Frequency-Domain Continuous-Time AR Modeling Using Non-Uniformly Sampled Measurements

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Abstract

A frequency domain approach to continuous-time auto regressive (AR) signal modeling is proposed. The algorithm allows for data pre-filtering as opposed to conventional AR modelling in the time domain. We illustrate the method by extracting resonance frequencies from data from a real-life application.

Keywords: continuous-time, AR model, pneumatic tire, frequency-Domain, resonance Frequency

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1 Introduction

A characteristic problem with signal processing using tachometer measurements on rotating axles is that the measurements are uniform in the angle domain but non-uniform (speed dependent) in the time domain. This comes from the fact that most common sensors for such applications measure the time between certain angle displacements, which is thus speed dependent. One can for instance illustrate this with the ABS sensors in a car, which give between 50 and 100 pulses per revolution for each wheel. If vibration analysis and other similar problems are to be approached in the time sampled domain, one either has to rely on data interpolation to uniform time sampling or derive dedicated algorithms. Motivated by the recent advances in system identification in the frequency domain [1, 2], we present a frequency domain approach and compare it at a theoretical level to the time domain algorithm proposed in [3, 4].

The main specifications on a procedure aimed at high-sensitivity vibration analysis are as follows:

1. Being based on parametric physical models of the vibration.
2. Operate on short data batches in a pre-specified speed interval where the data pass several quality checks.
3. Potential to reject wide band disturbances that are non-interfering with the vibration.
4. Potential to reject narrow band disturbances that are interfering with the vibration.

The method given in [3, 4] successfully solves the first three specifications, but not the last one. The method here on the other hand can easily be modified for robustness to narrow band interference. This general problem occurs in several applications, such as in an automotive drive-line where the vibration indicates engine knocks, or in robotics where vibrations come from the load, just to mention a few. The text will however focus on Point 1, 2 and 3 in the list above. There will only be a short discussion of time and frequency domain methods with respect to Point 4.

2 Time and frequency domain algorithms

2.1 Notation

Table 1 summarizes the notation and signal model that are used in the time and frequency domain. Basically, vibration analysis is approached by a continuous-time autoregressive (CAR) model motivated by a spring-damper model of the axel and its contact paths with the surrounding. Superimposed on this signal are other vibrations and external disturbances and the speed signal itself. Measurements are taken each time t_k as a pulse is received from the ABS sensor. These pulses represent a certain fixed angle displacement, which explains the special appearance of $y[k] = y(t_k)$ in Table 1.

Time domain	Frequency domain
$$y(t) = \frac{1}{A(p)} e(t) + d(t) \quad (1)$$ $$y[k] = y(t_k) = \frac{2\pi k}{L} + \int_{t_{k-1}}^{t_k} d(t)dt \quad (2)$$	$$\Phi_y(i\omega) = \Phi_H(i\omega)\Phi_e(i\omega) + \Phi_d(i\omega) \quad (3)$$ $$\Phi_H(i\omega) = \frac{\sigma^2}{\|A(i\omega)\|^2} \quad (4)$$
$e(t)$ white noise, $A(p)$ AR model, θ parameters in the AR model, $d(t)$ disturbance, $y[k]$ measured non-uniform samples of angle, L number of cogs per revolution, Angle uniform sampling, not time uniform sampling.	$\Phi_e(i\omega) = 1$ white noise spectrum, $A(i\omega)$ AR model, θ parameters in the AR model and noise, $\Phi_d(i\omega)$ disturbance spectrum, $\Phi_y(i\omega)$ 'measured' spectrum.

Table 1: Signal models and assumption in time and frequency domains, respectively.

2.2 Algorithms

The time domain algorithm proposed in [3, 4] contains the following steps: (1) interpolate data to a a high sampling rate to avoid aliasing, (2) band-pass filter the signal to get rid of broad band disturbances and to focus on mode 2 in Table 3, (3) down-convert the signal utilizing deliberate aliasing, (4) estimate a discrete time AR model and (5) extract vibration data from this model. It is not easy to modify this algorithm to cope with narrow-band interference, so the only practical solution is to turn off the algorithm when such a disturbance is detected.

