Asymptotic methods applied to some oceanography-related problems
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Doctoral thesis
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Abstract

In this thesis a number of issues related to oceanographic problems have been dealt with on the basis of applying asymptotic methods.

The first study focused on the tidal generation of internal waves, a process which is quantified by the conversion rates. These have traditionally been calculated by using the WKB approximation. However, the systematic imprecision of this theory for the lowest modes as well as turbulence at the seabed level affect the results. To handle these anomalies we introduced another asymptotic technique, homogenization theory, which led to significant improvements, especially for the lowest modes.

The second study dealt with the dynamical aspects of a nonlinear oscillator which can be interpreted as a variant of the classical two-box models used in oceanography. The system is constituted by two connected vessels containing a fluid characterised by a nonlinear equation of state and a large volume differences between the vessels is prescribed. It is recognised that the system, when performing relaxation oscillations, exhibits almost-discontinuous jumps between the two branches of the slow manifold of the problem. The lowest-order analysis yielded reasonable correspondence with the numerical results.

The third study is an extension of the lowest-order approximation of the relaxation oscillations undertaken in the previous paper. A Mandelstam condition is imposed on the system by assuming that the total heat content of the system is conserved during the discontinuous jumps.

In the fourth study an asymptotic analysis is carried out to examine the oscillatory behaviour of the thermal oscillator. It is found that the analytically determined corrections to the zeroth-order analysis yield overall satisfying results even for comparatively large values of the vessel-volume ratio.
Je dédie cette thèse de doctorat à mes très chers parents ; Abbès et Martine.

À toi Papa, mon meilleur ami, mon idole, mon héros, mon conseiller, mon enseignant, toi mon Sid, toi qui m’as initié aux raisonnements abstraits, toi qui m’as appris à éviter le contentement de soi et à explorer ce qui se trouve au-delà de la barrière imposée et réductrice, toi qui as su m’alerter contre la douceur griseante du succès, toi qui m’as inculqué l’idée de la persévérance face aux pentes qui peuvent parfois être raides.

À toi Maman, ma protectrice, ma tendre amie, mon inconditionnel soutien, toi la battante, toi celle qui m’as initié à la lecture, toi qui as répondu et réponds à tant de mes questions, toi qui as su faire épanouir en moi cette curiosité naturelle que chaque enfant possède, toi qui m’as permis malgré ta crainte maternelle d’errer et rêvasser sur les rochers et les plages de La Pointe Pescade à Alger, toi qui es solidement ancrée auprès de Papa dans ces moments pénibles.
List of papers

This thesis consists of an introduction and the following papers:


**Paper III:** M. Zarroug, 2009: Improving the lowest-order analysis of a nonlinear convective oscillator by applying a Mandelstam condition, *MISU report.*

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Chapter 1

Introduction

The strongly stratified Scandinavian waters are highly conducive for internal waves, and in 1907 tidally generated motion of this type was discovered during an investigation of the Great Belt conducted from the double-anchored Swedish research vessel Skagerrak. The internal waves occurred on the interface between the brackish surface-water outflow from the Baltic and waters of higher salinity originating from the Kattegatt. The field survey was led by Otto Pettersson of Stockholms Högskola (the predecessor of Stockholm University), one of the founding fathers of physical oceanography in Sweden. Pettersson’s curiosity was aroused by these observations, which he followed up by constructing an internal-wave rider mounted below the head of the observation pier belonging to the research station at Bornö on the west coast of Sweden, cf. the sketch in Fig. 1.1. This work was undertaken in collaboration with the Bornö chief hydrographer Nils Zeilon, who later summarized their findings in two ground-breaking papers [Zeilon, 1912, 1914]. Here it also deserves mention that Zeilon continued to pursue this line of research, as not least manifested by his frequently quoted laboratory experiments [Zeilon, 1934].

The primary drawback with the Pettersson-Zeilon instrument was that it was stationary in that it had to be mounted on land adjacent to a water column of considerable depth. This limitation was overcome by Börje Kullenberg, who designed and built a mobile internal wave rider in the early 1930s [Kullenberg, 1932]. Fig. 1.2 shows this
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Figure 1.1: An artistic impression of the Petersson-Zeilon internal-wave rider suspended beneath the hanging bridge at Bornö research station in Bohuslän.

bottom-moored instrument being deployed during an early survey in the Kattegatt, the results of which were later reported in a classical study [Kullenberg 1935] showing the presence of the semi-diurnal $M_2$ and $S_2$ tides as well as inertial oscillations.

