
Technical report, IDE1124 , September 28, 2011

Meshfree Methods in Option Pricing

Master's Thesis in Financial Mathematics

Anna Belova, Tamara Shmidt



School of Information Science, Computer and Electrical Engineering
Halmstad University

Meshfree Methods in Option Pricing

Anna Belova, Tamara Shmidt

Halmstad University
Project Report IDE1124

Master's thesis in Financial Mathematics, 15 ECTS credits

Supervisor: Prof. Matthias Ehrhardt
Examiner: Prof. Ljudmila A. Bordag
External referees: Prof. Mikhail Babich

September 28, 2011

Department of Mathematics, Physics and Electrical engineering
School of Information Science, Computer and Electrical Engineering
Halmstad University

Preface

First of all, I would like to thank our supervisor Prof. Matthias Ehrhardt for his continuous help, advice and professional guidance during the writing of this thesis.

I am also thankful to my partner Tamara Shmidt for the time we studied and worked together.

A special thank you is directed to the head of program Prof. Ljudmila A. Bordag for this interesting opportunity to study Financial Mathematics at Halmstad University, her constant responsiveness and indispensable help.

Last but not least, I would like to express my gratitude to my parents for their strong support, love and patience and to my friends for being there for me whenever I needed them.

Anna Belova

First of all I would like to thank our supervisor, Prof. Matthias Ehrhardt, for proposing this interesting theme of research and for his useful comments and hints during the work. I am grateful for helpful advices from the course coordinator Prof. Ljudmila A. Bordag during all period of our study. I am also thankful to all professors from the Financial Mathematics group for their interesting lectures. Especial thanks to my partner, Anna Belova, for interesting discussions and strong support during our study and work together, especially during the last two weeks. And of course I would like to thank my family and friends for support, that they give me everyday.

Tamara Shmidt

Abstract

A meshfree approximation scheme based on the radial basis function methods is presented for the numerical solution of the options pricing model. This thesis deals with the valuation of the European, Barrier, Asian, American options of a single asset and American options of multi assets. The option prices are modeled by the Black-Scholes equation. The θ -method is used to discretize the equation with respect to time. By the next step, the option price is approximated in space with radial basis functions (RBF) with unknown parameters, in particular, we consider multiquadric radial basis functions (MQ-RBF). In case of American options a penalty method is used, i.e. removing the free boundary is achieved by adding a small and continuous penalty term to the Black-Scholes equation. Finally, a comparison of analytical and finite difference solutions and numerical results from the literature is included.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 1 |
| 2 | Option Pricing | 3 |
| 2.1 | European Options | 3 |
| 2.2 | American Options | 5 |
| 2.3 | Barrier Options | 7 |
| 2.4 | Asian Options | 7 |
| 2.5 | Basket Options | 9 |
| 3 | Meshfree methods | 11 |
| 4 | Discretization and Algorithms | 17 |
| 4.1 | Discretization | 17 |
| 4.1.1 | The case of European Options | 17 |
| 4.1.2 | The case of Barrier Options | 19 |
| 4.1.3 | The case of Asian Options | 19 |
| 4.1.4 | The case of American Options and multi-asset American Options | 21 |
| 4.2 | Algorithms | 28 |
| 4.2.1 | European Options | 28 |
| 4.2.2 | Barrier Options | 30 |
| 4.2.3 | Asian Options | 32 |
| 4.2.4 | American Options and multi-asset American Options | 34 |
| 5 | Results | 37 |
| 5.1 | European Options | 37 |
| 5.2 | Barrier Options | 41 |
| 5.3 | Asian Options | 45 |
| 5.4 | American Options | 49 |
| 6 | Conclusions | 53 |

| | |
|--------------|----|
| Notation | 55 |
| Bibliography | 57 |
| Appendix | 59 |

Chapter 1

Introduction

An option is a derivative product representing a contract, which gives the buyer a right to buy (call) or sell (put) the underlying asset at prescribed price (the strike price) pending the certain period of time or on a prescribed date (exercise date).

There are plenty of kinds of options in the market. In this thesis we consider *Vanilla options*, i.e. the option without any special or unusual properties. A vanilla option can belong to different styles of options, namely the *European option* or the *American option*. The difference between these two styles of Vanilla options is the date of exercise: the European option can only be exercised at the end of its life on the maturity date, while the American option allows the holder the early exercise before the maturity date.

An analytical formula exists for the evaluation European call and put options. By assuming a risk-neutrality of the underlying asset price, Black and Scholes [1] showed that the European call option value satisfies a lognormal partial differential equation of diffusion type, which is known as the Black-Scholes equation. However, there is no analytical formula for the American option due to the free boundary. Until recently, there are a few grid-based numerical methods for the valuation of the American option value, but in this thesis we focus our attention on a "*meshfree radial basis function*" (RBF) approach as a spatial approximation for the numerical solution of the options value and its derivatives in the Black-Scholes equation.

Recently the meshfree RBF approximation for solving the Black-Scholes equation for both European and American options has been examined by a couple of authors. For instance, the meshfree RBF approach has been considered as a spatial approximation for the numerical solution of American option by Fasshauer *et al.* [2, 3]; Hon *et al.* [8, 9] examined application the global RBFs to transform the Black-Scholes equation into a system of first-order equations in time in order to approximate the numerical solution by

known numerical schemes like, for example, the fourth-order Runge-Kutta method; optimizing the method parameters has been investigated by Pettersson *et al.* [13]. A RBF approximation for options value was also studied by Koc *et al.* [10], Goto *et al.* [7] and Marcozzi *et al.* [12].

Hon and Mao [9] developed a numerical scheme by applying the global radial basis functions, particularly Hardy's multiquadric, as a spatial approximation for the numerical solution of the options value and its derivatives in the Black-Scholes equation. They showed that the method does not require the generation of a grid in contrast to the finite difference method. Moreover, the computational domain is composed of scattered collocation points. As we can see, the RBFs are infinitely continuously differentiable and this is the reason why the higher order partial derivatives of the options value can directly be computed by using the derivatives of the basis function.

Fasshauer, Khaliq and Voss [3] considered the Black-Scholes model for American basket options with a nonlinear penalty source term. A basket option is an option whose price is based on multiple underlying assets. A penalty method replaces a constrained optimization problem by a series of unconstrained problems whose solutions ideally converge to the solution of the original constrained problem. The unconstrained problems are formed by adding a term to the objective function that consists of a penalty parameter and a measure of violation of the constraints. The measure of violation is nonzero when the constraints are violated and is zero in the region where constraints are not violated. The problem can be solved on a fixed domain. The Gaussian radial function was used in this approach with a user-selectable shape parameter in the numerical tests.

In Chapter 2 the foundations of the option pricing are presented. Mesh-free methods are considered in Chapter 3. In Chapter 4 the procedure of discretization is described and algorithms are given. All results are presented in Chapter 5 and discussed and interpreted in Chapter 6.

Chapter 2

Option Pricing

An option is a derivative product representing a contract, which gives the buyer a right to buy (call) or sell (put) the underlying asset at prescribed price pending the certain period of time or on a prescribed date (exercise date). The prescribed price of the asset is called a strike price.

There are plenty of kinds of options in the market, namely an *European option*, an *American option* and so on.

The European option can only be exercised at the end of its life time on the maturity date, while the American option allows the early exercise before the maturity date. An exotic option is an option which possesses more features than European and American options. An example of an exotic option is a Barrier option, which become valuable (or worthless) when the asset price reaches some barrier. Another example of an exotic option is an Asian options, whose payoff which depends on the some kind of average asset price for certain period of time. If the option can be based on multiple underlying assets it is called a basket option.

In this thesis we consider the European, the American, the Barrier, the Asian and the basket options.

2.1 European Options

An analytical formula exists for the evaluation of European call and put options [17]. By assuming a risk-neutrality of the underlying asset price, Black and Scholes [17] showed that the European call option value satisfies a backward-in-time lognormal partial differential equation (PDE) of diffusion type, which is known as the Black-Scholes equation.

We consider an option, whose price $V(S, t)$ satisfies the following Black-

Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad 0 \leq t \leq T, \quad (2.1)$$

where r is the risk-free interest rate, σ is the volatility of the asset price S , $V(S, t)$ is the option value at time t and asset price S . The terminal condition at the time of expiry T is given as

$$V(S, T) = \begin{cases} \max\{E - S, 0\}, & \text{for a put option } P(S, t) = V(S, t), \\ \max\{S - E, 0\}, & \text{for a call option } C(S, t) = V(S, t), \end{cases} \quad (2.2)$$

where E is the strike price of the option.

The boundary conditions for a European call option are given in the following way

$$C(0, t) = 0, \quad C(S, t) \sim S \text{ as } S \rightarrow \infty, \quad (2.3)$$

where $C(S, t)$ is the value of the European call option satisfying the corresponding equation (2.1).

The boundary conditions for a European put option are given in the following way

$$P(0, t) = Ee^{-r(T-t)}, \quad P(S, t) \sim 0 \text{ as } S \rightarrow \infty, \quad (2.4)$$

where $P(S, t)$ is the value of the European put option satisfying the corresponding equation (2.1) with a constant interest rate r .

By the simple exponential substitution $S = e^y$ the PDE (2.1) and condition (2.2) can be changed to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial y^2} + (r - \frac{1}{2}\sigma^2) \frac{\partial U}{\partial y} - rU = 0, \quad (2.5)$$

$$U(y, T) = \begin{cases} \max\{E - e^y, 0\}, & \text{for a put option,} \\ \max\{e^y - E, 0\}, & \text{for a call option.} \end{cases} \quad (2.6)$$

It is well known that the explicit solution of the European call option problem (2.1) with corresponding conditions (2.2) and (2.3), when the interest rate and the volatility are constant, can be given as [17]

$$C(S, t) = SN(d_1) - Ee^{r(T-t)}N(d_2), \quad (2.7)$$

where $N(\cdot)$ is the cumulative distribution function for a standardised normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy.$$

Here

$$d_1 = \frac{\log \frac{S}{E} + (r + \frac{1}{2}\sigma_2)(T - t)}{\sigma\sqrt{T - t}}, \quad (2.8)$$

and

$$d_2 = \frac{\log \frac{S}{E} + (r - \frac{1}{2}\sigma_2)(T - t)}{\sigma\sqrt{T - t}}. \quad (2.9)$$

The corresponding explicit solution for the European put option (2.1) with conditions (2.2) and (2.4) is

$$P(S, t) = Ee^{r(T-t)}N(-d_2) - SN(-d_1). \quad (2.10)$$

2.2 American Options

The typical feature of an American option is that it allows for an early exercise before the maturity date. There is no explicit solution known for this case due to the free boundary.

A valuation of the American option is difficult because at each moment of time we have to determine not only the option value, but also the decision whether or not the option should be exercised for each value of S . This problem is known as a free boundary problem. In case of the American option it is unknown a priori where the boundary conditions should be applied since the optimal exercise price S_f is unknown.

The American option valuation problem can be specified by a few constraints. The first constraint says that the option value has to be greater than or equal to the payoff function since the arbitrage profit, which is given from early exercise, should not be greater than zero. To avoid arbitrage opportunity the option should be exercised in the region where the option value is equal to the payoff function, or it has to satisfy the corresponding Black-Scholes equation where it transcends the payoff. Therefore another constraint requires that the Black-Scholes equation is replaced by an inequality. From the arbitrage it also follows that the option value has to be a continuous function of S .

The value $V(S, t)$ of the American option satisfies the following inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0. \quad (2.11)$$

The terminal condition at the time of expiry T is given as

$$V(S, T) = \begin{cases} \max\{E - S, 0\}, & \text{for a put option,} \\ \max\{S - E, 0\}, & \text{for a call option,} \end{cases} \quad (2.12)$$

where E is the strike price of the option.

By the same exponential substitution $S = e^y$ as in the European option case the inequality (2.11) and condition (2.12) can be changed to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial y^2} + (r - \frac{1}{2}\sigma^2) \frac{\partial U}{\partial y} - rU \leq 0, \quad (2.13)$$

$$U(y, T) = \begin{cases} \max\{E - e^y, 0\}, & \text{for a put option,} \\ \max\{e^y - E, 0\}, & \text{for a call option.} \end{cases} \quad (2.14)$$

In the region $0 \leq S < S_f(t)$, where early exercise is optimal, the value of the American put option $P(S, t)$ satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP < 0, \quad (2.15)$$

and

$$P = E - S. \quad (2.16)$$

In the other region, $S_f(t) < S < \infty$, early exercise is not optimal and the value of the American put option $P(S, t)$ satisfies the Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0, \quad (2.17)$$

and

$$P > E - S. \quad (2.18)$$

The boundary condition at $S = S_f(t)$ are that P and its slope are continuous

$$P(S_f(t), t) = \max(E - S_f(t), 0), \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1. \quad (2.19)$$

The second condition in (2.19) is called "*smooth pasting condition*" and can be motivated by arbitrage arguments [16].

The value $C(S, t)$ of the American call option satisfies the corresponding equality in the holding region $0 \leq S < S_f(t)$

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0, \quad (2.20)$$

and inequality

$$P > S - E. \quad (2.21)$$

In the region $S_f(t) < S < \infty$ the early exercise is optimal for the American call option, therefore the value $C(S, t)$ satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC < 0, \quad (2.22)$$

and

$$P = S - E. \quad (2.23)$$

2.3 Barrier Options

Barrier options can be "knock-out" or "knock-in" options. If the barrier price of the option equal the barrier K , the option is called knock-out in case it can be exercised unless the asset price S achieves the barrier K before expiry. The option is called knock-in in case it can be exercised if the asset price S passes the barrier K before expiry.

The knock-out options can be classified into "up-and-out" and "down-and-out" options. The up-and-out option becomes worthless if the barrier K is reached from below before expiry. The down-and-out option becomes worthless if the barrier K is reached from above before expiry. The knock-in options can be classified into "up-and-in" and "down-and-in" options. The up-and-in option is worthless unless the barrier K is reached from below before expiry. The down-and-in option is worthless unless the barrier K is reached from above before expiry.

