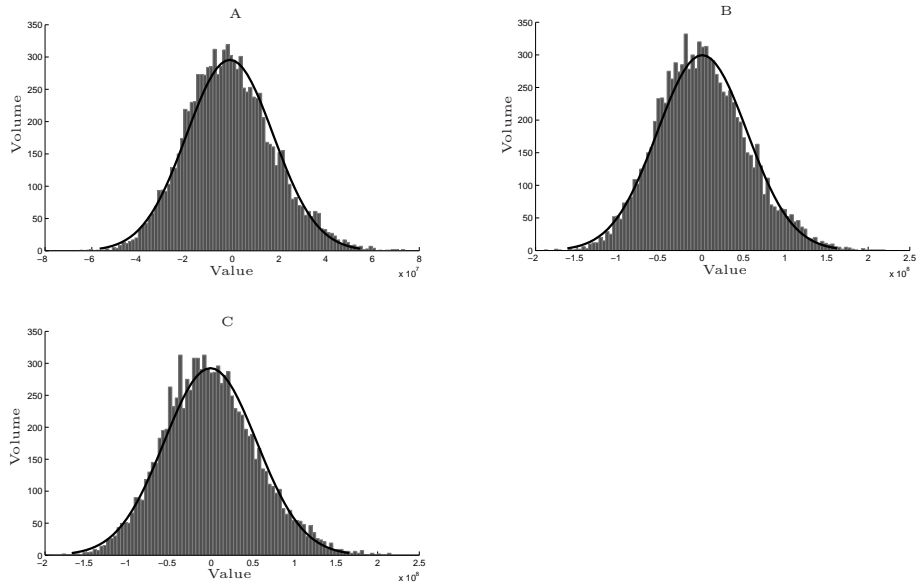


Figure 5.2: In the figure, the figures shows a histogram resembling the distribution for the different portfolios of 10 000 individuals. The simulated data is fitted to a normal distribution displayed by the black line.

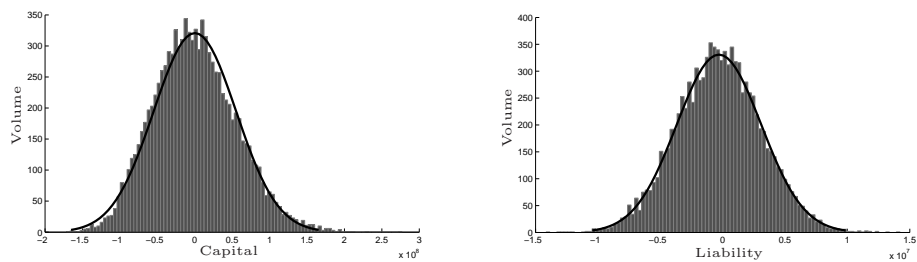


Figure 5.3: The left figure shows the change in capital for the time period of one year. The right figure shows distribution for the change in liability over one year. The simulation was performed for a portfolio of 10000 individuals with policy B.

CHAPTER 6

Conclusions

In this thesis we have implemented and analyzed two models for calculating the capital requirement in the Solvency II framework. The two models are the SST model [17] and the stochastic model presented in this thesis. The SST model assumes that the changes due to risk factors have a multivariate normal distribution and the SCR is calculated explicitly from the aggregated one dimensional normal distribution. In our model we do not make any assumptions of the distribution of the Δ_{RBC} .

In the implementation we have considered portfolios of different sizes consisting of three different insurance policies. In the calculation of the FTA two different mortality models were used, see (2.9) and (4.1). We see that when the mortality is described as in (4.1) we get a higher FTA for policies with lifelong pension and a lower FTA for the policy with limited pension.

The second part of the implementation we see that for a small number of individuals in the portfolio the stochastic model yields a higher SCR than the SST model. This is because the SST model does not consider the size of the portfolio. The stochastic model on the other hand takes the stochastic mortality risk into consideration which for small portfolios yields a higher SCR. Considering large portfolios the stochastic model gives a lower SCR. For policy A and B the SCR is about 70% of the SCR calculated in the SST model and for policy C this relation is about 50%. Compared to the capital requirement for the Solvency I we get a significantly higher value of the SCR in the new framework.

