Photonic quantum information
and experimental tests of
foundations of quantum
mechanics

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Abstract

Entanglement is a key resource in many quantum information schemes and in the last years the research on multi-qubit entanglement has drawn lots of attention. In this thesis the experimental generation and characterisation of multi-qubit entanglement is presented. Specifically we have prepared entangled states of up to six qubits. The qubits were implemented in the polarisation degree of freedom of single photons. We emphasise that one type of states that we produce are rotationally invariant states, remaining unchanged under simultaneous identical unitary transformations of all their individual constituents. Such states can be applied to e.g. decoherence-free encoding, quantum communication without sharing a common reference frame, quantum telecloning, secret sharing and remote state preparation schemes. They also have properties which are interesting in studies of foundations of quantum mechanics.

In the experimental implementation we use a single source of entangled photon pairs, based on parametric down-conversion, and extract the first, second and third order events. Our experimental setup is completely free from interferometric overlaps, making it robust and contributing to a high fidelity of the generated states. To our knowledge, the achieved fidelity is the highest that has been observed for six-qubit entangled states and our measurement results are in very good agreement with predictions of quantum theory.

We have also performed another novel test of the foundations of quantum mechanics. It is based on an inequality that is fulfilled by any non-contextual hidden variable theory, but can be violated by quantum mechanics. This test is similar to Bell inequality tests, which rule out local hidden variable theories as possible completions of quantum mechanics. Here, however, we show that non-contextual hidden variable theories cannot explain certain experimental results, which are consistent with quantum mechanics. Hence, neither of these theories can be used to make quantum mechanics complete.
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|           | M. Rådmark, M. Żukowski, and M. Bourennane                                                                                 |

| Paper IV  | State-Independent Quantum Contextuality with Single Photons                                                                  |
|           | E. Amselem, M. Rådmark, M. Bourennane, and A. Cabello                                                                      |
Quantum information is an interdisciplinary field of science connected to quantum physics, information theory, computer science and many other fields. One important goal is the quantum computer, which would be able to computing certain tasks (like factoring large numbers) much faster than conventional computers, as well as simulating other quantum systems and processes. Another application, closer to an every day use, is quantum key distribution (QKD) or quantum cryptography, allowing unconditionally secure communication based on the pure randomness allowed by quantum mechanics.

Although the journey toward a fully functioning, practically useful quantum computer may be long, I’m sure many new applications and inventions will be found on the way. For sure we need a better and deeper understanding of fundamental quantum mechanics. Such as how quantum and classical physics differ, and how quantum effects can be used to our advantage. Another question that arises is how optimal quantum mechanics is for describing the world around us. So far quantum theory has been able to predict experimental outcomes. However, other so called realistic theories, in which every property of a system is properly defined, and that to a large extent make the same predictions as quantum mechanics, can be constructed. Maybe such a theory could describe the world more accurate than quantum mechanics, and the randomness used in QKD isn’t really random, but only due to our current inability to predict the properties of quantum systems. Nevertheless, a subset of these realistic theories, based on locality has more or less (some loopholes remain) been disproved by experiments. Now another subset, based on non-contextuality, is for the first time being experimentally challenged.

I think that my contributions to the research field of quantum information, is a small, but valuable step for improving the generation, control, manipulation and detection of quantum states and processes, as well as for improving our understanding of fundamental quantum mechanics.
I have really appreciated the width of my working field, ranging from abstract mathematics to handicraft in the workshop and everything in between including lots of optics, programming, electronics design, and computer simulations, to mention a few things. I have learnt a lot, not only about quantum mechanics, but also in a broader spectrum covering large areas of science and technology.

My contributions to the accompanying papers

As recommended, I am below commenting on my contributions to the accompanying papers. When I started my Ph.D. studies in the summer of 2005, the lab was more or less empty. Hence, I have been involved in all aspects of building up a modern research lab, including purchasing and characterising optical components and other necessary instruments. I have also developed and built several coincidence units for real-time data sampling, and developed various computer programs for data communication and for controlling the experiments.

**Paper I:** I made major contributions to designing the experiment. I calculated the correlation functions and performed all the experimental work, as well as all data analysis. The paper was written by all co-authors.

**Paper II:** I made major contributions to designing the experiment. I derived the reduced entanglement witness and performed all the experimental work, as well as all data analysis. The paper was written by all co-authors.

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**Paper IV:** I designed the experiment and performed all the laboratory work, as well as all data analysis with equal contributions together with Elias Amselem. The paper was written by all co-authors.

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and confidence in me has really encouraged me during my time at Fysikum. Moreover, he has given me many wisdoms concerning the life as a scientist and in general.

I would like to express my deep appreciation to all present and former members of the Kiko group, who have been part of creating a very nice atmosphere combining playfulness and seriousness. I especially want to mention my lab mates Jan Bogdanski, Hatim Azzouz, Elias Amselem, Christian Kothe, Johan Ahrens, and Muhammad Sadiq, as well as my office mate Kate Blanchfield who also has helped me proof-reading the thesis.

I would like to acknowledge Attila Hidvegi for the very fruitful discussions we had during the development of the coincidence unit.

Thanks goes to Marek Żukowski and Adán Cabello for our great collaborations, and to Hoshang Heydari, Piotr Badziag, Gunnar Björk and Ingemar Bengtsson for discussions, and the whole KoF group for all cakes, buns and cookies at our coffee breaks.

Finally I would like to thank my lovely Martha and my family for always being there for me!
Sammanfattning på svenska

Kvantmekanisk sammanflätning är en viktig tillgång i många tillämpningar inom kvantinformation och de senaste åren har forskning inom fler-partikel-sammanflätning fått mycket uppmärksamhet. I denna avhandling presenteras arbete med experimentellt framställande och karakterisering av kvantmekanisk sammanflätning av ett flertal kvantbitar. Specifikt har vi lyckats generera sammanflätade kvanttillstånd med upp till sex kvantbitar. Dessa kvantbitar realiserades i polarisationen av enstaka fotoner. En klass av tillstånd som vi genererat är s.k. rotationsinvariantha tillstånd, som inte ändras när varje enskild kvantbit utsätts för samma unitära transformation. Sådana tillstånd kan användas för exempelvis dekoherensfri kodning av kvantinformation, kvantkommunikation utan gemensamma referensramar, kvanttelektron, kvantsekretessdelning och avlägsen tillståndspreparering. De innehår också egenskaper, som gör dem intressanta för att undersöka kvantmekanikens grundvalar.


Part I

Background material and results
Chapter 1

Introduction

1.1 Background

The concept of information is often seen as something abstract and unphysical. Nevertheless all information needs to be encoded in different states of some physical system. A few examples are positions of ink or graphite clusters on a sheet of paper, electric potential on a wire, and the sequence of nucleotides (adenine, guanine, thymine, cytosine) in a DNA molecule.

In quantum information, the physical system used for encoding the information is governed by the laws of quantum mechanics. This gives rise to properties, which are classically not possible. One of these is entanglement, which is an essential resource in many quantum information schemes. Especially entangled quantum states of two photons have been proved to be useful in quantum communication protocols like quantum key distribution [1–3], quantum dense coding [4, 5], and for violating Bell inequalities [6–10].

In the last two decades, entanglement of more than two photons has started to be explored too, e.g. by demonstrating quantum teleportation [11–13] and entanglement swapping [14]. Multiphoton entanglement [15–17] has opened a rich, promising and also complex subfield of quantum information science, as it enables applications intended for multi-user processes, such as telecloning [18] and secret sharing [19], and new protocols such as decoherence-free encoding [20] and reduction of communication complexity [21]. Moreover it can be used for further studying and testing the foundations of quantum mechanics.

We start this thesis by briefly describing quantum mechanics, emphasising the qubit and entanglement. Next, useful optical components as well as the non-linear process of parametric down-conversion are described. These
components are needed to generate the entangled photons and to engineer the desired state. In chapter 3, the quantum state analysis, detectors and a self-developed coincidence unit are described. The experimental results of six-photon entangled states are presented in chapter 4. Two of many possible applications of these states, namely decoherence-free encoding and quantum teleportation are discussed in chapters 5 and 6. In the appendix the reader will find the derivation of the entanglement witness operator used in chapter 4 as well as the experimental results of four- and two-photon entangled states.

1.2 The qubit

In the field of quantum information the fundamental element is the qubit, or the quantum bit [22]. It is a state in a two-dimensional Hilbert space and it is a generalisation of the classical bit. Whereas a classical bit can take one of the binary values 0 or 1, the qubit can analogously be in one of the two orthogonal basis states $|0\rangle$ or $|1\rangle$. Additionally the qubit can be in any superposition of the two basis states:

$$|Q\rangle = c_0 |0\rangle + c_1 |1\rangle,$$

(1.1)

where $c_0$ and $c_1$ are arbitrary complex numbers that satisfy the normalisation condition $|c_0|^2 + |c_1|^2 = 1$. This has no classical analogy and is one of the main properties making quantum information schemes like quantum key distribution, quantum teleportation, and quantum computation possible. In Eq. (1.1) $c_0$ is often assumed to be real-valued. This assumption is equivalent to factoring out and ignoring any global phase in the state. Since global phases have no observable effects this assumption is usually valid.