Barrier options are attractive because they give more flexibility: the option premium can be reduced through the barrier option by not paying a premium to cover scenarios which are regarded as unlikely.

The value $C(S, t)$ of the down-and-out call barrier option with the barrier K satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0, \quad S > K, \quad (2.24)$$

$$C(S, t) = 0, \quad S \leq K. \quad (2.25)$$

The terminal condition on the expiration date T is given as

$$C(S, T) = \max\{S - E, 0\}. \quad (2.26)$$

If the asset price S reaches K , the option is invalid. Therefore,

$$C(K, t) = 0, \quad S = K. \quad (2.27)$$

Consequently, the payoff X at the expiry date T satisfies

$$X = \begin{cases} \max\{S - E, 0\}, & \text{if } S > K \text{ for all } t < T, \\ 0, & \text{if } S \leq K \text{ at } t < T. \end{cases} \quad (2.28)$$

2.4 Asian Options

Asian options are averaging options whose terminal payoff depends on some form of averaging of the price of the underlying asset over a part or the whole of the option's life. For details we refer the reader to Kwok [11].

Asian options have the following advantages: Asian options reduce the risk of market manipulation; Asian options are typically cheaper than European or American options, because of the reducing of the volatility inherent in the option.

There are two main classes of Asian options, the "fixed strike" (average rate) and the "floating strike" (average strike) options. An average rate option is a cash settled option whose payoff is based on the difference between the average value of the asset during the period from the day of purchase and the expiration date and a strike price. An average strike option is a cash settled whose payoff is based on the difference between the average value of the asset during the period and the asset price at the expiration date. The terminal call payoff X is given as

$$X = \begin{cases} \max(A_T - E, 0), & \text{for a fixed strike,} \\ \max(S_T - A_T, 0), & \text{for an average strike.} \end{cases} \quad (2.29)$$

Here S_T - the asset price at expiry, E - the strike price, A_T denotes some form of average of the price of the underlying asset over the averaging period $[0, T]$.

In the discrete case it can be used an arithmetic average

$$A_T = \frac{1}{n} \sum_{i=1}^n S_{t_i}, \quad (2.30)$$

or the geometric one

$$A_T = \left[\prod_{i=1}^n S_{t_i} \right]^{\frac{1}{n}}, \quad (2.31)$$

where, S_{t_i} - the asset price at discrete time $t_i, i = 1, 2, \dots, n$.

In the limit $n \rightarrow \infty$ the discrete sampled averages become the continuous sampled averages and could be arithmetic

$$A_T = \frac{1}{T} \int_0^T S_t dt, \quad (2.32)$$

or geometric

$$A_T = \exp\left(\frac{1}{T} \int_0^T \ln S_t dt\right). \quad (2.33)$$

Consider that the payoff of an option depends on an average strike of an asset

$$\frac{1}{t} \int_0^t S(\tau) d\tau. \quad (2.34)$$

Setting

$$I = \int_0^t S(\tau) d\tau, \quad (2.35)$$

we can obtain the following governing PDE for the value of the Asian option [11]

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - rV = 0, \quad (2.36)$$

where r is the risk free interest rate, σ is the volatility of the stock price S , $V(S, t)$ is the option value at time t and stock price S .

The terminal payoff X is given by the following expression for put and call options

$$X = \begin{cases} \max(S - \frac{1}{T} \int_0^t S(\tau) d\tau, 0), & \text{for a call option,} \\ \max(\frac{1}{T} \int_0^t S(\tau) d\tau - S, 0), & \text{for a put option.} \end{cases} \quad (2.37)$$

2.5 Basket Options

A basket option is an option whose price is based on several underlying assets. The basket option is a good opportunity for a corporation to reduce the several different risks at the same time and for this reason it is cheaper [2].

Consider an American basket option. The price of d assets at time t is denoted by

$$S(t) = (S_1(t), \dots, S_d(t)). \quad (2.38)$$

For the American option early exercise is allowed, therefore this problem can be formulated as a free boundary problem that can be stated by a the Black-Scholes equation for multi-asset problems

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^d r S_i \frac{\partial P}{\partial S_i} - rP = 0, \quad (2.39)$$

$$S_i > \underline{S}_i(t), \quad i = 1, \dots, d, \quad 0 \leq t < T, \quad (2.40)$$

where $P(S, t)$ - the value of the American put option, $\underline{S}(t) = (S_1(t), \dots, S_d(t))$ - the moving boundary, T - time of expiry, σ_i - the volatility of the i -th underlying asset, r - the risk free interest rate (assumed to be fixed throughout the time period of interest), ρ_{ij} - correlation between assets i and j .

The payoff function is given by

$$F(S) = \max(E - \sum_{i=1}^d \alpha_i S_i, 0), \quad (2.41)$$

where E - the exercise price of the option, α_i - are given constants. The terminal condition

$$P(S, T) = F(S), \quad S \in \Omega = \{(S_1, \dots, S_d) : S_i > 0, i = 1, \dots, d\}. \quad (2.42)$$

Along the free boundary

$$P(\underline{S}(T), t) = F(\underline{S}(t)), \quad (2.43)$$

$$F(\underline{S}(T)) = 0. \quad (2.44)$$

The smooth pasting condition is given by

$$\frac{\partial P}{\partial S_i}(\underline{S}, t) = -\alpha_i, \quad i = 1, \dots, d. \quad (2.45)$$

The boundary conditions read

$$\lim_{S_i \rightarrow \infty} P(S, t) = 0, \quad S \in \Omega, \quad i = 1, \dots, d, \quad (2.46)$$

$$P(S, t) = g_i(S, t), \quad S \in \Omega_i, \quad i = 1, \dots, d, \quad (2.47)$$

where the Ω_i denote the boundaries of Ω along which the price S_i vanishes.

For the American option early exercise is allowed, therefore we have the following positivity constraint [2]

$$P(S, t) - F(S) \geq 0, \quad S \in \Omega. \quad (2.48)$$

Chapter 3

Meshfree methods

Computation with high-dimensional data is an important issue in many areas of science but a lot of traditional grid based numerical methods can not handle such problems. Meshfree methods are better opportunity when we deal with changes in the geometry of the domain of interest than classical discretization techniques such as finite differences, finite elements or finite volumes. Moreover, the meshfree discretization is independent from a mesh, because these techniques are based only on a set of independent points.

The scattered data fitting problem is one of the fundamental problems in approximation theory and data modelling in general. According to [4] the meshfree approximation method can be applied to the PDE.

Problem 3.1 [4] *Given data $(x_j, y_j), j = 1, \dots, N$ with $x_j \in \mathbb{R}^s, y_j \in \mathbb{R}$ find a continuous function $\mathcal{P}f$ such that $\mathcal{P}f(x_j) = y_j, j = 1, \dots, N$.*

Here the x_j are the measurement location (or data sites), and the y_j are the corresponding measurements (or data values). And these values are obtained by sampling a data function f at the data sites, $y_j = f(x_j), j = 1, \dots, N, x_j$ lies in a s -dimensional space \mathbb{R}^s and it means that we can cover many different types of problems.

We assume that the function $\mathcal{P}f$ is a linear combination of certain "basis functions" B_k

$$\mathcal{P}f(x) = \sum_{k=1}^N c_k B_k(x), \quad x \in \mathbb{R}^s. \quad (3.1)$$

Hence, we have to solve the following linear system

$$Ac = y, \quad (3.2)$$

where the entries of the interpolation matrix $A \in \mathbb{R}^{N \times N}$ are given by $a_{jk} = B_k(x_j)$, $j, k = 1, \dots, N$, $c = [c_1, \dots, c_N]^T$, $y = [y_1, \dots, y_N]^T$.

Problem 3.1 is well-posed, i.e. a solution to the problem will exist and be unique, if and only if the matrix A is non-singular.

However, there is the following result for multivariate setting:

Theorem 3.1 [4] *If $\Omega \subset \mathbb{R}^s$, $s \geq 2$, contains an interior point, then there exist no Haar spaces of continuous functions except for one-dimensional ones.*

A Haar space is a space of functions in which the interpolation matrix $(B_k(x_j))_{j,k=1}^N$ has a property of invertibility, i.e. there exists a matrix $(\hat{B}_k(x_j))_{j,k=1}^N$ such that

$$(B_k(x_j))_{j,k=1}^N (\hat{B}_k(x_j))_{j,k=1}^N = (\hat{B}_k(x_j))_{j,k=1}^N (B_k(x_j))_{j,k=1}^N = I,$$

where I is an identity matrix.

From Theorem 3.1. we have that the basis needs to depend on the data locations. There is a consideration of the positive definite matrices and functions for this purpose.

Definition 3.1 [4] *A real symmetric matrix A is called positive semi-definite if its associated quadratic form is non-negative, i.e.,*

$$h^T A h = \sum_{j=1}^N \sum_{k=1}^N h_j h_k a_{jk} \geq 0, \quad (3.3)$$

for $h = [h_1, \dots, h_N]^T \in \mathcal{R}^N$. If the only vector h that turns (3.3) into an equality is the zero vector, then A is called positive definite.

All eigenvalues of positive definite matrices are positive which means that a positive definite matrix is non-singular. Therefore, if the basis function B_k in the expansion (3.1) generate a positive definite interpolation matrix, the scattered data fitting problem is a well-posed interpolation problem.

Definition 3.2 [4] *A real-valued continuous function Φ is positive definite on \mathbb{R}^s if and only if it is even and*

$$\sum_{j=1}^N \sum_{k=1}^N h_j h_k \Phi(x_j - x_k) \geq 0, \quad (3.4)$$

for any N pairwise different points $x_1, \dots, x_N \in \mathbb{R}^s$, and $h = [h_1, \dots, h_N]^T \in \mathbb{R}^N$. The function Φ is strictly positive definite on \mathbb{R}^s if the only vector h that turns (3.4) into an equality is the zero vector.

It means that the basis functions in (3.1) should be positive definite functions

$$B_k(x) = \Phi(x - x_k),$$

or

$$\mathcal{P}f(x) = \sum_{k=1}^N h_k \Phi(x - x_k), \quad x \in \mathbb{R}^s. \quad (3.5)$$

This function will yield an interpolant that is translation invariant. Positive definite functions, which are also radial functions, are invariant under all Euclidean transformations: translations, rotations and reflections.

Definition 3.3 [4] *A function $\Phi : \mathbb{R}^s \rightarrow \mathbb{R}$ is called radial provided there exists a univariate function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(x) = \varphi(r)$, where $r = \|x\|$, and $\|\cdot\|$ is some norm on \mathbb{R}^s - usually the Euclidean norm.*

The definition implies that a radial function Φ has a following property

$$\text{If } \|x_1\| = \|x_2\| \text{ then } \Phi(x_1) = \Phi(x_2), \quad x_1, x_2 \in \mathbb{R}^d.$$

The interpolation problem with radial functions becomes insensitive to the dimension s of the space in which the data sites lie.

The univariate function φ is called a positive definite radial function on \mathbb{R}^s if and only if the associated multivariate function Φ is positive definite on \mathbb{R}^s and radial.

An important part of the theoretical analysis of the radial basis functions is the integral characteristics.

Definition 3.4 [4] *The Fourier transform of $f \in L_1(\mathbb{R}^s)$ is given by*

$$\hat{f}(\omega) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(x) e^{-i\omega \cdot x} dx, \quad \omega \in \mathbb{R}^s, \quad (3.6)$$

and its inverse Fourier transform is given by

$$\check{f}(x) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(\omega) e^{ix \cdot \omega} d\omega, \quad x \in \mathbb{R}^s.$$

Theorem 3.2 [4] *Let $\Phi \in L_1(\mathbb{R}^s)$ be continuous and radial, $\Phi(x) = \varphi(\|x\|)$. Then its Fourier transform $\hat{\Phi}$ is also radial, i.e., $\hat{\Phi}(\omega) = \mathcal{F}_s \varphi(\|\omega\|)$ with*

$$\mathcal{F}_s \varphi(r) = \frac{1}{\sqrt{r^{s-2}}} \int_0^\infty \varphi(t) t^{\frac{s}{2}} J_{(s-2)/2}(rt) dt,$$

where $J_{(s-2)/2}$ is the Bessel function of the first kind of order $(s-2)/2$.

The transform in the Theorem 3.2 is also known as a Bessel transform.

Definition 3.5 [4] *The Laplace transform of a piecewise continuous function f that satisfies $|f(t)| \leq Me^{at}$ for some constants a and M is given by*

$$\mathcal{L}f(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad s > a.$$

The following theorem is one of the most important results on positive definite functions.

Theorem 3.3 (*Bochner's Theorem*)

[4] *A function $\Phi \in C(\mathbb{R}^s)$ is positive definite on \mathbb{R}^s if and only if it is the Fourier transform of a finite non-negative Borel measure μ on \mathbb{R}^s ,*

$$\Phi(x) = \hat{\mu}(x) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} e^{-ix \cdot y} d\mu(y), \quad x \in \mathbb{R}^s.$$

The following theorem gives extension of Bochner's characterization to strictly positive definite functions.

Theorem 3.4 [4] *Let μ be a non-negative finite Borel measure on \mathbb{R}^s whose carrier is not a set of Lebesgue measure zero. Then the Fourier transform of μ is strictly positive definite on \mathbb{R}^s .*

Corollary 3.1 [4] *Let f be a continuous non-negative function in $L_1(\mathbb{R}^s)$ which is not identically zero. Then the Fourier transform of f is strictly positive definite on \mathbb{R}^s .*

This corollary describes the way to construct strictly positive definite functions.

The criterion to check whether a given function is strictly positive definite contained in the following theorem.

