Correlation Based Single Tone Frequency Estimation

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Abstract — The single tone frequency estimation problem is studied. Starting with the maximum likelihood estimator and an assumption on high SNR, a generic estimator utilizing a sequence of estimated correlations is derived. Its relation to several existing estimators is given, and an expression for its asymptotic error variance is derived. Simulation results which lend support to the theoretical findings are included.

I. INTRODUCTION

Frequency estimation from noise corrupted discrete time measurement is an important signal processing problem with applications in several engineering areas. In the single tone case the maximum likelihood estimator (MLE) is given by the peak of the periodogram. This method is often implemented using the FFT of the observations followed by a search for the bin corresponding to the maximum of the spectrum. In order to get an unbiased estimate with variance close to the Cramér-Rao lower bound some refinement is required. This can be done by padding the FFT with a large number of zeros, or by using an iterative minimization technique. Both of these approaches may be computationally intensive, and may not be feasible for real-time implementation.

From an assumption on high signal-to-noise ratio, closed-form estimators can be derived. In this paper, such estimators are derived and their performance is characterized.

II. SIGNAL MODEL AND THE MLE

Consider a complex-valued sinusoidal signal buried in noise

\[ x_k = s_k + v_k, \quad k = 0, \ldots, N - 1, \]
\[ s_k = Ae^{i\omega k}, \]

where \( A = |A|e^{i\mu} \) is a complex-valued amplitude, and \( \omega \in (-\pi, \pi) \) is the normalized (angular) frequency. The noise \( v_k \) is zero mean complex-valued circular white Gaussian with variance \( \sigma^2 \). The parameters \((|A|, \mu, \omega, \sigma^2)\) are all unknown, but in this paper the frequency \( \omega \) is the parameter of main interest.

The MLE of \( \omega \) in (1) is well known, given by the location at which the periodogram \( P(\omega) \) attains its maximum, that is, [1]

\[ \hat{\omega} = \arg \max_{\omega} P(\omega), \]
\[ P(\omega) = \left| \sum_{k=0}^{N-1} x_k e^{-i\omega k} \right|^2. \]  

An equivalent formulation of \( P(\omega) \) in (2) is known as the Blackman-Tukey spectral estimator (or as the correlogram), [2]

\[ P(\omega) = \sum_{m=-\lfloor(N-1)/2\rfloor}^{\lfloor(N-1)/2\rfloor} \hat{r}(m)e^{-i\omega m}. \]  

In (3), \( \hat{r}(m) \) is the biased autocovariance estimator, that is

\[ \hat{r}(m) = \frac{1}{N} \sum_{k=m}^{N-1} x_k x^*_{k-m}, \quad m = 0, \ldots, N - 1, \]

and \( \hat{r}(-m) = \hat{r}(m)^* \), where the * denotes complex conjugate.

From the MLE and an assumption on high SNR, approximate MLEs can be derived starting from the observation that a necessary condition for the MLE is

\[ \frac{dP(\omega)}{d\omega} = 0. \]  

Starting with the formulation (2), the condition (5) results in an estimate \( \hat{\omega} \) that is a weighted sum of \( \arg[x_k] \). The resulting estimator is equivalent to the one derived in [3], and thus an alternative derivation of that estimator is provided (the details are given in the Appendix). In order to avoid
the need for phase unwrapping, the estimator in [3] was re-formulated as a weighted sum of \( L[x_k x_{k-1}^*] \) in [4]; See also [5]. These estimators have similar performance as the MLE for signal-to-noise-ratios (SNR) above some threshold.

III. CURRELOGRAM BASED FREQUENCY ESTIMATION

From (3) and (5),

\[
\frac{dP(\omega)}{d\omega} = 2 \text{Im} \left[ \sum_{m=1}^{N-1} m \hat{r}(m) e^{i\omega m} \right] = 0, \tag{6}
\]

where \( \text{Im}[\cdot] \) denotes the imaginary part of the complex-valued quantity within the brackets. Let,

\[
\hat{R}(m) = \sum_{k=m}^{N-1} x_k x_{k-m}^*, \quad m = 1, \ldots, N - 1. \tag{7}
\]

Then, at high SNR, \( \hat{R}(m) \approx R(m) = (N - m)|A|^2 e^{i\omega m} \) where \( R(m) \) stands for (7) evaluated for noise-free data. Thus, \( \hat{R}(m) \approx (N - m)|A|^2 e^{i\omega \hat{R}(m)} \) that inserted into (6) gives

\[
\sum_{m=1}^{N-1} m(N - m) \text{Im} \left[ e^{i\omega \hat{R}(m)} - \omega m \right] = 0. \tag{8}
\]

Using the Taylor series expansion \( \text{Im}[e^{ix}] \approx \text{Im}[1 + ix] = x \) gives

\[
\sum_{m=1}^{N-1} m(N - m)(\omega \hat{R}(m) - \omega m) = 0. \tag{9}
\]

