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Performance Analysis of Noisy Message-Passing Decoding of Low-Density Parity-Check Codes

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Abstract— **A noisy message-passing decoding scheme is considered for low-density parity-check (LDPC) codes over additive white Gaussian noise (AWGN) channels. The internal decoder noise is motivated by the quantization noise in digital implementations or the intrinsic noise of analog LDPC decoders. We modelled the decoder noise as AWGN on the exchanged messages in the iterative LDPC decoder. This is shown to render the message densities in the noisy LDPC decoder inconsistent. We then invoke Gaussian approximation and formulate a two-dimensional density evolution analysis for the noisy LDPC decoder. This allows for not only tracking the mean, but also the variance of the message densities, and hence, quantifying the threshold of the LDPC code. According to the results, a decoder noise of unit variance, increases the threshold for a regular $(3, 6)$ code by 1.672dB.**

Keywords——**Low-Density Parity-Check (LDPC) codes, decoder noise, density evolution, consistency, Gaussian approximation.**

I. INTRODUCTION

Low-Density Parity-Check (LDPC) codes– first discovered by Gallager [1], [2] and more recently rediscovered by Spielman *et al.* [3] and MacKay *et al.* [4], [5]– has attracted much research interest due to their outstanding performance. These codes have also been included in recent wireless communication standards ・, [7]. Effective decoding of LDPC codes is accomplished by iterative message-passing schemes such as the sum-product or the belief-propagation (BP) algorithm and the min-sum algorithm [8].

The sum-product algorithm is set up based on elementary computations using sum-product modules [9]. The digital realization of sum-product modules for LDPC decoding involves implementation of approximate real-number arithmetic using quantization. In [10], a digital hardware implementation of LDPC codes is described and a low complexity encoder implementation is suggested. It is experimented in [11] that the sum-product algorithm requires a larger number of quantization bits when compared to its logarithmic alternative for a digital implementation. The quantization effect in LDPC decoding is also considered in [12], where it is shown that the decoder performance is strongly influenced by the quantization. The quantization error is often modelled by an additive noise and under certain conditions is assumed Gaussian [13].

Loeliger *et al.* in [9] introduced sum-product modules based on simple analog transistor circuits using soft XOR gates. Specifically, they have shown that the entire family of sum-product modules can be implemented by variations of a single simple circuit. With these circuits, any network of sum-product modules, in particular, iterative decoding of LDPC codes can be directly implemented in analog VLSI. They expressed that such decoding networks could outperform comparable digital implementations in terms of speed or power consumption.

In analog decoders, the exchanged messages are in general subject to an additive intrinsic noise which depends in part on the chip temperature [14]. To capture this phenomenon, in [14], a channel is considered that is subject to additive white Gaussian noise whose power is a weighted sum of the power of the channel input prior to the current time instant. This channel is motivated by point-to-point communication between two terminals that are embedded in the same chip. The variance of this noise can be considered constant when the heat-sink is ideal [22]. Therefore, in addition to the communication channel noise, the internal decoder noise may affect the communication of soft gates, and hence degrade the performance of the iterative analog decoder.

The performance of LDPC codes is characterized by a threshold on the communication channel quality. For channels whose quality is better than the threshold, the code performs well; otherwise, it incurs a non-negligible probability of error. This is first observed in the case of binary symmetric channels (BSC) and regular LDPC codes in [1] by Gallager. Luby *et al.* [15] have shown that irregular codes outperform regular codes, while their performance still exhibit a threshold effect. Richardson and Urbanke [16] generalized and analyzed this concept using density evolution (DE) and considered various channels including binary erasure, binary symmetric, Laplace and AWGN channels and message-passing decoding algorithms.

As discussed, practical digital or analog LDPC decoders are in general subject to quantization or thermal noise, respectively. Therefore, the effects of noise in iterative decoding process must be investigated. Varshney analyzed the performance of LDPC bit-flipping decoding over a BSC using a noisy message-passing algorithm, whose messages are subject to another binary symmetric channel [17][18]. Specifically, it is observed that the performance degrades

smoothly as the decoder noise increases. In this work, we analyze the performance of LDPC codes over an AWGN communication channel, when a sum-product decoding algorithm is employed whose exchanged messages are degraded by an independent additive white Gaussian noise. For such a setting, we invoke a density evolution analysis to track the probability distribution of exchanged messages during decoding. Specifically, we observe and quantify a threshold in decoding performance that depends both on the noise power of the communication channel and the decoder.