Theorem 3.5 [4] *Let Φ be a continuous function in $L_1(\mathbb{R}^s)$. Φ is strictly positive definite if and only if Φ is bounded and its Fourier transform is non-negative and not identically equal to zero.*

The following theorems gives us a criterion to check whether a given function is positive definite and radial.

Theorem 3.6 [4] *A continuous function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is positive definite and radial on \mathbb{R}^s if and only if it is the Bessel transform of a finite non-negative Borel measure μ on $[0, \infty)$,*

$$\varphi(r) = \int_0^\infty \Omega_s(rt) d\mu(t), \quad (3.7)$$

where

$$\Omega_s = \begin{cases} \cos r & \text{for } s = 1, \\ \Gamma(\frac{s}{2}) (\frac{2}{r})^{(s-2)/2} J_{(s-2)/2}(r) & \text{for } s \geq 2, \end{cases} \quad (3.8)$$

and $J_{(s-2)/2}$ is the classical Bessel function of the first kind of order $(s-2)/2$.

Theorem 3.7 [4] *A continuous function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is positive definite and radial on \mathbb{R}^s for all s if and only if it is of the form*

$$\varphi(r) = \int_0^\infty e^{-r^2 t^2} d\mu(t), \quad (3.9)$$

where μ is a finite non-negative Borel measure on $[0, \infty)$.

There are some facts below about completely monotone functions which leads to simple characterization of positive definite radial functions.

Definition 3.6 [4] *A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ which is in $C[0, \infty) \cap C^\infty(0, \infty)$ and which satisfies*

$$(-1)^l \varphi^{(l)}(r) \geq 0, \quad r > 0, l = 0, 1, 2, \dots, \quad (3.10)$$

is called completely monotone on $[0, \infty)$.

An integral characterization of completely monotone functions may be found in the Appendix.

Examples

There are some examples of different types of functions.

1. The strictly positive definite function.

The *Gaussian*

$$\Phi(x) = e^{-\alpha \|x\|^2}, \quad \alpha > 0, \quad (3.11)$$

is strictly positive definite on \mathbb{R}^s for any s the reason by the Fourier transform of a Gaussian is again a Gaussian.

2. The strictly positive definite and radial functions on \mathbb{R}^s with restrictions on the space dimension s .

(a) The *truncated power function*

$$\varphi_l(r) = (1 - r)_+^l, \quad l \geq \lfloor \frac{s}{2} + 1 \rfloor, \quad (3.12)$$

where

$$(x)_+ = \begin{cases} x & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

(b)

$$\varphi(r) = \int_0^\infty (1 - rt)_+^{k-1} f(t) dt, \quad k \geq \lfloor \frac{s}{2} + 2 \rfloor, \quad (3.13)$$

where $f \in C[0; \infty)$ - non-negative and identically equal to zero.

3. The completely monotone and not constant functions. These functions can be used as basic functions to generate bases for (3.5), since they lead to strictly positive definite radial functions on any \mathbb{R}^s .

(a) *Inverse multiquadrics*

$$\varphi(r) = (r + \alpha^2)^{-\beta}, \quad \alpha, \beta > 0. \quad (3.14)$$

Therefore

$$\mathcal{P}f(x) = \sum_{j=1}^N h_j (\|x - x_j\|^2 + \alpha^2)^{-\beta}, \quad x \in \mathbb{R}^s, \quad (3.15)$$

can be used to solve the scattered data interpolation problem.

(b) *Gaussian radial basis function.*

$$\varphi(r) = e^{-\alpha r}, \quad \alpha > 0. \quad (3.16)$$

We can use the following function for solving the interpolation problem

$$\mathcal{P}f(x) = \sum_{j=1}^N h_j e^{-\alpha \|x - x_j\|^2}, \quad x \in \mathbb{R}^s. \quad (3.17)$$

(c) *Quadratic Matern RBF.*

The Quadratic Matern RBF is given as

$$\varphi(r) = e^{-\alpha r} (3 + 3\alpha r + (\alpha r)^2), \quad \alpha > 0. \quad (3.18)$$

This function is C^4 at the origin.

(d) *Wendland's RBF.*

Wendland's RBF is strictly positive definite in \mathbb{R}^3 and given as

$$\varphi(r) = (1 - \alpha r)_+^6 (35(\alpha r)^2 + 18\alpha r + 3), \quad \alpha > 0. \quad (3.19)$$

This function is C^4 at the origin.

Chapter 4

Discretization and Algorithms

4.1 Discretization

For evaluating the prices of European, Barrier, Asian, American and multi-asset American options with radial basis functions we consider discretization methods which were proposed in Goto *et al.* [7] and Fasshauer *et al.* [2].

4.1.1 The case of European Options

It is well-known that the following Black-Scholes equation holds for the option price $V(S, t)$ with asset price S at time t

$$\frac{\partial V(S, t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S, t)}{\partial S^2} + rS \frac{\partial V(S, t)}{\partial S} - rV = 0, \quad (4.1)$$

where the volatility σ is assumed to be constant between the date of purchase and the expiration date.

If the differential operator is abbreviated as

$$F_1 = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r, \quad (4.2)$$

the PDE (4.1) becomes

$$\frac{\partial V(S, t)}{\partial t} + F_1 V(S, t) = 0, \quad t \in [0, T]. \quad (4.3)$$

The backward-in-time parabolic PDE (4.3) is supplied with the terminal conditions

$$V(S, T) = \begin{cases} \max\{E - S(T), 0\} & = (E - S(T))^+, \text{ for a put,} \\ \max\{S(T) - E, 0\} & = (S(T) - E)^+, \text{ for a call.} \end{cases} \quad (4.4)$$

An application of the theta-method for the time discretization of (4.3) leads to

$$\frac{V(S, t + \Delta t) - V(t)}{\Delta t} + (1 - \theta)F_1V(S, t + \Delta t) + \theta F_1V(S, t) = 0, \quad (4.5)$$

where $0 \leq \theta \leq 1$ denotes the implicitness parameter. After rearranging terms in (4.5) we obtain

$$[1 + (1 - \theta)\Delta t F_1]V(S, t + \Delta t) = [1 - \theta\Delta t F_1]V(S, t), \quad (4.6)$$

i.e.

$$H_1V(S, t + \Delta t) = G_1V(S, t), \quad (4.7)$$

where

$$H_1 = 1 + (1 - \theta)F_1,$$

$$G_1 = 1 - \theta\Delta t F_1.$$

The multi-quadric radial basis function (MQ - RBF), which will be used for the approximation of the option price $V(S, t)$, is given as [7]

$$\phi(S, S_j) = \sqrt{c^2 + \|S - S_j\|^2}, \quad (4.8)$$

where S_j is the asset price at the collocation point j for approximating the option price V . $\|S - S_j\|$ denotes the radial distance of each of the N scattered data points S_j . The parameter c is a positive and it is known as shape parameter. The value of c has dual effect on stability and accuracy of the approximation: as c is increased, so does the accuracy, but only at the cost of ill-conditioning of the matrix of the RBF. This effect is known as the trade-off principle.

Therefore, the approximation for the option price $V(S, t)$ by the radial basis function is given as

$$V(S, t) \simeq \sum_{j=1}^N \lambda_j^t \phi(S, S_j), \quad (4.9)$$

where N is the total number of the collocation points at the date t , λ_j^t - the unknown parameters depending on time t , $\lambda_j^t = \lambda_j(t)$, $\lambda_j^{t+\Delta t} = \lambda_j(t + \Delta t)$, where Δt is the time-step size.

After substituting the ansatz (4.9) into equation (4.7) we get

$$\sum_{j=1}^N \lambda_j^{t+\Delta t} H_1 \phi(S, S_j) = \sum_{j=1}^N \lambda_j^t G_1 \phi(S, S_j). \quad (4.10)$$

Starting from the terminal condition (4.4), the coefficients λ are determined from the numerical result by using any backward time integration scheme at each time step $T - \Delta t$.

4.1.2 The case of Barrier Options

Consider the down-and-out option with the expiration price E and the barrier K . It means that the option becomes worthless if the barrier K is reached from above before expiry. The price of the option satisfies

$$\frac{\partial V}{\partial t} + F_1 V = 0 \quad (S > K), \quad (4.11)$$

$$V = 0 \quad (S \leq K). \quad (4.12)$$

The terminal condition is given as

$$V(S, T) = \max(S(T) - E, 0) = (S(T) - E)^+. \quad (4.13)$$

If S reaches K , the following condition for the option value is satisfied

$$V(K, t) = 0 \quad (S = K). \quad (4.14)$$

Consequently, the payoff function X for the barrier option is given as:

$$X = \begin{cases} (S(T) - E)^+, & \text{for } S > K, \\ 0, & \text{for } S \leq K. \end{cases} \quad (4.15)$$

The discretized equation derived from equation (4.11) is the same as for the European option, equation (4.10), because the PDE is identical.

4.1.3 The case of Asian Options

For Asian options the payoff depends on an average strike of an asset S , given as

$$\frac{1}{t} \int_0^t S(\tau) d\tau. \quad (4.16)$$

Let us set

$$I = \int_0^t S(\tau) d\tau, \quad (4.17)$$

therefore, the following PDE for the Asian option price V holds

$$\frac{\partial V}{\partial t} + S \frac{\partial V}{\partial I} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (4.18)$$

By using the substitution [7]

$$V(S, R, t) = S \cdot H(R, t), \quad (4.19)$$

where H denotes the option price and R is defined as

$$R = \frac{1}{S} \int_0^t S(\tau) d\tau = \frac{I}{S}, \quad (4.20)$$

equation (4.18) leads to the backward-in-time convection-diffusion equation

$$\frac{\partial H}{\partial t} + F_2 H = 0. \quad (4.21)$$

Here, the operator F_2 is defined as

$$F_2 = \frac{1}{2} \sigma^2 R^2 \frac{\partial^2}{\partial R^2} + (1 - rR) \frac{\partial}{\partial R}. \quad (4.22)$$

The payoff function on the expiration date $t = T$ for a call option is given as

$$V(S, R, T) = \left(S - \frac{1}{T} \int_0^t S(\tau) d\tau \right)^+.$$

Using the equation (4.19) and (4.20) in expression for the payoff leads to

$$S \cdot H(R, T) = S \cdot \left(1 - \frac{R}{T} \right)^+.$$

The terminal condition for equation (4.21) is given as

$$H(R, T) = \left(1 - \frac{R}{T} \right)^+. \quad (4.23)$$

The approximation for the option price $H(R, t)$ by the radial basis function is given as

$$H(R, t) \simeq \sum_{j=1}^N \lambda_j^t \phi(R, R_j), \quad (4.24)$$

where N is the total number of the collocation points at the date t , λ_j - the unknown parameters, and MQ-RBF $\phi(R, R_j)$ is given as

$$\phi(R, R_j) = \sqrt{c^2 + \|R - R_j\|^2}. \quad (4.25)$$

As in the case of the European option the discretized equation reads

$$\sum_{j=1}^N \lambda_j^{t+\Delta t} H_2 \phi(R, R_j) = \sum_{j=1}^N \lambda_j^t G_2 \phi(R, R_j), \quad (4.26)$$

where

$$H_2 = 1 + (1 - \theta)\Delta t F_2, \quad G_2 = 1 - \theta\Delta t F_2.$$

4.1.4 The case of American Options and multi-asset American Options

We apply the meshfree approximation schemes for the solution of the American option problem and for the solution of the multi-asset American option problems. According to Fasshauer, Khaliq and Voss [2], we will use a penalty method to remove the free and moving boundary and a linearly implicit θ - method for the time discretization.

The Black-Scholes equation is satisfied for multi-asset problems

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^d r S_i \frac{\partial P}{\partial S_i} - rP = 0, \quad (4.27)$$

where the additional condition is given as

$$S_i > \bar{S}_i(t), i = 1, \dots, d, \quad 0 \leq t \leq T,$$

where d denotes the amount of assets, which have the following prices at time t : $S(t) = (S_1(t), \dots, S_d(t))$, $P(S, t)$ - the value of the American put option, $\bar{S}(t) = (\bar{S}_1(t), \dots, \bar{S}_d(t))$ denotes the moving boundary, T - the time of expiry, σ_i - the volatility of the i -th underlying asset, r - the risk free interest rate, which is fixed throughout the time period of interest, and ρ_{ij} denotes the correlation between the assets i and j .

The American put option has the following payoff function

$$F(S) = \left(E - \sum_{i=1}^d \alpha_i S_i \right)^+, \quad (4.28)$$

where E is the exercise price of the option and α_i are given constants. The terminal condition is given as

$$P(S, T) = F(S), \quad S \in \Omega = (S_1, \dots, S_d) : S_i > 0, i = 1, \dots, d. \quad (4.29)$$

To ensure that the exercise value and the continuation value of the option are the same along the exercise boundary, the following conditions are prescribed

$$P(\bar{S}(t), t) = F(\bar{S}(t)), \quad (4.30)$$

$$F(\bar{S}(T)) = 0. \quad (4.31)$$

The smooth pasting condition for a smooth transition is given as

$$\frac{\partial P}{\partial S_i}(\bar{S}, t) = -\alpha_i, \quad i = 1, \dots, d. \quad (4.32)$$

The boundary conditions are given as

$$\lim_{S_i \rightarrow \infty} P(S, t) = 0, \quad S \in \Omega, i = 1, \dots, d, \quad (4.33)$$

$$P(S, t) = g_i(S, t), \quad S \in \Omega_i, i = 1, \dots, d, \quad (4.34)$$

where Ω_i are the boundaries of Ω along which the price $S_i = 0$.

