Determination of a gravimetric geoid model of Kazakhstan using the KTH-Method

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To the memory of my dear friend

Arman Makhmudov

(1986-2008)

“Your golden heart stopped beating,
hard working hands put to rest.
God broke our hearts to prove to us,
He only takes the best...”
Abstract

This study work deals with the determination of the gravimetric geoid model for Kazakhstan by using the KTH-method.

A number of data sets were collected for this work, such as the gravity anomalies, high-resolution Digital Elevation Model (DEM), Global Geopotential Models (GGMs) and GPS/Levelling data. These data has been optimally combined through the KTH approach, developed at the Royal Institute of Technology (KTH) in Stockholm. According to this stochastic method, Stokes' formula is being used with the original surface gravity anomaly, which combine with a GGM yields approximate geoid heights. The corrected geoid heights are then obtained by adding the topographic, downward continuation, atmospheric and ellipsoidal corrections to the approximate geoid heights.

To compute the geoid model for Kazakhstan as accurately as possible with available data set different numerical tests have been performed:

- Choice of the best fit geopotential model in the computation area
- Investigations for the best choice of the initial condition for determination of the least-squares parameters
- Selection of the best parametric model for reducing the effect of the systematic error and data inconsistencies between computed geoid heights and GPS/Levelling heights.

Finally, 5' x 5' Kazakh gravimetric geoid (KazGM2010) in the area 40° ≤ φ ≤ 56° and 46° ≤ λ ≤ 88° has been modelled. The differences between the geoid undulations from GPS/Levelling and KazGM2010 change from −0.295 m to 0.327 m and the standard deviation is 0.177 m. The relative accuracy achieved was of the order of 1.219 ppm for baselines between 15 and 1633 kilometers.

**Key words:** geoid model, least-squares modification of Stokes' formula, the KTH-method, GPS/ Levelling, Kazakhstan
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### Abbreviations

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<th>AC</th>
<th>Additive corrections</th>
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<tr>
<td>B</td>
<td>BVP</td>
<td>Boundary Value Problem</td>
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<td>C</td>
<td>CGIAR-CSI</td>
<td>Consortium for Spatial Information of the Consultative Group for International Agricultural Research</td>
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<tr>
<td></td>
<td>CHAMP</td>
<td>CHAllenging Minisatellite Payload</td>
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<td>D</td>
<td>DEM</td>
<td>Digital Elevation Model</td>
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<td>DWC</td>
<td>Downward Continuation</td>
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<td>G</td>
<td>GETECH</td>
<td>Geophysical Exploration Technology</td>
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<td></td>
<td>GGM</td>
<td>Global Gravitional Model</td>
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<td></td>
<td>GNSS</td>
<td>Global Navigation Satellite System</td>
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<td></td>
<td>GOCE</td>
<td>Gravity field and Steady State Ocean Circulation Exporter</td>
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<td></td>
<td>GPS</td>
<td>Global Positioning System</td>
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<td></td>
<td>GRACE</td>
<td>Gravity Recovery and Climate Experiment</td>
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<td></td>
<td>GRS80</td>
<td>Geodetic Reference System 1980</td>
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<td>International Centre for Global Earth Models</td>
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<td>L</td>
<td>LS</td>
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<td>LSM</td>
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<td>MSE</td>
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<td>MSL</td>
<td>Mean Sea Level</td>
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<td>S</td>
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<td>Shuttle Radar Topography Mission</td>
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<td>Standard Deviation</td>
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<td>T-TLS</td>
<td>Truncated Total Least Squares</td>
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<td>USA</td>
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<td>USGS</td>
<td>USA Geological Survey</td>
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Chapter 1

Introduction

The geoid is the equipotential surface which coincides on average with the Mean Sea Level (MSL) of the Earth. According to C.F. Gauss, who first described it, it is the "mathematical figure of the Earth". The surface of the geoid is irregular, unlike the reference ellipsoid which is a mathematical idealized representation of the physical Earth, but considerably smoother than Earth's physical surface. Determination of the geoid model requires extensive gravity measurements and calculations.

![Figure 1.1](image.png)

**Figure 1.1**. The Earth surface, ellipsoid and geoid

**Geoid determination is very important, for various reasons:**

- Knowledge on the gravity field of the Earth is important for geodetic reference systems;
- Gravity field information is necessary for description of the positions of satellites and ground stations in suitable reference frames. The fact that the geoid reflects gravity field irregularities means that, a better understanding of the geoid enables refinement of satellite orbits.
- The geoid is valuable for oceanographers, hydrographic surveyors and maritime industries in general as vessels navigate vast bodies of water. The knowledge of the geoid is essential to better model ocean currents and undersea mapping especially in soundings. Sea-going vessels can take advantage of the currents and characteristics of the ocean to plan faster and safer routes, which in turn will use less fuel (conservation) and cost less.
- Knowledge of the geoid is important to model geodynamical phenomena (e.g., polar motion, Earth rotation, crustal deformation). Geoid is useful in interpretation of precursors to geo-hazards researches such as the study of...
post-glacial rebound, earthquakes, volcanoes, landslides, tsunamis etc. and mitigation.
- The fluctuations in the gravity field created by varying density of matter could be a valuable source of information for locating precious natural resources, such as ore deposits, oil, gas, etc.

The list above is not exhaustive by any means.

As it was mentioned above, the geoid plays an important role in the geodetic infrastructure, since it serves as the reference surface of other measurements and phenomena. Thus, many applications in geodesy, geophysics, oceanography and engineering require geoid-related heights. Traditionally, spirit levelling has been applied for accurate height determination. But the levelling survey is slow, expensive and labour intensive process. This is especially evident in large countries like Kazakhstan, its territory of 2,727,300 km² is greater than Western Europe. The establishment of a high resolution levelling network covering all parts of the country would be impractical from the financial point of view. Moreover levelling over areas with rough terrain, like the Ural Mountains in west and Tian - Shan, Alatau, Altay mountain ranges in east and south or central deserts of Kazakhstan, is very strenuous and time consuming.

On the other hand, the combined use of GNSS and geoid heights presents an alternative potential to the classic geometric levelling.

The heights directly obtained by GNSS measurements are geodetic heights that refer to the reference ellipsoid. These heights are fundamentally different from traditionally obtained heights which are given with respect to the geoid. Mathematically, there is a simple relation between these two heights and it can be expressed by the following equation:

\[ H = h - N \] (1.1.1)

In practice, the Eq. (1.1.1) reflects the possibility of GNSS-Levelling, because it states that if the geoidal height \( N \) is known, the orthometric height \( H \) can be obtained from ellipsoidal height \( h \) determined by GNSS. Obtaining orthometric heights by this way could in certain circumstances, depending on the required accuracy, replace conventional spirit levelling. Thus make the levelling procedure cheaper and faster which is necessary for both practical surveying and scientific applications.

To this extend many studies have been performed in different places all over the world using various methods and as examples we mention Schwarz, Sideris and Forsberg (1987), Véronneau (2000), Erol and Çelik (2004), among others.

The theoretical and practical aspects of the KTH-method are developed mainly by Lars E. Sjöberg since 1984. This method has proven to be the most viable option for
precise gravimetric geoid model determination as it is gradually developed and verified in Fan (1989), Nsombo (1996), Nahavandchi (1998), Hunegnaw (2001), Ellmann (2004), Ågren (2004), Kiamehr (2006), Daras (2008), Ulotu (2009) and many others. In all study cases the KTH-method gives good results due to the well-known potential of the LSM kernel, which matches the errors of the terrestrial gravity data, GGM and the truncation error in an optimum way. From the practical point of view, the use of precise additive correction terms, the KTH approach is accurate, simple and computationally efficient.

1.1 Research work of this thesis

The main purpose of this study is to determine the gravimetric geoid model for Kazakhstan by using the KTH - method which combines different heterogeneous data in an optimum way.

In this study work the following subjects are highlighted:

- Collecting the necessary database for the computation (terrestrial gravity data set, DEM, GGMs and the GPS/ Levelling points)
- Choice of the best fit geopotential model in the computation area
- Investigations for the best choice of the initial condition for determination of the least-squares parameters
- Computation of the geoid undulations using the KTH-method
- Reducing the effect of the systematic error and data inconsistencies between computed geoid heights and GPS/ Levelling heights.
- Evaluation of the gravimetric geoid model with GPS/ Levelling data.
Chapter 2

The KTH - method

Another name of the KTH - method that is often used is the Least Squares Modification of Stokes' formula (LSMS) with additive corrections (AC). Several different versions of this method have been modified and presented through the years. The aim of LSMS is to reduce in a least squares sense, errors of geoid height springing from truncation, terrestrial gravity and GGM potential coefficients.

2.1 The Least Squares Modification of Stokes' formula

The original Stokes' formula for gravimetric geoid is:

$$N = \frac{R}{4\pi\gamma} \int_\sigma S(\psi) \Delta g d\sigma$$  \hspace{1cm} (2.1.1)

where $R$ – mean radius of the Earth;
$\gamma$ – normal gravity on the reference ellipsoid;
$\psi$ – geocentric angle;
$\Delta g$ – gravity anomaly on the geoid;
$d\sigma$ – infinitesimal surface element of integration over a unit sphere $\sigma$;
$S(\psi)$ – Stokes' function.

