POST-MINKOWSKIAN ANALYSIS OF SLOWLY ROTATING STARS

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Master's thesis

September 2010

Ohana means family.
Family means nobody gets left behind, or forgotten.

— Lilo & Stitch

Dedicated to the loving memory of Jonas Jonsson.

1929 – 2010

ABSTRACT

The formalism developed by Hartle is used to set up the field equations of slowly and rigidly rotating stars valid to second order in the rotation parameter. These equations are then solved in the exterior vacuum region. In the interior region the field equations cannot be solved analytically and an expansion to second order in the weak-gravity limit is instead made.

The quadrupole moment of the vacuum metric is then calculated. This will depend on a constant that is determined in the matching procedure, which means that it is dependent on the interior of the star.

The quadrupole moment is found to deviate from that of the Kerr metric in the weak-gravity limit.

SAMMANFATTNING

Formalismen som utvecklats av Hartle används för att ställa upp fältekvationerna för långsamt och stelt roterande stjärnor som gäller till andra ordningen i rotationsparametern. Dessa ekvationer löses sedan i den yttre vakuumregionen. I den inre regionen kan inte ekvationerna löses analytiskt utan istället antas att gravitationen är liten och en expansion till andra ordningen görs i den.

Kvadrupolmomentet för vakuummetriken beräknas sedan. Det kommer att bero på en konstant som bestäms i matchningsproceduren, vilket betyder att den kommer att bero på stjärnans inre.

Kvadrupolmomentet visar sig skilja sig från Kerrmetrikens när gravitationen är liten.

Put your hand on a hot stove for a minute, and it seems like an hour.
Sit with a pretty girl for an hour, and it seems like a minute.
THAT'S *relativity.*

— *Albert Einstein*

Spacetime grips mass, telling it how to move,
and mass grips spacetime, telling it how to curve.

— *John Archibald Wheeler*

But my love this cannot be,
For so many years have gone
Though I'm older but a year
Your mother's eyes from your eyes cry to me.

— *Brian May*

ACKNOWLEDGMENTS

Thanks to Michael Bradley for his help and supervision.
Thanks to Sara Rydberg for all her support and for proof reading the report.
Thanks to all my wonderful friends in Umeå that made my time there so wonderful.

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INTRODUCTION

1 INTRODUCTION

In the beginning of the 20^{th} century Albert Einstein caused a small upheaval in the scientific world with his articles concerning the nature of time and space.

First, in 1905, he put forth the special theory of relativity which changed how the notions of motion, time and space where viewed. In essence what he did in his article was to postulate that the speed of light, c, was a constant independent of the reference frame of the observer. This means that no matter how fast an observer is moving he would measure the same value for the speed of light as an observer at rest, that value being c.

Then, in 1915, he completed the theory of general relativity which built on special relativity but also included gravity. In it gravity is no longer treated as a force but rather as a curvature of spacetime. This means that an object traveling along a straight line in absence of gravity would still be traveling along a straight line in the presence of gravity. The thing that would have changed is the curving of spacetime, like how a straight line on a ball is in fact curved. The "straight" line the object would follow in the curved space is called a geodesic.

The curving of space is described by Einstein Field Equations (EFE).

1.1 EINSTEIN FIELD EQUATIONS

The most important features of general relativity can be found from EFE, which mathematically can be written like

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.1}$$

where μ, ν are spacetime indices ranging from zero to three, $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the Energy-Momentum tensor.

The left hand side of the equation represents the geometry, i.e. the curving, of spacetime and the right hand side represents the presence of matter.

Both $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric so there exists at most ten field equations to be used to determine the metric components $g_{\mu\nu}$.

This seemingly simple structure of EFE is somewhat misleading, in reality EFE are probably impossible to solve in general and when they can be solved for some special case knowledge of both tensor calculus and differential geometry is needed.

1.2 CONVENTIONS

The conventions used in this thesis are:

- The signature of the metric will be $(+,-,-,-)$, where x^0 is a timelike coordinate and x^1, x^2, x^3 are spacelike.
- Einstein sum convention will be used.
- The speed of light, c, will be set to unity and $G = \frac{1}{8\pi}$ so that Einstein's equations, eq (1.1), will be simplified as $G_{\mu\nu} = T_{\mu\nu}$.
- Greek indices, $\mu, \nu, \ldots = 0, 1, \ldots$ will be used as coordinate indices.
- Terminology:
  ρ : energy density.
  p : pressure.
  Ω : angular velocity measured by an observer at rest at infinity.
  ω : rotation function.
  r_0 : radius of unperturbed star; zeroth order radius of the rotating star.

1.3 PERFECT FLUIDS

The interior of a star is complicated and hard to model. But ignoring thermodynamic effects, e.g. heat conduction and viscosity, the interior of the star can be modelled by a perfect fluid. For a star in equilibrium this is a reasonable approximation.

The Energy-Momentum tensor for a perfect fluid looks like:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}. \tag{1.2}$$

1.4 LINE ELEMENT

The distance between two neighbouring points in spacetime are measured by the line element, or the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.3}$$

Next follows a few examples of different line elements to make the reader a bit more familiar with the concept.

1.4.1 *The Schwarzschild Metric*

Perhaps the most important metric solution is the one discovered by Karl Schwarzschild in 1915 [17]. It describes the field of spherically symmetric vacuum spacetimes, or in other words the gravitational field around a spherical, non-rotating, mass. Its line element looks like:

$$ds^2 = A^2 dt^2 - \frac{dr^2}{B^2} - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1.4}$$

where

$$A^2 = B^2 = 1 - \frac{2M}{r}.$$

As proven by Birkhoff's theorem the Scwarzschild metric is the unique static, spherically symmetric vacuum solution to EFE.

The Schwarzschild metric can for example be used as a good approximation for the gravitational field around the Earth or the Sun.

1.4.2 *The Kerr Metric*

The Kerr metric describes the gravitational field outside an uncharged mass M rotating with angular momentum $J = Ma$ and was discovered by Roy Kerr in 1963[12]. Using Boyer-Lindquist coordinates the line element can be expressed as:

$$ds^2 = dt^2 - \frac{\Sigma}{\Delta} dr^2 - \Sigma^2 d\theta^2 - (r^2 + a^2)\sin^2\theta d\varphi^2 - \frac{2Mr}{\Sigma}(a\sin^2\theta d\varphi - dt)^2,$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$

The Kerr metric reduces to the Schwarzschild metric when $a = 0$, that is when the rotation stops, and it can also be shown that the Kerr metric is asymptotically flat. Despite these facts attempts to match the exterior Kerr metric to a rotating interior perfect fluid metric has proven mostly to be failures. The only known exact solution is the rotating disk of dust with maximal allowed angular momentum [15].

1.5 FURTHER READING

General relativity is a difficult subject to give a comprehensive introduction to. A lot can be said and still a lot would be left to say.

This thesis is aimed for readers that have taken at least an introductory course in general relativity.

For the interested reader there exists numerous books that covers the subject. The author of this thesis would recommend the following books
A short course in General Relativity by J. Foster et al.[7]

and
Relativity: An Introduction to Special and General Relativity by H. Stepahni et al.[18].

2 HARTLE'S APPROXIMATION SCHEME

Since this thesis will follow the scheme of Hartle, [10], on how to set up field equations for slowly rotating stars, next follows a summary of his work.

2.1 ASSUMPTIONS

Some assumptions are made about the rotating perfect fluid

(a) One-parameter equation of state, $\rho = \rho(p)$.

(b) Axial and reflection symmetry. Considerations are limited to configurations in equilibrium. In general relativity a configuration can only be in equilibrium if it is not radiating gravitational waves. Absence of gravitational waves are guaranteed by not having any time-dependent moments of the mass distribution, see e.g. [13]. Axial symmetry guarantees this. Reflection symmetry means that the configuration is symmetric about a plane perpendicular to the axis of rotation.

(c) Uniform(rigid) rotation. This means that the 4-velocity of the fluid will have the form $u^\mu = (u^t, 0, 0, u^\varphi) = (u^t, 0, 0, \Omega u^t)$, where Ω is the angular velocity of the fluid as measured by a stationary observer at infinity.

(d) Slow rotation, i.e. the angular velocity Ω is small enough so that the fractional change in pressure, energy density and gravitational field due to the rotation are all much less than unity. This condition implies that $R\omega << 1$, that is that every particle in the fluid must move at non-relativistic velocities.