We can also conclude that using this stochastic model the Δ_{RBC} is approximately normally distributed for large portfolios. Also, the change in capital as well as liability seems to have a normal distribution. Moreover we see that the change in capital dominates the change in liability and therefore contributes the most to the size of SCR.

For large portfolios, which is usually the case for life insurance portfolios, the stochastic model yields a lower SCR than the SST model. The stochastic model should also give the insurer a better overview of the specific risks. From this viewpoint the use of an internal stochastic model is preferred.

CHAPTER 7

Future work

In this thesis we have used general models to describe the risk factors, resulting in a normal distribution. The choice of model is arbitrary so the same simulations and analysis can be done using more heavy tailed distributions.

An immediate extension of the stochastic model is to include the calculation of the MCR in the model to obtain a complete capital requirement model for the Solvency II framework. This can be done by implementing a version of the *Cost of Capital* approach from the SST model. Moreover the mortality intensity can be modeled as in [9] to expand the mortality model further. This approach also gives an opportunity to introduce a new type of contract called mortality linked contracts.

The new demand for capital requirement in the Solvency II framework also gives the possibility to evaluate the pricing of insurance contracts. For example, given that the insurance company would like to keep a certain SCR level for a type of contract he can calculate a risk sensitive premium for that contract.

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Theorems and input data

Change of numeraire

Theorem 1 Let Q^N and Q^M be the martingale measures connected with the numeraires $N(t)$ and $M(t)$. The Radon-Nikodym derivative that changes the measure Q^M into Q^N is given by

$$\frac{dQ^N}{dQ^M} = \frac{N(T)/N(t)}{M(T)/M(t)}$$

Girsanov's theorem

Theorem 2 For any stochastic process $k(t)$ such that

$$P\left(\int_0^t k^2(s)ds < \infty\right) = 1,$$

consider the Radon-Nikodym derivative $\frac{dQ^*}{dQ} = p(t)$ given by

$$p(t) = \exp\left\{\int_0^t k(s)dW(s) - \frac{1}{2}\int_0^t k^2(s)ds\right\},$$

where W is a Brownian motion under the measure Q . Then, the process

$$W^*(t) = W(t) - \int_0^t k(s)ds$$

is a Brownian motion under the Q^* measure.

Input data

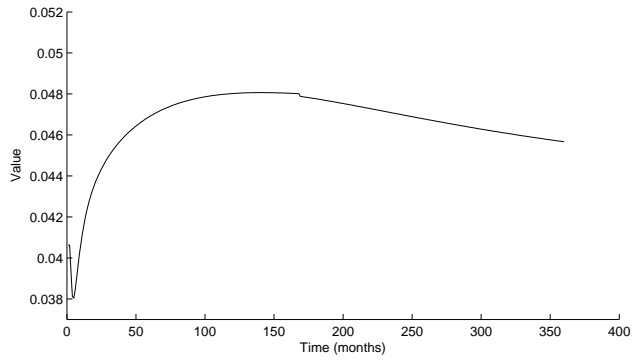


Figure A.1: In the figure the initial yield curve is shown. Monthly forward rates are calculated using the no arbitrage approach and future values are simulated in the LMM model.

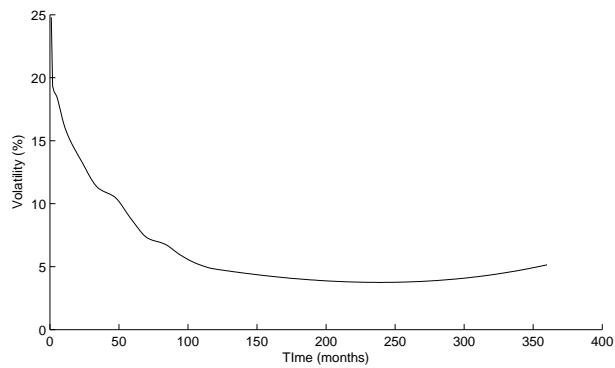


Figure A.2: The figure shows the volatility used for the simulations in the LMM model. Time steps are in months up to 30 years and the volatility values are in percent.