A Comparison Between Variational Inequality and Elrod-Adams Simulations of Reynolds Flow with Cavitation

Master’s Thesis in Computational Science and Engineering

Lei Sun, Xing Liang
A Comparison Between Variational Inequality and Elrod-Adams Simulations of Reynolds Flow with Cavitation

Master’s thesis in Computational Science and Engineering

School of Information Science, Computer and Electrical Engineering
Halmstad University
Box 823, S-301 18 Halmstad, Sweden

October 2011
Abstract

Despite considerable research related to lubrication with cavitation, significant simulation results and relative comparisons remain. To accomplish that, we make use of the finite element method which is seen as a better method to simulate lubrication with cavitation models such as the variational inequality model and the Elrod-Adams model. Different techniques are used during the process of resolution by the finite element method. A penalty method is used for the variational model and an iterative method is used for the Elrod-Adams model. Finally, some simulation results are presented to show different aspects of the two models.
Acknowledgements

The project, a comparison between the variational inequality and the Elrod-Adams methods for simulations of Reynolds flow with cavitation, took three months to finish. It was a challenge and hard work for us. We want thank some people for helping us to succeed in doing the project. First we thank our supervisor, Peter Hansbo. He is always ready to help us and give us some very important materials and suggestion. When we had troubles in programming, he gave us important advice and taught us a lot. He helps us to find the right direction to finish the project. He is so kind and strict. We couldn’t have finished the thesis without him. We also want to thank our classmates and parents. They always cared about our project and give us some advice.

Key words: Lubrication, cavitation, penalty method
1 INTRODUCTION

1.1 General View

Lubrication is fundamental for manufacture machines which separate two relative moving surfaces by use of a fluid film. The advantage of lubrication lies in reducing the friction and preventing the two surfaces from direct contact in order to protect the components and parts. The research in lubrication has three branches: fluid film lubrication, boundary film lubrication and dry friction. Fluid film lubrication is the study of two surfaces and the intermediate fluid, boundary film lubrication is to study the critical state between the fluid film lubrication and dry friction, whereas dry friction means friction without fluid medium. Further, fluid film lubrication has two research fields: hydrodynamic lubrication and hydrostatic lubrication. The dividing basis is the underlying theory of the fluid lubrication film which is caused by the relative action of the two surfaces, called hydrodynamic lubrication, or by the external force, called hydrostatic lubrication.

The evaluation and the influencing factors of the hydrodynamic lubrication work performance is a subject of considerable practical significance. In terms of that subject, an important phenomenon of fluid film lubrication is cavitation. If this occurs, it will cause deterioration of the hydraulic system which includes vibration, noise and bubble formation. Bubble formation can upset the continuity of the fluid, cause obstacles in oil pipes, reduce efficiency of volume, break the parts, shorten the life of the element or oil pipe, and contribute to the vibration of flow and pressure.

Therefore, the cavitation phenomenon of the hydrodynamic lubrication will be addressed. The Reynolds equation that is a general tool to resolve the hydrodynamic lubrication problems is not always established with regards to cavitation. To see the effect of subatmospheric pressure in hydrodynamic lubrication with cavitation, two popular cavitation models for lubrication, the variational inequality model and Elrod and Adams’ $p - \theta$ model are considered. We aim to apply the finite element method to simulate the two models and compare the results.

1.2 Outline

The paper is divided into six parts: this part covers the introduction and general view of the paper and some basic definitions such as hydrodynamic lubrication; the second part involves the basic model (Reynolds Equation) and some application conditions that will be addressed in the paper; the third part introduces the cavitation models including the variational inequality model, the Elrod and Adams’ $p - \theta$ model and the finite element method; the fourth part of the paper illustrates the numerical simulation of the two models and compares the results; the last part is conclusions.
2 LUBRICATION: BASIC MODEL

2.1 Reynolds Equation

Since hydrodynamic lubrication is an important field in engineering and manufacture, lots of research methods and techniques concerning hydrodynamic lubrication have appeared in the scientific literature over a long time. However, the Reynolds equation presented by Reynolds in 1886 is always viewed as the milestone, the foundation of which is the fluid continuity equation[1].