2.2 METRIC AND DRAGGING OF INERTIAL FRAMES

The metric of a general stationary axially symmetric system can be written as

$$ds^2 = H^2 dt^2 - M^2 dr^2 - r^2 K^2 [d\theta^2 + \sin^2(\theta)(d\varphi - L dt)^2], \quad (2.1)$$

where H, M, K, L are functions of r and θ.

The quantity $L(r, \theta)$ is the angular velocity acquired by an observer who falls freely from infinity to the point (r, θ). This is a completely relativistic property which means that an inertial frame, the freely falling observer, is "dragged" along with the rotation of a massive, stationary, rotating object. This dragging contributes a non-vanishing (t, φ)-component to the metric. This effect will be somewhat like the classical effect where air next to a rotating ball will be rotated along with the ball.

The density and metric of a stationary, axially symmetric system will behave in the same way under a reversal in time or a reversal in direction of rotation. An expansion of the density or of H, M, K in powers of the angular velocity Ω can then only contain even powers; while an expansion of L only can contain odd powers. So to order Ω^2 an expansion of the metric will look like

$$ds^2 = A(r)^2(1 + 2h(r,\theta))dt^2 - \frac{1 + 2m(r,\theta)}{B(r)^2}dr^2 -$$
$$r^2(1 + 2k(r,\theta))[d\theta^2 + \sin^2(\theta)(d\varphi - \omega(r,\theta)dt)^2] + \mathcal{O}(\Omega^3),$$

where h, m and k all are functions at least second order in the angular velocity Ω and ω is at least first order. The zero order expansion, when $h = m = k = \omega = 0$, is the non-rotating state.

Looking at $G_t{}^\varphi = T_t{}^\varphi$ up to first order a separation of variables can be performed if ω is expanded in vector spherical harmonics

$$\omega(r,\theta) = \sum_{l=1}^{\infty} \omega_l(r) \left(-\frac{1}{\sin(\theta)} \frac{dP_l(\cos(\theta))}{d\theta} \right), \tag{2.2}$$

where $P_l(\cos(\theta))$ is the l-th order Legendre polynomial. Using this the different values of ω decouples. By examining solutions at very large and very small r some properties of the angular velocity can be found. At small r space is required to be regular and at large r space is required to be flat. It can then be shown that all ω_l with $l > 1$ must vanish. This gives the major simplification

$$\omega(r,\theta) = \omega_1(r) \left(-\frac{1}{\sin(\theta)} \frac{dP_1(\cos(\theta))}{d\theta} \right) = \omega_1(r) \equiv \omega(r). \tag{2.3}$$

From the remaining field equations we have partial differential equations for the second order functions h, m, k. By expanding them in spherical harmonics

$$h(r,\theta) = \sum_{l=0}^{\infty} h_l(r) P_l(\cos(\theta)) \tag{2.4}$$

$$m(r,\theta) = \sum_{l=0}^{\infty} m_l(r) P_l(\cos(\theta)) \tag{2.5}$$

$$k(r,\theta) = \sum_{l=0}^{\infty} k_l(r) P_l(\cos(\theta)) \tag{2.6}$$

and, as the case for ω, the equations for coefficients with different values of l decouple. Since the equations for h_l,m_l and k_l with

$l > 2$ do not contain ω these terms must vanish to ensure that h, m, k vanish when ω vanishes (in the static limit). By symmetry in rotation and time the $l = 1$ terms must vanish. Since transformations of the type $r \to f(r)$ do not change the form of the metric, such a transformation may be used to guarantee that one of the $l = 0$ terms vanishes. Thus, by use of spherical harmonics, h, m, k and ω are simplified to

$$h(r, \theta) = h_0 + h_2(r)P_2(\cos(\theta)) \tag{2.7}$$

$$m(r, \theta) = m_0 + m_2(r)P_2(\cos(\theta)) \tag{2.8}$$

$$k(r, \theta) = k_2(r)P_2(\cos(\theta)) \tag{2.9}$$

$$\omega(r, \theta) = \omega(r). \tag{2.10}$$

The set of partial differential equations obtained from Einstein's equations has now been reduced to a system of ordinary, inhomogeneous, linear differential equations. The full line element of the slowly rotating configuration can now be obtained by solving this system.

2.3 COORDINATE SYSTEM

An expansion of the metric as a function of r and θ in powers of Ω is only valid if the fractional change in any metric function is sufficiently small throughout space. This condition cannot hold, for an example, for an expansion of the density. For such an expansion to be valid the change in density would have to be small at each point in space. Near the surface of the star this would not be true since the surface of the star will be displaced from it's non-rotating position and the change in density may be finite where the non-rotating density vanishes.

To avoid this problem a new coordinate system is set up for the rotating configuration. In it r_0 labels a point on the surface of constant pressure for the unperturbed configuration which is equivalent to the point r in the perturbed. The coordinate Θ is unchanged from the non-rotating θ. For a pictorial description of the new coordinate system see Figure 1 on page 10.

Or mathematically the transformation to the new coordinate system is given as

$$\Theta = \theta \tag{2.11a}$$

$$p(r, \theta) = p(r(r_0, \Theta), \Theta) = p(r_0) + \delta p(r_0). \tag{2.11b}$$

$\delta p(r_0)$ is a second order small shift of the pressure that changes from $p_{20}(0)$ (the 2nd order r dependent term of $p(r, \theta)$) at the centre to zero at the zero pressure surface. The function $r(r_0, \Theta)$ can be written as an expansion in angular velocity

$$r(r_0, \Theta) = r_0 + \xi(r_0, \Theta) + \mathcal{O}(\Omega^4). \tag{2.12}$$

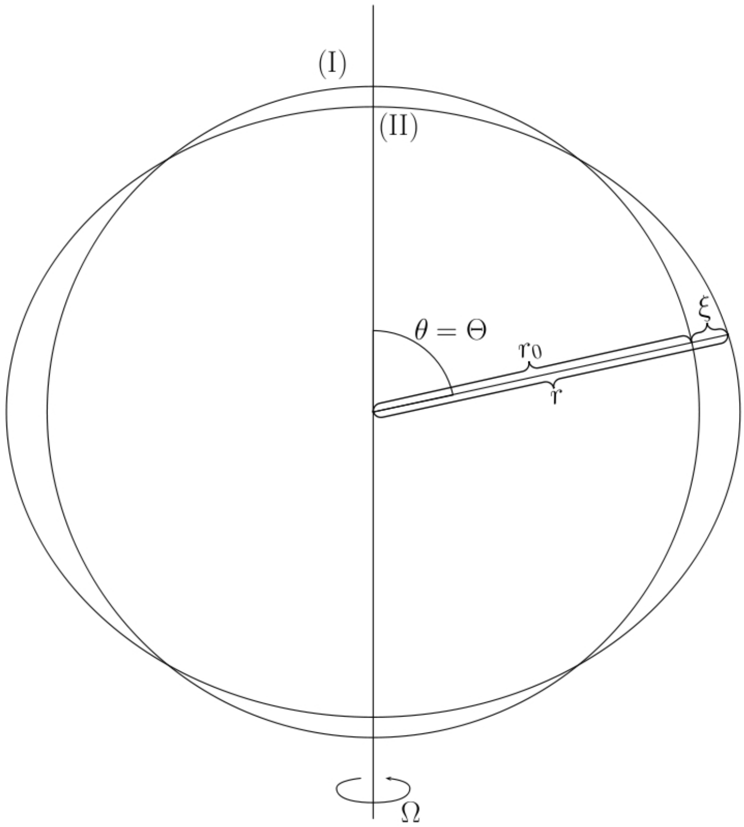

Figure 1: Definition of the coordinates r, θ in terms of the non-rotating coordinates r_0, Θ and the displacement ξ at a surface of constant pressure $p(r_0)$. The surface (I) is the surface of the non-rotating configuration and (II) is the surface of the rotating.