The state of a qubit can conveniently be represented graphically by a point on the surface of the Bloch sphere. In this case it is appropriate to use the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ according to Figure 1.1, as parameters. The quantum state $|Q\rangle$ from Eq. (1.1) can then be written as

$$|Q\rangle = c_0 |0\rangle + c_1 |1\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle = |Q(\theta, \phi)\rangle.$$

(1.2)

It is worth noting that two orthogonal states are always opposite to each other on the Bloch sphere. Hence the state orthogonal to $|Q(\theta, \phi)\rangle$ is:

$$|Q(\theta, \phi)\rangle_\perp = |Q(\pi - \theta, \pi + \phi)\rangle.$$

(1.3)
1.2 The qubit

The state of a qubit can also be represented by a vector as follows:

\[
|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(1.4)

\[
|Q\rangle = c_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}.
\]

(1.5)

In this context operators are represented by matrices. Any operator \( \hat{O} \) acting on a single qubit can be decomposed in a superposition of the three Pauli matrices and the identity matrix

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(1.6)

as

\[
\hat{O} = ic_z \sigma_z + ic_x \sigma_x + ic_y \sigma_y + c_1 1,
\]

(1.7)

where \( c_z, c_x, c_y, c_1 \) are complex numbers. In addition, by only allowing real coefficients \( (c_z, c_x, c_y, c_1 \in \mathbb{R}) \) satisfying \( c_z^2 + c_x^2 + c_y^2 + c_1^2 = 1 \), any unitary Hermitian operator can be obtained. The subset of dichotomic observables with eigenvalues \( \pm 1 \) is obtained by setting \( c_1 = 0 \).

The eigenstates of \( \sigma_z \) are \( |0\rangle \) and \( |1\rangle \) and these two states form a basis that is often referred to as the computational basis. Also the eigenstates of...
\( \sigma_x \) and \( \sigma_y \) form a complete basis and can likewise be used as basis states. The eigenstates and eigenvalues of the Pauli matrices are presented in Table 1.1.

**Table 1.1: The eigenvalues and eigenstates of the Pauli matrices.**

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<tr>
<th>Observable</th>
<th>Eigenvalue</th>
<th>Eigenstate</th>
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<tr>
<td>( \sigma_z )</td>
<td>+1</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>(</td>
</tr>
<tr>
<td>( \sigma_x )</td>
<td>+1</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>(</td>
</tr>
<tr>
<td>( \sigma_y )</td>
<td>+1</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>(</td>
</tr>
</tbody>
</table>

Another important property of quantum mechanics is that a measurement of a state will change the state itself. Measurements in quantum mechanics are represented by observables, i.e. Hermitian operators. The state will during a measurement be projected into one of the eigenstates of the measurement observable and the measurement value will be the corresponding eigenvalue. When measuring the state of a qubit the eigenvalue will always be +1 or −1. The probability to obtain each measurement result can be calculated as the expectation value of the corresponding projection operator. As an example, let us measure the spin in \( x \)-direction (by acting with the observable \( \sigma_x \)) on the state \( |0\rangle \). The probability \( p_{+x} \) for \( |0\rangle \) to be projected into the eigenstate \( |+x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and hence to measure the value +1 is given by

\[
p_{+x} = \langle 0 | \hat{P}_{+x} | 0 \rangle = \langle 0 | +x \rangle \langle +x | 0 \rangle = \frac{1}{2} , \tag{1.8}
\]

where \( P_{+x} = |+x\rangle \langle +x | \) is the projector onto \( |+x\rangle \). Similarly the probability for measuring −1 is given by

\[
p_{-x} = \langle 0 | \hat{P}_{-x} | 0 \rangle = \langle 0 | -x \rangle \langle -x | 0 \rangle = \frac{1}{2} . \tag{1.9}
\]

The expectation value of the observable \( \sigma_x \) acting on \( |0\rangle \) is now given by

\[
\langle 0 | P_{+x} - P_{-x} | 0 \rangle = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0 . \tag{1.10}
\]
The Pauli operators can in general be expressed in terms of their eigenstates as

\[ \sigma_i = |+i\rangle \langle +i| - |-i\rangle \langle -i| \quad \text{for } i = x, y, z . \]  

(1.11)

### 1.2.1 Mixed states and the density operator

Quantum states like \(|Q\rangle\) in Eq. (1.1), that are completely described by their state vector are called pure states. In contrast, mixed states are composed of statistical mixtures of two or more different pure states. Such states cannot be properly described by a state vector, but can conveniently be represented mathematically by density operators. The density operator representing a pure state with state vector \(|Q_i\rangle\) is given by its projector

\[ \rho_i = |Q_i\rangle \langle Q_i| . \]  

(1.12)

The density operator of a mixed state is a normalised sum of pure states given by

\[ \rho = \sum_i p_i \rho_i = \sum_i p_i |Q_i\rangle \langle Q_i| , \quad \text{with} \quad \sum_i p_i = 1 , \quad p_i \geq 0 \quad \forall \quad i . \]  

(1.13)

Some properties of the density operator are

- \( \rho \) is normalised: \( \text{Tr}(\rho) = 1 \)
- \( \rho \) is positive-semidefinite (has real positive eigenvalues)
- \( \text{Tr}(\rho^2) \leq 1 , \) with equality if and only if \( \rho \) represents a pure state.

The expectation value of an operator \( \hat{O} \) acting on a quantum state \( \rho \) (pure or mixed) is obtained by taking the trace over the product of the operator and the state density

\[ E(\hat{O}) = \text{Tr}(\hat{O}\rho) . \]  

(1.14)

Further, in contrast to the pure states that are represented by points on the surface of the Bloch sphere, all mixed states are interior points in the Bloch sphere, with the maximally mixed state in the center of the sphere. In general a single qubit state can be written as

\[ \rho = \frac{1}{2} (1 + \mathbf{r} \cdot \vec{\sigma}) , \]  

(1.15)

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) and \( \mathbf{r} = (x, y, z) \) is the Bloch vector, defining the state. Note that the \( \mathbf{r} \)-coordinates represent the expectation values of the Pauli operators [22].
1.2.2 Systems of several qubits

For larger systems consisting of more than one qubit, the Hilbert space to use is a direct (Kronecker) product of the Hilbert spaces for each qubit. As an example, the composite Hilbert space for one qubit in mode $a$ and one qubit in mode $b$ is given by

$$\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b .$$  (1.16)

This space is of dimension four and is for example spanned by the following set of four states:

$$\{ \lvert 0 \rangle_a \otimes \lvert 0 \rangle_b , \lvert 0 \rangle_a \otimes \lvert 1 \rangle_b , \lvert 1 \rangle_a \otimes \lvert 0 \rangle_b , \lvert 1 \rangle_a \otimes \lvert 1 \rangle_b \} .$$  (1.17)

In general, the dimension of a composite Hilbert space of $N$ qubits is $D = 2^N$. A product state $\lvert \Psi_N \rangle$ in this space will be written in one of the following ways:

$$\lvert \Psi_N \rangle = \lvert \psi_1 \rangle_a \otimes \lvert \psi_2 \rangle_b \otimes \ldots = \lvert \psi_1 \rangle \otimes \lvert \psi_2 \rangle \otimes \ldots = \lvert \psi_1 \rangle_{\psi_2} \ldots \ldots .$$

For numerical calculations it may be appropriate to denote the basis states from Eq. (1.17) as column vectors. This is naturally done by expanding the Kronecker products of the states:

$$\begin{align*}
\lvert 0 \rangle \otimes \lvert 0 \rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (1.18) \\
\lvert 0 \rangle \otimes \lvert 1 \rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (1.19) \\
\lvert 1 \rangle \otimes \lvert 0 \rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (1.20) \\
\lvert 1 \rangle \otimes \lvert 1 \rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (1.21)
\end{align*}$$
1.2 The qubit

A product of two local operators will in this context also be described by a
Kronecker product
\[
\hat{U} \otimes \hat{V} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.
\] (1.22)

This can also be generalised to higher dimensions.