In view of the mathematical formulation, the computation of hydrodynamic lubrication is generally to resolve the transition form of Navier-Stokes, in other words Reynolds equation is an approximation of the Stokes equations, as reported by Guy and Michele[2]. By resolving Reynolds equation, the represented through pressure distribution in a thin lubrication oil film between two surfaces could be approximated by.

\[
\frac{\partial}{\partial X} \left( \frac{H^3}{\mu} \frac{\partial P}{\partial X} \right) + \frac{\partial}{\partial Y} \left( \frac{H^3}{\mu} \frac{\partial P}{\partial Y} \right) = 6 \left( \frac{\partial}{\partial X} ((U_0 + U_H)H) + \frac{\partial}{\partial Y} ((V_0 + V_H)H) + 2(W_H - W_0) \right)
\]  

(1)

where \( P(x, y) \) is the kinematic pressure in the oil film and \( H(x, y) \) is the gap between two surfaces.

Assume \( u \) is the velocity of the fluid in the \( x \) direction and the boundary condition is \( u(0) = U_0 \) and \( u(H) = U_H \) and set

\[ U = U_0 + U_H \]

Further, assume \( v \) is the velocity of the fluid in the \( y \) direction and the boundary condition is \( v(0) = V_0 \) and \( v(H) = V_H \) and set

\[ V = V_0 + V_H \]

The density is an another physical variable in hydrodynamic lubrication and it can be seen as a constant when the lubricant isn’t compressed or affected by environment variable such as temperature. Otherwise the equation could read:

\[
\frac{\partial}{\partial X} \left( \rho H^3 \frac{\partial P}{\partial X} \right) + \frac{\partial}{\partial Y} \left( \rho H^3 \frac{\partial P}{\partial Y} \right) = 6 \left( \frac{\partial}{\partial X} (U \rho H) + \frac{\partial}{\partial Y} (V \rho H) + 2(W_H - W_0) \right)
\]

(2)

The left-hand side of Reynolds’ Equation signifies that the lubrication film pressure on the lubrication surface changes with the coordinates \( x \) and \( y \) and the right-hand side presents the different effect of lubrication film pressure. Expanding the right-hand side, its physical meanings are as follows:

\( (a) \). Dynamic pressure effect:

\[ U \rho \frac{\partial H}{\partial X}, V \rho \frac{\partial H}{\partial Y} \]

These two terms illustrate that the film exists due to relative motion of the two surfaces and the height changing in \( x \) or \( y \) direction contributes to the dynamic pressure effect. In this condition, the shape of the lubrication film is different from the original shape.
According to the flow continuity conditions, the system automatically decreases the input flow and increases the output flow in order to balance the volume flow of each cross section though the shape of film is different from the original state. Later, cavitation will be considered based on this effect.

(b). Stretch effect:

\[ \rho H \frac{\partial U}{\partial X}, \rho H \frac{\partial V}{\partial Y} \]

If the surface velocity varies in space because of elastic transformation of solid surface or other reasons, the different volume flow of each cross section contributes to the flow pressure.

(c). Variable density effect:

\[ U H \frac{\partial \rho}{\partial X}, V H \frac{\partial \rho}{\partial Y} \]

In this condition, the flow pressure is caused by the different mass flow for the lubrication density decreasing gradually along the moving direction, though the volume flow of each cross section is the same.

Among the three effects, the first one is usually the main factor contributing to the lubrication film pressure [23].

2.2 Boundary Conditions

One of the central problems in resolving Reynold’s equation is to set the boundary conditions. A large number of research exists on overlooking the precise details of the cavitation phenomenon and make simplifying assumptions such as Sommerfeld boundary conditions, which assume the bearing gap is filled with oil and the oil film is continuous and includes a negative pressure region, which is not quite reasonable.

Semi-Sommerfeld boundary conditions assume that the bearing gap is separated into an oil film region and a cavitation region. In this condition, the pressure in the negative region is set zero, which we define as atmospheric pressure. The theory considers the condition that the oil film could include the cavitation part. However, this viewpoint has no general meaning because the position of the cavitation zone should be established firstly.

