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THE THEOREMS OF INTERNATIONAL TRADE

WITH FACTOR MOBILITY

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1. Introduction

This paper addresses the relation between commodity trade and international
factor mobility in general terms. There are two motivations. The first is the
intrinsic importance of the subject relative to the limited attention it has thus
far received. Multi-commodity and multi-factor generalizations of the standard
factor proportions theory of international trade deal mostly with goods trade only
and ignore international mobility of factors of production (see Ethier (1984) for
a recent survey). Although there exists an extensive literature on various
aspects of international factor mobility (surveyed in Jones and Neary (1984)),
there are few attempts to discuss systematically the trade pattern in both goods
and factors when some factors are traded. Such a discussion is contained in
Svensson (1984), who extends previous work by Dixit and Woodland (1982) to trade
in factors as well as in goods. But Svensson's analysis deals only with marginal
factor endowment differences in the neighborhood of an autarky equilibrium. Thus
a more general treatment is needed.

Our second motivation is the dimensionality issue. As is well known, the
standard theorems of factor-endowments trade theory are very sensitive to whether

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1 We are grateful to Avinash Dixit and two anonymous referees for comments. A first version of this paper was written while Lars Svensson was visiting the NBER. We thank the NBER for its hospitality and for providing secretarial services. We are also grateful to and Lotten Bergström for typing and editorial assistance.
the number of goods equals the number of factors or not. This is widely regarded as very damaging to the theory, since the relative abundance of goods and factors is an arbitrary feature of nature and technology, and one about which most of us have limited intuition. Ethier (1984) presents a countervailing argument assigning factor mobility a key role. There are two parts to the argument. (i) Those traditional results which are otherwise quite general, with equal numbers of goods and factors, are weakened only slightly when goods outnumber factors but substantially when factors outnumber goods. Thus the key requirement is that there be at least as many goods as factors, not that they be precisely equal in number. (ii) The main reason dimensionality matters is not the technological distinction between goods and factors, but the assumption that the former are internationally traded while the latter are not. Thus the standard results are preserved when factors outnumber goods if enough factors are traded. This interpretation leaves the basic propositions sensitive mainly not to an arbitrary feature of nature but to whether enough markets exist -- to which most substantive results in economics are sensitive. In any event, this discussion implies a central role for factor mobility in an understanding of the significance of the basic propositions of factor endowments trade theory.

Our paper is organized as follows. Sections 2 and 3 set out the basic framework, and then Section 4 offers a full treatment of factor-price equalization, exposing the role of dimensionality. Section 5 derives some Rybczynski type and Stolper-Samuelson type results. The relation of commodity trade and factor trade to each other and to factor endowments is the topic of the Heckscher-Ohlin type results of Section 6, which thereby also extend the previous

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2 This point is stressed in Jones and Scheinkman (1977).

3 Earlier treatments of factor-price equalization and factor mobility may be found in Rodriguez (1975), Neary (1980), Svensson (1984), and Ethier (1984).

2. International Equilibrium

Consider a world of two countries, home and foreign. First we describe the home country. There are $M$ goods, indexed $i=1,\ldots, M$, all of which are traded. They are produced by $N$ factors, $N^f$, of which are traded and $N^n$ of which are not. Factor endowments are fixed.

There are $J$ (production) sectors, indexed $j=1,\ldots, J$. Each production sector is characterized by a convex technology $T^j$ of feasible combinations $(y^j,v^j)$ of net output $M$-vectors $y^j$ of goods and non-negative input $N$-vectors $v^j$ of factors. In particular, there are no externalities between sectors. (Indeed, sectors are defined as the finest partition of the overall production technology for which there are no intersectoral externalities). For the special case of no joint production, we can identify sectors with goods, and for sector $j$ all $(y^j,v^j)$ in $T^j$ will have $y^j_j > 0$, and $y^j_i = 0$ for all goods $i$ other than $j$.

For given goods prices $p$ and factor inputs $v^j$, the sector $j$ product function is defined as $\Sigma^j(p,v^j) = \max \{ p'y^j : (y^j,v^j) \in T^j \}$, the maximum value added obtainable, where $p'y^j$ denotes the inner product $\Sigma^j p'y^j_i$ (or, equivalently, the matrix product between the row vector $p'$ and the column vector $y^j$; we let all vectors without a prime be column vectors, and let a prime denote transpose). For a given domestic factor input $v$, the domestic product function is defined as $\tilde{G}(p,v) = \max \{ \Sigma^j G^j(p,v^j) : \Sigma^j v^j_j \leq v \}$, the maximum value of domestic output when factors are freely mobile between sectors. We let $v = (\tilde{K}',L')'$ denote home factor endowments, where the $N^f$-vector $\tilde{K}$ denotes ownership/endowments of traded factors and the $N^n$-vector $L$ denotes endowments of nontraded factors. For

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See Chang, Ethier and Kemp (1980) for a discussion of how joint production affects the basic theorems of international trade in goods.
simplicity we shall call these "capital" and "labor", respectively. Let the \( \mathbf{N}_T \)-vector \( \mathbf{K} \) denote capital input in production (as distinct from capital endowments) in the home country, and let the \( \mathbf{N}_T \)-vector \( \mathbf{r} \) denote rentals, the price of capital. Then, for given goods prices and rentals, and given factor endowments, we define the national product function \( G(p,r,v) \) as

\[
G(p,r,\tilde{K},L) = \tilde{G}(p,K(p,r,L),L) + r'(\tilde{K} - K(p,r,L)).
\]

Here the capital input vector \( K(p,r,L) \) is a solution to \( \max \{ G(p,K,L) + r'(\tilde{K}-K) : K > 0 \} \), which then fulfills\(^5\)

\[
\tilde{G}_K(p,K(p,r,L),L) = r.
\]

Hence we assume each sector behaves competitively, and takes goods prices and rentals as given. The first term on the right-hand side of (1) is domestic product from the use of factors \( (K,L) \) at home, and the second term is factor income from abroad, due to the net export of capital \( (\tilde{K}-K) \), the difference between endowments and domestic input of capital.

