Optimal stopping and incomplete information in finance

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This thesis contains three papers:

1. Recovering a piecewise constant volatility from perpetual put option prices,
2. Optimal selling of an asset under incomplete information,
3. Optimal selling of an asset with jumps under incomplete information.

In the first paper we propose an exact and efficient algorithm for calibration of a piecewise constant volatility model to a discrete set of perpetual put option prices. In the last two papers we study when to optimally liquidate an asset under incomplete information in two different settings. In the second paper it is assumed that the asset price process follows a geometric Brownian motion with unknown drift, which takes one of two given values. In the third paper we assume that the price process satisfies a jump diffusion model with unknown jump intensity taking one of two given values. We show that for both cases the optimal liquidation strategy is to sell the asset the first time the price falls below a time-dependent boundary. We describe the three papers in more detail below.

One of the best known implied volatility models is Dupire’s formula which writes the level- and time-dependent volatility in terms of derivatives of European option prices with respect to strike price and maturity. In parallel to Dupire’s formula, a level-dependent volatility is expressed by Ekström and Hobson in terms of perpetual put option prices and their derivatives. However, both models are built on the assumption of a continuum of given option prices, which is rather unrealistic. In practice, one needs to interpolate between a discrete set of strike prices, and the volatility is very sensitive to the interpolation procedure. In fact, there are plenty of time-homogeneous models that can reproduce the same finite set of perpetual put prices. A natural candidate is the piecewise constant function of the underlying stock price, which is studied in the first paper. We start with the forward problem, in which the piecewise constant volatility is given and the prices of the perpetual put option for different strikes are calculated. Then we consider the inverse problem, in which we construct a piecewise constant volatility which reproduces the given finite set of option prices. The results are also illustrated in a numerical example.

The drift of an asset price process is very difficult to estimate from historical data. To obtain a decent accuracy in the estimate for the drift, the observations of the price process for hundreds of years are needed. Thus it is not very realistic to assume that the drift of an asset price process is given. In the second paper we study the optimal liquidation problem under the assumption that the asset price follows a geometric Brownian motion.
with unknown drift, which takes one of two given values. If the drift is known, then the optimal stopping strategy can be determined immediately by simply comparing the drift with the interest rate. In our setting the drift is not known at the beginning, but an initial estimate of the probabilities of both values is given. As time goes by, one can observe the asset price fluctuations and hence update one’s beliefs about the probabilities for the drift distribution. First we formulate an optimal stopping problem under incomplete information about the drift, and then it is converted into a much easier problem through the filtering techniques and equivalent measure transformation for Brownian motion. An early application of the filtering techniques is the sequential testing of two alternative hypothesis about the drift of a Brownian motion. The simplified optimal stopping problem, the auxiliary problem, is studied and the optimal liquidation time is the first time the point process falls below a certain time-dependent, monotonically increasing and continuous boundary. We also derive an integral equation for the optimal stopping boundary and study the optimal liquidation problem of closing a short position in the asset.

In option pricing, jump-diffusion processes are widely used to model market fluctuations. In the third paper we consider the problem of optimal liquidation where the asset price satisfies a jump diffusion model with unknown jump intensity. It is assumed that the intensity takes one of two given values and we initially have an estimate of the probabilities of either of them. Notice that the problem would be trivial if the jump intensity was known. Although complete information of the intensity is not available at the beginning, one can observe the asset price process, and learn more about the distribution of the intensity. The equivalent measure transformations for jump processes with stochastic intensity as well as for compound Poisson processes and the filtering techniques for point processes are used here to simplify the problem. An early application of the filtering techniques is the sequential testing of two alternative hypothesis about the jump intensity of a Poisson process. The best liquidation strategy is to sell the asset the first time the counting process falls below or goes above a time-dependent monotone boundary, which depends on the distribution of the jump size of the compound Poisson process.
Abstract. In this paper we present a method to recover a time-homogeneous piecewise constant volatility from a finite set of perpetual put option prices. The whole calculation process of the volatility is decomposed into easy computations in many fixed disjoint intervals. In each interval, the volatility is obtained by solving a system of nonlinear equations.

1. Introduction

One of the most studied problems in mathematical finance is to calculate the price of an option if the diffusion coefficient of the underlying asset is given. In practice, it is often natural to consider the inverse problem: how to compute the volatility of the underlying stock price if a set of option prices is provided. For example, in the classical Black-Scholes model there is a unique correspondence between the constant volatility and the price of a European option. Thus, the implied volatility can be obtained by the Black-Scholes formula if one option price is known. In general, however, if more than one option price is given, a richer model is needed for the underlying process. One such model is the local volatility model, in which the volatility depends on the current stock price and the current time. Also this model can be calibrated to fit given option data perfectly. Indeed, Dupire [3] showed that the level- and time-dependent volatility can be written in terms of derivatives of European option prices with respect to strike price and maturity.

In the present paper we are interested in calibration of models from the prices of perpetual American put options. The volatility of the underlying is considered to be time-homogeneous. Similar to the European case mentioned above, if the Black-Scholes model is offered as the process of the underlying stock price, then it is straightforward to compute the constant volatility if one option price is given. In parallel to the Dupire’s equation, a level-dependent model for the stock price is created by Ekström and Hobson (2009), see also Alfonsi and Jourdain [1]. Ekström and Hobson assume that the prices of the perpetual put options are given for all different strike prices and they express the diffusion coefficient in terms of the option prices and their derivatives. This volatility is uniquely determined at the price level below the current stock price.

As noted above, both Dupire’s formula for the volatility and the level-dependent volatility recovered from prices of perpetual put options are calculated under an assumption of a (possibly double) continuum of given option prices. In reality, however, option prices are only given for a discrete set of strike prices, so one then needs to interpolate between them. Moreover, since the volatility is calculated using

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derivatives of the option prices, it is very sensitive to the interpolation procedure. On the other hand, the constant volatility defined in the Black-Scholes model is easy to calculate, but generically it is impossible to fit one constant volatility if several option prices are given.

Motivated by the discussion above, we consider a situation in which prices of perpetual American put options are given for a finite set of strike prices. To rule out arbitrage possibilities, the option price has to be increasing and convex in the strike. Moreover, we assume that the option price is strictly convex in strikes. Since there does not exist a continuum of option prices, one can create plenty of time-homogeneous models to reproduce the option data (one for each choice of interpolation procedure). A natural candidate for the time-homogeneous volatility model is the piecewise constant function of the stock price. In the present paper we prove the existence of a piecewise constant volatility that reproduces the given option prices. Given $n$ option prices, the whole calculation process is decomposed into elementary computation in $n$ fixed disjoint intervals. To obtain the volatility in each interval, one just needs to solve two nonlinear equations with two unknown variables. Moreover, since it does not involve differentiation of the option price, we believe that it is more stable with respect to small changes in the input than the model by Ekström and Hobson.

The paper is organized as follows. In Section 2, we study the forward problem. Provided that the volatility is a piecewise constant function of the underlying stock price, we can calculate the price of the perpetual put option for different strike prices. Section 3 treats the inverse problem and contains our main results. Given a finite set of prices of the perpetual put options, we present a method to construct a piecewise constant volatility which reproduces the option prices. In Section 4, the results are illustrated in a numerical example.

2. The Forward Problem

We consider a model where the process of the stock price $X$ solves the stochastic differential equation

$$dX_t = rX_t dt + \sigma(X_t)X_t dW_t.$$ 

Here $r$ is the constant interest rate, $W$ is the standard Brownian motion and $\sigma(X_t)$ is a positive function. Given the current stock price $x_0$, the price of a perpetual American put option with strike $K$ is

$$P(K) = \sup_{\tau} \mathbb{E}^x[e^{-r\tau}(K - X_\tau)^+],$$

where $\tau$ is any stopping time with respect to the filtration generated by $W$. The solution to the optimal stopping problem (1) is closely related to the ordinary differential equation

$$\frac{1}{2} \sigma^2(x)x^2u_{xx} + rxu_x - ru = 0.$$ 

There are two linearly independent solutions to this ODE. If one of them is positive and increasing and the other one positive and decreasing, then they are unique up to positive multiplicative constants, compare [2] pages 18-19. We denote these solutions by $\psi$ and $\varphi$, respectively. Without loss of generality, we choose the decreasing solution to satisfy $\varphi(x_0) = 1$. Define the hitting times $H_z = \inf\{t \geq 0 : X_t = z\}$. 
Since $e^{-rt}\varphi(X_t)$ is a local martingale and $\varphi(x)$ is decreasing in $x$, we have

$$E^x[e^{-rH_z}] = \frac{\varphi(x)}{\varphi(z)} \quad \text{if } x > z.$$  

Due to the time-homogeneity of problem (1), it suffices to take the supremum over stopping times that are exit times from an interval. Moreover, since put options are considered, we only need to take the supremum over hitting times $H_z$ for some level $z$, compare the proof of Lemma 2.2 in [4]. We thus find

$$P(K) = \sup_{\tau} E^x_0[e^{-r\tau}(K - X_\tau)^+]$$
$$= \sup_{z:z \leq x_0 \land K} E^x_0[e^{-rH_z}(K - X_{H_z})^+]$$
$$= \sup_{z:z \leq x_0 \land K} (K - z)E^x_0[e^{-rH_z}]$$
$$= \sup_{z:z \leq x_0} \frac{K - z}{\varphi(z)}.$$  

![Figure 1](image_url)

**Figure 1.** The picture shows how to determine the optimal exercise level $z$ for a given strike $K$.

It is easy to check that the function $\varphi(z)$ is strictly convex and decreasing in $z$. In Figure [1], let $\theta$ denote the acute angle between $x$-axis and the line passing through $(K,0)$ and $(z, \varphi(z))$. Here $z$ can be any price level below $x_0$ and $K$. It is easy to see that $(K - z)/\varphi(z) = \cot(\theta)$. Thus we just need to find the smallest $\theta$ for $z \leq x_0$. Define

$$(3) \quad \hat{K} = x_0 - \frac{\varphi(x_0)}{\varphi'(x_0)} = x_0 - \frac{1}{\varphi'(x_0)}$$

so that the line passing through $(\hat{K},0)$ and $(x_0, \varphi(x_0))$ is tangent to the curve $\varphi$. 


When $K \leq \hat{K}$, it is not optimal to exercise the option immediately. Instead, the investors should wait until the stock price hits the optimal stopping level to exercise the option. Thus we have

$$P(K) = \sup_z \frac{K - z}{\varphi(z)},$$

where the optimal $z$, which is the optimal exercise level for strike $K$, is chosen so that

$$(K - z)\varphi'(z) + \varphi(z) = 0.$$ \hfill (5)

Equation (5) indicates that the straight line passing through $(K, 0)$ and $(z, \varphi(z))$ is tangent to the function $\varphi$. We also obtain $P(K) = K - x_0$ for $K > \hat{K}$, which implies that it is optimal for investors to exercise the option immediately.

We now specialize to the case of piecewise constant volatility. More precisely, it is assumed that the interval $[0, x_0]$ is divided by a mesh consisting of $n$ points $a_1, a_2, a_3, \ldots, a_n$, which satisfy $0 < a_1 < a_2 \ldots a_{n-1} < a_n \leq x_0$. The volatility function $\sigma(X_t)$ is defined as

$$\sigma(x) = \begin{cases} 
\sigma_0, & 0 < x < a_1 \\
\sigma_i, & a_i \leq x < a_{i+1}, \ 1 \leq i \leq n - 1 \\
\sigma_n, & a_n \leq x
\end{cases},$$ \hfill (6)

where $\sigma_0, \ldots, \sigma_n$ are positive constants. On an interval $(a_i, a_{i+1})$, the volatility is constant and therefore the fundamental solution $\varphi$ is $C^\infty$. However, at a jump point $a_i$ of $\sigma$, the function $\varphi$ is merely $C^1$ and the second derivative has a jump. Given the piecewise constant volatility function $\sigma(x)$ defined above, the two independent positive solutions of the ODE (2) are

$$\psi(x) = x$$

and

$$\varphi(x) = \begin{cases} 
A_0 x^{-\beta_0} + B_0 x, & 0 < x < a_1 \\
A_i x^{-\beta_i} + B_i x, & a_i \leq x < a_{i+1}, \ 1 \leq i \leq n - 1 \\
A_n x^{-\beta_n} + B_n x, & a_n \leq x
\end{cases},$$ \hfill (7)

where $\beta_i = 2 r / \sigma_i^2$ for $i \in \{0, \ldots, n\}$. Here $A_i$ and $B_i$ for $i \in \{0, \ldots, n\}$ are chosen so that $\varphi(x)$ is $C^1$ everywhere. Without loss of generality, we let $\varphi(x_0) = 1$, thus

$$A_n = x_0^{\beta_n}, \ B_n = 0,$$

since $\varphi(x)$ is decreasing and non-negative for $x \geq a_n$. Due to the $C^1$-regularity, we have

$$\begin{cases} 
A_i a_{i+1}^{-\beta_i} + B_i a_{i+1} = A_{i+1} a_{i+1}^{-\beta_{i+1}} + B_{i+1} a_{i+1} + A_i \beta_i a_{i+1}^{-\beta_i} - B_i = -A_{i+1} \beta_{i+1} a_{i+1}^{-\beta_{i+1} - 1} + B_{i+1}.
\end{cases}$$ \hfill (8)

It follows that

$$\begin{cases} 
A_i = A_{i+1} a_{i+1}^{-\beta_i - \beta_{i+1} + \beta_{i+1}} + B_i = A_{i+1} a_{i+1}^{-\beta_i} - \beta_{i+1} - 1 + \beta_{i+1} + B_{i+1}.
\end{cases}$$ \hfill (9)

for $i \in \{0, \ldots, n - 1\}$. 

