Measurement-Integrated simulations and Kalman filter applied to a turbulent co-flowing jet

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This paper deals with the experimental evaluation of a flow analysis system based on the integration between an under-resolved Navier-Stokes simulation and experimental measurements with the mechanism of feedback (referred to as Measurement-Integrated simulation), applied to the case of a planar turbulent co-flowing jet. The experiments are performed with inner-to-outer-jet velocity ratio around 2 and the Reynolds number based on the inner-jet heights about 10000. The measurement system is a high-speed PIV, which provides time-resolved data of the flow-field, on a field of view which extends to 20 jet heights downstream the jet outlet. The experimental data can thus be used both for providing the feedback data for the simulations and for validation of the MI-simulations over a wide region. The effect of reduced data-rate and spatial extent of the feedback was investigated. Then, to deal with the reduced data different feedback strategies were tested. It was found that for small data-rate reduction the results are basically equivalent to the case of full-information feedback but as the feedback data-rate is reduced further the error increases and tend to be localized in regions of high turbulent activity. Moreover, it is found that the spatial distribution of the error looks qualitatively different for different feedback strategies. Feedback gain distributions calculated by optimal control theory are presented and proposed as a mean to make it possible to perform MI-simulations based on localized measurements only. So far, we have not been able to low error between measurements and simulations by using these gain distributions.
1. Introduction

A device that given an input (e.g., boundary conditions) is able to estimate the evolution of a system, is in classic system theory called an observer. Thus, any Navier-Stokes equation solver can be consider as an observer. However, (i) uncertainty regarding the boundary and initial condition, (ii) disturbances at which the real system is subjected and (iii) inaccuracy of the model can generate considerable errors in the estimation of the state of the flow.

To stabilize the error dynamics in such a way that the estimated state converges to the actual state of the system, a forcing term proportional to the error between experimental observations and estimated variables can be introduced in the simulation. Despite the fast increase in available computational power and the improvements of the mathematical models, how to perform fast and accurate estimation of a complex flow is still a open issue. In fact, even when the model is exact, like in the case of Direct Numerical Simulations (DNS), (i) uncertainty regarding the boundary and initial condition and (ii) disturbances at which the real system is subjected can generate considerable errors in the estimation of the state of the flow. Furthermore, fully resolved DNS are limited to simple geometry and even then they are extremely time-consuming, therefore not suitable in cases where real time estimation is required (ex. flow control applications). In most practical application a reduced order model is therefore required and inaccuracy of the models are another source of error (iii).

In order to compensate for (i), (ii) and (iii) closed loop estimators are often used. In closed loop estimation a forcing term proportional to the error between experimental observations and estimated variables can be introduced in the simulation. This introduces another challenge in estimator design: the availability of measurements of the physical system. In fact it is not uncommon that access to measurements is very limited (e.g. medical applications) and the scarcity of available measured states is a problem that has to be taken into account.

The ability to estimate the states of a complex flow from limited experimental observations is a recurring problem in many practical applications. Flow estimation can be either a necessary step in the direction of actively controlling the flow (e.g. to delay transition to turbulence, to control flow separation Pinier et al. (2007), etc), or it can be a source of valuable information as such, as in medical applications, when the knowledge of the exact flow field can yield extremely valuable informations for the diagnosis of potentially fatal diseases (Hölscher et al. 2006; Funamoto et al. 2008).

In such cases it is often desirable that the estimation occurs in a short time. Real-time requirements exclude the use of high-dimensional direct numerical simulations (DNS) as models for the flow. This additional constraint introduces
the problem of reducing the order of the system without losing information on the dynamics of the flow.

The mathematical foundation of many low-dimensional models can be found in the Galerkin models, although the dimension and properties of the resulting model strongly depend on the choice of the expansion modes. Based on the Karhunen-Loève decomposition a very compact empirical model can be derived from a reference solution of the Navier-Stokes equation (Deane et al. 1991). This type of Galerkin approximation is a very efficient representation of the system dynamics. To construct this model a detailed reference solution of the given problem is required and this might not always be available. Furthermore, the accuracy of the model prediction deteriorates quickly when the system departs from the reference solution (Noack et al. 2003).

Hayase & Hayashi (1997) proposed the idea to use under-resolved simulations as a model of the real flow. This methodology has been referred to as Measurement-Integrated simulation (MI-simulation). Several numerical and laboratory experiments have shown the validity of this simple and flexible tool. The concept has been applied to several flow cases such as the wake behind a square cylinder (Nisugi et al. 2004; Yamagata et al. 2008) as well as unsteady blood flow (Funamoto et al. 2008).

MI-simulations have demonstrated a capability of reproducing real flow characteristics, including fluctuations. However, in the above mentioned studies the flow case was characterized by a well defined dominant frequency (e.g. von Karman vortex shedding or heart pulse) and their potential to more complex flows such as high Reynolds number turbulent flows remains to be verified experimentally.

The aim of this work is to apply this method to a flow characterized by a broad range of scales such as a planar turbulent jet in a co-flowing stream at a Reynolds number close to that of realistic applications.