Stokes' function can be expressed as a series of Legendre polynomials:

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n + 1}{n - 1} P_n(\cos \psi)$$  \hspace{1cm} (2.1.2)

If we introduce a new variable $t = \cos \psi$, then Stokes' function can be also written in a closed form:

$$S(t) = \sqrt{\frac{2}{1 - t}} - 6 \sqrt{\frac{1 - t}{2}} + 1 - 5t - 3 \ln \left( \sqrt{\frac{1 - t}{2}} + \frac{1 - t}{2} \right)$$  \hspace{1cm} (2.1.3)
The facts are:

- Prerequisites and conditions for the original Stokes' method difficult to implement;
- Terrestrial gravity data usually contains long-wavelength systematic errors;
- Gravity data from dedicated satellite gravity missions like CHAMP and GRACE, has global coverage and small commission and omission errors;
- There exist dense and accurate global DEM models;
- Global gravity anomaly signal and error degree variance models which have given reliable results exist.

The above facts require a kernel modification, which could optimize in an optimum way, all deterministic and stochastic error sources, without prejudice to the basic assumptions and conditions set out by the Stokes' method. Thereby the KTH LSMS method aims at a kernel which confirms the following:

- Filter out the often erroneous long-wavelength component of gravity from terrestrial gravity data and replace it with one from the GGM;
- Admit the presence of terrestrial and GGM gravity data errors and account for them; this includes the restriction of GGM to degree M;
- Minimize bias due to truncation of integration to a cap by using the GGMs and global gravity anomaly error models;
- Minimum *mean square error* (MSE) of computed geoid height.

By taking into account the above facts and that the observed terrestrial gravity anomaly $\Delta \hat{g}$ is limited to a cap $\sigma_0$, with corresponding geocentric angle $\psi_0$, the GGM gravity anomaly $\Delta g_n^{GGM}$ is known up to degree and order $M$, the modified Stokes' method provides approximate geoid height $\tilde{N}$:

$$\tilde{N} = \frac{c}{2\pi} \int_{\sigma_0} S^L(\psi) \Delta g d\sigma + c \sum_{n=2}^{M} b_n \Delta g_n^{GGM}$$  \hspace{1cm} (2.1.4)

where $c = \frac{R}{2\gamma}$
- $S^L(\psi)$ – modified Stokes' function;
- $b_n$ – modification parameters;
- $L$ – degree of the kernel modification.
The modified Stokes’ function is expressed as:

$$S^L(\psi) = S(\psi) - \sum_{n=2}^{L} \frac{2n + 1}{2} s_n P_n(\cos \psi) \quad L \geq M$$

(2.1.5)

where $s_n$ - modification parameters with the assumption that $s_1$ and $s_0$ are zero.

$$b_n = (Q_n^L + s_n^*) \frac{c_n}{c_n + d c_n} \quad \text{for} \quad 2 \leq n \leq M$$

(2.1.6)

The truncation coefficients are further expressed as:

$$Q_n^L = Q_n - \sum_{k=2}^{L} \frac{2k + 1}{2} s_k e_{nk}$$

(2.1.7)

where $Q_n$ - Molodensky truncation coefficients:

$$Q_n = \int_{\psi_0}^{\pi} S(\psi) P_n(\cos \psi) \sin(\psi) d\psi$$

(2.1.8)

and $e_{nk}$ - functions of $\psi_0$ called Paul’s coefficients:

$$e_{nk}(\psi_0) = \int_{\psi_0}^{\pi} P_n(\cos \psi) P_k(\cos \psi) \sin(\psi) d\psi$$

(2.1.9)

The GGM gravity anomaly:

$$\Delta g_n^{GGM} = \frac{G M}{a^2} \left(\frac{a}{r}\right)^{n+2} (n - 1) \sum_{m=-n}^{n} C_{nm}Y_{nm} (\bar{\phi}, \lambda)$$

(2.1.10)

where $a$ - equatorial radius of the reference ellipsoid;
$r$ - geocentric radius of the computation point;
$G M$ - adopted geocentric gravitational constant;
$C_{nm}$ - fully normalized spherical harmonic coefficients of the disturbing
potential provided by GGM;
\( Y_{nm} \) — fully normalized spherical harmonics.

By utilizing the error estimates of the data, and some approximations (both theoretical and computational), we arrive at an estimate of the geoid height that we call the approximate geoidal height, which can be written in the following spectral form:

\[
\bar{N} = c \sum_{n=2}^{\infty} \left( \frac{2}{n-1} - \frac{Q_n^L - S_n^*}{n} \right) (\Delta g_n + \varepsilon_n^T) + c \sum_{n=2}^{M} (Q_n^L + S_n^*) (\Delta g_n + \varepsilon_n^S) \tag{2.1.11}
\]

where \( \varepsilon_n^T, \varepsilon_n^S \) — spectral errors of the terrestrial and GGM derived gravity anomalies, respectively.

The modification parameters are:

\[
S_n^* = \begin{cases} 
S_n & \text{if } 2 \leq n \leq L \\
0 & \text{if } n > L 
\end{cases} \tag{2.1.12}
\]

Based on the spectral form of the “true” geoidal undulation \( N \):

\[
N = c \sum_{n=2}^{\infty} \frac{2 \Delta g_n}{n - 1} \tag{2.1.13}
\]

The expected global MSE of the geoid estimator \( \bar{N} \) can be written as:

\[
m_N^2 = E \left\{ \frac{1}{4\pi} \int_{\sigma} (\bar{N} - N)^2 d\sigma \right\} \tag{2.1.14}
\]

Or

\[
m_N^2 = c^2 \left\{ \sum_{n=2}^{M} (b_n^2 c_n) + \sum_{n=2}^{\infty} [b_n^2 - Q_n^L(\psi_0) - S_n^*]^2 c_n + \sum_{n=2}^{\infty} \left[ \frac{2}{n-1} - Q_n^L(\psi_0) - S_n^* \right]^2 \sigma_n^2 \right\} \tag{2.1.15}
\]

where \( E\{\} \) — statistical expectation operator;
\( c_n \) — gravity anomaly degree variances;
\( \sigma_n^2 \) — terrestrial gravity anomaly error degree variances;
\( d_c n \) — GGM derived gravity anomaly error degree variances.
\[ b^*_n = \begin{cases} b_n & \text{if } 2 \leq n \leq L \\ 0 & \text{otherwise} \end{cases} \quad (2.1.16) \]

The gravity anomaly degree variance \( c_n \) is also known as the power spectrum component:

\[ c_n = \frac{1}{4\pi} \int_\sigma \Delta g_n^2 d\sigma \quad (2.1.17) \]

Whereas the terrestrial gravity anomaly error degree variance \( \sigma_n^2 \) and GGM-derived anomaly error degree variance \( dc_n \) are computed by:

\[ \sigma_n^2 = E \left\{ \frac{1}{4\pi} \int_\sigma (\varepsilon_n^T)^2 d\sigma \right\} \quad (2.1.18) \]

\[ dc_n = E \left\{ \frac{1}{4\pi} \int_\sigma (\varepsilon_n^S)^2 d\sigma \right\} \quad (2.1.19) \]

where \( \varepsilon_n^T \) – spectral error of the terrestrial gravity anomaly.

\( \varepsilon_n^S \) – spectral error of the GGM-derived gravity anomaly.

Since the true values of the error components are unknown, their estimation could be based on some standard approaches and stochastic models. The stochastic models used in this study will be discussed in Chapter 4.

The key factor to minimize the global MSE and to reduce all relevant errors in geoid modelling is a suitable selection of the LS parameters. For obtaining the LSM parameters Eq. (2.1.15) is differentiated with respect to \( s_n \). The resulting expression is then equated to zero and the modification parameters \( s_n \) are solved in the least squares sense form the linear system of equations:

\[ \sum_{r=2}^{L} a_{kr} s_r = h_k, \quad k = 2,3,\ldots,L \quad (2.1.20) \]

where \( a_{kr} \), \( h_k \) – modification coefficients.
\[
a_{kr} = d_k \delta_{kr} - \frac{2r + 1}{2} d_k e_{kr} - \frac{2k + 1}{2} d_k e_{kr} + \frac{2k + 1}{2} \frac{1}{12}\sum_{n=2}^{\infty} e_{nk} e_{nr} d_n \quad (2.1.21)
\]

\[
h_k = \frac{2\sigma_k^2}{k - 1} - Q_k d_k + \frac{2k + 1}{2} \sum_{n=2}^{\infty} \left( Q_n e_{nk} d_n - \frac{2}{n - 1} e_{nk} \sigma_n^2 \right) \quad (2.1.22)
\]

where

\[
d_n = \sigma_n^2 d c_n^* \quad , \quad d c_n^* = \begin{cases} 
    dc_n & \text{if } 2 \leq n \leq L \\
    c_n & \text{if } n > L
\end{cases}
\quad \text{and} \quad \delta_{kr} = \begin{cases} 
    1 & \text{if } k = r \\
    0 & \text{if } k \neq r
\end{cases}
\]

The system of equations in Eq. (2.1.20) involves inversion of the matrix \( [a_{kr}] \), which becomes ill-conditioned with increasing size \( l \). This calls for the application of mathematical regularisation procedure. Investigations made by Ellmann (2003) and Ågren (2004) have shown that while not all regularization methods work well with the inversion, but Singular Value Decomposition (SVD) technique is more effective and efficient.