The rest of this paper is organized as follows. In section II, we present the definitions and models for noisy message-passing decoder. Then, in section III, the consistency of the messages passed in the noisy decoder is investigated. In section IV, using Gaussian approximation we derive the density evolution equations for the noisy message-passing decoder. Numerical and simulation results are presented in section V. Finally, section VI concludes the paper.

II. NOISY MESSAGE-PASSING DECODING-BACKGROUND

Consider a regular binary (d_v, d_c) LDPC code with length n and parity check matrix, H. This parity check matrix and the LDPC decoding can be described using a bipartite graph with n variable nodes and $n(d_v/d_c)$ check nodes. Each variable node is connected to d_v check nodes and each check node is connected to d_c variable nodes, corresponding to the columns and rows of H, respectively. Fig. 1 shows a bipartite graph for a regular $(2,4)$ DLPC code with length 6. Any two connected nodes are known as *neighbors*.
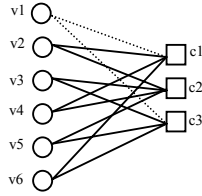

Figure 1. Bipartite graph of the regular (2,4) LDPC code, where □ denotes a check node and O denotes a variable node.

The message-passing decoding algorithm involves exchanging of the outputs of the check nodes and variable nodes in an iterative manner. Specifically, every check node receives messages from its d_c neighbors and after processing sends them back the results after processing. Similarly, the variable nodes do this by receiving the messages from their d_v check node neighbors.

We consider log likelihood ratios (LLRs) as variable nodes and check nodes messages, where the sign of the variable node messages specifies the bit estimate and the magnitude indicates its level of reliability. For the output message of a variable node we have

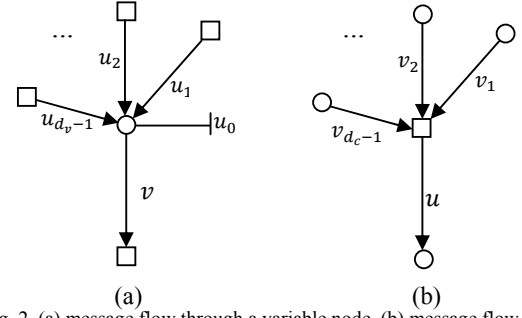$$v = \log \frac{p(y|x = 1)}{p(y|x = -1)}$$

Fig. 2. (a) message flow through a variable node, (b) message flow through a check node.

where x and y are respectively, the bit value of the node and all the information available to the variable node up to the present iteration obtained from edges other than the one carrying v [8]. Similarly, for a check node with u as the output, we have

$$u = \log \frac{p(y'|x' = 1)}{p(y'|x' = -1)}$$

with equivalent definitions for x' and y'.

According to the sum-product decoding algorithm, the message $v^{(l)}$ at iteration l from a variable node to a check node is given by

$$v^{(l)} = \sum_{i=0}^{d_v-1} u_i^{(l-1)} \tag{1}$$

where u_i, $i = 1, \ldots, d_v - 1$, are incoming LLRs from the neighbors of the variable node, except the check node that is to receive the message v, and u_0 is the incoming LLR message from the channel. Also, the updating rule for check node can be obtained as follows:

$$\tanh \frac{u^{(l)}}{2} = \prod_{j=1}^{d_c-1} \tanh \frac{v_j^{(l)}}{2} \tag{2}$$

where v_j's are defined similar to u_i's for check nodes. Figures 2(a) and (b) shows message-passing through a variable node and a check node, respectively.

For a noisy decoding system that is subject to additive white Gaussian noise, the model can be updated as in Figure 3, where n_i^u and n_i^v denote the additive white Gaussian noise affecting the output messages of the soft gates of check nodes and variable nodes, respectively. Hence, γ_i's and μ_j's are noisy versions of v_i's and u_j's, respectively. Therefore, the incoming LLRs to variable nodes and check nodes are now given by

$$\mu_i^{(l)} = u_i^{(l)} + n_i^u \tag{3}$$

$$\gamma_i^{(l)} = v_i^{(l)} + n_i^v \tag{4}$$

where, n_i^u and n_i^v are independent and identically distributed (iid) as $N(0, \sigma_d^2)$.
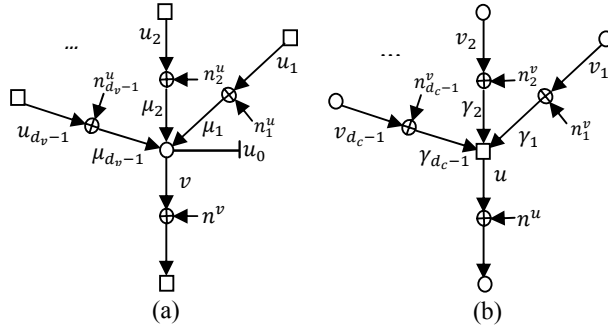
Fig. 3. The model of (a) a variable node (b) a check node in noisy belief propagation decoder.