Mixing layers and co-flowing jets have been widely studied in the literature (Nickels & Perry 1996; Gokarn et al. 2006; Örlü et al. 2008) both because they are widely present in industrial applications (engines, mixing chambers, etc.) and for their importance from the understanding of turbulence transport phenomena. This flow also represents a good benchmark for the MI-simulations, since it is necessary to capture the dynamics of the turbulent momentum transfer in order to correctly reproduce the main flow characteristics.

In this paper, we first test proportional feedback with zero off-diagonal gains, then, in order to address the need of localizing the measurements we calculate the extended Kalman filters. Physical-space convolution kernels (gain distributions or Extended Kalman filters) are computed from the linearized Navier-Stokes equations. This approach is similar to the one described by (Högberg 2001) and it has proven to be effective for the estimation of transitional flows, but the results on turbulent flows has not been fully explored yet.
Here, gain distributions are calculated using this methodology. However, a full evaluation of their usability is left for future work.

In order to test the two different approaches we perform high-speed PIV measurements, which yields time-resolved 2-D data of the flow-field, on a field of view which extends to 20 heights downstream the jet outlet. The experimental data can thus be used both for providing the feedback data for the simulations and for validation of the MI simulations over a wide region. The inner-to-outer-jet velocity ratio is close to 2 and the Reynolds number based on the inner-jet height is about 10000.

At first MI-simulations have been performed with a diagonal feedback gain matrix and full-information feedback (i.e. feedback applied at each point and at each time step). The feedback gain has been found by parameter variation until the error norm was minimized. This simulation should establish an upper limit to the performance of the MI-simulations for this case. Then, the effect of reduced feedback (i) data-rate and (ii) area is investigated. For (i) three different strategies are investigated in this configuration. Two were used by Funamoto et al. (2008) and Yamagata et al. (2008) and one is introduced here. For (ii) we test the effect of reducing feedback area and the density of feedback points. In order to relate the MI-simulation performance to turbulent quantities, a time-averaged error norm is introduced, and the spatial distribution of the error norm is also investigated.

Finally, the performance of proportional feedback with diagonal feedback gain matrix is complemented to the extended Kalman Filter approach, which instead uses only local measurements at 10 streamwise stations.

The paper is thus divided in the following parts: in the next section the experimental setup and measurement system are described. Thereafter we define the numerical schemes used for the MI-simulations and the different feedback strategies. Following this, the results of MI-simulations and the Kalman filters are presented and discussed. Finally the conclusions are summarized.

2. Experimental setup

The experiments are performed in a square channel with a cross section of $50 \times 50$ mm$^2$. The first 300 mm of the main channel are divided into three sections of dimensions 19, 10 and 19 mm by means of two horizontal walls that extend over the entire width of the channel (see figure 1c). The flow in the three ducts is supplied by two independent pumps through two radial distributors, one connected to the inner and one to the two outer channels (see fig 1d). This configuration allows us to control the flow rates of the inner and the outer sections independently, and to realize a planar co-flowing jet issuing from the end of the inner duct. The end of the splitter walls correspond to the beginning of the measurement domain (see figure 1a-b).
Two non-dimensional parameters can be varied in the present setup: the velocity ratio $\lambda_r = U_j/U_s$, where $U_j = 1.15$ m/s and $U_s = 0.54$ m/s are the centerline velocities of the inner and the outer jets respectively, and the Reynolds number $Re = U_jd/\nu$, where $d$ is the inner duct height and $\nu$ is the kinematic viscosity of the fluid. In this work the values for these parameters are $\lambda_r = 2.1$ and $Re = 11500$.

2.1. Measurement system

The time-resolved flow field was measured with a high-speed Particle Image Velocimetry (PIV) system. The PIV system used in this work consist in a double cavity 10mJ Nd:YLF laser (repetition rate 2-20000 Hz) as a light source, and two high-speed cameras (up to 3000 fps at full resolution) with resolution of $1024 \times 1024$ pixels.

The arrangement of the cameras and the laser-sheet is shown in figure 1.1(b). The two cameras were used to acquire two 2-D velocity fields, each of them of $50 \times 100$ mm$^2$. The second camera was tilted by $5^\circ$ in order to reach an overlap between the two fields of about 20 mm, so that the total measurement area is 180 mm.
The calibration of the two cameras was done by taking images of a calibration plate with known reference points in situ, and the calibration parameters were extracted using a pinhole-based model Willert (2006).

The flow was seeded with 10 µm silver-coated tracer particles, and in order to capture all the relevant structures of this particular flow, series of double-frame, single-exposure images were acquired at a rate of 1500 Hz for a total time of 6 s. From the images, the velocity fields were calculated using the commercial PIV software DaVis 7.2 from LaVision GmbH. The algorithm used is a multi-pass correlation with continuous windows deformation and shift, which allowed to achieve a final interrogation window size of 8×8 pixels with more than 95% of valid vectors. The size of the interrogation window is of about 0.75×0.75 mm² in physical space, which sets the lower limit for the spatial resolution. The window overlap was 50%, therefore the velocity field is sampled at a rate of about 2.6 samples/mm. Detailed information about the performance of PIV algorithms can be found in the literature Stanislas et al. (2008).