The second of these terms can then be expanded in spherical harmonics

$$\xi(r_0, \Theta) = \sum_{l=0}^{\infty} \xi_l(r_0) P_l(\cos(\Theta)) \tag{2.13}$$

where, once again, $P_l(\cos(\Theta))$ are Legendre polynomials. The axial symmetry of the star ensures that all the odd terms of the expansion vanish. Keeping terms up to order Ω^2

$$\xi = \xi_0(r_0) + \xi_2(r_0) P_2(\cos(\Theta)). \tag{2.14}$$

Expanding the pressure in spherical harmonics, keeping only terms up to order Ω^2 (remember that ξ already is second order in Ω) and setting equal to equation (2.11b) yields

$$\begin{aligned} p(r,\theta) &= p_0(r) + p_2(r,\theta) = \\ &= p_0(r) + p_{20}(r) + p_{22}(r) P_2(\cos(\theta)) = \\ &= p_0(r_0) + \xi \left.\frac{dp_0}{dr}\right|_{r_0} + p_{20}(r_0) + p_{22}(r_0) P_2(\cos(\theta)) = \\ &= p_0(r_0) + \delta(p(r_0)). \end{aligned} \tag{2.15}$$

This can now be separated into two parts

$$\xi_0 = \frac{-p_{20} + \delta(r_0)}{\frac{dp_0}{dr}|_{r_0}} \tag{2.16a}$$

$$\xi_2 = \frac{-p_{22}}{\frac{dp_0}{dr}|_{r_0}}. \tag{2.16b}$$

3 MATCHING PROCEDURE

Now it is established how the line element can be calculated, both in the vacuum region and the perfect fluid region. But to be a global solution to Einstein's field equations the solutions in the two regions must also be smooth and continuous over the intersection between the regions. The matching surface has previously been defined to be the zero pressure surface. From the point of view of the fluid this is a physically reasonable surface to speak of. However, in the vacuum region this definition is not valid. Instead the two surfaces must be identified by the fact that they must have the same structure. A suitable surface in the vacuum region for the matching can be shown, [16], to be determined by the condition

$$\tilde{\Omega}^2 g_{\varphi\varphi} + 2\tilde{\Omega} g_{\varphi t} + g_{tt} = 1 - \tilde{C}, \tag{3.1}$$

where $\tilde{\Omega}$ and $\tilde{C}$ are constants. This condition gives surfaces in the vacuum region with the same structure as the zero pressure surface in the fluid region.

To match two solutions with each other they must satisfy the Darmois-Israel matching conditions, [11, 5]. They state that both the extrinsic and intrinsic curvature (1st and 2nd differential form) must be continuous over some hypersurface $\mathcal{S}$. That is,

$$ds^2_{(pf)}\big|_{\mathcal{S}} = ds^2_{(v)}\big|_{\mathcal{S}} \tag{3.2a}$$

$$K_{(pf)}\big|_{\mathcal{S}} = K_{(v)}\big|_{\mathcal{S}}. \tag{3.2b}$$

To explain what the tensor K represents the concept of *embedding* has to be explained, [14, 19].

3.1 EMBEDDING

Imagine a sheet of paper. It is flat. If said paper was to be rolled into a cylinder it would still be flat intrinsically. Distances on the surface of the cylinder would still be the same as they were on the flat sheet of paper. Clearly something has changed with it though, but observers trapped on the surface would not be able to notice the difference since all the geodesics and distances on it would be the same. The difference between them is the way that they are embedded in space.

Now imagine a normal to both of the surfaces. The normal to the flat paper does not change as the normal is moved over the surface, but the normal to the cylinder does. This is what can be used to tell two surfaces apart. If the normal to two surfaces change in the same way they are said to be *isometric*. This is where the tensor K gets useful. The tensor K is a measure of how the normal to a hypersurface changes as it moves over the hypersurface, see Figure 2.

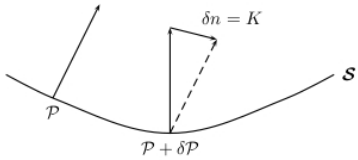

Figure 2: The extrinsic curvature of a hypersurface.

It is defined as

$$K \equiv K_{ab} dx^a dx^b \equiv h_a{}^c h_b{}^d n_{(c;d)} dx^a dx^b, \tag{3.3}$$

where

$$h_a{}^b = n_a n^b + \delta_a{}^b \tag{3.4}$$

is the projection operator onto the hypersurface orthogonal to the spacelike (normalized by $n_a n^a = -1$) normal vector n_a, and $n_{(c;d)}$ is the symmetric part of the covariant derivative of the normal.

3.2 COORDINATE TRANSFORMATION

To be able to match the interior and exterior solutions to each other the coordinates must be the same in both regions. To ensure this the interior coordinates are transformed,

$$\varphi \to \varphi + \Omega t \tag{3.5}$$

$$t \to c_4(1 + c_3) \tag{3.6}$$

where c_4, Ω, and c_3 are constants of zeroth, first and second order, respectively.

Part II

CALCULATIONS

4 FIELD EQUATIONS

The field equations are now calculated in orders of the rotation parameter,[1].

4.1 ZEROTH ORDER

To zeroth order in the rotation parameter the line element has the form:

$$ds^2 = A(r)^2 dt^2 - \frac{1}{B(r)^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\varphi^2. \quad (4.1)$$

Since the interior of the star is modelled by a perfect fluid the energy and pressure of the nonrotating configuration is given by $G^t{}_t = T^t{}_t$ and $G^r{}_r = T^r{}_r$ so that:

$$\rho_0 = \frac{1}{r^2}\left[1 - \frac{d(rB^2)}{dr}\right], \quad (4.2)$$

$$p_0 = \frac{1}{r^2}\left[\frac{B^2}{A^2}\frac{d(rA^2)}{dr} - 1\right]. \quad (4.3)$$

By using the isotropic pressure condition ,$G^r{}_r = G^\theta{}_\theta$, one gets:

$$B\frac{d^2A}{dr^2} + \left[\frac{d}{dr}(Ar)\right]\left[\frac{d}{dr}\left(\frac{B}{r}\right)\right] + \frac{A}{Br^2} = 0. \quad (4.4)$$

4.2 FIRST ORDER

To first order in the rotation parameter the line element has the form:

$$ds^2 = A(r)^2 dt^2 - \frac{1}{B(r)^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\varphi^2 + 2\omega(r) r^2 \sin^2(\theta) dt d\varphi, \quad (4.5)$$

which has one extra cross-term and contains $\omega(r)$.

To first order the only expression follows from $G^\varphi{}_t = 0$:

$$\frac{d}{dr}\left(r^4 \frac{B}{A}\frac{d\omega}{dr}\right) + 4r^3\omega\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (4.6)$$

4.3 SECOND ORDER

Four new equations follow from the full second order metric. From $G^r{}_\theta = 0$,

$$r\frac{d}{dr}(h_2 + k_2) + \frac{r}{A}\frac{dA}{dr}(h_2 - m_2) - (h_2 + m_2) = 0. \qquad (4.7)$$

Pressure isotropy in the angular direction gives $G^\theta{}_\theta = G^\varphi{}_\varphi$ which, by use of (4.6), can be written as:

$$6(h_2 + m_2) - r^4\left(\frac{B}{A}\right)^2\left(\frac{d\omega}{dr}\right)^2 + 4r^3\omega^2\frac{B}{A}\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (4.8)$$

Equality of pressure in the radial and angular direction, $G^r{}_r = G^\theta{}_\theta$, give two more equations. Collecting the θ-independent part give

$$6r^3B\frac{d}{dr}\left(\frac{1}{r}A^2B\frac{dh_0}{dr}\right) - 3B^2\frac{d(r^2A^2)}{dr}\frac{dm_0}{dr} + 12A^2m_0 - 3r^4B^2\left(\frac{d\omega}{dr}\right)^2 + 4r^3\omega^2AB\frac{d}{dr}\left(\frac{B}{A}\right) = 0 \quad (4.9)$$

while the θ-dependant part, after eliminating dh/dr with help of equation (4.7), take the form

$$2r\frac{B^2}{A}\frac{dA}{dr}\left(r\frac{dk_2}{dr} - m_2\right) - 2r^2Bh_2\frac{d}{dr}\left(\frac{B}{r}\right) + m_2 - 4k_2 - 5h_2 - \frac{1}{3}r^4\frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2 = 0. \quad (4.10)$$

By making a decomposition of the energy density function and the pressure like $\rho = \rho_0 + \rho_2$ where $\rho_2 = \rho_{20} + \rho_{22}P_2(\cos(\theta))$ and $p = p_0 + p_2$ where $p_2 = p_{20} + p_{22}P_2(\cos(\theta))$ the higher order terms can be calculated from $G^t{}_t$ and $G^r{}_r$ respectively. The second order small functions of the coordinate r are given for the energy density as

$$\rho_{20} = \frac{B}{6r^2A}\left[8r^3\omega^2\frac{d}{dr}\left(\frac{B}{A}\right) + 12\frac{A}{B}\frac{d}{dr}(rB^2m_0) - r^4\frac{B}{A}\left(\frac{d\omega}{dr}\right)^2\right] \quad (4.11)$$

and

$$\rho_{22} = -\frac{2(3A^2h_2 + r^2\omega^2)}{3r^3A\frac{dA}{dr}}\left[1 - B^2 + r^2\frac{d}{dr}\left(B\frac{dB}{dr}\right)\right] \quad (4.12)$$

and for the pressure as

$$p_{20} = \frac{B^2}{6r^2A^2}\left[12rA^2\frac{dh_0}{dr} - 12m_0\frac{d}{dr}(rA^2) + r^4\left(\frac{d\omega}{dr}\right)^2\right] \quad (4.13)$$

and

$$p_{22} = \frac{2B}{3rA}(3A^2h_2 + r^2\omega^2)\frac{d}{dr}\left(\frac{B}{A}\right). \tag{4.14}$$