Ever since these pioneering internal-wave studies were conducted more than seventy years ago, research in this field has continued, and presently the generation and subsequent breaking of internal waves is regarded as being mainly responsible for oceanic diapycnal mixing and turbulent dissipation. Stommel and Arons [1960] formulated one of the first theories showing the global consequences of the dissipative processes, which were assumed to be responsible for the upwelling. Although more sophisticated models dealing with the global oceanic circulation have been presented since then, the mixing associated with
breaking internal waves still plays a prominent role for the analysis. A better understanding of the generation of internal waves and their breaking is thus of considerable importance for more accurate insights concerning the distribution of the temperature and salinity in the sea, since thermohaline properties plays an essential role for oceanic dynamics.

A path-breaking investigation by Stommel [1961] was dedicated to a better understanding of the behaviour of the temperature and salinity in a fluid system constituted by two connected vessels. This set-up, taking the form of what is known as a box model, is characterised by conceptual simplicity and has the advantage of highlighting important and complex properties related to the non-linear dynamical behaviour of the system. This classical study paved the way for a simplified approach in oceanography where water masses of different temperature and salinity were assigned to different interconnected reservoirs, hereby making it possible to subject the equations governing the behaviour of the system to a methodical non-linear analysis.

In this thesis summary an introduction to the physics of the internal waves generated when tidal barotropic flow interacts with bottom topography in the deep ocean will first be given. This is followed by a
discussion of some general aspects of the box models frequently used for modelling certain aspects of the global circulation. The summary is concluded by an overview of possible future work.

The thesis is constituted by a set of papers, where paper I deals with internal-wave generation due to oceanic tides. Traditionally, the problem of quantifying the energy released when a barotropic flow meets a Gaussian topography has been dealt with using the Wenzel-Kramers-Brillouin method (generally denoted the WKB technique), an asymptotic procedure based on the assumption that the vertical density profile varies “slowly” compared to the length-scale of the waves. This method has, however, proved to be inappropriate for certain oceanographical applications, and in paper I homogenization theory is introduced as an alternative asymptotic approach for dealing with the problem.

Papers II, III, and IV are focused on a two-vessel box model that differs from those introduced by Stommel [1961] and Rooth [1982] in that it is only forced thermally, and that the working fluid furthermore has a nonlinear equation of state. Previous results have shown that for a suitable choice of parameters this system can manifest periodic behaviour, which in the limit of small values of the vessel-volume ratio $\delta$ assumes the character of a relaxation oscillation, cf. van der Pol [1921]. In paper II an asymptotically valid phase-plane analysis of the problem in the limit $\delta \to 0$ is undertaken, an investigation which in paper III is somewhat refined by imposing a Mandelstam condition on the solution during its phases of rapid motion. Paper IV, finally, subjects the mathematical problem posed by the two governing ordinary differential equations to an asymptotic analysis valid to order $\delta$. 
Chapter 2

Generation of internal waves; a local phenomenon

The Swedish oceanographer Sandström [1908] was a pioneer in realising the importance of vertical mixing for the more-or-less stationary state of the ocean. This local phenomenon allows cold deep waters originally formed at high latitudes to return to the surface [Munk and Wunsch, 1998]. Sustaining this upwelling requires a vertical eddy diffusivity with an inferred value of $10^{-4} \text{m}^2\text{s}^{-1}$, approximately one order of magnitude larger than those observed [Ledwell et al., 2000; Kunze and Sanford, 1996]. Polzin et al. [1997] measured the turbulent diffusivity in two different regions: the central Brazil Basin with a comparatively smooth bathymetry, which showed values around $0.1 \times 10^{-4} \text{m}^2\text{s}^{-1}$, and the Mid-Atlantic Ridge characterised by a rough topography, where the diffusivity was estimated to be $10^{-4} \text{m}^2\text{s}^{-1}$, with some values exceeding $10^{-3} \text{m}^2\text{s}^{-1}$. This enhanced diffusivity occurred throughout much of the water column, up to thousands of metres above the seabed. These results suggest the presence of a mechanism that transports the energy vertically, the properties of which depend on the characteristic length-scale of the bathymetry. Internal waves are the obvious candidate. Satellite observation have also shown that these waves can propagate thousands of kilometres from the regions of generation [Ray and Mitchum, 1997; Ray and Cartwright, 2001].
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2.1 Oceanic tides