For the purposes of this study, a specially designed MATLAB program made by Ellmann (2005) was used to obtain the LS parameters \( s_n, b_n \) of the optimum LSM method by Sjöberg (2003c).

### 2.2 Additive corrections

The application of Stokes' formula for the computation of the geoid necessitates that there are no masses outside the geoid. This simply means that there are no masses outside the geoid surface. However, those criteria are not fulfilled, due to the presence of topography and atmospheric masses above the geoid surface. External masses must be moved inside the geoid or totally removed (direct effect). The masses are then restored after applying Stokes' integral (indirect effect). Geoid determination by Stokes' formula also requests that gravity anomalies \( \Delta g \) have to refer to the geoid. For satisfying this second condition, as the gravity anomalies \( \Delta g \) are available on the surface of the Earth, reducing them from the surface of the Earth to the geoid in order (DWC).

In the KTH computational approach for geoid determination (Sjöberg 2003b) on the surface, gravity anomalies and GGM are used to determine the approximate geoid height, then all corrections are added to separately:
\[ \tilde{N} = \tilde{N} + AC \]  \hspace{1cm} (2.2.1)

\[ AC = \delta N^\text{Topo}_{\text{tot}} + \delta N^\text{atm}_{\text{comb}} + \delta N_{\text{ell}} \]  \hspace{1cm} (2.2.2)

where \( \delta N^\text{Topo}_{\text{tot}} \) — total topographic effect, which includes topographic effect and DWC effect to the model;

\( \delta N^\text{atm}_{\text{comb}} \) — combined atmospheric correction, which includes the sum of the direct and indirect atmospheric effects;

\( \delta N_{\text{ell}} \) — ellipsoidal correction for the spherical approximation of the geoid in Stokes’ formula to ellipsoidal reference surface.

### 2.2.1 Total topographic effect on the geoid model \( \delta N^\text{Topo}_{\text{tot}} \)

Total topographic effect \( \delta N^\text{Topo}_{\text{tot}} \) is the sum of the combined topographic correction \( \delta N^\text{Topo}_{\text{comb}} \) and downward continuation effect \( \delta N_{\text{dwc}} \).

#### Combined topographical correction \( \delta N^\text{Topo}_{\text{comb}} \)

The combined topographical correction is the summation of direct and indirect topographical impacts on the geoid. This correction can be determined as follows:

\[ \delta N^\text{Topo}_{\text{comb}} = \delta N_{\text{dir}} + \delta N_{\text{indir}} \approx -\frac{2\pi G \rho}{\gamma} H^2 \]  \hspace{1cm} (2.2.3)

where \( \rho \) — mean topographic mass density (\( \rho = 2.67 \text{ g/cm}^3 \));

\( H \) — orthometric height.

By summation of direct and indirect effects the reduction effect mainly decreases. Moreover, the direct topographic effect which is usually affected by a significant terrain effect is cancelled in the combined effect on the geoid.

The combined topographic correction depends on the density of topographic masses. As Sjöberg (1994) highlighted, if the density of topographic masses ranges within 5%, the propagated geoid error could be as large as a few decimetres, globally. As an example, for areas below 1300 m the combined topographic correction is within 1 cm, hence for some areas the knowledge of topographical densities is not a problem. In this study a constant mass density \( \rho = 2.67 \text{ g/cm}^3 \) is considered because of the difficulties of obtaining reliable density information, as
normally a constant density is used in many traditional approaches. More accurate results can be obtained by Ellmann and Sjöberg (2002, Eq. 10 and 11), if lateral density variation of topographic masses is well known.

The Eq. (2.2.3) is very simple and computer efficient as it is valid with slopes of topography less than 45°. Due to the fact that rough surface gravity anomalies are integrated in the KTH approach, some important comments must be considered in using the method. Errors of Stokes' integration (discretisation error) when sampling the mean surface anomalies from gravity point data due to loss of shortwave-length information. These errors can be significantly reduced by using special interpolation technique, for more details see (Ågren 2004). Moreover a good DEM should be available with at least the same resolution as the interpolated grid.

The Downward Continuation correction

The Downward Continuation correction is an essential correction in application of Stokes’ formula for geoid estimation, because the observed surface gravity anomalies must be downward continued to the geoid when the topographic effect is reduced.

The DWC has been done by various methods, but the most commonly used method is the inversion of Poisson’s integral. According to this method the surface gravity anomaly for direct topographic effect is reduced and then continued downward to the sea level. Sjöberg (2003a) introduced a new method for DWC. This method avoids the downward continued gravity anomaly and considers directly the DWC effect on the geoidal height. Accordingly, the DWC effect on the geoidal height can be expressed as:

\[
\delta N_{dwc} = \frac{c}{2\pi} \int_{\sigma_0} S_L(\psi) (\Delta g^* - \Delta g) \, d\sigma
\]  

(2.2.4)

where \(\Delta g\) – gravity anomaly at the surface computation point \(P\);
\(\Delta g^*\) – corresponding quantity downward continued to the geoid.

The final formulas for Sjöberg’s DWC method for any point of interest based on the LSM parameters are as follows (for more details, see Ågren (2004)):

\[
\delta N_{dwc}(P) = \delta N_{dwc}^{(1)}(P) + \delta N_{dwc}^{L1,Far}(P) + \delta N_{dwc}^{L2}(P) 
\]  

(2.2.5)
where

$$\delta N_{dwc}^{(1)}(P) = \Delta g(P) \gamma H_r + 3 \frac{\xi_r^0}{r_p} H_r - \frac{1}{2\gamma} \frac{\partial \Delta g}{\partial r} \bigg|_P H_r^2$$  \hspace{1cm} (2.2.6)

where $\xi_r^0$ – approximate value of height anomaly.

Due to diminutive value of $\delta N_{dwc}(P) = 1 \ mm$ that corresponds to an error of 1 m for $H_r = 2 \ km$ and $r_p = 6375 \ km$, it is comfortable to adopt:

$$\xi_r^0 \approx \frac{c}{2\pi} \int_{\sigma_0} S^L(\psi) \Delta g d\sigma + c \sum_{n=2}^{M} (s_n + Q_n^L) \Delta g_n^{GGM}$$  \hspace{1cm} (2.2.7)

$$\delta N_{dwc}^{L_1, Far}(P) = c \sum_{n=2}^{M} (s_n^* + Q_n^L) \left( \left( \frac{R}{r_p} \right)^{n+2} - 1 \right) \Delta g_n(P)$$  \hspace{1cm} (2.2.8)

$$\delta N_{dwc}^{L_2}(P) = \frac{c}{2\pi} \int_{\sigma_0} S^L(\psi) \left( \frac{\partial \Delta g}{\partial r} \bigg|_P (H_r - H_Q) \right) d\sigma_Q$$  \hspace{1cm} (2.2.9)

where $r_p = R + H_r$;

$\sigma_0$ – spherical cap with radius $\psi_0$ centered around $P$ and it should be the same as in modified Stokes’ formula;

$H_r$ – ortometric height of point $P$;

$\frac{\partial \Delta g}{\partial r} \bigg|_P$ – gravity gradient in point $P$:

$$\frac{\partial \Delta g}{\partial r} \bigg|_P = \frac{R^2}{2\pi} \int_{\sigma_0} \frac{\Delta g_q - \Delta g_p}{l_0^3} d\sigma_q - \frac{2}{R} \Delta g(P)$$  \hspace{1cm} (2.2.10)

where $l_0 = 2R \sin \frac{\psi_p \rho}{2}$

Eq. (2.2.9) can be adequately treated in the same way of the evaluation of the modified Stokes’ integration.
2.2.2 The atmospheric correction

The atmospheric masses outside the geoid surface cannot be completely removed. Therefore the effect of the forbidden atmospheric masses should be considered and added as additional term to fulfill boundary condition in Stokes' formula. In the International Association of Geodesy (IAG) approach, the Earth is supposed as a sphere with spherical atmospheric ring and the topography of the Earth is not taken into account. Consequently some direct and indirect effects to gravity anomaly should be taken into account;

The application of IAG approach using a limited cap size especially in Stokes' formula can cause a very significant error in zero order term ($> 3m$) (Sjöberg (2006)). In the KTH - method, the combined atmospheric effect can be approximated to order $H$:

$$
\delta N_{\text{comb}}^{a}(P) = c^{*} \left\{ \sum_{n=2}^{M} \left( \frac{2}{n-1} - s_{n} - Q_{n}^{l} \right) H_{n}(P) + \sum_{n=M+1}^{\infty} \left( \frac{2}{n-1} - \frac{n+2}{2n+1} Q_{n}^{l} \right) H_{n}(P) \right\}
$$

... (2.2.11)

where $\rho_{0}$ - density at sea level $\rho_{0}^{0} (\rho_{0}^{0} = 1.23 \cdot 10^{-3} \text{g/cm}^{3})$ multiplied by the gravitational constant $G (G = 6.673 \cdot 10^{-11} \text{m}^{3}/\text{kg} \cdot \text{s}^{2})$;

$\gamma$ - mean normal gravity on the reference ellipsoid;

$c^{*} = -\frac{2\pi R \rho_{0}}{\gamma}$.