2.2. Mean flow
In order to establish the characteristics of the flow in the channel, measurements at several spanwise positions were performed. In this paper we refer to the direction parallel to the X-coordinate as the streamwise direction, the direction parallel to Y as the cross-stream and the one parallel to Z as the spanwise direction (see fig. 1a). The streamwise and cross-stream velocity components are u and v, respectively. Time averages of these velocity components will be denoted with the capital letters U and V, respectively.

Figure 2 shows the streamwise development of the time-average and normal stresses ⟨u'²⟩ at the channel centerline (Z = 0), where the brackets denote the operation of ensemble average and u' are the turbulent fluctuations defined as u' = u − U. The profiles are scaled to fit the figure, and two main features can be observed: the first one is the two wakes generated by the blunt end of the splitter plates which might dominate the dynamics of the shear layer Örlü et al. (2008), the second is a slight asymmetry of the normal stress profiles due to asymmetries in the geometry of the channel. The spreading rate appears to be linear as typically found for these type of flows, and the normal stresses profiles show the characteristic local maxima at the two shear layers. In figure 3 the same profiles at the inlet are shown in a dimensional form.

The spanwise evolution of the velocity profiles have also been investigated to assess to what extent the flow can be considered to be two-dimensional. Figure 4 show the cross-stream profiles of U at 5 spanwise stations (Z/d = −2, −1, 0, 1, 2) at X/d = 0 and 7.5 in (a) and (b), respectively and figure 5 shows the spanwise profiles at the same streamwise positions. The latter has been measured by rotating the channel 90⁰ around the X-axis, so that the light sheet was parallel to the Z-axis.
Figure 2. Streamwise evolution of the mean (thick line) and normal stresses $< u'^2 >$ (thin line) profiles of $U$ are at the channel centerline ($Z/d=0$). The profiles are scaled to fit the figure.

Figure 3. Mean velocity (left) and turbulent fluctuations intensity (right) profiles at the inlet ($X/d = 0$). The symbols $\circ$, $\triangle$, and $\nabla$ show the centerline velocity and the centerline velocities of the upper and lower jet, respectively.

The results clearly show that there is a large degree of inhomogeneity in the spanwise direction ($\pm 10\%$) of the outer flows, which however does not cause the inner jet to deviate in a particular direction. A careful investigation of the experimental setup showed that the origin of this inhomogeneity is due to a deformation of the channel walls which causes the flow to accelerate or decelerate according to the local curvature.

Although the present flow case shows 3-dimensional effects and asymmetries that deviate from the canonical case of a planar jet in a co-flowing stream, these deviations represent an additional challenge for the design of an observer that aims at estimating such flow. It could therefore be argued that it offers a good benchmark case for the evaluation of the performance of the observer.

2.3. Turbulent fluctuations: spectral analysis and characteristic scales

In order to evaluate the performance of the observer, especially when considering reduced feedback information in space and time, it is important to characterize
Figure 4. Figures (a) and (b) show the cross-stream velocity profiles at 5 spanwise stations \((Z/d = -2, -1, 0, 1, 2)\). (a) is measured at \(X/d = 0\), whereas (b) is taken at \(X/d = 7.5\). All the profiles are normalized by the centerline inlet velocity \(U_j\).

Figure 5. Figures (a) and (b) show the spanwise profiles of \(U\) at \(Y/d = +1, 0, -1\). Figure (a) is measured at \(X/d = 0\), whereas (b) is taken at \(X/d = 7.5\). All the profiles are normalized by the centerline inlet velocity \(U_j\).

the relevant time and length scales of this particular flow. We do that by analyzing the energy spectra both in frequency \((\Phi(f))\) and in wavelength space \((\Theta(\lambda_x))\). The spectra are defined as:

\[
\Phi(f) = |\int_{-\infty}^{\infty} u(x,t)e^{-if\tau} dt|^2 \tag{1}
\]

\[
\Theta(\lambda_x) = |\int_{-\infty}^{\infty} u(x,t)e^{-i\lambda_x x} dx|^2 \tag{2}
\]

where \(f\) is the frequency and \(\lambda_x\) is the streamwise wavelength. \(\Phi\) and \(\Theta\) have been computed at the inlet at each \(Y/d\) position and the results are shown in figures 6 as function of the non-dimensional length scale \(d/\lambda_x\) and the Strouhal number \(St = \frac{f(d/2)}{(U_j - U_s)}\), respectively. Since the figures are shown in a logarithmic
Figure 6. Contours of the pre-multiplied spectra $\Theta$ and $\Phi$ as a function of $Y/d$ and the dimensionless wavelength $\lambda_x/d$ and Strouhal number $St = \frac{f(d/2)}{(U_j - U_s)}$, respectively, at the inlet $X/d = 0$.

The downstream evolution of $\Phi$ is shown in figure 7. As we move downstream the two peaks in the shear layers migrate towards lower frequencies, until they merge at $St = 0.1$ in the self-similar region.

