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***Study of the risk-adjusted pricing methodology model with
methods of Geometrical Analysis***

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RESEARCH ARTICLE

Study of the risk-adjusted pricing methodology model with methods of Geometrical Analysis

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Abstract. Families of exact solutions are found to a nonlinear modification of the Black-Scholes equation. This risk-adjusted pricing methodology model (RAPM) incorporates both transaction costs and the risk from a volatile portfolio. Using the Lie group analysis we obtain the Lie algebra admitted by the RAPM equation. It gives us the possibility to describe an optimal system of subalgebras and the corresponding set of invariant solutions to the model. In this way we can describe the complete set of possible reductions of the nonlinear RAPM model. Reductions are given in the form of different second order ordinary differential equations. In all cases we provide exact solutions to these equations in an explicit or parametric form. Each of these solutions contains a reasonable set of parameters which allows one to approximate a wide class of boundary conditions. We discuss the properties of these reductions and the corresponding invariant solutions.

Keywords: transaction costs; invariant reductions; exact solutions; singular perturbation

AMS Subject Classification: 35K55, 34A05, 22E60

1. Introduction

One of the most important problems at present is how to incorporate both the transaction costs and the risk from a volatile (unprotected) portfolio into the governing Black-Scholes equation. In the pioneering work of Leland [13], devoted to the problem of option pricing in the presence of transaction costs, the idea of a periodic revision of a hedging portfolio was introduced. Leland assumed that the level of transaction costs is constant, i.e. we have a market with proportional transaction costs. He reduced this problem to a nonlinear partial differential equation with an adjusted volatility. Leland claimed that the terminal value of the portfolio approximates the payoff as the length of a revision interval tends to zero. Later, Kabanov and Safarian [10] proved that Leland's conjecture based on approximate replication fails and his model has a non-trivial limiting hedging error relative to simulated marked prices (see as well the detailed discussion in [11]). Mathematical problems arise in the limiting cases as revisions become unboundedly frequent. As a practical matter, extremely frequent revisions will not be desirable and the average errors are less than one-half of one per cent of the price suggested by Leland's formula [14]. Within the framework of the Leland's model, Kratka [9] has suggested a mathematical method for pricing derivative securities in the presence of proportional transaction costs and he additionally took into account the risk

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of the unprotected portfolio in between the revisions. Jandačka and Ševčovič [8] modified Kratka’s approach in order to derive a scale-invariant model.

In the model introduced in [8] the risk from the volatile portfolio is described by the average value of the variance of the synthesized portfolio. The mathematical model was referred to as the risk-adjusted pricing methodology (RAPM) model. The RAPM model generalizes the famous Black-Scholes model for the pricing of derivative securities. In the model setting both the transaction costs and the unprotected portfolio risk depend on the time interval between two transactions and minimizing of the total risk leads to the RAPM model. The model was studied recently with numerical methods in the case of European and American options [19]. We describe briefly the model settings.

The authors of [8] assume that the stock price dynamics is given by the geometric Brownian motion

$$S_t = S_0 \exp((\rho - \sigma^2/2)t + \sigma W_t), \tag{1}$$

where $\{W_t, t \geq 0\}$ is the Wiener process, $\rho \in \mathbb{R}$ is the drift and $\sigma > 0$ is the instantaneous volatility of the asset, ρ, σ are constants. It is assumed that the risk-free bond earns at a continuously compounded constant rate r .

The time-steps Δt at which the portfolio can be hedged against the price change of the underlying asset S_t are non-infinitesimal and fixed. Additionally, the authors introduce the idea of a switching time t^* for the last revision of the portfolio. This means that the time interval $(0, T)$ is divided in two parts, in the first part $(0, t^*)$ the revisions of portfolio will be done regularly, and in the second one (t^*, T) there are no revisions and correspondingly no transaction costs. It is assumed that the interval (t^*, T) is very small and in this interval the price of the contingent claim $u(S, t)$, $t \in [t^*, T]$ is defined as in the classical Black-Scholes formula (here T is the maturity time). It is assumed that the model (similar to Leland’s model) does not include the cost of establishing the initial investor’s portfolio composition.

At time t the value of the dynamically hedged portfolio V_t is $V_t^\phi = \delta_t S_t + \beta_t B_t$, where δ_t is a number of units of the stock (a constant on each time interval Δt), B_t is the value of the bond and β_t is a number of units of the bond. We can put $B_0 = 1$ without loss of generality and rewrite the previous relation in the form $V_t^\phi = \delta_t S_t + \beta_t e^{rt}$. The pair $\phi = (\delta_t, \beta_t)$ defines the self-financing hedging strategy that maintains the portfolio.