Assume that the demand side of the home country can be represented by a standard \( M \)-vector demand function \( D(p,l) \), where \( l \) is national income. We define net export of goods, \( x \), and of capital, \( z \), as

\[
x(p,r,v) = y - c \quad \text{and} \quad z = \tilde{K} - K,
\]

the difference between output \( y \) and consumption (\( M \)-vector) \( c \), and between capital endowments and capital input. By standard properties of the national product function, the net export of goods and capital will be given by the functions\(^6\)

\[
x(p,r,v) = G_p(p,r,v) - D(p,G(p,r,v)) \quad \text{and}
\]

\[
z(p,r,v) = G_r(p,r,v) = \tilde{K} - K(p,r,L).
\]

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\(^5\) We shall let subindices denote (the vector of) partial derivatives, throughout. We disregard corner solutions.

\(^6\) We assume that these functions exist and are differentiable. However \( G_p \) and \( G_r \) -- and therefore \( x \) and \( z \) -- may not be uniquely defined (for example, if the technology possesses constant returns to scale and there are at least as many goods as factors). We address this below when it becomes relevant.
Furthermore, the prices of nontraded factors, wages $w$, will be given by

\[ w = g_L(p,r,v). \]  

The foreign country has factor endowments $v^* = (K^*,L^*)$ and analogous national product and demand functions, which give rise to net export functions for goods and capital, denoted by $x^*(p,r,v^*)$ and $z^*(p,r,v^*)$. A world equilibrium will satisfy

\[ x(p,r,v) + x^*(p,r,v^*) = 0 \]  

and

\[ z(p,r,v) + z^*(p,r,v^*) = 0, \]

that is, both goods and traded-factor markets are in equilibrium.

3. Cost Functions

Before investigating properties of the international equilibrium, we will describe some concepts and notations to be used subsequently. In particular, we now introduce the unit value added cost function.

Let the $N$-vector $W = (r',w')$ denote the prices of traded and nontraded factors. For each sector $j$, the unit value added cost function $c^j(p,W)$ is defined as

\[ c^j(p,W) = \min \{ W'v^j : p'y^j = 1, (y^j, v^j) \in T^j \}. \]

This cost function gives, for given goods prices (the $M$-vector $p$) and factor prices (the $N$-vector $W$), the minimum value of inputs for which value added, the value of the net output vector, is equal to unity.\(^7\) Let $Y^j$ denote value added in sector $j$; let $\tilde{Y} = (Y^j)$ be the corresponding $J$-vector; and let $C(p,W)$, the national (unit-value) cost (vector) function, be the vector whose components are the cost functions for the sectors with positive value added, and let $Y$ be the

\[ ^7 \text{See Woodland (1977).} \]

\[ ^8 \text{Alternatively, one can define unit "net revenue" cost functions according to } \tilde{c}^j(p,r,w) = \min \{ w'L^j : p'y^j - r'K^j = 1, (y^j,K^j,L^j) \in T^j \}. \text{ For our purpose unit (goods) value added functions are equally practical.} \]
vector of the positive components of \( \tilde{Y} \). That is,\(^9\) \( J(Y) = \{j: j \in J, y^j > 0\} \) (the set of sectors with positive value added) and \( C(p,w) = (c^j(p,w))_{j \in J(Y)} \). (We suppress the argument \( Y \), or \( J(Y) \), of the national cost function). The national cost function is homogeneous of degree zero in goods prices and factor prices.

Then, by Euler's theorem
\[
(10) \quad p'C_p + w'C_w = 0,
\]
where \( p'C_p \) denotes post-multiplication of the (row) \( N \)-vector \( p' \) by the \( (N \times J(Y)) \)-matrix \( C_p = [\delta c^j/\delta p^1] \), etc.

Finally, recall that the price derivatives of the cost function, in equilibrium, are conditional unit value added input functions.\(^\text{10}\) Then we can write the home countries' output, capital input and labor input as
\[
(11) \quad y = -C_p Y, \quad K = C_r Y \text{ and } L = C_w Y,
\]
where \( C_p' \) denotes post-multiplication of the matrix \( C_p = [\delta c^j/\delta p^1] \) by the (column) vector \( Y \), etc. We are now equipped for the basic tasks of this paper.

4. Factor Price Equalization

We first determine when free trade in goods and capital will internationally equalize the rewards of nontraded factors. To this end we now assume that the two countries have identical technology with constant returns to scale. Then \( G(p,r,v) \) is linearly homogeneous in \( v \), \( K(p,r,L) \) is linearly homogeneous in \( L \), and the relations apply to both countries.

\(^9\) We let \( J \) and \( J(Y) \) denote both sets of sectors and the number of sectors in each set.

\(^\text{10}\) As is well known, the production equilibrium can either be described with a domestic product function \( G(p,K,L) \) or with a national cost vector function with the zero profit condition \( C(p,r,w) = 1 \) and (11). See Woodland (1977) for properties of the unit value cost function.
We know that the standard factor-price equalization arguments can accommodate intermediate goods and joint production,\textsuperscript{11} so it should come as no surprise that international factor movements also cause no problems. For this reason we do not present a complete detailed proof (which would necessarily consist largely of slight modifications of well known arguments) but instead rely on the following indirect argument.\textsuperscript{12}

Imagine the existence of $N_T$ additional sectors, each of which uses one unit of one of the traded factors as its sole input and the service of that factor as its sole output. Then we can imagine the factors themselves as immobile, but their services as traded goods, so that we have a model of $M + N_T$ 'goods' and $N$ factors, all the latter 'nontraded'. Obviously this imaginary model is identical to our actual one, differing only in verbal description. We can then appeal to the standard analysis (with no international factor mobility) to conclude that, provided a dimensionality condition is fulfilled, to any equilibrium price system $(p, w)$ there will exist a $(M$ dimensional) diversification cone having the property that all countries with endowments in the cone can share the same factor prices $(r, w)$ if freely trading at the commodity prices $(p, r)$. The dimensionality condition is that the number of linearly independent sectors be at least as large as the number of factors: $J(Y) + N_T \geq M$. But this is equivalent to

\begin{equation}
J(Y) \geq N_T = \text{rank } \mathbf{C}_W.
\end{equation}

To understand the equality $N_T = \text{rank } \mathbf{C}_W$ in (12), we observe that the diversification cone referred to is in terms of all $M$ factors, but we are interested in one in terms of the $N_T$ nontraded factors only. So let us now define the generalized diversification cone $\Omega(p, r, w)$ as

\textsuperscript{11} See McKenzie (1955) and, also, Chang, Ethier and Kemp (1980).