Ever since antiquity, tides have been one of the most noteworthy manifestations of celestial phenomena affecting the earth known to man. For obvious reasons, tides of the atmosphere and the “solid” earth were first observed when adequate instrumentation was available. Oceanic tides, however, affected the well-being even of communities of cave-dwellers and have hence been the object of inquiry since immemorial times. In Fig. 2.1, month-long tidal gauge records [Defant, 1953] from four coastal stations in the Atlantic and Pacific Oceans are shown. The ones from Immingham in the North Sea and from Do-Son in the Gulf of Tonkin represent classical examples of pure semi-diurnal and diurnal tides, respectively. The records from San Francisco on the West coast of North America and from Manilla in the Philippines show "mixed" cases where either the semi-diurnal or the diurnal component dominates.

The main characteristics of the differing types of tides shown in Fig. 2.1 were accounted for qualitatively already in the 17th century by Newton on the basis of his equilibrium tidal theory. A dynamical approach to the tidal problem was taken in Napoleonic times by Laplace, whose considerably more subtle approach is that followed by modern research on tidal motion, an activity which since the 1960s has relied heavily on the use of global numerical modelling. In this context it deserves to be emphasised that the analysis of global tides, which has mainly been undertaken on a theoretical basis, has been given a triumphant vindication from satellite-borne altimetry during the two most recent decades.
From this somewhat cavalier review of the evolution of our insights concerning internal tides the reader may have been given the erroneous impression that the study of tides has been a slightly sedentary activity conducted on a theoretical basis within the confines of academic chambers. That this is not always has been the case is, perhaps, most dramatically illustrated by the fate of the notable French naturalist Jean Honoré de Lamanon. In the late 1780s this researcher accompanied de la Pérouse on his South-Sea expedition on board the
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frigate la Boussole. During this cruise de Lamanon made the first barometric observations of atmospheric tides, but subsequently met a tragic end at the hands of the inhabitants of Samoa in the course of executing his scientific duties:

(The excerpt above originates from de la Pérouse’s log-book)

2.2 Dynamical aspects of the internal-wave problem

In a stratified ocean, internal waves are generated when barotropic currents interact with ocean topography; this phenomenon is known to be the most important sink of tidal energy in the deep ocean. Egbert and Ray [2000] estimated that this power loss is on the order of 1 TW, viz. 25-30% of the global tidal energy dissipation. This is a high value that casts doubt on the older picture, where these losses are mainly associated with shallow shelf waters. The stratification subjects a displaced fluid parcel to a restoring force, which gives rise to internal waves with frequencies $\omega$ in the range $f < \omega < N(z)$, where $f(\theta) = 2\Omega \sin \theta$ is the Coriolis parameter and $N(z)$ the vertical distribution of the buoyancy frequency. When $f \sim \omega$, the waves are inertial and move towards smaller values of $f(\theta)$, i.e. towards the equator, where they will be trapped. Here non-linear processes take place, leading to a transfer of energy from the low-frequency part of the spectrum to the higher-frequency range.

In contrast to electromagnetic waves, internal waves do not obey the Snellius law of reflection. The conserved quantity when internal
waves are reflected is the angle $s$:

$$s = \sqrt{\frac{\omega_0^2 - f^2(\theta)}{N^2(z) - \omega_0^2}},$$

(2.1)

where $\omega_0$ is the fundamental tidal frequency. When an internal-wave packet propagating from deeper to shallower water encounters a bottom topography, three situations can be distinguished with respect to the steepness $\alpha$ of the topography. For a subcritical topography ($\alpha < s$) the waves are scattered towards smaller depths. When the topography is supercritical, $\alpha > s$, the beam is reflected back into the deep ocean. When $s \sim \alpha$, the wavelength becomes smaller while the amplitude and the energy increase, causing the wave to become unstable and break down, which initiates local mixing and turbulence.