Replacing the parabolic window in (9) with the generic window \( V_m \), truncating the sum after \( M \) terms, and solving for \( \omega \) yields the estimator

\[
\hat{\omega} = \frac{\sum_{m=1}^{M} V_m \omega \hat{R}(m)}{\sum_{m=1}^{M} V_m}. \tag{10}
\]

In (10), replacing the unnormalized sum (7) with the biased or unbiased ACF estimator results in equivalent estimators. Especially, the choice \( V_m = m(N - m) \) and \( M = N - 1 \) is expected to have nearly optimal performance at high SNR. For \( M < N - 1 \), there is no optimality associated with the parabolic window.

Estimators of the form (10) for specific \( V_m \) have been discussed in the literature. In [6], \( V_m = \delta_{m,M} \) (where \( \delta_{m,M} \) is the Kronecker delta) was considered, and in [7] \( V_m = m \) was investigated. The special case \( M = 1 \) is known as the linear prediction estimator, [8]; See also [4].

Since \( \omega \hat{R}(m) \) for \( m > 1 \) holds without ambiguity only when \( \text{arg} \hat{R}(m) \in (-\pi, \pi) \), the mapping \( \{x_k\}_{k=0}^{N-1} \) to \( \{\omega \hat{R}(m)\}_{m=1}^{M} \) should be carried out with a suitable phase unwrapping algorithm. Alternatively, the estimator (10) may be reformulated as given below.

IV. FREQUENCY ESTIMATION WITHOUT PHASE UNWRAPPING

In order to derive a correlation based frequency estimator, that is independent of any phase unwrapping algorithm, the estimator (10) has to be reformulated. Introduce,

\[
\hat{\Phi}(m) = \omega \hat{R}(m) \hat{R}(m - 1)^*, \quad m = 1, \ldots, M. \tag{11}
\]

By definition, \( \hat{\Phi}(0) \) is real-valued, and thus \( \hat{\Phi}(1) = \omega \hat{R}(1) \).

The following summation-by-part formula is easily verified

\[
\sum_{m=0}^{M-1} \Delta G_m F_m = F_0 G_M - \sum_{m=0}^{M-1} G_{m-1} \Delta F_m. \tag{12}
\]

where \( \Delta F_m = F_m - F_{m-1} \) and \( \Delta G_m = G_m - G_{m-1} \). With \( F_m = \omega \hat{R}(m) \), \( \Delta F_m = \hat{\Phi}(m) \), and \( \Delta G_m = V_m \) it follows that

\[
\sum_{m=0}^{M-1} V_m \omega \hat{R}(m) = \omega \hat{R}(M)|G_M| - \sum_{m=0}^{M-1} \omega \hat{R}(m) \hat{\Phi}(m). \tag{13}
\]

where \( F_0 = 0 \) is used. Similarly, with \( F_m = m \)

\[
\sum_{m=1}^{M} V_m m = MG_M - \sum_{m=1}^{M} G_{m-1}. \tag{14}
\]

Inserting (13) and (14) into (10), using \( \omega \hat{R}(M) = \sum_{m=1}^{M} \omega \hat{R}(m) \hat{\Phi}(m) \), gives

\[
\omega = \frac{\sum_{m=1}^{M} (G_M - G_{m-1}) \hat{\Phi}(m)}{MG_M - \sum_{m=1}^{M} G_{m-1}}. \tag{15}
\]

The quantity \( G_m \) follows from \( \Delta G_m = V_m \) and \( G_{-1} = 0 \), that is

\[
G_m = \sum_{i=1}^{m} V_i. \tag{16}
\]

Some special cases of the estimator (15)-(16) are studied below.

A. An approximate MLE

An approximate MLE is given by \( V_m = m(N - m) \) and \( M = N - 1 \). A straightforward calculation gives

\[
\hat{\omega} = \frac{2}{(N^2 - 1)N^2} \sum_{m=1}^{N-1} \alpha_m \hat{\Phi}(m), \quad \alpha_m = (N^2 - 1)N - (m - 1)m(3N - 2m + 1), \tag{17}
\]

where the sequence \( \{\hat{\Phi}(m)\}_{m=1}^{N-1} \) is calculated according to (7) and (11).

B. Fitz’ frequency estimator, \( V_m = m \)

In [7], the estimator (10) with \( V_m = m \) was proposed. From (15) an alternative implementation is

\[
\omega = \frac{3}{M(M + 1)(2M + 1)} \sum_{m=1}^{M} \beta_m \hat{\Phi}(m), \tag{18}
\]

\[
\beta_m = M(M + 1) - m(m - 1). \]
C. Rectangular lag window, \( V_m = 1 \)

For \( V_m = 1 \), the resulting estimator is

\[
\hat{\omega} = \frac{2}{M(M+1)} \sum_{m=1}^{M} (M + 1 - m) \hat{\Phi}(m).
\]  

(19)

For a given \( M \) the estimators (18)-(19) have similar performance in terms of error variance. An estimator similar to (19) was proposed in [9].