\textsuperscript{12} We were persuaded to follow this strategy by the anonymous referee who suggested it.
\( \Omega(p, r, w) = \{L \geq 0: L = C_w Y, Y \geq 0\} \).

It is, for given goods and factor prices, the set of labor inputs consistent with non-negative value added in all sectors. The diversification cone is of dimension \( N_N \) if \( N_N = \text{rank } C_w \). Let us now assume that the left-hand side inequality in (12) indeed holds with equality. (It is not difficult to modify the following argument for the case \( J(v) > N_N \)). Then \( C_w \) is square and can be inverted, and we can use (11) to get
\[
y = -c_{p_w}^{-1}L, \quad \text{and } K = c_{r_w}^{-1}L.
\]

Then any equilibrium featuring free trade and factor mobility between both countries at prices \((p,r)\) can also feature equalization of the prices of nontraded factors at \( w \), if both \( L \) and \( L^* \) are in \( \Omega(p,r,w) \). For example, consider a different allocation between the countries of nontraded factors, \( L^1 \) and \( L^*^1 \), with \( L^1 + L^*^1 = L + L^* \), \( w^1 L^1 = w L \) and \( w^1 L^*^1 = w^* L^* \), having both \( L \) and \( L^* \) in the diversification cone. Is this consistent with a new equilibrium with unchanged goods and factor prices? If, in fact, these do not change, we have by (14)
\[
(y^1 + y^1^*) = -c_{p_w}^{-1}(L^1 + L^*^1) = -c_{p_w}^{-1}(L + L^*) = y + y^* \quad \text{and}
\]
\[
(K^1 + K^1) = c_{r_w}^{-1}(L^1 + L^*^1) = c_{r_w}^{-1}(L + L^*) = K + K^*.
\]

Thus world output of goods and world input of capital remain unchanged. Since world demand for goods and world endowments of capital are unchanged, world markets for goods and capital remain in equilibrium. It follows that there exists a new equilibrium with unchanged goods and factor prices and hence factor price equalization.

Are we sure that there do not still exist other equilibria, with the same prices \((p,r)\) for goods and traded factors, where factor price equalization does not obtain? If so, we have not in fact proved that factor price equalization will
take place if \( L \) and \( L^* \) are both in \( \Omega(p,r,w) \). To dispose of this possibility,\(^{13}\) and so establish factor price equalization, assume there exist wage vectors \( w^0 \) and \( w^1 \), with \( w^1 \neq w^0 \), and let \( L \) be in both \( \Omega(p,r,w^0) \) and \( \Omega(p,r,w^1) \). Then there exist \( y^0 \) and \( y^1 \) such that

\[
L = C^0_w y^0 \quad \text{and} \quad L = C^1_w y^1 ,
\]

where \( C^0_w \) and \( C^1_w \) denote \( C_w(p,r,w^0) \) and \( C_w(p,r,w^1) \). Furthermore, by zero homogeneity of the national cost function, we have

\[
-\frac{p}{r} C^0_w = r' C^0_w + w^0 C^1_w \quad \text{and} \quad -\frac{p}{r} C^1_w = r' C^1_w + w^1 C^1_w ,
\]

with obvious notation.

Recalling that the derivatives are conditional input demands, since the input demands minimize cost, we also have

\[
r' C^0_w + w^1 C^0_w \geq -\frac{p}{r} C^0_w = r' C^0_w + w^0 C^0_w ,
\]

with at least one inequality strict if \( w^1 \neq w^0 \). Thus \( w^1 C^0_w \geq w^0 C^0_w \), so

\[
(w^1 - w^0)' C^0_w y^0 = (w^1 - w^0)' L \geq 0 ,
\]

with strict inequality if \( w^1 \neq w^0 \). By a symmetric argument, we can show

\[
(w^0 - w^1)' L \geq 0 ,
\]

with strict inequality if \( w^0 \neq w^1 \). It follows that \( w^1 = w^0 \), contrary to what was assumed at first.

Hence, factor price equalization must obtain. The crucial condition is (12). If there is no joint production, this reduces to

\[
M + N_T \geq N .
\]

That is, the total number of international markets (goods and traded factors) must be at least as great as the number of factors. This indicates that at bottom factor price equalization depends not on an arbitrary aspect of nature (the

\[^{13}\text{See McKenzie (1955) for an analysis of the case of trade in goods only.}\]
relative numbers of goods and factors) but rather, just like most interesting propositions in economics, on the existence of enough markets. Note, however, that the number of markets which is sufficient depends upon the number of factors (so that reducing the number of goods and increasing the number of traded factors a like amount is not neutral: the required number of international markets rises).

Of course, with joint production, adding goods markets that do not increase the number of sectors is of no help. Also, what if some outputs are not traded? At this point we therefore say a word about nontraded goods. They should of course be allowed if commodities are to be treated analogously to factors. We exclude them for expository reasons, since they do not affect our basic argument. If included, conditions for equilibrium in nontraded goods markets would be solved for nontraded goods prices as functions of the present state variables, and these functions would simply be embodied in the form of the national product functions, and so forth. Then, provided proper care were taken in the use of the diversification-cone concept, the argument of this section would proceed essentially unchanged. In particular, (12) would remain the key dimensionality condition. Note that this indicates a way in which factors and goods differ: an additional nontraded factor makes (12) more stringent, but an additional nontraded good does not.

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14 For discussions of the new issues which nontraded goods introduce, and of the changes they imply for standard propositions, see Ethier (1972), Flam (1979), and Woodland (1982, Ch. 5).