3. Measurement-Integrated simulations

The governing equations for the dynamic behavior of the flow are governed by the incompressible Navier-Stokes equations, which can be written in the form:

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla) u + \nu \nabla^2 u - \nabla P + f,$$

(3)

where $u$ is the velocity vector, $P$ is the pressure and $f$ is the body force. To close the system, the continuity equation is needed:

$$\nabla \cdot u = 0,$$

(4)

as well as boundary and initial conditions. From equations (3) and (4), an equation for pressure can be derived:

$$\nabla^2 p = -\text{div}(u \cdot \nabla)u + \nabla f.$$  

(5)

Equations (3) and (5) are used in this work as fundamental equations. The basic idea of the MI-simulations is to introduce in the body force term a feedback...
signal proportional to the difference between the measured output signal of the physical flow and the simulations.

3.1. Discretized equations and proportional control

The discretized forms of equations (3) and (5) can thus be written as:

\[
\frac{du_N}{dt} = -(u \cdot \nabla) u_N + \nu \nabla^2 u_N - \nabla p_N + f_N \tag{6}
\]

\[
\nabla^2 p_N = -\nabla \left( (u \cdot \nabla) u_N \right) + \nabla f_N, \tag{7}
\]

where \( u_N \) and \( p_N \) are the 2N-dimensional velocity and N-dimensional pressure vector, respectively, and \( N \) is the number of grid points used in the simulation. According to the methodology of MI-simulation stated above, the discretized body force \( f_N \) is defined as a linear function of the difference between the measured and simulated velocity vector:

\[
f_N = -K(u_N - u_N^*), \tag{8}
\]
where the asterisk denotes the measured velocity vector. \( \mathbf{K} \) is a \( 2N \times 2N \) matrix containing the feedback gains. In this approach, the feedback gains are optimized by trial and error method on the assumption that the matrix \( \mathbf{K} \) is diagonal and all the non-zero elements have the same value \( k \). Hereafter, the orthogonal component \( k \) will be referred to as the feedback gain. The fact that the matrix \( \mathbf{K} \) is diagonal implies that (unlike the case of the optimal control mentioned above) the error in a velocity component in a given grid point only effects the same velocity component in the same grid point.

### 3.2. Kalman filter

The feedback can also be designed by assuming linear dynamics and the use Linear Quadratic Gaussian theory (Högberg 2001). This gives so-called Extended Kalman filters and they are obtained by assuming:

1. that the flow is parallel and constant in the streamwise direction (local analysis),
2. that the fluctuations from the mean are governed by the Orr-Sommerfeld equation (i.e. the linearised Navier Stokes for such a mean flow)
3. a covariance matrix in \( y \),
4. a spectrum for the streamwise wavenumber \( \alpha \) and
5. a signal to noise ratio.

As interpreted here, they are used together with a discrete set of predefined sensor points, denoted \((x^k_{sp}, y^k_{sp})\) and the idea is to use the velocity at these sensor points only when determining the feedback. The result are then four feedback gain distributions for each sensor point: \( L^k_{uu}(x, y) \), \( L^k_{uv}(x, y) \), \( L^k_{vu}(x, y) \) and \( L^k_{vv}(x, y) \). The first index is the property sensed at \((x^k_{sp}, y^k_{sp})\) and the second is the property forced everywhere. An important comment is that the feedback gains are different for different choices of sensor points at a given streamwise position. For example, assume that \( x = 0 \) has been chosen as the streamwise position of sensor points. The feedback gain for sensor point \((0, 0.6)\) will then depend on the other \( y \)-positions that have been chosen at \( x = 0 \).

The gain distributions are calculated a-priori given the mean flow profiles \( \bar{U}(x^k_{sp}, y) \) and covariance matrices. From these distributions and the (measured) fluctuating velocity signals at the sensor points, \( \tilde{u}(x^k_{sp}, y^k_{sp}, t) \) and \( \tilde{v}(x^k_{sp}, y^k_{sp}, t) \), the \( x \)- and \( y \)-components of the feedback gain are obtained as:

\[
\begin{align*}
    f_x(x, y, t) &= \sum_{k=1}^{M} L^k_{uu}(x, y) \tilde{u}(x^k_{sp}, y^k_{sp}, t) + L^k_{v\bar{u}}(x, y) \tilde{v}(x^k_{sp}, y^k_{sp}, t) \\
    f_y(x, y, t) &= \sum_{k=1}^{M} L^k_{uv}(x, y) \tilde{u}(x^k_{sp}, y^k_{sp}, t) + L^k_{v\bar{v}}(x, y) \tilde{v}(x^k_{sp}, y^k_{sp}, t)
\end{align*}
\]
in the continuous setting. Above, $M$ is the number of sensor points. In this paper, the Kalman filters have been computed at 10 streamwise stations and $M = 2$, so that a total of 20 time-resolved measurements are used.

### 3.3. Numerical schemes and boundary conditions

The governing equations are discretized by the finite volume method. The resulting discretized system is solved with an algorithm similar to the SIMPLER method (Patankar 1980). The MI simulations reported in this paper are 2D and is a first attempt to build an observer for this flow since our experimental flow is not completely two dimensional. In the context of system observers, this imperfection can be seen as additional system noise.