Reynolds boundary conditions assume that the oil film will rupture when the gradient value of the pressure decreases to zero. Jacoboson-Floberg and Olsson[3] boundary conditions are based on the continuity of mass flow through the entire bearing is composed of two regions: one is the conserving mass region at the contact between the full film in which the pressure \( p > 0 \) and another one is the cavitation region at rupture and reformation boundary in which the pressure \( p \leq 0 \). The JFO boundary conditions have enriched the Reynold conditions and evaluate the performance of bearing more precisely, because they not only supply the conditions of the oil film rupture but also regeneration. The JFO boundary conditions could be seen as a reasonable boundary condition, but also have drawback that is not easy to simulate in computer codes because of the complication caused by the numerical treatment of the non-linear boundary conditions [4].
In later research, some other boundary conditions have been proposed such as Separation boundary conditions and Double Reynolds boundary conditions[5].

Figure 1: Here is the Boundary conditions figure: The upper left figure presents the Sommerfeld boundary conditions; The upper right figure shows the Semi-Sommerfeld boundary conditions; the second line two figures are Reynolds boundary conditions and Jacoboson-Floberg and Olsson boundary conditions

2.3 Reynolds Equation Formulation

Assume between two surfaces $\Gamma_1$ and $\Gamma_2$ which moves relatively there is a thin film with viscosity $\mu$. Where $\Gamma_1$ is stationary and $\Gamma_2$ moves with velocity $U = (U, 0, 0)$. The modified Reynolds’ equation is as follows:

$$-\nabla \cdot (H^3 \nabla P) = -6\mu U \frac{\partial H}{\partial x}$$

(3)

As show above modified Reynolds equation could be seen as Poisson function in $\Omega = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$, which is formulated as:

General form:

$$\nabla \cdot (k \nabla P) = f \quad \text{in} \quad \Omega$$

(4)

Dirichlet boundary conditions:

$$P = 0 \quad \text{on} \quad \partial \Omega$$

Neumann boundary conditions:

$$\frac{\partial P}{\partial n} = 0 \quad \text{on} \quad \partial \Omega$$
3 CAVITATION MODELS AND METHODS

3.1 Cavitation

Cavitation, in the sense of empty cavities in a liquid, is an important factor of wear in some engineering parts. Considering the cavitation in the lubricated devices, two models occupy a decisive position till now. The first model is the variational inequality model on the basis of the Reynolds equation introduced above. The second model is the mass-conserving Elrod-Adams model which could be seen as a useful tool to simulate hydrodynamic lubrication involving cavitation.

Reynolds equation is well known to resolve fluid pressure of the lubrication film, but the pressure is not constant because of the moving of the surfaces between the lubrication film and the load enforced it. Therefore, the pressure causes the film to rupture where the cavitation phenomenon happens, which makes the lubrication analysis complicated and plays a significant role in lubrication researches.

When cavitation shows up in the normal lubrication problem, the two models assume the whole region $\Omega$ is divided into two regions: an active region ($\Omega^+$) and a cavitation region ($\Omega^0$). In the active region the pressure is positive; in the cavitation region the pressure is zero. The problem will be complicated if the boundary ($\Sigma$) between the active region ($\Omega^+$) and the cavitation region ($\Omega^0$) is not defined firstly. For the two models the position of the boundary ($\Sigma$) is given implicitly and the pressure in the active region approximated to zero on the boundary.

\[ \Omega^+ = \{ x, y \in \Omega; \quad P(x, y) > 0 \} \]

Cavitation region

\[ \Omega^0 = \{ x, y \in \Omega; \quad P(x, y) = 0 \} \]

\[ \Sigma = \{ x, y \in \partial \Omega^+; \quad \lim_{\partial \Omega^+} P(x, y) = 0 \} \]

Reynolds equation is a second order partial differential equation, and it is hard to get the exact solution without significant error. However with the rapid progress in computer technology, it is reasonable to propose some methods to solve Reynolds equation at an affordable cost.

The finite difference method, the finite element method and the boundary element method are three popular numerical methods to resolve partial differential equations. The theoretical bases, the mode of discretization and the strength of the three numerical methods are different:

1. The finite difference method is based on Taylor’s formula, and replaces the solution region with finite different stencil on a grid of nodes

2. The finite element method is a generally used numerical method that is based on the variational principle and the weighted residual method. The method divides the solution region into finite elements which are not overlapped. Comparing with the finite difference method, this method can handle problems with complex geometries not limiting to the rectangle or simple geometries.