The change of V_t^ϕ in any time-step Δt is equal to $\Delta V_t^\phi = V_{t+\Delta t}^\phi - V_t^\phi = \beta_t e^{rt}(e^{r\Delta t} - 1) + \delta_t(S_{t+\Delta t} - S_t) - r_R S_t \Delta t$. The total risk premium r_R contains two parts $r_R = r_{TC} + r_{VP}$. The transaction costs (TC) in this case are modeled by the expression

$$r_{TC} = \frac{C\sigma S|u_{SS}|}{\sqrt{2\pi}}, \quad C = \frac{S_{ask} - S_{bid}}{S}, \tag{2}$$

where C is the round trip transaction costs per unit dollar of transaction [13], [6], [12] and $u(S, t)$ is the value function of the contingent claim with respect to the asset price S and time t . During the time-step Δt the portfolio is unprotected and the risk connected with a volatile portfolio (VP) is modeled by

$$r_{VP} = \frac{1}{2} R \sigma^4 S^2 (u_{SS})^2 \Delta t, \tag{3}$$

where R is a risk premium coefficient introduced in [9] and [8] and represents the

marginal value of investor's exposure to a risk. The total risk premium depends on the time-lag Δt and it is a strong convex function between two consecutive portfolio revisions [19]. To obtain a risk-adjusted Black-Scholes equation the authors minimize the total risk premium $r_R = r_{TC} + r_{VP}$. They then obtain for the optimal time-lag the following value

$$\Delta t_{opt} = \frac{C^{2/3}}{\sigma^2(R\sqrt{2\pi}|Su_{SS}|)^{2/3}}.$$

Using Ito's formula the authors of [8] finally obtain the risk-adjusted pricing methodology model

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 - \mu(Su_{SS})^{\frac{1}{3}}) - ru + rSu_S = 0, \quad \mu = 3 \left(\frac{C^2 R}{2\pi} \right)^{1/3}, \quad (4)$$

where $t \in (0, t^*)$ and the value t^* is determined by the implicit equation $T - t^* = \min_{S>0} \Delta t_{opt}(S, t^*)$. The equation represents a well-posed parabolic problem under the condition that

$$Su_{SS}(S, t) < \left(\frac{3}{4\mu} \right)^3. \quad (5)$$

The condition (5) will not be fulfilled for usual Call and Put options at $S = E$ and $t \rightarrow T^-$, where E is the strike price of the corresponding option. To avoid the singularities in the model the authors introduced the switching time t^* such that condition (5) is satisfied by $t = t^*$. The equation for t^* which can be reduced to the form $T - t^* = CR^{-1}\sigma^{-2}$ (for European Call and Put options) has a positive solution and the condition (5) is satisfied if

$$\frac{C}{R} < \sigma^2 T, \quad CR < \frac{\pi}{8}. \quad (6)$$

From the analytical point of view this model is represented by a fully nonlinear parabolic differential equation (PDE). In addition, equation (4) possesses a non-trivial singular perturbed algebraic structure.

One of the few methods that exist to study such fully nonlinear equations with a singular perturbed algebraic structure is the method of Lie group analysis. Our goal is to study the RAPM model with this method of analysis equation (4).

The analytical solutions which we will obtain using this method can be used as a benchmark for numerical or other methods. We will show that the RAPM model possesses four-dimensional symmetry algebras both when $r = 0$ and when $r \neq 0$; both algebras are isomorphic. We list in both cases the complete set of symmetry reductions of equation (4). It is possible to provide exact solutions to all reduced equations in an explicit or parametric form. Due to the exact form of solutions it is possible to compare different structures of these solutions in both cases (where the interest rate is $r = 0$ and $r \neq 0$).

From which it can be seen that each case should be studied in their own right, we cannot simply replace $r \neq 0$ by $r = 0$ in the formulas developed for the case $r \neq 0$. In addition to the value of the interest rate, each of these solutions contains two integration parameters and up to three free parameters which are nontrivially embedded in the solutions. The variation of these parameters can help to approximate different types of boundary conditions.

The same method of the Lie group analysis was used earlier in [1] – [4] to study the symmetry groups of nonlinear PDEs arising from the modeling of feedback effects of large traders on the market price of the underlying and on the price of the corresponding derivative product. In [1] and [3] we studied the symmetry properties of the model introduced by Frey in [5]. In [2] and [4] we studied the model introduced by Sircar and Papanicolaou in [18]. In all cases it was possible to provide symmetry reductions and to study the properties of invariant solutions.