15 For some issues, say those connected with purchasing power parity or real exchange rates, it may be of interest to discuss equalization of nontraded-goods prices as well. It is not difficult to derive a condition analogous to (12) that holds with joint production. If factor prices are equalized, a sufficient condition for equalization of nontraded goods prices is simply that the number of distinct sectors must be at least as large as the number of nontraded goods.
5. Comparative Statics

In factor-endowments models of commodity trade, the Rybczynski theorem describes the effects of endowment changes on outputs, and the Stolper-Samuelson theorem addresses the implications of commodity-price changes for factor rewards.\textsuperscript{16} Both propositions exploit the technological relation between goods and factors and do not depend upon whether factors are internationally traded or not (although the latter might help determine the circumstances under which the theorems can in fact be applied). For example, the Rybczynski theorem's description of how commodity outputs respond to changes in a nation's employment of factors is not sensitive to whether the latter changes are due to domestic factor accumulation or to the import of traded factors. Thus we need say little about the influence of factor trade on the standard propositions. Instead we focus on the new questions that arise.

There are three relevant aspects to the two theorems. First, they are linked together by the "reciprocity relations". Next, each of the theorems contains two assertions, one relating to magnitudes and one relating to directions. For example, the Stolper-Samuelson theorem asserts that commodity price changes produce unambiguous changes in real factor rewards, and also that relative factor intensities help to predict the direction of the latter. We examine in turn each of the three aspects.

(i) The reciprocity relations

The standard reciprocity relations follow from differentiation of the domestic product function. We have $\tilde{G}_{vp} = \tilde{G}_{vp}'$, where again a prime denotes transpose. Hence, for good $i$ and factor $j$
\(\frac{\partial y^j}{\partial v^j} = (\partial/\partial v^j)(\partial G/\partial p^i) = (\partial/\partial p^i)(\partial G/\partial v^j) = \partial w^j/\partial p^i\)

These relations hold whenever the respective terms are well defined. They reflect only the technology (and the optimization implicit in the domestic product function) and they hold regardless of whether factor \(j\) is traded or nontraded. But the presence of traded factors introduces the questions of how changes in the endowments of nontraded factors influences a country's use of traded factors, and of how changes in the (international) price of traded factors affect the rewards of nontraded factors. We therefore derive an appropriate set of reciprocity relations. This is not difficult. For we have, using twice-differentiability of the national product function, \(G_{rL} = G_{rL'}\), and since, by (5) and (6),

\[w_r = G_{Lr}\text{ and } -K_r = G_{rL},\]

we get the desired reciprocity relation

\[-\partial k^i/\partial L^j = \partial w^j/\partial r^i,\]

for traded factor \(i\) and nontraded factor \(j\).

(ii) Magnitudes

It is well known that the following results hold under very general circumstances when \(M = N\) and there is no joint production: (a) an increase in the price of any good causes a more-than-proportional rise in some factor reward and a decline in some other factor reward; (b) at given commodity prices, an increase in the economy-wide use of any factor requires a more-than-proportional rise in the output of some good and an absolute fall in the output of some other good; (c) application of the reciprocity relations (22) to the "Stolper-Samuelson" result (a) yields further "Rybczynski" results and application to the "Rybczynski" result (b) yields further "Stolper-Samuelson" results. If there are more goods than
factors these results are almost completely preserved, but they are weakened substantially when the number of factors exceeds the number of goods.\textsuperscript{17}

The presence of traded factors has little effect on these propositions: the results follow whenever the conditions are met. But two points should be made. First, the above results are weakened when there are more factors than goods because endowment changes at constant prices then require changes in factor rewards for factor markets to clear. Thus the analysis of the previous section implies that, with traded factors, condition (21) replaces $M \geq N$ as the dimensionality requirement for the results to hold in full strength: there must be at least as many international markets as factors.

The second point concerns the relation between traded and nontraded factors. Suppose an exogenous rise in the reward of some traded factor, all other international prices remaining fixed. Then some factor reward must fall, else no sector would be able to earn non-negative profits at unchanged commodity prices. With the rewards of traded factors fixed, it must be a nontraded factor that becomes cheaper. Furthermore, this is a real decline since commodity prices have not changed. Thus each traded factor is an "enemy" to some nontraded factor. The reciprocity relations (35) then imply that the demand for any traded factor is reduced by a rise in the endowment of some single nontraded factor. Note that these results do not require condition (31) and that they are fully compatible with joint production.

(iii) Directions

In factor-endowments trade models, factor intensities predict, in an average sense, the response of factor rewards to commodity-price changes and the response of outputs to factor-endowment changes. For example, commodity-price changes are

\textsuperscript{17} See Ethier (1984) for details.
positively correlated with changes in the rewards of those factors used relatively most intensively (see Ethier (1982, 1984)). Such predictions are not sensitive to whether some factors are traded or not. But now we are interested instead in predicting the direction of change of commodity outputs and traded-factor usage jointly in response to changes in the endowments of nontraded factors, and also in predicting changes in nontraded factor rewards in response to changes in the vector of commodity prices and traded-factor rewards. We have
\[ y = G_p(p, r, v) \quad \text{and} \quad z = G_r(p, r, v) = \tilde{K} - K(p, r, L). \]
Define \( q = (y', z')' \) and \( \pi = (p', r')' \). Then we have
\[ q = G_\pi(\pi, \tilde{K}, L). \]
Consider the effects of an endowment change from \( L^0 \) to \( L^1 \), at given international prices \( \pi \). Let \( q^1 = G_\pi(\pi, \tilde{K}, L^1) \), \( q^0 = G_\pi(\pi, \tilde{K}, L^0) \) and define the real-valued function
\[ a(L) = (q^1 - q^0)'G_\pi(\pi, \tilde{K}, L). \]
By the mean-value theorem there exists an \( \bar{L} \) on the line segment connecting \( L^0 \) and \( L^1 \) such that
\[ a(L^1) - a(L^0) = a_{\bar{L}}(L^1 - L^0), \]
where
Substituting the definitions of these terms yields
\[ (q^1 - q^0)'(q^1 - q^0) = (q^1 - q^0)'R(L^1 - L^0) \geq 0, \]
where
\[ R = G_{\pi L}(\pi, \bar{L}) = \begin{bmatrix} G_{pL}(p, r, \bar{L}) \\
-K_r(p, r, \bar{L}) \end{bmatrix}. \]

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18 Except, of course, where the validity of Rybczynski-type correlations depends upon dimensionality, in which case condition (21) becomes relevant as discussed above.