The existence of a barotropic equation of state, $\rho = \rho(p)$, is equivalent to saying

$$\frac{\partial\rho}{\partial\theta}\frac{\partial p}{\partial r} = \frac{\partial\rho}{\partial r}\frac{\partial p}{\partial\theta}. \tag{4.15}$$

By substituting the decomposition of energy density and pressure into this relation yields to second order

$$\rho_{22}\frac{\partial p_0}{\partial r} = p_{22}\frac{\partial\rho}{\partial r}. \tag{4.16}$$

But this condition is already identically satisfied in terms of the above field equations so this does not give any new information. By assuming that the equation of state remains the same up to second order gives

$$\rho_2 = \frac{d\rho}{dp}p_2 \tag{4.17}$$

and since ρ_2 and p_2 already are second order this simplifies to

$$\rho_2\frac{dp_0}{dr} = p_2\frac{d\rho_0}{dr}. \tag{4.18}$$

The θ-dependent part of equation (4.18) is the same as equation (4.16), but the θ-independent part gives the relation

$$\rho_{20}\frac{dp_0}{dr} = p_{20}\frac{d\rho_0}{dr}. \tag{4.19}$$

By substituting ρ_0 and p_0 in equation (4.19) with the expressions given in equations (4.2) and (4.3) and ρ_{20} and p_{20} with the expressions given in equations (4.11) and (4.13) a final expression for h_0 and m_0 is derived

$$\begin{aligned}
&24rA^4\left[r^2\frac{d^2}{dr^2}(B^2) + 2(1-B^2)\right]\frac{dh_0}{dr} + 24rAm_0\frac{dA}{dr}\times \\
&\left[4A^2(B^2-1) - 4r^2AB\frac{dA}{dr}\frac{dB}{dr} - 4r^2A^2B\frac{d^2B}{dr^2}\right] + \\
&8r^3A^5\frac{dA}{dr}\frac{d}{dr}\left(\frac{B^2}{A^2}\right)\left[3\frac{dm_0}{dr} + \frac{r^2\omega^2}{B^2}\frac{d}{dr}\left(\frac{B^2}{A^2}\right)\right] + \\
&A^2\left[2r^2\frac{d^2}{dr^2}(B^2) - 2r^2A\frac{dA}{dr}\frac{d}{dr}\left(\frac{B^2}{A^2}\right) + 4(1-B^2)\right]\times \\
&\qquad\left[r^4\left(\frac{d\omega}{dr}\right)^2 - 12A^2m_0\right] = 0.
\end{aligned} \tag{4.20}$$

4.4 VACUUM SOLUTION

For the region outside the star, i.e. the exterior vacuum region, the above field equations can be solved analytically,[4], by imposing $\rho = p = 0$. The zero order field equation can be solved for A and B, which yields the expected, and well-known, see section 1.4.1, stationary vacuum solution

$$A^2 = B^2 = 1 - \frac{2M}{r} \tag{4.21}$$

where M is the mass of the star in length units.

In the vacuum exterior the first order equation simplifies as

$$\frac{d}{dr}\left(r^4 \frac{d\omega}{dr}\right) = 0.$$

Solving for ω yields

$$\omega = \frac{2aM}{r^3} - C_2. \tag{4.22}$$

Demanding asymptotic flatness the constant C_2 vanishes, since the metric should reduce to the non-rotating at infinity.

The first and second order field equations can then be solved for the other functions[1]

$$h_0 = \frac{1}{r-2M}\left(\frac{a^2M^2}{r^3} + c_2 + d_1(r-2M)\right) \tag{4.23}$$

$$m_0 = \frac{1}{2M-r}\left(\frac{a^2M^2}{r^3} + c_2\right) \tag{4.24}$$

$$h_2 = 3c_1(2M-r)\log\left(1-\frac{2M}{r}\right) + a^2\frac{M}{r^4}(M+r)$$
$$+2c_1\frac{M}{r}(3r^2-6Mr-2M^2)\frac{r-M}{2M-r}$$
$$+\left(1-\frac{2M}{r}\right)r^2q_1 \tag{4.25}$$

$$k_2 = 3c_1(r^2-2M^2)\log\left(1-\frac{2M}{r}\right) + a^2\frac{M}{r^4}(2M+r)$$
$$-2c_1\frac{M}{r}(2M^2-3Mr-3r^2) + (2M^2-r^2)q_1 \tag{4.26}$$

$$m_2 = 6\frac{a^2M^2}{r^4} - h_2. \tag{4.27}$$

The solution is determined by the mass M, the first order small rotation parameter a and the second order small constants c_1, c_2 and q_1. These are to be determined during the matching procedure, when the interior and exterior solution are matched on the zero pressure surface. The exterior is asymptotically flat if $q_1 = 0$ [2], and if $q_1 = c_1 = 0$ the metric is equivalent to the Kerr metric [3].

1 The result of the funciton h_0 looks slightly different from that obtained in [2], but by making a small coordinate transformation of the form $t \to t(1+\epsilon)$,where ϵ is a constant of order Ω^2, the same result is obtained.

5 MATCHING

Using equation (3.2) the constants can then be solved for. Both the first and second fundamental form can be separated into parts of different order

$$g_{ab} = g^{(0)}_{ab} + g^{(1)}_{ab} + g^{(2)}_{ab} \tag{5.1a}$$

$$K_{ab} = K^{(0)}_{ab} + K^{(1)}_{ab} + K^{(2)}_{ab}, \tag{5.1b}$$

where the superscript denotes the order of the term. A suitable matching surface can be written as $r_0 + \xi$ in the inner region and as $r_0 + \chi$ the outer region. To second order in the inner region equation (5.1) looks like

$$\begin{aligned} g_{ab}(r) &= g^{(0)}_{ab}(r) + g^{(1)}_{ab}(r) + g^{(2)}_{ab}(r) \\ &= g^{(0)}_{ab}(r_0 + \xi) + g^{(1)}_{ab}(r_0) + g^{(2)}_{ab}(r_0) \\ &= g^{(0)}_{ab}(r_0) + \left.\frac{\partial g^{(0)}_{ab}(r)}{\partial r}\right|_{r_0} \xi + g^{(1)}_{ab}(r_0) + g^{(2)}_{ab}(r_0) \end{aligned} \tag{5.2a}$$

$$\begin{aligned} K_{ab}(r) &= K^{(0)}_{ab}(r) + K^{(1)}_{ab}(r) + K^{(2)}_{ab}(r) \\ &= K^{(0)}_{ab}(r_0 + \xi) + K^{(1)}_{ab}(r_0) + K^{(2)}_{ab}(r_0) \\ &= K^{(0)}_{ab}(r_0) + \left.\frac{\partial K^{(0)}_{ab}(r)}{\partial r}\right|_{r_0} \xi + K^{(1)}_{ab}(r_0) + K^{(2)}_{ab}(r_0) \end{aligned} \tag{5.2b}$$

and a similar result hold in the outer region. So setting the equations equal to each other according to equation (3.2) the different order functions can be solved for. Below $^{(\nu)}$ stand for vacuum and the functions without superscript refer to the quantities in the interior.
Zeroth order:

$$A^{(\nu)} = c_4 A\,, \quad B^{(\nu)} = B\,, \quad A^{(\nu)}{}_{,r} = c_4 A_{,r}\,. \tag{5.3}$$

First order:

$$\omega^{(\nu)} = c_4(\omega - \Omega)\,, \quad \omega^{(\nu)}{}_{,r} = c_4 \omega_{,r}\,. \tag{5.4}$$