The function $\varphi(x)$ defined in (7), (8) and (9) is the decreasing fundamental solution to the ODE (2). Hence, for $K \leq \hat{K} = x_0(1 + 1/\beta_n)$ the option price is

$$P(K) = \sup_z \frac{K - z}{\varphi(z)},$$

where the optimal $z$ is determined by

$$K - z)\varphi'(z) + \varphi(z) = 0. \tag{11}$$

For $K > \hat{K}$ we have $P(K) = K - x_0$. Since $\varphi(x)$ is $C^1$ and strictly convex in $x$, equation (11) defines a one-to-one correspondence between strike prices $K \in (0, \hat{K}]$ and optimal exercise levels $z \in (0, x_0]$. Now let $K_i^*$ be the strike price for which $a_i$ is the optimal exercise level. By (11), we have

$$K_i^* = -\frac{\varphi(a_i)}{\varphi'(a_i)} + a_i = \frac{(1 + \beta_i)A_i}{\beta_ia_i^{-1} - B_i\beta_i}. \tag{12}$$

for $1 \leq i \leq n$. It is straightforward to find that

$$K_n^* = a_n(1 + \beta_n)/\beta_n. \tag{13}$$

Since $K$ is strictly increasing as a function of $z$, we have that $K_i^*$ is increasing in $i$. Moreover, for $K \in [K_i^*, K_{i+1}^*)$, the optimal exercise level $z$ belongs to $[a_i, a_{i+1})$. By summarizing all our findings in this section, we obtain the following theorem.

**Theorem 1.** Given the piecewise constant volatility defined in (6), the price of the perpetual American put option defined by (1) is given by

$$P(K) = \begin{cases} 
K - x_0, & K \geq \hat{K} \\
\frac{x_0^n(1 + \beta_n)^{1 + \beta_n}}{A_i^n x_0^n + B_i^n}, & K^*_i \leq K < \hat{K} \\
\frac{K - z}{A_i^n x_0^n + B_i^n}, & K_i^* \leq K < K_{i+1}^*, \ 1 \leq i \leq n - 1 \\
0, & 0 < K < K_1^*. 
\end{cases} \tag{14}$$

Here $A_i$ and $B_i$ for $i \in \{0, ..., n - 1\}$ are defined by (9). The optimal exercise level $z$ in (14) is determined implicitly by

$$(K - z)(-A_i\beta_i z^{-\beta_i - 1} + B_i) + A_i z^{-\beta_i} + B_i z = 0 \tag{15}$$

if $K \in [K_i^*, K_{i+1}^*)$ for $i \in \{1, ..., n - 1\}$ or if $K < K_1^*$ for $i = 0$.

By the theorem, the option price $P(K)$ can be computed explicitly if a piecewise constant volatility $\sigma(x)$ is given as in equation (6).

### 3. The Inverse Problem

In this section, we take the point of view that option prices for a discrete set of strikes written on a certain underlying asset can be recorded from the market. We construct a piecewise constant volatility function of the underlying stock price, which is calibrated to perfectly fit the finite set of option prices.

Assume that $n$ strike prices and the corresponding $n$ perpetual put option prices are given from the market data. Arbitrage considerations give that the put option price has to be non-decreasing and convex in $K$. Below we make the slightly stronger assumptions that it is increasing and strictly convex. Thus the option price $P(K_i)$ has to satisfy

$$P(K_1) < P(K_2) < ... < P(K_n) \tag{16}$$
for the strike prices $0 < K_1 < K_2 < \ldots < K_n$. For the index level $n$, we assume that $K_n$ satisfies $P(K_n) = K_n - x_0$, where $x_0$ is the current stock price. Below $K_n$ will correspond to $\hat{K}$ in the forward problem, so any option with strike price that is bigger than or equal to $K_n$ should be exercised immediately. Later in this section we will discuss the case when $K_n$ can not be observed from the market. We also assume that the natural bounds

$$ (K_i - x_0)^+ < P(K_i) < K_i $$

for $i \in \{1, \ldots, n-1\}$ are fulfilled, where the first strict inequality implies that it is not optimal to exercise the option immediately. We also assume that the value function $P(K)$ is strictly convex in the strikes, so that

$$ \frac{P(K_2) - P(K_1)}{K_2 - K_1} > \frac{P(K_1)}{K_1} $$

and

$$ \frac{P(K_i) - P(K_{i-1})}{K_i - K_{i-1}} < \frac{P(K_{i+1}) - P(K_i)}{K_{i+1} - K_i} $$

for $i \in \{2, \ldots, n-1\}$. It follows that

$$ \frac{P(K_i) - P(K_{i-1})}{K_i - K_{i-1}} < 1 $$

for $i \in \{2, \ldots, n\}$.

Next we will draw a graph containing all the information given by (16), (17), (18), (19) and (20). In the $x$-$\varphi(x)$ coordinate system, draw the lines passing through $(K_i, 0)$ with slope $-1/P(K_i)$ for every $i \in \{1, \ldots, n\}$. We refer to those lines as “option lines” and denote the option line with index level $i$ by $l_i$. According to equation (19), we have

$$ \frac{K_i - K_{i-1}}{P(K_i) - P(K_{i-1})} > \frac{K_{i+1} - K_i}{P(K_{i+1}) - P(K_i)} $$

where the expression on the left hand side is the second coordinate of the intersection between $l_i$ and $l_{i-1}$, and the expression on the right hand side is the second coordinate of the intersection between $l_{i+1}$ and $l_i$.

According to equation (18), we obtain

$$ \frac{K_2 - K_1}{P(K_2) - P(K_1)} < \frac{K_1}{P(K_1)}, $$

which implies that the second coordinate of the intersection between $l_2$ and $l_1$ is smaller than the second coordinate of the intersection between $l_1$ and $\varphi(x)$-axis. Therefore, as $i$ decreases from $n$ to 2, the first and the second coordinate of the intersection between $l_i$ and $l_{i-1}$ are all positive and decreases and increases, respectively. Equation (17) gives

$$ \frac{K_i - x_0}{P(K_i)} < 1 $$

for any $i \in \{1, \ldots, n-1\}$, which implies that $(x_0, 1)$ is on the right side of all the option lines except $l_n$. Note that the point $(x_0, 1)$ is on $l_n$. According to (20), we have

$$ \frac{K_n - K_{n-1}}{P(K_n) - P(K_{n-1})} > 1. $$
This shows that the second coordinate of the intersection between $l_n$ and $l_{n-1}$ is larger than 1. Summing up all the information mentioned above, Figure [2] gives a simple version of option lines.

Theorem 2. Assume that $n$ strike prices $K_1, ..., K_n$ and the corresponding prices of perpetual put options $P(K_1), ..., P(K_n)$ satisfying the conditions (16), (17), (18), (19), (20) and $P(K_n) = K_n - x_0$ are given, where $x_0$ is the current stock price. Then there exists a time-homogeneous process with a piecewise constant volatility that recovers the option prices.

Proof. We are looking for a piecewise constant volatility of the form

$$\sigma(x) = \begin{cases} 
\sigma_0, & 0 < x < b_1 \\
\sigma_{i1}, & b_1 \leq x < c_i, 1 \leq i \leq n-1 \\
\sigma_{i2}, & c_i \leq x < b_{i+1}, 1 \leq i \leq n-1 \\
\sigma^*, & x \geq b_n = x_0,
\end{cases}$$

(21)

where $0 < b_1 < c_1 < ... < b_i < c_i < b_{i+1} < ... < b_n = x_0$. With this volatility, the decreasing fundamental solution to the ODE (2) is of the form

$$\phi(x) = \begin{cases} 
A_0 x^{-\beta_0} + B_0 x, & 0 < x < b_1 \\
A_{i1} x^{-\beta_{i1}} + B_{i1} x, & b_i \leq x < c_i, 1 \leq i \leq n-1 \\
A_{i2} x^{-\beta_{i2}} + B_{i2} x, & c_i \leq x < b_{i+1}, 1 \leq i \leq n-1 \\
A^* x^{-\beta^*} + B^* x, & x_0 \leq x
\end{cases}$$

(22)

where $\beta_0 = 2r/\sigma_0^2$ (and similarly for $\beta_{i1}, \beta_{i2}$ and $\beta^*$). The constants $A_0, A_{i1}, A_{i2}, A^*, B_0, B_{i1}, B_{i2}$ and $B^*$ should be chosen so that $\phi(x)$ is $C^1$ everywhere. If one
can find a function \( \varphi(x) \) of the form (22) with suitable parameters which is tangent to all the option lines and touching the point \((x_0, 1)\) and the first coordinates of those tangent points are not bigger than \(x_0\), then it satisfies

\[
P(K_i) = \sup_z \frac{K_i - z}{\varphi(z)},
\]

and the optimal \( z \) for each \( K_i \) is smaller than or equal to \( x_0 \). Hence, the corresponding piecewise constant volatility \( \sigma(x) \) is calibrated to fit the set of option prices perfectly and is the volatility that we are looking for.

Next we are going to determine \( b_i \) for \( i \in \{1, ..., n\} \) and \( c_i \) for \( i \in \{1, ..., n-1\} \). Let \( d_i \) be the first coordinate of the intersection of \( l_i \) and \( l_{i+1} \). Thus we have

\[
d_i = \frac{K_i P(K_{i+1}) - K_{i+1} P(K_i)}{P(K_{i+1}) - P(K_i)}
\]

for \( i \in \{1, ..., n-1\} \). Choose \( b_i \) to be

\[
b_i = \frac{d_{i-1} + d_i}{2}
\]

for \( i \in \{2, ..., n-1\} \), and \( b_1 = d_1/2 \) and \( b_n = x_0 \). We will construct \( \varphi \) so that \( b_i \) is the first coordinate of the tangent point where \( \varphi \) touches the \( l_i \). (In fact this tangent point could be chosen anywhere on the segment that connects the nearest two intersections, but we believe that the mid-point is a natural choice. For \( b_1 \), one may argue that \( b_1 = 2d_1 - b_2 \) is another natural choice in some cases.) The second coordinate of the tangent point corresponding to \( b_i \) is

\[
\varphi(b_i) = \frac{(K_i - K_{i-1})(P(K_{i+1}) - P(K_i))}{2(P(K_i) - P(K_{i-1}))(P(K_{i+1}) - P(K_i))}
+ \frac{(K_{i+1} - K_i)(P(K_i) - P(K_{i-1}))}{2(P(K_i) - P(K_{i-1}))(P(K_{i+1}) - P(K_{i-1}))}
\]

for \( i \in \{2, ..., n-1\} \), which can be easily computed by plugging in the option data. Additionally, we have

\[
\varphi(b_1) = \frac{2K_1 - d_1}{2P(K_1)}, \quad \varphi(b_n) = 1.
\]

For simplicity, we let \( c_i \) equal \( d_i \) defined in equation (23). For a graphic illustration of the choices of \( b_i \) and \( c_i \), see Figure [3].

In the interval \([b_i, b_{i+1}]\) for \( i \in \{1, ..., n-1\} \) the function \( \varphi(x) \) should be \( C^1 \), which implies the following equations

\[
\varphi(b_i) = A_{i1} b_i^{-\beta_{i1}} + B_{i1} b_i
\]

\[
\frac{\partial \varphi(x)}{\partial x} \bigg|_{x=b_i} = -\beta_{i1} A_{i1} b_i^{-\beta_{i1}-1} + B_{i1} = -\frac{1}{P(K_i)}
\]

\[
\varphi(b_{i+1}) = A_{i2} b_{i+1}^{-\beta_{i2}} + B_{i2} b_{i+1}
\]

\[
\frac{\partial \varphi(x)}{\partial x} \bigg|_{x=b_{i+1}} = -\beta_{i2} A_{i2} b_{i+1}^{-\beta_{i2}-1} + B_{i2} = -\frac{1}{P(K_{i+1})}
\]

\[
\varphi(c_i) = A_{i1} c_i^{-\beta_{i1}} + B_{i1} c_i = A_{i2} c_i^{-\beta_{i2}} + B_{i2} c_i
\]

\[
\frac{\partial \varphi(x)}{\partial x} \bigg|_{x=c_i} = -\beta_{i1} A_{i1} c_i^{-\beta_{i1}-1} + B_{i1} = -\beta_{i2} A_{i2} c_i^{-\beta_{i2}-1} + B_{i2}.
\]
Let $\varphi(x, \beta_i) = \varphi(x)$ be a function of the variables $x$ and $\beta_i$ for $x \in [b_i, c_i]$ and $\varphi(x, \beta_i) = \varphi(x)$ be a function of the variables $x$ and $\beta_i$ for $x \in [c_i, b_{i+1}]$. Note that $\varphi(b_i)$ and $\varphi(b_{i+1})$ are constants that we can compute, $\varphi(x)$ is a function of the variable $x$ and $\varphi(x, \beta_i)$ ($\varphi(x, \beta_{i+1})$) is a function with variables $x$ and $\beta_i$ ($\beta_{i+1}$).