$H_{n}$ - Laplace harmonic of degree $n$ for the topographic height:

$$
H_{n}(P) = \sum_{m=-n}^{n} H_{nm} \bar{y}_{nm}(P)
$$

(2.2.12)

The elevation $H$ of the arbitrary power $v$ can be presented to any surface point with latitude and longitude $(\phi, \lambda)$ as follows:

$$
H^{v}(\phi, \lambda) = \sum_{m=0}^{\infty} \sum_{m=-n}^{n} H_{nm}^{v} Y_{nm}(\phi, \lambda)
$$

(2.2.13)

where $H_{nm}^{v}$ - normalized spherical harmonic coefficient of degree $n$ and order $m$.

It can be determined by the spherical harmonic analysis:
\[ H_{nm}^\nu = \frac{1}{4\pi} \int \int_{\sigma} H^\nu(\varphi, \lambda) Y_{nm}(\varphi, \lambda) \, d\sigma \]  

(2.2.14)

In this study we used the normalized spherical harmonic coefficients \( H_{nm}^\nu \) of degree and order 360, which were computed by Fan (1998).

### 2.2.3 The ellipsoidal correction

Stokes' formula is derived for a spherical boundary surface (geoid), but the geoid is closer to an ellipsoid than the sphere. Therefore the ellipsoidal correction has to be applied.

The deviation between the ellipsoid and the geoid is caused the relative error of 0.3% in geoid determination. This deviation is a consequence of geoid irregularities. Therefore for accurate geoid model it is important to estimate the ellipsoidal correction. Ellipsoidal correction has been studied by different authors through the years, e.g., Molodensky (1962), Moritz (1980), Fei and Siders (2000) and Sjöberg (2003d).

The ellipsoidal correction for the original and modified Stokes' formula is derived by Sjöberg (2003d) and Ellmann and Sjöberg (2004) in a series of spherical harmonics to the order of \( e^2 \), where is the first eccentricity of the reference ellipsoid. The approximate ellipsoidal correction can be determined by a simple formula:

\[ \delta N_e \approx \psi_0 \left[ (0.12 - 0.38 \cos^2 \theta) \Delta g + 0.17 \bar{N} \sin^2 \theta \right] \]  

(2.2.15)

where \( \psi_0 \) – cap size (in units of degree of arc);  
\( \theta \) – geocentric co-latitude.

\( \Delta g \) is given in mGal and \( \bar{N} \) in m.

It is concluded by Ellmann and Sjöberg (2004) that the absolute range of the ellipsoidal correction in LSMS does not exceed the cm level with a cap size within a few degrees.
Chapter 3
Data Acquisition

This study aims at computing the geoid model for Kazakhstan, which is located in the western Asia and bordered by the Russian Federation, China, Kyrgyzstan, Uzbekistan and Turkmenistan. The target area is defined as the area with geographical boundaries from $40^\circ$ to $56^\circ$ northern latitudes and from $46^\circ$ to $88^\circ$ eastern longitudes. Figure 3.1 shows the borders of the target area and the outer borders at the fixed spherical distance of $\psi = 3^\circ$.

![Map of Kazakhstan and surrounding countries](image)

**Figure 3.1.** Location of the target area (fenced by bold rectangle). External rectangle enclosed the target area at a spherical distance of $\psi = 3^\circ$

To compute 5’ x 5’ Kazakh geoid model in the target area the following data sets were collected:

- Terrestrial gravity data bounded by $37^\circ \leq \phi \leq 59^\circ$ and $43^\circ \leq \lambda \leq 91^\circ$
- DEM data in the same area
- GPS/Leveling points
- GGMs
3.1 Terrestrial gravity anomaly data

For this study work the pre-processed Bouguer anomaly grid has been provided by the Geophysical Exploration Technology (GETECH) group through a confidentiality agreement.

The GETECH is leading petroleum and minerals consultancy best known for its unique global gravity and magnetic data holdings and services. They have data covering almost every country in the world at a variety of scales and resolutions. Their expanding library of complementary geological and geophysical datasets currently includes: geological maps, topography/bathymetry, drainage, geochemistry and radiometric data.

The gravity anomaly data set, which was supplied by GETECH, included 5’ x 5’ Bouguer gravity anomalies and topography/bathymetry grids for an area defined as longitude $43E - 91E$ and latitude $37N$ to $59N$.

The Bouguer anomaly grid contains 265 x 577 blocks, but does not have any information on the accuracy of different values. This grid was made by digitizing the 1:1 000 000 scale Bouguer anomaly maps, on which gravity stations were not marked.

Since GETECH gravity data processing is not clearly known and due to the lack of control points, the accuracy of the gridded anomalies remains unknown.

Figure 3.2 portrays the surface plot of the Bouguer anomaly grid.

![Figure 3.2](image.png)

Figure 3.2. Surface plot of the Bouguer gravity anomaly data set [mGal]
3.2 The GPS/ Levelling data

One of the most preferred and independent assessment of gravimetric geoid model is by use of geoid height obtained from co-located GPS and spirit levelling heights. In this study this kind of data has been observed and will be used to estimate the accuracy of the DEM and computed geoid heights. Figure 3.3 portrays distribution of the GPS/Levelling data over the study area (some points are too close and appear as one).

![Figure 3.3: Distribution of GPS/Levelling points over the Kazakhstan](image)

The geodetic coordinates of the points were derived from GPS measurements performed for different individual projects from 2004 to 2009. Unfortunately there are many inconsistencies to the way the coordinates of those points were computed, like datum and epoch discrepancies, tectonic motion and general geodynamic effects not considered in the solution. Concerning the orthometric heights of the points, they were taken from the old geodetic network and they vary from 1st to 3rd order.

GPS/Levelling data is one of the constraints for this study because of the difficulties with releasing data.

Absolute accuracies of the GPS/Levelling data are estimated to:

- \( \sigma_h \approx \pm 0.1 \, m \) for ellipsoidal heights
- \( \sigma_H \approx \pm 0.1 \, m \) for orthometric heights

Table 3.1 shows the coordinates of the GPS/Levelling points as well as their ellipsoidal height and geoid height values \( N_{GPS/Levelling} \) computed by the following equation:
\[ N_{GPS/Levelling} = h - H \]  

(3.3.1)

where \( h \) — ellipsoidal height;
\( H \) — orthometric height.

Table 3.1: The GPS/Levelling data

<table>
<thead>
<tr>
<th></th>
<th>( \phi^\circ )</th>
<th>( \lambda^\circ )</th>
<th>( h \text{ [m]} )</th>
<th>( N_{GPS/Levelling} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>50.09843</td>
<td>71.98765</td>
<td>425.85875</td>
<td>-33.16269</td>
</tr>
<tr>
<td>2.</td>
<td>48.93754</td>
<td>77.00233</td>
<td>990.67519</td>
<td>-38.57730</td>
</tr>
<tr>
<td>3.</td>
<td>49.56433</td>
<td>82.04814</td>
<td>551.44981</td>
<td>-43.44099</td>
</tr>
<tr>
<td>4.</td>
<td>44.01493</td>
<td>68.74625</td>
<td>354.56770</td>
<td>-39.41261</td>
</tr>
<tr>
<td>5.</td>
<td>54.08833</td>
<td>70.89258</td>
<td>92.51193</td>
<td>-25.93592</td>
</tr>
<tr>
<td>6.</td>
<td>49.08775</td>
<td>53.98363</td>
<td>24.82157</td>
<td>-15.30957</td>
</tr>
<tr>
<td>7.</td>
<td>45.99147</td>
<td>75.08763</td>
<td>299.74621</td>
<td>-46.88395</td>
</tr>
<tr>
<td>8.</td>
<td>47.62357</td>
<td>81.35879</td>
<td>1078.44522</td>
<td>-46.46428</td>
</tr>
<tr>
<td>9.</td>
<td>49.76482</td>
<td>66.82375</td>
<td>381.71123</td>
<td>-29.15675</td>
</tr>
<tr>
<td>10.</td>
<td>52.75302</td>
<td>73.54378</td>
<td>44.46766</td>
<td>-30.84814</td>
</tr>
<tr>
<td>11.</td>
<td>50.83575</td>
<td>72.43958</td>
<td>441.62468</td>
<td>-32.25541</td>
</tr>
<tr>
<td>12.</td>
<td>52.87853</td>
<td>66.25789</td>
<td>181.39139</td>
<td>-22.33816</td>
</tr>
<tr>
<td>13.</td>
<td>47.23988</td>
<td>66.29838</td>
<td>295.05342</td>
<td>-33.67192</td>
</tr>
<tr>
<td>14.</td>
<td>49.23887</td>
<td>60.29398</td>
<td>175.39184</td>
<td>-19.42760</td>
</tr>
<tr>
<td>15.</td>
<td>47.93244</td>
<td>69.93473</td>
<td>326.85317</td>
<td>-36.44512</td>
</tr>
<tr>
<td>16.</td>
<td>47.89424</td>
<td>56.98722</td>
<td>171.86418</td>
<td>-19.72903</td>
</tr>
<tr>
<td>17.</td>
<td>45.74094</td>
<td>53.27211</td>
<td>-40.65799</td>
<td>-15.33784</td>
</tr>
<tr>
<td>18.</td>
<td>45.80178</td>
<td>53.46814</td>
<td>-42.48544</td>
<td>-15.73046</td>
</tr>
<tr>
<td>19.</td>
<td>45.66443</td>
<td>53.47863</td>
<td>-41.86413</td>
<td>-15.76917</td>
</tr>
<tr>
<td>20.</td>
<td>45.55838</td>
<td>53.23912</td>
<td>-40.41912</td>
<td>-15.16772</td>
</tr>
</tbody>
</table>
3.3 The Digital Elevation Model (DEM)