D. Lank et. al.’s estimator, \( V_m = \delta_m,M \)

For \( V_m = \delta_m,M \) it directly follows

\[
\hat{\omega} = \frac{1}{M} \sum_{m=1}^{M} \hat{\Phi}(m).
\]  

(20)

The estimator (20) is an alternative implementation of the estimator in [6].

V. Performance analysis

A. Cramér-Rao bound

The lower bound on the variance of any unbiased estimate of \( \omega \) is given by the Cramér-Rao bound (CRB), that for this estimation problem has the form, [1]

\[
\text{CRB}[\hat{\omega}] = \frac{6}{\text{SNR} \cdot (N^2 - 1)},
\]  

(21)

where the signal-to-noise-ratio is defined by \( \text{SNR} = |A|^2/\sigma^2 \).

Truncating the estimator (10) after \( M \) covariance elements results in an inferior performance, that is the error variance is expected to be larger than (21). A tighter bound on the achievable accuracy for this class of estimators is derived below.

B. Minimum variance frequency estimation

In [10], it was shown that the lag window \( V_m \) that minimizes the error variance of the frequency estimate is given by

\[
V_m = \frac{1}{m} \sum_{k=1}^{M} R_{k,m}^{-1},
\]  

(22)

where \( R_{k,p}^{-1} \) denotes the \( k,p \)-th element of \( R^{-1} \). The elements of \( R \) are given by, [10]

\[
R_{k,p} = \frac{1}{\text{SNR}} \frac{1}{kp} \left( \frac{\min(k, p, N - p, N - k)}{(N-k)(N-p)} \right) + \frac{1}{2\text{SNR}} \frac{1}{N-k},
\]  

(23)

The window (22)-(23) depends on SNR that in general is unknown. Thus, this estimator merely has a theoretical relevance. The window (22)-(23) in combination with the expression for the error variance derived below form a lower bound on the accuracy for a given \( M, 1 \leq M < N - 1 \).

C. Asymptotic error variance

Consider the estimator (10). Then the following result holds true.

\[
\text{var}[\hat{\omega}] = \frac{1}{S_3(M, V_m)^2} \left( \frac{S_1(M, V_m)}{\text{SNR}} + \frac{S_2(M, V_m)}{2\text{SNR}^2} \right),
\]  

(24)

where \( \text{var}[\hat{\omega}] \) denotes the asymptotic error variance, and

\[
S_1(M, V_m) = \sum_{m=1}^{M} \sum_{n=1}^{M} V_m V_n \min(m,n,N-m,N-n),
\]  

(25)

\[
S_2(M, V_m) = \sum_{m=1}^{M} \frac{V_m^2}{N-m},
\]  

\[
S_3(M, V_m) = \sum_{m=1}^{M} m V_m.
\]

The proof of (24)-(25) is a straightforward generalization of the proof in [11] where \( \text{var}[\hat{\omega}] \) for \( V_m = m \) (the Fitz’ estimator) was derived.

Not surprisingly, for the parabolic window \( V_m = m(N-m) \) the error variance is minimized for \( M = N - 1 \) for which \( \text{var}[\hat{\omega}] \to \text{CRB}[\hat{\omega}] \) as \( \text{SNR} \to \infty \). The error variance of (18) as function of \( M \) is approximately minimized for \( M = 17/20N \), [11], whereas for (20) the error variance is minimized for \( M = 2N/3 \). For the latter estimator the relative efficiency, that is the quotient of the error variance divided by the CRB, tends to 1.125 as \( \text{SNR} \to \infty \).

D. Numerical complexity

The numerical complexity of the proposed estimators is evaluated as follows: a multiplication of two complex-valued scalars requires 6 floating point operations (flops) (4 multiplications and 2 additions), and a complex addition valued scalars requires 6 floating point operations (flops) (4 multiplications and 2 additions), and a complex addition

\[ 4 \text{M}(2N - M) + 4M - 7 \text{flops.} \]

result is also valid for (17) with \( M = N - 1 \). For (20) the required complexity is \( 4M(2N - M) + 3M - 6 \).

VI. Numerical Examples

A. Performance versus SNR

The performance of the AMLE in (17), and the estimator (20) by Lank et. al. is studied in Figure 1. For the latter estimator, \( M = 2N/3 \) is used, [6]. As reference, the performance of MLE (implemented as a grid search in the periodogram) and the weighted phase averager (WPA), [4], are included in this study. The performance of estimators (18) and (19) is close to the performance of AMLE and is therefore omitted.