1.2.3 The no-cloning theorem

An important property of qubits is that an arbitrary and unknown quantum
state of a qubit cannot be copied onto two qubits, without disturbing the
original qubit [22]. To investigate this theoretically let us imagine a machine,
represented by the unitary operator \(\hat{U}\), which can perfectly perform the
following operation:
\[
\hat{U}(|\psi\rangle \otimes |B\rangle) = |\psi\rangle \otimes |\psi\rangle.
\] (1.23)

That is copying the qubit \(|\psi\rangle\) onto a qubit initially in a “blank” state \(|B\rangle\).
Of course the copying machine should also be able to copy other states, e.g.
\(|\phi\rangle\):
\[
\hat{U}(|\phi\rangle \otimes |B\rangle) = |\phi\rangle \otimes |\phi\rangle.
\] (1.24)

Taking the inner product of Eq. (1.23) and (1.24) now leads to two different
expressions
\[
(\langle B|_{b} \otimes \langle \psi|_{a}) \hat{U}^{\dagger} \hat{U}(|\phi\rangle_{a} \otimes |B\rangle_{b}) = \begin{cases} 
\langle \psi|_{b} \otimes \langle \psi|_{a} |\phi\rangle_{a} \otimes |\phi\rangle_{b} = \langle \psi, \phi \rangle^2 \\
\langle B|_{b} \otimes \langle \psi|_{a} |\phi\rangle_{a} \otimes |B\rangle_{b} = \langle \psi, \phi \rangle
\end{cases},
\] (1.25)

which are both fulfilled only if \(|\psi\rangle\) and \(|\phi\rangle\) are either equal or orthogonal.
Therefore a perfect general quantum state copying machine cannot exist.
It has nevertheless been shown that approximate quantum cloning is pos-
sible [23] and an upper bound for the fidelity for such processes has been
derived. For the simplest case in which one additional copy of an original
qubit is obtained, the upper bound of the fidelity is \(F = \frac{5}{6}\) [24].

1.2.4 Implementing qubits with photons

There are several physical implementations of qubits, e.g. the spin of an
electron or a nucleus, energy levels of an atom, ion, or a superconducting
phase qubit, the current in a superconducting flux qubit or the charge of a
superconducting charge qubit. Here, another implementation will be used, namely the polarisation of a photon, where horizontal (H) and vertical (V) polarisation of a single photon represent the states $|0\rangle$ and $|1\rangle$, respectively. The horizontal polarisation direction is usually defined to be parallel to the optical table in the lab. Two other common basis sets apart from the $H/V$-basis are the $+/-$- and $L/R$-bases with basis states defined as

$$|+\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle),$$  \hspace{1cm} (1.26)  
$$|\rangle = \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle),$$  \hspace{1cm} (1.27)  
$$|L\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle),$$  \hspace{1cm} (1.28)  
$$|R\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i|V\rangle).$$  \hspace{1cm} (1.29)  

$|+\rangle$ and $|\rangle$ are linearly polarised at $\pm 45^\circ$, whereas $|L\rangle$ and $|R\rangle$ are left- and right-circularly polarised. The three bases mentioned here are so called mutually unbiased bases, meaning that all scalar products between basis states from different bases have the same absolute value, namely $D^{-1/2}$. $D$ is the dimension of the Hilbert space, which in the qubit case is two ($D = 2$).
1.3 Entanglement

Entanglement is an interesting concept of quantum mechanics concerning quantum systems of two (or more) qubits, which can only be described by one single quantum state although the qubits themselves may be separated by vast distances. Taking e.g. a superposition of the first and the last of the basis states in Eq. (1.17)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a \otimes |0\rangle_b + |1\rangle_a \otimes |1\rangle_b) ,$$

we see that this state is divided into two separate modes $a$ and $b$. Nevertheless can the system not be described by a product of one state for the first part (in mode $a$) and one state for the second part (in mode $b$)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a \otimes |0\rangle_b + |1\rangle_a \otimes |1\rangle_b) \neq |\psi_1\rangle_a \otimes |\psi_2\rangle_b ,$$

but only one pure state ($|\Phi^+\rangle$) would correctly describe the system. States that, like $|\Phi^+\rangle$, are not separable into two products are called entangled states. In the two-qubit Hilbert space there are four so-called maximally entangled states spanning an orthonormal basis. These states are known as the Bell states and are expressed as:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) ,$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) ,$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) ,$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) .$$

Theoretically there is no upper bound on how far from each other entangled qubits can travel and still only be described by one single quantum state. Experimentally, entanglement between two photons have been observed with a distance of 144 km between the two qubits [25].

In order to entangle two qubits there always has to be some kind of interaction between them. Using only local operations ($\hat{O}$) and classical communication, one would end up with a series of operators acting on one qubit and another series of operators acting on the second qubit. The result would always remain separable.

$$(\hat{O}_a \otimes \hat{O}_b)(|\psi_a\rangle \otimes |\psi_b\rangle) = \hat{O}_a |\psi_a\rangle \otimes \hat{O}_b |\psi_b\rangle$$
Ways to entangle photons include e.g. simultaneous creation in a non-linear process or interferometric overlaps on a beam splitter, combined with post-selection. One of the most extraordinary properties of entangled states is that they can exhibit perfect correlations between their constituents, regardless of which basis they are measured in.

### 1.3.1 Correlation functions

The joint probabilities for measurement results of multi-qubit states are calculated in the same way as for single qubit states. I.e. by the use of projection operators. Accordingly, the expectation value of e.g. \( \sigma_x \otimes \sigma_x \) acting on a two-qubit state \( |\psi\rangle \) is given by

\[
E(\sigma_x, \sigma_x) = \langle \psi | \sigma_x \otimes \sigma_x | \psi \rangle = \langle \psi | P_{++} - P_{+-} - P_{-+} + P_{--} | \psi \rangle ,
\]

where \( P_{++} = |+, +\rangle \langle +, +|, \) etc.

The correlation function \( C(X, Y) \) between two observables \( X \) and \( Y \) can now be defined as

\[
C(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E(X)^2} \sqrt{E(Y^2) - E(Y)^2}} .
\]

When \( C(X, Y) = 1 \), the state is said to be perfectly correlated in the observables \( X \) and \( Y \). If \( C = -1 \) the two qubits are perfectly anti-correlated, and with \( C = 0 \) there is no correlation between the measurement outcomes. Examples of correlations for three different types of quantum states (product, mixed, and entangled states) in three different bases are

\[
|\psi_p\rangle = |0\rangle \otimes |1\rangle = |01\rangle \Rightarrow \begin{cases} C(\sigma_z, \sigma_z) = 0 \\ C(\sigma_x, \sigma_x) = 0 \\ C(\sigma_y, \sigma_y) = 0 \end{cases} .
\]

\[
\rho_m = \frac{1}{2} (|01\rangle \langle 01| + |10\rangle \langle 10|) \Rightarrow \begin{cases} C(\sigma_z, \sigma_z) = -1 \\ C(\sigma_x, \sigma_x) = 0 \\ C(\sigma_y, \sigma_y) = 0 \end{cases} .
\]

\[
|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \Rightarrow \begin{cases} C(\sigma_z, \sigma_z) = -1 \\ C(\sigma_x, \sigma_x) = 1 \\ C(\sigma_y, \sigma_y) = 1 \end{cases} .
\]

The first two states are in a sense more classical, as they show no correlation at all or are correlated in only one basis, which can be explained by classical physics. The maximally entangled state, on the other hand, shows perfect (anti-) correlations in any basis.
1.3 Entanglement

The square of operators that are unitary and Hermitian, which is the case for most physically meaningful operators, equals the identity matrix. This simplifies the denominator of Eq. 1.38. Moreover, a single qubit of a maximally entangled state is completely mixed, yielding zero expectation value for the Pauli operators \( E(\sigma_i) = 0, \ i = z, x, y \). Hence, the correlation function can be greatly simplified according to \( C(X, Y) = E(XY) \) and in quantum information the term quantum correlation has often come to mean the expectation value. In general the correlation function for an \( n \)-qubit state is defined as the expectation value of the product of \( n \) local observables (one for each qubit). Admitting any dichotomic observables (any real superpositions of Pauli matrices) would give \( 2^n \) degrees of freedom in the measurement setup, and hence the correlation function is a function of \( 2^n \) independent variables.

Correlation functions are often used experimentally to show the coherence of entangled states. In such cases one usually rotates one local observable such that its eigenstates rotate around a great circle of the Bloch sphere, holding the other observables fixed. The correlation function will then have one degree of freedom and can easily be plotted against the rotated angle of the variable observable.

1.3.2 Bell inequalities

According to quantum mechanics a system cannot have definite values of two non-commuting physical properties. For example a qubit can never have definite values of the spin in both \( z \)- and \( x \)-direction, only probabilities for the different measurement outcomes are provided by quantum mechanics. This was a big concern for Einstein, Podolsky and Rosen, who argued that quantum mechanics could not be complete, being unable to predict the two properties [26]. Later on, so called, local hidden variable theories were proposed to complete quantum mechanics. These hidden variables, connected to each particle, should carry information about the result of any measurement that could be made on the particle. However, Bell’s theorem states that the correlations shown by entangled states, which are predicted by quantum mechanics, cannot be properly described by local hidden variable theories. There are two assumptions in these theories, of which at least one must be abandoned, due to Bell’s theorem. The first assumption, which is often referred to as realism, is that all physical properties have definite values independent of observation. The second assumption is that a measurement of one particle should not be able to influence the measurement result of another space-like separated particle. This follows from the special theory
of relativity and is called locality.