Striking examples of the latter phenomenon [Frederiksen et al., 1992; Klitgaard et al., 1995] have been found around the Faroe Islands in the North Atlantic, where there are areas of the bottom near the shelf-break with a slope that is more-or-less critical with regard to internal waves characterised by the predominant semidiurnal tidal frequency and where consequently wave-breaking can be expected to take place. In the immediate proximity of these well-defined regions, there are zones which are distinguished by exceptionally high abundances of the scleractinian coral *Lophelia Pertusa* as well as the demosponges *Poriferia*. Both of these species feed on suspended matter, which is available in particularly high concentrations adjacent to areas characterised by strong near-bottom turbulence. As a final curiosity in this context it may be noted that although the anomalously high coral densities were first recorded on the basis of marine-biological surveys, Faroese trawler skippers had long been aware of the dangers that the "sponge zones" posed to bottom trawls, since these tended to become overloaded which in turn could lead to mechanical failures of either the towing gear or the trawl-sacks themselves.

The two-dimensional equations of motion describing internal-wave generation can be formulated as:

$$\partial_t u(x, z, t) + \partial_x p(x, z, t) = 0,$$

(2.2)

$$\partial_z p(x, z, t) - \partial_y b(x, z, t) = 0,$$

(2.3)

$$\partial_t b(x, z, t) + w N^2(z) = 0,$$

(2.4)
whereas the continuity equation is

$$\partial_x u(x, z, t) + \partial_z w(x, z, t) = 0.$$  \hfill (2.5)

Here $u$, $w$ is the horizontal and vertical velocity component, respectively, $b(x, z, t)$ the buoyancy and $p(x, z, t)$ the pressure. The nonlinear bottom-boundary condition is given by

$$w(-H + h) = (U(t) + u(x, z, t)) \frac{dh(x)}{dx},$$  \hfill (2.6)

where $U(t)$ is a barotropic velocity field, here assumed to have the form $U(t) = U_0 \cos(\omega_0 t)$, and $h(x)$ is the topographic height. Note that although the problem has been formulated as a two-dimensional one, the physics remain unchanged in three dimensions. To obtain the problem in this case it suffices to instead choose two-dimensional topographies $h(x, y)$. Linearization around the constants $U_0$ and $z = -H$ for the subcritical regime with topographic slopes $\alpha$ that are smaller than $s$ yields

$$w(-H) = U(t) \frac{dh(x)}{dx},$$  \hfill (2.7)

the other boundary condition being $w = 0$ at $z = 0$. Assuming that the dynamical variables have the time-dependent form

$$X = X_0 \exp(-i\omega t),$$  \hfill (2.8)

and expanding the vertical velocity $w$ vertically as

$$w = \sum_{n=1}^{\infty} w_n(x, t) \psi_n(z),$$  \hfill (2.9)

leads, after some algebraic manipulation, to the projection onto the $z$-axis of the following eigenvalue problem;

$$\frac{d}{dz} \left( \frac{1}{N(z)^2} \frac{d \Psi_n}{dz} \right) + \frac{1}{c_n^2} \Psi_n = 0,$$  \hfill (2.10)

where

$$\Psi_n(z) = \frac{\partial \psi_n(z)}{\partial z},$$  \hfill (2.11)
and \( c_n \) is the \( n \)th eigenvalue.

The dynamics of the internal waves are fully characterised by solving the Sturm-Liouville problem given by Eq. (2.10). However, for realistic oceanographical buoyancy-frequency profiles \( N(z) \), this eigenvalue problem can only be solved numerically. For cases such that

\[
\frac{d \ln N(z)}{dz} \ll \frac{1}{\lambda_n},
\]  

(2.12)

WKB theory can be utilised to yield an analytical approximation of the eigenvalue problem. It may, however, be shown that this method is inaccurate for the lowest modes. In paper I, homogenisation theory has been introduced with the aim of improving the results for the first modes. This method yielded a good correspondence between the results for the lowest modes and those obtained numerically.

### 2.3 Calculation of energy conversion

The rate of energy conversion from the barotropic tide to internal waves by subcritical topography was first calculated analytically by Bell [1975a, b]. He assumed the ocean to be infinitely deep, and the buoyancy frequency to be constant. The infinite-depth assumption implies that there is no reflection from the surface, and that the internal-wave spectrum is continuous. A more realistic model [Llewellyn Smith and Young, 2002] has later been proposed for taking into account the nonuniform stratification \( N(z) \) and the finite depth of the ocean; in this case the spectrum of the internal waves becomes discrete.