19 We noted earlier that under some circumstances \( \tilde{G}_\pi \) will not be well defined. Whenever \( \tilde{G}_\pi \) can assume alternative values set it equal to one of them by some arbitrary method. The following arguments apply to all such methods that have \( \tilde{G}_\pi \) differentiable.
where we suppress $\tilde{K}$ since $G_{KL}$ is independent of it. Thus we can write

$$(26a) \quad (y^1 - y^0)'G_{PL}(L^1 - L^0) \geq 0 \quad \text{and}$$

$$(26b) \quad z'K_L(L^1 - L^0) \leq 0.$$ 

This is the general (directional) Rybczynski theorem for an economy with traded factors. Note that $G_{PL}$ and $K_L$ depend only upon technology. Using the term "relative factor intensities" to refer to the relative magnitudes of the elements of $G_{PL}$, correlation (26a) says that any change in the endowment of nontraded factors will, at constant prices of goods and traded factors, tend on average to raise the most the outputs of goods that are relatively most intensive in the use of those nontraded factors that have increased the most. Note that we look at the various commodities' intensity of use of nontraded factors only.

In relation (26b), $K_L$ is the matrix of partial derivatives (at the intermediate point) of traded factor inputs with respect to nontraded factor endowments.\(^{20}\) In Svensson (1984), two factors are said to be cooperative (non-cooperative) if the corresponding element of $K_L$ is positive (negative). Then (26b) has the natural interpretation that any change in the endowment of nontraded factors will, at constant prices of goods and traded factors, tend on average to increase the most the input and import of traded factors that are most cooperative with those nontraded factors that have increased the most.\(^{21}\) (Nontraded factors can be thought of as "employed" in the national usage of traded factors in the sense that, given commodity outputs, an increased endowment of nontraded factors allows the economy to reduce its usage of traded factors. This interpretation

\(^{20}\) We have $K_L = -G_{KK}^{-1}G_{KL}$ if the relevant derivatives exist. The elements of $G_{KL}$ consist of the effects of nontraded factors on the demand price of traded factors. With only one output and one traded factor they equal the cross derivative of the production function.

\(^{21}\) Relations (26a) and (26b) are generalizations of results in Svensson (1984).
makes it clear that the notion of "relative cooperation" used in this paragraph is strictly analogous to that of "relative factor intensity" as used in the preceding one.)

Our result (26) is extremely general in that it allows joint production, applies to arbitrary endowment changes, and is independent of the relative numbers of goods, traded factors, and nontraded factors. Thus the movement from \( L^0 \) to \( L^1 \) may or may not change \( w \). But for (26) to be more useful we need to explore the nature of the key matrix \( R \). In particular, we would like to be able to relate the matrix \( G_{PL} \) to direct factor inputs. To this end assume that there are the same number of goods as sectors. Then, if the matrix \( -C_p \) in (11) is of full rank, it can be inverted.\(^{22}\) Doing so, (11) becomes

\[
K = -C_r C_p^{-1} y = -C_r C_p^{-1} G_p \quad \text{and} \quad L = -C_w C_p^{-1} y = -C_w C_p^{-1} G_p .
\]

To proceed further we need further restrictions. Assume that condition (21) holds, so that our results from the previous section imply that a small deviation of \( L \) produces no change in \( w \). Then from (27)

\[
K_L = -C_r C_p^{-1} G_{PL} \quad \text{and} \quad I = -C_w C_p^{-1} G_{PL} ,
\]

where \( I \) is the identity matrix. If, furthermore, (21) holds with equality, \( C_w C_p^{-1} \) is square so that (assuming full rank) it can be inverted to yield

\[
G_{PL} = -C_p C_w^{-1} \quad \text{and} \quad K_L = C_r C_w^{-1} .
\]

Here \( -C_w C_p^{-1} \) is simply the matrix of direct nontraded factor requirements (at the intermediate point) for the respective goods. Hence, the elements of \( G_{PL} \) cannot be regarded as direct input coefficients - as utilized by the textbook version of the Rybczynski theorem - but we see that they are based on the same information.

\(^{22}\) If joint production is excluded, \( -C_p \) is the diagonal matrix whose diagonal elements are the inverse of the goods prices and hence can each be made equal to unity by the proper choice of units of measurement of goods.
If the inverse $C_w C_r^{-1}$ (equals $[-C_w C_p^{-1}][-C_r C_p^{-1}]^{-1}$) of the matrix $C_r C_w^{-1}$ exists (which at least requires as many traded factors as nontraded), the negative of it can be interpreted as denoting nontraded factor "requirements" for "producing" (decreasing input of) the respective traded factor (service), in analogy with the direct nontraded factor input matrix for the respective goods.

Continuing to suppose that (21) holds -- but now not necessarily with equality -- suppose that the hypothetical endowment change leaves the economy within the original generalized diversification cone. Thus no change in $w$ will take place and the production techniques will likewise be unaltered. In this case the matrices $G_{pL}$ and $K_L$ can simply be calculated from the observed techniques.

Note also that (27) now, under the assumption of no change in $w$, gives

\begin{align}
(28a) \\ (L^1 - L^0)'[-C_w C_p^{-1}](y^1 - y^0) = (L^1 - L^0)'(L^1 - L^0) > 0 \quad \text{and} \\
(28b) \\ (K^1 - K^0)'[-C_r C_p^{-1}](y^1 - y^0) = (K^1 - K^0)'(K^1 - K^0) > 0.
\end{align}

Inequality (28a) relates output and nontraded factor endowments in terms of the direct nontraded-factor input requirements. Inequality (28b) relates traded-factor inputs and goods outputs in terms of the direct traded-factor input requirements, and hence is only an indirect relation between traded-factor inputs and nontraded factor endowments.