The DEM used in this research is the gridded topography with block size of 3”x3” from Shuttle Radar Topography Mission (SRTM) reprocessed by the Consortium for Spatial Information of the Consultative Group for International Agricultural Research (CGIAR-CSI). In its original release, SRTM data contained numerous voids (areas without data), specifically over water bodies (lakes and rivers), and in areas where insufficient textural detail was available in the original radar images to produce three-dimensional elevation data. Therefore the original SRTM data has been subjected to a number of processing steps to provide seamless and complete elevational surfaces for the globe.

The DEM extends to cover the target area with the proposed 3° adapted offset as \(37° \leq \phi \leq 59°\) and \(43° \leq \lambda \leq 91°\). The DEM grid was resampled to \(5’x\ 5’\) to meet agreement in resolution with the grid of computation points and gravity anomaly grid. In present study the absolute vertical accuracy of the DEM has been estimated to be 23 m, based on the comparison with 20 levelling points.

Figure 3.4 shows the surface plot of the SRTM DEM of the study area.

![Figure 3.4](image)

**Figure 3.4**  SRTM digital elevation model of Kazakhstan [m]
3.4 The Global Gravitational Models (GGMs)

For the determination of the long wavelength part of the Earth’s gravity field more than 100 GGMs, which express the Earth’s gravity field and thus geoid heights in terms of spherical harmonic basis functions, have been computed by various groups. The International Centre for Global Earth Models (ICGEM) conducted comparison of quasigeoid heights derived from the models with GPS/Levelling derived geoid values from USA, Canada, Europe and Australia. The Root Mean Square (RMS) differences for the best five GGMs of ICGEM made evaluation are presented in the Table 3.2.

Table 3.2 RMS values of GPS/Levelling minus GGM derived geoid heights [m]

<table>
<thead>
<tr>
<th>Gravity Model</th>
<th>Nmax</th>
<th>USA</th>
<th>Canada</th>
<th>Europe</th>
<th>Australia</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>6169 points</td>
<td>1930 points</td>
<td>1235 points</td>
<td>201 points</td>
</tr>
<tr>
<td>EGM 2008</td>
<td>2159</td>
<td>0.248</td>
<td>0.126</td>
<td>0.208</td>
<td>0.217</td>
</tr>
<tr>
<td>EIGEN-5C</td>
<td>360</td>
<td>0.341</td>
<td>0.251</td>
<td>0.303</td>
<td>0.244</td>
</tr>
<tr>
<td>EIGEN-GL04C</td>
<td>150</td>
<td>0.339</td>
<td>0.253</td>
<td>0.336</td>
<td>0.244</td>
</tr>
<tr>
<td>ITG-GRACE2010S</td>
<td>180</td>
<td>0.548</td>
<td>0.459</td>
<td>0.595</td>
<td>0.523</td>
</tr>
<tr>
<td>EIGEN-GRACE02S</td>
<td>150</td>
<td>0.739</td>
<td>0.643</td>
<td>0.828</td>
<td>0.538</td>
</tr>
</tbody>
</table>

The five GGMs are as follows:

EGM 2008 Combined high resolution gravity field model complete to spherical harmonic degree and order 2159, which incorporates improved 5’x5’ gravity anomalies and has benefited from the latest GRACE based satellite solutions, released on April 2008

EIGEN-5C Combined gravity field model complete to degree and order 360 from GRACE, Lageos and surface gravity data, released on September 29, 2008

EIGEN-GL04C Combined gravity field model complete to degree and order 360 from GRACE, Lageos and with high resolution 0.5° x 0.5° gravimetry and altimetry surface data, released on March 31, 2006
ITG-GRACE2010S  Long-term mean gravity field model covering the time span August, 2002 to August, 2009 calculated from GRACE data only up to degree and order 180.

EIGEN-GRACE02S  GRACE satellite-only Earth gravity field model complete to degree and order 150 released on February 13, 2004 to the GRACE Science Team and August 9, 2004 to the public.
Chapter 4

Preparation for geoid model determination

The ultimate goal of this study work is to determine the Kazakh gravimetric geoid model as accurately as possible with available data set. And it is well known that accuracy depends on the initial parameters for the computation.

4.1 Free-air gravity anomaly

One problem with Stokes’ approach is that it requires the gravity observations be downward-continued from the Earth’s surface to the geoid. The simple free-air reduction may be considered as giving approximate boundary values at the geoid, to be used in Stokes’ formula.

By free-air gravity reduction, a gravity value measured at point A on the surface of the Earth is reduced to the geoid along the plumb line under the assumption that there are no masses (or just free-air) between the geoid and the surface.

![Diagram of free-air gravity reduction](image)

**Figure 4.1.** Free-air gravity reduction
Hence, the free-air gravity anomaly at point A:

\[ \Delta g_F = g_A - \gamma_B \]  \hspace{1cm} (4.1.1)

where \( g_A \) — gravity value at a surface point A; 
\( \gamma_B \) — normal gravity on the telluroid;

To determine the free-air anomalies from available Bouguer anomalies the following formula was used:

\[ \Delta g_F = \Delta g_B + B \]  \hspace{1cm} (4.1.2)

where \( B \) — gravitational force of the Bouguer plate along the plumb line:

\[ B = 2\pi G \rho_0 H_A \]  \hspace{1cm} (4.1.3)

\( H_A \) — orthometric height of point A [m]; 
\( \rho_0 \) — standard value for crustal density \( (\rho_0 = 2.67 \text{g/cm}^3) \).

4.2 The least-squares modification parameters

For the computation of three stochastic modifications of Stokes' formula special program \textit{LS\_coeff.m} by Dr. Artu Ellman has been used.

To compute the LS parameters by this program the following initial information is needed:

- GGM
- Terrestrial data error variance \( C(0) \)
- Integration cap size \( \psi_0 \)
- Modification degrees \( L, M \)

4.2.1 Models for gravity anomaly and data error degree variances

The program \textit{LS\_coeff.m} utilizes some well-known a priori and empirical models for gravity anomaly and data error degree variances, which can be used for estimating the global MSE by Eq.\,(2.1.15) and the LSM parameters. In general, the data errors
contain the random and systematic error contribution. For example the terrestrial data may be affected by various systematic errors caused by uncertainties between different geodetic datums, the usage of different data acquisition and processing specifications. As it is mentioned in Section 2.1, possible systematic errors are neglected for the stochastic models of this study. This is supported by common practice, that all known systematic errors are removed from computations, rather than accounted for separately. “Concerning unknown systematic discrepancies, these remain unknown indeed and estimation of their actual range could be insurmountable challenge (or just a speculation)” (Ellman (2004)).

The Gravity anomaly error degree variances $c_n$ can be obtained by using coefficients $\Delta\tilde{\sigma}_{nm}, \tilde{\sigma}_{nm}$ of disturbing potential and fundamental constants (gravity mass constant GM and equatorial radius a) of the used GGM as follows:

$$c_n = \frac{(GM)^2}{a^4} (n - 1)^2 \sum_{m=0}^{n} (\Delta\tilde{\sigma}_{nm}^2 + \tilde{\sigma}_{nm}^2) \quad (2.2.1)$$

In practice the infinite sum in Eq. (2.1.15) must be truncated at some upper limit of expansion (in this study $n_{max} = 2000$). This may be far beyond of the harmonic degrees of the used GGM. In the program LS_coeffs the $c_n$ values for degrees $n > M$ are calculated by a formula in Tscherning and Rapp (1974):

$$c_n = A't^{n+2} \frac{(n - 1)}{(n - 2)(n + 24)} \quad (2.2.2)$$

with the parameters chosen to $A = 425.28 \text{ mGal}^2$ and $t = 0.999617$, it should be noted that this model is valid just for the gravity field uncorrected for any topographic effects.