2. Symmetry properties

Equation (4) is the main subject of our investigations. The equation possesses a complicated analytical and algebraic structure. In this section we provide the Lie group analysis of this equation with the goal of describing the complete set of symmetries of equation (4) and to obtain possible reductions. Using the invariants of the subgroups of the symmetry group of the studied equation we reduce the partial differential equation to ordinary differential equations (ODEs). Solutions to these ODEs give us the invariant solutions to the nonlinear RAPM model in an analytical form.

We obtain the symmetry group of the RAPM model in the way suggested by Sophus Lie and developed further in [16], [15] and [7]. We first find, using the Lie determining equations, the Lie algebra L_r of a dimension r admitted by the equation. Then we use an exponential map $\exp : L_r \rightarrow G_r$ and obtain the transformations of the symmetry group G_r . To each subalgebra $h_i \subset L_r$ there corresponds a subgroup H_i of G_r [7], [15], [16]. In most cases we do not need the explicit form of the group transformations and use directly the subalgebras h_i of L_r in order to reduce the RAPM model.

In this way we prove the following theorem.

Theorem 2.1: *The equation (4) admits a four dimensional Lie algebra L_4 with the following infinitesimal generators*

$$U_1 = S \frac{\partial}{\partial S} + u \frac{\partial}{\partial u}, \quad U_2 = e^{rt} \frac{\partial}{\partial u}, \quad U_3 = \frac{\partial}{\partial t}, \quad U_4 = S \frac{\partial}{\partial u}. \tag{7}$$

The commutator relations are

$$[U_1, U_2] = -U_2, \quad [U_2, U_3] = -rU_2, \tag{8}$$

$$[U_1, U_3] = [U_1, U_4] = [U_2, U_4] = [U_3, U_4] = 0. \tag{9}$$

The commutator relations (8) depend on the parameter r , i.e. on the interest rate included in the model. Depending on whether $r = 0$ or $r \neq 0$, we obtain different commutation relations for the algebra generators of the Lie algebra L_4 . After the proper choice of generators we obtain, in both cases, isomorphic algebras.

All four-dimensional real Lie algebras were classified by Patera and Winternitz [17]. We will use this classification and the corresponding notations for generators of L_4 . The algebra is spanned by the following generators $L_4 = \langle e_1, e_2, e_3, e_4 \rangle$, which will have different meaning depending on the value of r . We denote a two dimensional Lie algebra spanned by two operators e_1, e_2 with the unique non-trivial commutator $[e_1, e_2] = e_2$ as L_2 . The algebra L_4 is a decomposable Lie algebra and can be written as a semi-direct sum

$$L_4 = L_2 \oplus e_3 \oplus e_4, \quad L_2 = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_2. \tag{10}$$

Table 1. [17] The optimal system of subalgebras h_i of the algebra L_4 where $a \in \mathbb{R}$, $\epsilon = \pm 1$, $\phi \in [0, \pi]$.

Dimension	Subalgebras
1	$h_1 = \langle e_2 \rangle$, $h_2 = \langle e_3 \cos(\phi) + e_4 \sin(\phi) \rangle$, $h_3 = \langle e_1 + a(e_3 \cos(\phi) + e_4 \sin(\phi)) \rangle$, $h_4 = \langle e_2 + \epsilon(e_3 \cos(\phi) + e_4 \sin(\phi)) \rangle$
2	$h_5 = \langle e_1 + a(e_3 \cos(\phi) + e_4 \sin(\phi)), e_2 \rangle$, $h_6 = \langle e_3, e_4 \rangle$, $h_7 = \langle e_1 + a(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi) \rangle$, $h_8 = \langle e_2 + \epsilon(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi) \rangle$, $h_9 = \langle e_2, e_3 \sin(\phi) - e_4 \cos(\phi) \rangle$
3	$h_{10} = \langle e_1, e_3, e_4 \rangle$, $h_{11} = \langle e_2, e_3, e_4 \rangle$, $h_{12} = \langle e_1 + a(e_3 \cos(\phi) + e_4 \sin(\phi)), e_3 \sin(\phi) - e_4 \cos(\phi), e_2 \rangle$

Case $r \neq 0$. In the case $r \neq 0$ the generators take the form

$$\begin{aligned}
 e_1 &= (r - 1)U_1 + U_3 = (r - 1)S \frac{\partial}{\partial S} + (r - 1)u \frac{\partial}{\partial u} + \frac{\partial}{\partial t}, & e_2 &= U_2 = e^{rt} \frac{\partial}{\partial u}, \\
 e_3 &= rU_1 + U_3 = rS \frac{\partial}{\partial S} + ru \frac{\partial}{\partial u} + \frac{\partial}{\partial t}, & e_4 &= U_4 = S \frac{\partial}{\partial u}.
 \end{aligned} \tag{11}$$