After some manipulation the six equations become

$$
\begin{align*}
\hat{\varphi}(x, \beta_i) &= \varphi(b_i) P(K_i) + b_i \beta_i x, \\
\hat{\varphi}(x, \beta_{i+1}) &= \varphi(b_{i+1}) P(K_i) + b_{i+1} \beta_{i+1} x, \\
\frac{\partial \hat{\varphi}(x, \beta_i)}{\partial x} |_{x = c_i} &= \frac{\partial \varphi(x, \beta_i)}{\partial x} |_{x = c_i} = \varphi(c_i, \beta_i), \\
\frac{\partial \hat{\varphi}(x, \beta_{i+1})}{\partial x} |_{x = c_i} &= \frac{\partial \varphi(x, \beta_{i+1})}{\partial x} |_{x = c_i}.
\end{align*}
$$

for $i \in \{1, ..., n - 1\}$. The option prices $P(K_i)$ and $K_i$ for $i \in \{1, ..., n\}$ are given and it is straightforward to compute $c_i$, $b_i$ and $\varphi(b_i)$ for each $i$, so in each interval $[b_i, b_{i+1}]$ there are only two unknown parameters $\beta_i$ and $\beta_{i+1}$ to be determined.

Next we will prove the existence of a solution to the system of equations (27).

It is easy to check that

$$
\lim_{\beta_i \to 0} \hat{\varphi}(x, \beta_i) = \varphi(b_i) - \frac{x - b_i}{P(K_i)}.
$$
which implies that $\hat{\varphi}(x, \beta_{1i})$ tends to $l_i$ as $\beta_{1i}$ goes to zero. Similarly, one can show that $\hat{\varphi}(x, \beta_{1i})$ tends to $l_{i+1}$ as $\beta_{1i}$ goes to zero. We can also check that
\[
\lim_{\beta_{1i} \to \infty} \hat{\varphi}(x, \beta_{1i}) = \frac{\varphi(b_i)x}{b_i}, \quad \lim_{\beta_{1i} \to \infty} \hat{\varphi}(x, \beta_{1i}) \to \infty.
\]

Claim 1. The functions $\hat{\varphi}(x, \beta_{1i})$ and $\hat{\varphi}(x, \beta_{1i})$ are increasing in $\beta_{1i}$ and $\beta_{2i}$, respectively.

Proof of the claim. For $x \in [b_i, c_i]$, define $z = b_i/x$. Then
\[
\hat{\varphi}(x, \beta_{1i}) = \frac{\varphi(b_i)P(K_1) + b_i}{(1 + \beta_{1i})P(K_1)} z^{\beta_{1i}} + \frac{P(K_1)\beta_{1i}\varphi(b_i) - b_i}{zP(K_1)(1 + \beta_{1i})}.
\]
Taking derivative of $\hat{\varphi}(x, \beta_{1i})$ with respect to $\beta_{1i}$ yields
\[
\frac{1 + \beta_{1i}}{\varphi(b_i) + \frac{b_i}{P(K_1)}} \frac{\partial \hat{\varphi}(x, \beta_{1i})}{\partial \beta_{1i}} = z^{-\beta_{1i}} \ln z + \frac{1}{1 + \beta_{1i}} (1 - z^{-\beta_{1i}}) = f(z).
\]
Note that $0 < z \leq 1$ and $f(1) = 0$. Thus in order to show that $\hat{\varphi}(x, \beta_{1i})$ is increasing in $\beta_{1i}$, it suffices to show that $f(z)$ is positive for $0 < z < 1$. Next differentiating $zf(z)$ with respect to $z$ gives
\[
\frac{\partial (zf(z))}{\partial z} = (1 + \beta_{1i})z^{\beta_{1i}} \ln z < 0
\]
for $0 < z < 1$. Note that $1 \times f(1) = 0$, thus $zf(z) > 0$ for $0 < z < 1$. It follows that $f(z) > 0$ for $0 < z < 1$, which implies that $\partial \varphi(x, \beta_{1i})/\partial \beta_{1i} > 0$. One can also show that $\partial \varphi(x, \beta_{2i})/\partial \beta_{2i} > 0$ by a similar argument as above. This finishes the proof of the claim.

Therefore as $\beta_{1i}$ increases from zero to infinity, $\hat{\varphi}(x, \beta_{1i})$ increases from $\varphi(b_i) - \frac{b_i}{P(K_1)}$ to $\frac{\varphi(b_i)x}{b_i}$. Again as $\beta_{2i}$ increases from zero to infinity, $\hat{\varphi}(x, \beta_{2i})$ also increases from $\varphi(b_{i+1}) - \frac{x-b_{i+1}}{P(K_{i+2})}$ to infinity. Let $A$ be the point $[c_i, \frac{K_i - c_i}{P(K_{i+2})}]$ and $B$ be the point $[c_i, \frac{\varphi(b_i)(b_{i+1} - c_i) + \varphi(b_{i+1})(c_i - b_{i})}{b_{i+1} - b_i}]$, compare Figure [4]. If we choose a point on the segment $AB$ except for the point $A$, then there exist $\beta_{1i}$ and $\beta_{2i}$ such that $\hat{\varphi}(x, \beta_{1i})$ and $\hat{\varphi}(x, \beta_{2i})$ pass through that point. For $\beta_{1i}$ and $\beta_{2i}$ sufficiently small, $\hat{\varphi}(x, \beta_{1i})$ and $\hat{\varphi}(x, \beta_{2i})$ will meet at some point at the segment $AB$ which is very close to the point $A$. Clearly, the angle between $\hat{\varphi}(x, \beta_{1i})$ and $\hat{\varphi}(x, \beta_{2i})$ is smaller than $180^\circ$, compare Figure [4]. If $\hat{\varphi}(x, \beta_{1i})$ and $\hat{\varphi}(x, \beta_{2i})$ meet at point $B$, the angle between the two $\hat{\varphi}$ functions is larger than $180^\circ$ due to the convexity of $\hat{\varphi}(x, \cdot)$ in $x$. As the intersection of the two $\hat{\varphi}$ functions moves from $A$ to $B$ along the segment $AB$, the angle between the two $\hat{\varphi}$ functions increases from less than $180^\circ$ to more than $180^\circ$. Hence, by a continuity argument, there exists a point on the segment $AB$ such that the angle between the two $\hat{\varphi}$ functions is exactly $180^\circ$. For this particular choice of $\beta_{1i}$ and $\beta_{2i}$, define $\varphi$ by $\varphi(x) = \hat{\varphi}(x, \beta_{1i})$ for $x \in [b_i, c_i]$ and $\varphi(x) = \hat{\varphi}(x, \beta_{2i})$ for $x \in [c_i, b_{i+1}]$. In this way, $\varphi$ is $C^1$ at $c_i$. This finishes the proof of the existence of $\beta_{1i}$ and $\beta_{2i}$ for $i \in \{1, \ldots, n - 1\}$.

Note that $\beta_0$ only influences the prices of options with $K < K_1$ which are not given. Thus $\beta_0$ can be defined arbitrarily. For simplicity, we let $\beta_0 = \beta_{11}$. It follows that $A_0 = A_{11}$ and $B_0 = B_{11}$. Since $\varphi(x)$ is $C^1$ and decreasing in $x$ and $\varphi(x_0) = 1$, it is easy to calculate that $\beta^* = x_0/P(K_n)$, $B^* = 0$ and $A^* = x_0^{\beta^*}$. □
According to (27), in the interval \([b_i, b_{i+1})\) for \(i \in \{1, \ldots, n-1\}\), we just need to solve the two nonlinear equations

\[
\varphi(b_i)P(K_i) + \frac{P(K_i)\beta_{i1}\varphi(b_i) - b_i c_i}{(1 + \beta_{i1})P(K_i)} + \frac{P(K_{i+1})\beta_{i2}\varphi(b_{i+1}) - b_{i+1} c_i}{(1 + \beta_{i2})P(K_{i+1})} = \varphi(b_{i+1})P(K_{i+1}) + \frac{P(K_{i+1})\beta_{i2}\varphi(b_{i+1}) - b_{i+1} c_i}{(1 + \beta_{i2})P(K_{i+1})}
\]

and

\[
-\beta_{i1}\frac{\varphi(b_i)P(K_i) + \frac{P(K_i)\beta_{i1}\varphi(b_i) - b_i c_i}{(1 + \beta_{i1})P(K_i)} + \frac{P(K_{i+1})\beta_{i2}\varphi(b_{i+1}) - b_{i+1} c_i}{(1 + \beta_{i2})P(K_{i+1})}}{b_i c_i} = -\beta_{i2}\frac{\varphi(b_{i+1})P(K_{i+1}) + \frac{P(K_{i+1})\beta_{i2}\varphi(b_{i+1}) - b_{i+1} c_i}{(1 + \beta_{i2})P(K_{i+1})}}{b_{i+1} c_i}
\]

to obtain the volatility.

**Theorem 3.** Assume that \(n\) strike prices \(K_1, \ldots, K_n\) and the corresponding prices of perpetual put options \(P(K_1), \ldots, P(K_n)\) satisfying the conditions specified in Theorem 2 are given. The piecewise constant volatility that recovers the option prices can be computed by the following procedure:

1. Calculate \(c_i, b_i\) and \(\varphi(b_i)\) for each \(i\) by equations (23), (24), (25) and (26).
2. Additionally, let \(b_1 = c_1/2\) and \(b_n = x_0\).
3. Plug the numbers computed above into (28) and (29) to obtain \(\beta_{i1}\) and \(\beta_{i2}\) for \(i \in \{1, \ldots, n-1\}\). The tail volatility \(\beta^* = x_0/P(K_n)\) and let \(\beta_0 = \beta_{i1}\).

Then the piecewise constant volatility that recovers the given option prices is given
by

\[ \sigma(x) = \begin{cases} \sqrt{\frac{2r}{\beta_0}}, & 0 \leq x < b_1 \\ \sqrt{\frac{2r}{\beta_i}}, & b_i \leq x < c_i \text{ for } 1 \leq i \leq n - 1 \\ \sqrt{\frac{2r}{\beta_i}}, & c_i \leq x < b_{i+1} \text{ for } 1 \leq i \leq n - 1 \\ \sqrt{\frac{2r}{\beta_*}}, & x \leq x_0. \end{cases} \]

**Remark 1.** Note that the volatility model described above is not the unique piecewise constant volatility model that reproduces the option data. In fact, the choice of the break points \( b_i \) and \( c_i \) is arbitrary to some extent. For example, our assumption that the tangent point where \( \varphi \) touches the option line is located in the middle of two intersections could easily be changed, thereby giving rise to a different volatility. Also note that we obtain \( 2n \) constant volatilities from \( n \) option prices, so the degree of freedom is \( n \). However, it seems difficult for us to both decrease the degree of freedom and at the same time ensure the solvability of the problem.

It might happen that \( K_n \) satisfying \( P(K_n) = K_n - x_0 \) can not be observed from the market. In such a case, to apply the calibration method described above, we must make up a proper \( K_n \) with \( P(K_n) = K_n - x_0 \) using the given option data \( P(K_1), \ldots, P(K_{n-1}) \). Since \( P(K) \) is strictly convex with respect to \( K \), the strike price \( K_n \) has to satisfy

\[ K_n - x_0 > P(K_{n-1}) + (K_n - K_{n-1}) \frac{P(K_{n-1}) - P(K_{n-2})}{K_{n-1} - K_{n-2}}. \]

There are many ways of choosing \( K_n \) that satisfies (30). For example, one way would be to use a second order Taylor expansion of \( P \) at the point \( K_{n-1} \). However, we omit the details of this procedure.

4. **Numerical illustration**

In the numerical test, we assume that option data observed in the market are actually calculated using a CEV-model with \( \sigma(x) = x^{-\frac{1}{2}} \). Let the current stock price be \( x_0 = 10 \), the interest rate be \( r = 0.1 \), then the \( \varphi \) function (compare [2] pages 18-19) is given by

\[ \varphi(x) = 26.6423x \int_x^\infty \frac{1}{y^2}e^{-0.2y} dy. \]

According to (3) and (31), we obtain that \( \hat{K} = 13.8778 \). A series of strike prices \( \{K_i\} \) between 3 and 13.8778 are selected as option data, and the corresponding series of option prices \( \{P(K_i)\} \) can be computed by equation (10) and (11). The volatility (the solid curve) produced by Theorem 3 is plotted comparing to the real volatility (the dotted curve) in Figure [5]. As the number of option prices given increases, our predicted volatility fits better and better with the given volatility.

**References**


Figure 5. The dotted line is the real volatility $\sigma(x) = x^{-1/2}$, and the solid line is the computed volatility.


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OPTIMAL SELLING OF AN ASSET UNDER INCOMPLETE INFORMATION

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Abstract. We consider an agent who wants to liquidate an asset with unknown drift. The agent believes that the drift takes one of two given values, and has initially an estimate for the probability of either of them. As time goes by, the agent observes the asset price and can therefore update his beliefs about the probabilities for the drift distribution. We formulate an optimal stopping problem that describes the liquidation problem, and we demonstrate that the optimal strategy is to liquidate the first time the asset price falls below a certain time-dependent boundary. Moreover, this boundary is shown to be monotonically increasing, continuous and to satisfy a non-linear integral equation.