The following positivity constraint is also necessary, because the early exercise is permitted

$$P(S, t) - F(S) \geq 0, \quad S \in \Omega. \quad (4.35)$$

For eliminating the moving boundary, we will use a penalty term, which was considered in the article by Fasshauer, Khaliq and Voss [2]. The penalty term was chosen so that the solution stays above the payoff function as the solution approaches expiry and small enough so that the PDE still resembles the Black-Scholes equation very closely. Therefore, the penalty term has the following form [2]

$$\frac{\epsilon C}{P_\epsilon + \epsilon - q}, \quad (4.36)$$

where $0 < \epsilon \ll 1$ is a small regularization parameter, $C \geq rE$ is a positive constant, and the barrier function is given as

$$q(S) = E - \sum_{i=1}^d \alpha_i S_i. \quad (4.37)$$

After adding the penalty term (4.36) to the equation (4.27) the PDE becomes

$$\frac{\partial P_\epsilon}{\partial t} + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 P_\epsilon}{\partial S_i \partial S_j} + \sum_{i=1}^d r S_i \frac{\partial P_\epsilon}{\partial S_i} - r P_\epsilon + \frac{\epsilon C}{P_\epsilon + \epsilon - q} = 0, \quad (4.38)$$

here

$$S \in \Omega, \quad 0 \leq t \leq T.$$

The same terminal and boundary conditions are satisfied

$$P_\epsilon(S, T) = F(S), \quad S \in \Omega, \quad (4.39)$$

$$P_\epsilon(S, t) = g_i(S, t), \quad S \in \Omega_i, i = 1, \dots, d, \quad (4.40)$$

$$\lim_{S_i \rightarrow \infty} P_\epsilon(S, t) = 0, \quad S \in \Omega, i = 1, \dots, d. \quad (4.41)$$

By using the radial basis function approach the following expression for the value of the option is obtained

$$P(S, t) = \sum_{j=1}^N a_j(t) \phi(\|S - x_j\|).$$

Here the MQ-RBF is used

$$\phi(\|S - x_j\|) = \sqrt{c^2 + \|S - x_j\|^2}. \quad (4.42)$$

The Single Asset Case

For this case the following Black-Scholes equation with a penalty term is considered

$$\frac{\partial P_\epsilon}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P_\epsilon}{\partial S^2} + r S \frac{\partial P_\epsilon}{\partial S} - r P_\epsilon + \frac{\epsilon C}{P_\epsilon + \epsilon - q(S)} = 0. \quad (4.43)$$

The boundary conditions are given as

$$P_\epsilon(0, t) = E, \quad (4.44)$$

$$\lim_{S \rightarrow \infty} P_\epsilon(S, t) = 0. \quad (4.45)$$

The terminal condition is given by

$$P(S, T) = (E - S)^+. \quad (4.46)$$

By using the collocation approach the following expressions for the value of the option and for the partial derivatives are obtained

$$P_\epsilon(S, t) = \sum_{j=1}^N a_j(t) \phi(\|S - x_j\|), \quad (4.47)$$

$$\frac{\partial P_\epsilon}{\partial t} = \sum_{j=1}^N \dot{a}_j(t) \phi(\|S - x_j\|), \quad (4.48)$$

$$\frac{\partial P_\epsilon}{\partial S} = \sum_{j=1}^N a_j(t) \phi'(\|S - x_j\|), \quad (4.49)$$

$$\frac{\partial^2 P_\epsilon}{\partial S^2} = \sum_{j=1}^N a_j(t) \phi''(\|S - x_j\|), \quad (4.50)$$

where

$$\phi'(\|S - x_j\|) = (c^2 + \|S - x_j\|^2)^{-\frac{1}{2}} (S - x_j), \quad (4.51)$$

$$\phi''(\|S - x_j\|) = -(c^2 + \|S - x_j\|^2)^{-\frac{3}{2}} (S - x_j)^2 + (c^2 + \|S - x_j\|^2)^{-\frac{1}{2}}. \quad (4.52)$$

After substitution (4.47)- (4.50) into (4.43), we have [2]

$$\begin{aligned} & \sum_{j=1}^N \dot{a}_j(t) \phi(\|S - x_j\|) + \frac{1}{2} \sigma^2 S^2 \sum_{j=1}^N a_j(t) \phi''(\|S - x_j\|) \\ & + rS \sum_{j=1}^N a_j(t) \phi'(\|S - x_j\|) - r \sum_{j=1}^N a_j(t) \phi(\|S - x_j\|) \end{aligned}$$

$$+ \frac{\epsilon C}{\sum_{j=1}^N a_j(t) \phi(\|S - x_j\|) + \epsilon - q(S)} = 0. \quad (4.53)$$

After collocation at the points x_i , $i = 1, \dots, N$ the following system is satisfied for the coefficients a_j , collected in the vector \mathbf{a}

$$\Phi \dot{\mathbf{a}} + R\mathbf{a} + Q(\mathbf{a}) = 0, \quad (4.54)$$

where $R = (\hat{r}_{ij})$ such that

$$\hat{r}_{ij} = \frac{1}{2} \sigma^2 \Phi''_{S,ij} + r \Phi'_{S,ij} - r \Phi_{ij}, \quad (4.55)$$

and

$$\Phi_{ij} = \phi(\|x_i - x_j\|), \quad \Phi'_{S,ij} = x_i \phi'(\|x_i - x_j\|), \quad \Phi''_{S,ij} = x_i^2 \phi''(\|x_i - x_j\|). \quad (4.56)$$

Here vector $Q(\mathbf{a})$ is determined by

$$Q_i(\mathbf{a}) = \frac{\epsilon C}{\Phi_i \mathbf{a} + \epsilon - q(x_i)}, \quad i = 1, \dots, N, \quad (4.57)$$

where Φ_i denoting the i -th row of the matrix Φ .

Again, the theta-method is used for the time discretization

$$\Phi \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\Delta t} + \theta R \mathbf{a}^{n+1} + (1 - \theta) R \mathbf{a}^n + \theta Q(\mathbf{a}^{n+1}) + (1 - \theta) Q(\mathbf{a}^n) = 0, \quad (4.58)$$

where $\mathbf{a}^n = \mathbf{a}(n\Delta t)$ with Δt the time step chosen for discretization in time interval.

By replacing \mathbf{a}^n in the penalty term by \mathbf{a}^{n+1} this method is turned into a linearly implicit method, which is well studied, however the order of accuracy in time is limited to first-order

$$\Phi \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\Delta t} + \theta R \mathbf{a}^{n+1} + (1 - \theta) R \mathbf{a}^n + Q(\mathbf{a}^{n+1}) = 0, \quad (4.59)$$

or

$$[\Phi - (1 - \theta)\Delta t R] \mathbf{a}^n = [\Phi + \theta \Delta t R] \mathbf{a}^{n+1} + \Delta t Q(\mathbf{a}^{n+1}). \quad (4.60)$$

In case $\theta = \frac{1}{2}$ (the Crank-Nicolson method) the scheme is

$$[\Phi - A] \mathbf{a}^n = [\Phi + A] \mathbf{a}^{n+1} + \Delta t Q(\mathbf{a}^{n+1}), \quad (4.61)$$

where

$$A = \frac{1}{2} \Delta t R.$$

From the terminal condition (4.46) and the fact that

$$P_\epsilon(S, T) = \sum_{j=1}^N a_j(T) \phi(\|S - x_j\|),$$

after collocation at the points x_i , $i = 1, \dots, N$ the following system can be obtained for the coefficients $a_j(T)$

$$\Phi \mathbf{a}(T) = \mathbf{P}, \quad (4.62)$$

where $\mathbf{P} = [P_\epsilon(x_1, T), \dots, P_\epsilon(x_N, T)]^T$, $\mathbf{a}(T) = [a_1(T), \dots, a_N(T)]$.

The Multi-Asset Case

The value of the multi-asset option satisfies the same expression like the single asset option does

$$P_\epsilon(S, t) = \sum_{j=1}^N a_j(t) \phi(\|S - x_j\|). \quad (4.63)$$

Since here the MQ - RBF are used, the partial derivatives of the radial basis functions are given by

$$\begin{aligned} \frac{\partial \phi(\|S - x_l\|)}{\partial S_i} &= (c^2 + \|S - x_l\|^2)^{-\frac{1}{2}} (S_i - x_{l,i}), \\ \frac{\partial^2 \phi(\|S - x_l\|)}{\partial S_i \partial S_j} &= -(c^2 + \|S - x_l\|^2)^{-\frac{3}{2}} (S_i - x_{l,i})(S_j - x_{l,j}), \end{aligned} \quad (4.64)$$

where $x_{l,i}$ denote the i -th component of the center x_l .

The system of ODEs is also the same as for single asset options

$$\Phi \dot{\mathbf{a}} + R \mathbf{a} + Q(\mathbf{a}) = 0. \quad (4.65)$$

Here R denotes the following matrix

$$R = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j \Phi_S^{(i,j)} + \sum_{i=1}^d r \Phi_S^{(i)} - r \Phi, \quad (4.66)$$

with the matrix Φ determined by $\Phi_{kl} = \phi(\|x_k - x_l\|)$, and with the following expressions for the $\Phi_S^{(i)}$ and $\Phi_S^{(i,j)}$

$$\Phi_{S,kl}^{(i)} = x_{k,i} \frac{\partial \phi(\|S - x_l\|)}{\partial S_i} \Big|_{S=x_k}, \quad (4.67)$$

$$\Phi_{S,kl}^{(i,j)} = x_{k,i} x_{k,j} \frac{\partial^2 \phi(\|S - x_l\|)}{\partial S_i \partial S_j} \Big|_{S=x_k}, \quad (4.68)$$

where $x_{k,i}$ denotes the i -th component of the k -th center, $i = 1, \dots, d, k = 1, \dots, N$.

The vector $Q(\mathbf{a})$ is determined by

$$Q_k(\mathbf{a}) = \frac{\epsilon C}{\Phi_k \mathbf{a} + \epsilon - q(x_k)}, \quad k = 1, \dots, N. \quad (4.69)$$

For the multi-asset case we have d single asset problems with the following boundary conditions

$$\frac{\partial g_i}{\partial t} + \frac{1}{2} \sigma^2 S_i^2 \frac{\partial^2 g_i}{\partial S_i^2} + r S_i \frac{\partial g_i}{\partial S_i} - r g_i + \frac{\epsilon C}{g_i + \epsilon - q(S_i)} = 0, \quad (4.70)$$

$$0 \leq S_i \leq S_\infty, \quad 0 \leq t < T,$$

$$g_i(S_i, T) = (E - \alpha_i S_i)^+, \quad (4.71)$$

$$g_i(0, t) = \frac{E}{\alpha_i}, \quad (4.72)$$

$$g_i(S_\infty, t) = 0, \quad (4.73)$$

where $i = 1, \dots, d$, d - the number of underlying assets and $q(S_i) = E - \alpha_i S_i$.

4.2 Algorithms

In this section we consider algorithms for evaluating the option's value.

4.2.1 European Options

The results of the discretization, which were obtained above, will be used for evaluating the fair price of the European option. The following procedure were proposed in Goto *et al* [7].

1. S_{max} is chosen big enough and N collocation points are taken uniformly for the asset price S in the interval $[0, S_{max})$. When we choose the number of the collocation points we have to remember that the total number of the collocation points strongly affects the computational accuracy. Computational error decreases and the condition number increases according to the increase of the total number of the collocation points.
2. The time-step size $\Delta t = T/M$ is chosen and the time interval $[0, T]$ is discretized with the time-step Δt , where $t = 0$ is the date of purchase, $t = T$ is the exercise date and M denotes the number of time steps. In different papers we can see that the computational error is very huge for $\Delta t > 0.02$ and it is getting less at $\Delta t = 0.005$.
3. The option price $V(S, T)$ at the expiration date $t = T$ is calculated from the corresponding terminal condition (4.4).
4. The parameter $\lambda_j^T = \lambda_j(T)$ on the expiration date T is calculated from the equation (4.9) on $V(S, T)$.
5. $t \leftarrow T - \Delta t$.
6. To obtain the unknown coefficients λ_j^t equation (4.10) is solved.
7. $t \leftarrow t - \Delta t$.
8. If $t > 0$, the process goes to the step 6.
9. To obtain the fair price of the option $V(S, 0)$, $\lambda_j^0 = \lambda(0)$ is substituted into equation (4.9).

Algorithm 1 European Call Option evaluation

Choose $N, S_{max}, T, M, E, r, \sigma, \theta$ Calculate $\Delta S, \Delta t$ $S_1 = \Delta S$ **for** $j = 1$ to N **do** Calculate $S_j = S_j + \Delta S$ **end for** $t \leftarrow T$ $V_T \leftarrow \max(S - E, 0)$ **for** $j = 1$ to N **do** Calculate $\phi(S, S_j) = \sqrt{c^2 + \|S - S_j\|^2}$ Calculate $\frac{\partial \phi}{\partial S}(S, S_j) = (c^2 + \|S - S_j\|^2)^{-\frac{1}{2}}(S - S_j)$ Calculate $\frac{\partial^2 \phi}{\partial S^2}(S, S_j) = (c^2 + \|S - S_j\|^2)^{-\frac{3}{2}}(c^2 + \|S - S_j\|^2 + (S - S_j))$ **end for**Calculate λ_j^T from $V(S, T) \simeq \sum_{j=1}^N \lambda_j^T \phi(S, S_j)$ as $\lambda^{\mathbf{T}} = \phi^{-1} \mathbf{V}_{\mathbf{T}}$ $t \leftarrow T - \Delta t$ **for** $j = 1$ to N **do** Calculate $F_1 \phi(S, S_j) = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2}(S, S_j) + r S \frac{\partial \phi}{\partial S}(S, S_j) - r \phi(S, S_j)$ Calculate $H_1 \phi(S, S_j) = 1 + (1 - \theta) \Delta t F_1 \phi(S, S_j)$ Calculate $G_1 \phi(S, S_j) = 1 - \theta \Delta t F_1 \phi(S, S_j)$ **end for****while** $t > 0$ **do** Calculate $\lambda^{\mathbf{t}-1} = G_1 \phi(S, S_j) \setminus (\lambda^{\mathbf{t}} H_1 \phi(S, S_j))$ $t \leftarrow t - \Delta t$ **end while**Calculate V_0 from $V(S, 0) \simeq \sum_{j=1}^N \lambda_j^0 \phi(S, S_j)$

4.2.2 Barrier Options

Here we consider an algorithm suggested in Goto *et al* [7] for evaluating the option fair price of the down-and-out option of the expiration price E and the barrier K . Since the results of the discretization for the barrier option are the same as for the European one, so the valuation algorithm is as follows.