For the model (1), 1000 realizations of length \( N = 24 \) are generated. The parameters are \( A = e^{j\mu} \) where \( \mu \) is uniformly distributed in \([0, 2\pi]\) and \( \omega = 2\pi f \) with \( f = 0.05 \) being the normalized frequency. The noise \( v_k \) is zero mean complex-valued Gaussian.
TABLE I
SNR threshold (in dB) as function of number of samples N.

<table>
<thead>
<tr>
<th>N</th>
<th>MLE</th>
<th>AMLE</th>
<th>WPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>32</td>
<td>-4</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>64</td>
<td>-5</td>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td>128</td>
<td>-7</td>
<td>-4</td>
<td>7</td>
</tr>
<tr>
<td>256</td>
<td>-10</td>
<td>-4</td>
<td>8</td>
</tr>
</tbody>
</table>

For high enough SNR the empirical mean-square-error (MSE) of MLE, WPA, and AMLE attains the CRB, whereas the MSE of (20) is approximately 0.5dB above the CRB. The performance curves of MLE and AMLE exhibit similar threshold values around 0dB, whereas the threshold for WPA is approximately 7dB higher.

B. Performance versus M

The performance versus M is studied in Figure 2 where the empirical MSE and the theoretical error variance for (18) and (20) are shown. As reference, the CRB and the empirical MSE for (17) are included in Figure 2.

Here, the SNR is given by SNR = 20dB and the number of runs is 10000. An excellent agreement between the empirical results and the results predicted by theory may be noted. In general there are two minima of the variance curves, where the minimum corresponding to the lower M-value results in slightly inferior performance.

C. Performance versus N

In Table I, the SNR threshold as function of N is shown for MLE, AMLE, and WPA. The same experiment conditions as above are considered. The threshold is measured as the location where the MSE starts to deviate from the CRB, based on performance curves similar to the ones in Figure 1 averaged out of 100 simulation runs. From the figures in Table I, one may note that the thresholds for MLE and AMLE are shifted to lower SNR as N increases. For the WPA, however, the threshold is shifted to higher SNR as N increases. See [5] for an explanation of this behavior.

VII. CONCLUSIONS

A class of correlation based single tone frequency estimators has been derived utilizing an assumption on high SNR. The estimators are formulated in terms of differentiated autocovariances, and thus no additional unwrapping of the phase is required. A novel approximate maximum likelihood (AMLE) estimator has been derived, and alternative implementations of existing suboptimal estimators have been proposed. It has been shown that the performance of AMLE is nearly optimal, that is the error variance is close to the CRB for high and moderate SNR.

APPENDIX

I. TRETTER’S FREQUENCY ESTIMATOR

Here, an alternative derivation of the estimator in [3] is given. From (2) and (5),

\[
\frac{dP(\omega)}{d\omega} = 2 \text{Im} \left( \sum_{k=0}^{N-1} a_k^* e^{i\omega k} \right) \left( \sum_{k=1}^{N-1} k x_k e^{-i\omega k} \right) = 0 \tag{26}
\]

For high SNR, \( x_k \approx s_k = Ae^{i\omega k} \), and thus \( x_k \approx |A|e^{i\omega k} \). Inserting the latter expression for \( x_k \) into (26) gives

\[
\text{Im} \left( \sum_{k=0}^{N-1} |A|e^{i(\omega k - \omega k s)} \right) \left( \sum_{k=1}^{N-1} |A|ke^{i(\omega k s - \omega k)} \right) = 0 \tag{27}
\]

Using the Taylor series expansion \( e^{ix} \approx 1 + ix \), a tedious but straightforward calculation solving for \( \omega \) results in the estimator

\[
\hat{\omega} = \frac{6}{N(N^2-1)} \sum_{k=1}^{N-1} k \hat{\omega}(x_k) - (N-1) \sum_{k=0}^{N-1} \hat{\omega}(x_k) \tag{28}
\]

The estimator in (28) is the estimator in [3]. In [3] it was observed that \( \hat{\omega}(x_k) \approx \omega k + \mu + v_k \) where \( \mu = \text{arg}[A] \) and \( v_k \) is a real-valued white Gaussian noise. Thus the MLE of \( \omega \) and \( \mu \) is given by a least squares fit, that results in the estimator (28).

REFERENCES

Fig. 1. The frequency error variance as a function of SNR for $N = 24$. The diagrams show empirical mean-square-error for the MLE, the weighted phase averager (WPA), the approximate MLE (17), and the estimator (20) with $M = 16$. In the lower diagrams the theoretical error variance is indicated by dashed lines.

Fig. 2. Error variance versus number of correlations used for (18) (dashed line) and (20) (dashed-dotted line). For (18) and (20) the empirical MSEs are depicted by "*" and "x", respectively. As reference, the CRB (solid line), the lower bound (22)-(25) (solid line), and empirical MSE of AMLE ("o") are shown.