Operator Splitting Techniques for American Type of Floating Strike Asian Option

Master’s Thesis in Financial Mathematics

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Preface

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Abstract
In this thesis we investigate Asian floating strike options. We particularly focus on options with early exercise - American options. This type of options are very lucrative to the end-users of commodities or energies who are tend to be exposed to the average prices over time. Asian options are also very popular with corporations, who have ongoing currency exposures. The main idea of the pricing is to examine the free boundary position on which the value of the option is depending. We focus on developing a efficient numerical algorithm for this boundary. In the first Chapter we give an informative description of the financial derivatives including Asian options. The second Chapter is devoted to the analytical derivation of the corresponding partial differential equation coming from the original Black - Scholes equation. The problem is simplified using transformation methods and dimension reduction. In the third and fourth Chapter we describe important numerical methods and discretize the problem. We use the first order Lie splitting and the second order Strang splitting. Finally, in the fifth Chapter we make numerical experiments with the free boundary and compare the result with other known methods.
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Chapter 1

Introduction

1.1 Financial Derivatives

In terms of derivatives in financial context one can refer to a contract which price at given time depends on the value of the underlying asset i.e. any financial contract. An example of an underlying asset can be stocks, exchange rates, commodities such as crude oil, gold, etc. or interest rates. In the last decades, there was a huge expansion of derivative trading on financial markets. Derivative securities have became a successful trading instrument all over the world.

Financial derivatives are used as a main securing tool against unpredictable movements of financial markets. Examples of derivatives are forwards, futures, swaps and options. In the case of forward and futures the asset must be exercised, while in the case of the option this is just a right. A swap is a derivative in which counter-parties exchange certain benefits from their financial instruments for a predefined period of time. Combination of these types are also possible. They might include compound options, which are options on options; or futures options, where the underlying is a future contract.

1.1.1 Options

An European call (put) on an underlying asset gives the holder the right, but not the obligation, to buy (sell) the underlying at a predefined price $E$ (strike price or exercise price) at a certain future date $T$ (the maturity). At this time the writer of the options is obliged to sell (buy) the underlying from the holder of the options. The purchase value of the option is called the premium, and it is payed by the holder to a writer when the contract
Chapter 1. Introduction

is sold. The European option can be exercised only at the maturity time. Mathematically, it can be expressed by the following payoff function

\[ V(S, T) = \begin{cases} 
V^{CE} = [S(T) - E]^+, \\
V^{PE} = [E - S(T)]^+, 
\end{cases} \]

where \( V^{CE} \) (\( V^{PE} \)) denotes the European call (put) option and by \([S(T) - E]^+\) we define \( \max\{S(T) - E, 0\} \).

An American option is an option which can be exercised at any time up to maturity. In the case of the American options the payoff functions, when exercised, are identical with the European type. Because of that, the prices of the American call (\( V^{CA} \)) and put (\( V^{PA} \)) are bounded from below

\[ V(S, t) = \begin{cases} 
V^{CA} \geq [S(t) - E]^+, \\
V^{PA} \geq [E - S(t)]^+. 
\end{cases} \]

Option writers and buyers also, called option traders, complete the market together. Therefore one can take four different positions, Figure 1.1, on the market namely:

- long call - buy call option,
- short call - sell call option,
- long put - buy put option,
- short put - sell put option.

Simple options as call and put are commonly called plain vanilla options. Even though, the plain vanilla options are widely known and used, there are also many different types of options demanded on the market called by the common name exotic options.

Exotic option are commonly traded over-the-counter (OTC) and their features are making them more complex compared to the plain vanilla options. They can differ in many ways such as they can depend on more underlying assets (basket options), the price can depend not just on the the current asset price (path-dependent options) etc. From above mentioned, the most commonly used are the path-dependent options. They most frequent are

- The Asian options - the price of the option depends on the averaged asset price during the lifetime of the option,
Figure 1.1: The graphical illustration of option positions. In order: long call, long put, short call and short put.

- The Barrier option - the option is either activated or extinguished upon the occurrence of the event of the underlying price reaching a predefined barrier,

- The Chooser option - gives the holder a predefined time to decide the type of the option (call or put),

- The Lookback options - the price of the option depends on the maximum (minimum) of asset price through the options lifetime.

In this work we focus on Asian options which are briefly described in Section 1.2.

1.1.2 Option Pricing

An inseparable part of derivative products is their pricing procedure. The model developed by Black and Scholes [2] and independently by Merton [7] has brought a completely new perspective to the financial world. In spite of the strict assumptions the model and its variations are widely used as a main mathematical model of financial markets and derivative instruments.

Assuming that the movement of the underlying asset follows the Geometrical Brownian Motion (GBM)
\[
\frac{dS}{S} = \mu dt + \sigma dX,
\]

where \(S\) is the asset price, \(\mu\) is the drift term and \(\sigma\) the volatility of the stock return. By the term \(X\) we denote the standard Wiener’s process. Taking into consideration the following assumptions

- the risk-free interest rate \(r\) and the volatility \(\sigma\) are known functions,
- there are no transaction costs associated with hedging a portfolio,
- the underlying asset pays no dividends during the life of the option,
- there are no arbitrage possibilities,
- trading of the underlying asset can be take place continuously,
- short selling is permitted,
- the assets are infinitely divisible,

the following backward-in-time partial differential equation can be derived

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad 0 \leq t \leq T.
\]

The solution of this PDE using the final condition \(V(S,T) = (S - E)^+\), i.e. the price of the European non-dividend paying call option is given in the explicit form

\[
V^{PE}(S,t) = SN(d_1) - Xe^{-(T-t)}N(d_2),
\]

\[
d_{1,2} = \frac{\ln \frac{S}{E} + (r \pm \frac{\sigma^2}{2})T - t}{\sigma \sqrt{T-t}},
\]

where \(N(\cdot)\) is a cumulative normal distribution function with \(\mu = 0\) and \(\sigma = 1\). The price of the European non-dividend paying put is calculated similarly with the final condition \(V(S,T) = (E - S)^+\).

In the last years many extensions have been made to the model. The model is versatile and capable to adapt for the case of the dividend paying underlying asset, variable interest rates and volatilities and also for the American case, even though their valuation is more difficult.
1.2 The Asian Options

The Asian options are path-dependent options. The payoff of these options depends not only on the current price of underlying asset but on some of its average over a specified time period. The main advantage of Asian options is their price, which is less than its plain vanilla alternative. Asian options are often used as a hedge tool against unexpected movements in asset prices i.e averaging reduces the susceptibility to price manipulation. An example could be a crude oil consumer who is afraid of price increase in future. He prefers to have his crude oil supplies for the price equal to the average of last few weeks. His requirements can be satisfied by a special type of Asian options. These option were first introduced in Tokyo of Banker’s Trust in 1987 issued on already mentioned crude oil contracts [13].

There are many variations of Asian options depending on how the payoff function is defined and what input variables are used. In the first case the type of averaging should be discussed which can be either arithmetic or geometric. It is convenient to use geometric average if the underlying asset behaves according to the geometrical Brownian motion. In this case the problem can be transformed into the classical heat equation. On the other hand it is the arithmetic average which is used in real world, even though its valuation is more difficult. The sampling of both arithmetic and geometric average can be either continuous or discrete. While the discrete type of sampling is used in the real world, it is more convenient to use the continuous from a mathematical point of view. Then the average in the case of continuous-time models for geometric and arithmetic average are given respectively by

$$\bar{S}^A(t) = \frac{1}{t} \int_0^t S(u)du,$$
$$\bar{S}^G(t) = \exp \left[ \frac{1}{t} \int_0^t \ln S(u)du \right].$$

For the discrete average, where N denotes the number of equidistant averaging points, the average process are expressed respectively by

$$\bar{S}^A_d(N) = \frac{1}{N} \sum_{n=1}^N S(t_n),$$
$$\bar{S}^G_d(N) = \exp \left[ \frac{1}{N} \sum_{n=1}^N \ln(S(t_n)) \right].$$
Figure 1.2: The Example of the time development of Microsoft Corporation stock price and corresponding type of continuous average (on the left) and the difference between continuous and discrete average (on the right). Source: http://finance.yahoo.com

We can particularly divide Asian options depending on the form of the payoff function into two main categories:

- **the average strike** options, also known as the floating strike options, payoff is given by a difference between the spot price at maturity time and the strike price calculated as an average of the underlying during the specified time interval

\[
V(S, A, T) = \begin{cases} 
V^{CS} = [S(T) - \overline{S}(T)]^+, \\
V^{PS} = [\overline{S}(T) - S(T)]^+
\end{cases}
\]

- **the average rate (fixed strike)** options payoff is defined as a difference between the average price of underlying and a predefined strike price

\[
V(S, A, T) = \begin{cases} 
V^{CS} = [\overline{S}(T) - E]^+, \\
V^{PS} = [E - \overline{S}(T)]^+
\end{cases}
\]

As for the plain vanilla European option, there exist as well an American alternative for the European Asian options. **the Hawaiian options** are options with the early-exercise feature, also called as American-style Asian options. The holder of these options can exercise not just in the maturity time but at any time during the lifetime of the contract. The received payoff
is derived from the average up to the exercise time. Unsurprisingly, all the characteristic features introduced for the European case can be adapted to the American style options.
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Chapter 2

The Analytical Derivation

In this Chapter we discuss the transformation methods for the Asian option proposed by Ševčovič [3], [10]. We particularly focus on the American type of Asian options and a free boundary problem arising from this problem.