Case $r = 0$. Using the previous notations we can represent L_4 in the case $r = 0$ in the form

$$\begin{aligned}
 e_1 &= -U_1 = -S \frac{\partial}{\partial S} - u \frac{\partial}{\partial u}, & e_2 &= U_2 = \frac{\partial}{\partial u}, \\
 e_3 &= U_3 = \frac{\partial}{\partial t}, & e_4 &= U_4 = S \frac{\partial}{\partial u}.
 \end{aligned} \tag{12}$$

Patera and Winternitz [17] looked for classifications of the sub-algebras into equivalence classes under their group of inner automorphisms. They also used the idea of normalization which guarantees that the constructed optimal system of subalgebras is unique up to the isomorphisms.

This classification allows us to divide the invariant solutions into non-intersecting equivalence classes. In this way it is possible to find the complete set of essential different invariant solutions to the equation under consideration. We use this classification and give a list of all non-conjugate one-, two- and three-dimensional subalgebras. The optimal normalized system of subalgebras to the algebra L_4 is listed in Table 1.

In Table 1 we use the operators e_1, e_2, e_3, e_4 given by (12) if $r = 0$ and by (11) if $r \neq 0$.

In correspondence with the set of subalgebras listed in Table 1, we obtain the complete set of invariant functions and reduce equation (4) to different ODEs using these functions as dependent and independent variables.

3. Group-invariant reductions provided by the one-dimensional symmetry subgroups in the case $r \neq 0$

In this chapter we study the symmetry reductions of the RAPM model (4) which we obtain using one of the one-dimensional symmetry subgroups $H_i, i = 1, \dots, 4$. These symmetry subgroups $H_i \subset G_4$ are generated by the corresponding subalgebras $h_i, i = 1, \dots, 4$ listed in Table 1 by a usual exponential map. We skip the study of invariant reductions to the two and three dimensional subgroups listed in Table 1 because they only give trivial results for the RAPM model.

Case H_1 . This one-dimensional subgroup H_1 is generated by the subalgebra

$$h_1 = \langle e_2 \rangle = \langle e^{rt} \frac{\partial}{\partial u} \rangle .$$

It describes a gauge (or evolutionary) symmetry of the equation. It means that to each solution to equation (4) we can add a term αe^{rt} , where α is arbitrary constant. The new function $u(t, S) \rightarrow u(t, S) + \alpha e^{rt}$ is then still a solution to the equation. This symmetry does not give rise to any invariant reductions of equation (4).

Case H_2 . We look for the invariants of the subalgebra $h_2 = \langle e_3 \cos(\phi) + e_4 \sin(\phi) \rangle$. In the variables (t, S, u) we obtain that h_2 has the form

$$h_2 = \langle \cos(\phi) \frac{\partial}{\partial t} + r \cos(\phi) S \frac{\partial}{\partial S} + (\cos(\phi) r u + \sin(\phi) S) \frac{\partial}{\partial u} \rangle . \quad (13)$$

The invariants z, w of the corresponding subgroup $H_2 \subset G_4$ can be chosen in the form

$$z = S e^{-rt}, \quad w = \frac{u}{S} - \frac{\tau}{r} \ln S, \quad r \neq 0, \quad \tau = \tan(\phi), \quad \phi \in [0, \pi], \quad \phi \neq \pi/2. \quad (14)$$

We take the invariants z, w as the new independent and dependent variables, respectively, then the PDE (4) is reduced to the ordinary differential equation of the following form

$$(\tau + rz(zw_{zz} + 2w_z)) \left(1 - \mu r^{-\frac{1}{3}} (\tau + rz(zw_{zz} + 2w_z))^{\frac{1}{3}} \right) + \frac{2r\tau}{\sigma^2} = 0, \quad (15)$$

$$r \neq 0, \quad \tau = \tan(\phi), \quad \phi \in [0, \pi], \quad \phi \neq \pi/2.$$

This second order differential equation can be reduced to a first order equation by the substitution $w_z(z) = v(z)$ which has the form

$$(\tau + r(z^2 v)_z) \left(1 - \mu r^{-\frac{1}{3}} (\tau + r(z^2 v)_z)^{\frac{1}{3}} \right) + \frac{2r\tau}{\sigma^2} = 0. \quad (16)$$

From this equation it follows that the expression $(z^2 v)_z$ is a constant. If we denote $(\tau + r(z^2 v)_z)^{1/3} = p(z)$, then for the value $p(z)$ we obtain an algebraic equation of the fourth order

$$p^3 \left(1 - \mu r^{-\frac{1}{3}} p \right) + \frac{2r\tau}{\sigma^2} = 0. \quad (17)$$