Second order:

$$h^{(\nu)}_2 = h_2\,, \quad k_2 = k_2\,, \quad h^{(\nu)}_0 = h_0 + c_3 \tag{5.5}$$

$$c_4 A (h^{(\nu)}_0{}_{,r} - h_{0,r}) + \xi_0 (A^{(\nu)}{}_{,rr} - c_4 A_{,rr}) = 0 \tag{5.6}$$

$$c_4 A(h_2^{(v)}{}_{,r} - h_{2,r}) + \xi_2 (A^{(v)}{}_{,rr} - c_4 A_{,rr}) - c_4 r A_{,r} (k_2^{(v)}{}_{,r} - k_{2,r}) = 0 \quad (5.7)$$

$$B(m_0^{(v)} - m_0) = \xi_0 (B^{(v)}{}_{,r} - B_{,r}) \quad (5.8)$$

$$B(m_2^{(v)} - m_2) = \xi_2 (B^{(v)}{}_{,r} - B_{,r}) + rB(k_2^{(v)}{}_{,r} - k_{2,r}) \quad (5.9)$$

$$\chi_0 = \xi_0 \,, \quad \chi_2 = \xi_2 . \quad (5.10)$$

Inserting the vacuum functions listed in section 4.4, the constants can be solved for. All quantities below are evaluated at the zero order radius, $r = r_0$. The zero order equations yield

$$M = \frac{r_0}{2}(1 - B^2) \,, \quad c_4 = \frac{B}{A} \,, \quad r_0 = \frac{A}{2B^2 \frac{dA}{dr}}(1 - B^2) \quad (5.11)$$

the first order

$$a = \frac{B r_0^3}{3A(B^2 - 1)} \frac{d\omega}{dr} \,, \quad \Omega = \frac{r_0}{3} \frac{d\omega}{dr} + \omega \quad (5.12)$$

and finally the second order

$$c_1 = \frac{B^2}{9 r_0^2 A^2 (B^2 - 1)^6} \left[r_0^4 B^2 (B^4 - 3) \left(\frac{d\omega}{dr} \right)^2 + 36 A^2 h_2 (1 - B^4) + 72 A^2 B^2 (h_2 + k_2) \right] \quad (5.13)$$

$$c_2 = \frac{\xi_0}{2} \left(B^2 - 1 + 2 r_0 B \frac{dB}{dr} \right) - \frac{r^5}{36} \frac{B^2}{A^2} \left(\frac{d\omega}{dr} \right)^2 - r_0 B^2 m_0 \quad (5.14)$$

$$c_3 = \frac{r_0^4}{36 A^2} \left(\frac{d\omega}{dr} \right)^2 + \frac{c_2}{r_0 B^2} - h_0 \quad (5.15)$$

$$\begin{aligned} q_1 = & \frac{1}{9 r_0^2 A^2 (B^2 - 1)^6} \Big\{ 18 k_2 A^2 (B^4 - 1)(B^4 - 8B^2 + 1) \\ & + 216 A^2 B^2 \log B \left[2B^2 (h_2 + k_2) + h_2 (1 - B^4) \right] \\ & + 36 h_2 A^2 B^2 (B^2 - 1)(B^4 + B^2 - 8) \\ & + r_0^4 \left(\frac{d\omega}{dr} \right)^2 B^2 \left[(B^2 - 1)(2 + 11 B^2 - 7 B^4) \right. \\ & \left. + 6 B^2 (B^4 - 3) \log B \right] \Big\} \end{aligned} \quad (5.16)$$

$$\chi_0 = \xi_0 \,, \quad \chi_2 = \xi_2 . \quad (5.17)$$

Only six of the nine second order equations were needed to solve for the constants c_1, c_2, c_3, χ_0 and χ_2. Luckily, the remaining three equations are identically satisfied.

6 RELATIVISTIC MULTIPOLE MOMENTS

A lot of solutions to EFE exists but no clear physical interpretation can be given to most of them. This is why the concept of relativistic multipole moments are important. For stationary asymptotically flat solutions the relativistic multipole moments can be compared to the corresponding Newtonian multipole moments and a physical interpretation of the solution can be given. And for the case of this thesis the multipole moment can be calculated for the general metric and compared to the multipole moment of the Kerr metric. For the Kerr metric to really describe the exterior of a rotating star the multipole moment of it should be exactly the same as for the general rotating metric calculated in this thesis.

The basic idea of a relativistic multipole moment was first introduced by Geroch[8] for the case of static asymptotically flat vacuum spacetimes, and extended to the stationary case by Hansen[9] and Thorne[20].

6.1 DEFINITION

The relativistic multipole moments are defined on the 3-space of the timelike Killing trajectories at spatial infinity.

A 3-space $(\mathcal{M}, h)$ with a positive definite metric h is said to be asymptotically flat if there exists a manifold $(\tilde{\mathcal{M}}, \tilde{h})$ and a conformal factor $\tilde{\Omega}$ satisfying

(i) $\tilde{\mathcal{M}} = \mathcal{M} \cup \Lambda$, where Λ is a single point.

(ii) $\tilde{h}_{ij} = \tilde{\Omega}^2 h_{ij}$

(iii) $\tilde{\Omega}\big|_\Lambda = \tilde{D}_i\tilde{\Omega}\big|_\Lambda = 0$, $\tilde{D}_i\tilde{D}_j\tilde{\Omega}\big|_\Lambda = \tilde{h}_{ij}\big|_\Lambda$,

where $\tilde{D}$ is the derivative operator associated with $\tilde{h}_{ij}$.

From the timelike Killing field, K^a, one constructs the two potentials, f and ψ, from the norm and the curl of the field. From the vacuum EFE, $R_{\mu\nu} = 0$, the curl is locally a gradient. So we have that the two potentials are constructed from $f = K^a K_a$ and $\psi_{,a} = \epsilon_{abcd} K^b K^{c;d}$.

The complex gravitational potential will then look like

$$\xi = \frac{1-\mathscr{E}}{1+\mathscr{E}}, \tag{6.1}$$

where $\mathscr{E} = f + i\psi$ is the Ernst potential. The gravitational potential is given the conformal weight $-\frac{1}{2}$, so that

$$\tilde{\xi} = \tilde{\Omega}^{-1/2}\xi. \tag{6.2}$$

The multipole tensors are then defined recursively as

$$\begin{aligned} P^{(0)}(x^i) =& \xi \\ P^{(1)}_j(x^i) =& \xi_{,j} \\ P^{(n+1)}_{k_1 k_2 \ldots k_{n+1}}(x^i) =& D_{<k_{n+1}} P^{(n)}_{k_1 \ldots k_n>} \\ & - \frac{1}{2} n(2n-1) R_{<k_1 k_2} P^{(n-1)}_{k_3 \ldots k_{n+1}>}, \end{aligned} \tag{6.3}$$

where $< k_1 \ldots k_{n+1} >$ denotes the operation of taking the symmetric and trace-free part. D_i and R_{ij} are the covariant derivative and Ricci tensor, respectively, with respect to the 3-metric h_{ij}. The conformal image $\tilde{P}^{(n)}_{k_1 \ldots k_{n+1}}$ is defined correspondingly in terms of the tilded quantities. The multipole moments are given entirely in terms of the scalar moments for axisymmetric spacetimes

$$P_n = \frac{1}{n!} \tilde{P}^{(n)}_{k_1 \ldots k_n} n^{k_1} \ldots n^{k_n} \big|_\Lambda \tag{6.4}$$

in terms of the axis vector n^i.

6.2 QUADRUPOLE MOMENT OF THE VACUUM METRIC

Following the method of Fodor et al. [6] on how to calculate the multipole moments of a stationary axisymmetric spacetime, the metric is first transformed to the canonical form

$$ds^2 = f(dt - \tilde{\omega} d\phi)^2 - f^{-1} \left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right], \tag{6.5}$$

where f, γ and $\tilde{\omega}$ are functions of ρ and z by the coordinate transformations

$$\begin{aligned} \rho &= A r \sin\theta \left[1 + h_0 + h_2 + k_2 - \frac{3}{2} \sin^2\theta (h_2 + k_2) \right] \\ z &= \left[(r - M) \left(1 + 2h_0 - \frac{1}{2} \sin^2\theta (h_2 + 2k_2 - m_2) \right) \right. \\ & \left. + r^2 A^2 \left(h_{0,r} - \frac{1}{2} \sin^2\theta (h_{2,r} + k_{2,r}) \right) \right] \cos\theta \end{aligned} \tag{6.6}$$

that is valid to second order in Ω.