If we assume that the matrix $C_w C_r^{-1}$ exists, we can derive the relation

\[(L^1 - L^0)[C_w C_r^{-1}](K^1 - K^0) > 0\]

in terms of the direct nontraded-factor "requirements" in "producing" (decreasing input of) for the respective traded factor (service).

We also show a Stolper-Samuelson analogue to (26). Consider the effect on nontraded factor prices $w$ of a change from goods and traded factor prices $\pi^0 = (p^0, r^0)'$ to $\pi^1 = (p^1, r^1)'$. Starting from $w(\pi, v) = G_L(p, r, L)$, let $w^1 = w(\pi^1, L)$
and \( w^0 = w(\pi^0, L) \), and define \( b(\pi) = (w^1 - w^0)'G_L(\pi, L) \). By the mean-value theorem, there exist a \( \tilde{\pi} \) on the line segment between \( \pi^0 \) and \( \pi^1 \) such that

\[
b(\pi^1) - b(\pi^0) = b_{\tilde{\pi}}(\pi^1 - \pi^0).
\]

Substituting the definitions of these terms, we have the desired analogue to (26),

\[
\begin{align*}
(w^1 - w^0)'G_{LP}(p^1 - p^0) & \geq 0 \quad \text{and} \\
(w^1 - w^0)'G_{LR}(r^1 - r^0) & \geq 0.
\end{align*}
\]

Here we should note that these matrices, in spite of appearance, are not the same as in (26), since the intermediate points \( \tilde{\pi} \) and \( \tilde{L} \) are chosen independently.

6. Patterns of Trade in Goods and Factors

In this section we develop versions, appropriate in the presence of factor trade, of the Heckscher-Ohlin theorem.

(i) The price version of the Heckscher-Ohlin theorem

Assume that the number of goods equals the number of sectors and that \( C_p \) is of full rank so that (10) implies

\[
p' = r'[-C_r C_p^{-1}] + w'[-C_w C_p^{-1}].
\]

Furthermore, assume no joint production. Then the right-hand side above depends on factor prices only and indeed the right-hand side is identical to the (row) vector of the standard unit-cost functions. Then we can write

\[
\begin{align*}
\pi' = W'B(W) \quad \text{(30a)}
\end{align*}
\]

where

\[
B(W) = \begin{bmatrix}
-C_r C_p^{-1} & I \\
r_p & 0 \\
-C_w C_p^{-1} & 0
\end{bmatrix}.
\]

Here \(-C_r C_p^{-1}\) and \(-C_w C_p^{-1}\) are the direct capital input and labor input matrices in goods production, respectively. Let us note that we can interpret (30) in the following way, as we did when we discussed factor price equalization. We extend
the set of outputs by assuming that there are \( N_T \) additional sectors, each of which uses one unit of that factor as its sole input and the service of that factor as its sole output. Then we can interpret \([-C_{W}^{-1},0]\) as the direct capital input matrix for this extended set of outputs, where the identity matrix \( I \) is the direct capital input matrix in the \( N_T \) additional sectors. The matrix \([-C_{W}^{-1},0]\) is the direct labor input matrix for the extended set of outputs.

We follow the method used in Ethier (1982) and now define the real-valued function \( d(W) \) by
\[
d(W) = W'B(W)(x',z')'
\]
where \((x',z')\) denotes the actual free-trade vector of net exports of goods and factors. Let \( W^A \) and \( W^*A \) denote autarky factor prices at home and abroad. Then, for some \( \tilde{W} \) between \( W^A \) and \( W^*A \),
\[
d(W^A) - d(W^*A) = (W^A - W^*A)'d_{W}(\tilde{W}), \text{ or}
\]
\[
(x^A - x^*A)'(x',z')' = (W^A - W^*A)'D(\tilde{W})(x',z')'.
\]
Here we have used \( d_{W}(\tilde{W}) = D(W)(x',z')' \), since, by cost minimization, all terms \( \sum_{j} W^j \partial d_{Wj}(\tilde{W})/\partial W^i \) are zero. That the left-hand side of (31) is nonpositive follows directly from the generalized law of comparative advantage. Then, substituting for the right-hand side gives
\[
(r^A - r^*A)'[-C_{W}^{-1}]x + (W^A - W^*A)'[-C_{W}^{-1}]x + (r^A - r^*A)'z < 0,
\]
where we recall that the direct capital input and labor input matrices \(-C_{W}^{-1}\) and \(-C_{W}^{-1}\) are evaluated at \( \tilde{W} \). The inequality (32) is a weak result, relative to the goal of predicting something about \((x,z)\) solely from \((W^A - W^*A)\). Now we need to know \((r^A - r^*A)\) as well, that is, we do not use a concept of relative factor.

23 Such generalized results have been obtained by Deardorff (1980) and Dixit and Norman (1980), and surveyed in Ethier (1984). Since the theorems do not suppose international factor immobility they directly apply to our model.
abundance limited to nontraded factors only. The problem is that the latter do not in general give enough information if there are also traded factors.

In terms of the interpretation with the extended set of outputs referred to above, it is not surprising that we need to know \( r^A - r^*A \) as well as \( w^A - w^*A \), since then there are \( N = N_T + N_N \) "nontraded" factors and \( M + N_T \) traded "goods".