3. The boundary element method uses discretizations on surfaces, typically for use with objects in free space.

In this paper, the finite element method with certain techniques will be used to resolve the cavitation models.
3.2 Finite element method

The finite element method (FEM) is an effective method which is often applied to numerical problems over different domains described by partial differential equation. The earliest concept is what Courant mentioned in 1943, which is to define piecewise continuous functions on triangular regions[9]. Early engineering versions can be traced back to the research of Turner, Clough, Martin and Topp[10]. The formal name finite element method was adopted by Ray W. Clough who used the method to resolve problems in airplane and civil engineering domain in 1960[11]. The concept of FEM is to find an approximate solution comparing to the exact solution by transforming a complicated problem to a simple problem. The steps could be summarized as follows:

1. Discretize the whole solution region into element subdomains
2. Choose the appropriate interpolating function
3. Establish the FE equations in local coordinate system
4. Transform the FE equations into global coordinate system
5. Assemble the system matrix based on system connectivity
6. Impose boundary conditions and solve the FE equations

Since Reynolds equation is an elliptic partial differential equation, the finite element method has been used to obtain numerical solutions of journal bearing lubrication problems. In 1991 Kumar and Booker proposed a model to resolve complicated lubrication problems, which handled transient evolution of cavitation in lubrication. The model is based on mass-conservation and resolved by the finite element method with algorithms given for implementation in both direct problems (specified motion) and indirect problems (specified load). Later researchers deal with applications[12][13]. For the purpose of treating hull cavitation in underwater-shock problems, Felippa and Deruntz describe a theoretical formulation and computational implementation of a method in 1984[14]. In 1998, Tomomi applied an upwind finite element method to predict cavitating flows in arbitrarily shaped channels[15].

Here the finite element form of Reynolds equation (Eq. (4)) can be found. The finite element solution is the solution of the weak form of the partial differential function. For the active region, multiply any test function $v \in V$ firstly, because in the active region the Reynolds equation holds established. Then integrate in the region $\Omega^+$. General form:

$$\int_{\Omega^+} \nabla (k \nabla p) v d\Omega = \int_{\Omega^+} f v d\Omega$$

Using Greens' function:

$$\int_{\partial \Omega^+} n \cdot k \nabla p v_s - \int_{\Omega^+} k \nabla p \nabla v d\Omega = \int_{\Omega^+} f v d\Omega$$

Consider the boundary conditions:

$$-\int_{\Omega^+} k \nabla p \nabla v d\Omega = \int_{\Omega^+} f v d\Omega$$

(6)
3.3 Variational inequality model and method

3.3.1 Model

As seen in Eq.(5), we only talk about the active region ($\Omega^+$). For the active region, the value of the pressure could be seen as: $P = 0$ on the boundary $\Sigma$ and $P > 0$ in $\Omega_+$, because if $P < 0$ that means the pressure is lower than the atmospheric pressure (recall that we defined atmospheric pressure as zero pressure). In this case, the film will detach and cavitation will happen. The complete film has been changed into an incomplete film, which makes the whole system not able to fulfill Reynolds’ equation. However, the lubrication with cavitation problem can also be resolved by Reynolds equation by modifying some terms, which is the variational inequality, sometimes called Reynolds inequality.

Let $c$ be a typical thickness of the film and set $[6]

\begin{align*}
  p & := \frac{Pc^2}{6\mu U} \\
  d & := \frac{H}{c} \\
  f & := -\frac{\partial d}{\partial x}
\end{align*}

Seek $p \in K$ where $K = \{v \in H_0^1\Omega : v \geq 0\}$ to resolve the variational Reynolds inequality model

$$\int_{\Omega} d^3\nabla p \cdot \nabla(v - p)d\Omega \geq \int_{\Omega} f(v - p)d\Omega \quad v \in K$$

3.3.2 Penalty method

To resolve multivariate function minimum or maximum problems, some constraints should be satisfied. The penalty method which is proposed by Fiacco and Mccormick in 1968 is seemed as a technique transform constrained problem to unconstrained problem $[16]$. The basic model is as follows:

Objective function:

$$Y = F\{X\} \quad X = \{x_1, x_2, ..., x_n\}$$

Constrains functions:

$$g_k(X) = C_k \quad k = 1, 2, ..., l$$

Penalty term:

$$\sum_{i=1}^{N} p_k(g_k(X) - C_k)^2$$

Modified function:

$$\varphi = Y + \sum_{i=1}^{N} p_k(g_k(X) - C_k)^2$$

Here, $l$ is the number of constraints ($l < n$) and $p_k$ is an arbitrary large positive penalty parameter. The modified objective function is constructed from the original objective function and the penalty term. Without any constrains, the modified objective function could be resolved as an unconditional extreme value problem. During the computation, the extreme value show up at the point $X^* = \{(x_1)^*, (x_2)^*, ..., (x_n)^*\}$, which should satisfy such the condition $g_k(X^*) - C_k \approx 0$.
3.3.3 Finite element penalty method

The technique combining a penalty term with the variational form to define a finite element method is often used to resolve partial differential equation with complicated boundary. From the view of finite element method, the approximation will be penalized when it deviates from the value of the exact solution in the Dirichlet boundary conditions. Importing the penalty method into the variational form of the finite element method has also been researched more generally.

The penalty method used generally recently is closest to that one to solve the second order linear elliptic equation such as Possion equation and Laplace equation, which is presented by Wheeler[17] in 1973, Babuska used a penalty method to analyze the Possion model partial differential equation with homogeneous Dirichlet boundary condition[18]. Some authors present a penalty parameter which depends on the smoothness of problems to be resolved[19][20].

Later in Babuska’s research, he set up a new scheme which illustrates that the choice of the penalty parameter for some types of problems has no significant effect on the rate of convergence by constructing a new computational scheme[18]. According to this, the penalty parameter could be selected arbitrary. However, a parameter which has relationship with local mesh size and film height is chosen in this model[21][22]. This choice ensures optimal convergence for our finite element method.

3.3.4 Variational inequality penalty finite element formulation

Divide $\Omega$ into several quasiuniform quadrilateral elements. Let $T$ denote an element and let $\tau (\tau = \{T\})$ be the set of all of the elements. The local mesh size is $h$ which could be regarded as a piecewise constant function such that $h(x, y) = h|_T$ for $x, y \in T$.

Here $V_h$ is:

$$V_h = \{ v \in H^1_0(\Omega) : v|_T \in P^1(T), \forall T \in \tau \}$$

Consider cavitation modeled by the Reynolds inequality as in Eq.(6). By using a penalty term, the inequality could be transformed into an equality and the pressure could be gotten. Here importing the penalty term into the variational form:

$$\int_{\Omega} d^3 \nabla p_h \nabla v d\Omega + \int_{\Omega} 1/3 p^-_h v d\Omega = \int_{\Omega} f v d\Omega \quad (8)$$

Where the penalty parameter is piecewise function talked above, assume $\varepsilon = \gamma^{-1}s$, where $\gamma$ is a constant and the $s$ is the area of each element.

$$\beta(p_h) = \begin{cases} 0 & \text{if } p_h \geq 0 \\ p_h/\varepsilon & \text{if } p_h < 0 \end{cases}$$

We define $p^-_h := \min(p_h, 0)$ and write:

$$\int_{\Omega} d^3 \nabla p_h \nabla v d\Omega + \int_{\Omega} 1/3 p^-_h v d\Omega = \int_{\Omega} f v d\Omega \quad (9)$$
3.4 Elrod and Adams’ \( p-\theta \) model and its implementation

3.4.1 Model

The variational inequality model is based on a mathematical formulation and the physical meaning is not quite clear. In this part, the Elrod-Adam model, which has a clear physical meaning, will be introduced.

As shown above, the JFO boundary condition is based on the mass continuity flow which could be seen as an earlier mass-preserving cavitation theory. The Elrod-Adams model used generally [7] was presented by Elrod and Adam in 1974 and satisfies the conservation of mass flow across the rupture and reformation boundaries. The advantage of the model is using one equation to present the whole lubrication domain and one switch function to show the pressure in the complete film region and the incomplete film region (cavitation region).

The model includes two unknown parameters, one of which is the pressure \( p \) and the other is in the range of zero to one (the saturation of fluid in the mixture)[8]. In the complete film region, the pressure is larger than zero \((p > 0, \theta = 1)\); in the incomplete film region, \( \theta < 1 \).