1. Introduction

This paper treats the problem of optimal timing for an irreversible sale of an indivisible asset under incomplete information about its drift. The asset price is assumed to follow a geometric Brownian motion $X$ with unknown drift, and an agent who decides to sell at time $t$ receives at this time the amount $X_t$. The objective of the agent is to choose a liquidation time for which the expected value of the (discounted) asset price is maximised. Such problems are important for all types of investors with insufficient knowledge of the future trend of an asset.

In the case with complete information about the model parameters of $X$, the corresponding optimal liquidation problem is trivial. Indeed, if the drift is larger than the interest rate, then on average the asset price grows faster than money in a risk-free bank account, and the agent should keep the asset as long as possible. Similarly, a drift smaller than the interest rate implies that the agent should liquidate the asset immediately and instead deposit the money in the bank. However, we remark that the assumption of complete information about the parameters of $X$ is quite strong. The volatility of an asset can, at least in principle, be estimated instantaneously by observing the price fluctuations over an arbitrarily short time period. On the other hand, the drift is notoriously difficult to estimate from historical

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data, and to achieve a decent accuracy in the estimate for the drift one typically needs observations of the process from hundreds of years.

Instead, we allow for incomplete information by modelling the drift as a random variable which is not directly observable for the agent. Initially, the agent’s beliefs about the drift are summarised by a probability distribution. As time goes by, however, he observes the asset price process, and based on these observations his beliefs may change. Naturally, if the asset price rises quickly, then the agent will consider it more likely that the drift takes the larger of the two values, and he would postpone the liquidation. Similarly, if the asset falls drastically, then it is likely that the true drift is small, and the agent would be more inclined to liquidate his position early. We show below that this intuition is true, i.e. there exists a boundary between the continuation region and the stopping region such that the optimal liquidation time coincides with the first time the asset price falls below the boundary. We also derive monotonicity and continuity properties of the boundary, and we show that it satisfies a non-linear integral equation similar to the one which characterises the optimal stopping boundary for the American put option.

Related problems of liquidating an indivisible asset have been studied in [4] and [7]. These papers study a risk-averse agent who wants to sell an indivisible asset with the possibility of hedging some of the risk by investing in a correlated stock market. The paper [2], see also [6], studies a problem of optimal selling of an asset, where optimality is measured by closeness between the current asset price and its ultimate maximum over the whole time period. In all the papers referred to above, the agent is assumed to have complete information about the underlying price processes. The methods we use to treat the incomplete information in our setting are standard and based on filtering theory, see for example [12]. An early application of these techniques is the sequential testing of two alternative hypotheses about the drift of a Brownian motion; for further details and related references see Chapter VI.21 in [14]. Similar techniques to tackle investment problems in markets with incomplete information are also applied in [9] and [11], where the problem of maximising expected utility of terminal wealth by trading in different assets is studied. The papers [1] and [10] study the optimal timing for an investment under incomplete information. Mathematically, the investment problem in [1] is equivalent to the pricing of an American call option written on an asset with unknown drift. Using filtering techniques, the problem is reduced to an optimal stopping problem with complete information, but with two underlying spatial dimensions. A clever observation in [10] reduces the two-dimensional problem into a one-dimensional optimal stopping problem, but in general for a time-dependent pay-off function (for one specific choice of parameters, however, the time dependence disappears and the optimal stopping problem can be solved explicitly). In the present paper, the optimal liquidation problem has a linear pay-off, which implies that the problem can be reduced to a one-dimensional optimal stopping
problem for a time-homogeneous diffusion with an affine pay-off regardless of what parameters are chosen. Consequently, this reduced problem is straightforward to analyse using standard methods from optimal stopping theory.

The present paper is organised as follows. In Section 2 we formulate the liquidation problem with incomplete information, and we apply filtering techniques to write it as a two-dimensional problem with complete information. Moreover, we apply a Girsanov transformation that reduces the problem to a one-dimensional optimal stopping problem for a time-homogeneous diffusion with an affine pay-off function. We also provide the solution of the optimal liquidation problem (in terms of the boundary of the auxiliary optimal stopping problem), see Theorem 1. The auxiliary optimal stopping problem is treated in Section 3, where we demonstrate the existence of a monotonically increasing and continuous optimal stopping boundary. We also show that the boundary together with the value function solves a parabolic free boundary problem. In Section 4 we derive an integral equation for the optimal stopping boundary. Finally, in Section 5 we study a related situation in which the agent seeks an optimal time to close a short position in the asset.

2. The optimal liquidation problem and its solution

To model the situation with incomplete information, we assume that the asset price process $X$ follows a geometric Brownian motion with unknown drift $\mu$ and constant volatility $\sigma > 0$. More precisely,

$$dX_t = \mu X_t dt + \sigma X_t d\tilde{W}_t, \quad t \geq 0$$

where $\tilde{W}$ is a standard Brownian motion independent of $\mu$ on a probability space $(\Omega, \mathcal{F}, P)$. Here, for simplicity, we assume that the drift $\mu$ can only take two values $\mu_l$ and $\mu_h$ satisfying $\mu_l < r < \mu_h$, where the interest rate $r \geq 0$ is a constant, and the initial asset price $X_0$ is a positive constant. We consider an agent who owns the asset and wants to liquidate his position before a given future fixed time $T > 0$. At the initial time 0, the true value of the drift $\mu$ is not known, but we assume that the agent has an initial guess for the probabilities of the events $\{\mu = \mu_l\}$ and $\{\mu = \mu_h\}$. More explicitly, we assume that the agent’s initial estimate of the probability of the event $\{\mu = \mu_h\}$ is a constant $\Pi_0 \in (0, 1)$. Accordingly, the estimate of the probability of $\{\mu = \mu_l\}$ is $1 - \Pi_0$. Furthermore, we assume that the agent can observe the value process $X$, but neither the drift $\mu$ nor the Brownian motion $\tilde{W}$. This is a natural assumption since in a real world situation, no underlying Brownian motion can be observed, and to estimate the drift with a high precision is infeasible.

**Example** Consider a Brownian motion $Z_t = a t + b B_t$ with (unknown) drift $a$ and volatility $b$ (here $B$ is a standard Brownian motion). An estimate for the drift $a$ would be $\hat{a} = \frac{Z_t}{t}$, and a 95%-confidence interval is then
\( (\hat{a} - 1.96b/\sqrt{t}, \hat{a} + 1.96b/\sqrt{t}) \). Even if the volatility is small, say \( b = 0.1 \), in order for the confidence interval to be reasonably tight, say \( (\hat{a} - 0.02, \hat{a} + 0.02) \), one needs approximately 100 years of observations! Moreover, the observation time that is needed grows (inverse) quadratically in the length of the confidence interval.

The objective of this paper is to determine when to sell the stock in order to maximise the expected wealth. More precisely, let \( \{\mathcal{F}_t^X\}_{t \in [0,T]} \) be the completion of the filtration generated by the process \( X \). The agent then seeks an \( \mathcal{F}^X \)-stopping time \( \tau \) with \( 0 \leq \tau \leq T \) for which the supremum

\[
V = \sup_{0 \leq \tau \leq T} E[e^{-r\tau}X_\tau]
\]

is attained, where the supremum is taken over all \( \mathcal{F}^X \)-stopping times \( \tau \).

**Remark** Note that in the omitted cases \( \Pi_0 = 0 \) and \( \Pi_0 = 1 \), the problem is simply a problem with complete information, and the solution is trivial. Indeed, if \( \Pi_0 = 1 \), then \( \mu = \mu_h \) and \( e^{-r\tau}X_t \) is a submartingale, so optional sampling yields that \( V = X_0e^{(\mu_h - r)T} \). Similarly, if \( \Pi_0 = 0 \), then \( e^{-r\tau}X_t \) is a supermartingale, and \( V = X_0 \). Also note that it is necessary to have \( T < \infty \) in order to avoid a degenerate problem when \( \Pi_0 > 0 \). In fact, plugging in the stopping time \( \tau = n \) and letting \( n \) tend to infinity shows that in the perpetual case we would have an infinite value \( V \).

**Remark** Inserting \( \tau = 0 \) into (1) yields a lower bound \( V \geq X_0 \). Another lower bound can be found by comparing with the corresponding ‘European value’ \( X_0(\Pi_0 e^{(\mu_h - r)T} + (1 - \Pi_0) e^{(\mu_l - r)T}) \) determined by inserting \( \tau = T \) in (6). Moreover, an upper bound for \( V \) can be found by observing that increasing \( \mu_l \) to \( r \) simply gives a higher pay-off. In that case it is clear that \( e^{-r\tau}X_t \) is a submartingale, so the optional sampling theorem gives that \( V \leq X_0(\Pi_0 e^{(\mu_h - r)T} + 1 - \Pi_0) \). Consequently,

\[
\max\{X_0, X_0(\Pi_0 e^{(\mu_h - r)T} + (1 - \Pi_0) e^{(\mu_l - r)T})\} \leq V \leq X_0(\Pi_0 e^{(\mu_h - r)T} + 1 - \Pi_0).
\]

Naturally, if \( \Pi_0 \) is small, then the agent is rather confident that the true drift is \( \mu_l \), and he would liquidate immediately and rather deposit the money in the bank. On the other hand, if \( \Pi_0 \) is close to one, then he considers it likely that the drift is \( \mu_h \), and he would prefer to postpone the selling. By observing the process \( X \), however, the agent’s estimates for the probabilities of the events \( \{\mu = \mu_h\} \) and \( \{\mu = \mu_l\} \) may change. For \( t \geq 0 \), let

\[
\Pi_t = P[\mu = \mu|\mathcal{F}_t^X]
\]

be the probability at time \( t \) that \( \mu = \mu_h \) conditional on the observations of \( X \) up to time \( t \). From Theorems 7.12 and 9.1 in Lipster and Shiryaev [12], the value process \( X \) and the belief process \( \Pi \) satisfy

\[
\frac{dX_t}{d\Pi_t} = \begin{pmatrix} \mu_l + \Pi_t(\mu_h - \mu_l) \\ 0 \end{pmatrix} dt + \begin{pmatrix} \sigma \\ \omega \Pi_t(1 - \Pi_t) \end{pmatrix} d\bar{W}_t
\]
where $\omega = (\mu_h - \mu_l)/\sigma$ and $(\bar{W}, \mathcal{F}^X)$ is a $P$-Brownian motion defined by
$$d\bar{W}_t = d\tilde{W}_t + \frac{\mu - (1 - \Pi_t)\mu_l - \Pi_t\mu_h}{\sigma} dt.$$ 

Note that the drift of $X$ depends on $\Pi$, so the optimal stopping problem (1) has two underlying spatial dimensions. However, since $X$ and $\Pi$ are both expressed in terms of the same Brownian motion $\bar{W}$, the number of spatial dimensions can be reduced. Indeed, below we follow [10] and use a Girsanov transformation to reduce the problem to a one-dimensional stopping problem.

To do this, define a new process $W$ by
$$dW_t = (\omega \Pi_t - \sigma) dt + d\bar{W}_t$$
and a new measure $P^*$ by its Radon-Nikodym derivative
$$(2) \quad \frac{dP^*}{dP} = \exp\left\{-\frac{1}{2} \int_0^T (\sigma - \omega \Pi_t)^2 dt + \int_0^T (\sigma - \omega \Pi_t) d\bar{W}_t \right\}$$
with respect to $P$. By Girsanov’s Theorem, $W$ is a $P^*$-Brownian motion.

Next, define the likelihood ratio $\Phi$ by $\Phi_t = \Pi_t/(1 - \Pi_t)$. A straightforward application of Ito’s formula gives
$$d\Phi_t = \omega^2 \Pi_t \Phi_t dt + \omega \Pi_t d\bar{W}_t,$$
so both $X$ and $\Phi$ are geometric Brownian motions under $P^*$. Moreover, the filtration generated by $W$ coincides with the one generated by $X$.

Define the likelihood process
$$\eta_t = \exp\left\{-\frac{1}{2} \int_0^t (\sigma - \omega \Pi_s)^2 ds + \int_0^t (\omega \Pi_s - \sigma) dW_s \right\}.$$ 

Since $(W, \mathcal{F}^X)$ is a $P^*$-Brownian motion, the process $\eta$ is an $\mathcal{F}^X$-martingale under $P^*$. Let
$$F_t = \frac{1 + \Phi_t}{1 + \Phi_0}.$$

Then
$$\frac{dF_t}{F_t} = \frac{d\Phi_t}{1 + \Phi_t} = \sigma \omega \Pi_t dt + \omega \Pi_t dW_t,$$
so
$$F_t = \exp\left\{\frac{1}{2} \int_0^t (2\sigma \omega \Pi_s - \omega^2 \Pi_s^2) ds + \int_0^t \omega \Pi_s dW_s \right\}.$$ 

Consequently,
$$\eta_t X_t = e^{\mu_l t} F_t X_0.$$
Denote by \( E^* \) the expectation operator with respect to the new measure \( P^* \). Then, changing the measure in (1) yields

\[
V = \sup_{0 \leq \tau \leq T} E[e^{-r\tau} X_\tau] = \sup_{0 \leq \tau \leq T} E^*[e^{-r\tau} \eta T X_\tau] \\
= \sup_{0 \leq \tau \leq T} E^*[e^{-r\tau} \eta T X_\tau] = X_0 \sup_{0 \leq \tau \leq T} E^*[e^{(\mu - r)\tau} F_\tau] \\
= \frac{X_0}{1 + \Phi_0} \sup_{0 \leq \tau \leq T} E^*[e^{(\mu - r)\tau} (1 + \Phi_\tau)],
\]

where the third equality follows by conditioning upon \( F_\tau \) together with the martingale property of \( \eta \).