The best known version of Bell inequalities, which is very suitable for experiments, was proposed by Clauser, Horne, Shimony and Holt and is therefore called the CHSH inequality. It can be derived in the following way. Suppose Alice and Bob have one particle each to measure on. Alice can measure property $P_A$ of her particle to get their values $A_1$ and $A_2$ respectively, and Bob can measure $P_B$ of his particle in order to get their values $B_1$ and $B_2$. The result for each measurement is $A_1 = 1\pm 1$. If we now calculate the quantity $A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2$ we get

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 = (A_1 + A_2)B_1 + (A_1 - A_2)B_2 = \pm 2,$$  \hspace{1cm} (1.42)

since $A_1 + A_2 = \pm 2 \iff A_1 - A_2 = 0$ and $A_1 + A_2 = 0 \iff A_1 - A_2 = \pm 2$.

Suppose now that $p(1, a_2, b_1, b_2)$ is the probability for the system to be in a state where $A_1 = a_1$, $A_2 = a_2$, $B_1 = b_1$ and $B_2 = b_2$. The expectation value of the left hand side of Eq. (1.42) now obeys

$$\langle |A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2| \rangle = \left| \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot (a_1b_1 + a_1b_2 + a_2b_1 - a_2b_2) \right| =$$

$$\left| \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot (\pm 2) \right| \leq 2. \hspace{1cm} (1.43)$$

By rewriting the expectation value

$$\langle |A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2| \rangle = \left| \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot (a_1b_1 + a_1b_2 + a_2b_1 - a_2b_2) \right| =$$

$$\left| \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot a_1b_1 + \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot a_1b_2 + \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot a_2b_1 - \sum_{a_1, a_2, b_1, b_2} p(a_1, a_2, b_1, b_2) \cdot a_2b_2 \right| =$$

$$\langle |A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2| \rangle,$$  \hspace{1cm} (1.44)

and combining Eq. (1.43) and Eq. (1.44) we arrive at the CHSH inequality:

$$\langle |A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2| \rangle \leq 2. \hspace{1cm} (1.45)$$

As we will see now an entangled quantum state can violate this inequality. Let for example Alice and Bob share the maximally entangled Bell state
1.3 Entanglement

$|\Psi^{-}\rangle$, defined in Eq. (1.35), and measure the following observables:

$$A_1 = \sigma_x, \quad A_2 = \sigma_y, \quad B_1 = \frac{-\sigma_x - \sigma_y}{\sqrt{2}}, \quad B_2 = \frac{\sigma_y - \sigma_x}{\sqrt{2}}. \quad (1.46)$$

The first term in the inequality (1.45) yields

$$\langle A_1 B_1 \rangle = \langle \Psi^- | A_1 B_1 | \Psi^- \rangle = \langle \Psi^- | \frac{\sigma_x \sigma_y}{\sqrt{2}} | \Psi^- \rangle - \langle \Psi^- | \frac{\sigma_x \sigma_y}{\sqrt{2}} | \Psi^- \rangle = 0 - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \quad (1.47)$$

By similar calculations the other terms are found to be

$$\langle A_1 B_2 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_2 B_1 \rangle = \frac{1}{\sqrt{2}}, \quad \langle A_2 B_2 \rangle = \frac{1}{\sqrt{2}}. \quad (1.48)$$

Inserting the results of Eq. (1.47) and Eq. (1.48) into the CHSH inequality (Eq. (1.45)) yields

$$|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| = 2\sqrt{2} \not\leq 2. \quad (1.49)$$

Thus, according to quantum mechanics the Bell inequality can be violated, which is impossible in classical physics based on local hidden variable theories [22]. Many experiments have been performed, with results favouring the quantum mechanical predictions [6–9, 27].
Chapter 2

Quantum state engineering

In this chapter the experimental implementation of a scheme generating genuine multipartite entanglement will be described. We choose here to encode our quantum state in the polarisation of photons. Hence we need to engineer the photons in such a way that we can factor out the whole wave function describing the system of photons, except the polarisation part. The various techniques used to accomplish this and to engineer a six-photon entangled quantum state will be discussed here.

2.1 Optical components for photon processing

Here we start to describe a few of the basic optical components that are used in the experiment.

2.1.1 Wave plates

A wave plate consists of a birefringent media, usually quartz, and introduces a certain phase shift (retardation) between the two orthogonal linear polarisations parallel to, respectively, the slow and the fast optical axis of the wave plate (see Figure 2.1). As we will see, wave plates can be used for rotating the polarisation of transmitted light. The two most important and most common wave plates are the half wave plate (HWP) and the quarter wave plate (QWP), which introduce a phase shift of $\pi$ and $\pi/2$ respectively. A wave plate can be modelled with the use of the coordinate system rotation
matrix, $\hat{R}(\nu)$, and the phase shift matrix, $\hat{p}(\Phi)$, given by

$$\hat{R}(\nu) = \begin{pmatrix} \cos \nu & \sin \nu \\ -\sin \nu & \cos \nu \end{pmatrix}, \quad (2.1)$$

$$\hat{p}(\Phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Phi} \end{pmatrix}. \quad (2.2)$$

In these equations $\nu$ is the angle between the fast axis of the wave plate and the lab frame $H$-axis, and $\Phi$ is the retardation of the slow axis compared to the fast axis of the wave plate. The matrix representing a general wave plate, $\hat{W}(\nu, \Phi)$, can now be obtained as follows: First the coordinate system is rotated so that the $x$-axis is aligned to the fast axis of the wave plate, then the phase shift is added, and finally the coordinate system is rotated back.

$$\hat{W}(\nu, \Phi) = \hat{R}(-\nu)\hat{p}(\Phi)\hat{R}(\nu) =$$

$$= \begin{pmatrix} \cos^2 \nu + e^{i\Phi} \sin^2 \nu & \frac{1}{2}(1 - e^{i\Phi}) \sin(2\nu) \\ \frac{1}{2}(1 - e^{i\Phi}) \sin(2\nu) & \sin^2 \nu + e^{i\Phi} \cos^2 \nu \end{pmatrix}. \quad (2.3)$$

If one sends linearly polarised light into a wave plate the output light will in general be elliptically polarised, but there are important special cases:

- A HWP@45° ($\nu = 45° = \pi/4$) is represented by

$$\hat{W}\left(\frac{\pi}{4}, \pi\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \quad (2.4)$$

and turns $H$ to $V$ and vice versa.

- A HWP@22.5° ($\nu = 22.5° = \pi/8$) is represented by

$$\hat{W}\left(\frac{\pi}{8}, \pi\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z), \quad (2.5)$$
which is called the Hadamard operator and turns $H$ to $+$ and $V$ to $-$. 

- A QWP@45° ($\nu = 45° = \pi/4$) is represented by 

$$
\hat{W}(\frac{\pi}{4}, \frac{\pi}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (1 - i\sigma_x),
$$

and turns $H$ to $L$ and $V$ to $R$.

In the examples above any global phase shifts have been removed.

The introduced phase shift $\Phi$ is governed by the thickness ($t$) and the birefringence ($\Delta n$) of the plate according to $\Phi = 2\pi t \Delta n/\lambda$. Since $\Phi$ typically is of the order of $\pi$ and $\Delta n \sim 0.01$, the thickness will be very small. Such plates (called true zero-order wave plates) are however available, especially for telecom wavelengths (1300 nm - 1700 nm) with thicknesses of the order of 100 $\mu$m. More common are multiple order plates, which are designed to introduce phase shifts of several full waves plus the desired phase shift ($\Phi = m \cdot 2\pi + \Phi_{des}$). The multiple order wave plates are thus thicker and easier to handle, but they are also more sensitive to changes in wavelength, temperature, and angle of incidence. A third type of wave plates are the so called compound zero order plates or just zero order plates. They are constructed of two multiple order plates with their optical axes crossed. As a result most of the retardation in each plate is cancelled by the other one and only a small retardation corresponding to the difference of the two compound plates remains. In this way wave plates with $m = 0$ (zero order) can be manufactured for all optical wavelengths and the advantages of the true zero order plates can be combined with easy handling. The compound zero order wave plates are what we use in our lab.