The two potentials, f and ψ, can then be calculated from the timelike Killing vector $K^a = \delta^a_0$, giving

$$f = K^a K_a = g_{00} = (1 + 2h) A^2 - r^2 \omega^2 \sin^2\theta \tag{6.7}$$

$$\psi = -\frac{2aM}{r^2} \cos\theta \tag{6.8}$$

A suitable conformal factor is found to be $\tilde{\Omega} = \tilde{r}^2 \equiv \tilde{\rho}^2 + \tilde{z}^2$ after the coordinate transformations

$$\tilde{\rho} = \frac{\rho}{\rho^2 + z^2} \tag{6.9}$$

$$\tilde{z} = \frac{z}{\rho^2 + z^2} \tag{6.10}$$

$$\tilde{\varphi} = \varphi. \tag{6.11}$$

Which means, from equation (6.2), that

$$\tilde{\xi} = \frac{1}{\tilde{r}}\xi. \tag{6.12}$$

Even though it is not generally true, Fodor et al.[6] have shown that for $n \leqslant 3$ the scalar moments P_n are given by the coefficients in the expansion

$$\tilde{\xi}(\tilde{\rho} = 0) = \sum_{n=0}^{\infty} m_n \tilde{z}^n \tag{6.13}$$

$\tilde{\rho}$ equal to zero corresponds in the original coordinates to $\theta = 0, \pi$. An expansion of ξ, that is given by equation (6.1), gives to second order in the rotational parameter Ω

$$\xi = \frac{M}{r-M} + i\frac{Ma}{(r-M)^2} - \frac{M^2a^2}{r(r-M)^3} - (h_0 + h_2)A^2\left(\frac{r}{r-M}\right)^2. \tag{6.14}$$

Since the three last terms of equation (6.14) already are first and second order, respectively, it follows from equation (6.6) that $r = z + M$ can be used. However it can not be used for the first term since that only is of zeroth order. It must be expanded to second order in the rotational parameter, and from equation (6.6) it can be used that

$$r - M = z - 2h_0(r-M) - r^2A^2h_{0,r} \tag{6.15}$$

to second order on the axis ($\theta = 0$). Inserting this into the first term of equation (6.14) and only keeping terms that are up to second order in the rotational parameter gives

$$\frac{M}{r-M} = \frac{M}{z - 2h_0(r-M) - r^2A^2h_{0,r}} = \frac{M}{z} + \frac{M}{z^2}\left[2h_0(r-M) + r^2A^2h_{0,r}\right], \tag{6.16}$$

where once again $r = z + M$ can be used. Using at last equation (6.2) to express $\tilde{\xi}$ in $\tilde{z}$ one gets to second order for the asymptotically flat metric

$$\tilde{\xi} = M - c_2 + iMa\tilde{z} - M\left(a^2 + \frac{16}{5}M^4c_1\right)\tilde{z}^2. \tag{6.17}$$

This means that the mass, the angular momentum and the quadrupole moment are given by

$$\begin{aligned} \text{Mass} &= M - c_2 \\ J &= Ma \\ Q &= -M(a^2 + \frac{16}{5}M^4c_1). \end{aligned}$$

To compare this with the quadrupole moment of the Kerr metric($Q = -Ma^2$) the relative deviation from it is calculated

$$\frac{\Delta Q}{Q} \equiv \frac{Q - Q_{\mathrm{Kerr}}}{Q_{\mathrm{Kerr}}} = \frac{16M^4 c_1}{5a^2}. \tag{6.18}$$

7

THE POST-MINKOWSKIAN LIMIT

Since the equations for the fluid domain cannot be solved, an expansion will instead be made in the weak-gravity limit [1]. The expansion will be made in the small parameter $\lambda = \frac{M}{r_0}$, where M is the mass and r_0 the radius of the fluid ball. When writing the expansion parameter in SI-units it is more clear that this is just the gravitational potential, $\lambda = \frac{GM}{r_0 c^2}$, where the c^2 is added to make it dimensionless.

7.1 FIRST ORDER

First a first order expansion is made of the spherically symmetric configuration. The functions A and B has the expansions

$$A = 1 + A_1 + A_2 + \dots \tag{7.1a}$$

$$B = 1 + B_1 + B_2 + \dots, \tag{7.1b}$$

where, of course, the subscripts denote the order in λ. Because of the freedom of rescaling the time coordinate the zeroth order constant of the function A can be demanded to be one. The zeroth order constant of the function B must be one if the central pressure and density are to be finite.

Plugging the expansions, equation (7.1), into equation (4.6) yields to first order

$$\frac{d^2 A_1}{dr^2} + \frac{1}{r}\frac{d}{dr}(B_1 - A_1) - 2\frac{B_1}{r^2} = 0. \tag{7.2}$$

Solving this for B_1 yields

$$B_1 = -r\frac{dA_1}{dr} + r^2 C. \tag{7.3}$$

Making a similar expansion for the pressure and density

$$\rho = \rho_1 + \rho_2 + \dots \tag{7.4a}$$

$$p = p_1 + p_2 + \dots \tag{7.4b}$$

and plugging this into the equations for the pressure, equation (4.3), yields to first order in λ

$$p_1 = \frac{2}{r^2}\left(r\frac{dA_1}{dr} + B_1\right) = 2C. \tag{7.5}$$

So the first term in the expansion of the pressure is a constant. To be able to find a zero-pressure surface this constant must be put to zero, so $C = 0$, and therefore $p_1 = 0$. The first non-vanishing terms of the density and pressure are

$$\rho_1 = \frac{2}{r^2}\frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right) \tag{7.6}$$

$$p_2 = p_c - \left(\frac{dA_1}{dr}\right)^2 - 4\int\frac{1}{r}\left(\frac{dA_1}{dr}\right)^2 dr. \tag{7.7}$$

The fact that the pressure is one order higher than the density can be understood in a newtonian context by looking at the virial theorem applied to a spherically symmetric configuration. Is says that $\langle p\rangle = k\lambda\langle\rho\rangle$, where k is a constant of order unity and $\langle p\rangle$ and $\langle\rho\rangle$ are the time average of the pressure and density.

To complete the system an equation of state, $p = p(\rho)$, is required. Differentiating this equation gives

$$\frac{dp_2}{dr} = \frac{dp_2(\rho_1)}{d\rho_1}\frac{d\rho_1}{dr} = -\rho_1\frac{dA_1}{dr}. \tag{7.8}$$

Here ρ_1 is given by equation (7.7).

Next is ω. Looking at a newtonian fluid and the balance between the gravitational and the centripetal forces at the equator one sees that the angular velocity $\omega < \frac{\sqrt{\lambda}}{r_0}$. This means that the angular velocity should have the post-minkowskian expansion

$$\omega = \omega_{1/2} + \omega_{3/2} + \dots. \tag{7.9}$$

This also seems reasonable considering that ω, or its derivatives, only appear quadratically in equation (4.8) and (4.10). Using equation (4.6) one reaches the result

$$\frac{d\omega_{1/2}}{dr} = 0, \tag{7.10}$$

that is to say that $\omega_{1/2}$ is a constant. This constant cannot be put to zero in the same way as p_1 was put to zero earlier. This is because $\omega_{1/2}$ corresponds to the non-linear source term in the higher order equations. With it zero there would be no rotation of the system.

The second order quantities, h_2, k_2 and m_2, have the post-minkowskian expansions

$$h_2 = h_{21} + h_{22} + \dots \tag{7.11a}$$

$$k_2 = k_{21} + k_{22} + \dots \tag{7.11b}$$

$$m_2 = m_{21} + m_{22} + \dots. \tag{7.11c}$$

With the help of equations (4.7) and (4.8) one obtains that

$$k_{21} = m_{21} = -h_{21}, \tag{7.12}$$

and from equation (4.10)

$$\frac{d^2 h_{21}}{dr^2} + \frac{2}{r}\frac{dh_{21}}{dr} - \frac{1}{2}\left(\frac{d\rho_1}{dA_1} + \frac{12}{r^2}\right) h_{21} = \frac{r^2}{6}\frac{d\rho_1}{dA_1}\omega_{1/2}^2, \tag{7.13}$$

where

$$\frac{d\rho_1}{dA_1} = 2\frac{d^3A_1}{dr^3}\bigg/\frac{dA_1}{dr} - \frac{4}{r^2} + \frac{4}{r}\frac{d^2A_1}{dr^2}\bigg/\frac{dA_1}{dr}\,. \tag{7.14}$$

Equation (7.13) has the particular solution $h_{21p} = -\omega_{1/2}^2 r^2/3$. The constant c_1 is needed to compute the quadrupole moment, as described in section 6, and to first order in λ c_1 can be written as

$$c_1 = \frac{1}{16r^7\left(\frac{dA_1}{dr}\right)^6}\left[r\frac{dA_1}{dr}\frac{dh_{21h}}{dr} - \frac{h_{21h}}{r^3}\frac{d}{dr}\left(r^4\frac{dA_1}{dr}\right)\right], \tag{7.15}$$

where h_{21h} is a non-zero solution to the homogeneous version of equation (7.13). All functions in the expression for c_1 should of course be evaluated at the zero-pressure radius $r = r_0$.