Based on the sum-product algorithm, the decoding is now set up based on the following updating rules at iteration l

$$v^{(l)} = u_0 + \sum_{i=1}^{d_v-1} \mu_i^{(l-1)} \qquad (5)$$

$$\tanh \frac{u^{(l)}}{2} = \prod_{j=1}^{d_c-1} \tanh \frac{\gamma_j^{(l)}}{2}. \qquad (6)$$

In the next sections, we propose an approach for performance analysis of the presented noisy message-passing decoding algorithm for LDPC codes.

III. CONSISTENCY FOR GAUSSIAN APPROXIMATION

The density evolution is an analytical method to understand the limits and predict the performance of LDPC decoders [19]. For an AWGN channel and a LDPC sum-product decoder, we can approximate the densities of the messages exchanged between the check and variable nodes as Gaussian [8],[20]. Hence, these densities may be characterized only with their mean and variance. A Gaussian random variable whose variance is twice its mean is said to be consistent [19]. The assumption of consistency, allows for setting up density evolution as a one-dimensional equation based on the mean of the message densities. In [8], this assumption is used for DE analysis of a (noiseless) LDPC decoder over an AWGN channel, and subsequently, quantifying the threshold of the code.

In density evolution, the key assumption is that the code block length is sufficiently large, based on which it may be assumed that the LDPC code Tanner graph is cycle free. Since the code is linear and the channel is symmetric, considering the transmission of an all-one codeword using a BPSK modulation, the LLRs received over an AWGN channel are Gaussian distributed with mean $m_0 = 2/\sigma_n^2$ and variance $\sigma_0^2 = 4/\sigma_n^2$, where σ_n^2 is the variance of channel noise [8]. Now if u_i's are Gaussian, independent and identically distributed, then the result of their summation is also Gaussian, and for regular LDPC codes we can assume the variables u, v, u_i and v_i are all Gaussian.

First we check the purport of consistency for a noisy sum-product LDPC decoder. To this end, we consider the expected values of both sides of (3) and (5) and obtain

$$m_v^{(l)} = m_0 + (d_v - 1)m_u^{(l-1)} \qquad (7)$$

where m_v and m_u denote the mean of variable nodes and check nodes, respectively. The index i is omitted as u_i's are iid. Next, if we compute the variances of both sides of (3), we have

$$\sigma_{\mu_i^{(l)}}^2 = \sigma_{u_i^{(l)}}^2 + \sigma_d^2. \qquad (8)$$

And for the variance of variable node output we obtain

$$\sigma_{v^{(l)}}^2 = \sigma_0^2 + \text{var}\left(\sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right) + 2\text{cov}\left(u_0, \sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right). \qquad (9)$$

Since μ_i's are Gaussian and iid, we have

$$\text{var}\left(\sum_{i=1}^{d_v-1} \mu_i^{(l-1)}\right) = \sum_{i=1}^{d_v-1} \text{var}\left(\mu_i^{(l-1)}\right) = (d_v - 1)\sigma_{\mu^{(l-1)}}^2. \qquad (10)$$

The last term in (9) is zero, as the code Tanner graph is assumed cycle-free and u_0 is independent of the noisy messages which add to it to construct the outgoing v. Therefore, the variance of a variable node output is simplified to

$$\sigma_{v^{(l)}}^2 = \sigma_0^2 + (d_v - 1)\sigma_{u^{(l-1)}}^2 + (d_v - 1)\sigma_d^2 \qquad (11)$$

Note that the index i is omitted as the u_i's and μ_i's are iid.