Considering cavitation phenomenon and importing the saturation function, the Elrod-Adams model can be formulated by modifying the Eq.(3):

\[
-\nabla \cdot \left( H^3 \nabla P \right) = -6\mu U \frac{\partial (\theta H)}{\partial x} \quad \text{in} \quad \Omega
\]

For the above equation, when computing the pressure, some constraint conditions should be considered:

In the active region \((\Omega^+)\)

\[
P > 0 \quad \theta = 1
\]

(11)

In the cavitation region \((\Omega^0)\)

\[
P = 0 \quad 0 < \theta < 1
\]

(12)

3.4.2 Elrod and Adams’ \( p-\theta \) finite element iterative formulation

1. Compute the initial pressure: Without considering the cavitation, compute the pressure for each node \((p_{i,j})\) on \( \partial \Omega \) is zero and in \( \Omega \) using

\[
\int_{\Omega} (k \nabla p) \nabla v d\Omega = -\int_{\Omega} \theta f vd\Omega
\]

2. Modify the initial pressure: pick out cavitation nodes by checking whether \((p_{i,j}) < 0\), if it’s true \((p_{i,j})=0\) is set

3. Optimize model with iterative method: under a certain condition of convergence, use the relationship between \( p \) and \( \theta \) to optimize the model. For each step, update \( p \) or \( \theta \) by iterative method,

\[
\int_{\Omega} (k \nabla p^{n+1}) \nabla v d\Omega = -\int_{\Omega} \theta^n f vd\Omega
\]

\[
\int_{\Omega} (k \nabla p^n) \nabla v d\Omega = -\int_{\Omega} \theta^{n+1} f vd\Omega
\]
4 NUMERICAL SIMULATIONS

4.1 Density Distribution Comparisons

4.1.1 Density distribution comparison for different mesh sizes

The two figures are the pressure density figure, where the horizontal axis is the pressure value and the vertical axis is the density. The objective to draw the pressure density figure is to show the frequency of one pressure value. From the figure, the pressure density distribution could be gotten. In the view of the graph, each figure is composed of four curves which have similar curvilinear trends. With the pressure increasing, the density grows from zero to the maximum and then decreases to zero gradually at last. The key point in the graph is that the most parts of the different curves is coincide which shows that the density distribution is almost the same for different mesh size.

Comparing with the first model simulation, the shape of the density distribution figure is different. The first model density distribution does not follow some normal distribution rule, but the second model is uniformity just as a Gaussian distribution.
4.1.2 Density distribution comparison for different parameters

![Density distribution for variational inequality model](image1)

![Density distribution for variational inequality model](image2)

From the view of the Figs.4, 5, we show the density distribution changing with the parameters for 1024 elements mesh size. The x coordinate corresponds to the pressure, and the y coordinate corresponds to the density. Different colors present density with different parameters. For example, to the left in Fig.3, the red color has the least radius. Fig.4 shows the density distribution for the variational inequality model. From the view of the left graph, the density distributions with different radii R have similar curvilinear trends. The density increases to the maximum value rapidly and decreases to the zero gradually. The small radius, R=0.05, has the largest value about 11 among the others. Compared to the left graph, the right one shows that the density distributions with different height, which have the similar curvilinear trends like the radius. The difference between them is that the least height has the largest density value which is smaller than one generated by the least radius. The two graphs in the Fig.4 show that the density distribution has no significant different for various parameters.

The graph in Eq.5, shows the relationship with density distribution and parameters for Elrod and Adams model. They are different from that of the variational inequality model. The density grows from zero to the maximum 0.4 and then decreases to zero smoothly and gradually for each parameter combination. That means changing the parameters has no effect on the density distribution.
4.2 Pressure Distribution Comparisons

As shown above, the figures are the projection on $XOZ$ of the $3-D$ pressure figures for 64 elements, 256 elements, 1024 elements and 4096 elements. The horizontal axis is
the $x$–coordinate of each node and the vertical axis is the pressure of corresponding node. Considering the unit region $\Omega$, the $x$-coordinate is $x \in \{0, 1\}$.

The maximum value of the pressure for the variational inequality model could be gotten around the $1.6$, but the value for Elrod and Adams’ $p–\theta$ model is not stable for different mesh sizes. As shown in Fig.5 the value in the sub figures is about $1.6$, but that in intermediate two figures is less than $1.5$. The minimum value of the pressure could not be presented because the figure is just an outline of the $3–D$ pressure figure and the negative pressure value is tiny.

Comparing the two figures, the smooth curve degree of the different mesh size could be shown. Figures with more elements become smooth.