**Remark** Note that the measure change defined in (2) slightly differs from the one in [1] and [10], where the new measure instead is defined so that the Radon-Nikodym derivative coincides with

\[
\exp \left\{ -\frac{1}{2} \int_0^T \omega^2 \Pi_t^2 dt - \int_0^T \omega \Pi_t d\bar{W}_t \right\}
\]
on \( F^X_T \). To reduce the number of spatial dimensions, Klein [10] then employs the equality

\[
X_t = X_0 e^{\varepsilon t} (\Phi_t / \Phi_0)^\beta,
\]

where

\[
\beta = \frac{\sigma}{\omega} = \frac{\sigma^2}{\mu_h - \mu_l} \quad \text{and} \quad \varepsilon = (\mu_h + \mu_l - \sigma^2)/2.
\]

If \( \varepsilon = 0 \), then the obtained optimal stopping problem is time-homogeneous, and an explicit solution can be found. The measure change in (2) is tailor-made for the situation of a linear pay-off structure considered in the current paper. Thanks to the linearity of the pay-off, the optimal stopping problem on the right hand side of (4) is expressed in terms of a time-homogeneous diffusion \( \Phi \) with an affine pay-off function independent of time. Note that this is the case not only for \( \varepsilon = 0 \), but for all possible parameter values.

In view of (4), we introduce the auxiliary optimal stopping problem

\[
\Gamma(t, z) = \sup_{0 \leq \tau \leq T - t} E^*[e^{(\mu - r)\tau} (1 + Z_\tau)],
\]

where

\[
Z_u := z \exp \left\{ (\sigma \omega - \omega^2/2)u + \omega W_u \right\}, \quad u \geq 0,
\]

and the supremum is taken over stopping times with respect to the filtration generated by \( W \). Note that

\[
V = \frac{X_0 \Gamma(0, \Phi_0)}{1 + \Phi_0}.
\]

Moreover, an optimal stopping time for the problem (6) translates to an optimal stopping time for the original problem (1).
In the next section we study the optimal stopping problem (6). In particular, we prove the existence of a continuous and monotonically increasing function $b : [0, T] \rightarrow [0, \infty)$ such that the stopping time
\[ \tau_{t,z}^* := \inf \{ u \in [0, T-t] : Z_u \leq b(t+u) \} \wedge (T-t) \]
is an optimal stopping time, i.e. a stopping time for which the supremum in (6) is attained. The following result is then a direct consequence of relation (5).

**Theorem 1.** Let $b$ be the function mentioned above, the existence of which is proved in Proposition 2 below. Define the stopping time
\[ \tau^* = \inf \{ t : X_t \leq \frac{X_0}{\Phi_0} e^{t\beta} \} \wedge T. \]
Then $\tau^*$ attains the supremum in (1).

**Remark** The optimal stopping boundary and the optimal stopping time $\tau^*$ are illustrated in Figure 1. Note that it also follows from the analysis of the auxiliary problem below (in particular equation (12)) and relation (7) that $V$ is the solution of a free boundary problem. Indeed, straightforward calculations show that $V = U(0, \Phi_0)$, where the function $U(t, \phi)$ satisfies
\[
\begin{cases}
U_t + \frac{\omega^2 \phi^2}{2} U_{\phi\phi} + \frac{\phi(1+\phi)(\mu_h-\mu)}{1+\phi} U_{\phi} + \frac{\phi(\mu_h-r) + \mu - r}{1+\phi} U = 0 & \text{if } \phi > b(t) \\
U(t, \phi) = X_0 & \text{if } \phi \leq b(t) \text{ or } t = T \\
U_{\phi}(t, \phi) = 0 & \text{if } \phi = b(t). 
\end{cases}
\]
Remark Note that the value $V$ exhibits an easy monotone dependence on the model parameters $\mu_l$, $\mu_h$, $r$, $\Pi_0$ and $T$. The dependence on volatility is slightly more involved to analyse. However, it is a consequence of Corollary 2.7 in [3] that $\Gamma$ is monotonically increasing in the diffusion coefficient (i.e. in $\omega = (\mu_h - \mu_l)/\sigma$), so $V$ is decreasing in $\sigma$. The intuition behind this is that in the case of a small volatility, learning of the true value of the drift is fast, which is beneficial for the agent.

3. The auxiliary optimal stopping problem

In this section we study the optimal stopping problem (6). This problem is similar to the one arising in the valuation of American put options, compare [5] and Chapter 2.7 in [8]. We prove the existence of a monotone and continuous optimal stopping boundary, and we show that the boundary and the value function $\Gamma$ solves a related free boundary problem.

Recall that $Z$ satisfies
\[
\frac{dZ_u}{Z_u} = \sigma \omega du + \omega dW_u = (\mu_h - \mu_l)du + \omega dW_u, \quad u \geq 0.
\]

We will also use the representation $Z_u = zH_u$, where
\[
H_u = \exp\{(\sigma \omega - \frac{\omega^2}{2})u + \omega W_u\}.
\]

With this notation,
\[
\Gamma(t, z) = \sup_{0 \leq \tau \leq T-t} E^*[e^{(\mu_l-r)\tau}(1 + zH_\tau)].
\]

Proposition 1. The function $\Gamma : [0,T] \times (0, \infty) \to (0, \infty)$ is continuous.

Proof. Assume that $z_2 > z_1 > 0$, and let $\tau$ be an optimal stopping time for $\Gamma(t, z_2)$ in the sense that
\[
\Gamma(t, z_2) = E^*[e^{(\mu_l-r)\tau}(1 + z_2H_\tau)]
\]
(such an optimal stopping time exists, see for example Theorem D.12 in [8]). Then we have
\[
0 \leq \Gamma(t, z_2) - \Gamma(t, z_1) \leq E^*[e^{(\mu_l-r)\tau}(1 + z_2H_\tau)] - E^*[e^{(\mu_l-r)\tau}(1 + z_1H_\tau)] = (z_2 - z_1)E^*[e^{(\mu_l-r)\tau}H_\tau].
\]

Ito’s formula gives that the process $Y_t := e^{(\mu_l-r)t}H_t$ satisfies
\[
dY_t = (\mu_h - r)Y_t \, dt + \omega Y_t \, dW_t.
\]

Since the drift $(\mu_h - r)$ is strictly positive, $e^{(\mu_l-r)t}H_t$ is a submartingale, so the Optional Sampling Theorem gives that
\[
0 \leq \Gamma(t, z_2) - \Gamma(t, z_1) \leq (z_2 - z_1)E^*[e^{(\mu_l-r)(T-t)}H_{T-t}] = (z_2 - z_1)e^{(\mu_h-r)(T-t)},
\]
which shows that $\Gamma$ is Lipschitz continuous in $z$. 

Let $0 \leq t_1 < t_2 < T$ and $z \in (0, \infty)$. Let $\tau_1$ denote an optimal stopping time for $U(t_1, z)$, and set $\tau_2 = \tau_1 \wedge (T - t_2)$. Hence $0 \leq \tau_1 - \tau_2 \leq t_2 - t_1$. We then have

\begin{equation}
0 \leq \Gamma(t_1, z) - \Gamma(t_2, z)
\end{equation}

\begin{equation}
\leq E^*[e^{(\mu - r)\tau_1} (1 + zH_{\tau_1})] - E^*[e^{(\mu - r)\tau_2} (1 + zH_{\tau_2})]
\end{equation}

\begin{equation}
= E^*[e^{(\mu - r)\tau_1} - e^{(\mu - r)\tau_2}] + z E^*[e^{(\mu - r)\tau_1} H_{\tau_1} - e^{(\mu - r)\tau_2} H_{\tau_2}].
\end{equation}

Now

\begin{equation}
E^*[e^{(\mu - r)\tau_1} H_{\tau_1} - e^{(\mu - r)\tau_2} H_{\tau_2}]
\end{equation}

\begin{equation}
= E^*[e^{(\mu - r)\tau_2} H_{\tau_2} E^*[e^{(\mu - r)(\tau_1 - \tau_2)} H_{\tau_1} \frac{H_{\tau_1}}{H_{\tau_2}} - 1|\mathcal{F}_{\tau_2}]])
\end{equation}

\begin{equation}
\leq E^*[e^{(\mu - r)\tau_2} H_{\tau_2}] F(t_2 - t_1)
\end{equation}

\begin{equation}
\leq e^{(\mu_h - r)(T - t_2)} F(t_2 - t_1),
\end{equation}

where

\begin{equation}
F(t) := E^*[\sup_{u \in [0, t]} e^{(\mu_k - r - \omega^2/2)u + \omega W_u}] - 1
\end{equation}

and where we used the fact that $e^{(\mu - r)t}H_t$ is a submartingale. Note that $F(t) \to 0$ as $t \to 0$, which implies that the second term on the right hand side of (10) tends to zero as $t_2 - t_1 \to 0$. A similar argument applies to the first term in (10), thus showing that $\Gamma$ is continuous as a function of $t$. Since $\Gamma$ is also uniformly continuous with respect to $z$, this finishes the proof. \(\square\)

Choosing the stopping time $\tau = 0$ in (6), we find that $\Gamma(t, z) \geq G(z) := 1 + z$. Define the \textit{continuation region} $\mathcal{C}$ to be

\begin{equation}
\mathcal{C} = \{(t, z) \in [0, T) \times (0, \infty) : \Gamma(t, z) > G(z)\}
\end{equation}

and the \textit{stopping region} $\mathcal{D}$ by

\begin{equation}
\mathcal{D} = \{(t, z) \in [0, T] \times (0, \infty) : \Gamma(t, z) = G(z)\}.
\end{equation}

According to general theory for optimal stopping problems, see for example [14], the stopping time

\begin{equation}
\tau_D = \inf\{0 \leq s \leq T - t : (t + s, Z_s) \in \mathcal{D}\}
\end{equation}

is an optimal stopping time in (6). Therefore, to determine an optimal stopping time it suffices to determine the optimal stopping region $\mathcal{D}$.

Define $f(z) := \mathcal{L}G(z) - (r - \mu_l)G(z)$, where

\begin{equation}
\mathcal{L}G(z) := \frac{\omega^2 z^2}{2} G_{zz}(z) + (\mu_h - \mu_l)z G_z(z)
\end{equation}

is the infinitesimal operator of $Z$. A simple calculation shows that

\begin{equation}
f(z) = (\mu_h - r)z + \mu_l - r \begin{cases} 
\geq 0 & \text{if } z \geq \frac{r - \mu_l}{\mu_h - r} \\
< 0 & \text{if } z < \frac{r - \mu_l}{\mu_h - r}.
\end{cases}
\end{equation}
It therefore follows from Ito’s formula that $e^{(\mu l - r)s}G(Z_s)$ is a submartingale for $s \leq \inf\{ u: Z_u < \frac{T - \mu}{\mu h - r} \}$. By the Optional Sampling Theorem, all points $(z, t)$ with $z > \frac{T - \mu}{\mu h - r}$ belong to the continuation region $C$.

A better bound for the stopping region $D$ is easily derived by comparing $\Gamma$ with the corresponding 'European' value. More precisely, we have that

$$\Gamma(z, t) = \sup_{0 \leq \tau \leq T - t} E^*[e^{(\mu l - r)\tau}(1 + Z_\tau)]$$

$$\geq E^*[e^{(\mu l - r)(T - t)}(1 + Z_{T - t})]$$

$$= e^{(\mu l - r)(T - t)} + z e^{(\mu h - r)(T - t)}.$$

Therefore, all points $(z, t)$ such that $z > (1 - e^{(\mu l - r)(T - t)})/(e^{(\mu h - r)(T - t)} - 1)$ satisfy $\Gamma(z, t) > 1 + z$, i.e. they belong to the continuation region. Note that the function $b^E(t) := (1 - e^{(\mu l - r)(T - t)})/(e^{(\mu h - r)(T - t)} - 1)$ is increasing and satisfies $b^E(-\infty) = 0$ and $b^E(T) = \frac{T - \mu}{\mu h - r}$.

**Proposition 2.** There exists a non-decreasing and right continuous function $b : [0, T] \to [0, \frac{T - \mu}{\mu h - r}]$ such that

$$C = \{(t, z) \in [0, T) \times (0, \infty) : z > b(t)\}.$$

Moreover, the supremum in (6) is attained for the stopping time $\tau_D = \inf\{0 \leq u \leq T - t : Z_u \leq b(t + u)\}$.