### 2.1.2 Beam splitters

A beam splitter (BS) is an optical component that divides an incident beam of light into two output beams. The most common form of BS is the beam splitter cube and that is also what we use here. It is composed of two right angle triangular glass prisms glued together at their hypotenuse surfaces. These surfaces are coated with a dielectric coating, whose type and thickness determine the reflectance and transmittance of the BS. Another common form of BS is the beam splitter plate, which consists of a glass plate with a partially reflecting coating on one side. They are usually designed for an incident angle of 45°.
In quantum mechanics, a general lossless BS introduces the following transformations, where the creation operators ($\hat{a}^\dagger$ and $\hat{b}^\dagger$) correspond to the spatial modes $a$ and $b$ in Figure 2.2, respectively [28]:

\[
\begin{align*}
\hat{a}_H^\dagger & \to T_H \cdot \hat{a}_H^\dagger + e^{i\delta_{R,H}} R_H \cdot \hat{b}_H^\dagger, \\
\hat{b}_H^\dagger & \to T_H \cdot \hat{b}_H^\dagger - e^{-i\delta_{R,H}} R_H \cdot \hat{a}_H^\dagger, \\
\hat{a}_V^\dagger & \to e^{i\delta_{T,V}} T_V \cdot \hat{a}_V^\dagger + e^{i\delta_{R,V}} R_V \cdot \hat{b}_V^\dagger, \\
\hat{b}_V^\dagger & \to e^{-i\delta_{T,V}} T_V \cdot \hat{b}_V^\dagger - e^{-i\delta_{R,V}} R_V \cdot \hat{a}_V^\dagger.
\end{align*}
\]

$T_H$ ($R_V$) $\in \mathbb{R}$ is the absolute value of the transmission (reflection) amplitude for an incident photon of horizontal (vertical) polarisation. From unitarity follows that $T_H^2 + R_H^2 = T_V^2 + R_V^2 = 1$ (the normalisation relation). The phases $\delta_{T,V}$, $\delta_{R,H}$ and $\delta_{R,V}$ can take any values, but $\delta_{T,H} = 0$ corresponding to a reference phase. The differences of these phases lead to a phase shift between $H$ and $V$ in each mode, which we compensate for by using wave plates. HWPs and QWPs set at $0^\circ$ or $90^\circ$ (with their fast optical axes either horizontally or vertically) would usually introduce phase shifts between $H$ and $V$ of $\pm \pi$ and $\pm \pi/2$. However, by rotating the plates around their vertical axes, the effective thickness increases and hence also the phase shift increases. We have found that by appropriately choosing the type of wave plate (HWP or QWP) and the orientation of its fast optical axis ($0^\circ$ or $90^\circ$), one can compensate for any phase shift ($0$-$2\pi$) between $H$ and $V$ by rotating the wave plate up to around $30^\circ$. However, phase shifts of multiple wavelengths ($m \cdot 2\pi$) will not be fully compensated for.

There are two special cases of BSs that are very common, the non-polarising symmetric 50/50 beam splitters (NPBS or just BS) and the polar-
isising beam slitters (PBS). Additionally BSs can be designed to have basically any splitting ratio for both horizontal and vertical light.

The ideal 50/50 BS has the same reflectance and transmittance for both H and V, and thus \( T_H = R_H = T_V = R_V = 1/\sqrt{2} \). Real BSs are, however, never perfect but often have different reflectance and transmittance (i.e. spatial asymmetry) and/or different reflectance (and transmittance) for H and V (polarising). By aligning the BS properly one can often adjust it to behave polarisation independently with \( T_H \approx T_V \) and \( R_H \approx R_V \), but with different transmittance and reflectance. In our case the BSs had \( T_{H}^{2} \approx T_{V}^{2} \approx 0.55 \) and \( R_{H}^{2} \approx R_{V}^{2} \approx 0.45 \) after fine adjustments.

Polarising beam splitters (PBSs) are specified to have zero transmittance for V and zero reflectance for H, and are consequently completely polarising. For the PBSs in the lab we find after optimising alignment that \( T_{V}^{2} \approx R_{H}^{2} \approx 0.001 \). The polariser is another component for achieving polarised light. It transmits light of some linear polarisation \( \langle P \rangle \) and absorbs light of the perpendicular polarisation. Mathematically it is described with the projection operator \( \langle P \rangle \langle P \rangle \), where \( \langle P \rangle \) denotes the transmitted polarisation state.

### 2.1.3 Optical fibers

Optical fibers basically consist of a core surrounded by a cladding, where the cladding usually has a slightly lower index of refraction than the core. In addition there can be some protection surrounding the cladding. The core and the cladding are often made of silica (amorphous silicon dioxide, SiO\(_2\)). Despite this, optical fibers show a small amount of birefringence, due to mechanical stress and strain in the fibers. This causes the polarisation to be rotated in the fiber, depending on how it is bent and by temperature changes etc. There are two main types of optical fibers that we use in the experimental setup: Multi-mode and single-mode fibers.

#### Multi-mode fibers

In multi-mode fibers the core usually has a diameter of 50 \( \mu \)m or more, supporting several spatial propagation modes entering at different angles. Rays entering the fiber at a larger angle will then travel longer paths, causing intermodal dispersion.
Single-mode fibers

Single-mode fibers (SMF) are optical fibers with a very small core diameter (typically a few µm) and a small refractive index difference between core and cladding. For a given wavelength and polarization they only transmit one single spatial propagation mode, and hence they can be used to single out one spatial mode from a multi-mode light beam. An important property of SMFs is that the spatial distribution of light exiting the fiber carries no information whatsoever about the spatial distribution of the light in front of the fiber. When changing the input coupling, only the power being transmitted through the fiber is affected [29].

2.2 Pump laser

The first component in the chain leading to six entangled photons is a powerful femtosecond laser. The laser is pulsed at 80 MHz, with a pulse length around 140 fs and an average power of 3.3 W, corresponding to an average power of 0.3 MW within each pulse. The tunable wavelength is set to 780 nm with a full width at half maximum (FWHM) of about 5.5 nm. The horizontally polarised laser light is then focused onto a 1 mm thick BIBO (BiB₃O₆, bismuth triborate) crystal for second harmonic generation (SHG) [30], yielding an intense beam of vertical polarisation in the UV region at 390 nm and a bandwidth (FWHM) of 1.1 nm. A series of dichroic mirrors serve to separate the near IR from the UV light. From the bandwidth of the UV light it follows that the UV pulses must be prominently longer than the original near IR pulses. According to the Fourier transform limit, the UV pulses must be longer than 200 fs. The power of the UV light is about 1.3 W, which gives a conversion efficiency around 40%. The strong non-linearity of the BIBO crystal does not only give us a high conversion efficiency but it also causes transversal walk-off in the crystal reducing the beam quality unsymmetrically. Instead of having an almost perfect beam quality ($M^2_H = M^2_V = 1$) as from the laser, the UV beam after the SHG has $M^2_H = 1.70$ and $M^2_V = 1.24$ making it elliptical. $M^2$ is a measure of beam quality, see [31–33] for details. Spherical and cylindrical lenses are now used to shape the UV beam and to focus it onto a BBO (beta barium borate) crystal for parametric down-conversion. The two waists are about 168 µm (for $H$) and 153 µm (for $V$) and the Raileigh length of the foci are around 130 mm and 150 mm, respectively.
2.3 Parametric down-conversion

Parametric down-conversion (PDC) in non-linear BBO crystals has proved to be very efficient for generation of entangled photon pairs [34] and is also used in the experiment described in this thesis. The process involves the conversion of one pump photon into two less energetic photons historically called the signal and the idler photon. Energy and momentum conservation in the process introduces strong correlations in energy, momentum, emission time and polarisation between the signal and the idler photon. The conversion process of one pump photon is spontaneous (stimulated by vacuum fluctuations) and hence it is known as spontaneous parametric down-conversion (SPDC). Since the conversion of a pump photon is independent of previous conversions and is a dichotomic process (each pump photon is either converted or not converted), the number of created pairs within a time $\Delta t$, follows a binomial distribution $\text{Bin}(n, p)^1$, where $n$ is the average number of pump photons entering the crystal per $\Delta t$ and $p$ is the probability for each photon to be down-converted. In (S)PDC the probability $p$ is typically very small ($p \sim 10^{-12}$ as we will see in section 4.2) and the photon number $n$ is very large, making the Poisson distribution $\text{Po}(\mu = np)^2$ a good approximation to $\text{Bin}(n, p)$. Thus, the number of converted pairs follows a Poisson distribution with mean value $\mu = np$ [35].

There are two versions of PDC, simply called type-I and type-II. In type-I PDC in a BBO crystal the pump photons should be of extraordinary polarisation and the signal and idler will both be of ordinary polarisation. In type-II, on the other hand, the pump photons should again be of extraordinary polarisation, but one of the generated photons will be of ordinary and the other of extraordinary polarisation, making it possible to obtain polarisation-entangled photons. One can also obtain polarisation-entangled photons from type-I PDC, but then with the use of two crystals, with their optical axes orthogonal to each other, and the pump polarisation at an angle to the optical axes. In the work described here only type-II PDC is used, so type-I will not be discussed further.
2.3.1 Type-II PDC

Using type-II PDC, the down-converted photons are emitted onto two cones with orthogonal polarisation (ordinary and extraordinary). Due to its birefringence, tilting of the BBO crystal will increase the opening angle of the two cones and the cones will eventually intersect. See Figure 2.3 for the case of degenerate wavelengths, i.e. when the wavelengths of the signal and the idler are the same ($\lambda_s = \lambda_i$). In Figure 2.3(a) the two cones intersect at one line parallel to the pump beam. This is known as collinear type-II PDC. After more tilting of the crystal the situation depicted in Figure 2.3(b) is reached. In this case, called non-collinear type-II PDC, the two cones cross at two non-parallel lines. See Figure 2.4 for photos showing cross sections of the emission cones, taken with a single photon sensitive CCD camera. In the experiment the degenerate case of non-collinear type-II PDC is used and the upper cone has vertical polarisation and the lower cone has horizontal polarisation. Since the intersection lines between the two cones are symmetric around the pump, a signal photon in one of the crossings will always have its corresponding idler in the other crossing and vice versa, i.e. the signal and the idler are indistinguishable in their spatial modes. Their spectral indistinguishability in the degenerate case have already been mentioned and in the limit of a thin crystal they are also indistinguishable in time arrival, because of their simultaneous creation. Now, as a signal and an idler photon in one of the crossings are completely indistinguishable apart from their polarisations,

\[ p_X(n) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ for } k = 0, 1, \ldots, n \text{ and } 0 < p < 1 \]

\[ p_X(k) = \mu^k e^{-\mu}/k!, \text{ for } k = 0, 1, 2, \ldots \]
the whole wave function for the emitted pair, except its polarisation part, can be factored out. This yields the polarisation-entangled Bell-state $|\Psi^+\rangle$ given by

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|H_a V_b\rangle + |V_a H_b\rangle), \quad (2.11)$$

where $H_a$ ($V_b$) denotes a horizontally (vertically) polarised photon in spatial mode $a$ ($b$).