7.2 SECOND ORDER

Taking the expansion instead to second order,similarly as in the previous section, yields for equation (4.6), when using equation (7.3) to simplify

$$\frac{d^2A_2}{dr^2} - \frac{1}{r}\frac{dA_2}{dr} + \frac{1}{r}\frac{dB_2}{dr} - \frac{2}{r}B_2 + \frac{A_1}{r}\frac{dA_1}{dr} - A_1\frac{d^2A_1}{dr^2} - 2r\frac{d^2A_1}{dr^2}\frac{dA_1}{dr} + \left(\frac{dA_1}{dr}\right)^2 = 0 \tag{7.16}$$

and solving for B_2 results in

$$B_2 = -r\frac{dA_2}{dr} + rA_1\frac{dA_1}{dr} + r^2\left(\frac{dA_1}{dr}\right)^2 + r^2\frac{p_c}{2} - 2r^2\int\left(\frac{dA_1}{dr}\right)^2 dr. \tag{7.17}$$

The next terms for the pressure and the density are

$$\rho_2 = -3p_c - 2\left(\frac{dA_1}{dr}\right)^2 + \frac{2}{r^2}\frac{d}{dr}\left[r^2\left(\frac{dA_2}{dr} - A_1\frac{dA_1}{dr}\right)\right] -3\frac{d}{dr}\left[r\left(\frac{dA_1}{dr}\right)^2\right] + 12\int\frac{1}{r}\left(\frac{dA_1}{dr}\right)^2 dr \tag{7.18}$$

and

$$p_3 = 2p_c A_1 + 2A_1 \left(\frac{dA_1}{dr}\right)^2 - 2\frac{dA_1}{dr}\frac{dA_2}{dr} + 4r\left(\frac{dA_1}{dr}\right)^3$$
$$-8A_1 \int \frac{1}{r}\left(\frac{dA_1}{dr}\right)^2 dr - 2\int \frac{1}{r}\frac{dA_1}{dr}\left(4\frac{dA_2}{dr} - r\left(\frac{dA_1}{dr}\right)^2\right.$$
$$\left. -8A_1\frac{dA_1}{dr} + 3r^2\frac{dA_1}{dr}\frac{d^2A_1}{dr^2}\right) dr. \qquad (7.19)$$

From equation (4.6) one gets to second order in λ

$$\omega_{3/2} = 4\omega_{1/2}\int \frac{\int r^2 \frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right) dr}{r^4} dr$$
$$= 4\left(A_1 - \frac{2}{r^3}\int r^2 A_1\, dr\right)\omega_{1/2}. \qquad (7.20)$$

$\omega_{5/2}$ will also be needed but that equation is quite long and will be spared the reader.

From the field equations for the functions that are second order in the rotation parameter, Ω, equations (4.7),(4.8) and (4.10) one gets to second order in λ

$$k_{22} = -h_{22} - \frac{2}{3}\int\left(3h_{21}\frac{dA_1}{dr} - r\omega_{1/2}^2\frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right)\right) dr \qquad (7.21)$$

$$m_{22} = -h_{22} + \frac{2}{3}r^2\omega_{1/2}^2\frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right) \qquad (7.22)$$

and

$$\frac{d^2h_{22}}{dr^2} + \frac{2}{r}\frac{dh_{22}}{dr} - \frac{1}{2}\left(\frac{d\rho_1}{dA_1} + \frac{12}{r^2}\right)h_{22}$$
$$-\left\{\left[\frac{d\rho_2}{dA_1} + \frac{d\rho_1}{dA_1}\left(2r\frac{dA_1}{dr} + A_1\right)\right]\frac{h_{21}}{2}\right.$$
$$-5\frac{d^2A_1}{dr^2}h_{21} + \frac{2}{r}\frac{dA_1}{dr}h_{21} + r\frac{d^2A_1}{dr^2}\frac{dh_{21}}{dr}$$
$$\left.+\frac{dA_2}{dA_1}\left(\frac{6}{r^2}h_{21} - \frac{2}{r}\frac{dh_{21}}{dr} - \frac{d^2h_{21}}{dr^2}\right)\right\}$$
$$-\frac{r^2}{6}\omega_{1/2}^2\left[\left(-\frac{16}{r^3}\int r^2A_1\, dr + 7A_1 + 2r\frac{dA_1}{dr}\right)\frac{d\rho_1}{dA_1}\right.$$
$$\left.+\frac{d\rho_2}{dA_1} - \frac{14}{r^2}\frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right)\right] = 0. \qquad (7.23)$$

The constant c_1 will to second order yield a very long expression that will be spared the reader.

7.3 EQUATION OF STATE

The next thing is to impose an equation of state. An equation of state describes how the pressure and energy density are related.

7.3.1 *Schwarzschild Fluid*

The most simple equation of state would be an incompressible fluid, or a Schwarzshild fluid. A Schwarzschild fluid is characterized by the fact that its energy density is constant throughout the fluid ball. When the pressure changes the volume of the fluid ball does not, so the energy density is kept constant. This is not a very realistic equation of state, but it is good as a first approximation.

7.3.1.1 *First Order*

Demanding asymptotic flatness one gets from equation (7.7) and (7.13) that

$$A_1 = \rho_1 \frac{r^2}{12} + b_0 \quad \text{and} \quad h_{21} = \omega_{1/2}^2 \frac{r^2}{2} \tag{7.24}$$

Here b_0 is an integration constant that either can be transformed away with a first order rescaling of the time coordinate or be chosen to have the value $b_0 = -\rho_1 r_0^2/4$. The second choice would make the constant c_4 in equation (5.11) equal to unity. Plugging this into the other first order equations yields

$$\begin{aligned} B_1 &= \rho_1 \frac{r^2}{6}, \qquad\qquad p_2 = p_c - \rho_1^2 \frac{r^2}{12} \\ \omega_{3/2} &= \rho_1 \omega_{1/2} \frac{r^2}{5}. \end{aligned} \tag{7.25}$$

From the fact that the pressure should be zero at the radius $r = r_0$, r_0 can be solved for from the above equations

$$r_{01} = \frac{2\sqrt{3p_c}}{\rho_1}. \tag{7.26}$$

Using matching condition, equation (5.11), one gets that

$$\left.\frac{dA_1}{dr}\right|_{r=r_0} = \frac{M_1}{r_0^2}. \tag{7.27}$$

From this M can be solved for

$$M_1 = \frac{4}{3} \frac{(3p_c)^{3/2}}{\rho_1^2} \tag{7.28}$$

and then from equation (5.12)

$$a_1 = -\frac{24}{5} \frac{p_c \omega_{1/2}}{\rho_1^2}. \tag{7.29}$$

Plugging all this into equation (7.15) yields

$$c_{11} = \frac{5}{1024} \left(\frac{\rho_1}{p_c}\right)^5 \omega_{1/2}^2, \tag{7.30}$$

which finally gives a value for the difference of quadrupole moment to the Kerr metric

$$\frac{\Delta Q}{Q} = \frac{25\rho_1}{16p_c}. \tag{7.31}$$

By observing the order of λ of the different factors the difference of the quadrupole moment can be seen to diverge as $1/\lambda$.

7.3.1.2 *Second Order*

The procedure for calculating the second order terms will be much the same as calculating the first order.