The GGM derived gravity anomaly error degree variances $dc_n$ is also known as error degree variance, usually obtained from GGM potential coefficients variances $d^2c_{nm}$ and $d^2\tilde{\sigma}_{nm}$ as:

$$dc_n = \frac{(GM)^2}{a^4} (n - 1)^2 \sum_{m=0}^{n} (d^2c_{nm} + d^2\tilde{\sigma}_{nm}) \quad (2.2.3)$$

The terrestrial gravity anomaly error degree variances $\sigma_n^2$ can be estimated from an error degree covariance function $C(\psi)$. The following function could be used (cf. Sjöberg (1986));
\[
C(\psi) = c_1 \left[ \frac{1 - \mu}{\sqrt{1 - 2\mu \cos \psi + \mu^2}} - (1 - \mu) - (1 - \mu)\mu \cos \psi \right]
\]

(2.2.4)

And the degree variances \( \sigma_n^2 \) for the reciprocal distance type function are given by:

\[
\sigma_n^2 = c_1 (1 - \mu)\mu^n, \quad 0 < \mu < 1
\]

(2.2.5)

where \( c_1, \mu \) can be estimated from a knowledge of the function \( C(\psi) \). Two pre-defined quantities are needed for estimation of \( \sigma_n^2 \). These are: the error variance \( C(0) \) (the value of the covariance function at \( \psi = 0 \)) and the correlation length \( \psi_{1/2} \) (to the point where \( C(\psi) \) is degraded to half of \( C(0) \), thus \( C(\psi_{1/2}) = \frac{1}{2} C(0) \)).

For \( \psi = 0 \) and \( \psi_{1/2} = 0.1^0 \) the variance by Eq. (2.2.4) becomes

\[
C(0) = c_1 \mu^2 \quad \Rightarrow \quad C(\psi_{1/2}) = \frac{1}{2} c_1 \mu^2
\]

(2.2.6)

The solution for \( \mu = 0.99899012912 \) (associated with \( \psi_{1/2} = 0.1^0 \)) is used in the software. Inserting \( \mu \) into Eq. (2.2.6) yields that function \( C(\psi) \) is completely determined. Thereafter the degree variances \( \sigma_n^2 \) are computed from Eq. (2.2.5). Hence in order to determine the degree variances \( \sigma_n^2 \) we need to adopt optimal \( C(0) \), which corresponds to the local data involved in geoid computation.

### 4.2.2 The modification degrees

Choosing the upper limit of the geopotential model \( M \) and the upper limit of the harmonics \( L \) to be modified in Stokes' function is important in the geoid modelling.

The selection of the limit \( M \) is directly related to the quality of the GGM, which will be used. Practically, due to restricted access to terrestrial data and for necessity to increase the computational efficiency the integration radius \( \psi_0 \) is often limited to a few hundred kilometers. This implies that a relatively high \( M \) should compensate this limitation. On the contrary, error stemming from potential coefficients increases proportionally with increasing in GGMs degree. Therefore an equalized point between the GGM and the terrestrial gravity data should be found.

Upper limit of geopotential model \( M \) is arbitrary and generally not equal to \( L \), the LSM method assumes that the upper limit of modification \( L \) is at least as high as \( M \), \((L \geq M)\). However, the special choice \( L = M \) is often applied in practice
(Ellman (2004), Kiamenr (2006), Ulotu (2009), etc). It should be noted that in the original publication (Sjöberg (1984)) the biased LS method was designed to the case $L = M$. In this study the upper limit of the modification is considered to be equal to the upper modification limit, $L = M$.

4.3 Transformation parameters

As it was mentioned in Section 1.1, one of the most important purposes of the precise geoid model determination is to replace the costly conventional geometric levelling measurements with GNSS measurements during geodetic and surveying work relying on the relationship between ellipsoidal and orthometric heights (Eq. (1.1.1)). However in practice Eq.(1.1.1) does not work properly due to the presence number of factors, such as:

- Random errors in the heights $H$, $h$, $N$
- Datum inconsistencies inherit among the height types
- Systematic effects and distortions in the height data
- Assumptions and theoretical approximations made in processing observed data
- Different type of deformations which can affect the coordinate reference systems, datum and monuments

The relations between different height types can be portrayed by Figure 4.3.

![Figure 4.3](image-url)

**Figure 4.2.** Relations between orthometric height, ellipsoidal height and geoid undulation


One may see from Figure 4.3, if we know the ellipsoidal height \( h \) of a point on the terrain as measured by a GPS receiver, and we want to convert this to a height in the national datum \( H \), then we need to know the height of the national datum above the ellipsoid at that point.

\[
H_i = h_i - (N_i + H_{CS})
\]  

(4.4.1)

In the Eq. (4.4.1) \( N_i + H_{CS} \) models a "corrector surface", which allows us to toggle between the national datum and ellipsoidal heights (Krynski and Lyszkowicz(2006)).

Even as the corrector surface implied by the known benchmarks values is not physically coincident with the geoid its physical shape is likely to be similar. Hence, the method adopted to model the corrector surface was to fit the computed gravimetric geoid model to geometric geoid model.

The computed geoid heights can be fitted to the set of GPS/Levelling points by the following procedure:

1. Calculate the discrepancies between \( N_{GPS/L levelling} \) and \( N \):

\[
\Delta N_i = (h_i - H_i) - N_i
\]  

(4.4.2)

2. Make the regional evaluation of the optimal parametric model for geoid fitting.

Corrector surface can be formed from parametric model as:

\[
H_{CS} = a^T x
\]  

(4.4.3)

where \( a \) — \( n \times 1 \) vector of known coefficients;

\( x \) — \( n \times 1 \) vector of unknown parameters.

The parametric model supposed to describe the mentioned systematic errors and datum inconsistencies inherit in the different height data sets. Its type varies in form and complexity depending on a number of factors. 3-, 5- and 7- similarity transformation parameter models are often used to map the situation. Equations of these models are as follows:
3 - parameter model:

\[
 a_i = \begin{pmatrix}
\cos \phi_i \cos \lambda_i \\
\cos \phi_i \sin \lambda_i \\
\sin \phi_i
\end{pmatrix} \quad x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} \quad (4.4.3a)
\]

5 - parameter model:

\[
a_i = \begin{pmatrix}
\cos \phi_i \cos \lambda_i \\
\cos \phi_i \sin \lambda_i \\
\sin \phi_i \\
\sin^2 \phi_i \\
1
\end{pmatrix} \quad x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} \quad (4.4.3b)
\]

7 - parameter model:

\[
a_i = \begin{pmatrix}
\cos \phi_i \cos \lambda_i \\
\cos \phi_i \sin \lambda_i \\
\sin \phi_i \\
(cos \phi_i \sin \phi_i \cos \lambda_i) / W_i \\
(cos \phi_i \sin \phi_i \sin \lambda_i) / W_i \\
\sin^2 \phi_i / W_i \\
1
\end{pmatrix} \quad x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{pmatrix} \quad (4.4.3c)
\]

\[
W_i = \sqrt{1 - e^2 \sin \phi_i} \quad (4.4.3c - 1)
\]

where \(\phi_i, \lambda_i\) – horizontal geodetic coordinates of the network;
\(e^2\) – first eccentricity of the reference geodetic ellipsoid.

Matrix system of the observation equations:

\[
Ax = \Delta N - \varepsilon \quad (4.4.4)
\]

where \(A - m \times n\) vector of known coefficients \(a\);
\(\varepsilon\) – the combined random error for the triple heights \((H_i, h_i, N_i)\).

The solution for the vector of unknown parameters is obtained under the least squares conditions \(\varepsilon^T \varepsilon = \min\). LS give the best estimates for the parameters and errors:
\[ \hat{\varepsilon} = (A^T A)^{-1} A^T \Delta N \] (4.4.5)

\[ \hat{\bar{\varepsilon}} = (I - A(A^T A)^{-1} A^T) \Delta N \] (4.4.6)

where \( I \) – unit matrix.

The adjusted errors \( \hat{\varepsilon} \) show the level of absolute agreement between gravimetric and GPS/Levelling geoid models.

The variance-covariance matrices of the estimated parameters and are as follows:

\[ C_{\hat{\varepsilon}\hat{\varepsilon}} = \sigma_0^2 (A^T A)^{-1} \] (4.4.7)

\[ C_{\bar{\varepsilon}\bar{\varepsilon}} = \sigma_0^2 [I - A(A^T A)^{-1} A^T] \] (4.4.8)

where \( \sigma_0^2 \) – a posteriori unit weight variance factor:

\[ \sigma_0^2 = \frac{\hat{\varepsilon}^T \hat{\varepsilon}}{n - m} \] (4.4.9)

where \( n \) – number of GPS/levelling points;
\( m \) – number of estimated parameters.