1. S_{max} is chosen big enough and N collocation points are taken uniformly for the asset price S in the interval $[0, S_{max})$.
2. The time-step size $\Delta t = T/M$ is chosen and the time interval $[0, T]$ is discretized with the time-step Δt , where $t = 0$ is the date of purchase, $t = T$ is the exercise date and M denotes the number of time steps.
3. The option price $V(S, T)$ at the expiration date $t = T$ is calculated from the terminal condition (4.13).
4. The parameter λ_j^T on the expiration date T is calculated from the equation (4.9) on $V(S, T)$.
5. $t \leftarrow T - \Delta t$.
6. To obtain the unknown coefficients λ_j^t equation (4.10) is solved.
7. $t \leftarrow t - \Delta t$.
8. If $t > 0$, the process goes to the step 6.
9. To obtain the fair price of the option $V(S, 0)$, λ_j^0 is substituted into equation (4.9).

Algorithm 2 Barrier Option evaluation

Choose $N, S_{max}, T, M, E, r, \sigma, \theta, K$
 Calculate $\Delta S, \Delta t$
 $S_1 = \Delta S$
for $j = 1$ to N **do**
 Calculate $S_j = S_j + \Delta S$
end for
 $t \leftarrow T$
for $j = 1$ to N **do**
 if $S_j > K$ **then**
 $V(S_j, T) \leftarrow \max(S_j - E, 0)$
 else
 $V(S_j, T) \leftarrow 0$
 end if
end for
for $j = 1$ to N **do**
 Calculate $\phi(S, S_j) = \sqrt{c^2 + \|S - S_j\|^2}$
 Calculate $\frac{\partial \phi}{\partial S}(S, S_j) = (c^2 + \|S - S_j\|^2)^{-\frac{1}{2}}(S - S_j)$
 Calculate $\frac{\partial^2 \phi}{\partial S^2}(S, S_j) = (c^2 + \|S - S_j\|^2)^{-\frac{3}{2}}(c^2 + \|S - S_j\|^2 + (S - S_j))$
end for
 Calculate λ_j^T from $V(S, T) \simeq \sum_{j=1}^N \lambda_j^T \phi(S, S_j)$ as
 $\lambda^T = \phi^{-1} \mathbf{V}_T$
 $t \leftarrow T - \Delta t$
for $j = 1$ to N **do**
 Calculate $F_1 \phi(S, S_j) = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2}(S, S_j) + r S \frac{\partial \phi}{\partial S}(S, S_j) - r \phi(S, S_j)$
 Calculate $H_1 \phi(S, S_j) = 1 + (1 - \theta) \Delta t F_1 \phi(S, S_j)$
 Calculate $G_1 \phi(S, S_j) = 1 - \theta \Delta t F_1 \phi(S, S_j)$
end for
while $t > 0$ **do**
 Calculate $\lambda^{t-1} = G_1 \phi(S, S_j) \setminus (\lambda^t H_1 \phi(S, S_j))$
 $t \leftarrow t - \Delta t$
end while
 Calculate V_0 from $V(S, 0) \simeq \sum_{j=1}^N \lambda_j^0 \phi(S, S_j)$

4.2.3 Asian Options

Here we consider an algorithm suggested in Goto *et al* [7] for evaluating of option fair price of the Asian average strike call option. The algorithm is similar to one for the European option, but it uses another formulation of the discretized equation and the terminal condition.

1. H_{max} is chosen big enough and N collocation points are taken uniformly for the asset price H in the interval $[0, H_{max})$.
2. The time-step size $\Delta t = T/M$ is chosen and the time interval $[0, T]$ is discretized with the time-step Δt , where $t = 0$ is the date of purchase, $t = T$ is the exercise date and M denotes the number of time steps.
3. The option price $H(R, T)$ at the expiration date $t = T$ is calculated from the terminal condition (4.23).
4. The parameter λ_j^T on the expiration date T is calculated from the equation (4.24) on $H(R, T)$.
5. $t \leftarrow T - \Delta t$.
6. To obtain the unknown coefficients λ_j^t equation (4.26) is solved.
7. $t \leftarrow t - \Delta t$.
8. If $t > 0$, the process returns to the step 6.
9. To obtain the fair price of the option $H(R, 0)$, λ_j^0 is substituted into equation (4.24).

Algorithm 3 Asian Average Strike Call Option evaluation

Choose $N, R_{max}, T, M, r, \sigma, \theta$ Calculate $\Delta R, \Delta t$ $R_1 = \Delta R$ **for** $j = 1$ to N **do** Calculate $R_j = R_j + \Delta S$ **end for** $t \leftarrow T$ $H(R_j, T) \leftarrow \max(1 - \frac{R}{T}, 0)$ **for** $j = 1$ to N **do** Calculate $\phi(R, R_j) = \sqrt{c^2 + \|R - R_j\|^2}$ Calculate $\frac{\partial \phi}{\partial R}(R, R_j) = (c^2 + \|R - R_j\|^2)^{-\frac{1}{2}}(R - R_j)$ Calculate $\frac{\partial^2 \phi}{\partial R^2}(R, R_j) = (c^2 + \|R - R_j\|^2)^{-\frac{3}{2}}(c^2 + \|R - R_j\|^2 + (R - R_j))$ **end for**Calculate λ_j^T from $H(R, T) \simeq \sum_{j=1}^N \lambda_j^T \phi(R, R_j)$ as $\lambda^T = \phi^{-1} \mathbf{H}_T$ $t \leftarrow T - \Delta t$ **for** $j = 1$ to N **do** Calculate $F_2 \phi(R, R_j) = \frac{1}{2} \sigma^2 R^2 \frac{\partial^2 \phi}{\partial R^2}(R, R_j) + (1 - r) R \frac{\partial \phi}{\partial R}(R, R_j)$ Calculate $H_2 \phi(R, R_j) = 1 + (1 - \theta) \Delta t F_2 \phi(R, R_j)$ Calculate $G_2 \phi(R, R_j) = 1 - \theta \Delta t F_2 \phi(R, R_j)$ **end for****while** $t > 0$ **do** Calculate $\lambda^{t-1} = G_2 \phi(R, R_j) \setminus (\lambda^t H_2 \phi(R, R_j))$ $t \leftarrow t - \Delta t$ **end while**Calculate H_0 from $H(S, 0) \simeq \sum_{j=1}^N \lambda_j^0 \phi(R, R_j)$

4.2.4 American Options and multi-asset American Options

Using the discretization obtained above the following algorithm was suggested by Fasshauer *et al* [2] in case of the single asset American option.

1. Choose a time step Δt and a value of θ .
2. Assemble the matrices Φ and R from (4.55) and (4.56).
3. Calculate the matrices $R_1 = \Phi - (1 - \theta)\Delta t R$ and $R_2 = \Phi + \theta\Delta R$.
4. Factor the matrices Φ and R_1 .
5. Initialize the solution vector \mathbf{P} via

$$P(x_i, T) = \max(E - x_i, 0), \quad i = 1, \dots, N.$$
6. For each time step
 - (a) Update the coefficients by solving $\Phi \mathbf{a} = \mathbf{P}$ using the factorization obtained in step 4.
 - (b) Compute $\mathbf{b} = R_2 \mathbf{a}$ and the vector $Q(\mathbf{a})$ from (4.57).
 - (c) Find the next coefficients by solving the linear system

$$R_1 \mathbf{a} = \mathbf{b} + \Delta t Q(\mathbf{a})$$
 using the factorization computed in step 4.
 - (d) Update the solution vector \mathbf{P} via $P(x_i, t) = \Phi \mathbf{a}$, $i = 2, \dots, N - 1$.
 - (e) Apply the boundary conditions $P(x_1, t) = E$ and $P(X_N, t) = 0$.

The algorithm for the multi-asset option, which consists of d assets, is similar for the previous one with only difference, that one time step for each of the $d - 1$ -dimensional boundary problems are included into the time-stepping loop of the d - dimensional problem.

Algorithm 4 American Put Option evaluation

Choose $N, S_{max}, T, M, r, \sigma, \theta$
 Calculate $\Delta S, \Delta t$
 $S_1 = \Delta S$
for $j = 1$ to N **do**
 Calculate $S_j = S_j + \Delta S$
end for
for $i = 1$ to N **do**
 for $j = 1$ to N **do**
 Calculate $\Phi_{ij} = \phi(\|S_i - S_j\|)$
 Calculate $\Phi'_{S,ij} = S_i \phi'(\|S_i - S_j\|)$
 Calculate $\Phi''_{S,ij} = S_i^2 \phi''(\|S_i - S_j\|)$
 end for
end for
 $R \leftarrow \frac{1}{2} \sigma^2 \Phi''_S + r \Phi'_S - r \Phi$
 $R_1 \leftarrow \Phi - (1 - \theta) \Delta t R$
 $R_2 \leftarrow \Phi + \theta \Delta R$
for $i = 1$ to N **do**
 $P(S_i, T) = \max(E - S_i, 0)$
end for
while $t > 0$ **do**
 Calculate \mathbf{a} from $\Phi \mathbf{a} = \mathbf{P}$
 Calculate $\mathbf{b} = R_2 \mathbf{a}$
 for $i = 1$ to N **do**
 Calculate $Q_i(\mathbf{a}) = \frac{\epsilon C}{\Phi_i \mathbf{a} + \epsilon - q(S_i)}$
 end for
 Calculate \mathbf{a} from $R_1 \mathbf{a} = \mathbf{b} + \Delta t Q(\mathbf{a})$
 for $i = 2$ to $N - 1$ **do**
 $P(S_i, t) \leftarrow \Phi \mathbf{a}$
 end for
 $P(S_1, t) \leftarrow E$
 $P(S_N, t) \leftarrow 0$
 $t \leftarrow t - \Delta t$
end while

Chapter 5

Results

Here we will present results of numerical experiments according to the algorithms described before.

5.1 European Options

We consider a European call Option with strike E and expiration date T . Parameters used in numerical example are presented in Table 5.1. We use a few different values of the support radius c and a few kinds of RBF for the radial basis function approximation of the option price V_t : the Multiquadric (MQ) RBF (4.8) [7], the Quadratic Matern (QMat) RBF (3.18) [5], Wendland's (Wend) RBF (3.19) [5] and Inverse multiquadric (IMQ) RBF (3.14) [5].

Table 5.1: Parameters for numerical analysis

| | |
|-----------------------------|---------------------------|
| Maximum asset price value | $S_{max} = 30$ |
| Number of asset data points | $N = 121$ |
| Number of time steps | $M = 100$ |
| Time-step size | $\Delta t = 0.005$ |
| Expiration date | $T = 0.5$ (<i>year</i>) |
| Exercise price | $E = 10.0$ |
| Risk free interest rate | $r = 0.05$ |
| Volatility | $\sigma = 0.2$ |
| Crank-Nicolson method | $\theta = 0.5$ |
| Support radius | $c = 0.01$ |

Numerical results for the RBF approximation with different RBF are shown in Table 5.2 in comparison with analytical solution for the European call

Option. Here V_{MQ} denotes the result obtained with the MQ-RBF, V_{QMat} is the result obtained with the Quadratic Matern RBF, V_{Wend} is the result obtained with the Wendland's RBF and V_{IMQ} is the result obtained with the IMQ-RBF.

Table 5.2: Results for the European call Option, $c = 0.01$

| Asset price S | $V_{Analytical}$ | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|------------------|----------|------------|------------|-----------|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0288 |
| 2.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0415 |
| 4.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.2104 |
| 6.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0015 |
| 8.0 | 0.0456 | 0.0415 | 0.0000 | 0.0000 | 0.0967 |
| 10.0 | 0.6888 | 0.5866 | 0.0000 | 0.0000 | 0.1152 |
| 12.0 | 2.2952 | 2.0421 | 2.0000 | 2.0000 | 2.3662 |
| 14.0 | 4.2496 | 3.9125 | 4.0000 | 4.0000 | 4.2607 |
| 16.0 | 6.2470 | 5.8556 | 6.0000 | 6.0000 | 5.9834 |
| 18.0 | 8.2469 | 7.8056 | 8.0000 | 8.0000 | 8.2129 |

The results obtained with the value of the support parameter $c = 0.01$ are illustrated by Figure 5.1.

Numerical results obtained after increasing of the parameter c to $c = 1.0$ are shown in Table 5.3.

Table 5.3: Results for the European call Option, $c = 1.0$

| Asset price S | $V_{Analytical}$ | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|------------------|----------|------------|------------|-----------|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 4.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 6.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 8.0 | 0.0456 | 0.0014 | 0.0000 | 0.0000 | 0.0000 |
| 10.0 | 0.6888 | 0.0508 | 0.0000 | 0.0000 | 0.0000 |
| 12.0 | 2.2952 | 1.9544 | 2.0000 | 2.0000 | 2.0000 |
| 14.0 | 4.2496 | 3.9044 | 4.0000 | 4.0000 | 4.0000 |
| 16.0 | 6.2470 | 5.8551 | 6.0000 | 6.0000 | 6.0000 |
| 18.0 | 8.2469 | 7.8059 | 8.0000 | 8.0000 | 8.0000 |

The results obtained with the value of the support parameter $c = 1.0$ are illustrated by Figure 5.2.