2.1 The Transformational Method

In this section we shall consider the price dynamics driven by the GBM in the following form

\[ dS = (r - q)Sdt + \sigma SdX, \quad (2.1) \]

where \( r \) is the risk free interest rate, \( q > 0 \) is the continuous dividend yield and \( \sigma \) denotes the volatility. By the term \( X \) we denote the standard Wiener’s process. As we already discussed, the floating strike Asian option with arithmetic average is a financial instrument, which depends not only on the stock price \( S \) and maturity time \( T \), but also on the average \( A \). Thus, the price can be written as a function \( V(S, A, t) \). Applying Itô’s lemma (see Appendix 2) we get the following expression

\[ dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial A}dA + \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (2.2) \]

The arithmetic average \( A = \frac{1}{t} \int_0^t S(u)du \) gives the differential equation

\[ \frac{dA}{dt} = \frac{dA_t}{dt} = \frac{1}{t} S_t - \frac{1}{t^2} \int_0^t S_\tau d\tau = \frac{S_t - A_t}{t} = \frac{S - A}{t}. \quad (2.3) \]
Substituting (2.1) and (2.3) into equation (2.2) we obtain the differential equation for the price process \( V(S, A, t) \)

\[
dV = \left( \frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{\partial^2 V}{\partial S^2} + \frac{S - A}{t} \frac{\partial V}{\partial A} \right) dt + \sigma S \frac{\partial V}{\partial S} dX. \tag{2.4}
\]

We consider now a portfolio \( \Pi \), which consists of one derivative (option) and \(-\Delta\) of underlyings. The derivative \( d\Pi \) of this portfolio, so the one time step change in the case of the dividend paying underlying asset is

\[
d\Pi = dV - \Delta dS - q\Delta S dt. \tag{2.5}
\]

We consider here \( \Delta \) being a constant during one-time step. Now, finally putting (2.1), (2.4), (2.5) together we obtain

\[
d\Pi = \left[ \frac{\partial V}{\partial t} + (r - q)S \left( \frac{\partial V}{\partial S} - \Delta \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{S - A}{t} \frac{\partial V}{\partial A} - q\Delta S \right] dt + \sigma S \left[ \frac{\partial V}{\partial S} - \Delta \right] dX. \tag{2.6}
\]

In order to get rid off the uncertainty caused by the term \( dX \) in our portfolio we shall choose \( \Delta = \frac{\partial V}{\partial S} \). By this setting we can eliminate the randomness present in our portfolio through the asset price process, which is driven by the Brownian motion. This move is a so called *delta hedging*. Because of the fact of arbitrage opportunities we shall consider a risk-free investment into a riskless asset. An investment of the amount \( \Pi \) into this asset would bring a growth

\[
d\Pi = r\Pi dt \tag{2.7}
\]

in one time step. Any other deterministic growth would arise in an arbitrage opportunity. Thus (2.7) and (2.6) should be equal. Using this equality, and dividing by \( dt \) we obtain a PDE for the floating strike Asian option with arithmetic average

\[
\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{S - A}{t} \frac{\partial V}{\partial A} - rV = 0. \tag{2.8}
\]

For the American type of options we have to develop also boundary conditions. According to Kwok [5] we denote the early exercise boundary of the call option as \( S_f(A, t) \) and describe the early exercise region by

\[
\varepsilon = \{(S, A, t) \in [0, \infty) \times [0, \infty) \times [0, T), S \geq S_f(A, t)\}. \tag{2.9}
\]
For the call option the first two conditions arise from the European types. The terminal condition at time $T$ and the homogeneous Dirichlet condition at $S = 0$

$$V(S, A, T) = (S - A)^+, \quad V(0, A, t) = 0. \tag{2.10}$$

As the option price reaches the early exercise (free) boundary one can determine the price of the option from the payoff function at that moment. The slope of the option with respect to the price $S$, $\frac{\partial C}{\partial S}$ at the free boundary $S_f(A, t)$ should be equal 1. This guarantees that the option value is connected to the payoff function arising from the early exercise of the option smoothly, ensuring us no arbitrage opportunity. The boundary conditions following this arguments can be written as

$$V(S_f(A, t), A, t) = S_f(A, t) - A, \quad \frac{\partial V}{\partial S}(S_f(A, t), A, t) = 1. \tag{2.11}$$

Thus we obtain a two-dimensional PDE. Fortunately, there exist a transformation method using similarities for floating strike Asian option, which reduces the dimension of this problem. Using the new variables

$$x = \frac{S}{A}, \quad \tau = T - t, \quad W(x, \tau) = \frac{1}{A} V(S, A, t), \tag{2.12}$$

the equation (2.8) can be transformed to the following parabolic PDE:

$$\frac{\partial W}{\partial \tau} - (r - q)x \frac{\partial W}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - \frac{x - 1}{T - \tau} \left( W - x \frac{\partial W}{\partial x} \right) + rW = 0. \tag{2.13}$$

The early exercise boundary $S_f$ can be also reduced to a one dimensional variable $x_f(t) = S_f(A, t)/A$. To obtain a spatial domain for the equation (2.13) we introduce a new variable $\rho(\tau) = x_f(T - \tau)$. Further $W(x, \tau)$ is the solution of this equation for $x \in (0, \rho(\tau))$, $\tau \in (0, T)$. From (2.10) and (2.11) we can determine the new initial and boundary conditions

$$W(x, 0) = (x - 1)^+, \quad \forall x > 0, \tag{2.14}$$

respectively

$$W(0, \tau) = 0, \quad W(\rho(\tau), \tau) = \rho(\tau) - 1, \quad \frac{\partial W}{\partial x}(\rho(\tau), \tau) = 1. \tag{2.15}$$
2.1.1 The Fixed Domain Transformation

In this section we present a fixed domain transformation of the free boundary problem. The idea is to transform the problem into a nonlinear parabolic equation on a fixed domain. Following Ševčovič [3] we use a new variable ξ and a new auxiliary function representing a synthetic portfolio

\[ \xi = \ln \frac{\rho(\tau)}{x}, \quad \Pi(\xi, \tau) = W(x, \tau) - x \frac{\partial W}{\partial x}(x, \tau). \] (2.16)

Now, if we assume that \( W(x, \tau) \) is a smooth solution of (2.13) we can differentiate it with respect to \( x \) and multiply the result by \( x \). In the following we subtract the result from (2.13) and obtain

\[ \frac{\partial W}{\partial \tau} - x \frac{\partial^2 W}{\partial x \partial \tau} - (r - q + \frac{1}{2} \sigma^2) x^2 \frac{\partial^2 W}{\partial x^2} - \frac{x - 1}{T - \tau} x^2 \frac{\partial W}{\partial x} + \frac{1}{2} \sigma^2 x^3 \frac{\partial^3 W}{\partial x^3} + \]

\[ + \frac{1}{T - \tau} \left( W - x \frac{\partial W}{\partial x} \right) + r \left( W - x \frac{\partial W}{\partial x} \right) = 0. \] (2.17)

From the used new variables (2.16), we can derive the following equations

\[ \frac{\partial \Pi}{\partial \xi} = x^2 \frac{\partial^2 W}{\partial x^2}, \quad \frac{\partial^2 \Pi}{\partial \xi^2} + 2 \frac{\partial \Pi}{\partial \xi} = -x^3 \frac{\partial^3 W}{\partial x^3}, \quad \frac{\partial \Pi}{\partial \tau} + \frac{\dot{\rho}}{\rho} \frac{\partial \Pi}{\partial \xi} = \frac{\partial W}{\partial \tau} - x \frac{\partial^2 W}{\partial x \partial \tau}. \]