**Proof.** First note that

$$C = \{(t, z) : \Gamma(t, z) > G(z)\} = \{(t, z) : \Gamma(t, z) > 1 + z\}.$$

For some fixed $t \in [0, T)$ and $z' > z > 0$, suppose that $(t, z)$ is in $C$. Then there exists a stopping time $\tau$ such that

$$E^*[e^{(\mu l - r)\tau}(1 + z H_\tau)] > 1 + z.$$  

Consequently,

$$\Gamma(t, z') \geq E^*[e^{(\mu l - r)\tau}(1 + z'H_\tau)]$$

$$= E^*[e^{(\mu l - r)\tau}(1 + z'H_\tau)] + (z' - z)E^*[e^{(\mu l - r)\tau}H_\tau]$$

$$> 1 + z + (z' - z)E^*[e^{(\mu l - r)\tau}H_\tau].$$

Since the process $Y_t := e^{(\mu l - r)t}H_t$ is a submartingale, compare (9) above, the Optional Sampling Theorem gives

$$\Gamma(t, z') > 1 + z + (z' - z)E^*[e^{(\mu l - r)\tau}H_\tau]$$

$$\geq 1 + z + z' - z = 1 + z'.$$

Therefore, $(t, z')$ also belongs to the continuation region $C$, proving the existence of a function $b : [0, T] \to [0, \infty]$ such that

$$C = \{(t, z) : t \in [0, T) \text{ and } z > b(t)\}.$$

The fact that $b$ only takes values smaller than $\frac{T - \mu}{\mu h - r}$ follows from the discussion before Proposition 2, and the monotonicity of $b$ follows from the
monotonicity of \( t \mapsto \Gamma(t, z) \). Finally, the right continuity of \( b \) follows from the fact that the continuation region \( C \) is an open set, and the optimality of \( \tau_D \) is already established.

**Remark** In view of the discussion preceding Proposition 2, the optimal stopping boundary \( b(t) \) satisfies

\[
b(t) \leq b^E(t) = (1 - e^{(\mu_l - r)(T-t)})/(e^{(\mu_b - r)(T-t)} - 1).
\]

A similar bound is then valid also for the optimal stopping boundary in Theorem 1.

**Proposition 3.** The value function \( \Gamma(t, z) \) satisfies the boundary value problem

\[
\begin{cases}
\Gamma_t(t, z) + L \Gamma(t, z) + (\mu_l - r) \Gamma(t, z) = 0 & \text{if } z > b(t) \\
\Gamma(t, z) = G(z) = 1 + z & \text{if } z \leq b(t) \text{ or } t = T \\
\Gamma_z(t, z) = G'(z) = 1 & \text{if } z = b(t)
\end{cases}
\]

**Proof.** Since \( \Gamma(t, z) > 1 + z \) for \( z > b(t) \) and \( \Gamma(t, b(t)) = 1 + b(t) \) for \( t \in [0, T] \), it follows that

\[
\liminf_{\rho \to 0} \frac{\Gamma(t, b(t) + \rho) - \Gamma(t, b(t))}{\rho} \geq 1.
\]

Thus, it remains to show that

\[
\limsup_{\rho \to 0} \frac{\Gamma(t, b(t) + \rho) - \Gamma(t, b(t))}{\rho} \leq 1.
\]

For any \( \rho > 0 \), denote by \( \tau_\rho := \tau^*_{t, b(t) + \rho} \) the optimal stopping time for the starting point \( (t, b(t) + \rho) \) as defined in (8). We have

\[
\Gamma(t, b(t) + \rho) - \Gamma(t, b(t)) \\
\leq E^*[e^{(\mu_l - r)\tau_\rho} (1 + (b(t) + \rho) H_{\tau_\rho})] - E^*[e^{(\mu_l - r)\tau_\rho} (1 + b(t) H_{\tau_\rho})] \\
= \rho E^*[e^{(\mu_l - r)\tau_\rho} H_{\tau_\rho}].
\]

We know that the optimal stopping boundary \( s \mapsto b(s) \) is increasing on \([t, T]\) and \( s \mapsto (\frac{\mu_l}{2} - \sigma) s \) is a lower function of the Brownian motion \( W \) at zero. It follows that \( \tau_\rho \to 0 \) \( P^* \)-a.s. as \( \rho \to 0 \), which tells us that

\[
E^*[e^{(\mu_l - r)\tau_\rho} H_{\tau_\rho}] \to 1
\]

as \( \rho \to 0 \) by the dominated convergence theorem. Hence (13) holds, so \( z \mapsto \Gamma(t, z) \) is \( C^1 \) at \( z = b(t) \), and \( \Gamma_z = 1 = G' \).

The proof that \( \Gamma \) satisfies the partial differential equation in (12) relies on the continuity of \( \Gamma \) and follows along the same lines as for example in the case of the American put option, compare page 72 in [8]. We omit the details.

**Remark** Furthermore, it can be shown that the pair \((\Gamma, b)\) is the unique solution to the free boundary problem (12) (within some appropriate class of functions). We leave this and instead refer to Chapter 2.7 in [8] where this is shown for the American put option.
Proposition 4. The boundary $b(t)$ is continuous on $[0, T)$ and $b(T^-) = \frac{r - \mu_l}{\mu_h - r}$.

Proof. It follows from Proposition 2 that $b$ is right-continuous on $[0, T)$. To prove the left-continuity, define $b(T^-) = r - \mu_l - \mu_h - r$, and assume that the boundary $b(t)$ has a jump at $t_\ast \in (0, T]$, i.e. $b(t_\ast) > b(t_\ast^-)$.

By (11) and a continuity argument, there exists a $\delta < 0$ and a one-side open rectangle $R := [t', t_\ast) \times [c, d] \subseteq C$ with $b(t_\ast) \leq c < d < b(t_\ast)$ such that

\begin{equation}
\Delta G(z) - (r - \mu_l)G(z) < \delta < 0
\end{equation}

and

\begin{equation}
0 \leq \Gamma_z(t, z) - G_z(t, z) < \frac{-\delta}{(\mu_h - \mu_l)b(t_\ast)}
\end{equation}

for all $(t, z) \in R$. Since $R$ is contained in $C$, we also have

$L\Gamma(t, z) - (r - \mu_l)\Gamma(t, z) = -\Gamma_t(t, z) \geq 0$.

Together with (14), this yields

\begin{equation}
(\mu_h - \mu_l)z(\Gamma_z(t, z) - G_z(t, z)) + \frac{1}{2}\omega^2 z^2 (\Gamma_{zz} - G_{zz}) = L\Gamma - LG \geq (r - \mu_l)(\Gamma - G) - \delta.
\end{equation}

Using (15) it follows that

$\Gamma_{zz} - G_{zz} \geq 2\omega^2 z^2 (r - \mu_l)(\Gamma - G) \geq 2\omega^2 b(t_\ast)^2 (r - \mu_l)(\Gamma - G) =: \eta > 0$

in $R$. Therefore

\begin{align*}
\Gamma(t, z) - G(z) &= \int_{b(t)}^{z} \int_{b(t)}^{u} (\Gamma_{zz}(t, v) - G_{zz}(v)) \, dv \, du \\
&\geq \frac{1}{2} \eta (z - b(t))^2 \geq \frac{1}{2} \eta (c - b(t_\ast))^2 > 0
\end{align*}

for any $(t, z)$ in the rectangle. Since both the value and the gain functions are continuous, this leads to $\Gamma(t_\ast, z) > G(z)$ for any $z \in [c, d]$, which contradicts the fact that $(t_\ast, z)$ is in the stopping region. Therefore $b(t)$ is continuous at $t \in [0, T)$ and $b(T^-) = \frac{r - \mu_l}{\mu_h - r}$.

\[\square\]

4. An integral equation for the optimal stopping boundary

In this section we derive an integral equation for the optimal stopping boundary. The derivation follows along similar lines as for the American put option, see [5].
Theorem 2. The optimal stopping boundary $b(t)$ satisfies the integral equation
\begin{equation}
1 + b(t) = e^{(\mu - r)(T - t)} + b(t) e^{(\mu_h - r)(T - t)} \\
- \int_0^{T-t} (\mu - r) e^{(\mu - r)u} N\left( \frac{1}{\omega \sqrt{u}} \left[ \ln \frac{b(t + u)}{b(t)} - \omega u + \frac{\omega^2 u}{2} \right] \right) \\
+ b(t) (\mu_h - r) e^{(\mu_h - r)u} N\left( \frac{1}{\omega \sqrt{u}} \left[ \ln \frac{b(t + u)}{b(t)} - \omega u - \frac{\omega^2 u}{2} \right] \right) du,
\end{equation}
where $N(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} dy$ is the cumulative distribution function of the standard normal distribution.

Proof. Fix a $t \in [0, T]$ and $Z_0 = z \in (0, \infty)$. Applying Ito’s formula to $e^{(\mu - r)s} \Gamma(t + s, Z_s)$ and taking the expected value gives
\begin{equation}
e^{(\mu - r)(T-t)} E^*[G(Z_{T-t})] = e^{(\mu - r)(T-t)} E^*[\Gamma(T, Z_{T-t})] \\
= \Gamma(t, z) + \int_0^{T-t} e^{(\mu - r)u} E^*[F(Z_u) I(Z_u \leq b(t + u))] du
\end{equation}
where $G(y) = 1 + y$ and $F = \mathcal{L} G - (r - \mu) G$ as before. (The use of Ito’s formula can be motivated by similar arguments as for the American put option, compare [13].) Straightforward calculations give
\begin{equation}
e^{(\mu - r)(T-t)} E^*[G(Z_{T-t})] = E^*[e^{(\mu - r)(T-t)}] + E^*[e^{(\mu - r)(T-t)} Z_{T-t}]
= e^{(\mu - r)(T-t)} + z e^{(\mu_h - r)(T-t)}.
\end{equation}
The integrand of the right hand side in (17) is
\begin{equation}
e^{(\mu - r)u} E^*[F(Z_u) I(Z_u \leq b(t + u))] \\
e^{(\mu - r)u} (\mu_h - r) E^*[H_u I(Z_u \leq b(t + u))]
+ e^{(\mu - r)u} (\mu - r) E^*[I(Z_u \leq b(t + u))].
\end{equation}
We have
\begin{equation}
E^*_{t,z} [H_u I(Z_u \leq b(t + u))] \\
= E^*_{t,z} \left[ \exp \left( (\sigma \omega - \omega^2/2)u + \omega W_u \right) I \left( \frac{W_u}{\sqrt{u}} \leq \frac{\ln(b(t + u)/z) + (\omega^2/2 - \sigma \omega)u}{\omega \sqrt{u}} \right) \right] \\
= e^{\sigma \omega u} N(d_1),
\end{equation}
where
\begin{equation}
d_1 = \frac{1}{\omega \sqrt{u}} \left[ \ln \frac{b(t + u)}{z} - \omega u - \frac{\omega^2 u}{2} \right].
\end{equation}
Similarly
\begin{equation}
E^*_{t,z} [I(Z_u \leq b(t + u))] = N(d_2)
\end{equation}
where $d_2 = d_1 + \omega \sqrt{u}$. Using (17)–(21) and inserting $z = b(t)$ yields the integral equation (16).
Remark Using local time-space calculus, it was proved in [13] that the optimal stopping boundary of the American put option is the unique solution to the corresponding integral equation. Using similar techniques, uniqueness for equation (16) can be established. We omit the details.

5. Closing a Short Position

In this section we consider an agent with a short position in the asset, and who seeks an optimal time to close the position. To study this situation we formulate the optimal stopping problem

\[
v = \inf_{0 \leq \tau \leq T} E[e^{-r\tau}X_\tau],
\]

where the infimum is taken over \(\mathcal{F}\)-stopping times \(\tau\). All the assumptions about the model are as described in Section 2 above.

By exactly the same arguments provided above, we find that

\[
v = \frac{X_0}{1 + \Phi_0} \gamma(0, \Phi_0),
\]

where \(\gamma\) is defined through the auxiliary optimal stopping problem

\[
\gamma(t, z) = \inf_{0 \leq \tau \leq T-t} E^*[e^{(\mu_l-r)\tau}(1+Z_\tau)],
\]

where

\[
Z_u := z \exp\left\{(\sigma \omega - \frac{\omega^2}{2})u + \omega W_u\right\}, \quad u \geq 0,
\]

and the infimum is taken over stopping times with respect to the filtration generated by \(W\). Moreover, an optimal stopping time for the problem (23) translates to an optimal stopping time for the original problem (22).

The following results parallel those for the optimal liquidation problem for a long position, and the proofs are omitted.

**Theorem 3.** There exists a non-increasing and continuous function \(b : [0, T) \to [\frac{r-\mu_l}{\mu_l}, \infty)\) with \(b(T^-) = \frac{r-\mu_l}{\mu_l}\) such that the continuation region \(C = \{(x, t) \in [0, T) \times (0, \infty) : \gamma(t, z) < 1 + z\}\) satisfies

\[
C = \{(t, z) \in [0, T) \times (0, \infty) : z < b(t)\}.
\]

Moreover, the infimum in (23) is attained for the stopping time

\[
\tau_D := \inf\{0 \leq u \leq T-t : Z_u \geq b(t+u)\}.
\]

The function \(\gamma : [0, T] \times (0, \infty) \to (0, \infty)\) is continuous and satisfies the boundary value problem

\[
\begin{cases}
\gamma(t, z) + L\gamma(t, z) + (\mu_l - r)\gamma(t, z) = 0 & \text{if } z < b(t) \\
\gamma(t, z) = G(z) = 1 + z & \text{if } z \geq b(t) \text{ or } t = T \\
\gamma_z(t, z) = G'(z) = 1 & \text{if } z = b(t).
\end{cases}
\]
The optimal stopping boundary $b(t)$ satisfies the integral equation

\[
1 + b(t) = e^{(\mu - r)(T-t)} + b(t)e^{(\mu_h - r)(T-t)}
- \int_0^{T-t} (\mu_l - r)e^{(\mu_l - r)u}\left\{1 - N\left(\frac{b(t + u)}{b(t)} - \omega\sigma u + \frac{\omega^2 u}{2}\right)\right\} du.
\]

Corollary 1. The infimum in (22) is attained for the stopping time

\[
\tau^* := \inf\{t : X_t \geq X_0 \Phi_0 e^{\epsilon t} \beta^\lambda(t)\} \land T.
\]

Remark Unlike the optimal liquidation problem for a long position, the problem of this subsection also makes sense to study with an infinite horizon.