This equation has four roots $q_i, i = 1, \dots, 4$. In dependence on the values of the constants μ and τ some of these roots are real. We denote the real roots by k_i . To find solutions to the ODE (15) we have just to integrate two simple first order differential equations

$$\tau + r(z^2 v)_z = k_i^3, \quad w_z(z) = v(z). \quad (18)$$

Then to each root k_i the corresponding solutions to equation (15) are given as two parametric families of functions

$$u(S, t) = \frac{k_i^3}{r} S \ln S - (k_i^3 - \tau) t S + c_1 S + c_2 e^{rt}, \quad (19)$$

where $c_1, c_2 \in \mathbb{R}, r \neq 0, \tau = \tan(\phi), \phi \in [0, \pi], \phi \neq \pi/2$.

Case H_3 . The subalgebra h_3 is spanned by the generator $e_1 + a(e_3 \cos(\phi) + e_4 \sin(\phi))$. In the variables (t, S, u) it means that we have to do with the subalgebra of the form

$$h_3 = \left\langle (1 + a \cos(\phi)) \frac{\partial}{\partial t} + ((r - 1) + ar \cos(\phi)) S \frac{\partial}{\partial S} + \right. \tag{20}$$

$$\left. ((r - 1)u + a(\cos(\phi)ru + \sin(\phi)S)) \frac{\partial}{\partial u} \right\rangle .$$

The two first invariants of the corresponding subgroup H_3 are given by z, w which are connected to variables (t, S, u) by

$$z = Se^{-(r+\gamma)t}, \quad u(S, t) = Sw(z) + \zeta S \log S, \tag{21}$$

where the constants are $\gamma = (1 + a \cos(\phi))^{-1}, \zeta = \frac{a \sin(\phi)}{r(1+a \cos(\phi)) - 1}, a \in \mathbb{R}, \phi \in [0, \pi]$. Using these expressions we reduce the RAPM equation to an ordinary differential equation of the form

$$\frac{\sigma^2}{2} (z(zw)_{zz} + \zeta) \left(1 - \mu(z(zw)_{zz} + \zeta)^{\frac{1}{3}} \right) + r\zeta - \gamma zw_z = 0. \tag{22}$$

The solutions to this equation can be given in the parametric form

$$z(\theta) = \exp \left(\int \frac{d\theta}{k_i(\theta)^3 - \theta - \zeta} \right),$$

$$w(\theta) = \int \frac{\theta d\theta}{k_i(\theta)^3 - \theta - \zeta}, \tag{23}$$

where $\theta \in \mathbb{R}$ is a parameter and $q_i(\theta)$ is one of the real roots of the fourth order algebraic equation

$$\frac{\sigma^2}{2} k_i(\theta)^3 (1 - \mu k_i(\theta)) + r\zeta - \gamma\theta = 0. \tag{24}$$

Case H_4 . The subalgebra h_4 is spanned by the generator $e_2 + a(e_3 \cos(\phi) + e_4 \sin(\phi))$. In terms of the variables (t, S, u) it means that we are dealing with the subalgebra of the form

$$h_4 = \left\langle \epsilon \cos(\phi) \frac{\partial}{\partial t} + \epsilon r \cos(\phi) S \frac{\partial}{\partial S} + (e^{rt} + \epsilon(\cos(\phi)ru + \sin(\phi)S)) \frac{\partial}{\partial u} \right\rangle . \tag{25}$$

The invariants of the corresponding subgroup H_4 are z and w , where

$$z = Se^{-rt}, \quad u(S, t) = Sw(z) + \left(\frac{\tau}{r} + \frac{\epsilon}{r \cos(\phi)} z^{-1} \right) S \log S, \tag{26}$$

with $\tau = \tan(\phi)$, $\phi \in [0, \pi]$, $\phi \neq \pi/2$ and $\epsilon = \pm 1$. We take these invariants as new invariant variables and reduce equation (4) to an ODE of the following form

$$\frac{\sigma^2}{2} \left(z(zw)_{zz} + \frac{\tau}{r} + \frac{\epsilon}{rz \cos(\phi)} \right) \left(1 - \mu \left(z(zw)_{zz} + \frac{\tau}{r} + \frac{\epsilon}{rz \cos(\phi)} \right)^{\frac{1}{3}} \right) + \tau + \frac{\epsilon}{z \cos(\phi)} = 0. \tag{27}$$

If we denote $p(z) = \left(z(zw)_{zz} + \frac{\tau}{r} + \frac{\epsilon}{rz \cos(\phi)} \right)^{\frac{1}{3}}$ then for the value $p(z)$ we obtain an algebraic equation of the fourth order

$$p^3(z) (1 - \mu p(z)) + \frac{2\tau}{\sigma^2} + \frac{2\epsilon}{z\sigma^2 \cos(\phi)} = 0. \tag{28}$$

This equation has four roots which we denote $q_i, i = 1, \dots, 4$ as in the case H_2 .