So demanding asymptotic flatness one gets

$$A_2 = \rho_2\frac{r^2}{12} + \frac{\rho_1^2 r^4}{144} + \frac{r^2}{4}\left(\frac{\rho_1 a_0}{3} + p_{c2}\right) + b_1$$

and

$$h_{22} = -\frac{5}{56}\rho_1\omega_{1/2}^2 r^4 - r^2\omega_{1/2}^2\left(\frac{1}{4}\frac{p_{c2}}{\rho_1} + a0\right) \tag{7.32}$$

and inserting this into the other second order equations give

$$\begin{aligned} B_2 &= -\rho_2\frac{r^2}{6} - \rho_1^2\frac{r^4}{72} \\ p_3 &= -\rho_1\rho_2\frac{r^2}{6} - p_{c2}\rho_1\frac{r^2}{3} + p_{c3} \\ \omega_{5/2} &= \rho_2\omega_{1/2}^2\frac{r^2}{5} + \omega_{1/2}\frac{r^2}{5}\left(\frac{11}{56}\rho_1^2 r^2 + p_{c2}\right). \end{aligned} \tag{7.33}$$

Now it is time to determine the zero-pressure surface, mass and the rotation parameter. For that purpose they are expanded in λ

$$r_0 = r_{01} + r_{02} + \dots \tag{7.34}$$

$$M = M_1 + M_2 + \dots \tag{7.35}$$

$$a = a_1 + a_2 + \dots \tag{7.36}$$

The first order terms are the ones given in the previous section.

The second order terms can then be calculated. Since the pressure is zero at the radius r_0 one get

$$r_{02} = \sqrt{\frac{6p_{c3}}{\rho_1(\rho_2 + 2p_{c2})}} \tag{7.37}$$

and using the other matching conditions yield

$$\begin{aligned} M_2 =& \sqrt{6}\rho_2\left(\frac{\sqrt{\rho_1 p_{c3}(\rho_2 + 2p_{c2})}}{\rho_1(\rho_2 + 2p_{c2})}\right)^3 \\ a_2 =& -\frac{12p_{c3}\omega_{1/2}}{35\rho_1^2(\rho_2 + 2p_{c2})}\times \\ & \left(\frac{6\rho_1 p_{c3}}{\rho_2 + 2p_{c2}} - 14p_{c2} - 7b_0\rho_1 + 21p_{c2}\right). \end{aligned} \tag{7.38}$$

The constant c_1 is expanded in orders of λ

$$c_1 = c_{11} + c_{12} + \dots \tag{7.39}$$

c_1 is needed to compute the relative difference of the quadrupole moment to the Kerr metric and to second order it is given by

$$\begin{aligned} c_{12} = &\left[-\frac{\omega_{1/2}^2 (\rho_2 + 2p_{c2})^5}{6720 \rho_1 p_{c3}^5} \right] \times \\ &\left[\frac{3648 p_{c3} \rho_1}{\rho_2 + 2p_{c2}} + 525 p_{c1} + 2100 \rho_1 b_0 + 5250 \rho_2 \right]. \end{aligned} \tag{7.40}$$

This finally gives a correction to the relative difference of the quadrupole moment given by

$$\frac{\Delta Q}{Q}_{\text{corr}} = -\frac{1816 \rho_1 p_{c3} + 1575 \rho_2 p_{c2} + 1750 p_{c2}^2 + 350 \rho_2^2}{112 \rho_1 p_{c3}}, \tag{7.41}$$

which is of order λ^0. This means that this term actually diverges to a constant. But a constant added to infinity will mean no big difference.

7.3.2 *Newtonian Polytropes*

Newtonian polytropes have an equation of state that look like

$$p = p_c \left(\frac{\rho}{\rho_c} \right)^{\gamma}, \tag{7.42}$$

where p_c and ρ_c are the central pressure and density, respectively. These kind of equation of state is a more realistic approximation than the previous and is often used in astrophysics to model the interior of neutron stars.

The equations cannot be integrated completely for this equation of state, but some things can still be said from it.

7.3.2.1 *First Order*

Plugging the polytropic equation of state into equation (7.8) yields

$$\begin{aligned} \frac{dp}{dr} &= \frac{\gamma p_c}{\rho_c^{\gamma}} \rho^{\gamma - 1} \frac{d\rho}{dr} \\ &= -\rho \frac{dA_1}{dr}. \end{aligned}$$

Integrating and solving for ρ one gets the following equation for A_1

$$\rho = k(C - A_1)^{\frac{1}{\gamma - 1}} = 2 \frac{d^2 A_1}{dr^2} + \frac{4}{r} \frac{dA_1}{dr}, \tag{7.43}$$

where

$$k = \left(\frac{\rho_c^\gamma}{p_c}\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}}.$$

Plugging this back into the equation of state the constant C can be identified by the fact that the pressure should be zero at the radius $r = r_0$, so $C = A_1(r_0)$.

This value of C also means that the density is zero at the zero-pressure radius. So one gets that

$$\frac{d^2A_1}{dr^2}\Big/\frac{dA_1}{dr}\bigg|_{r_0} = \frac{-2}{r_0}. \tag{7.44}$$

The homogeneous version of the differential equation for h_{21} now looks like

$$\frac{d^2h_{21h}}{dr^2} + \frac{2}{r}\frac{dh_{21h}}{dr} - \frac{1}{2}\left(\frac{-k}{\gamma-1}\left(A_1(r_0) - A_1\right)^{\frac{2-\gamma}{\gamma-1}}\frac{12}{r^2}\right)h_{21h} = 0. \tag{7.45}$$

Using matching condition equations (5.11) and equation (7.44) in equation (7.15) gives

$$c_1 = -\frac{h_{21h}r^4}{16M^5}\left(\frac{dh_{21h}}{dr}\Big/\frac{dh_{21h}}{dr} - \frac{2}{r}\right)\bigg|_{r_0}. \tag{7.46}$$

This mean that for c_1 to be zero either h_{21h} or $\frac{dh_{21h}}{dr}\Big/\frac{dh_{21h}}{dr} - \frac{2}{r}$ must be zero at the zero-pressure radius. For the metric to be asymptotically flat q_1, equation (5.16), must be zero. Substitution of $h_{21} = h_{21h} - \omega_{1/2}^2 r^2/2$ then gives an equation for h_{21h} if the metric are to be asymptotically flat.

$$q_1 = \frac{1}{15r^2}\left(9h_{21h} + 3r\frac{dh_{21h}}{dr} - 5\omega_{1/2}^2 r^2\right)\bigg|_{r_0} = 0. \tag{7.47}$$

Note that if $h_{21h} = 0$ then $q_1 \neq 0$ so the only way c_1 could be zero is if $\frac{dh_{21h}}{dr}\Big/\frac{dh_{21h}}{dr} - \frac{2}{r}$ would be zero at the zero-pressure radius.

Unfortunately Bradley et al.[1] have shown that this not is the case, but instead that

$$\left(\frac{dh_{21h}}{dr}\Big/h_{21h} - \frac{2}{r_0}\right)\bigg|_{r_0} < 0. \tag{7.48}$$

The conclusion must then be drawn that $c_1 \neq 0$ so for polytropes to first order in λ there is a difference in the quadrupole moment to the Kerr metric, so

$$\frac{\Delta Q}{Q} > 0. \tag{7.49}$$

7.4 SECOND ORDER

Of course it would be desirable to be able to say something of newtonian polytopes to second order. Unfortunately, the equations become to difficult for this meagre author to make something of. This was perhaps not so surprising considering that the first order equations only could be studied qualitatively.

Part III

CONCLUSIONS

8 CONCLUSIONS

The problem of finding solutions to EFE for slowly rotating stars has proven to be a most formidable one. In this thesis two simplifying steps were taken. Firstly the rotation was assumed to be slow and an expansion to second order was made in it. Secondly the gravitation was assumed to be small and an expansion to second order was made in it.

Even though the problem is simplified in doing these expansions the equation of state can still make it analytically unsolvable.

In the case for an incompressible fluid the equations can be solved completely and the result shows a difference in the quadrupole moment to that for the Kerr metric. For newtonian polytropes the equation of state makes the problem analytically unsolvable but things can still be said about the solution. In the end the same conclusions must be drawn for a polytrope as for an incompressible fluid, that is that the quadrupole moment differs from that of the Kerr metric.

That the quadrupole moment differs from that of the Kerr metric means that rotating stars with the considered equation of states not can be sources of the Kerr metric. Sources for the Kerr metric have been sought a long time and continue to be elusive.

In writing this thesis the author feels that a deeper appreciation for the difficulty to solve EFE has been gained. This difficulty is the reason that general relativity as a whole is a field of modern physics that have lagged much behind other ones that are as strange, like quantum mechanics. Most development in the field has been made in the last decades thanks to the development of computers. In fact much of the results found in this thesis would have been impossible to find without computers[1].

Some future work that could be done is to investigate other equation of state to try to find an internal solution that could be matched to the Kerr vacuum solution. A numerical study of the equations that are impossible to solve analytically could also be interesting.

1 In this thesis the program Maple 9.5 was used together with the package GRTensorII for tensor algebra.

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