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OPTIMAL SELLING OF AN ASSET WITH JUMPS UNDER INCOMPLETE INFORMATION

BING LU

Abstract. We study the optimal liquidation strategy of an asset with price process satisfying a jump diffusion model with unknown jump intensity. It is assumed that the intensity takes one of two given values, and we have an initial estimate for the probability of both of them. As time goes by, by observing the price fluctuations, we can thus update our beliefs about the probabilities for the intensity distribution. We formulate an optimal stopping problem describing the optimal liquidation problem. It is shown that the optimal strategy is to liquidate the first time the point process falls below (goes above) a certain time-dependent boundary.

1. Introduction

Jump diffusion models capturing discontinuity property of asset price processes are widely used for modelling market fluctuations. In this paper, we treat the optimal selling problem of an asset whose price process follows a jump diffusion model. It is assumed that the jump intensity can take one of two given values. The task is to choose a selling time less than or equal to a fixed time $T$ for which the expected value of the discounted price is maximised.

This problem is trivial if the jump intensity is known, since one can easily compute the expected rate of return of the asset with the intensity given. If the rate of return is greater than the interest rate, which implies that the asset price is more likely to increase faster than money in the bank, then the agent prefers to keep the asset as long as possible. On the other hand, if the rate of return is less than the interest rate, then we expect that the asset price grows slower than money in the bank. Thus the asset should be liquidated immediately. If the jump intensity is unknown, then the optimal stopping problem is not trivial.

Although the intensity is not known at the beginning, it is assumed that the agent has an initial estimate of the distribution of the intensity. As time goes by, the agent learns more and more about the intensity by observing the asset price fluctuations. Therefore, based on his/her continuously updated estimate of the intensity, the agent chooses the optimal stopping strategies.

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Optimal liquidation of an asset whose price process follows a geometric Brownian motion with two possible values of the drift has been studied in [6], and it is a similar problem to the current one. In fact, the methods used in [6] are similar to the methods used in this paper. A major difference between the two papers is that the information of the drift is mixed with noise from the Brownian motion in the former, while in the current paper there is no Brownian noise since the jump process is purely discontinuous and the Brownian path is continuous. In this paper we use the filtering techniques for point processes, of which an early application is the sequential testing of a Poisson process with two alternative hypothesis about the intensity, compare [5]. Related problems of liquidating an asset have been studied in [4] and [9].

The present paper is organised as follows. In section 2 we formulate an optimal stopping problem with incomplete information about the jump intensity of the point process contained in the jump diffusion model. By the filtering techniques, the estimated distribution of the jump intensity can be expressed in terms of the point process. Applying the Girsanov theorem for jump process with stochastic intensity, we perform a measure transformation so that the intensity under the new measure is a constant. Next we perform another Girsanov transformation for compound Poisson process to simplify the payoff function, which finally becomes affine and only depends on the estimated distribution of the jump intensity. To solve the optimal stopping problem, an auxiliary problem which can easily translate to the original problem is defined. In section 3 we show that there exists a monotone and right-continuous optimal stopping boundary for the auxiliary problem and the auxiliary value function is the unique viscosity solution of the pricing equation. In section 4 we let the jump diffusion model be the Merton model and find the optimal stopping boundary.

2. Optimal Stopping Problem with a Jump Diffusion Model

Assume that the optimal stopping problem is built on a complete probability space \((\Omega, \mathcal{F}, P)\), which will be described explicitly later. For a stochastic process \(X = \{X_t; t \geq 0\}\) defined on \((\Omega, \mathcal{F}, P)\), let \(\mathcal{F}^X = \{\mathcal{F}^X_t; t \geq 0\}\) be the \(P\)-augmentation of the filtration generated by \(X\).

The price process of an asset is modeled by the jump diffusion model

\[
X_t = x \exp\{\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\},
\]

where \(W\) is a Brownian motion, the jump sizes \(Y_i\) are i.i.d. with distribution \(f\) and \(N\) is a point process on the complete probability space \((\Omega, \mathcal{F}, P)\). The drift \(\mu\) and the volatility \(\sigma\) are constants. The compensator of the point process is \(A_t = \lambda t\) and the intensity \(\lambda = \lambda(\omega)\) takes two values \(\lambda_1\) and \(\lambda_0\) with probabilities \(\pi_0\) and \(1 - \pi_0\). It is assumed that \(\lambda_1 > \lambda_0 > 0\) and
$\pi_0 \in [0, 1]$. Let $\nu^0$ and $\nu^1$ be the Levy measures for compound Poisson processes with intensity $\lambda_0$ and $\lambda_1$, respectively.

We consider the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}_{0,T}} E[e^{-r\tau}X_\tau],$$

where $r$ is the constant interest rate and $\mathcal{T}_{0,T}$ denotes the set of all $\mathcal{F}^X$-stopping times between 0 and $T$. The supremum is taken over all stopping times $\mathcal{T}_{0,T}$.

If the jump intensity is known, so is the expected rate of return of the asset. To be explicit, if the intensity is $\lambda_i$ for $i \in \{0, 1\}$ fixed, then the rate of return is given by

$$R_i := \mu + \frac{1}{2}\sigma^2 + \int_R (e^x - 1)\nu^i(dx).$$

The optimal exercise strategy can be made instantly by comparing the interest rate with the rate of return.

If the jump intensity is unknown and both $R_0$ and $R_1$ are greater or less than the interest rate, the optimal stopping strategy is also trivial. Therefore, in order to prevent the optimal stopping problem (2) from being trivial, we assume that the condition

$$(r - R_0)(R_1 - r) > 0$$

is satisfied.

Now we will discuss the range of possible values of $V$. Two lower bounds can be found by letting $\tau = T$ and $\tau = 0$, respectively. For the upper bound, notice that if we let $\min\{R_1, R_0\} \geq r$, then $e^{-rt}X_t$ is a submartingale. The optional sampling theorem implies that the optimal stopping time should be $T$. Thus the value function $V$ satisfies

$$x \max\{1, \pi_0 e^{(R_1-r)T} - (1 - \pi_0) e^{(R_0-r)T}\} \leq V(x) \leq x \max\{\pi_0 e^{(R_1-r)T} + (1 - \pi_0) e^{(R_0-r)T}\},$$

where $z^+ = \max\{z, 0\}$. It is not hard to see the monotone dependence of the value function $V$ on the parameters $r$, $\mu$, $\sigma$, $\lambda_0$, $\lambda_1$, $T$ and $\pi_0$.

To have a better understanding of the problem, it is useful to discuss the underlying probability space. Define $\Omega = \{\lambda_0, \lambda_1\} \times \Omega^*$, where $\Omega^*$ supports Brownian motions and compound Poisson processes with jump size distribution $f$ and jump intensities $\lambda_0$ or $\lambda_1$. Let $\mathcal{F} = 2^{\{\lambda_0, \lambda_1\}} \otimes \mathcal{B}(\Omega^*)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra on $\Omega^*$. There are two filtrations on $(\Omega, \mathcal{F}, P)$ worth noticing: one is the filtration generated by $X$ and $\lambda$, $\mathcal{F}^{X,\lambda} = \{\mathcal{F}_t \otimes \sigma(\lambda); t \geq 0\}$, with which complete information is available, then the optimal stopping problem becomes trivial; the other is the filtration generated only by $X$, i.e., $\mathcal{F}^X$, this is the information known for the decision-maker, who does not know the exact value of $\lambda$. For the rest of the text, we will only consider the latter filtration.
For $t \geq 0$, let
\[
\pi^*_t = P[\lambda = \lambda_1|\mathcal{F}_t^X]
\]
be the probability at time $t$ that $\lambda = \lambda_1$ conditional on the observation of $X$ up to time $t$. By the filtering techniques, compare Chapter VI.23 in [11] or Theorem 19.7 in [10], the probability $\pi^*$ can be expressed by
\[
\pi^*_t = \frac{\pi_0 \exp \left(N_t \log \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0)t\right)}{1 - \pi_0 + \pi_0 \exp \left(N_t \log \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0)t\right)}.
\]
Clearly, $\pi^*$ is an $\mathcal{F}_t^X$-adapted cadlag (right-continuous with left limit) process. We define a new process by
\[
\pi_t = \pi^*_t - \pi_0.
\]
Thus $\pi_t$ is left continuous then $\mathcal{F}_t^X$-predictable. Next we find the predictable $\mathcal{F}_t^X$-intensity of $N$ using the definition of point process with stochastic intensity, compare [2].

**Definition 1.** Let $N_t$ be a point process adapted to some history $\mathcal{F}_t$, and let $\lambda_t$ be a nonnegative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$
\[
\int_0^t \lambda_s ds < \infty \quad P\text{-a.s.}
\]
If for all nonnegative $\mathcal{F}_t$-predictable processes $C_t$, the equality
\[
E[\int_0^\infty C_s dN_s] = E[\int_0^\infty C_s \lambda_s ds]
\]
is verified, then $N_t$ admits the $\mathcal{F}_t$-intensity $\lambda_t$.

**Claim 1.** The point process $N_t$ admits the predictable $\mathcal{F}_t^X$-intensity $\lambda_t =: \lambda_0 + (\lambda_1 - \lambda_0)\pi_t$.

**Proof.** It is easy to see that $\lambda_t$ is nonnegative and it satisfies
\[
\int_0^t \lambda_s ds \leq \int_0^t \lambda_1 ds < \infty, \quad P\text{-a.s.}
\]
for all $t \geq 0$. Since the process $\lambda_t$ is $\mathcal{F}_t^X$-adapted and left-continuous, it is $\mathcal{F}_t$-progressive according to T33 in [2]. By the definition above, it remains to prove that
\[
E[\int_0^\infty C_t (\lambda_0 + (\lambda_1 - \lambda_0)\pi_t) dt] = E[\int_0^\infty C_t dN_t]
\]
for every non-negative $\mathcal{F}_t^X$-predictable process $C_t$. 
The right hand side satisfies
\[
E\left[ \int_{0}^{\infty} C_t dN_t \right] = E\left[ \int_{0}^{\infty} C_t (I(\lambda = \lambda_1)N_t^{\lambda_1} + I(\lambda = \lambda_0)N_t^{\lambda_0}) \right]
\]
\[
= E\left[ \int_{0}^{\infty} C_t (I(\lambda = \lambda_1)\lambda_1 dt + I(\lambda = \lambda_0)\lambda_0 dt) \right]
\]
\[
= E\left[ \int_{0}^{\infty} C_t E\left[ (I(\lambda = \lambda_1)\lambda_1 + I(\lambda = \lambda_0)\lambda_0)|\mathcal{F}_t^X \right] dt \right]
\]
\[
= E\left[ \int_{0}^{\infty} C_t (\pi_t \lambda_1 + (1 - \pi_t)\lambda_0) dt \right],
\]
where the second last equality is obtained by taking conditional expectation on \( \mathcal{F}_t^X \) and using Fubini’s theorem. The predictability of \( \lambda_t \) follows from \( \pi_t \). The proof is complete. \( \square \)

Define a new process \( L \) by
\[
(4) \quad dL_t = L_t - h_t \{ dN_t - (\pi_t \lambda_1 + (1 - \pi_t)\lambda_0) dt \} \quad \text{and} \quad L_0 = 1,
\]
where
\[
h_t = \frac{(\lambda_0 - \lambda_1)\pi_t}{\pi_t \lambda_1 + (1 - \pi_t)\lambda_0} > -1.
\]
Note that \( L \) is a \( P \)-martingale. Define a new measure \( \hat{P} \) by
\[
\frac{d\hat{P}}{dP} = L_T \quad \text{on} \quad \mathcal{F}_T.
\]
By the Girsanov theorem for point process with stochastic intensity (compare T2, T3 on Chapter VI in [2]), one can show that under the measure \( \hat{P} \) the intensity of the point process \( N \) is given by
\[
\hat{\lambda} = (\pi_t \lambda_1 + (1 - \pi_t)\lambda_0)(1 + h_t) = \lambda_0.
\]
Define a likelihood ratio \( \phi \) by
\[
\phi_t = \frac{\pi_t^*}{1 - \pi_t^*} \quad \text{and} \quad \phi_0 = \frac{\pi_0}{1 - \pi_0}.
\]
Define a new process \( \eta \) by
\[
(5) \quad \eta_t = \frac{1 + \phi_t}{1 + \phi_0} = (1 - \pi_0)(1 + \phi_t).
\]
Recall the fact that \( \pi^* \) is cadlag, it follows that both \( \phi \) and \( \eta \) are cadlag. The following claim shows that the Radon-Nikodym derivative \( L \) can be expressed in terms of \( \eta \), which plays an important role in the process of simplifying the optimal stopping problem. In the setting of two alternative drifts for geometric Brownian motion, a similar trick can be performed, compare [9] and [6].