To examine whether a noisy LDPC decoder is consistent, we consider $\sigma_{v^{(l)}}^2 = 2m_v^{(l)}$ and $\sigma_{u^{(l-1)}}^2 = 2m_u^{(l-1)}$ in (11) and compare it with (7). It is clear that as long as σ_d^2 is non-zero, the two are not equivalent and hence the noisy LDPC sum-product decoder is not consistent. As a result, it does not suffice to track only the mean values of the nodes outgoing messages in iterations, and it is required to quantify both the mean and variance of random variables. The simulation results in [21] show a similar situation, when there is an incorrect estimate of the channel signal-to-noise ratio (SNR) at a (noiseless) LDPC decoder; i.e., the ratio of the variance to the mean changes with iteration and therefore, the consistency is violated.
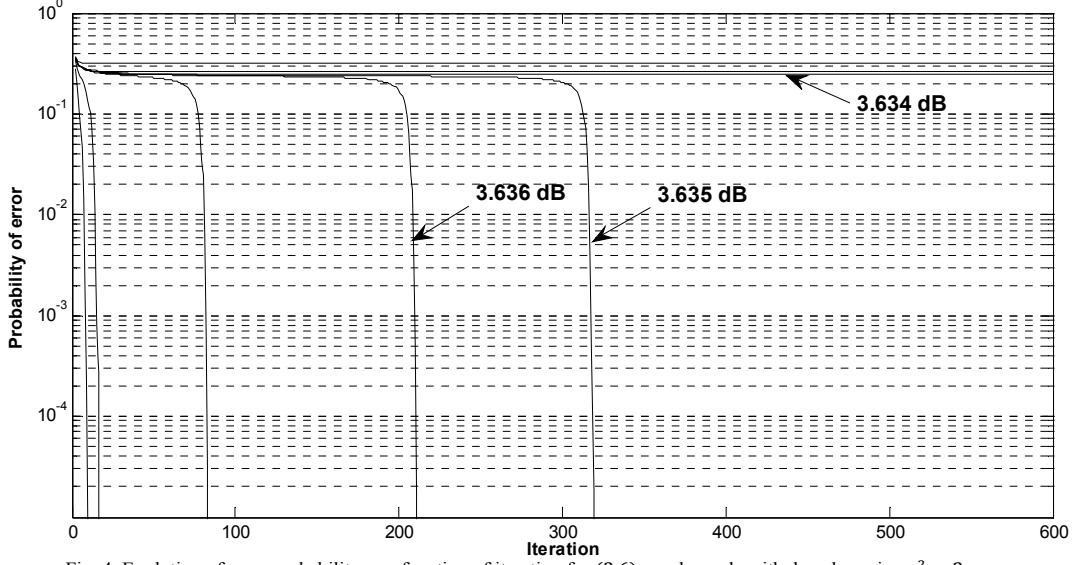
Fig. 4. Evolution of error probability as a function of iteration for (3,6) regular code with decoder noise $\sigma_d^2 = 2$

IV. DENSITY EVOLUTION FOR NOISY MESSAGE-PASSING

In the case of a noisy LDPC decoder, we have shown that consistency does not hold and we should track both the mean and the variance of outgoing LLRs. To this end, we use the key equations (3)-(8) and (11).

By computing the expected value of both sides of tanh rule (6), and noting the independence of γ_j's, we have

$$E\left(\tanh\frac{u^{(l)}}{2}\right) = E\left(\prod_{j=1}^{d_c-1}\tanh\frac{\gamma_j^{(l)}}{2}\right) = \left(E\left(\tanh\frac{\gamma^{(l)}}{2}\right)\right)^{d_c-1}$$

(12)

where, from (4) we have

$$\gamma^{(l)} \sim N(m_v^{(l)}, \sigma_{v^{(l)}}^2 + \sigma_d^2).$$

(13)

Next, by computing the expected value of squared tanh rule, and noting the independency of γ_j's, we obtain the second major equation as follows

$$E\left(\tanh^2\left(\frac{u^{(l)}}{2}\right)\right) = \left(E\left(\tanh^2\left(\frac{\gamma^{(l)}}{2}\right)\right)\right)^{d_c-1}$$

(14)

The density evolution can be obtained by simultaneously solving equations (12) and (14). Specifically, representing $\gamma^{(l)}$ using (13) and $m_v^{(l)}, \sigma_{v^{(l)}}^2$ from (7) and (9), we obtain the DE for check nodes in (15). Therefore, the equations in (15) can be used to track $m_{u^{(l)}}$ and $\sigma_{u^{(l)}}^2$ in decoding iterations of a

regular (d_v, d_c) LDPC and for given values of channel and decoder noise variance. As evident in (15), the effect of decoder noise appears in the check-node DE, with degree of variable nodes, d_v, as a factor.