### 4.4 Numerical tests

To compute the geoid model for Kazakhstan as accurately as possible with available data set different numerical tests have been performed. We tried different GGMs with different degree and order and also with different accuracy for the gravity data.

#### 4.4.1 Selection of the GGM

The GGM is the most important components in the determination of gravimetric geoid. Therefore, an optimal geopotential model should be selected.

Kiamehr (2006a) has verified that the global evaluations of GGMs sometimes tend to be too optimistic and global statistics are not necessary true representatives of a
region. That is why GGMs should be evaluated specifically for the geographic region concerned.

As it was explained in Section 3.3 five geopotential models were selected from the list of available GGMs in nowadays. However EGM 2008 will not be used in the nomination of the optimal GGM in the Kazakhstan due to the technical problems occurred with running the software.

To find out which of the remained four geopotential models is more suitable for the study area GGM derived geoid heights $\bar{N}_{GGM}$ were compared with $N_{GPS/Levelling}$.

**Table 4.1** Statistics of residuals between $\bar{N}_{GGM}$ and $N_{GPS/Levelling}$

7-parameter transformation is used to remove systematic effects.

<table>
<thead>
<tr>
<th>GGM</th>
<th>$L = M$</th>
<th>Residuals, $\Delta N$ [m]</th>
<th>min</th>
<th>max</th>
<th>mean</th>
<th>St.Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITG-GRACE2010S</td>
<td>180</td>
<td>Pre-fit</td>
<td>-3.217</td>
<td>3.967</td>
<td>1.489</td>
<td>1.976</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Post-fit</td>
<td>-1.672</td>
<td>1.515</td>
<td>0.000</td>
<td>0.952</td>
</tr>
<tr>
<td>EIGEN-GRACE02S</td>
<td>120</td>
<td>Pre-fit</td>
<td>-7.489</td>
<td>7.234</td>
<td>-0.651</td>
<td>4.759</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Post-fit</td>
<td>-7.791</td>
<td>6.819</td>
<td>0.000</td>
<td>3.274</td>
</tr>
<tr>
<td>EIGEN-5C</td>
<td>360</td>
<td>Pre-fit</td>
<td>-8.027</td>
<td>16.296</td>
<td>3.186</td>
<td>6.293</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Post-fit</td>
<td>-10.710</td>
<td>12.136</td>
<td>0.000</td>
<td>5.754</td>
</tr>
<tr>
<td>EIGEN-GL04C</td>
<td>120</td>
<td>Pre-fit</td>
<td>-11.679</td>
<td>11.618</td>
<td>-0.960</td>
<td>6.717</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Post-fit</td>
<td>-6.752</td>
<td>7.765</td>
<td>0.000</td>
<td>3.458</td>
</tr>
</tbody>
</table>

The discrepancies between $N_{GPS/Levelling}$ and $\bar{N}_{GGM}$ was calculated as:

$$\Delta N = N_{GPS/Levelling} - \bar{N}_{GGM}$$  \hspace{1cm} (4.2.1)

From the results in Table 4.1 we see that ITG-GRACE2010S model has best agreement with GPS/Levelling points. Therefore, this model has been chosen for geoid computation in Kazakhstan.

Looking back on the global evaluation (Table 3.2), we observe that the ITG-GRACE2010S was ranked as 3rd (if EGM 2008 is not taken into account). So if we had
gone by the global evaluation, this GGM would not have been chosen for the geoid determination in Kazakhstan.

### 4.4.2 Error degree variances

Recall that the main objective of the modification procedure is to minimize the influence of errors in the geoid computation. The KTH LSMS allows minimization of the truncation error, the influence of erroneous gravity data and geopotential coefficients in the least-squares sense. We need to assume signal and error degree variances (GGM and terrestrial gravity) to be able to estimate the least-squares modification coefficients. The ground gravity observations distributed in Kazakhstan are heterogeneous and often affected by different systematic errors. When the recently published, very accurate GRACE GGMs are used, it becomes important to use a kernel modification that effectively filters out the long-wavelength errors from the terrestrial gravity anomalies (e.g., Featherstone (2003)). For this reason we need a proper weighting scheme for the data as an a priori or empirical stochastic model. However, the true errors for the gravity data are not well known and we can only estimate their accuracy from a model.

Therefore computed geoid heights $\tilde{N}$ (Eq(2.2.1)) were tested versus GPS/Leveling points by variation of the upper limit of geopotential model ($M = 120, 150, 180$) and the terrestrial data error variances $C(0) = 3, 6, 9, 12 \text{ mGal}^2$. The results of the performed test are summarized in Table 4.2.

**Table 4.2** The standard deviation between $\tilde{N}$ and $N_{\text{GPS/Leveling}}$ with different initial conditions for computation of the LSM parameters after 7-parameter fitting.

- **ITG-GRACE2010S**
- Integration cap size $\psi_0 = 3^\circ$

<table>
<thead>
<tr>
<th>Terrestrial data error variance $C(0)$, [mGal$^2$]</th>
<th>Standard deviation [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L = M = 120$</td>
</tr>
<tr>
<td>3</td>
<td>0.4354</td>
</tr>
<tr>
<td>6</td>
<td>0.4181</td>
</tr>
<tr>
<td>9</td>
<td>0.4051</td>
</tr>
<tr>
<td>12</td>
<td>0.3993</td>
</tr>
</tbody>
</table>

Table 4.2 indicates that the ITG-GRACE2010S model fits the GPS/Leveling points better at its maximum degree 180 and with $C(0) = 9 \text{ mGal}^2$
4.4.3 Fitting parameters

As mentioned earlier, different types of the parametric models can be used in order to fit the computed geoid heights to the set of GPS/Levelling points. These parametric model types varied in form and complexity depending on a number of factors. And the selection of the proper model type depends on the data distribution, density and quality. In this study work 3-, 5- and 7-parameter fitting models were tested. The results of this test is given in Tables 4.3-4.4 and portrayed in Figure 4.4.

Table 4.3 Fitting parameters and their standard errors (\(\sigma\))

| Model | 3-parameters | | 5-parameters | | 7-parameters | \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Adjusted Parameter</td>
<td>Value [m]</td>
<td>(\sigma) ± [m]</td>
<td>Value [m]</td>
<td>(\sigma) ± [m]</td>
</tr>
<tr>
<td>(x_1)</td>
<td></td>
<td>2,819</td>
<td>0,563</td>
<td>-1,811</td>
<td>3,415</td>
</tr>
<tr>
<td>(x_2)</td>
<td></td>
<td>7,612</td>
<td>1,136</td>
<td>-2,738</td>
<td>7,641</td>
</tr>
<tr>
<td>(x_3)</td>
<td></td>
<td>-8,114</td>
<td>0,974</td>
<td>-40,711</td>
<td>90,502</td>
</tr>
<tr>
<td>(x_4)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x_7)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.3 Post-fit absolute agreement between computed geoid heights and GPS/Levelling
Table 4.4 Statistics of fitting the computed geoid heights to the set of GPS/Levelling points [m]

<table>
<thead>
<tr>
<th>Residuals</th>
<th>Pre-fit</th>
<th>Post-fit</th>
<th>3 - parameters</th>
<th>5 - parameters</th>
<th>7 - parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>min</td>
<td>-0.1306</td>
<td>-0.3515</td>
<td>-0.3780</td>
<td>-0.2946</td>
<td></td>
</tr>
<tr>
<td>max</td>
<td>1.7594</td>
<td>0.4532</td>
<td>0.4065</td>
<td>0.3266</td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>0.7870</td>
<td>-0.0004</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta N}$</td>
<td>0.5664</td>
<td>0.2576</td>
<td>0.2451</td>
<td>0.1773</td>
<td></td>
</tr>
</tbody>
</table>

From Table 4.4 we can see that the accuracy of gravimetric geoid model gradually improves as the level of accounting for systematic effects gets better (from 3-parameters to 5-parameters account), the overall accuracy moves from about 57 cm to about 18 cm.

Of the three systematic models used in this study, the 7-parameter transformation model has the smallest standard error of the geoid height residual vector.

4.5 Accuracy estimate

The accuracy of the computed geoid heights with respect to GPS/Levelling has been established by the residual vector $\hat{\epsilon}$. Note that this accuracy estimate incorporates residuals from all the three height systems:

$$\sigma_{\Delta N}^2 = \sigma_h^2 + \sigma_H^2 + \sigma_N^2 \quad (4.6.1)$$

By removing the influence of the other two systems using the standard error estimates arrived at earlier, i.e. $0.028$, $\sigma_h \approx \pm 0.1 \text{ m}$ and $\sigma_H \approx \pm 0.1 \text{ m}$ as per Eq. (4.6.1), accuracy estimate of computed geoid undulations is obtained as $\sigma_N = \pm 0.107 \text{ m}$.