The computational error ϵ_{RMSE} was determined as the root mean square error (RMSE) and calculated in the form

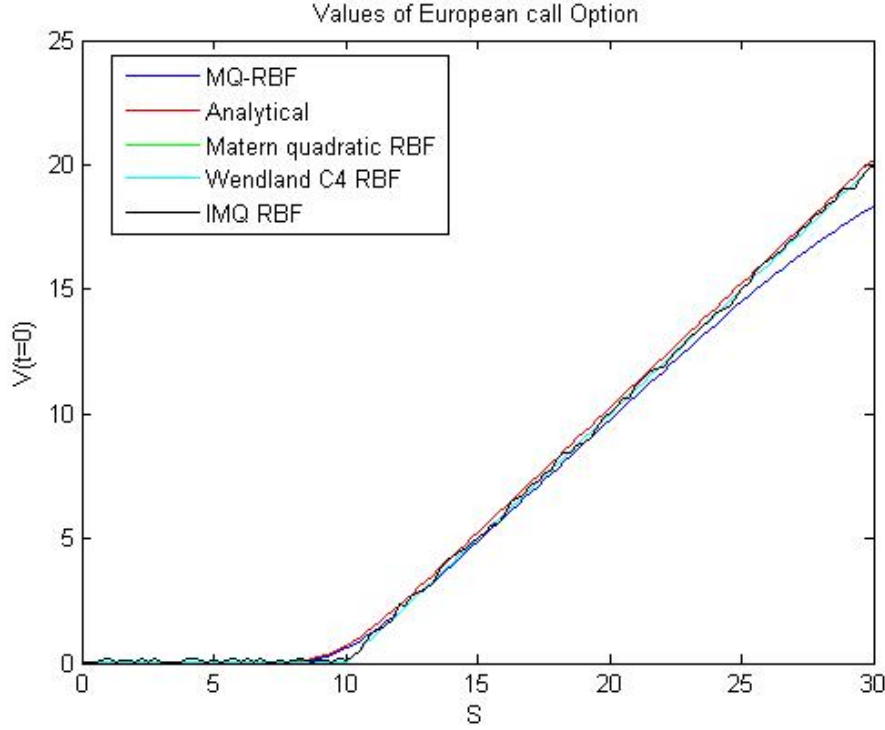


Figure 5.1: Values of the European call Option, obtained by meshfree method in comparison with the analytical solution for the European call Option with parameters listed in Table 5.1. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter $c = 0.01$.

$$\epsilon_{RMSE} = \frac{1}{\sqrt{N}} \sqrt{\sum_{j=1}^N |V(S_j, t)_{RBF} - V(S_j, t)_{Analytical}|^2}, \quad (5.1)$$

where $V(S_j, t)_{RBF}$ and $V(S_j, t)_{Analytical}$ are the numerical solution by using the RBF approximation and the theoretical solution, respectively.

The maximum computational error ϵ_{max} is calculated as

$$\epsilon_{max} = \max(|V(S_j, t)_{RBF} - V(S_j, t)_{Analytical}|). \quad (5.2)$$

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 5.4.

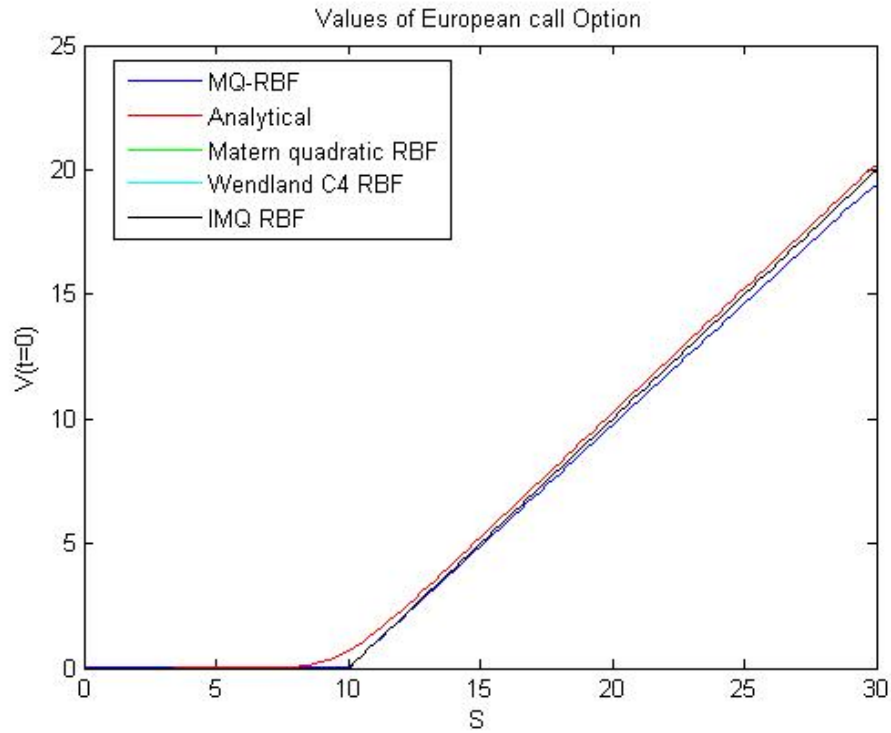


Figure 5.2: Values of the European call Option, obtained by meshfree method in comparison with the analytical solution for the European call Option with parameters listed in Table 5.1. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter $c = 1.0$

Table 5.4: The RMSE and the maximum computational error

| c | ϵ_{RMSE}^{MQ} | ϵ_{RMSE}^{QMat} | ϵ_{RMSE}^{Wend} | ϵ_{RMSE}^{IMQ} | ϵ_{max}^{MQ} | ϵ_{max}^{QMat} | ϵ_{max}^{Wend} | ϵ_{max}^{IMQ} |
|------|------------------------|--------------------------|--------------------------|-------------------------|-----------------------|-------------------------|-------------------------|------------------------|
| 0.01 | 0.6205 | 0.2413 | 0.2413 | 0.2276 | 1.9245 | 0.6889 | 0.6889 | 0.5947 |
| 1.0 | 0.4416 | 0.2414 | 0.2414 | 0.2414 | 0.8284 | 0.6889 | 0.6889 | 0.6889 |

5.2 Barrier Options

We consider a down-and-out call option with strike E and barrier K . Parameters used in numerical example are listed in Table 5.5. We use a various values of the support parameter c and a few kinds of RBF for the radial basis function approximation of the option price V_t : the MQ-RBF (4.8) [7], the Quadratic Matern RBF (3.18) [5], the Wendland's RBF (3.19) [5] and the IMQ RBF (3.14) [5].

Table 5.5: Parameters for numerical analysis

| | |
|-----------------------------|---------------------------|
| Maximum asset price value | $S_{max} = 30$ |
| Number of asset data points | $N = 121$ |
| Number of time steps | $M = 100$ |
| Time-step size | $\Delta t = 0.005$ |
| Expiration date | $T = 0.5$ (<i>year</i>) |
| Exercise price | $E = 10.0$ |
| Barrier | $K = 9.0$ |
| Risk free interest rate | $r = 0.05$ |
| Volatility | $\sigma = 0.2$ |
| Crank-Nicolson method | $\theta = 0.5$ |
| Support parameter | $c = 0.01$ |

Numerical results for the RBF approximation with different RBF are shown in Table 5.6 in comparison with analytical solution for the down-and-out call Option. Here V_{MQ} denotes the result obtained with the MQ-RBF, V_{QMat} is the result obtained with the Quadratic Matern RBF, V_{Wend} is the result obtained with the Wendland's RBF and V_{IMQ} is the result obtained with the IMQ RBF.

The results obtained with the value of the support parameter $c = 0.01$ are illustrated by Figure 5.3.

Numerical results obtained after increasing of the parameter c to $c = 1.0$ are shown in Table 5.7.

The results obtained with the value of the support parameter $c = 1.0$ are illustrated by Figure 5.4.

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 5.8.

Table 5.6: Results for the down-and-out call option, $c = 0.01$

| Asset price S | $V_{Analytical}$ | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|------------------|----------|------------|------------|-----------|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0288 |
| 3.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0415 |
| 5.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.2104 |
| 7.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0015 |
| 9.0 | 0.0000 | 0.2000 | 0.0000 | 0.0000 | 0.0967 |
| 11.0 | 1.3901 | 1.2235 | 1.0000 | 1.0000 | 0.1152 |
| 13.0 | 3.2587 | 2.9572 | 3.0000 | 3.0000 | 2.3662 |
| 15.0 | 5.2474 | 4.8819 | 5.0000 | 5.0000 | 4.2607 |
| 17.0 | 7.2469 | 6.8305 | 7.0000 | 7.0000 | 5.9834 |
| 19.0 | 9.2469 | 8.7803 | 9.0000 | 9.0000 | 8.2129 |

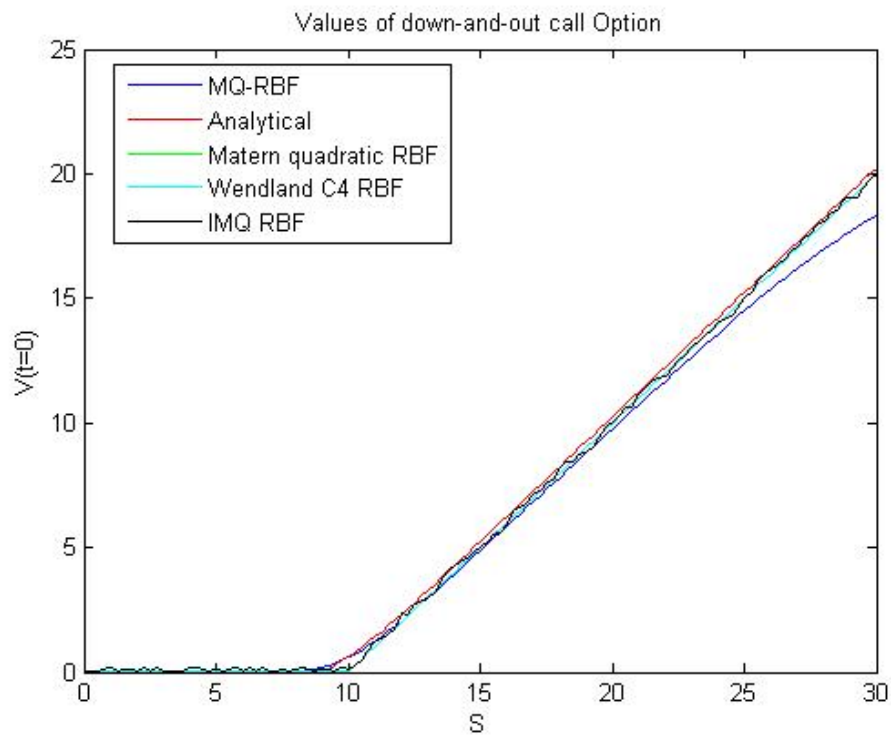


Figure 5.3: Values of the down-and-out call Option, obtained by meshfree method in comparison with the analytical solution for the down-and-out call Option with parameters listed in Table 5.5. Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the support parameter $c = 0.01$.

Table 5.7: Results for the down-and-out call option, $c = 1.0$

| Asset price S | $V_{Analytical}$ | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|------------------|----------|------------|------------|-----------|
| 0.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 5.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 7.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 9.0 | 0.0000 | 0.0033 | 0.0000 | 0.0000 | 0.0000 |
| 11.0 | 1.3902 | 0.9808 | 1.0000 | 1.0000 | 0.0000 |
| 13.0 | 3.2587 | 2.9293 | 3.0000 | 3.0000 | 2.0000 |
| 15.0 | 5.2474 | 4.8798 | 5.0000 | 5.0000 | 4.0000 |
| 17.0 | 7.2469 | 6.8305 | 7.0000 | 7.0000 | 6.0000 |
| 19.0 | 9.2469 | 8.7813 | 9.0000 | 9.0000 | 8.0000 |

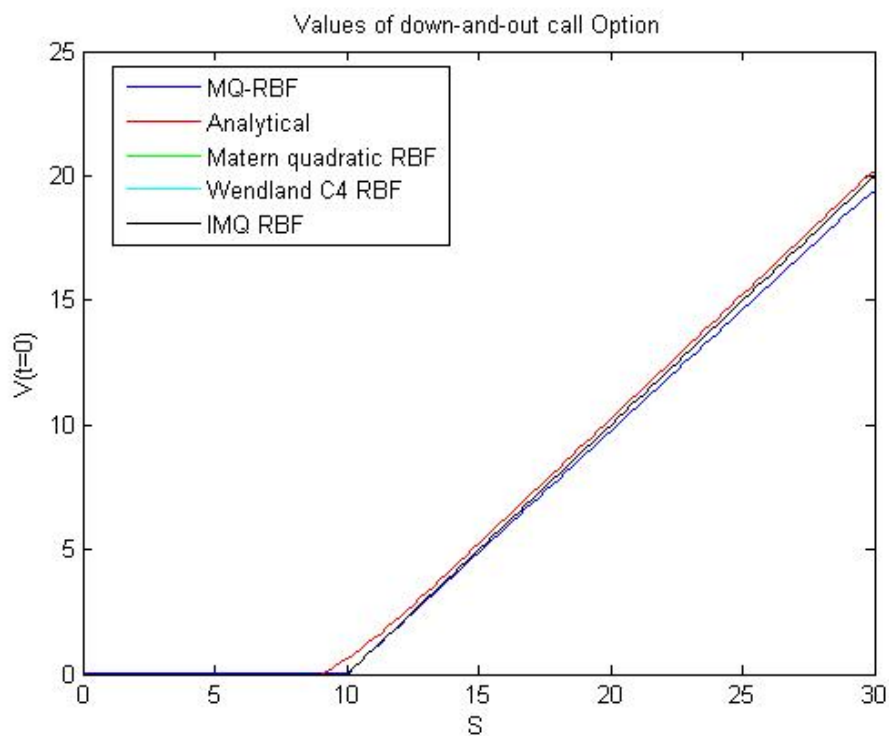


Figure 5.4: Values of the down-and-out call Option, obtained by meshfree method in comparison with the analytical solution for the down-and-out call Option with parameters listed in Table 5.5. Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the RBF parameter $c = 1.0$

Table 5.8: The RMSE and the maximum computational error

| c | ϵ_{RMSE}^{MQ} | ϵ_{RMSE}^{QMat} | ϵ_{RMSE}^{Wend} | ϵ_{RMSE}^{IMQ} | ϵ_{max}^{MQ} | ϵ_{max}^{QMat} | ϵ_{max}^{Wend} | ϵ_{max}^{IMQ} |
|------|------------------------|--------------------------|--------------------------|-------------------------|-----------------------|-------------------------|-------------------------|------------------------|
| 0.01 | 0.6207 | 0.2303 | 0.2303 | 0.2199 | 1.9245 | 0.6166 | 0.6166 | 0.5430 |
| 1.0 | 0.4358 | 0.2303 | 0.2303 | 0.2303 | 0.8284 | 0.6166 | 0.6166 | 0.6166 |

5.3 Asian Options

We consider an Asian average strike call option. Parameters used in numerical example are listed in Table 5.9. We use a various values of the support parameter c and a few kinds of RBFs for the radial basis function approximation of the option price V_t : the MQ-RBF (4.8) [7], the Quadratic Matern RBF (3.18) [5], the Wendland's RBF (3.19) [5] and the IMQ RBF (3.14) [5].