The convective terms were discretized with the reformulated QUICK scheme Hayase et al. (1992) and the time derivative with the second order implicit Euler scheme. Linear algebraic equations are solved using the MSI scheme Schneider & Zedan (1981). The discretized operators in equations (6) and (7), as well as supplementary pressure correction equations and the velocity correction procedure are described in Patankar (1980).

The simulation domain and the coordinate system are shown in figure 8. The simulation domain covers the region $-4 \leq X/D \leq 4$. The upstream boundary condition is three uniform velocity profiles of 0.5 m/s, 1 m/s and 0.5 m/s, respectively, whereas the downstream boundary is free. The initial condition is given as no flow at $t = 0$. The computational grid is an orthogonal equidistant staggered mesh with $N_x \times N_y = 200 \times 50$ grid points in the $X$ and $Y$ directions, and the time step is taken to be $\Delta t = 1/1500$ s. The Reynolds number of the simulation is the same as that for the experiments. The simulation parameters are summarized in Table 1.
Kalman filter applied to a turbulent co-flowing jet

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel length</td>
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</tr>
<tr>
<td>Channel height</td>
<td>50 mm</td>
</tr>
<tr>
<td>Splitter walls length</td>
<td>200 mm</td>
</tr>
<tr>
<td>Inner duct widths</td>
<td>19 mm, 10 mm, 19 mm</td>
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<tr>
<td>Reynolds number Re</td>
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</tr>
<tr>
<td>Grid points $N_x \times N_y$</td>
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</tr>
<tr>
<td>Grid spacing $\Delta x \times \Delta y$</td>
<td>2 mm × 1 mm</td>
</tr>
<tr>
<td>Time step $\Delta t$</td>
<td>$0.66 \times 10^{-3}$ s</td>
</tr>
</tbody>
</table>

Table 1. Simulation parameters

3.4. Choice of feedback gain for proportional feedback

The feedback term modifies the dynamics of the system described by the Navier-Stokes equations (eq.(3)), but the choice of the feedback gain should be such that the magnitude of the feedback term (as described in eq.(8)) decreases as the simulation time elapses, so that the MI-simulations converge to the original system. Moreover, the level at which the error term converges should be minimum.

Based on these considerations the feedback gain is obtained by trial-and-error method in the following fashion: (i) MI-simulations were performed for various values of the feedback gain, $k=0, 1, 2, 4, 8, 16, 32, 64, 96, 128, 256$ and (ii) evaluated by the error of the MI-simulation compared to the reference solution (i.e. measurements) with the error norm $e_u(t)$ defined by:

$$e_u(t_k) = \sqrt{\frac{1}{2N}((\mathbf{u}_N(t_k) - \mathbf{u}_N^*(t_k))^T(\mathbf{u}_N(t_k) - \mathbf{u}_N^*(t_k)))},$$

where $t_k$ is the simulation time step. Time evolutions of the error norm $e_u$ for different values of the feedback gain $k$ are shown in figure 9a. For the case of $k = 0$, corresponding to the case of an ordinary simulation without any feedback, the error norm stays almost constant ($e_u = 0.19 \pm 0.02$). As $k$ is increased, in the range of $1 < k < 96$, there is an initial transient where the error norm decreases rapidly to reach an asymptote whose final value depends on the particular value of $k$. The initial reduction rate increases with increasing $k$, but becomes negative (i.e., the error increases) for $k = 128$, exhibiting a typical instability of this type of feedback systems. Figure 9b shows $e_{st}$ (subscript $st$ for spatial-temporal averaged), defined as the time average of the error norm $e_u$, as a function of $k$. The figure shows that $e_{st}$ decreases with increasing $k$ for $k \leq 96$, and that the least value of $e_{st}$ is 0.064 for $k = 96$, a third of that for $k = 0$. (A large error norm for $k = 128$ is also shown). Therefore the value of $k = 96$ will be used in this work.
3.5. *Feedback schemes for reduced feedback data-rate and reduced density/area.*

So far the ideal case where measurements are available at every time step in all points has been considered. However, in many practical applications this might not be the case and time or spatial resolution may be limited. In this paper the case of reduced time-resolution is considered.

In such cases where the measurement time-step $\Delta t_{in}$ is larger than the simulation time-step $\Delta t$, there are several strategies of applying feedback. Two strategies have been proposed by Funamoto *et al.* (2008) and Yamagata *et al.* (2008). The first one is to apply the same measure as a feedback signal until
a new measurement is available. This strategy will be referred to as persistent feedback, and it is illustrated in figure 10b. The second strategy is to apply feedback only at the time-steps where the measurements are available (intermittent feedback), figure 10c. In this work we introduce a third strategy in which the gap between one measurement and the next one are filled with a linear interpolation between the two points. This is defined as continuous feedback, and it is shown in figure 10d. Results from the different strategies are shown in the next section. To simplify the discussion we introduce the notation for the feedback data-rate $\lambda = \Delta t_{in}/\Delta t$.