The introduction of nonuniform stratification into Bell’s theory was carried out using the WKB approximation. Nycander [2005] employed this procedure for calculating the global energy released from the interaction between tidally-induced barotropic currents and the bathymetry (Fig. 2.2). This approximation has the disadvantage of yielding conversion rates that are erroneously dependent on \( N_B \), the buoyancy frequency at the seabed. In paper I it was shown that this asymptotic analysis is misleading for the lowest modes or, equivalently, broad-ridge topographies. These authors instead employed
homogenization theory, which yields conversion rates that depend on the value of $N(z)$ averaged over a near-bottom vertical region of the same height as the vertical length-scale of the internal wave. This approach led to the conversion rates becoming more accurate for the lowest modes, which is particularly important since it is known that these modes are responsible for removing the energy from the generation site [Alford, 2003].

![Figure 2.2: Global distribution of the energy flux from the diurnal $M_2$ tidal component to internal waves. The unit is W/m$^2$. Logarithmic color scale, e.g. -4 means $10^{-4}$ W/m$^2$. Black for values $< 10^{-6}$ W/m$^2$.](image)

**Figure 2.2:** Global distribution of the energy flux from the diurnal $M_2$ tidal component to internal waves. The unit is W/m$^2$. Logarithmic color scale, e.g. -4 means $10^{-4}$ W/m$^2$. Black for values $< 10^{-6}$ W/m$^2$.

### 2.4 Supercritical topography

Much theoretical work on the generation of internal waves has focused on subcritical topographies. This simplification allows a linear superposition of the topographies, a prerequisite for the application of Fourier analysis, and yields conversion rates proportional to the spectral density of the topography. Models describing tidal generation from subcritical topographies are becoming more sophisticated with fewer constraints than in Bell’s theory. This is of great importance, since “gentle” slopes are ubiquitous in the world ocean. However, the
most important sites for internal-wave generation are those having a rough bathymetry, \textit{i.e.} supercritical topographies. These are considered as being responsible for about half of the global energy conversion that occurs at depths greater than 1000 m \cite{Nycander2005}. Presently, the theory of tidal generation for supercritical topographies is still very rudimentary. Attempts at clarification have been made by trying to model the scattering of internal waves \cite{Robinson1969, Larsen1969} and, more recently, their generation \cite{StLaurent2000, LlewellynSmith2003}. These investigators considered a knife-edge profile where the topography is defined as a vertical wall with a limited height $h_0$. It is the most straightforward profile for which solutions can be obtained in the context of internal-wave generation. More complex models have been introduced to take into account the more realistic situation of interference between reflecting internal-wave beams trapped between supercritical topographies \cite{Nycander2006, Balmforth2009}.
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Chapter 3

Box models; a global approach to oceanography

As previously discussed, internal waves probably represent the principal mechanism giving rise into the mixing required for balancing the global-ocean overturning circulation. The role and presence of mixing was first outlined in the Stommel and Arons [1960] model, where it took the form of western-boundary mixing. This model paved the way for increasingly sophisticated approaches to understanding the dynamics of the water masses and attempting to determine the engine responsible for the circulation.

Another elegant approach to the global-circulation problem has been the introduction of the box-model concept by Stommel [1961]. Within this framework, the ocean is considered as a dynamical system having mixed boundary conditions, an appellation referring to separate boundary conditions for the two dynamical variables characterising the system, viz. temperature and salinity. Despite the apparent simplicity of this approach compared to the extreme complexity of the global circulation, new and interesting phenomena can be discovered using these simple thermohaline box models.
3.1 The Stommel two-box model

In the model proposed by Stommel [1961] thermohaline convection has been modelled by the convection occurring between two well-mixed interconnected reservoirs, cf. Fig. 3.1.

Figure 3.1: Two vessels connected by a capillary tube with a flow rate $q$ and a capillary resistance $k$.