Claim 2.
\[
(6) \quad \eta_t = \frac{1}{L_t}.
\]
Proof. By the definition of $\phi$ and Itô’s formula, the dynamics of $\phi$ is given by

\begin{equation}
    d\phi_t = \frac{\lambda_1 - \lambda_0}{\lambda_0} \phi_t d(N_t - \lambda_0 t).
\end{equation}

Using (5) and (7) one can show that

\begin{equation}
    d\eta_t = (1 - \pi_0) d\phi_t = (\eta_t + \pi_0 - 1) \frac{\lambda_1 - \lambda_0}{\lambda_0} d(N_t - \lambda_0 t)
    = \pi_0 \eta_t \frac{\lambda_1 - \lambda_0}{\lambda_0} d(N_t - \lambda_0 t).
\end{equation}

Applying Itô’s formula to $1/L_t$ yields

\begin{equation}
    \frac{1}{L_t} \lambda_1 - \lambda_0 \lambda_0 \pi_0 d(N_t - \lambda_0 t).
\end{equation}

Comparing (8) and (9) and using $\eta_0 = L_0 = 1$, the equality (6) is attained.

Let $\tilde{E}$ denote the expectation taken under the measure $\tilde{P}$. We have

\begin{align*}
    (1 + \phi_0)V &= \sup_{\tau \in \mathbb{R}_0, T} E[e^{-r\tau}(1 + \phi_0)X_T] \\
    &= \sup_{\tau \in \mathbb{R}_0, T} \tilde{E}[e^{-r\tau}\eta_T(1 + \phi_0)X_T] \\
    &= \sup_{\tau \in \mathbb{R}_0, T} \tilde{E}[e^{-r\tau}\eta_T(1 + \phi_0)X_T] \\
    &= \sup_{\tau \in \mathbb{R}_0, T} \tilde{E}[e^{-r\tau}(1 + \phi_\tau)X_T],
\end{align*}

where the second equality follows from Claim 2, the third equality is obtained by taking conditional expectation on $\mathcal{F}_T^X$ as well as using the martingale property of $\eta$ under $\tilde{P}$ and last equality follows from (5).

Define

\[ \tilde{\lambda} := \int_R e^x \nu^0(dx) \quad \text{and} \quad \tilde{f}(dx) := \frac{e^x f(dx)}{\int_R e^x f(dx)}. \]

Notice that $\tilde{\lambda}$ is well defined as a result of assumption (3). It is easy to check that $\tilde{f}(dx)$ is a distribution function. Let $\tilde{\nu}$ be the Levy measure of compound Poisson processes with intensity $\tilde{\lambda}$ and jump size distribution $\tilde{f}(x)$. Define a new process $Z$ by

\begin{align*}
    Z_t &= \exp \left\{ -\frac{1}{2} \sigma^2 t + \sigma W_t + \sum_{i=1}^{N_t} \ln \frac{d\tilde{\nu}}{d\nu^0}(Y_i) + (\lambda_0 - \tilde{\lambda}) t \right\} \\
    &= \exp \left\{ -\frac{1}{2} \sigma^2 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i + (\lambda_0 - \tilde{\lambda}) t \right\}.
\end{align*}
It follows that the price process $X$ can be written in terms of $Z$

$$X_t = xZ_t \exp \left\{ \frac{1}{2}\sigma^2 t + \mu t + (\tilde{\lambda} - \lambda_0)t \right\}. \tag{10}$$

Define a new measure $\tilde{P}$ by

$$\frac{d\tilde{P}}{d\hat{P}} = Z_T \text{ on } \mathcal{F}_T.$$ 

Using (10) and performing a measure transformation, we get

$$V(x) = \sup_{\tau \in \tau_0,T} \tilde{E}\left[ e^{-r\tau} (1 + \phi_\tau) \exp \left\{ \frac{1}{2}\sigma^2 \tau + \mu \tau + (\tilde{\lambda} - \lambda_0)\tau \right\} Z_\tau \right].$$

According to the Girsanov theorem for compound Poisson process (compare Proposition 9.6 in [3]), under the measure $\tilde{P}$ the jump intensity of $N$ is $\tilde{\lambda}$, the jump sizes have the distribution $\tilde{f}$ and the process $(W_t - \sigma t)_{t \geq 0}$ is a Brownian motion.

In view of this, we introduce the auxiliary optimal stopping problem

$$U(t, z) = \sup_{\tau \leq T-t} \tilde{E}\left[ e^{(R_0 - r)\tau} (1 + Z_\tau) \right], \tag{11}$$

where the supremum is taken over all $\mathcal{F}^N$-stopping times $\tau$ and

$$dZ_t = \frac{\lambda_1 - \lambda_0}{\lambda_0} Z_t (dN_t - \lambda_0 dt) \quad \text{and} \quad Z_0 = z.$$ 

Clearly, $\mathcal{F}^N \subset \mathcal{F}^X$ and only the observation of the point process $N$ is useful for solving the auxiliary problem. Easy calculations give

$$dY_t := d\exp((R_0 - r)t) Z_t = (R_1 - r)Y_t dt + \frac{\lambda_1 - \lambda_0}{\lambda_0} (dN_t - \tilde{\lambda} dt)$$

Assumption (3) thus implies that $\exp((R_0 - r)t) (1 + Z_t)$ is neither a supermartingale nor a submartingale. Therefore the auxiliary problem is not trivial. Note that the auxiliary problem can easily translate to the original problem (2) by

$$V(x) = \frac{xU(0, \phi_0)}{1 + \phi_0}$$

and the optimal stopping times for the two problems can be translated too.

In the next section we show the existence of an optimal stopping boundary of the auxiliary problem denoted by $b(t)$, such that the stopping time

$$\tilde{\tau}_{t,z} := \inf\{v \in [0, T-t] : Z_v \leq (\geq)b(t+v) \} \wedge (T-t)$$

is an optimal stopping time for $r > (\leq)R_0$. Thus we have the following result.
Theorem 1. For $r > (\leq) R_0$, the stopping time

$$\tau = \inf \{ t : N_t \leq (\geq) \frac{\ln b(t) - \ln \phi_0 + (\lambda_1 - \lambda_0)t}{\ln \lambda_1 - \ln \lambda_0} \} \land T$$

attains the supremum in (2).

Hence, to find the optimal stopping time, one only needs to observe the trajectory of the point process $N$ and ignore the continuous fluctuation of $X$.

3. The Auxiliary Problem

In this section we study the auxiliary optimal stopping problem (11). We prove the existence of a monotone and right-continuous optimal stopping boundary, and we show that the value function is the unique viscosity solution of the pricing equations. Recall the auxiliary problem

$$U(t, z) = \sup_{\tau \leq T-t} \tilde{E}[e^{(r-R_0)\tau} (1 + zH_\tau)]$$

where

$$dH_t = \frac{\lambda_1 - \lambda_0}{\lambda_0} H_t (dN_t - \lambda_0 dt) \quad \text{and} \quad H_0 = 1.$$ 

It is easy to see that $U(t, z) \geq G(z) := 1 + z$ by letting $\tau = 0$. Define the continuation region $C$ by

$$C = \{(t, \phi) \in [0, T) \times (0, \infty) : U(t, z) > G(z)\}$$

and the stopping region $D$ by

$$D = \{(t, \phi) \in [0, T] \times (0, \infty) : U(t, z) = G(z)\}.$$ 

Define $F(z) := \mathcal{L}G(z) + (R_0 - r)G(z)$, where

$$\mathcal{L}G(z) := (\lambda_0 - \lambda_1)zG_z(z) + \dot{\lambda}(G(z + \frac{\lambda_1 - \lambda_0}{\lambda_0} z) - G(z))$$

is the infinitesimal operator of $z$. For simplicity of notation, let

$$b^* := (r - R_0)/(R_1 - r).$$

Using assumption (3), simple calculations show that

$$F(z) \begin{cases} 
\geq 0 & \text{if } z \geq b^* \\
< 0 & \text{if } z < b^*
\end{cases} \quad \text{and} \quad F(z) \begin{cases} 
\geq 0 & \text{if } z \leq b^* \\
< 0 & \text{if } z > b^*
\end{cases},$$

for $R_0 < r$ and $R_0 > r$, respectively. For $R_0 < r$, applying Itô’s formula and using (12) shows that $e^{(R_0-r)s}G(Z_s)$ is a submartingale for $s \leq \inf\{u : Z_u < b^*\}$. Therefore, all points $(z, t)$ satisfying $z > b^*$ belong to the continuation region $C$. A similar result for $R_0 > r$ can be obtained easily, i.e., all points $(z, t)$ with $z < b^*$ belong to $C$. 
Proposition 1. For $R_0 < r$, there exists a non-decreasing and right-continuous function $b : [0, T) \to [0, b^*]$ such that
\[ C = \{(t, z) \in [0, T) \times (0, \infty) : z > b(t)\}. \]
For $R_0 > r$, there exists a non-increasing and right-continuous function $b : [0, T) \to [b^*, \infty]$ such that
\[ C = \{(t, z) \in [0, T) \times (0, \infty) : z < b(t)\}. \]

Proof. In the case of $R_0 < r$, for some fixed $t \in [0, T)$ and $z' > z > 0$, suppose that $(t, z')$ is in the continuation region $C$. Then there exists a stopping time $\tau$ such that
\[ \tilde{E}[e^{(R_0 - r)\tau}(1 + zH_\tau)] > 1 + z. \]
We also have
\[ U(t, z') \geq \tilde{E}[e^{(R_0 - r)\tau}(1 + z'H_\tau)] = \tilde{E}[e^{(R_0 - r)\tau}(1 + zH_\tau)] + (z' - z)\tilde{E}[e^{(R_0 - r)\tau}H_\tau] \geq 1 + z + (z' - z)\tilde{E}[e^{(R_0 - r)\tau}H_\tau] \geq 1 + z + z' - z = 1 + z', \]
where the last inequality follows from the submartingale property of $e^{(R_0 - r)t}H_t$ and the optional sampling theorem.

Thus, $(t, z')$ also belongs to $C$, which proves the existence of a function $b : [0, T) \to [0, \infty]$ such that $C = \{(t, z) : z > b(t)\}$. The fact that $b$ only takes values smaller than $b^*$ follows from the discussion above the Proposition, and the monotonicity of $b$ follows from the monotonicity of $t \mapsto U(t, z)$. Since $C$ is an open set, the boundary $b(t)$ is right-continuous. The second part of the theorem can be proved in a similar way, so the proof is omitted. \Box

In order to show that the value function $U(t, z)$ is a classical solution of the pricing equation, one needs to prove that $U$ is continuously differentiable with respect to both variables. However, in the case of American type of option with a degenerate diffusion coefficient, there is no guarantee that the $C^{1,1}$-regularity holds, compare (page 401) [3]. Instead, the weaker notion viscosity solution is used. The following proposition is a direct result of Proposition 3.3 and Theorem 3.1 in [12].

Proposition 2. The function $U : [0, T] \times (0, \infty) \to (0, \infty)$ is continuous and it is the unique viscosity solution of:
\[ \begin{cases} 
\min \{-U_t(t, z) - LU(t, z) + (r - R_0)U(t, z), U(t, z) - (1 + z)\} = 0 \\
U(T, z) = 1 + z, 
\end{cases} \]
where
\[ LU(t, z) = -(\lambda_1 - \lambda_0)zU_z(t, z) + \tilde{\lambda}[U(t, \frac{\lambda_1}{\lambda_0}z) - U(t, z)]. \]
The uniqueness of the viscosity solution is proved by Soner [7] and Sayah [1] using the comparison principles. A very similar result for American put options is shown in Proposition 12.9 in [3].

4. The Merton Model

In this section we let the jump diffusion model be the Merton model and find the optimal stopping boundary.

For the Merton model, we assume that the jump sizes are normally distributed with mean $\alpha$ and variance $\rho^2$, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi\rho^2}} \int_{-\infty}^{x} e^{-\frac{(y-\alpha)^2}{2\rho^2}} dy.$$  

For simplicity of notation, let $\beta = \alpha + \rho^2/2$. Easy calculations show that the intensity of $N$ is $\hat{\lambda} = e^\beta \lambda_0$ and

$$R_i = \mu + \frac{1}{2} \sigma^2 + \lambda_i (e^\beta - 1) \quad \text{for} \quad i \in \{0, 1\}.$$  

**Remark 1.** Notice that optimal stopping problems with different jump diffusion models can have the same optimal liquidation strategy.

The inequality $\beta > 0$ implies that $\hat{\lambda} > \lambda_0$. By the assumption 3, it must satisfy that $R_0 < r$, then according to Proposition 1 the optimal stopping time is the first time the point process $N$ falls below a certain time-dependent boundary. When $\beta < 0$, we get $\hat{\lambda} < \lambda_0$ and $R_0 > r$, thus the optimal stopping time is the first time $N$ goes above a certain time-dependent boundary.

![Figure 1](image-url)  
**Figure 1.** The optimal stopping boundary of the problem (2) with the Merton model, a simulated path of the point process $N$, and the optimal stopping time $\tau$. The values of the parameters are $r = 0.04$, $\mu = -1$, $\sigma = 0.2$, $\lambda_0 = 2$, $\lambda_1 = 4$, $\alpha = 0.2$, $\rho = 0.4$, $T = 2$, $\pi_0 = 0.375$ and $\beta = 0.28$.  

References


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