Substituting into equation (2.16) we finally obtain the parabolic PDE in terms of \( \Pi(\xi, \tau) \)

\[ \frac{\partial \Pi}{\partial \tau} + a(\xi, \tau) \frac{\partial \Pi}{\partial \xi} - \frac{1}{2} \sigma^2 \frac{\partial^2 \Pi}{\partial \xi^2} + \left( r + \frac{1}{T - \tau} \right) \Pi = 0, \] (2.18)

where \( \xi \in (0, \infty), \tau \in (0, T) \) and \( a(\xi, \tau) \) is the function of the \( \rho \) in the form

\[ a(\xi, \tau) = \frac{\dot{\rho}(\tau)}{\rho(\tau)} + (r - q) - \frac{1}{2} \sigma^2 - \frac{\rho e^{-\xi} - 1}{T - \tau}. \] (2.19)

In the process of determining initial conditions we use (2.14) and obtain

\[ \Pi(\xi, 0) = \begin{cases} -1, & \xi < \ln \rho(0), \\ -0, & \xi < \ln \rho(0). \end{cases} \] (2.20)
For the case of boundary conditions we use our knowledge from (2.15) and impose Dirichlet conditions for \( \Pi(\xi, \tau) \)

\[
\Pi(0, \tau) = -1, \quad \Pi(\infty, \tau) = 0. \tag{2.21}
\]

Since \( W(\rho(\tau), \tau) = \rho(\tau) - 1 \) and \( \frac{\partial W}{\partial x}(\rho(\tau), \tau) = 1 \), we can easily conclude, that \( \frac{\partial W}{\partial \tau}(0, \tau) \rightarrow \rho(\tau) \) \( \text{as} \ x \rightarrow \rho(\tau) \). (2.22)

Passing to the limit \( x \rightarrow \rho(\tau) \) in equation (2.13), we end up with an algebraic relation between the free boundary position \( \rho(\tau) \) and the boundary trace \( \frac{\partial \Pi}{\partial \xi}(0, \tau) \)

\[
-(r-q)\rho(\tau) - \frac{1}{2}\sigma^2\frac{\partial \Pi}{\partial \xi}(0, \tau) + \rho(\tau) - 1 = 0. \tag{2.23}
\]

2.1.2 An Equivalent Form of the Free Boundary

Ševčovič [10] used the expression (2.23) to determine a nonlocal algebraic formula for the free boundary position. This result contains the value of \( \frac{\partial \Pi}{\partial \xi}(0, \tau) \), which causes in case of small inaccuracy an computational error in the whole domain of \( \xi \in (0, \infty) \). Therefore, this equation is not suitable for a robust numerical approximation scheme. Bokes and Ševčovič [3] suggested an equivalent form of the free boundary \( \rho(\tau) \), which was proved to be a more robust scheme from the numerical point of view. They integrated the equation (2.18) with respect to \( \xi \) on the domain \( \xi \in (0, \infty) \)

\[
\frac{d}{d\tau} \int_0^\infty \Pi d\xi + \int_0^\infty a(\xi, \tau) \frac{\partial \Pi}{\partial \xi} d\xi - \frac{1}{2}\sigma^2 \int_0^\infty \frac{\partial^2 \Pi}{\partial \xi^2} d\xi + \int_0^\infty \left[ r + \frac{1}{T-\tau} \right] d\xi = 0.
\]

Now, using boundary conditions (2.21) and the algebraic equation (2.23) they derived the following differential equation

\[
\frac{d}{d\tau} \left[ \ln \rho(\tau) + \int_0^\infty \Pi d\xi \right] + q \rho(\tau) - q - \frac{1}{2}\sigma^2 + \int_0^\infty \left[ r - \rho(\tau)e^{-\xi} - \frac{1}{T-\tau} \right] \Pi d\xi = 0. \tag{2.24}
\]
Chapter 2. The Analytical Derivation

The limit of the early exercise boundary close to expiry \( \rho(0^+) \) has been derived in \([8], [6]\) and the result for the arithmetic average call option can be given as

\[
\rho(0^+) = \max \left[ \frac{1 + rT}{1 + qT}, 1 \right].
\]

The proof of this formula following the idea of Kwok & Dai \([6]\) is presented in the Appendix 1.

2.1.3 The Backward Transformation

The pricing equation for the American type of Asian call option can be derived using a backward transformation of the equation (2.16). This equation can be modified to

\[
\frac{\partial}{\partial x} \left( \frac{W(x, \tau)}{x} \right) = -x^2 \Pi(\xi, \tau). \tag{2.25}
\]

Integrating this equation with respect to \( x \) on the domain \([x, \rho(\tau)]\) yields

\[
\frac{\rho(\tau) - 1}{\rho_T} - \frac{W(x, \tau)}{x} = - \int_x^{\rho(\tau)} x^{-2} \Pi(\xi, \tau) dx. \tag{2.26}
\]

Let us recall, that a transformation \( \xi = \ln \frac{\rho(\tau)}{x} \) was used from where \( x = e^{-\xi} \rho(\tau) \). Substituting back we have

\[
W(x, \tau) = \frac{1}{\rho(\tau)} \left[ \rho(\tau) - 1 + \int_0^{\ln \frac{\rho(\tau)}{x}} e^{\xi} \Pi(\xi, \tau) d\xi \right]. \tag{2.27}
\]

Finally, applying the series of transformations (2.12), the price of the contract depending on the position of the free boundary \( \rho(T - t) \) follows the equation

\[
V(S, A, t) = \frac{A}{\rho(T - t)} \left[ \rho(T - t) - 1 + \int_0^{\ln \frac{\rho(T - t)}{S}} e^{\xi} \Pi(\xi, \tau) d\xi \right]. \tag{2.28}
\]
Chapter 3

The Numerical Methods

In this Chapter the used numerical techniques are described in more details. We introduce the finite difference method, splitting techniques, interpolation and numerical integration.

3.1 The Finite Difference Methods

In general, the finite difference methods are used to solve differential equations using finite difference quotients to numerically approximate the derivative terms. These techniques are used especially for boundary values problems. The finite-differences can be obtained either from the limiting behaviour or from Taylor’s expansion of the function. To construct and solve a finite-difference scheme for a differential equation we need to define and generate a set of points, where the numerical approximation will exist. It is usually done by (dividing) the domain $-\infty < a < b < \infty$ into $N+1$ subintervals as following: $a = x_0 < x_1 \ldots < x_N = b$. The set $\{x_0, x_1 \ldots x_N\}$ is called the grid. We denote the step size between two points by $h_i = x_i - x_{i-1}$. If all step size have the same length we refer to the discrete uniform grid of the interval $[a, b]$. Therefore, we can write $h = (b - a)/N$. In this work, all the discretizations are uniform.

The error of the solution is defined as a difference between the exact and numerical solution. The error term is caused either by computer rounding (round-off error) or the discretization procedure (truncation error). We are particularly interested in the local truncation error which refers to the error arising from a single application of the method. To determine the truncation error the reminding term from the Taylor’s expansion can be used. Usually, it is written in terms of $O(h^i)$ where $i = 1, 2, \ldots, N$ is the order of the truncation error.
Chapter 3. The Numerical Methods

The most commonly used first order finite-difference quotients to approximate the first order derivatives of the function $u(x)$ are:

- The forward finite-difference
  
  \[ D_{+}u(x) = \frac{u(x + h) - u(x)}{h} + O(h), \quad (3.1) \]

- The backward finite-difference
  
  \[ D_{-}u(x) = \frac{u(x) - u(x - h)}{h} + O(h), \quad (3.2) \]

- The central finite-difference
  
  \[ D_{0}u(x) = \frac{u(x + h) - u(x - h)}{2h} + O(h^2). \quad (3.3) \]

From the family of higher order derivatives we mention just the most common finite difference formula of the second order derivative. The formula can be derived using (3.1) and (3.2):

- The central finite-difference
  
  \[ D_{0}^{2}u(x) = D_{+}D_{-}u(x) = \frac{u(x - h) - 2u(x) + u(x + h)}{h^2} + O(h^2). \quad (3.4) \]

3.2 The Operator Splitting Methods

The idea of the method is to solve complex models by splitting it into a sequence of sub-models, which are comparably simpler to solve. Physical processes like convection or diffusion are usually simulated. As every numerical treatment the operator splitting produces an error term as well. By increasing the order of the splitting we can obtain higher numerical precision linked with higher computational time. In this we work refer to a time splitting techniques often called as fractional steps method [12].