In a spatial sense, the same percentage of the total area can be covered by the feedback points in several ways. We can introduce feedback forcing every $n$ grid points in the streamwise and the cross-stram direction, or we can uniformly cover a given area with feedback forcing and leave the rest unforced. In order to test what is the most efficient way of reducing spatial feedback information we tested 3 different cases. In Case 1 $n = 9$ in the streamwise direction and $n = 1$ in the cross-stream direction. In Case 2 $n = 3$ for both streamwise and cross-stram direction. In Case 3 $n = 1$ in all directions but feedback was applied only only the first 11 points in the streamwise direction. In all cases the total area reduction was 89% compared to the full information case.

4. Results

In this section the results from the MI-simulations will be shown and compared to the experimental results. All the results presented in this section are in a non-dimensional form. To distinguish between measured and estimated (i.e. output from MI-simulations) variables, the latter will be denoted by the (‘) symbol.

4.1. Full-information feedback

At first, MI-simulations are performed by introducing the feedback signal at every point of the domain and at every time-step ($\lambda = \Delta t_{in}/\Delta t = 1$). The results show the capability of MI-simulations to estimate first and second order statistics of the velocity field with a good accuracy. The asymmetries of the real flow are also well reproduced, even if the simulation domain is perfectly symmetric.

For comparison, simulations on the same grid are performed setting the feedback gain to zero (i.e. no feedback is applied). Due the high numerical dissipation introduced by the under-resolved computational grid, they completely fail to reproduce any of the characteristic features of the flow, such as, for example, the spreading rate of the mixing layer. This is illustrated by figure 11, where the contours of the mean and r.m.s. fields for $u$ are shown.

Figure 12 shows a direct comparison between experimental and MI-simulation ($k = 96$), and ordinary simulations ($k = 0$) results. It can be seen that, after
an initial overshoot due to a mis-match in the boundary conditions at the inlet between experiments and simulations, the MI-simulations perfectly reproduce the centerline mean velocity decay. The centerline velocity is directly connected to the dynamics of the shear-layer, which is governed by the Reynolds stress distribution (i.e., second order statistics). It is therefore not surprising that the r.m.s. profiles are also well predicted, although slightly underestimated, by the MI-simulations (see figures 12b and 12d).

It is also interesting to quantify and localize the error. One way to do it is to introduce the time-averaged spatial norm, defined as in (12):

\[ e_t(x, y) = \sqrt{\frac{\sum_{k=1}^{N_t} (\hat{u}(x, y, t_k) - u(x, y, t_k))^2}{N_t}}, \]

where \( N_t \) is the number of time-steps in the simulations. Figure 13 shows the contours of \( e_t \). It can be seen that the spatial distribution is rather uniform, except for a slight increase of the local average error in the shear layer regions, around \( x = 0, y = \pm 0.6 \). The relative error remains below 5% of the centerline velocity \( U_j \) in almost the all simulation domain, in agreement with figure 9b.

4.2. Effect of inlet boundary condition and of the interpolation to the computational grid

A possible concern in these MI-simulations is the effect of the inlet boundary condition, and especially the flow-rates in the inner and outer flows. In order to study the sensitivity to these two parameters, MI-simulations have been performed for different values.

The results are shown in figure 14 where the error as a function of overall mean velocity \( u_m \) is shown for \( \lambda_r = 2 \) in (a) and as a function of \( \lambda_r \) for \( u_m = 0.58 \text{ m/s} \) (the optimal value in (a)) in (b). Clear minima are seen, and the values used above are the optimal ones. Furthermore we can observe that the MI-simulations are more sensitive to the variations in \( u_m \), which can be explained by the fact that setting a different flow rate than the experimental one, will conflict in the continuity constraint, since the simulations are 2-D.

Another important issue to be taken into account is the effect of the interpolations scheme of the experimental data onto the numerical grid. In fact, as shown in figure 15, while the mean quantities remain unchanged, the fluctuation level have been reduced by about 20% in the interpolation. This means that the scales that have been smoothed out in the interpolation cannot be recovered in the MI-simulations. Therefore the numerical grid has to be chosen as function of the scales one wants to resolve since the grid will set the lower limit to the resolved scales.
4.3. Reduced feedback data-rate

The fact that the MI-simulation performs so well for feedback time-ratio $\lambda = 1$ is not surprising, since the dynamics of the real flow is entirely embedded in the feedback signal. Therefore, it is interesting to check how MI-simulations perform when the information of the feedback signal is reduced.

Simulations with reduced feedback data-rate (i.e. increasing $\lambda$) were performed with the three strategies described in figure 10. The first evaluation of the results can be done by looking at the space-time-averaged error norm $e_{st}$, defined in the previous section. Figure 16 shows the results of this analysis.

As it appears from the figure, $e_{st}$ stays fairly constant for all feedback methods until $\lambda \leq 10$ (corresponding $St > 1$, i.e. the simulation time step is smaller than any measured time scale), but after this value the error norm starts to increase for all methods. The space-time average error norm is minimum for the continuous method for all $\lambda$. However, an observer based on the continuous strategy can never be run in real-time since it has to be at least $\Delta t - \Delta t$ behind the real flow in order to have access to all information needed. This delay might or not be critical, depending on application.