Table 2 outlines the proposed frequency domain algorithm. Here, the narrow band disturbances can be interpreted as outliers in the estimate of the spectrum. Time domain outliers have traditionally been dealt with by introducing more robust norms in the estimation criteria e.g [5]. It is our opinion that that this approach will be transferrable to the case of frequency domain data. In this text, on the other hand, the focus will be on rejection of wide band disturbances and short data batches. Managing these two issues is a necessary condition for the overall usefulness of the method in the context of vibration analysis.

1. Approximate the truncated continuous-time Fourier transform with $$\hat{Y}(i\omega_k) = \int_0^T \hat{y}(t)e^{i\omega_k t}dt, \quad (5)$$ $$\hat{y}(t) = \sum_{i=1}^{N} y(t_i)\phi_i(t-t_i), \quad (6)$$ $$w_k = \frac{2\pi}{T}k, \quad k \in \mathcal{N}. \quad (7)$$	2. Average the periodogram $\hat{\hat{\Phi}}_y(i\omega) = \left\|\hat{Y}(i\omega)\right\|^2$ over batches with similar vehicle speed. 3. Maximum likelihood estimate the CAR-model $$\hat{a} = \arg\min_a \sum_{k\in\mathcal{N}} \frac{\hat{\hat{\Phi}}_y(i\omega_k)}{\Phi_H(i\omega_k,\theta)} + \log\Phi_H(i\omega_k,\theta)$$ $$\Phi_H(i\omega_k,\theta) = \frac{\sigma^2}{\|(i\omega_k)^2 + 2\gamma i\omega_k + \omega_0^2\|^2}$$

Table 2: Frequency-Domain algorithm.

3 Frequency Domain Estimation

Let us define the truncated Fourier transformation of the continuous time output $\{y(t) : t \in [0,T]\}$ in expression (1) in Table 1 above as

$$Y_T(i\omega) = \frac{1}{\sqrt{T}}\int_0^T y(t)e^{-i\omega t}dt.$$

A complicating element in a practical estimation procedure is that we do not have access to the entire continuous time realization of the output. Instead we have, as pointed out in expression (2) in Table 1, a finite number of samples of the continuous output y_t at time instances $\{t_1, t_2, \ldots, t_N\}$. Therefore it is in some way necessary to approximate or reconstruct the continuous time realization. In this paper the output is reconstructed as

$$\hat{y}(t) = \sum_{i=1}^{N} y(t_i)\phi_i(t-t_i)$$

where ϕ_i are interpolation kernels. In this text we will use piecewise-constant interpolation, which will often go under the name Zero-Order Hold (ZOH).

From the interpolated output it is possible to compute an approximation of the Fourier transform which is

$$\hat{Y}_T(i\omega) = \frac{1}{\sqrt{T}} \sum_{k=1}^{N-1} y(t_k) \frac{e^{-i\omega t_{k-1}} - e^{-i\omega t_k}}{i\omega}$$

in the piecewise constant case. From the expressions above we can then calculate the approximate periodogram which is denoted as

$$\hat{\hat{\Phi}}_y(i\omega) = \left| \hat{Y}_T(i\omega) \right|^2$$

In order to reduce the variance of the periodogram, the data batch is split into N_b sub-batches of duration T_n, $n = 1, \ldots, N_b$. Then a periodogram $\hat{\hat{\Phi}}_y^{(n)}$ is calculated for each batch and an estimate is formed as a direct average

$$\hat{\hat{\Phi}}_y(i\omega) = \frac{1}{N_b - 1} \sum_{n=1}^{N_b} \hat{\hat{\Phi}}_y^{(n)}(i\omega).$$

This method is analogous to the method by Welch [6] for the smoothing of spectral estimates.

When an estimate of the power spectrum is available a CAR model can be identified by solving the following Maximum-Likelihood (ML) procedure described in [7] and [8]

$$\hat{\theta} = \arg\min_{\theta} \sum_{k \in \mathcal{N}} \frac{\hat{\hat{\Phi}}_y(i\omega_k)}{\Phi_H(i\omega_k, \theta)} + \log \Phi_H(i\omega_k, \theta).$$

Here Φ_H is defined as in (4) in Table 1. The frequencies $\omega_k,\ k = 1, \ldots, N_\omega$ where $\omega_k = 2\pi k/T,\ k \in \mathcal{N}$ have been selected such that the Fourier transforms of the output at different frequencies are asymptotically uncorrelated. The index set $\mathcal{N}$ denotes those frequencies we wish to use in the estimation procedure.