Remark. The roots q_i in this equation differ from the roots of equation (24) or (17). Still, we denote here (and later) all real roots of a fourth order algebraic equation by k_i to show the similar structure of solutions.

Then to each root $k_i(z)$ the corresponding solutions to equation (4) are given as two-parametric families of functions

$$u(S, t) = e^{rt} \int \left(\int \frac{k_i(z)^3}{z} dz \right) dz + S(\tau t + c_1) + e^{rt} \left(\frac{\epsilon}{\cos(\phi)} t + c_2 \right), \tag{29}$$

where $\tau = \tan(\phi)$, $z = Se^{-rt}$, $\phi \in [0, \pi]$, $\phi \neq \pi/2$, $c_1, c_2 \in \mathbb{R}$ and $\epsilon = \pm 1$.

The special case of invariant solutions.

In some cases it is more rewarding not to take one of the classical representatives listed in Table 1 of the non-conjugated subalgebras but rather turn to an equivalent one which gives us a simpler ODE. Let us take a one-dimensional subalgebra of the form $h = \langle e_1 + \alpha e_2 \rangle$, where e_1, e_2 are defined by (11). The invariants of the corresponding subgroup H are defined by the infinitesimal generator

$$U = e_1 + \alpha e_2 = (r - 1)U_1 + U_3 + \alpha U_2, \tag{30}$$

and can be chosen in the form

$$z = Se^{-(r-1)t}, \quad w = u(S, t)e^{-(r-1)t} - \alpha e^t. \tag{31}$$

Remark. In the case $r = 1$ the dependence of the invariants z, w on t will be trivial. It means then that $z = S$ is an invariant and $w = u + \alpha e^t$. On the other hand, the value $r = 1$ implies that on the market 100 per cent interest rates are accepted. This is certainly a case which can not be modeled with the RAPM model. We can, therefore, exclude the case $r = 1$.

We use these invariant functions z and w to reduce the original equation (4) to

the ODE of the form

$$-w + zw_z + \frac{1}{2}\sigma^2 z^2 w_{zz} (1 - \mu(zw_{zz})^{1/3}) = 0. \tag{32}$$

It is easy to see that this equation does not depend on the arbitrary parameter α which is included in (30). The second order ODE (32) can be reduced to a first order one

$$v_z - \mu v_z^{4/3} = -\frac{2v}{\sigma^2 z} \tag{33}$$

by the substitution

$$v(z, w) = zw_z - w. \tag{34}$$

Equation (33) has a parametric solution. We obtain this solution in the following way. We rewrite equation (33) in the form

$$v(z) = -\frac{\sigma^2}{2} z (v_z - \mu v_z^{4/3}) = G(z, v_z), \tag{35}$$

then the parametric solution to this equation is given by the solution to the system of equations

$$v(\theta) = G(z(\theta), \theta), \quad z_\theta = \frac{G_\theta(z, \theta)}{\theta - G_z(z, \theta)} = -\frac{\sigma^2}{2} \frac{z \left(1 - \frac{4}{3}\mu\theta^{\frac{1}{3}}\right)}{\theta \left(1 - \frac{\sigma^2}{2} \left(1 - \mu\theta^{\frac{1}{3}}\right)\right)}, \tag{36}$$

where $\theta \in \mathbb{R}$ is a parameter. The system (36) and correspondingly equation (33) have the following solution

$$v(\theta) = -\frac{\sigma^2}{2} z(\theta)(\theta - \mu\theta^{4/3}), \quad z(\theta) = c_1 \left(1 - \frac{\sigma^2}{2} \left(1 - \mu\theta^{\frac{1}{3}}\right)\right)^{1+3\gamma} \theta^{-\frac{\sigma^2}{2}\gamma}, \tag{37}$$

where $\gamma = \left(1 - \frac{\sigma^2}{2}\right)^{-1}$ and $c_1 = \text{const}$. Using the parametric solution (37) to (33) we obtain the parametric solution to (32). We used the substitution (34) which now takes the form

$$v(\theta) = z(\theta)w_z - w = (\ln z(\theta))_\theta^{-1} w_\theta - w. \tag{38}$$