The intermittent feedback method performs as well as the continuous method until $\lambda \leq 20$ ($St \approx 1$). The intermittent method produces better results than the persistent method for $\lambda < 100$ ($St \approx 0.1$). This can be explained by the fact that the persistent feedback method introduces a phase delay increasing with increasing $\lambda$. A similar conclusion was found in Funamoto et al. (2008). However in the present work it has been found that for higher values of $\lambda$, the error norm for the case of the intermittent feedback starts to diverge and persistent feedback performs better. One possible reason for this behavior is that for the case of intermittent feedback, the total energy of the feedback forcing decreases with increasing $\lambda$. As before, an analysis of the spatial distribution of the time-averaged error norm is performed. It is found that with increasing $\lambda$, the error distribution becomes less homogeneous and it tends to concentrate in the shear layer regions, where the turbulent activity is higher. An example of spatial distribution of the error norm for $\lambda = 50$ is shown in figure 17. Besides the time average error norm the spatial distribution of the error norm of the mean fields $E$ is shown. This gives an idea of the DC error compared to the r.m.s. error.

This figure also reveals a qualitative difference in the error-norm distribution of the intermittent feedback and the other two methods. The difference is that while for the cases of persistent and continuous feedback the error is very well correlated with the shear layer regions, the intermittent feedback gives an error norm that decreases downstream. In this case, the correlation with the shear layer regions is less obvious. This difference can be explained by looking at the DC error distribution. While the DC error for the persistent and continuous feedback strategy is very low, the intermittent feedback strategy does
not reproduce the mean field as well as the other two strategies. One possible explanation for this behavior can be that for the intermittent case the total feedback energy decreases with increasing $\lambda$.

Summarizing, in agreement with previous studies Yamagata et al. (2008) we have found that reducing the feedback data-rate reduces the accuracy of the MI-simulations. In addition we have found that this happens as soon as the simulation time steps becomes larger than the smallest scales observed in the measurements. It was also shown that the three feedback strategies will perform differently in different regions of the flow field and that the error increases more in regions of high turbulence activity as $\lambda$ is increased. We believe that this is due to the fact that turbulent flows are characterized by a broad range of time and length scales and we can speculate that finding a way of including off-diagonal elements in the feedback-gain matrix $K$ could be a possible improvement to the MI-simulations.

4.4. Reduced feedback area

Figure 18 shows the time evolution of the error norm with reduced spatial information from the measurements. In all cases, 1/9 (11%) of the full information is used. The cases are case 1 (black): every 9th point in the streamwise direction (at these streamwise positions all vertical points are used), case 2 (red): every third point in both directions and case 3 (blue): the first 1/9 of the streamwise extent. The reference full-information control is case 4 (green).

The results in figure 18 show that the strategy with which the feedback area is reduced has a huge effect on the result. The best result is obtained for case 1 and this can be understood from the fact that the flow variations are propagating downstream in the computational box. As long as the distance (in the streamwise direction) is shorter than the dominant wavelengths, the increase of the error norm can be expected to be limited. In the other cases, considerable information of the flow variations is disregarded by the reduction of the feedback area and consequently, the error norm is increased.
Kalman filter applied to a turbulent co-flowing jet

Figure 11. Contours of $u$ and $u_{rms}$. (a), (b): experimental results, (c), (d): MI-simulation full-information, $k = 96$, (e), (f): Simulations, $k=0$. The mean and r.m.s fields are computed from a set of 1500 snapshots, acquired over a time span of 6 seconds.
Figure 12. Mean velocity decay and $u_{r.m.s.}$ profile at $Z/d = 0$. Experimental results (continuous, black line) and MI-simulation full-information (continuous, red line).

Figure 13. Spatial distribution of the error norm.
Figure 14. Effect of inlet boundary condition in the MI-simulation. (a): Variation of the overall mean velocity ratio with a constant velocity ratio between the outer and inner flows. (b): Variation of the velocity ratio between inner and outer flows at $u_m = 0.58$ m/s.
Figure 15. Mean velocity decay and $u_{r.m.s.}$ profile at $Z/d = 0$. Experimental results (continuous, black line), and interpolated result (continuous, red line).

Figure 16. Space-time-averaged error norm $e_{st}$ as a function of the feedback time-ratio $\lambda$ for the three feedback methods.
Figure 17. Spatial distribution of the error norm for $\lambda = 50$: Intermittent feedback $e_t$ (a) and DC error norm (b), Persistent feedback $e_t$ (c) and DC error norm (d), Continuous feedback $e_t$ (d) and DC error norm (e).
Figure 18. Space averaged error norm $e_s(t)$ for Case 1 ($i = 9, j = 1$), Case 2 ($i=3, j=3$) Case 3 ($i=1, j=1$, total feedback area 11%). Case 4 is the full-information feedback.)
Kalman filter applied to a turbulent co-flowing jet

4.5. Extended Kalman filter gains

Examples of (normalized) feedback gain distributions of extended Kalman filters are shown in figures 19, 20 and 21. In each figure, the sensor point is shown as a black circle and the gains have been calculated assuming a second sensor point at the same streamwise position, which is marked with a light gray circle. As mentioned, the local approach implies that sensor points at other streamwise positions do not affect the distributions.