Table 5.9: Parameters for numerical analysis

| | |
|-----------------------------|---------------------------|
| Maximum R | $R_{max} = 1.0$ |
| Number of asset data points | $N = 101$ |
| Number of time steps | $M = 1000$ |
| Time-step size | $\Delta t = 0.0005$ |
| Expiration date | $T = 0.5$ (<i>year</i>) |
| Risk free interest rate | $r = 0.1$ |
| Volatility | $\sigma = 0.4$ |
| Crank-Nicolson method | $\theta = 0.5$ |
| Support parameter | $c = 0.06$ |

Numerical results for the RBF approximation with different RBF are shown in Table 5.10 in comparison with solution obtained by finite difference method (FD), particularly, Crank-Nicolson method. Here H_{MQ} denotes the result obtained with the MQ-RBF, H_{QMat} is the result obtained with the Quadratic Matern RBF, H_{Wend} is the result obtained with the Wendland's RBF and H_{IMQ} denotes the result obtained with the IMQ RBF.

Table 5.10: Results for the Asian average strike call option, $c = 0.06$

| Asset price R | H_{FD} | H_{MQ} | H_{QMat} | H_{Wend} | H_{IMQ} |
|-----------------|----------|----------|------------|------------|-----------|
| 0.0 | 0.9990 | 0.9922 | 1.0000 | 1.0000 | 0.9810 |
| 0.1 | 0.7990 | 0.7911 | 0.8000 | 0.8000 | 0.7842 |
| 0.2 | 0.5990 | 0.5912 | 0.6000 | 0.6000 | 0.5820 |
| 0.3 | 0.3990 | 0.3914 | 0.4000 | 0.4000 | 0.3501 |
| 0.4 | 0.1990 | 0.1920 | 0.2000 | 0.2000 | 0.1702 |
| 0.5 | 0.0014 | 0.0115 | 0.0000 | 0.0000 | 0.0305 |
| 0.6 | 0.0000 | 0.0015 | 0.0000 | 0.0000 | 0.0269 |
| 0.7 | 0.0000 | 0.0012 | 0.0000 | 0.0000 | 0.0239 |
| 0.8 | 0.0000 | 0.0014 | 0.0000 | 0.0000 | 0.0085 |
| 0.9 | 0.0000 | 0.0027 | 0.0000 | 0.0000 | 0.0156 |

The results obtained with the value of the support parameter $c = 0.06$ are illustrated by Figure 5.5.

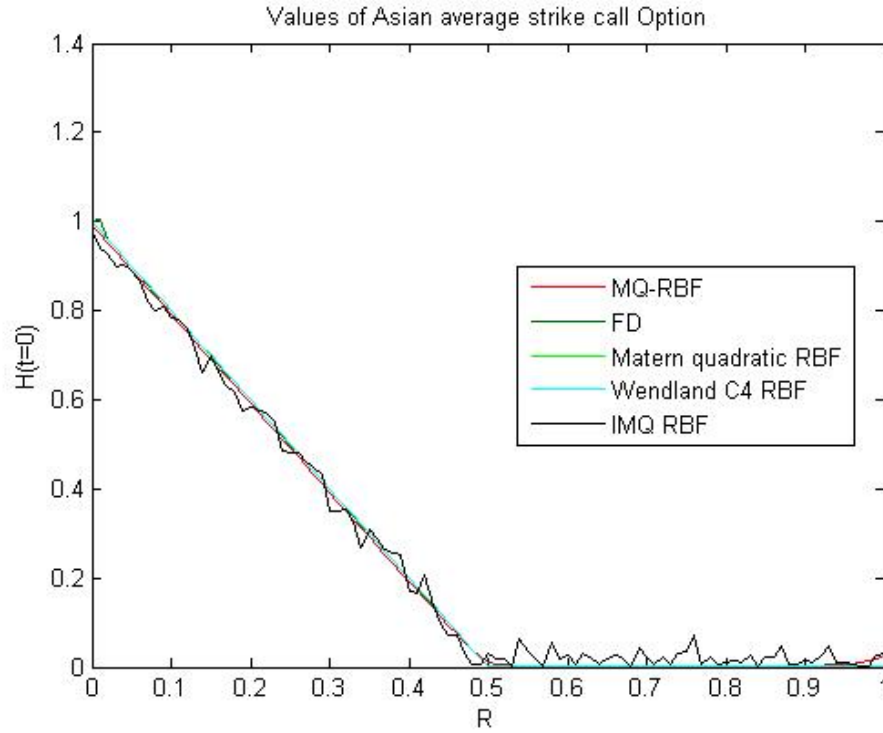


Figure 5.5: Values of the Asian average strike call Option, obtained by mesh-free method with parameters listed in Table 5.9, in comparison with the finite difference method with implicitness parameter $\theta = 0.5$ (Crank-Nicolson scheme). Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the RBF parameter $c = 0.06$.

Numerical results obtained after increasing of the parameter c to $c = 0.09$ are shown in Table 5.11.

The results obtained with the value of the support parameter $c = 0.09$ are illustrated by Figure 5.6.

The RMSE and maximum computational error for different kinds of RBFs and the various values of the parameter c are represented in Table 5.12.

Table 5.11: Results for the Asian average strike call option, $c = 0.09$

| Asset price R | H_{FD} | H_{MQ} | H_{QMat} | H_{Wend} | H_{IMQ} |
|-----------------|----------|----------|------------|------------|-----------|
| 0.0 | 0.9990 | 0.9919 | 1.0000 | 1.0000 | 0.9980 |
| 0.1 | 0.7990 | 0.7911 | 0.8000 | 0.8000 | 0.7764 |
| 0.2 | 0.5990 | 0.5912 | 0.6000 | 0.6000 | 0.5920 |
| 0.3 | 0.3990 | 0.3914 | 0.4000 | 0.4000 | 0.3828 |
| 0.4 | 0.1990 | 0.1919 | 0.2000 | 0.2000 | 0.1543 |
| 0.5 | 0.0014 | 0.0095 | 0.0000 | 0.0000 | 0.0071 |
| 0.6 | 0.0000 | 0.0014 | 0.0000 | 0.0000 | 0.0024 |
| 0.7 | 0.0000 | 0.0012 | 0.0000 | 0.0000 | 0.0215 |
| 0.8 | 0.0000 | 0.0014 | 0.0000 | 0.0000 | 0.0066 |
| 0.9 | 0.0000 | 0.0025 | 0.0000 | 0.0000 | 0.0337 |

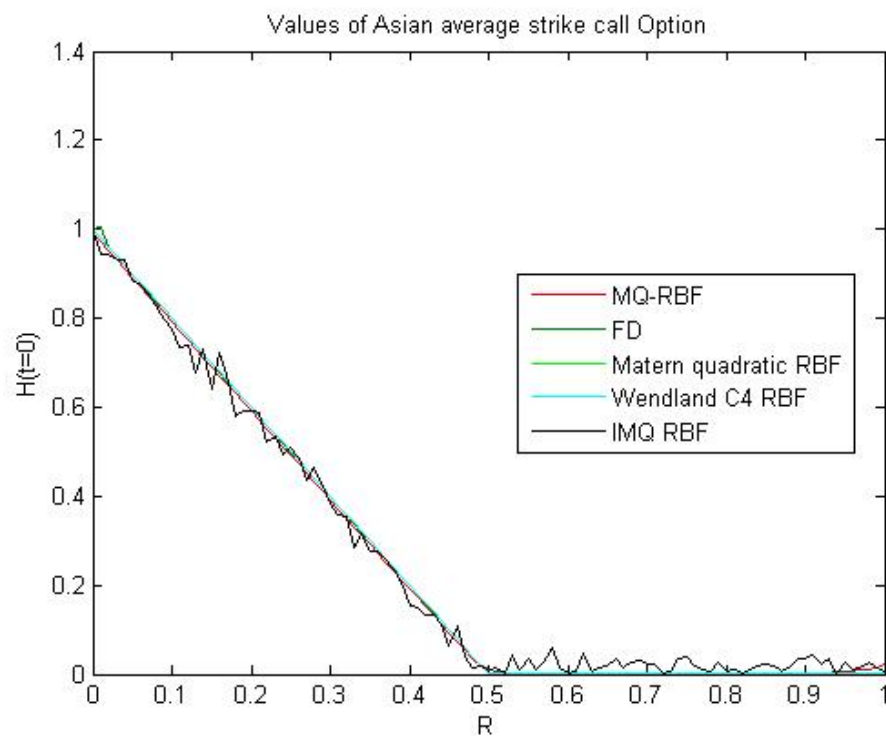


Figure 5.6: Values of the Asian average strike call Option, obtained by mesh-free method with parameters listed in Table 5.9, in comparison with the finite difference method with implicitness parameter $\theta = 0.5$ (Crank-Nicolson scheme). Here the MQ RBF, Quadratic Matern RBF, Wendland's RBF and IMQ RBF were used with the RBF parameter $c = 0.09$.

Table 5.12: The RMSE and the maximum computational error

| c | ϵ_{RMSE}^{MQ} | ϵ_{RMSE}^{QMat} | ϵ_{RMSE}^{Wend} | ϵ_{RMSE}^{IMQ} | ϵ_{max}^{MQ} | ϵ_{max}^{QMat} | ϵ_{max}^{Wend} | ϵ_{max}^{IMQ} |
|------|------------------------|--------------------------|--------------------------|-------------------------|-----------------------|-------------------------|-------------------------|------------------------|
| 0.06 | 0.0075 | 0.0025 | 0.0025 | 0.0262 | 0.0334 | 0.0239 | 0.0239 | 0.0735 |
| 0.09 | 0.0071 | 0.0025 | 0.0025 | 0.0253 | 0.0333 | 0.0239 | 0.0239 | 0.0627 |

5.4 American Options

We consider an American put option, parameters used in numerical example are listed in Table 5.13. We use a few kinds of RBF for the radial basis function approximation of the option price V_t : the MQ-RBF (4.8) [7], the Quadratic Matern RBF (3.18) [5], the Wendland's RBF (3.19) [5] and the IMQ RBF (3.14) [5].

Table 5.13: Parameters for numerical analysis

| | |
|--|--------------------|
| Maximum asset price | $S_{max} = 30$ |
| Number of asset data points | $N = 101$ |
| Number of time steps | $M = 100$ |
| Time-step size | $\Delta t = 0.005$ |
| Expiration date | $T = 0.5$ (year) |
| Exercise price | $E = 10.0$ |
| Risk free interest rate | $r = 0.05$ |
| Volatility | $\sigma = 0.2$ |
| Crank-Nicolson method | $\theta = 0.5$ |
| Constant in the penalty term | $C = 0.9$ |
| Regularization parameter in the penalty term | $\epsilon = 0.001$ |

Numerical results for the RBF approximation with different RBF are shown in Table 5.14 in comparison with solution obtained by finite difference method (FD), particularly, Crank-Nicolson method. Here V_{MQ} denotes the result obtained with the MQ-RBF, V_{QMat} is the result obtained with the Quadratic Matern RBF, V_{Wend} is the result obtained with the Wendland's RBF and V_{IMQ} is the result obtained with the IMQ RBF.

The results obtained with the value of the support parameter $c = 0.02$ are illustrated by Figure 5.7.