### 2.3.2 Multipartite entanglement

When moving toward more advanced quantum information schemes, with multiple parties, multipartite entanglement (entanglement between more than two qubits) is a valuable resource. Multipartite entangled states encoded with photons are most frequently obtained by taking independent entangled pairs, like in Eq. (2.11) and overlapping one photon from each pair in a beam splitter [12, 13, 36], utilising quantum interference [10, 37]. With techniques like this, one needs to carefully match path lengths and spatial overlaps. It is possible though, to obtain multipartite entanglement directly from one PDC source, without the use of fragile overlaps, making the setup more robust [38–40]. In order to achieve this two or more pump photons should be down-converted coherently. How this is done will be described in the next section (2.4).
In contrast to the creation of two-partite entangled pairs, the coherent creation of two or more pairs is not completely spontaneous, due to interference effects (stimulated emission) in the BBO crystal. This will be shown below when we write out the states and their probabilities for the non-collinear emission that we have used in our experiment.

**Non-collinear emission**

The state of the multiphoton emission, originating from a single pulse, in non-collinear type-II PDC can be expressed as

\[
|\text{PDC}\rangle = \frac{1}{\cosh^2 K} \sum_{p=0}^{\infty} \tanh^p K \sum_{m=0}^{p} e^{im\phi} |mH_a, (p-m)V_a, (p-m)H_b, mV_b\rangle ,
\]

(2.12)

where \( |mH_a\rangle \) denotes the Fock state with \( m \) horizontally polarised photons in mode \( a \), etc \[41,42\]. \( K \) is a function of non-linearity and length of the crystal, pump power and filtering bandwidth, and \( \phi \) is the phase difference between horizontal and vertical polarisation due to birefringence in the crystal. This is a good quantum optical description provided that all photon pairs are indistinguishable. The \( n \)-th order PDC emission is obtained by only taking the terms corresponding to \( p = n \) from Eq. (2.12).

With the phase \( \phi = 0 \) the first order term, corresponding to the emission of two photons, is proportional to

\[
|H_aV_b\rangle + |V_aH_b\rangle ,
\]

(2.13)

which agrees with Eq. (2.11). Now we will continue to look at four-partite entanglement, which yields the simplest multipartite entangled state obtained directly from PDC. As the down-converted photons always come in pairs, three-partite entanglement is omitted here. The second order term of Eq. (2.12), with \( p = 2 \), corresponding to the emission of four photons is proportional to

\[
|2H_a, 2V_b\rangle + |1H_a, 1V_a, 1H_b, 1V_b\rangle + |2V_a, 2H_b\rangle ,
\]

(2.14)

It should be noted here that the weights of the different terms in Eq. (2.14) are equal. This is not what one would expect for a product of two pairs. In this case one would get the following biseparable state

\[
(|H_aV_b\rangle + |V_aH_b\rangle) \otimes (|H_aV_b\rangle + |V_aH_b\rangle) =
\]

\[
= |2H_a, 2V_b\rangle + 2|1H_a, 1V_a, 1H_b, 1V_b\rangle + |2V_a, 2H_b\rangle .
\]

(2.15)
Comparing this to Eq. (2.14) shows that multi-order PDC is fundamentally and intrinsically different than products of several entangled pairs. Due to the bosonic nature of photons the emission of completely indistinguishable photons are favoured compared to photons that have orthogonal polarisation but are otherwise indistinguishable.

Another interesting point to look into is the rate of higher order emission events compared to the rate of first order processes. This can be of importance when calculating noise originating from higher order terms, as well as when comparing multiphoton sources based on multiorder emission from one crystal and sources with several crystals. In general the probability for an $n$:th order process in one pulse is given by

$$P_n = \frac{\tanh^{2n} K}{\cosh^4 K} \cdot (n + 1),$$

which is derived from Eq. (2.12). In section 4.2 we will use this and the measurement data to assign a numerical value to $K$.

### 2.4 Processing PDC photons

The parametric down-conversion process described in the previous section is the process in which the entanglement is created. It is however not trivial to realise a specific polarisation entangled state, which we want to do here. The photons from the PDC source must be processed further as will be discussed now.

The product state in Eq. (2.15) is not only obtained when a product state is desired, e.g. by the use of two BBO crystals in series. Also when striving for a state like in Eq. (2.14) attained through a higher order process, a mixture with the product state will be obtained if there is the slightest distinguishability (temporal, spectral or spatial) between down-converted photons from the different pairs. In this case $a'$ and $b'$ in Eq. (2.15) would denote some kind of distinguishability from $a$ and $b$, respectively. For this reason much effort must be made to achieve good indistinguishability in order to get a final state of high quality.

#### 2.4.1 Spatial overlapping and transversal walk-off

In the PDC section we assumed that the photons were emitted into single spatial modes. In practice, however, the photons are emitted in a continuum of directions, so in order to obtain our final state the next step is to pick out
only the two relevant single spatial modes. To do this we use single-mode fibers (SMFs), which are convenient for selecting only one spatial mode.

The two spatial single modes are precisely defined by carefully coupling the intersections of the two frequency-degenerate cones from the down-conversion into two single-mode fibers. However, when doing this, transversal walk-off effects cause difficulties in coupling $H$ and $V$ optimally at the same time. Transversal walk-off is an effect due to birefringence, where an extraordinary polarised ray (e-ray) will travel through the birefringent media at an angle, but leave the media parallel to the incoming ray [43, 44]. The result is that the e-ray is transversally displaced, see Figure 2.5. Due to the orientation of the BBO crystal in our experiment, the extraordinary polarisation is the same as vertical, causing a displacement of $V$ relative to $H$. One way of suppressing this effect is to use another identical BBO crystal with half the thickness (1 mm) of the PDC-BBO, preceded by a half wave plate at 45°, rotating $H$ to $V$ and vice versa. Since a pair of down-converted photons has equal probability to be created at any depth in the crystal, the average creation depth is in the middle, i.e. at 1 mm for a 2 mm thick crystal. The walk-off angle is 73 mrad, so in the down-conversion crystal $V$ is displaced by a distance between zero and 146 $\mu$m (on average 73 $\mu$m) relative to $H$. When entering the compensation crystal $H$ and $V$ have been interchanged and the ‘new’ $V$ is displaced 73 $\mu$m. Transversal walk-off will also affect the pump beam, which also is of extraordinary polarisation. The walk-off angle for the pump is 77 mrad and it will therefore closely follow the e-ray of the down-conversion. However, as the pump traverses the crystal at an angle, the down-conversion of ordinary polarisation will proceed parallel to the incoming beam. This can cause blurring of the ordinary (horizontally) polarised light, which is sometimes seen in photos of down-conversion rings.

Another complication that occurs when working with fibers is the unstable polarisation, briefly discussed in section 2.1.3. In our experiment it is of great importance that the photons that are guided through the SMFs keep their polarisation, since that is what we encode the qubits in. To deal with this we have made passive polarisation controllers, that allow us to set the effective optical axes of the SMFs parallel or orthogonal to the $H$ direction. Doing this will ensure that coupled $H$- and $V$-photons will retain their polarisation, but there will also be a phase introduced between $H$ and $V$. This phase can be compensated for by tilting one of the compensation crystals. The polarisation controllers are designed to behave as a QWP followed by a HWP and another QWP [45]. This arrangement should make it possible to rotate an arbitrary pure polarisation state into any other pure polarisation state. In practice however the controllers are far from behaving as ideal
2.4 Processing PDC photons

Figure 2.5: Transversal walk-off and its compensation.

QWPs and HWPs, but nevertheless they are surprisingly good for adjusting the effective optical axis along $H$- or $V$-direction as described above.