This is a linear first order differential equation for the function $w(t)$ and together with the parametric representation of $z(\theta)$ (37) the solution to this equation gives us the parametric solution to (32)

$$w(\theta) = z(\theta) (c_2 + g(\theta)), \quad c_2 = \text{const}, \tag{39}$$

where the function $g(\theta)$ is given by

$$g(\theta) = \frac{\sigma^2}{2\mu^2}\theta^{\frac{1}{3}} \left(-4 + \frac{\sigma^2}{2} \left(5 + 2\mu\theta^{\frac{1}{3}} \right) - \frac{\sigma^4}{4} \left(1 + \frac{\mu}{2}\theta^{\frac{1}{3}} + \frac{4}{3}\mu^2\theta^{\frac{2}{3}} \right) \right. \\ \left. - \frac{\mu^2\sigma^6}{8}\theta^{\frac{2}{3}} \left(1 - \mu\theta^{\frac{1}{3}} \right) \right) + \left(\frac{\mu\sigma^2}{2} \right)^{-3} \left(4 - \frac{\sigma^2}{2} \right) \left(1 - \frac{\sigma^2}{2} \right)^2 \ln \left(1 - \frac{\sigma^2}{2} \left(1 - \mu\theta^{\frac{1}{3}} \right) \right).$$

Expressions (39) and (37) give a parametric representation of a solution $w(z)$ to equation (32).

4. Group-invariant reductions provided by one-dimensional symmetry subgroups in the case $r = 0$

We repeat the procedure of constructing the invariant solutions to the RAPM model in the case $r = 0$. The general structure of the optimal system of subalgebras is the same in both cases but the form of infinitesimal generators differ. The invariants and the reductions therefore take another forms.

Case H_1^0 . The generator of the subalgebra h_1^0 has a very simple form $e_2 = \frac{\partial}{\partial u}$ in the case $r = 0$. This means that we are dealing with a subgroup of translations in the u -direction. Hence, to each solution to equation (4) with $r = 0$, we can add an arbitrary constant without destroying the property of the function to be a solution.

This subgroup does not provide any reduction.

Case H_2^0 . The subalgebra h_2^0 has the form $h_2^0 = \langle e_3 \cos(\phi) + e_4 \sin(\phi) \rangle$, it means that in terms of the variables (t, S, u) we have the subalgebra of the following type

$$h_2^0 = \langle \cos(\phi) \frac{\partial}{\partial t} + \sin(\phi) S \frac{\partial}{\partial u} \rangle. \tag{40}$$

The invariants of the subgroup H_2^0 are given by

$$z = S, \quad w = u(S, t) - \tau t S, \quad \tau = \tan(\phi), \quad \phi \in [0, \pi], \quad \phi \neq \pi/2. \tag{41}$$

If we use the variables z, w as new independent and dependent variables we obtain the following reduction of the RAPM model (4) with $r = 0$

$$\frac{\sigma^2}{2} z w_{zz} \left(1 - \mu (z w_{zz})^{\frac{1}{3}} \right) + \tau = 0, \quad \tau = \tan(\phi), \quad \phi \in [0, \pi], \quad \phi \neq \pi/2. \tag{42}$$

We denote $(z w_{zz})^{\frac{1}{3}} = p(z)$ and obtain for the value $p(z)$ an algebraic fourth order equation

$$p^3 (1 - \mu p) + \frac{2\tau}{\sigma^2} = 0. \tag{43}$$

As before we denote the real roots of this equation by k_i . To find solutions to the ODE (42) we have just to integrate twice

$$z w_{zz} = k_i^3. \tag{44}$$

Then the corresponding solutions to equation (42) are given by

$$u(S, t) = k_i^3 S (\ln S - 1) + \tau t S + c_1 S + c_2, \tag{45}$$

where $\tau = \tan(\phi)$, $c_1, c_2 \in \mathbb{R}, \phi \in [0, \pi], \phi \neq \pi/2$.

Case H_3^0 . The subalgebra h_3^0 for $r = 0$ has the form

$$h_3^0 = \langle a \cos(\phi) \frac{\partial}{\partial t} - S \frac{\partial}{\partial S} + (a \sin(\phi) S - u) \frac{\partial}{\partial u} \rangle, \tag{46}$$

where $a \in \mathbb{R}, \phi \in [0, \pi]$ are parameters. The invariants z, w of the group H_3^0 are given by the expressions

$$z = S e^{\delta t}, \quad u(S, t) = S w(z) + \zeta S \log S,$$

where the parameters are defined as

$$\delta = (a \cos(\phi))^{-1}, \quad \zeta = a \sin(\phi), \quad a \in \mathbb{R}, \quad a \neq 0, \quad \phi \in [0, \pi], \quad \phi \neq \pi/2, \tag{47}$$

and the reduced equation takes the form

$$\frac{\sigma^2}{2} (z(zw)_{zz} + \zeta) \left(1 - \mu(z(zw)_{zz} + \zeta)^{\frac{1}{3}} \right) + \delta z w_z = 0. \tag{48}$$

The solutions to this equation can be represented in the parametric form (23), where $k_i(v)$ is one of the real roots of the equation

$$\frac{\sigma^2}{2} k_i(v)^3 (1 - \mu k_i(v)) + \delta v = 0, \tag{49}$$

and the parameter δ is defined in (47).