4 Properties of Bias and Variance

The bias or disturbances present in the periodogram will translate into bias in the parameter estimates. Therefore it is important to a user of the method to know how these relate to each other in order to minimize bias. He/she would also like to tune the estimation procedure in order to minimize the variance of the parameter estimates. In [7] and [8] it is possible to show that in the case of bias

$$E(\hat{\theta} - \theta_0) \approx \sum_{k \in \mathcal{N}} S(i\omega_k) \Delta\Phi_y(i\omega_k).$$

where $\hat{\theta}$ are the estimated parameters and θ_0 are the true parameter values. The *relative bias* in the periodogram estimate of the power spectrum is defined as

$$\Delta\Phi(i\omega_k) = \frac{E\hat{\hat{\Phi}}_y(i\omega_k) - \Phi(i\omega_k, \theta_0)}{\Phi(i\omega_k, \theta_0)}$$

Here we have for the sake of simplicity defined $\Phi = \Phi_H$. The *sensitivity* of the parameter estimates to the relative bias in the periodogram is

$$S(i\omega_k) = \Psi(\theta_0, \Phi)^{-1}\Psi_k(\theta_0, \Phi).$$

The so called *relative* sensitivity is defined as

$$\Psi_k(\theta_0, \Phi) = \frac{\Phi'_\theta(i\omega_k, \theta_0)}{\Phi(i\omega_k, \theta_0)}.$$

and

$$\Psi(\theta_0, \Phi) = \sum_{k \in \mathcal{N}} \Psi_k(\theta_0, \Phi)\Psi_k(\theta_0, \Phi)^T$$

Non-interfering disturbances occur in areas where the model spectrum Φ_H is small and the relative bias can therefore be quite large. Hence in order to avoid parameter bias it is necessary to ignore information from frequencies where the relative bias and sensitivity are large.

In an online automotive application, computational power and available memory will always pose important design constraints. Calculating the periodogram can be cumbersome and it is important to know which frequencies carry the most information. In the case of variance it is shown in [7] and [8] that

$$E(\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)^T \to \Psi(\theta_0, \Phi)^{-1}.$$

as $T \to \infty$ and the largest sampling interval goes to zero. Again the relative sensitivity plays an important role. In order to reduce the variance information from frequencies where the relative sensitivity is large should be prioritized.

5 Experimental Results

In this section the theory presented above is applied to the estimation of the resonance peak of the torsional vibrations of a pneumatic tire. The samples $y(t_k)$ are pre-processed measurements from an axel angle measurement device. The frequency spectrum of $y(t)$ is approximately divided as summarized in Table 3. The vibrations in the range 30-60 Hertz can be modelled as a spring-damper

0-10	10-15	15-30	30-60	60-80	80-100	100–
Speed	Mode 1	Noise	Mode 2	Noise	Mode 3	Noise
Narrow-band noise components						

Table 3: Frequency spectrum with approximate limits in Hz

system excited by white noise $e(t)$

$$y(t) = H(p)e(t)$$

with transfer operator

$$H(p) = \frac{\sigma}{p^2 + 2\gamma p + \omega_0^2}.$$

The output will therefore have the continuous-time spectrum

$$\Phi_H(i\omega) = \frac{\sigma^2}{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}. \tag{8}$$

with a resonance peak located at the frequency

$$\omega_{res} = \sqrt{\omega_0^2 - 2\gamma^2}$$

For the special parameterization of the spectrum in expression (8) the relative sensitivity functions for the respective parameters are shown in Figure 1. Here we have chosen $\gamma = 33.88$ and $\omega_0 = 289.687$. This means that $w_{res} = 285\ rad/s$ or $f_{res} = 45.47Hz$. From this figure, we conclude that γ is sensitive near the

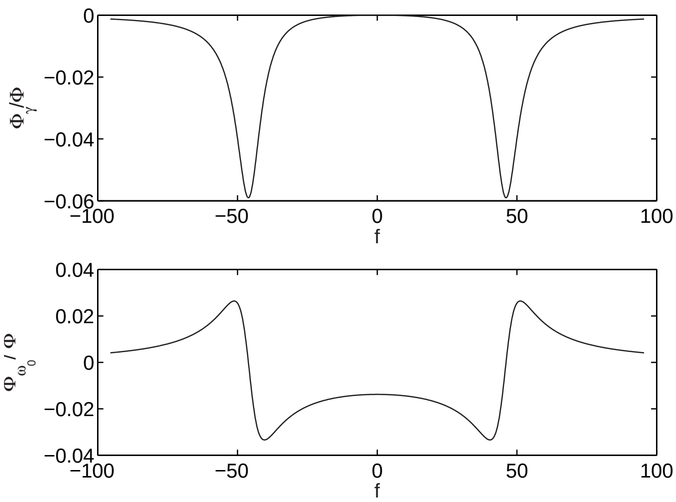

Figure 1: Relative sensitivities for γ (upper) and ω_0 (lower).