2.4.2 Spectral filtering and coherence time

Already by doing the spatial filtering a rough spectral filtering is automatically obtained. This is due to the correlations between the wavelength and the opening angle of the emission cones. This coarse filtering is, however, usually too broadband to achieve a good visibility of the state. Here we use additional 3 nm (FWHM) interference filters to narrow the spectral width, and hence to increase the coherence time (length) of the down-converted photons. The relation between coherence time $\tau_c$ (FWHM) and spectral width ($\Delta \lambda$, FWHM) is given by

$$\tau_c = \frac{2 \ln 2 \cdot \lambda^2}{\pi c \Delta \lambda} \approx \frac{1.47 \cdot 10^{-9} \cdot \lambda^2}{\Delta \lambda},$$ \hspace{1cm} (2.17)

leading to $\tau_c \approx 300$ fs for $\lambda=780$ nm and using the 3 nm filter. In the calculation of the coherence time in Eq. (2.17) the interference filter is assumed to have Gaussian spectral transmission, which is a good approximation of the filters we use in the lab. There is a trade-off to take into account when choosing a narrow band filter. The narrower it is, of course the better spectral indistinguishability one will get, but the measurement count rate will drop. It should be noted that since the pump is broadband it is likely that many down-conversion pairs will have one of its photons outside the spectral transmission range of the filter, leading to a decreasing ratio of coincidence counts to single counts, when moving toward narrower filters.
2.4.3 Indistinguishability by time of arrival

Since the PDC processes in the BBO crystal can occur anywhere within the pump pulse, it has to be short in order to yield indistinguishability between photons from different pairs with respect to time of arrival. The near IR laser has a specified pulse length of 140 fs ± 20 fs, but we know that the pulse length of the UV pulses obtained from the SHG are at least 180 fs due to the Heisenberg uncertainty (Δλ = 1.2 nm). However, the UV pulses are longer than that due to chromatic dispersion in the crystal. A more likely UV pulse length would be between 200 fs and 250 fs [35, 46, 47]. Depending on if the down-conversion happens at the front or at the back edge of the pulse, the temporal difference could be more than 300 fs. The relative power, however, belonging to these outer parts of the pulse is small, leading to low probabilities of such big time differences. The major part of the power is concentrated within a shorter time period.

Chromatic dispersion in the BBO (the UV light travels slower than the near IR light) can also cause time differences between different pairs. The average time difference between two down-converted pairs from two different conversions is 126 fs and the maximum time difference is 377 fs.

Moreover, since the crystal is birefringent H- and V-polarised photons will travel with different speeds through the crystal. In this case the traversing time for the ordinary (H-) polarised photons is 200 fs/mm longer than for extraordinary (V-) photons, causing V-photons to always arrive to the detectors before the H-photons. This effect is called longitudinal walk-off. Conveniently enough, the arrangement to reduce the effects of transversal walk-off mentioned above also reduces the longitudinal walk-off in an optimal way. With this arrangement the maximum time difference between H- and V-photons originating from the same conversion process is 200 fs and the average time difference is zero. The extra crystals used for walk-off compensation can also serve as a convenient instrument to set the phase $\phi$ between H and V in Eq. (2.12). Due to their birefringence, just by slightly tilting one of the compensator crystals the effective path length difference between H and V is altered, making it possible to change $\phi$ to any desired value.

All together, the maximum time difference between different photons from higher order PDC processes within one pulse is in the order of the coherence time. I.e. the photons will be in a coherent superposition and can thus be used to show multi-photon entanglement.
2.5 Distributing the qubits

In the case of generating a multi-qubit state from a higher order process of PDC, more than one qubit is obtained in each mode. Thus we have a superposition of photon number states, that in a general form can be written as

\[ A_1 |n_{H_a},0_{V_a},n_{H_b},0_{V_b}\rangle + A_2 |n_{H_a},0_{V_a},(n-1)_{H_b},1_{V_b}\rangle + \ldots \\
\cdots + A_{n^2} |0_{H_a},n_{V_a},0_{H_b},n_{V_b}\rangle, \]  

(2.18)

where \( n \) is the number of photons in each mode (\( a \) and \( b \)) and \( A_i \) is the amplitude connected to the \( i \):th term. We want, however, to prepare a multi-qubit state, where the qubits are somehow separated, so each qubit can be studied independently of the others. Here, further splitting into different spatial modes is used to distinguish the qubits, i.e. the qubit detected in spatial mode \( a \), will be called qubit \( a \) and so on. This makes it easy to perform local operations on each qubit, which is crucial in many quantum information schemes. It is also necessary for the detection, since we do not have photon number resolving detectors. The qubit distribution into different modes is the last step in the state preparation. Conditioned on that after the distribution there is one photon in each mode, the final polarisation entangled multi-photon state is formed. The general form of the state then looks like

\[ \tilde{A}_1 |H,\ldots,H,H\rangle_{abc\ldots} + \tilde{A}_2 |H,\ldots,H,V\rangle_{abc\ldots} + \ldots \\
\cdots + \tilde{A}_{2^n} |V,\ldots,V,V\rangle_{abc\ldots}, \]  

(2.19)

where \( \tilde{A}_i \) is the amplitude connected to the \( i \):th term, and \( a, b, c, \ldots \) are the new spatial modes after the beam splitters.
Chapter 3

Quantum state analysis

3.1 Polarisation analysis

The quantum states now need to be analysed and characterised. Since we have used polarisation encoding, we also need to perform polarisation analysis. In our experiment we perform local projective measurements on each qubit. In general the measurement results will depend on the choice of projectors, or equivalently the choice of measurement basis.

The polarisation analysis setup, depicted in Figure 3.1, consists of a HWP, a QWP, a PBS and finally two single photon detectors coupled through multi-mode fibers at the two output ports of the PBS. With this setup the polarisation can be measured in an arbitrary basis $|Q\rangle / |Q\rangle_\perp$, determined by the angular settings of the two wave plates. I.e. any polarisation state $|Q\rangle$ can be rotated to linear polarisation in $H$-direction by operating with a HWP and a QWP in specific angles. The state $|Q\rangle_\perp$ orthogonal to $|Q\rangle$, will then be rotated to linear polarisation in $V$-direction. Let now the unitary operator $\hat{U}_{\text{QH}}(\nu_q, \nu_h)$ be the product of the QWP and the HWP according to

$$\hat{U}_{\text{QH}}(\nu_q, \nu_h) = \hat{W}_{\text{QWP}}(\nu_q, \pi/2) \cdot \hat{W}_{\text{HWP}}(\nu_h, \pi),$$

(3.1)

where $\nu_q$ and $\nu_h$ are the angular settings for the QWP and the HWP. The wave plate operator $\hat{W}$ is described in section 2.1.1. As was stated above when $|Q\rangle$ is rotated to $|H\rangle$, $|Q\rangle_\perp$ is rotated to $|V\rangle$. Additionally there will be an unimportant phase difference ($\Delta$) between the two cases:

$$\hat{U}_{\text{QH}}(\nu_q, \nu_h) |Q\rangle = |H\rangle,$$

(3.2)

$$\hat{U}_{\text{QH}}(\nu_q, \nu_h) |Q\rangle_\perp = e^{i\Delta} |V\rangle.$$

(3.3)
If Eq. (3.2)-(3.3) are multiplied from the right with $\langle Q \rangle$ and $\langle Q \rangle_\perp$, respectively, we obtain

$$\hat{U}_{QH}(\nu_q, \nu_h) \langle Q \rangle \langle Q \rangle = \langle H \rangle \langle Q \rangle,$$  (3.4)

$$\hat{U}_{QH}(\nu_q, \nu_h) \langle Q \rangle_\perp \langle Q \rangle_\perp = e^{i\Delta} \langle V \rangle \langle Q \rangle_\perp.$$  (3.5)

Adding Eq. (3.4) and Eq. (3.5) leads to

$$\hat{U}_{QH}(\nu_q, \nu_h) \cdot (\langle Q \rangle \langle Q \rangle + \langle Q \rangle_\perp \langle Q \rangle_\perp) = \hat{U}_{QH}(\nu_q, \nu_h) \cdot 1 =$$

$$= \hat{U}_{QH}(\nu_q, \nu_h) = \langle H \rangle \langle Q \rangle + e^{i\Delta} \langle V \rangle \langle Q \rangle_\perp.$$  (3.6)

The angular settings for the two wave plates in order to perform measurements in the most common bases ($H/V$, $+/\$ and $L/R$) are given in Table 3.1.

<table>
<thead>
<tr>
<th>Basis</th>
<th>$\nu_h$ (HWP)</th>
<th>$\nu_q$ (QWP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H/V$</td>
<td>0°</td>
<td>0°</td>
</tr>
<tr>
<td>$+/$</td>
<td>22.5°</td>
<td>0°</td>
</tr>
<tr>
<td>$L/R$</td>
<td>0°</td>
<td>45°</td>
</tr>
</tbody>
</table>

Let us now turn to the next component of the analysis station, the PBS.
3.1 Polarisation analysis

A PBS used with one input port introduces the following transformations

\[
\hat{a}_H^\dagger \rightarrow \hat{h}_H^\dagger ,
\]
where we have named the two spatial output modes \(h\) and \(v\), respectively. In this application it is not really important which polarisation the photon has after the PBS, so we can trace that out and since we only use one input port, the input mode can be factored out. Hence the single input PBS operator \(\hat{U}_{PBS}\) can be expressed as

\[
\hat{U}_{PBS} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta_{R,V}} \end{pmatrix} = |h\rangle \langle H| + e^{i\delta_{R,V}} |v\rangle \langle V| .
\]

Note that this operator does not represent a qubit rotation, but maps a qubit encoded in photon polarisation to a qubit encoded in path.