The Lie - Trotter Splitting Method

The Lie - Trotter splitting method is a first order splitting which solves two sub-problems sequentially. Suppose we have given the Cauchy problem

\[ \frac{\partial u(t)}{\partial t} = Au(t) + B(t), \quad t \in [0, T], \quad u(0) = u_0. \quad (3.5) \]
Splitting techniques assume that the problem can be split into two or more sub-problems. By these assumptions we can introduce the Lie splitting on the interval \([t^n, t^{n+1}]\) in the following way:

\[
\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^n, t^{n+1}], \quad u(t^n) = u^n, 
\]

(3.6)

\[
\frac{\partial v(t)}{\partial t} = Bv(t), \quad t \in [t^n, t^{n+1}], \quad v(t^n) = u(t^{n+1}), 
\]

(3.7)

for \(n = 0, 1, \ldots, N - 1\) and \(u^n\) is given as a initial condition for time step \(n\) from (3.5). Then, we refer to \(u^{n+1} = v(t^{n+1})\) as the solution and a new starting point for \(t \in [t^{n+1}, t^{n+2}]\). One can show using Taylor series that the Lie splitting method gives first order accuracy.

The Strang Splitting

One of the widely used and very popular operator splitting technique is the second-order Strang splitting [9]. The idea is to solve (3.6) for time step \(\Delta t/2\), then to solve (3.7) for a full time step \(\Delta t\) and finally a half time step solution \(\Delta t/2\) for the equation (3.6). The algorithm is given in this way:

\[
\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^n, t^{n+\frac{1}{2}}], \quad u(t^n) = u^n, 
\]

(3.8)

\[
\frac{\partial v(t)}{\partial t} = Bv(t), \quad t \in [t^n, t^{n+1}], \quad v(t^n) = u(t^{n+\frac{1}{2}}), 
\]

(3.9)

\[
\frac{\partial u(t)}{\partial t} = Au(t), \quad t \in [t^{n+\frac{1}{2}}, t^{n+1}], \quad u(t^{n+\frac{1}{2}}) = v(t^{n+1}), 
\]

(3.10)

for \(n = 0, 1, \ldots, N - 1\) and \(u^n\) is given as a initial condition for time step \(n\) from (3.5). Again, \(u^{n+1} = v(t^{n+1})\) is used as starting point for the next approximation interval \([t^{n+1}, t^{n+2}]\). The order of the accuracy is two. This can be shown using Taylor series.

3.3 The Interpolation Technique

Interpolation is a process of defining a function at every point that has given preliminary values at discrete set of known points. It is an useful technique to determine unknown function values needed in numerical analysis where we calculate with just a discrete set of points. There are many different interpolation techniques and some of them are explained below.
• Probably the simplest method is **linear interpolation**. Generally, it takes two data points \((x_n, f(x_n))\) and \((x_{n+1}, f(x_{n+1}))\) and the interpolation is

\[
f(x) = f_n + (f_{n+1} - f_n) \frac{x - x_n}{x_{n+1} - x_n}, \quad \text{for the pair } (x, y).
\]

This interpolation is not very precise though it is quick and easy to use.

• **Polynomial interpolation** is the generalization of the linear interpolation which based on a first order polynomial. Let us suppose we want a interpolation polynomial in the form

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.
\]

The function \(p(x)\) interpolates the set of data points \(f(x_i)\) for \(i = 0, 1, \ldots, n\). That means that we have to solve the system of equations which is produced from the two above mentioned equations

\[
\begin{bmatrix}
x_0^n & x_0^{n-1} & \ldots & x_0 & 1 \\
x_1^n & x_1^{n-1} & \ldots & x_1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_n^n & x_n^{n-1} & \ldots & x_n & 1
\end{bmatrix}
\begin{bmatrix}
a_n \\ a_{n-1} \\ \vdots \\ a_0
\end{bmatrix}
= \begin{bmatrix}
f_{x_0} \\ f_{x_1} \\ \vdots \\ f_{x_n}
\end{bmatrix}.
\]

This matrix is commonly called **Vandermonde matrix**. The calculation of the polynomial interpolation is rather expensive as to mention some disadvantages. Also, oscillation problems can occur at the end points of the data set. This can be avoided using **spline interpolation**.

• A **spline** is a piecewise defined polynomial function. It means that every small interval can be a different interpolant of \(n\)-th order polynomial, while the smoothness of the curve at connecting points is preserved. This type of interpolation is preferred over the classical polynomial interpolation because it requires a usage of low-degree polynomials. The most commonly used are the splines of the 3\textit{rd} order, the **cubic splines**.

### 3.4 The Numerical Integration

In this work, the numerical integration of the definite integral based on the Newton-Cotes method is used. We use the first order method based on linear
interpolation often called as \textit{trapezoidal method}. The integral on the spatial domain \([x_n, x_{n+1}]\) is calculated as follows

\[
\int_{x_n}^{x_{n+1}} f(x) \, dx \approx \int_{x_n}^{x_{n+1}} \left[ f(x_n) + \frac{x - x_n}{x_{n+1} - x_n} (f(x_{n+1}) - f(x_n)) \right] \, dx \\
= \frac{f(x_n) + f(x_{n+1})}{2} (x_{n+1} - x_n). \tag{3.11}
\]
Chapter 4

The Numerical Treatment of The Problem

In Chapter 2, we described a transformation procedure for the two-dimensional Black-Scholes equation solving the price of the floating strike call option. We have been able to reduce the dimension of the model and also eliminate the dependence of the free boundary on the computational domain. Moreover, in Chapter 3 we introduced useful numerical techniques. Hence, we have all the tools now, we can move to the numerical treatment of our model. To sum up, we present for convenience the problem with respective boundary conditions for the call option once more

\[ \frac{\partial \Pi}{\partial \tau} + a(\xi, \tau) \frac{\partial \Pi}{\partial \xi} - \frac{1}{2} \sigma^2 \frac{\partial^2 \Pi}{\partial \xi^2} + \left( r + \frac{1}{T - \tau} \right) \Pi = 0, \]

(4.1)

where the coefficient \( a(\xi, \tau) \) is given in the form

\[ a(\xi, \tau) = \frac{\dot{\rho}(\tau)}{\rho(\tau)} + (r - q) - \frac{1}{2} \sigma^2 - \frac{\rho e^{-\xi} - 1}{T - \tau}. \]

The set of initial and boundary conditions have been derived for the call option contract as follows

\[
\begin{align*}
\Pi(\xi, 0) &= \begin{cases} 
-1, & \xi < \ln \rho(0), \\
-0, & \xi \geq \ln \rho(0), 
\end{cases} \\
\Pi(0, \tau) &= -1, \\
\Pi(\xi, \tau) &= 0, & \xi \to \infty.
\end{align*}
\]

(4.2)
Our problem is also closely connected with the equivalent form of the free boundary position \( \rho(\tau) \) and with a value of this boundary close to expiry:

\[
0 = \frac{d}{d\tau} \left[ \ln \rho(\tau) + \int_0^\infty \Pi d\xi \right] + q \rho(\tau) - q - \frac{1}{2} \sigma^2 \\
+ \int_0^\infty \left[ r - \frac{\rho(\tau)e^{-\xi} - 1}{T - \tau} \right] \Pi d\xi, \tag{4.3}
\]

\[
\rho(0^+) = \max \left[ \frac{1 + rT}{1 + qT}, 1 \right].
\]

It is important to mention that instead of the spatial domain \( x \in (0, \infty) \) we consider a finite range \( x \in (0, R) \), where \( R \) is sufficiently large for our purposes. This artificial boundary limits the computation domain and speeds up the numerical computation. The optimal value of this boundary will be examined in the upcoming part of our work. The time domain \( \tau \in (0, T) \) is finite as well and depends on the maturity time of the option contract.

### 4.1 The Lie Splitting Procedure

The finite difference method is used in the discretization process of the equation (4.1). We use the time step \( k = \Delta \tau \) for the time domain \( \tau \in (0, T) \) and \( h = \Delta \xi \) correspondingly for the spatial domain \( \xi \in (0, R) \). We may define \( N = \frac{T}{k} \) and \( M = \frac{R}{h} \) as a finite amount of time and space steps in our discretization. Hence, \( \tau_j = jk, j \in [0, N] \) and \( \xi_i = ih, \) where \( i \in [0, M] \).