V. NUMERICAL AND SIMULATION RESULTS

We solved the density evolution equations in (15) iteratively for a (3,6) regular code considering $m_u^{(0)} = 0$ and $\sigma_{u^{(0)}}^2 = 0$ as initial conditions. This provides us with the mean and variance of the check nodes and allows for the computation of the threshold for the given variances of decoder and channel noise.

Figure 4 shows the evolution of probability of error as a function of iteration for several values of channel SNR and a noisy decoder with $\sigma_d^2 = 2$. It is evident that the threshold is 3.635dB in which for values above it, the probability of error converges to zero and for values below it, the probability of error converges to a non-zero value. The threshold for a noiseless decoder is 1.163dB.

Figure 5 shows the relation between the threshold and the LDPC decoder noise standard deviation. It is observed that the SNR threshold increases as the decoder noise standard deviation increases. This is in line with a similar observation in [17] and [18] on the performance of bit-flipping LDPC decoding in the presence of noisy message-passing over BSC channels, where it deteriorates as the cross over probability of decoder noise increases. The interpolation of numerical results follows a linear function as $Tr = 1.163 + 1.7448\,\sigma_d$.

$$E\left[\tanh\left(\frac{N\left(m_{u^{(l)}}, \sigma_{u^{(l)}}^2\right)}{2}\right)\right] = \left(E\left[\tanh\left(\frac{N\left(m_0 + (d_v-1)m_u^{(l-1)},\ \sigma_0^2 + (d_v-1)\sigma_{u^{(l-1)}}^2 + d_v\sigma_d^2\right)}{2}\right)\right]\right)^{d_c-1}$$

$$E\left[\tanh^2\left(\frac{N\left(m_{u^{(l)}}, \sigma_{u^{(l)}}^2\right)}{2}\right)\right] = \left(E\left[\tanh^2\left(\frac{N\left(m_0 + (d_v-1)m_u^{(l-1)},\ \sigma_0^2 + (d_v-1)\sigma_{u^{(l-1)}}^2 + d_v\sigma_d^2\right)}{2}\right)\right]\right)^{d_c-1}$$
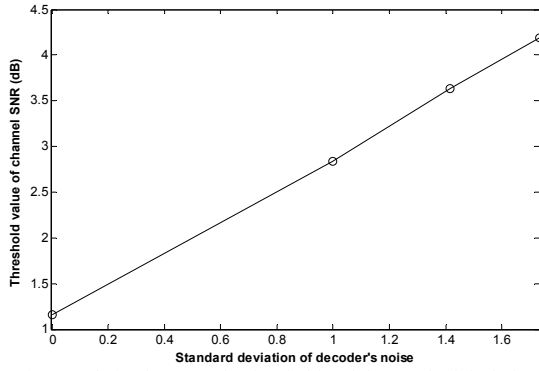
(15)

Fig. 5. Relation between the threshold and the standard deviation of decoder noise σ_d.

Fig. 6. Performance of (3,6) regular finite length code with decoder noise variance σ_d^2.

Figure 6 depicts the simulation results for the performance of a (3,6) regular LDPC code with length 1008. It is evident that the simulation results follow the analytical results reasonably closely even for finite length codes. Also, the thresholds obtained analytically in Figure 5 coincide with those in simulations of Figure 6 with good approximation.

VI.   CONCLUSIONS

The performance of a noisy belief propagation scheme for the decoding of LDPC codes over AWGN channels is investigated. Observing the inconsistency of exchanged message densities in the iterative decoder, a density evolution scheme is formulated to track both the mean and variance of message densities. The results quantify the increase of the decoding threshold as a consequence of the decoder internal noise. In this work, we modeled the decoder noise as AWGN on the exchanged messages, however, an interesting future step is to incorporate other noise models possibly directly obtained from practical decoder implementations. We are also considering extending the current results for regular codes to irregular LDPC codes.

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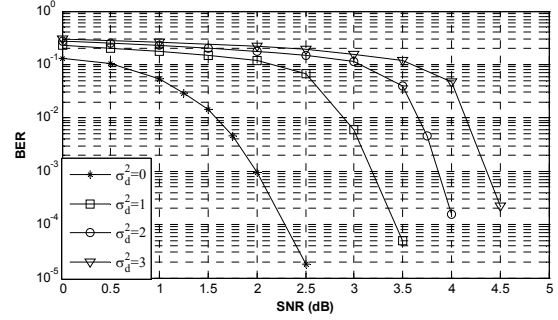
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