The system is governed by the following set of differential equations:

$$\frac{dT}{dt} = \kappa (T^{ex} - T) - |2q| T,$$

(3.1)

$$\frac{dS}{dt} = \kappa_s (S^{ex} - S) - |2q| S,$$

(3.2)

where $\kappa$, $\kappa_s$ are the temperature and salinity transfer coefficients, $T^{ex}$ and $S^{ex}$ the fixed exterior temperature and salinity, respectively, and $q$ the flow rate. By the change of variables $\tau = \kappa t$, $\lambda = \kappa_s / \kappa$, $x = T/T^{ex}$, $y = S/S^{ex}$, and $f = 2q/\kappa_s$, Eqs. (3.1) (3.2) can be recast into the following non-dimensional form:

$$\frac{dx}{d\tau} = (1 - x) - |f| x,$$

(3.3)

$$\frac{dy}{d\tau} = \lambda(1 - y) - |f| y,$$

(3.4)
together with
\[ qk = \alpha (Ry - x), \quad (3.5) \]
the latter equation being the capillary linear law that reflects the
dynamics of the flow through the tube connecting the two vessels.
The flow resistance is given by \( k \), \( \alpha \) being a constant.

![Diagram of Stommel box model](image)

**Figure 3.2:** A schematic picture of two states of the Stommel box model. a) Circulation driven by a thermal gradient, b) Circulation driven by a salinity gradient.

The situation studied here is limited to \( \lambda < 1 \), *viz.* when the
salinity-transfer coefficient is smaller than that of the temperature.
The equilibrium points are determined by the solutions of \( dx/d\tau = dy/d\tau = 0 \) together with Eq. (3.5). These steady-state solutions rep-
resent either a single stable-state regime in the form of a node or a
three-state regime. In the latter case a stability analysis shows the
presence of an unstable saddle point, a stable node, and a stable
spiral. Two different situations can be distinguished for these sta-
ble equilibria (Fig. 3.2): a regime characterised by the domination
of the thermal difference, *i.e.*, a convective flow from the cold vessel to
the warm one, and another regime where the salinity-induced den-
sity difference dominates the thermal gradient, the flow in this case
being directed from the warm vessel to the cold one. A dramatic
consequence of even a slight increase of the value of either \( k \) or \( q \) is
an irreversible shift of the system from the three-state regime to one
characterised by a single stable state.
This simple model can be used to represent a two-layer thermohaline system (Fig. 3.3) where the upper layer is warm and the salinity is gradually increased. The bottom layer is kept cold and heavy, and at some point, when salinity effects become dominant in the upper layer, the system becomes unstable which leads to an overturning. The new upper layer is now cold and fresh.

![Two-layer thermohaline system with mixed boundary conditions.](image)

### 3.2 The Rooth three-box model

The Stommel model discussed above can be generalised to a three-box variety where two “poles” and the “equator” are included. Rooth [1982] introduced a system where three vessels are connected to each other as shown in Fig. 3.4; the poles (outer boxes) are joined by a single connection and the equator is linked to the poles via the two upper connections permitting poleward transports of fresh water.

A simplified version of the Stommel-Rooth model [Bryan, 1986] will here be considered, where the direct pole-to-pole connection has been eliminated, and only the pole-equator connections are retained. This simplification does not in any way alter the qualitative results. The polar salinities are governed by

\[ V_p \dot{S}_1 = -F + |q_1| (S_2 - S_1), \]  
\[ V_p \dot{S}_3 = -F + |q_3| (S_2 - S_3), \]

where \( q_1(T_1, T_2, S_1, S_2) \) and \( q_3(T_2, T_3, S_2, S_3) \) are the equator-to-pole flows. It should be noted here that the equatorial salinity \( S_2 \) can
be eliminated due to conservation of the total salt content of the system. These equations may be shown to exhibit a total of nine steady states, \( \dot{S}_1 = \dot{S}_3 = 0 \). Five are unstable and four are stable. Among the four stable states, two are symmetric, \( S_1 = S_3, q_1 = q_3 \), one of which is driven by the thermal gradient (Fig. 3.5a). This state can be interpreted as a linkage of two “Stommel-box” configurations (Fig. 3.2a). The remaining stable state is driven by the salinity difference (Fig. 3.5b) and can also be viewed in Stommel-box terms.

The remaining two stable states are antisymmetric, \( S_1 = \)}
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\(-S_3, q_1 = -q_3\). These can be seen as cross-combinations of the two Stommel-boxes illustrated in Figs. 3.2a, b. Under these conditions the system behaves as if it were solely the pole-to-pole connection which is operating. In one of these cases the circulation is driven by the thermal gradient (Fig. 3.6a), in the other by the salinity difference (Fig. 3.6b).