4.3 Contour Pressure Comparisons

4.3.1 Contour pressure comparison for different mesh size

![Figure 8: Different mesh size for variational inequality model](image)

The two figures are contour figures for different mesh sizes. The values of the horizontal axis and the vertical axis give the information of the position of the nodes and the curves with value number are the contour lines for the same pressure value.

The four sub contour line figures for each figure present the contour lines changing with different mesh size. From the coarse mesh size to small mesh size the figure becomes from sparse to tight, from irregular to uniform. Especially the contour line for $4096$ elements is obviously clear.

Here we not only illustrate the pressure contour curves but also the separation dots (red dots) where the pressure is lower than the atmospheric pressure. The symbols present where the cavitation phenomenon happens and where the penalty function is enforced. For the each curve with same value such as $0.2$ means each nodes on the curve has the same pressure value. With the mesh size increasing, the number of red nodes
(the nodes with negative pressure) increases progressively.

Compare Fig.9 with Fig.8, the separation dots (red dots) where the pressure is lower than the atmosphere pressure can’t be seen in any figure. That doesn’t mean the cavitation phenomenon has not happened during the process of the two surface relative moving. It only means that the final results are without negative pressure. That is a significant difference from the first model numerical simulation.

4.3.2 Contour pressure comparison for different parameters

There now follows, the two figures with $3 \times 3$ sub figures for each. For each figure, the h value increases for constant R value in the horizontal direction and the R value increases for constant h value in vertical direction. Obviously, the sub figures becomes denser with the R value increasing in vertical line, which means the pressure increment speed becomes faster. The R value and the h value have some relationships with the initial positive pressure value.
Figure 10: Counter Pressure for Variational inequality model

Figure 11: Counter Pressure for Elrod and Adams model
4.4 Load Comparisons

4.4.1 Load comparison for different mesh size

Fig. 12: Load distribution for different mesh size

Fig. 12 shows load value trends for different mesh sizes. Here the load means the counter force when the two surfaces move with the extral forces. The left one is the simulation for the variational inequality model and the right one is the Elrod and Adams’ $p - \theta$ model.

The $x$ – coordinate shows the cycle-index (convergence rate), in which the maximum number of the left one is 9, the right one is 1000. That means that the convergence rates for the two simulations are obviously different.

The $y$ – coordinate present the load value, where the minimum number of the left one is less than zero, the right one is larger than 0.06 and the maximum number of the two figure is almost same around the 0.2.

Generally we see that the trend of the three lines is similar except the line for 64 elements in the left figure. All of the four lines go sharply for the first two or three cycles and the gently for the last period. However, the trend of the right one is like zigzag and the fluctuations 1024 elements is larger than 256 elements but the two both larger than 4096 elements.
4.4.2 Load comparison for different parameters

As shown in the left figure of Fig.13 or Fig.14, it gives some information about how the R value influences the load distribution. From the view of the trend of the curves, the R value has no obvious effect on the shape. However, load value increases with the increasing of the R value which means the load value and the R value are positive correlation. For the variational inequality model, the growing rate decreases gradually; for the Elrod and Adams’ $p-\theta$ model, the counterpart is stable except from the first step.

The right figure of Fig.13 or Fig.14 reveals the relationship between the h value and the load distribution. The changing of the h value has no significant effect on load value and tendency. When h=0.15, those two index show unstable as curves with h=0.2,h=0.25,h=0.30.
5 CONCLUSIONS

By comparing simulations coming from the two models, conclusions could be drawn from four perspectives: density distribution, pressure distribution, contour pressure comparisons and load comparisons.

1. Density distribution: The cavitation for the Elrod-Adams model has a curve which is similar to a normal distribution. It has a stable maximum density value. However, the variational inequality model is a little different. The model has a larger maximum density value and sharp curvilinear trend.

2. Pressure distribution: The maximum value of the pressure for variational inequality model is close to 1.6. It is still stable. For the Elord-Adams model, the maximum value of the pressure varies from about 1.4 to 1.6. It fluctuates a lot.

3. Contour pressure comparison: They have similar contour pressure distribution trend and variation tendency for different parameters.

4. Load comparisons: For the variational inequality model, all curves are smooth and increase to maximum value gradually. However, the four line of the Elrod-Adams model are zigzaging and fluctuate a lot. Further work should focus on methods for stabilizing the numerical model to eliminate the zigzaging pattern.
References