(ii) The quantity version of the Heckscher-Ohlin theorem

In factor-endowments trade theory, the quantity version of the Heckscher-Ohlin theorem is related to the Rybczynski theorem. We now wish to use our versions of the latter to develop explanations of the pattern of joint trade in goods and factors. To that end, we now define the vectors \( x^P \) and \( x^I \) of home country "preference trade" and "induced trade", respectively, as follows

\[
x^P = g(y^1 + y^*) - c \quad \text{and} \quad x^I = x - x^P,
\]

where \( c \) denotes the home consumption vector, \( g \) the home-country share of the value of world output, and \( y^1 \) and \( y^* \) home and foreign outputs respectively. Note that if the home and foreign countries share identical homothetic tastes, \( x^P = 0 \) and \( x = x^I \). Define \( L^0 = g(L^1 + L^*) \), where \( L^1 \) and \( L^* \) denote endowments of nontraded factors at home and abroad. Then, if \( p \) and \( r \) equal the free trade equilibrium prices, we have from (26)

\[
(y^1 - y^0)'G_{pL}(L^1 - L^0) > 0 \quad \text{and} \quad (K^1 - K^0)'K_{L}(L^1 - L^0) > 0,
\]

where \( y^i = G_p(p,r,K,L^i) = G_p(p,K^i,L^i) \) and \( K^i = K(p,r,L^i) \) for \( i = 0,1 \). Now we let \( Y^1, Y^* \) and \( Y^0 = g(Y^1 + Y^*) \) denote the respective vectors of value added. Using (11) we have \( y^0 = -C_p y^0 = -gC_p Y'_p - gC_p Y'_p = g(y^1 + y^0) \), and hence

\[
y^1 - y^0 = [y^1 - c] + [c - g(y^1 + y^*)] = x - x^P = x^I.
\]

Also, since similarly \( K^0 = C_r Y^0 = gC_r Y'_r + gC_r Y'_r \),

\[
K^1 - K^0 = [K^1 - K] + [K - g(K + K^*)] = -z + z. \quad E
\]
Here $z$ denotes the vector of home-country capital exports and $z^E$ denotes the vector each component of which shows the excess of home ownership of the respective traded factor above the fraction $g$ of the world supply. Let $z^I$ be defined by

$$z^I = z - z^E.$$ 

Thus we have

$$(34a) \quad x^I G_{pL} [L^L - g(L^L + L^*)] \geq 0 \text{ and}$$

$$(34b) \quad z^I K[L^L - g(L^L + L^*)] \leq 0.$$ 

This is the generalized quantity version of the Heckscher-Ohlin theorem. Relation $(34a)$ says that the induced net exports of goods are positively correlated with the country's relative abundance of the nontraded factors utilized relatively intensively. Equation $(33)$ decomposes total factor trade $z$ into what we call "endowment factor trade" $z^E$ and "induced factor trade" $z^I$. The endowment trade component is the direct result of the extent to which the country's ownership of traded factors is not proportional to the world supply of these factors. This trade is independent of factor intensities (and technology generally), of tastes, of prices, and of endowments of nontraded factors, except in the indirect sense that all these help to determine the factor of proportionality $g$, which depends upon the actual equilibrium. By contrast, induced factor trade $z^I$, which alone enters into $(34b)$, is determined by factor intensities and by endowments of nontraded factors. That is, it must be such that the net induced factor import ($-z^I$) is on average higher (lower) for traded factors that are cooperative (non-cooperative) with the nontraded factors that are abundant relative to world factor proportions. Ownership of traded factors has no influence (except, again, if it helps determine the general equilibrium). Broadly speaking, countries conduct factor-endowment trade directly to export abundant traded factors and to import scarce traded factors, and they conduct induced factor trade indirectly to export
abundant nontraded factors and to import scarce nontraded factors — in the form of those traded factors which are non-cooperative with (i.e. best substitute for) them in the production process. Both components of total factor trade thus have a factor-endowment base.

Note the analogy between the two types of goods trade and the two types of factor trade. Induced goods trade, like induced factor trade, is determined by technology and relative nontraded factor endowments as summarized in (34). Preference (goods) trade reflects taste differences just as endowment (factor) trade reflects ownership differences. If tastes are identical and homothetic, and if the home and foreign countries own traded factors in identical proportions, (34) describes all trade. It is common to use the term "demand reversal" to describe a situation where taste differences cause the opposite pattern of commodity trade (in a 2x2 world) to that predicted on the basis of relative factor endowments. We can analogously use the term "demand-endowment reversal" to refer to the case where $x^P$ and $z^E$ are such that $x'G_{pL}[L^1 - g(L^1 + L^s)] < 0$ and $z'K_{L}[L^1 - g(L^1 + L^s)] < 0$.

The basic result (34) allows joint production, is valid no matter how great the difference between home and foreign endowments, requires no restrictions on dimensionality, and imposes no additional restrictions on technology (such as ruling out higher dimensional analogs of factor-intensity reversals). To proceed further, suppose that the number of sectors equals the number of goods, that there are at least as many goods and traded factors as there are factors, and that in the free trade equilibrium the home and foreign endowments of nontraded factors lie in a common generalized diversification cone (so that there is factor price equalization). If we now apply to (28a) the same logic that we applied to (26) in order to derive (34) we obtain

$$[L^1 - g(L^1 + L^s)]'[-C_wC_p^{-1}]x^I > 0.$$
Thus a country will on average export those goods which make relatively intensive use of the country's relatively abundant nontraded factors. (This of course applies only to induced trade. We describe as a "demand reversal" the case where substitution of $x$ for $x^I$ in (35) reverses the direction of the inequality).

Note two aspects of this result. First, the concept of relative factor intensity employed here is the most natural one: simply the relative sizes of the direct nontraded-factor input requirements of the techniques actually in use. Second, note that only the endowments of nontraded factors, and only that part of the input matrix pertaining to nontraded factors, enter into (35). Changes in the ownership of traded factors, or technological changes which do not influence the use of nontraded factors, will produce no effect. That is, (35) establishes a sense in which the pattern of induced commodity trade, on average, cannot be reversed by such changes, even if the trade vector $x^I$ changes. But such a reversal could result from a change in the traded status of a factor,\(^24\) since such a change would add or subtract from the matrix in (35).

If we presume that there are as many traded factors as nontraded and that the matrix of nontraded factor "requirements" for the respective traded factor, $-C_w^{-1}C_r$, is well-defined, we can derive the analogue to (35),

\begin{equation}
[L^1 - g(L^1 + L^*)]'[-C_w^{-1}C_r]z^I \geq 0,
\end{equation}

It can be interpreted as stating that a country will on average export those traded factor services that make relatively intensive use of the country's relatively abundant factors.