Table 5.14: Results for the American put option, $c = 0.02$

| Asset price S | V_{FD} | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|----------|----------|------------|------------|-----------|
| 0.0 | 9.7531 | 10.0000 | 10.0000 | 10.0000 | 10.0000 |
| 3.0 | 7.0000 | 6.8160 | 7.0000 | 7.0000 | 7.0000 |
| 6.0 | 4.0000 | 3.8859 | 4.0000 | 4.0000 | 4.0000 |
| 9.0 | 1.0652 | 1.0269 | 1.0000 | 1.0000 | 1.0000 |
| 12.0 | 0.0508 | 0.0426 | 0.0000 | 0.0000 | 0.0000 |
| 15.0 | 0.0007 | 0.0024 | 0.0000 | 0.0000 | 0.0000 |
| 18.0 | 0.0000 | 0.0019 | 0.0000 | 0.0000 | 0.0000 |
| 21.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 24.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 27.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

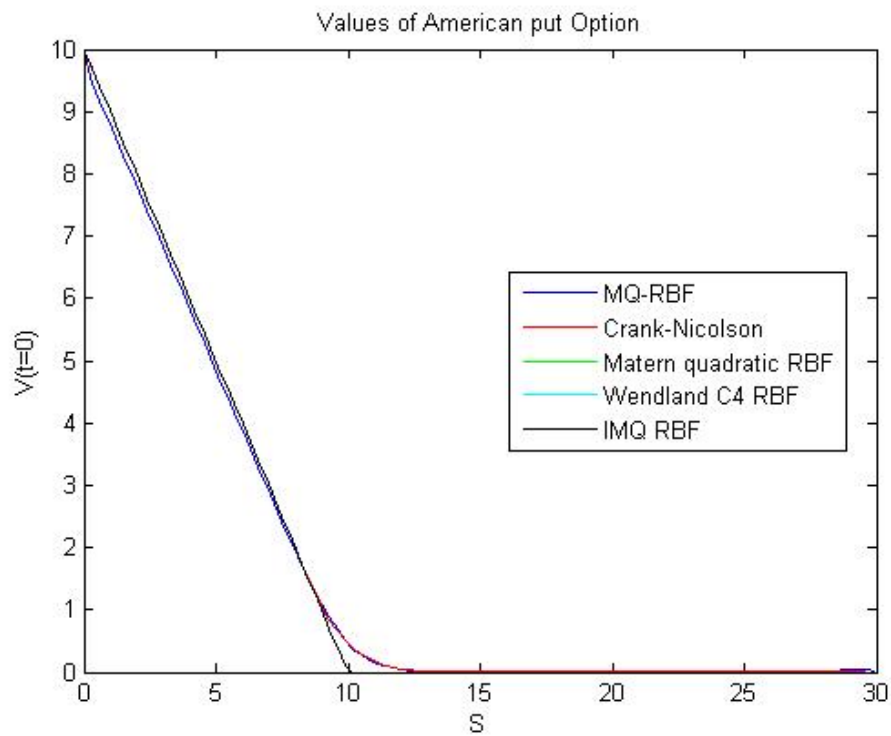


Figure 5.7: Values of the American put Option, obtained by meshfree method with parameters listed in Table 5.13, in comparison with the finite difference method with implicitness parameter $\theta = 0.5$ (Crank-Nicolson scheme). Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the support parameter $c = 0.02$.

Numerical results obtained after increasing of the parameter c to $c = 1.0$ are shown in Table 5.15.

Table 5.15: Results for the American put option, $c = 1.0$

| Asset price S | V_{FD} | V_{MQ} | V_{QMat} | V_{Wend} | V_{IMQ} |
|-----------------|----------|----------|------------|------------|-----------|
| 0.0 | 9.7531 | 10.0000 | 10.0000 | 10.0000 | 10.0000 |
| 3.0 | 7.0000 | 6.8160 | 7.0000 | 7.0000 | 7.0000 |
| 6.0 | 4.0000 | 3.8859 | 4.0000 | 4.0000 | 4.0000 |
| 9.0 | 1.0652 | 0.9537 | 1.0000 | 1.0000 | 1.0000 |
| 12.0 | 0.0508 | 0.0034 | 0.0000 | 0.0000 | 0.0000 |
| 15.0 | 0.0007 | 0.0020 | 0.0000 | 0.0000 | 0.0000 |
| 18.0 | 0.0000 | 0.0019 | 0.0000 | 0.0000 | 0.0000 |
| 21.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 24.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 27.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

The results obtained with the value of the support parameter $c = 1.0$ are illustrated by Figure 5.8.

The RMSE and maximum computational error for different kinds of RBFs relative to FD method and the various values of the parameter c are represented in Table 5.16.

Table 5.16: The RMSE and the maximum computational error

| c | ϵ_{RMSE}^{MQ} | ϵ_{RMSE}^{QMat} | ϵ_{RMSE}^{Wend} | ϵ_{RMSE}^{IMQ} | ϵ_{max}^{MQ} | ϵ_{max}^{QMat} | ϵ_{max}^{Wend} | ϵ_{max}^{IMQ} |
|------|------------------------|--------------------------|--------------------------|-------------------------|-----------------------|-------------------------|-------------------------|------------------------|
| 0.02 | 0.0899 | 0.0796 | 0.0796 | 0.0796 | 0.2469 | 0.4093 | 0.4093 | 0.4093 |
| 1.0 | 0.1180 | 0.0796 | 0.0796 | 0.0796 | 0.4186 | 0.4093 | 0.4093 | 0.4093 |

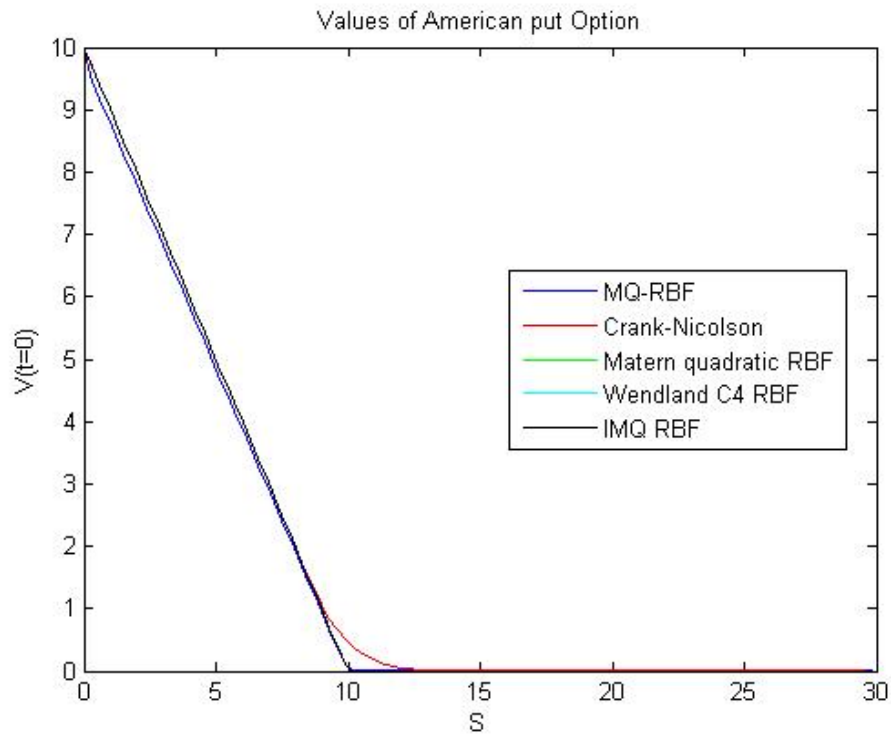


Figure 5.8: Values of the American put Option, obtained by meshfree method with parameters listed in Table 5.13, in comparison with the finite difference method with implicitness parameter $\theta = 0.5$ (Crank-Nicolson scheme). Here the MQ RBF, the Quadratic Matern RBF, the Wendland's RBF and the IMQ RBF were used with the RBF parameter $c = 1.0$

Chapter 6

Conclusions

In this thesis, we analyzed the European, Barrier, Asian and American options via the RBF approach to obtain the approximate value of the option price. It is necessary to use accurate, fast methods with very low memory requirements, because the financial markets are becoming more and more complex.

The RBF approximation with infinitely smooth RBFs can be spectrally accurate, meaning that the required number of node points for a certain desired accuracy is relatively small. Moreover, the method has a meshfree nature that makes it easy to use in higher dimensions. These methods in contrast to the traditional simulation algorithms use the geometry of the simulated object directly for calculations. Interpolation problem with radial basis functions becomes insensitive to the dimension of the space in which the data sites lie, instead a multivariate function, whose complexity will increase with increasing space dimension, we can use the same univariate function for all choices of dimension. We analyzed the results obtained by Goto *et al.* [7] and by Fasshauer *et al.* [2]. In [7] it was shown that the results of the RBF approximation agreed well with the theoretical solution. Fasshauer [2] got results for American options and multi-asset American options problems comparable to the finite difference method with fewer degrees of freedom.

In our study, we examined and implemented the RBF approximation to obtain the fair price of the European, Barrier, Asian and American options. As a new development, besides MQ RBF, we investigated different kinds of RBFs such as the Quadric Matern RBF and Wendland's RBF.

The meshfree approach can be more accurate and stable method compared with the Finite Difference method and can be applied to solve partial differential equations. In our work, the best results were obtained in case of the Asian Option for the solution of the reduced equation, where the fair

price of the option is purely comparable with the values, obtained by the Finite Difference method.

The results obtained by the meshfree method strongly depends on the choice of the RBF, implemented for the approximation of the value of the option. In our investigation, the least root-mean-square error was obtained with the MQ RBF in comparison with Quadric Matern and Wendland's RBF, however the greatest maximum error was shown in the case of the MQ RBF.

The choice of the RBF's shape parameter c also affects to the accuracy of the method, however it was observed, that the influence of the change of the RBF's parameter c is the most intense for the MQ RBF while for the another examined RBFs the RMSE and the maximum error remain almost unaltered.

In the further research, we can examine the influence of the different parameters of the method to the accuracy, particularly, to find the optimal value of the RBF's parameter c . It would also be very interesting for us to find the RBF that gives better results then the RBFs examined before.

Notation

| | |
|------------------------|--|
| $V(S, t)$ | The price of the option. |
| $C(S, t)$ | The price of the call option. |
| $P(S, t)$ | The price of the put option. |
| r | The risk-free interest rate. |
| σ | The volatility of the asset price. |
| S | The asset price. |
| T | The expiration date of the option. |
| E | The strike price of the option. |
| K | The barrier of the Barrier option. |
| S_f | The optimal exercise price (a free boundary) of the American option. |
| S_j | The collocation point. |
| θ | The implicit parameter. |
| c | The RBF parameter. |
| $\phi_j, \phi(S, S_j)$ | The RBF at the collocation point j . |

Bibliography

- [1] F. Black, M. Scholes (1973)
The pricing of options and corporate liabilities. Political Economy, **81**, 637 – 655.
- [2] G.E. Fasshauer, A.Q.M. Khaliq, D.A. Voss (2004)
Using Meshfree Approximation for Multi-Asset American Options. J. Chinese Inst. Eng., **27**, 563 – 571.
- [3] G.E. Fasshauer, A.Q.M. Khaliq, D.A. Voss (2004)
A parallel time stepping approach using meshfree approximations for pricing options with non-smooth payoffs. Proceedings of Third World Congress of the Bachelier Finance Society, Chicago.
- [4] G.E. Fasshauer (2006)
Meshfree Methods. Handbook of Theoretical and Computational Nanotechnology, M. Rieth and W. Schommers (eds.), American Scientific Publishers, **27**, 33 – 97.
- [5] G.E. Fasshauer (2007)
Meshfree Approximation Methods with MATLAB (Interdisciplinary Mathematical Sciences). World Scientific Publishing Co.Pte.Ltd., 500.
- [6] B. Fornberg, E. Larsson, N. Flyer (2011)
Stable computations with Gaussian radial basis functions. SIAM J. Sci. Comput., **33**, pp. 869-892.
- [7] Y. Goto, Z. Fei, S. Kan, E. Kita (2007)
Options valuation by using radial basis function approximation. Engrg. Anal. Bound. Elem., **31**, 836 – 843.
- [8] Y.C. Hon (2002)
A quasi-radial basis functions method for American options pricing. Comput. Math. Appl., **43**, 513 – 524.

- [9] Y.C. Hon, X.C. Mao (1999)
A radial basis function method for solving options pricing model. J. Financial Engineering, **8**, 1 – 24.
- [10] M.B. Koc, I. Boztosum, D. Boztosum (2003)
On the numerical solution of Black-Scholes equation. Proceedings of international workshop on Meshfree method, Lisbon, Portugal, 6 – 11.
- [11] Y.K. Kwok (1998)
Mathematical Models of Financial Derivatives. Springer, Singapore.
- [12] M. D. Marcozzi, S. Choi, C. S. Chen (1994)
RBF and optimal stopping problems; an application to the pricing of vanilla options on one risky asset. Boundary Element Technology XIV, C.S. Chen et al. (eds.), Computational Mechanics Publications, 345 – 354.
- [13] U. Pettersson, E. Larsson, G. Marcusson, J. Persson (2005)
Option Pricing using Radial Basis Functions. ECCOMAS Thematic Conference on Meshless Methods, <http://www.math.ist.utl.pt/meshless/>.
- [14] M.J.D. Powell (1992)
The Theory of Radial Basis Function Approximation. Advances in Numerical Analysis III, Oxford: Clarendon Press, 105 – 210.
- [15] S. Rippa (1999)
An algorithm for selecting a good value for the parameter c in radial basis function interpolation. Adv. Comput. Math., **11**, 193 – 210.
- [16] R. Seydel (2006)
Tools for Computational Finance. Springer, Berlin.
- [17] P. Wilmott, S. Howison, J. Dewynne (1995)
The Mathematics of Financial Derivatives: A Student Introduction. Cambridge University Press.
- [18] Z. Wu, Y.C. Hon (2003)
Convergence error estimate in solving free boundary diffusion problem by radial basis functions method. Engrg. Anal. Bound. Elem., Barking, Essex, England: Elsevier, **27**, 73 – 79.
- [19] Z. Wu (2003)
Compactly Supported Positive Definite Radial Functions. J. Adv. Comput. Math., **4**, 283 – 292.

Appendix

Here some integral characterizations of completely monotone functions are presented.

Theorem 6.1 [4] (*Hausdorff-Bernstein-Widder Theorem*)

A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is completely monotone on $[0, \infty)$ if and only if it is the Laplace transform of a finite non-negative Borel measure μ on $[0, \infty)$

$$\varphi(r) = \mathcal{L}\mu(r) = \int_0^\infty e^{-rt} d\mu(t). \quad (6.1)$$

Theorem 6.2 [4] (*Schoenberg*)

A function ϕ is completely monotone on $[0, \infty)$ if and only if $\Phi = \varphi(\|\cdot\|^2)$ is positive definite and radial on \mathbb{R}^s for all s .

Theorem 6.3 [4] (*Schoenberg*)

If the function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is completely monotone but not constant, then $\varphi(\|\cdot\|^2)$ is strictly positive definite and radial on \mathbb{R}^s for any s .