The approximation \( \Pi_i^j \approx \Pi(\xi_i, \tau_j) \) and \( \rho(\tau_j) = \rho^j \) are used throughout all the thesis. Using the foregoing notations and the backward-in-time finite difference (3.2) the equation (4.1) is discretized such as

\[
\frac{\Pi^j - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^j}{\partial \xi} - \left( \frac{\sigma^2}{2} + \frac{\rho^j e^{-\xi} - 1}{T - \tau_j} \right) \frac{\partial \Pi^j}{\partial \xi} - \frac{1}{2} \sigma^2 \frac{\partial^2 \Pi^j}{\partial \xi^2} + \left( r + \frac{1}{T - \tau_j} \right) \Pi^j = 0,
\]

where \( c^j \) is the approximation of \( c(\tau_j) = \frac{\rho(\tau_j)}{\rho(\tau_j)} \). In the following we apply a Lie splitting technique to this equation separating it into two nonlinear parts. Using an auxiliary portfolio \( \Pi \), finite difference and a procedure (3.6)-(3.7) we obtain

\[
\frac{\Pi^j_{i+1} - \Pi^j_i}{k} + c^j_i(\rho^j) \frac{\Pi^j_{i+1} - \Pi^j_{i-1}}{2h} = 0, \quad \Pi^j_i = \Pi^j_i, \tag{4.5}
\]
\[ \frac{\Pi^{j+1}_i - \Pi^j_i}{h} - \left( \frac{\sigma^2}{2} + \frac{\rho^i e^{-\xi} - 1}{T - \tau_j} \right) \frac{\Pi^{j+1}_{i+1} - \Pi^{j+1}_{i-1}}{2h} - \]
\[ \frac{1}{2} \sigma^2 \frac{\Pi^{j+1}_{i+1} - 2\Pi^{j+1}_i + \Pi^{j+1}_{i-1}}{h^2} + \left( r + \frac{1}{T - \tau_j} \right) \Pi^{j+1}_i = 0, \quad \Pi_i = \Pi^{j+1}_i. \quad (4.6) \]

In the following the step-by-step algorithm for the Lie splitting is presented. We introduce two ways of solving equation (4.5), which is the transport equation where an analytical solution is available. Firstly, we need to set up the starting points. We recall the boundary conditions (4.2) and that \( \rho(0^+) = \max\left\{ \frac{1+r+q}{1+q}, 1 \right\} \). Defining \( p \) as the order of the inner loop for all \( j = 1, 2, \ldots, N \) we proceed to the successive iteration procedure. Supposing the pair \((\Pi^{j,p}, \rho^{j,p})\) converges to the value \((\Pi^{j,\infty}, \rho^{j,\infty})\) as \( p \to \infty \), we set \( \Pi^{j,0} = \Pi^{j-1} \) and \( \rho^{j,0} = \rho^{j-1} \) then the computation of the pair \((\Pi^{j,p+1}, \rho^{j,p+1})\) for all \( p = 0, 1, \ldots, N - 1, \ldots \) follows this three step algorithm:

1.) We use the forward-finite difference (3.1) to discretize the time step in the equivalent form of the free boundary

\[ \ln \rho^{j,p+1} = \ln \rho^{j,0} - \int_0^\infty \Pi^{j,0} d\xi + \int_0^\infty \Pi^{j,p} d\xi + k \left[ q + \frac{1}{2} \sigma^2 - q \rho^{j,0} - \int_0^\infty \left( r - \frac{\rho^{j,0} e^{-\xi} - 1}{T - \tau_{j,0}} \right) \Pi^{j,0} d\xi \right]. \quad (4.7) \]

Since the computation of the integral \( \int_0^\infty \Pi^n d\xi \) cannot be done analytically, we use the trapezoidal method to numerically integrate this expression introduced in Subsection 3.4.

2.a) Secondly, we proceed to equation (4.5), to the first part of the Lie splitting. Using the output \( \rho^{j,p+1} \) from step 1, we have \( b^{j,p}_i = \frac{D_s(\rho^{j,p})}{\rho^{j,p}} + r - q \). The value of our auxiliary portfolio can be calculated from the system of equations

\[ \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
-(\frac{k}{2h})b^{j}_1 & 1 & (\frac{k}{2h})b^{j}_2 & 0 & \ldots & 0 \\
0 & -(\frac{k}{2h})b^{j}_2 & 1 & (\frac{k}{2h})b^{j}_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & -(\frac{k}{2h})b^{j}_{n-1} & 1 & (\frac{k}{2h})b^{j}_{n-1} \\
0 & \ldots & \ldots & 0 & 0 & 1
\end{bmatrix} \Pi^{j,p+1} = \Pi^{j,0}, \quad (4.8) \]
where $\Pi^{j,0} = \Pi^{j-1}$ and for the sake of simplicity we set $b_i^j = b_i^{j,p}$.

(2.b) As we already mentioned, in analytical form the equation (4.5) is a classical transport equation $\partial_\tau \Pi + c(\tau) \partial_\xi \Pi$ for $\tau \in (\tau_{j-1}, \tau_j]$ subject to the boundary condition $\Pi(0, \tau) = -1$ and initial condition $\Pi(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$. It is known that the free boundary of this options need not to be monotone, therefore the velocity term $c(\tau)$ can be either positive or negative. The primitive function $C(\tau)$ of the velocity $c(\tau)$ can be easily determined as $C(\tau) = \ln \rho(\tau) + (r - q)\tau$. Then after some corrections the solution of the transport equation for the $p$-th loop can be presented by the formula

$$
\Pi_i^{j,p+1} = \begin{cases} 
\Pi_i^{j,0}(\eta_i), & \text{if } \eta_i = \xi_i - \ln \rho_i^{j,0} - (r - q)k > 0, \\
-1, & \text{otherwise.}
\end{cases}
$$

Thus, we obtained the value of the auxiliary portfolio $\Pi^{j,p+1}$ using both numerical (2.a) and analytical (2.b) approach. It will be interesting to observe how this methods will perform in a computational analysis. We expect the analytical method to operate better in terms of accuracy.

(3.) At last, the equation (4.6) was about to be examined. Since, we have a system of equations, to obtain the value of $\Pi^{j,p+1}$ we use our auxiliary portfolio $\Pi^{j,p+1}$ gained from step 2 to enter the system

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\alpha_i^j & \beta_i^j & \gamma_i^j & 0 & \cdots & 0 \\
0 & \alpha_2^j & \beta_2^j & \gamma_2^j & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_{n-1}^j & \beta_{n-1}^j & \gamma_{n-1}^j \\
0 & \cdots & \cdots & 0 & 0 & 1
\end{bmatrix}
\Pi^{j,p+1} = \Pi^{j,p+1},
$$

where we recall the boundary conditions $\Pi(0, \tau) = -1$, $\Pi(M, \tau) = 0$.
and

$$
\alpha^j_i = \alpha^j_i (\rho^{j,p+1}) = -\frac{k}{2h^2} \sigma^2 + \frac{k}{2h} \left( \frac{1}{2} \sigma^2 + \frac{\rho^{j,p+1} e^{-\xi_i} - 1}{T - \tau_j} \right),
$$

$$
\gamma^j_i = \gamma^j_i (\rho^{j,p+1}) = -\frac{k}{2h^2} \sigma^2 - \frac{k}{2h} \left( \frac{1}{2} \sigma^2 + \frac{\rho^{j,p+1} e^{-\xi_i} - 1}{T - \tau_j} \right), \quad (4.11)
$$

$$
\beta^j_i = \beta^j_i (\rho^{j,p+1}) = 1 + \left( r + \frac{1}{T - \tau_j} \right) k - \alpha^j_i (\rho^{j,p+1}) - \gamma^j_i (\rho^j).
$$

We set \( p = p + 1 \) and repeat step 1 - step 3. Once we obtained an acceptable tolerance for \( p \to \infty \) we set \( \Pi^j = \Pi^{j,\infty} \) and \( \rho^j = \rho^{j,\infty} \) and proceed on to the next time step \( j + 1 \).