Expression (35) can be interpreted, in the customary way, as saying that commodity trade substitutes for the exchange of nontraded factors. A similar relation can, surprisingly, be established between (induced) goods trade and

\(^{24}\) An example may be found in Svensson (1984).
actual (induced) factor trade. Using (28b) in the same way that we have used (28a) and (26) leads to

\[(37) \quad z^I [-C \cdot C^{-1}] x^I < 0.\]

This says that a country on average exports those goods that make relatively intensive use of those factors that are imported in excess of endowment trade. That is, induced trade in goods and the induced exchange of traded factors are, in a sense, substitutes. Note that, again, the relevant concept of factor intensity is the relative size of the direct traded-factor inputs of the techniques actually used. Also, the part of the technology pertaining to nontraded factors does not enter at all.

Indeed, under present assumptions, this basic substitution property can be made more exact. We first note that the well-known Travis-Vanek theorem extends to encompass factor trade in a straightforward way. Induced goods trade would enable the home country to consume the fraction \( g \) of the world output of each good. Factor price equalization then implies that these consumed goods would embody the services of the fraction \( g \) of the world’s stock of each factor, traded or nontraded. Let

\[x^L = -C \cdot C^{-1} x^I \quad \text{and} \quad x^K = -C \cdot C^{-1} x^I\]

denote the nontraded-factor and traded-factor content of induced goods trade. Then from the definition of \( x^I \) it follows

\[x^L = L^1 - g(L^1 + L^2) \quad \text{and} \quad x^K = (K^1 - z) - g(K^1 + K^2),\]

by our generalized Travis-Vanek theorem. Thus

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25 Note that we are referring to the structure of commodity trade and not to its aggregate size. The volume of goods trade may be either larger or smaller in the presence of factor trade than it would be without such trade. This alternative notion of substitutability/complementarity is analyzed in Markusen (1983) and Svensson (1984).

26 See Vanek (1968).
(38) \[ x^K = K^1 - g(K^1 + K^8) - z = z^E - z = -z^I. \]

That is, **induced factor trade is equal to and opposite in sign to the traded-factor content of induced goods trade.**

### 7. Summary and Conclusions

Factor price equalization for nontraded factors, with trade in goods and some factors, results if the number of distinct sectors with positive value added is at least as large as the number of nontraded factors and if each country's endowment of nontraded factors is in the generalized diversification cone. If there is no joint production, this requires the number of goods and traded factors to be at least as large as the total number of factors, that is, the number of international markets should be at least as large as the number of factors. The introduction of nontraded goods does not change this result, as long as we do not require nontraded goods price equalization.

Factor price equalization does not per se depend on the (rather arbitrary) relative number of goods and factors but instead on the (less arbitrary) relative number of international markets and factors.

Next we examined the Rybczynski and Stolper-Samuelson theorems. The usual reciprocity relations hold between the effect on output of factor input variations (at constant goods prices) and the effect on factor prices of goods price variations (at constant factor input), independently of whether some factors are traded or not. We also derived additional reciprocity relations, between the effect on traded factor inputs of nontraded factor input variations (at constant goods and traded factor prices) and the effect on nontraded factor prices of variations in traded factor prices (at constant nontraded factor input).

The "magnitude" aspects of the Rybczynski and Stolper-Samuelson theorems require the conditions for nontraded factor price equalization to hold in full
strength, that is, there should be at least as many international markets as factors. Even without nontraded-factor price equalization, each traded factor is an "enemy" to some nontraded factor, in that the nontraded factor's price must fall if the traded factor's price increases. By the reciprocity relations, the demand for any traded factor is reduced by the endowment of some single nontraded factor.

We derived a general (directional) Rybczynski theorem: any change in the endowment of nontraded factors will, at constant goods and traded factor prices, tend on the average to raise the most the output of those goods that use relatively intensively those nontraded factors which increase the most. Here, "relative intensity" is defined from the sign pattern of the Rybczynski matrix of the national product function. With regard to traded factors, the input is on the average raised the most of those traded factors that are cooperative with those nontraded factors which increase the most. Here, "cooperation" is defined from the sign pattern of the matrix of traded factor input derivatives with respect to nontraded factors. This result is extremely general in that it allows joint production, applies to arbitrary endowment changes, and is independent of the relative numbers of goods, traded factors, and nontraded factors.

If the conditions for factor price equalization hold, the above theorem holds for relative intensity defined from the direct nontraded factor input coefficients for goods production.

We also derived a (directional) Stolper-Samuelson theorem that a change in goods and traded-factor prices on average increases the most the prices of those nontraded factors used relatively intensively by the goods and being cooperative with traded factors whose prices increase the most.

Finally, we looked at patterns of trade in goods and factors. The price version of the Heckscher-Ohlin theorem is rather weak. The quantity version fares
better. Decomposing goods trade into "preference trade" and "induced trade", with the former reflecting taste differences, we derived a quantity version of the Heckscher-Ohlin theorem: the induced net export of goods is positively correlated with a country's relative abundance of the nontraded factors used relatively intensively. Decomposing factor trade into "endowment trade" and "induced trade", with the former due to the country's endowment of traded factors not being proportional to world endowments of traded factors, induced factor trade is positively correlated with the relative abundance of nontraded factors that are relatively cooperative in production. This result is again very general, allows for joint production, arbitrary differences between home and foreign endowments, and requires no restriction on dimensionality. Assuming nontraded factor price equalization, the theorem holds for relative intensities defined by direct nontraded-factor input coefficients. We also demonstrated that in general induced goods trade and induced factor trade are substitutes, and we established the precise sense in which this is so. The Travis-Vanek theorem was extended to factor trade, under nontraded-factor price equalization.

The basic theorems of international trade, suitably interpreted, hold with both goods trade and factor trade. In particular, with at least as many distinct sectors as goods, the crucial dimensionality condition under which the theorems hold in their strong versions is that the number of international markets for goods and traded factors be at least as large as the total number of factors. Hence, the crucial issue is not the relative number of goods and factors per se but rather the number of markets. Of the central propositions, only the price version of the Heckscher-Ohlin theorem fails to be essentially preserved by this condition.
References