First the sensor points are varied in figure 19. At the upstream position (a,c), the Kalman filter is seen to force short wavelength oscillations occurring due to the wake instabilities just downstream of the splitter plates whereas further downstream (b,d), large scale jet modes are forced. When comparing the sensor points at \( Y/d = 0.6 \) (a,b) to the sensor points at \( Y/d = -0.6 \), the gain distributions are mirrored but otherwise similar.
Figure 20. All four normalized extended Kalman filter feedback gains from sensor point \((x_{sp}, y_{sp}) = (1.8, 0.6)\) (marked with a black circle) assuming there is a second sensor point at \((x_{sp}, y_{sp}) = (1.8, -0.6)\) (marked with a light gray circle). The walls from which the jet is ejected are indicated with horizontal lines to the left. The mean velocity profile is shown with a black line. (a): \(L_{uu}^k(x, y)\), (b): \(L_{uv}^k(x, y)\), (c): \(L_{vv}^k(x, y)\) and (d): \(L_{ww}^k(x, y)\). Red is high amplitude and blue is low. The \(\alpha\) spectrum and covariance matrices were estimated from the experimental data.

In figure 20, the gain distributions from one sensor point \((X/d = 1.8, Y/d = 0.6)\) are shown for all choices of sensed and forced properties. In (a), where \(u\) is sensed and forced, it is seen that the forcing is mainly in the sensed property around the sensor point. In the other cases, the forcings are not localized to the sensor points but instead distributed. The symmetry and phase relations of the gain distributions for the different cases can, to some extent, be understood by considering the flow dynamics, although this will not be discussed at length here.

Finally, figure 21 shows that the gain distributions are different in detail but similar in general appearance when the choice of streamwise wave number spectrum and covariance matrix are varied.
Figure 21. Normalized extended Kalman filter feedback gains $L_{vu}$ for different choices of streamwise wave number spectra and covariance matrix. The sensor point is $(x_{sp}, y_{sp}) = (1.8, 0.6)$ (marked with a black circle) and there is a second sensor point at $(x_{sp}, y_{sp}) = (1.8, -0.6)$ (marked with a light gray circle). The walls from which the jet is ejected are indicated with horizontal lines to the left. The mean velocity profile is shown with a black line. (a): $\alpha$-spectrum and covariance matrix estimated from experimental data, (b): flat $\alpha$ spectrum and covariance matrix from experimental data, (c): $\alpha$ spectrum from experimental data and diagonal covariance matrix. Red is high amplitude and blue is low.

Up to this point, the Kalman filters have not been able to produce any error reduction when added to the full information MI-simulations (see figure 22). A full report of these efforts are left for future publications.

5. Conclusions
MI-simulations have been applied to a planar turbulent jet in a co-flowing stream. Time resolved measurements of the flow have been performed with high-speed PIV. The measurements have been used both as a feedback signal and as a reference for the evaluation of the MI-simulations. At first the ideal
The case where all the measurements are available at all times in every point (full-information feedback) has been considered and the optimal value of for the feedback gain \( k = 96 \) has been found by trial and error (assuming a diagonal form of the feedback gain matrix \( K \)).

The results show the capabilities of the MI-simulations of reproducing with decent accuracy first and second order statistics of a turbulent flow at high Reynolds number. The relative error of the full-information feedback method was found to be on the order of 5%. To deal with a reduced availability of feedback data, three feedback strategies have been tested, two that where introduced by Funamoto et al. (2008) (intermittent feedback and persistent feedback), and one that was proposed here, based on interpolation of feedback data (continuous feedback). It was found that for all methods the error norm does not increase for feedback data-rate \( \lambda < 10 \) and the error norm is minimum for all \( \lambda \) in the case of continuous feedback. With increasing \( \lambda \) the error norm increases for all feedback methods and the spatial error distribution becomes less homogeneous. Moreover, the spatial distribution of the error looks qualitatively different in the case of intermittent feedback with respect to the case of the other two methods. In the last two cases the error norm correlates with regions of high turbulent activity.

It is also possible to design extended Kalman filters based on the experimental data. This has been done and the gain distributions show that localized measurements can result in distributed forcing. The actual distributions depend on the local mean velocity profiles and thus for the streamwise position of the sensor. Far upstream, the aim becomes small wavelength (in the streamwise direction) wake instabilities whereas larger jet modes are targeted further downstream. Initial attempts of applying the Kalman filters to the MI-simulation have not resulted in distinct reductions of the estimation error.

**Figure 22.** Space averaged error norm \( e_s(t) \) for ordinary simulations (black line), MI-simulations with full information feedback (red line) and MI-simulations using Kalman filters (green line).
up to this point. However, this is work in progress and the final verdict is yet
to be made.
References