### 4.2 The Strang Splitting Procedure

Here, our own work the Strang splitting \cite{9} is presented. Now, using two auxiliary portfolios \( \Pi_i, \Pi^i \), the finite differences and procedures (3.8)-(3.10) we obtain a three step method:

$$
\frac{\Pi^{j+\frac{1}{2}}_i - \Pi^j_i}{k} + c^j_i (\rho^j) \frac{\Pi^{j+\frac{1}{2}}_i - \Pi^{j+\frac{1}{2}}_i}{2h} = 0, \quad \Pi^j_i = \Pi^i_i, \quad (4.12)
$$

$$
\frac{\Pi^{j+1}_i - \Pi^j_i}{k} - \left( \frac{\sigma^2}{2} + \frac{\rho^j e^{-\xi_i} - 1}{T - \tau_j} \right) \frac{\Pi^{j+1}_i - \Pi^{j+1}_i}{2h} =
$$

$$
\frac{1}{2} \sigma^2 \Pi^{j+1}_i \frac{2\Pi^{j+1}_i}{h^2} + \Pi^{j+1}_i - \Pi^{j+1}_i\left( r + \frac{1}{T - \tau_j} \right) \Pi^{j+1}_i = 0, \quad \Pi^j_i = \Pi^{j+\frac{1}{2}}_i, \quad (4.13)
$$

$$
\frac{\Pi^{j+1}_i - \Pi^{j+\frac{1}{2}}_i}{k} + c^j_i (\rho^j) \frac{\Pi^{j+1}_i - \Pi^{j+1}_i}{2h} = 0, \quad \Pi^{j+\frac{1}{2}}_i = \Pi^{j+1}_i. \quad (4.14)
$$

We introduce the Strang splitting algorithm as we did in the case of Lie splitting. We both study the numerical and analytical way of solving the transport equation. Hence, there are three time steps. We expect a higher computational time but higher precision since we talk about a second order method. We work with the same initial and boundary conditions \( \rho^0, \Pi^0 \).
Defining $p$ as the order of the inner loop for all $j = 1,2,\ldots, N$ we proceed to the successive iteration procedure. Supposing the pair $(\Pi_{j}^{p}, \rho_{j,p})$ as $p \to \infty$ converges to the value $(\Pi_{j,\infty}, \rho_{j,\infty})$. The computation of the pair $(\Pi_{j}^{p+1}, \rho_{j,p+1})$ for all $p = 0,1,\ldots, N-1,\ldots$ follows now a four step algorithm:

(1.) Firstly, we start with the same step as before. We rely on the forward-finite difference (3.1) to discretize the time step in the equivalent form of the free boundary

$$\ln \rho_{j,p+1} = \ln \rho_{j,0} - \int_{0}^{\infty} \Pi_{j}^{0} d\xi + \int_{0}^{\infty} \Pi_{j,p} d\xi + k \left[ q + \frac{1}{2} \sigma^2 - q \rho^j - 1 - \int_{0}^{\infty} \left( r - \frac{\rho_{j,0} e^{-\xi} - 1}{T - \tau_{j,0}} \right) \Pi_{j,0} d\xi \right].$$

(4.15)

We use the trapezoidal method to approximate the expressions $\int_{0}^{\infty} \Pi_{n} d\xi$.

(2.a) Secondly, we proceeded to equation (4.12). Here the computations starts to differ because we calculate with only a half-time step, thus we have to adjust step 2 from the Lie splitting. Using the output $\rho_{j,p+1}$ from step 1 we have $c_{j}^{p} = \frac{D_{x}(\rho_{j,p})}{\rho_{j,p}} + r - q$. The value of our auxiliary portfolio $\Pi$ in time step $n+\frac{1}{2}$ can be calculated from the set of equations

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
-\left( \frac{k}{4h} b_{1}^{j} \right) & 1 & \left( \frac{k}{4h} b_{1}^{j} \right) & 0 & \cdots & 0 \\
0 & -\left( \frac{k}{4h} b_{2}^{j} \right) & 1 & \left( \frac{k}{4h} b_{2}^{j} \right) & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & -\left( \frac{k}{4h} b_{n-1}^{j} \right) & 1 & \left( \frac{k}{4h} b_{n-1}^{j} \right) \\
0 & \cdots & \cdots & 0 & 0 & 1 \\
\end{bmatrix} \Pi_{j,p+\frac{1}{2}} = \Pi_{j,0},
$$

(4.16)

where $\Pi_{j,0} = \Pi_{j-1}$

(2.b) Similarly, as in step (2.b) dedicated to the Lie splitting the transport equation $\partial_{t} \Pi + c(\tau) \partial_{\tau} \Pi$ can be solved analytically with the difference that only a half-time step is performed. Therefore, equation (4.9) changes to

$$\Pi_{i}^{p+\frac{1}{2}} = \begin{cases} 
\Pi_{i}^{p,0}(\eta_{i}), & \text{if } \eta_{i} = \xi_{i} - \ln \frac{\rho_{i,0}}{\rho_{i,p+\frac{1}{2}}} - (r - q) \frac{k}{2} > 0, \\
-1, & \text{otherwise}. 
\end{cases}$$

(4.17)
Since the value $\rho_{j,p}^{i,p+\frac{1}{2}}$ is not known we obtain it using the interpolation from Section 3.3.

(3.) Next, equation (4.13) is solved. With $\Pi_{i,p}^{j,p+\frac{1}{2}}$ we enter the set of equations

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \alpha_1^j & \beta_1^j & \gamma_1^j & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \alpha_{n-1}^j & \beta_{n-1}^j & \gamma_{n-1}^j & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1}
\end{bmatrix}
= \begin{bmatrix}
\Pi_{i,p}^{j,p} \\
\Pi_{i,p}^{j,p} \\
\Pi_{i,p}^{j,p} \\
\Pi_{i,p}^{j,p} \\
\Pi_{i,p}^{j,p}
\end{bmatrix},
$$

(4.18)

where $\Pi_{i,p}^{j,p} = \Pi_{i,p}^{j,p+\frac{1}{2}}$, we recall the boundary conditions $\Pi(0,\tau) = -1$, $\Pi(M,\tau) = 0$ and

$$
\alpha_i^j = \alpha_i^j(\rho^j) = -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h} \left(\frac{1}{2} \sigma^2 + \frac{\rho^j e^{-\xi_i} - 1}{T - \tau_j}\right),
$$

$$
\beta_i^j = \beta_i^j(\rho^j) = 1 + \left(r + \frac{1}{T - \tau_j}\right)k - \alpha_i^j(\rho^j) - \gamma_i^j(\rho^j),
$$

$$
\gamma_i^j = \gamma_i^j(\rho^j) = -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h} \left(\frac{1}{2} \sigma^2 + \frac{\rho^j e^{-\xi_i} - 1}{T - \tau_j}\right),
$$

(4.19)

(4.a) In this step we repeat the step (2.a) with the difference that the auxiliary portfolio obtained in step 3 is the input into the system

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
-\frac{k}{2h} b_i^{j_1} & 1 & \frac{k}{2h} b_i^{j_1} & 0 & \cdots & 0 \\
0 & -\frac{k}{2h} b_i^{j_2} & 1 & \frac{k}{2h} b_i^{j_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & -\frac{k}{2h} b_i^{j_{n-1}} & 1 & \frac{k}{2h} b_i^{j_{n-1}} \\
0 & \cdots & \cdots & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1} \\
\Pi_{i,p}^{j,p+1}
\end{bmatrix}
= \begin{bmatrix}
\Pi_{i,p}^{j,p+\frac{1}{2}} \\
\Pi_{i,p}^{j,p+\frac{1}{2}} \\
\Pi_{i,p}^{j,p+\frac{1}{2}} \\
\Pi_{i,p}^{j,p+\frac{1}{2}} \\
\Pi_{i,p}^{j,p+\frac{1}{2}}
\end{bmatrix},
$$

(4.20)

where, $\Pi_{i,p}^{j,p+\frac{1}{2}} = \Pi_{i,p}^{j,p+1}$ and $c_i^j = c_i^{j,p}$ is given from (2.a).

(4.b) Repeating the step (2.b) with the auxiliary portfolio $\Pi_{i,p}^{j,p+1}$.
\[ \Pi_{i}^{j,p+1} = \begin{cases} \Pi_{i}^{j,p+1} (\eta_i), & \text{if } \eta_i = \xi_i - \ln \frac{\rho_{i,p+1}^{j,p+1}}{\rho_{i,p+1}^{j,p}} - (r - q)^{\frac{k}{2}} > 0, \\ -1, & \text{otherwise.} \end{cases} \] (4.21)

We set \( p = p + 1 \) and repeat step 1 - step 4. Once we have an acceptable tolerance for \( p \to \infty \) we set \( \Pi^j = \Pi^{j,\infty} \) and \( \rho^j = \rho^{j,\infty} \) and we move on to the next time step \( j + 1 \).
Chapter 5

The Numerical Experiments

In this, final part we present several numerical examples using the above mentioned four procedures.