The absolute accuracy estimate of the geoid model is very much dependent on the ability to assign weights to the observations or systems involved and also on how far the systematic effects have been removed. In turn, the reliability of systematic models to correctly account for the effects, is not only or always a function of the model type and the number of parameters involved, but also depends much on the GPS/Levelling network accuracy, distribution, coverage and density.
However, it is well known that the GPS and levelling observations have very good accuracies in the relative sense, because most of the systematic errors are cancelled or eliminated through the differencing of observations. This leads us to evaluating relative accuracy of the computed geoid heights based on $\Delta H_{\text{GPS/Geoid}}$ versus levelling data.

\[
\Delta H_{\text{GPS/Geoid}} = \Delta h - \Delta N
\]  

(4.6.3)

where $\Delta N$ – post-fitted geoid undulations.

\[
\Delta N = (N_2 - N_1) + (\varepsilon_2 - \varepsilon_1)
\]  

(4.6.4)

The relative accuracy of GPS derived orthometric heights and the Levelling heights were estimated in part per million as follows:

\[
ppm = \text{mean} \left| \frac{(\Delta H_{\text{GPS/Geoid}} - \Delta H)_{\text{mm}}}{D_{ij km}} \right|
\]  

(4.6.5)

where $D_{ij}$ – length of the baseline.

The results of this investigation shows that the computed geoid undulations fit the GPS/Levelling data set with 1.219 ppm (see Table 4.4).
Table 4.5  Relative accuracy of the computed geoid heights based on $\Delta H_{GPS/Geoid}$ versus Levelling heights

|   | $\Delta H_{GPS/Geoid}$ [m] | $\Delta H$ [m] | $D_{ij}$ [m] | $\left| \frac{(\Delta H_{GPS/Geoid} - \Delta H)_{mm}}{D_{ij \_km}} \right|$ |
|---|---------------------------|----------------|-------------|----------------------------------|
| 1. | 570,2982                  | 570,231        | 385548,27   | 0,174064                         |
| 2. | -434,426                  | -434,362       | 374005,89   | 0,17115                          |
| 3. | -201,514                  | -200,91        | 1195466,19  | 0,50446                          |
| 4. | -274,181                  | -275,532       | 1131693,21  | 1,19433                          |
| 5. | -78,3985                  | -78,3167       | 1305152,43  | 0,06266                          |
| 6. | 305,0919                  | 306,499        | 1633816,79  | 0,86125                          |
| 7. | 778,4728                  | 778,2793       | 512291,80   | 0,37755                          |
| 8. | -713,636                  | -714,042       | 1098857,44  | 0,36908                          |
| 9. | -334,866                  | -335,552       | 575809,27   | 1,19131                          |
| 10. | 398,0188                  | 398,5643       | 226536,60   | 2,4081                           |
| 11. | -269,72                   | -270,151       | 483188,23   | 0,89116                          |
| 12. | 124,3435                  | 124,9958       | 627194,13   | 1,04001                          |
| 13. | -133,538                  | -133,906       | 498780,40   | 0,73781                          |
| 14. | 167,9235                  | 168,4788       | 726673,70   | 0,76429                          |
| 15. | -171,316                  | -171,705       | 968970,48   | 0,40185                          |
| 16. | -217,715                  | -216,913       | 371305,09   | 2,15885                          |
| 17. | -1,40673                  | -1,43484       | 16679,79    | 1,68509                          |
| 18. | 0,574975                  | 0,660025       | 15287,74    | 5,56327                          |
| 19. | 0,785898                  | 0,84356        | 22090,12    | 2,61031                          |
| 20. | ppm           | 1,219  |
Chapter 5
Computation of the Kazakh gravimetric geoid model (KazGM2010)

The final 5'x5' Kazakh gravimetric geoid model (KazGM2010) in the area $40^\circ \leq \phi \leq 56^\circ$ and $46^\circ \leq \lambda \leq 89^\circ$ was computed based on the free-air gravity anomalies in the 5'x5' grid size, ITG-GRACE2010S with its maximum degree and order 180 and 5'x5' SRTM global DEM.

Based on the numerical investigation explained in Chapter 4 the LS parameters for the Kazakh geoid determination were calculated from the following initial conditions:

1. Geopotential model - ITG-GRACE2010S
2. Terrestrial data error variance $\mathcal{C}(0) = 9mGal^2$
3. Integration cap size $\psi_0 = 3^\circ$

The Kazakh gravimetric geoid model was calculated by the KTH-method following the procedure explained in Chapter 2.

Figures 5.1-5.2 show the contour maps and 3D view of the approximate geoid model as well as its long-wavelength and short-wavelength components.
a. The long wavelength component $\bar{N}_2$ [m]  
Contour interval – 2 m

b. The short wavelength component $\bar{N}_1$[m]  
Contour interval – 1 m

c. Kazakhstan geoid model without additive corrections [m]  
Contour interval – 2 m

Figure 5.1. Approximate geoid model determination results
Chapter 5. Computation of the Kazakh gravimetric geoid model

**Figure 5.2.** 3D-view of the approximate geoid model determination results

- **a.** The long wavelength component $\overline{N}_2 [m]$

- **b.** The short wavelength component $\overline{N}_1 [m]$

- **c.** Kazakhstan geoid model without additive corrections [m]
All AC terms were determined based on the methods, which explained in Section 2.2. 3D views of these corrections are represented below. (Figures 5.3 - 5.6).

**Figure 5.3.** The combined topographic correction for the KazGM2010 [m]

**Figure 5.4.** The DWC correction for the KazGM2010 [m]
Figure 5.5. The combined atmospheric correction for the KazGM2010 [mm]

Figure 5.6. The ellipsoidal correction for the KazGM2010 [mm]

Notice that all correction terms depend on the modification, cap size and maximum degree for the GGM being used. As mentioned before, we can see that the atmospheric and ellipsoidal corrections are small.
Finally the Kazakh gravimetric geoid model was assembled as a sum of the approximate geoid model $\tilde{N}$ and the additive corrections $AC$. The KazGM2010 is portrayed in Figure 5.7 and Figure 5.8 as surface and contour maps, respectively.

**Figure 5.7.** 3D model of the KazGM2010 [m]
Chapter 6

Conclusions

This thesis aims to compute a gravimetric geoid for Kazakhstan in the area $40^\circ \leq \phi \leq 56^\circ$ and $46^\circ \leq \lambda \leq 88^\circ$. The following data sets were collected:

- Gravity anomalies provided by GETECH ($37^\circ \leq \phi \leq 59^\circ$ and $43^\circ \leq \lambda \leq 91^\circ$)
- 3"x3" resolution DEM from SRTM in the same area
- 20 GPS/Levelling points distributed roughly all over the Kazakhstan
- GGMs

A detailed investigation of all GGMs for choosing the suitable model for geoid computation in Kazakhstan was made. Based on the international comparison of a number geopotential models five GGM candidates were chosen to be used for geoid computation in Kazakhstan. Thereafter, a regional investigation of the selected GGMs was done by GPS/Levelling approach. The investigation resulted in choosing the ITG-GRACE2010S model as it fits the GPS/Levelling data with accuracy of 0.596 m.

In order to compute the geoid model for Kazakhstan as accurately as possible with available data set different numerical tests have been performed. The geoid heights for these points were calculated using ITG-GRACE2010S model with different degrees and orders, and also different accuracies for the terrestrial gravity data. Based on these numerical investigations Kazakh gravimetric geoid model were calculated from the following initial parameters:

1. Geopotential model – ITG-GRACE2010S
2. Terrestrial data error variance $C(0) = 9 mGal^2$
3. Integration cap size $\psi_0 = 3^\circ$

Thereafter, with intention to reduce the effect of various systematic errors that have been encountered in the terrestrial gravity and GPS/Levelling data sets and data inconsistencies corrector surface idea was used. Several parametric models have been tested to choose the optimal one. The analysis of the results come out that the seven-parameter model transformation used for modelling the deviations between the gravimetric geoid and the available GPS/Levelling points gives the best results in the experimental area, which reduced the standard deviation from 0.57 m before fitting to 0.18 m.
Finally, a national geoid model at 5’ x 5’ resolution and designated as KazGM2010 has been modeled. The differences between the geoid undulations from GPS/Levelling and KazGM2010 change from \(-0.295\) m to \(0.327\) m and the standard deviation is \(0.177\) m. The relative accuracy achieved was of the order of \(1.219\) ppm for baselines between 15 and 1633 kilometers.

The theory described in this study work is the first step towards the realization of a one-centimetre accuracy geoid model for Kazakhstan.

The following factors should be taken into account in the future studies:

- One of the important factors, which should be considered, is the accuracy estimate of the terrestrial gravity measurements.

- For any future test it is recommended to use much denser high quality GPS/Levelling points. Well distributed GPS/Levelling data (especially in mountainous areas) is important in evaluating and refining the gravimetric geoid model.

- The potential of the Earth Gravitational Model (EGM2008) must be tested, as it is the most up to date and high resolution model up to degree 2160 which has been developed and published by National Geospatial-Intelligence Agency.
References


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