![Figure 3.6: Three-box model, antisymmetric case where \(S_1 = -S_3, T_1 = T_3\). a) The system is driven by a thermal gradient. b) The case when the system is driven by a salinity gradient.](image)

3.3 A thermal oscillator and its dynamical properties

The previous sections mainly dealt with characterising the behaviour of the Stommel as well as the Rooth/Bryan thermohaline box models, where it was found that these systems ultimately assume stable states corresponding to a stationary circulation. It is, however, also possible for a box-model circulation to manifest oscillatory characteristics, a topic that is explored in papers II, III, and IV. With this in mind, we reduce the scope of our analysis to a thermally forced system constituted by two interconnected vessels (Fig. 3.7) with a working fluid characterised by a quadratically nonlinear equation of state. As shown by Lundberg and Rahm [1984], the temperatures \(T_1\) and \(T_2\) in the two vessels are governed by the following set of autonomous differential equations:
Figure 3.7: Illustration of the physical system consisting of two thermally forced reservoirs of equal height containing a well-mixed fluid with a nonlinear equation state.

\[ F(T_1, T_2) = \delta \frac{dT_1}{dt} = Ra \mid T_2^2 - T_1^2 \mid (T_2 - T_1) + \]
\[ + \frac{1}{Pe} \left( \frac{\theta}{1 - \theta} - T_1 \right) + \frac{\kappa}{Pe} (T_2 - T_1), \quad (3.8) \]

\[ G(T_1, T_2) = \frac{dT_2}{dt} = -Ra \mid T_2^2 - T_1^2 \mid (T_2 - T_1) + \]
\[ + \frac{1}{Pe} \left( \frac{1}{1 - \theta} - T_2 \right) - \frac{\kappa}{Pe} (T_2 - T_1), \quad (3.9) \]

where \( Ra \) is a Rayleigh number proportional to the strength of the thermal forcing, \( Pe \) a Péclet-type number which characterises the time-scale, \( \delta = V_1/V_2 \) the volume ratio between the vessels, \( \theta = T_1^{ex}/T_2^{ex} \) the ratio of the prescribed external temperatures, and \( \kappa \) the ratio between the internal and external heat-exchange coefficients of the system.
The Bendixson-Poincaré theorem can be applied to determine the criteria for periodic behaviour of this two-dimensional system. For the theorem to be applicable, all solutions to the governing equations above must be bounded. That this indeed is the case can be proved using a Liapunov function [Lundberg and Rahm, 1984], but is also recognised on physical grounds since it is evident that the temperatures $T_1$ and $T_2$ cannot transgress the limits set by the prescribed forcing temperatures $T_{1ex}$ and $T_{2ex}$. If it hereafter can be established that the stationary solutions to the problem are unstable, the behaviour of the solutions in the phase-plane is given by a limit cycle (corresponding to an oscillatory state of the system). The analysis is thus initiated by determining the critical points $(T_1^c, T_2^c)$ of the system of equations, which are found to be

$$T_1^c = \frac{11 + \theta}{2(1 - \theta)} + \frac{1 + 2\kappa}{8RaPe} \frac{1 - \theta}{1 + \theta} \left[ 1 - \left( 1 + \frac{8RaPe}{(1 + 2\kappa)^2} \frac{1 + \theta}{1 - \theta} \right)^{1/2} \right],$$

$$T_2^c = \frac{11 + \theta}{2(1 - \theta)} - \frac{1 + 2\kappa}{8RaPe} \frac{1 - \theta}{1 + \theta} \left[ 1 - \left( 1 + \frac{8RaPe}{(1 + 2\kappa)^2} \frac{1 + \theta}{1 - \theta} \right)^{1/2} \right].$$

Hereafter a linear stability analysis of this stationary solution is carried out, which ultimately yields the characteristic equation

$$\text{det}(A - \lambda I) = 0. \quad (3.10)$$

Here

$$A = \begin{pmatrix} \frac{1}{\delta} \frac{\partial F}{\partial T_1} & \frac{1}{\delta} \frac{\partial F}{\partial T_2} \\ \frac{\partial G}{\partial T_1} & \frac{\partial G}{\partial T_2} \end{pmatrix} \quad (3.11)$$

is the Jacobian matrix of the system evaluated at the critical point $(T_1^c, T_2^c)$. The eigenvalues of the characteristic equation are

$$\lambda_{1,2} = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4 \det(A)}}{2}. \quad (3.12)$$
Chapter 3: Box models; a global approach to oceanography