In Figure 5.1 we show the behaviour of the early exercise boundary $\rho(\tau)$ depending on the value $R$. The other parameters are $T = 50, r = 0.06, q = 0.04, \sigma = 0.2$. The number of time steps is $m = 10^5$. The number of spatial steps on the grid is chosen according to the value of parameter $R$. It is visible, that for every case for this particular situation the $R = 3$ would be sufficient. Thus we work with this parameter and a number of spatial steps $n = 300$.

In Figure 5.2 we present a comparison of the early exercise for Lie splitting where the transport equation is calculated in both numerical and analytical ways. We name it analytical respectively numerical approach for the sake of simplicity. We show the same also for the Strang splitting. ($r = 0.06, q = 0.04, \sigma = 0.2, T = 50$).

In Figure 5.3 we plot the number of the inner loops $p$ needed in every time step to obtain an acceptable tolerance and convergence. The analytical Lie and Strang splitting methods are compared. We chose $m = 10^5$. As it is visible the Strang splitting requires more loops as the Lie splitting. And this goes hand-in-hand with a higher computational time, which was observed in the Strang splitting during the experiments.

In Figure 5.4 we show the position of the early exercise boundary $\rho(\tau)$ for different values of $r$. The algorithm is stable for all the cases. The higher the interest rate the bigger value we get for the early exercise boundary close to expiry $\tau = 0$ ($r = 0.06, q = 0.04, \sigma = 0.2, T = 50$). We plot all the cases for both the Strang and the Lie splitting methods.

In Figure 5.5 the comparison of the early exercise is shown for various values of the dividend yield $q$. The early exercise close to expiry $\tau = 0$ is behaving in an opposite way as in the case of the influence of the interest rate $r$, i.e. the higher the value of $q$ the lower the position of $\rho(\tau)$.
Chapter 5. The Numerical Experiments

Figure 5.1: The graphical presentation of the early exercise boundary depending on $R$ for the Lie Splitting (top) and the Strang splitting (bottom) for both analytical (left) and numerical (right) approach. The blue line represents $R = 1.4$, the green line represents $R = 1.6$, the red line represents $R = 2.0$, the light blue line $R = 3.0$ and the purple line $R = 5.0$.

Figure 5.2: The comparison of the early exercise boundary for analytical and numerical approach for both the Lie Splitting (left) and the Strang Splitting (right). The green line represent the Strang Splitting and the blue line represents the Lie Splitting.
Figure 5.3: The comparison of number of the inner loops \( (p) \) for the analytical approaches.

Figure 5.4: The comparison the free boundary for various \( r \) for the Lie Splitting (top) and the Strang splitting (bottom) for both analytical (left) and numerical (right) approach. The blue line represents \( r = 0.02 \), the green line represents \( r = 0.04 \), the red line represents \( r = 0.06 \) and the light blue line represents \( r = 0.10 \).
Figure 5.5: The comparison of the free boundary for various $q$ for the Lie Splitting (top) and the Strang splitting (bottom) for both analytical (left) and numerical (right) approach. The blue line represents $q = 0.04$, the green line represents $q = 0.06$, and the red line represents $q = 0.10$. 
In Figure 5.6 we plot the influence of the volatility $\sigma$ on the early exercise boundary $\rho(\tau)$ for the analytical approach in the Strang splitting. We use the parameter values $q = 0.1, 0.2, 0.3$.

In Figure 5.7 we present the time point where the option should be exercised. This moment is the first intersect of the early exercise boundary depending on real time $\rho(t)$ and $x(t)$ which is fact the ratio of the spot price $S(t)$ and its average over the time $A(t)$. The $\rho(t)$ is generated using the Strang splitting with analytical approach. As a spot price we use data for Microsoft Corp. from http://finance.yahoo.com. ($r = 0.005, q = 0.01, \sigma = 0.3, T = 2$).

In Table 5.1 we compare call option values. We compare our results using all the methods with the Forward Shooting Grid Method (FSG) used in [1], where authors calculated the option values for different $T$, $r$ and $q$.

In Table 5.2 we can find comparison of call option values. We match values obtained using our methods with values form Hansen and Jorgensen [4].
Chapter 5. The Numerical Experiments

Figure 5.7: A comparison of the free boundary position for different $\sigma$. The red line represents $\sigma = 0.1$, the green line represents $\sigma = 0.2$, the blue line represents $\sigma = 0.3$ and the light blue line $\sigma = 0.4$.

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<th>Lie/N</th>
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Table 5.1: The numerical comparison of the option values obtained using FSG and our methods (N denotes the numerical solution of the transport equation and A the analytical).
### Table 5.2: The comparison of option values obtained using our methods for various values of $T$, $\sigma$ and $r$ (N denotes the numerical solution of the transport equation and A the analytical).

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Chapter 6

Conclusions and Outlook

It this thesis we dealt with American type of Asian call options with floating strike and arithmetic average. The main goal of the work was to develop an effective and numerically stable method for the pricing of these type of options. Many authors have studied this problem with different approach and for this reason also with different results. This work was mainly inspired by the latest work of prof. Daniel Ševčovič. At first we took a shot tour in the world of financial derivatives. The main ideas and definitions were introduced briefly but rather precisely. Then our attention was focused on the options especially Asian ones. In the next part we introduced an analytical derivation of the PDE describing the options value and we made further dimension reduction and transformations to simplify the problem. The backward transformation of this reduced equation allowed us to calculate the options price simply by substituting the variables once calculated numerically. That is why we introduced some numerical methods such as finite differences, operator splitting techniques and many others. Next, we treated the numerical problem using the already mentioned finite difference methods. We made the numerical solution of the problem easier using the Lie splitting and the Strang splitting. The whole numerical treatment was described in a step by step procedure. There, the transport equation was the part of the problem. We decided to evaluate this PDE with a different approach then it was done in previous works. The usage of a fully numerical solution saved us a lot of calculation time without a loss of accuracy. We investigated especially the free boundary, which position is crucial in option pricing. Finally, we compared the results between each other and also with other used methods. Additionally, the sensitivity of the solution on interest rate, volatility and dividend rate was investigated. We found the Strang splitting method operating slower than the Lie splitting method. But of course the numerical approach for the transport equation combined with the
Strang splitting made the calculation faster, comparable to the classical Lie splitting approach.

In the future we plan to develop a more efficient numerical solution for the position of the free boundary, using the Strang splitting method and calculating the position of the free boundary in every half time step. This should make the computational time faster without loss of accuracy. This development has already started. We will also study the dependence of interpolation during the process of the numerical treatment. Our plan is to extend the thesis for the put options as well.
Notation

\( t \)  Time.

\( T \)  Expiration time.

\( E \)  Strike price.

\( S(t) \)  Spot price - Price of the underlying asset at time \( t \).

\( A(t) \)  Average at time \( t \)

\( V(S, A, t) \)  Option value - Price of the financial derivative depending on time \( t \), asset’s price \( S \) and average \( A \).

\( [S(T) − E] \)  Payoff function at \( T (= \max[S(T) − E, 0]) \).

\( r \)  Interest rate.

\( \sigma \)  Volatility.

\( q \)  Dividend yield.

\( x(t) \)  Transformed spatial variable\( (=\frac{S(t)}{A(t)}) \).

\( \tau \)  Time variable \( (=T−t) \).

\( \rho(\tau) \)  Transformed free boundary \( (=x(T−\tau)) \) .

\( W(x, \tau) \)  Transformed payoff function \( (=\frac{1}{\lambda}V(S, A, t)) \).

\( \xi \)  Transformed spatial variable \( (=\frac{\rho(\tau)}{x}) \).

\( \Pi \)  Synthetic portfolio \( (=W(x, \tau)−x\frac{\partial W(x, \tau)}{\partial x}) \).

\( k \)  Time step.

\( h \)  Spatial step.

\( j \)  Index for time step.