Case H_4^0 . The subalgebra h_4^0 for $r = 0$ has the form

$$h_4^0 = \langle \epsilon \cos(\phi) \frac{\partial}{\partial t} + (1 + \epsilon \sin(\phi) S) \frac{\partial}{\partial u} \rangle, \tag{50}$$

where $\epsilon = \pm 1, \phi \in [0, \pi]$ are parameters.

The invariants z, w of this subgroup H_4^0 are given by the expressions

$$z = S, \quad w(z) = u(S, t) - \tau t S - \frac{\epsilon t}{\cos(\phi)}, \quad \tau = \tan(\phi), \tag{51}$$

and the RAPM model is reduced to the ODE of the form

$$\frac{\sigma^2}{2} z^2 w_{zz} \left(1 - \mu (z w_{zz})^{\frac{1}{3}} \right) + \tau z + \frac{\epsilon}{\cos(\phi)} = 0, \tag{52}$$

where $\tau = \tan(\phi), \phi \in [0, \pi], \phi \neq \pi/2, \epsilon = \pm 1$. The structure of equation (52) is very similar to previous cases and we can use similar tools to solve it. We first

substitute $(zw_{zz})^{1/3} = p(z)$. Then for the function $p(z)$ we obtain a fourth order algebraic equation but now its coefficients depend on the variable z

$$p(z)^3 (1 - \mu p(z)) + \frac{2\tau}{\sigma^2} + \frac{2\epsilon}{z\sigma^2 \cos(\phi)} = 0, \tag{53}$$

where $\tau = \tan(\phi)$, $\phi \in [0, \pi]$, $\phi \neq \pi/2$, $\epsilon = \pm 1$. For each real root $k_i(z)$ of this equation we have then to solve a linear ODE

$$z w_{zz} = k_i(z)^3. \tag{54}$$

The corresponding invariant solutions to (4) then have the form

$$u(t, S) = \int \left(\int \frac{k_i(S)^3}{S} dS \right) dS + \tan(\phi) t S + \frac{\epsilon}{\cos(\phi)} + c_1 S + c_2, \tag{55}$$

where $c_1, c_2 \in \mathbb{R}$, $\phi \in [0, \pi]$, $\phi \neq \pi/2$, $\epsilon = \pm 1$.

The expressions for these solutions are rather lengthy and because of which they are omitted here.

5. Conclusion

In the previous sections we found the complete series of invariant reductions of the RAPM model. In each of these cases the partial differential equation (4) is reduced to an ordinary differential equation. Using the optimal system of subalgebras (Table 1) we are able to present the complete set of the non-equivalent reductions of equation (4) up to the transformations of the group G_4 . The reductions and the corresponding invariant solutions are presented in section 3 for $r \neq 0$ and in section 4 for $r = 0$. In both cases we obtain three non-trivial reductions to ODEs. In all six cases it is possible to solve these ODEs and to obtain the explicit or parametric representations of exact invariant solutions to the RAPM model. We deal with the very seldom case that we can compare structures of non-equivalent invariant solutions since they are given in explicit or parametric forms.

Each of these solutions contains two integration parameters and some free parameters connected with the corresponding subgroup. This reasonable set of parameters allows one to approximate a wide class of boundary conditions.

The RAPM model (4) possesses a non-trivial analytical and singular-perturbed algebraic structure. There exist rather few methods to study equations of such high complexity. An application of both analytical and numerical methods to singular-perturbed equations is a highly non-trivial task. The RAPM model was studied before in detail with numerical methods in [8] and in [19]. The authors of [8] derive a robust numerical scheme for solving equation (4) and perform extensive numerical testing of the model and compare the results to real market data. In [19] Ševčovič studies the free boundary problem for the RAPM model and provides a description of the early exercise boundary for American style Call options. Using the same numerical method he provides as well computational examples of the free boundary approximation for American style of Asian Call options with arithmetically average floating strike. He proposed a numerical method based on the finite difference approximation combined with an operator splitting technique for numerical approximation of the solution and computation of the free boundary condition position.

On the other hand the Lie group analysis of the RAPM model which we provide in this paper gives us a more general, alternative point of view on the structure of this equation. It opens the possibility to exploit the Lie algebraic structure of the equation and may be helpful to improve another methods of solution.

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