The eigenvalues $\lambda_{1,2}$ determine the character of the critical points. In the context of analysing the behaviour of Eqs. (3.8, 3.9), two general cases of interest can be distinguished: if $tr(A)^2 - 4 |A| < 0$ the stationary solution is a spiral point, stable if $tr(A) < 0$, unstable if $tr(A) > 0$. Fig. 3.8 illustrates the system behaviour for the parameters $Ra = 107.9$, $Pe = 1$, $\theta = -13/14$, $\delta = 1/2$, and $\kappa = 0$.

![Figure 3.8: Trajectory showing spiral behaviour when it approaches the stable critical point. $Ra = 107.9$, $Pe = 1$, $\theta = -13/14$, $\delta = 1/2$.](image)

For a slight decrease of $Ra$ to 107.5 the critical point becomes unstable, resulting in the limit cycle shown in Fig. 3.9.

In papers II, III, and IV, a characteristic feature of the general system governed by Eqs. (3.8, 3.9) is that it for small values of the parameter $\delta$ executes what is known as relaxation oscillations, characterized by their typical “jerkeness”. From a formal standpoint this implies that in the governing equations the time derivative preceded by $\delta$ can be disregarded, which implies that the behaviour of the system in the asymptotic limit of $\delta \to 0$ can be described in terms of phase-point motion along the slow manifold, interspersed by rapid jumps from one branch of the manifold to the other.

In paper II an analysis of this type is carried through, and the resulting approximations in the phase-plane were found to correspond reasonably well to the purely numerical results for small values of $\delta \to 0$. This lowest-order analysis is only strictly valid as $\delta \to 0$, but, as shown in paper III the results for a finite value of $\delta$ can be im-
Figure 3.9: Self sustained periodic solution for $Ra = 107.5$, $Pe = 1$, $\theta = -13/14$, $\kappa = 0$, and $\delta = 1/2$. The cross indicates the location of the unstable critical point.

proved by imposing a Mandelstam condition, which for the problem under consideration here assumes the form of prescribing the rapid jumps between the manifold branches as being adiabatic, i.e. requiring the total heat content of the system to be conserved during the phases of rapid motion. It should be underlined that this condition is not a consequence of the governing equations, but represents an introduction of additional information to deal with the problem.

From a mathematical standpoint the consistent way of dealing with the effects of a finite value of $\delta$ is to subject the problem to an asymptotically valid analysis. This task has been undertaken in paper IV wherein the slightly heterodox method of dealing with the problem in the phase-plane is necessitated by the fact that, due to the non-analytical right-hand sides of the governing equations, it is not possible to reduce the problem to one of the time-evolution of solely one variable. Consonant with the approach taken in papers II and III, the results of this analysis are reported in the phase-plane, and they are recognised as showing a good correspondence with the numerically determined limit cycles. The insights gained from a close examination of these results valid to the order $\delta$, furthermore, serve as an a posteriori justification of the Mandelstam condition applied in paper III.
Chapter 4

Conclusions and outlook

An overview of some challenging topics regarding internal-wave generation and quantification has been presented in this thesis, where a general description of the mechanisms of internal-wave generation was introduced. Strong evidence indicates that this local phenomenon probably must be regarded as being mainly responsible for energy dissipation in the ocean via upwelling and bottom “friction”. So far the generation of internal waves in nature is still poorly understood. In paper I a study of internal waves generated by a subcritical topography with a rather idealised profile has been carried out. Some qualitative as well as quantitative improvements were observed through the introduction of the homogenization technique. One of the most difficult and least tractable issues in this area of research is the quantification of the energy generated by critical topographies, in which field of inquiry a lot remains to be done.

The topic of internal-wave generation has been followed by a more global approach to oceanography which consists of examining the dynamics of the ocean on the basis of employing a box-model formalism. Despite its manifest simplicity, this qualitative approach, introduced by Stommel [1961], is capable of doing justice to fascinating phenomena of considerable complexity. In applied mathematics these problems constitute a field of their own, known as dynamical systems. A flavour of this topic has been presented here, where some results and ideas dealing with the fundamentals of dynamical-systems theory have
been introduced and form the general framework underlying papers II, III, and IV.
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