\( i \)  Index for spatial step.
Bibliography


Appendix

1. A Multidimensional Version of Itô’s lemma

Suppose $f(x_1, \ldots, x_n, t)$ is a multidimensional differentiable function, the stochastic process $Y_n$ is defined by $Y_n = f(X_1, \ldots, X_n, t)$, where the process $X_j$ follows

$$dX_j(t) = \mu(t)dt + \sigma_j(t)dW_j(t), \quad j = 1, 2, \ldots, n,$$

where $W_j(t)$ is a standard Wiener’s process. $W_j(t)$ and $W_i(t)$ are assumed to be correlated so that $dW_jdW_i = \rho_{ij}$, then we define the multidimensional Itô’s lemma:

$$dY_n = \left[ \frac{\partial f}{\partial t}(X_1, \ldots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \ldots, X_n, t) \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(t)\sigma_j(t)\rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_1, \ldots, X_n, t) \right] dt \\
+ \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \ldots, X_n, t)dW_j.$$

2. The limit of an early the exercise boundary close to expiry

We consider an American type floating strike Asian call options with arithmetic average. We shall recall the linear complementary problem for the American types of options, which for this case is defined as following

$$\left\{ \frac{\partial W}{\partial \tau} - (r - q)x \frac{\partial W}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - \frac{x - 1}{T - \tau} \left( W - x \frac{\partial W}{\partial x} \right) + rW \right\}_{x, \tau} = 0$$
\[ \times \left\{ \frac{W - (1 - x)}{g(x, \tau)} \right\} = 0, \]

while \( L_x W \leq 0 \) and \( g(x, \tau) \geq 0 \) in \((0, \infty) \times (0, T)\).

From the condition \( W(0, \tau) = (x - 1)^+ \) it is straightforward that in the exercise region \( W = x - 1 \). Substituting this to the equation (2.13), we obtain an inequality for the stopping region

\[
\frac{\partial(x - 1)}{\partial \tau} - (r - q)x \frac{\partial(x - 1)}{\partial x} - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2(x - 1)}{\partial x^2} \frac{x - 1}{T - \tau} \left( x - \frac{1}{x} \frac{\partial x - 1}{\partial x} \right) + rW
\]

\[= qx(T - \tau) + x - 1 - r(T - \tau) \geq 0.\]

Along with the non-negativity of the final exercise payoff we have

\[ x(0^+) \geq \max \left\{ \frac{1 + rT}{1 + qT}, 1 \right\}. \]

Now, to reduce the inequality to an equality we assume that there exist \( x \) in the continuation region such that \( x > \max \left\{ \frac{1 + rT}{1 + qT}, 1 \right\} \). In the continuation region \( W(x, 0^+) = x - 1 \) and

\[ \frac{\partial(W)}{\partial(\tau)} \bigg|_{\tau \to 0^+} = - \left[ qx + \frac{x - 1}{T - \tau} - r \right] < 0, \]

but this leads to a contradiction with \( \frac{\partial(W)}{\partial(\tau)} \bigg|_{\tau \to 0^+} > 0 \) from \( g(x, \tau) \geq 0 \). We finally deduce, that

\[ x(0^+) = \max \left\{ \frac{1 + rT}{1 + qT}, 1 \right\}. \]

3. Source Code

1. Main function

function [ xi, tau, rhos, pi, pm ] = StrangCallDiff( r, sigma, q, T, L, n, m, pmax, eps, interpolation)

pm=0;

h=L/n ; k=T/m;
\( \xi = 0 : h : L \);
\( \tau = 0 : k : T \);
\( \rho = \max \left( \frac{r+1/T}{q+1/T}, 1 \right) \);
\( \Pi = -(\xi < \log(\rho)) \);
\( \Pi_W = -(\xi < \log(\rho)) \);
\( \pi = \text{zeros}(n+1,m+1) \);
\( \pi(:,1) = \Pi \);
\( \rho_{01} = \rho \);
for \( i = 1 : m \)
    \( \rho_{old} = \rho \);
    \( \Pi_{old} = \Pi \);
    \( \Pi_W_{old} = \Pi_W \);
    for \( p = 1 : p_{max} \)
        \( \rho_{old} = \rho \);
        \( \Pi_{old} = \Pi \);
        \( \Pi_W_{old} = \Pi_W \);
        \( \Pi_{POL} = \text{Step2} (\rho, \rho_{old}, \Pi_{old}, \xi, r, q, k/2, h, \text{interpolation}) \);
        \( \Pi_W = \text{Step3}(\Pi_{POL}, \rho, \tau(i), \xi, r, h, k, \sigma, T) \);
        \( \Pi = \text{Step2}(\rho, \rho_{old}, \Pi_W, \xi, r, q, k/2, h, \text{interpolation}) \);
        \( \rho = \text{Step1}(\Pi, \rho_{old}, \Pi_{old}, \tau(i), k, \sigma, q, r, \xi, T, h) \);
        if (\text{norm}(\rho_{old} - \rho) < \epsilon) 
            break;
        end
    end
    \( \rho_{0i+1} = \rho \);
    \( \pi(:,i+1) = \Pi \);
end

2. Step 1

function \( \text{sol} = \text{Step1}(\Pi, \rho_{old}, \Pi_{old}, \tau(j), k, \sigma, q, r, \xi, T, h) \)
\( \text{integ} = (r - (\rho_{old}\exp(-\xi) - 1)/(T - \tau(j))).*\Pi_{old} \);
\( \text{part} = \log(\rho_{old}) + k*(q + \sigma^2/2 - q*\rho_{old} - (\text{sum(integ)} - \text{integ(1)/2 - integ(end)/2}*h) + (\text{sum(\Pi_{old}) - \Pi_{old}(1)/2 - \Pi_{old}(end)/2 - (\text{sum(\Pi)} - \text{PI(1)/2 - PI(end)/2})/2))) \);
\( x = \exp(\text{part}) \);

3. Step 2b and 4b

function \( \text{sol} = \text{Step2}(\rho, \rho_{old}, \Pi_{old}, \xi, r, q, k, h, \text{interpolation}) \)
n = \text{length(\xi)};
\( d1 = 1 \);
\( \alpha = -(k/(2*h))*(\rho_{old}/\rho - \rho_{old} + r - q) \);
\[ \gamma_1 = \frac{k}{2h} \left( \frac{\rho - \rho_{old}}{2k \rho} + r - q \right) \]

\[ d = \text{ones}(1,n) \]

\[ \alpha = d \cdot \alpha_1 \]

\[ \gamma = d \cdot \gamma_1 \]

\[ \gamma(1,1) = 0 \]

\[ \alpha(1,n) = 0 \]

\[ \text{sol} = \text{gallery} \left( \text{'tridiag'}, \alpha(3: end-1), d(2:end-1), \gamma(2:end-2) \right) \backslash \text{PI}_{old}(2:end-1) \]

\[ \text{sol2} = \text{zeros}(n,1) \]

\[ \text{sol2}(1) = -1 \]

\[ \text{sol2(end)} = 0 \]

\[ \text{sol2}(2:end-1) = \text{sol} \]

\[ \text{sol} = \text{sol2}' \]

4. Step 3

function sol = Step3 ( PIPOL, rho, tau, xi, r, h, k, sigma, T )

\[ n = \text{length}( \xi ) \]

\[ \alpha = \frac{-k}{2h^2} \sigma^2 + \frac{1}{2} \frac{k}{2h} \left( \frac{\rho \exp(-\xi) - 1}{T - tau} \right) \]

\[ \gamma = -\frac{k}{2h^2} \sigma^2 - \frac{1}{2} \frac{k}{2h} \left( \frac{\rho \exp(-\xi) - 1}{T - tau} \right) \]

\[ \text{PIPOL}(2) = \text{PIPOL}(2) + \alpha(2) \]

\[ \text{diago} = 1 + \frac{r + 1/(T - tau)}{h} - \alpha - \gamma \]

\[ \text{sol} = \text{gallery} \left( \text{'tridiag'}, \alpha(3: end-1), \text{diago}(2:end-1), \gamma(2:end-2) \right) \backslash \text{PI}_{POL}(2:end-1) \]

\[ \text{sol2} = \text{zeros}(n,1) \]

\[ \text{sol2}(1) = -1 \]

\[ \text{sol2(end)} = 0 \]

\[ \text{sol2}(2:end-1) = \text{sol} \]

\[ \text{sol} = \text{sol2}' \]

5. Price function

function sol = price (S,A, rho, xi , pi, h )

\[ b = \log( A \rho / S ) \]

\[ \text{sol} = \exp(xi) \]

\[ \text{PI}_{tmp} = \text{sol} \cdot \pi \]

\[ \text{PI} = \text{PI}_{tmp}(\xi <= b) \]

\[ \text{sol} = ( S / \rho ) \left( \rho - 1 + h \left( \text{sum}(\text{PI}) - \text{PI}(1) / 2 - \text{PI}(\text{end}) / 2 \right) \right) \]