natural resonance frequency of the system. The frequency ω_0 on the other hand is particularly sensitive at low frequencies. According to Table 3 there is noise between 15 and 30 Hertz and 60 and 80 Hertz. Therefore we restrict the frequencies used to those between 30 and 60 Hertz.

In Figure 2 we have estimated the resonance frequencies from the refined set of real life data from an ABS sensor. The data have been divided into four parts. These parts have then been subdivided into a set of batches with a duration of a certain number of revolutions/laps of the tire. The number of laps per sub-batch is indicated on the x-axis of the figure. Periodograms have been estimated using ZOH for each sub-batch and subsequently averaged in order to yield four estimates of the spectrum of each batch. The mean value and standard deviation are then plotted. The figure indicates that the method is feasible and that a batch size of about 10 laps is sufficient to yield a stable estimate of the resonance frequency with moderate variance. Below 10 laps per sub-batch the mean value of parameter estimates decrease rapidly due to leakage bias from the short observation time T_n of each sub-batch.

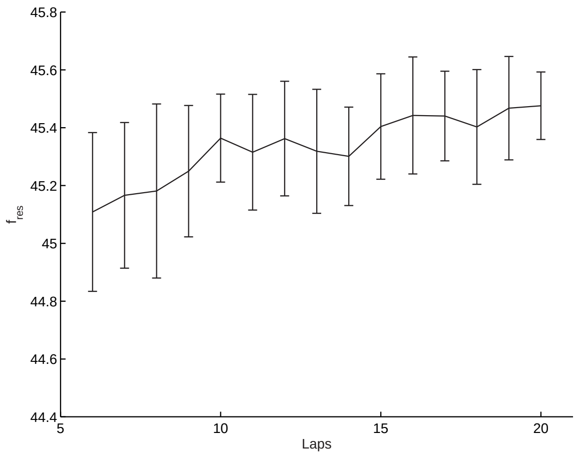

Figure 2: Resonance frequency f_{res} in Hertz versus batch size in number of tire laps. Bars indicates one standard deviation.

6 Conclusions

In this paper a frequency-domain alternative to the estimation of axel vibrations has been outlined. The method has proved to be feasible on real-life data. It is also thought to be easily extended in order to reject narrow band disturbances interfering with the vibration. An algorithm which is more robust to outliers could be acquired if the probability density function in the ML method is changed to a more robust one[5]. This would be a natural objective of future work.

References

[1] R. Pintelon and J. Schoukens, *System Identification - A Frequency Domain Approach*, IEEE Press, Piscataway, NJ, 2001.

[2] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, Upper Saddle River, 1999.

[3] N. Persson, "Event based sampling with applicaion to spectral estimation," Lic. Dissertation Linköping Studies in Science and Technology Thesis No. 981, Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden, 2002.

[4] N. Persson and F. Gustafsson, "Event based sampling with application to vibration analysis in pneumatic tires," in *Proceedings of IEEE Int Conf on Acoustics, Speech and Signal Processing in Salt Lake City, USA*, 2001.

[5] P. Huber, *Robust Statistics*, John Wiley & Sons, 1981.

[6] P.D Welch, "The use of the fast fourier transform for the estimaton of power spectra: A method based on time averaging over short modified periodograms.," *IEEE Trans. Audio Electroacoust.*, vol. AU-15, pp. 70–73, 1967.

[7] J. Gillberg and L. Ljung, "Frequency-domain identification of continuous-time arma models: Interpolation and non-uniform sampling," Tech. Rep. LiTH-ISY-R-2625, Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden, September 2004.

[8] J. Gillberg, "Methods for frequency domain estimation of continuous-time models," Lic. Dissertation Linköping Studies in Science and Technology Thesis No. 1133, Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden, 2004.

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