The operator corresponding to the polarisation analysis setup can now be written as

\[
\hat{U}_{PBS} \cdot \hat{W}_{QWP}(\nu_q, \pi/2) \cdot \hat{W}_{HWP}(\nu_h, \pi) = \hat{U}_{PBS} \cdot \hat{U}_{QH}(\nu_q, \nu_h) = |h\rangle \langle H| + e^{i\delta_{R,V}} |v\rangle \langle V| =
\]

where \(|Q\rangle = \hat{U}_{QH}^\dagger(\nu_q, \nu_h)|H\rangle\) and hence \(\langle Q\rangle = \langle H| \hat{U}_{QH}(\nu_q, \nu_h)\).

Consequently, an incident photon will be projected onto the measurement basis \(|Q\rangle / |Q\rangle_\perp\), and will after the PBS be in a superposition of being in output mode \(h\) and in output mode \(v\). What we actually have done here is a basis change from an arbitrary polarisation basis to the computational basis of a path qubit: the photon being in spatial mode \(h\) or \(v\). This basis is the natural choice for detection and finally the photon wave function will collapse in one of the detectors in spatial mode \(h\) or \(v\).

For multi-photon polarisation analysis we use one polarisation analysis station, depicted in Figure 3.1 that corresponds to the local operator in Eq. (3.10), for each qubit. A local operator can be described as a product of operators which act in single-qubit Hilbert spaces. Using only such local operators it is impossible to make a basis transformation from a basis with entangled basis states to a basis with separable basis states, and vice versa. Hence we cannot measure a multi-qubit state in an arbitrary basis with this setup. When analysing multi-qubit states we are using the subset of projections, constituting of local multi-qubit projections, which can
Quantum state analysis

be implemented by projections of each single qubit. In other words, we are only measuring in bases with separable basis states, such as the two-qubit $+/−$-basis with basis states $|++\rangle$, $|+-\rangle$, $|-+\rangle$ and $|--\rangle$. This also means that we cannot, by only one measurement setting, detect if a measured state is entangled or not. An example of a projective measurement that cannot be done only by local projections is a Bell-measurement, where we would like to make a basis change from the Bell basis, spanned by the Bell states given by Eq. (1.32)-(1.35), to e.g. the two-qubit spatial mode $h_a/v_a \otimes h_b/v_b$-basis. The reason for this is that the Bell states are entangled states and thus non-local operations are needed.

When analysing measured data one needs to take into account that the different polarisation analysis stations do not have equal efficiency. The main reasons for this are that the detectors have different quantum efficiencies and dark count rates, and that the multi-mode fiber couplings usually differ by a few percent. Since we are interested in events with maximally one detection per polarisation analysis station, it is the relative efficiency between the two outputs in each station that is important. In order to measure this relative efficiency we have used a polariser just in front of the analysis station to be tested. The polariser guarantees that we enter with linearly polarised light. Now the HWP is rotated until we have maximised (minimised) the count rate in mode $h$ ($v$). Then the HWP is rotated again until we have the opposite situation. The dark count rate is then subtracted from the maximum count rate for each arm, and we calculate the ratio between the two differences:

$$\eta^i_h = 1 \Rightarrow \eta^i_v = \frac{M_v - D_v}{M_h - D_h},$$

where $M$ and $D$ are the maximum and the dark count rates in the two output modes and the superscript $i$ specifies to which mode the current analysis station belongs. The corrected coincidence count rate $c(l_a,l_b,\ldots)$, can now be calculated from the measured coincidence count rate $\tilde{c}(l_a,l_b,\ldots)$ according to

$$c(l_a,l_b,\ldots) = \frac{\tilde{c}(l_a,l_b,\ldots)}{\eta^a_l \cdot \eta^b_l \cdot \ldots},$$

where $l_a,l_b,\ldots$ are the detection modes (either $h$ or $v$, corresponding to the eigenstates with eigenvalue $\pm 1$) in the different analysis stations ($a,b,\ldots$).

### 3.2 Avalanche photodiodes

We are using silicon based avalanche photodiodes (APDs) as single photon detectors. Their quantum efficiency is about 55% and their dark count rate
3.3 Multi-channel coincidence unit

typically lies between 300 Hz and 600 Hz. They are operated in Geiger mode, i.e. the reverse bias voltage lies above the breakdown voltage, in our case around 15 V above. This would normally lead to a large current flow through the diode, but to avoid this the current must be quenched. When a photon is absorbed an electron-hole pair is broken and the two charge carriers are accelerated in opposite directions by the high bias voltage (several hundred volts). Both the free electron and the hole can now break more electron-hole pairs, creating an avalanche current, which is easily detectable. The detection of the high current now triggers an active quenching circuit which decreases the bias voltage to roughly 15 V below breakdown voltage and the avalanche will stop [48, 49]. The electron-hole pairs can also be broken by thermal excitations, leading to so called dark counts. After a detection the APDs output a TTL signal of a pulse width around 21 ns and an amplitude of 4.1 V when a 50 Ω termination is used. The dead time is about 50 ns.

3.3 Multi-channel coincidence unit

As stated above the APDs output a TTL signal when detecting a photon and in order to keep track of and record all these detection events we have developed and built a coincidence unit. This device is able to count and record all possible single and coincidence events between 16 channels. Since we are observing up to six-photon states, there are six coincident photons that will be measured in one out of two eigenstates, i.e. 12 APDs and 12 channels of the coincidence unit are needed.

From section 2.3 we know that each pump pulse can give rise to any number of down-converted photon pairs (with different probabilities). The task of the multi-channel coincidence unit is to count how many down-converted photons, originated from one single pump pulse, that have been detected. It should also be able to record in which detectors the photons have been detected and by that means discriminate between different coincidence events. The maximal time, within which photon detections are considered to be coincident is called the time window. The theoretical upper bound for the time window is given by the pulse repetition frequency of the laser. In our case the laser works at 80 MHz pulse repetition frequency, which corresponds to 12.5 ns between the pulses. Due to jitter in the detectors the time window has to be shorter, and the shorter it is the smaller the probability of dark count detections within the time window will be. In this respect, the shorter time window the better. The detector time jitter also sets a lower limit for the duration of the time window. The time window must be longer than the
time jitter in order to find a signal in a specific time slot. Especially when many detectors are used, a longer time window might be preferred.

As a clock we use a crystal oscillator at 100 MHz, but with use of clock multiplying (3x) and DDR (double data rate) technology we are able to sample data at an effective rate of 600 MHz, corresponding to a time resolution and minimal time window of 1.67 ns. The window we use in the experiment is set to 3.3 ns. In this way we are sure that all detected photon coincidences within one time window originate from the same single pump pulse and we also know, from section 2.4, that down-converted photons which originate from one pulse will be in a coherent superposition. Hence, in principle all photons that are detected within the time window are coherent with each other.

In principle the coincidence unit works as follows. As the unit detects a signal, the time window is opened and the data on the inputs at the end of the time window is interpreted as an address, see Figure 3.2. In the figure only three channels are shown and the resulting address is 101 in its binary form. The signal outputs from the APDs are typically much longer than the time window. Next, the data at that address in the memory is incremented by one. When the integration time is over the data is read out from the memory and sent to the computer. Since the memory is of finite size it can only count up to a certain number for each address. The memory depth for all units except the 16-channel version is 32 bits corresponding to a maximal count of $2^{32} - 1 \approx 4.3 \cdot 10^9$. Due to the large amount of channels in the 16-channel version the memory depth is here limited to 27 bits, corresponding to a maximal count of $2^{27} - 1 \approx 1.3 \cdot 10^8$, i.e. 32 times smaller. We typically use an integration time of 5 minutes resulting in a maximal allowed count rate of about 14 million counts/sec. This count rate is definitely higher than what we couple from the PDC source.

To communicate with a computer we have implemented a UART (universal asynchronous receiver/transmitter) to handle serial port communication. We use a serial communication protocol with a rate of 115 200 bps (bits per second) and 1 start bit, 8 data bits, 1 stop bit and no parity bits. The communication is two-way where the length of the integration time is communicated from the computer to the coincidence unit, and data of recorded single and coincidence events going in the opposite direction.

The coincidence unit can also compensate for the different time delays between photon detection and the TTL output of the different detectors. By changing internal delays at every input port the input data can be delayed up to 4.7 ns in 74 ps increments. In cases when delays longer than 4.7 ns are needed, additional coaxial cable extensions are used.
Figure 3.2: Time window of the multi-channel coincidence unit.
Chapter 4

Six-photon entanglement

In this chapter the experimental results of two six-photon entangled states will be presented. To begin with, the efficiency of the source of parametric down-conversion as well as the coupling to single-mode fibers are examined. Then we will show how the methods and techniques described in chapter 2 are used to implement genuinely entangled six-qubit states, but we start by giving an overview of the experimental setup.

4.1 The experimental setup

An overview of the experimental setup is shown in Figure 4.1 and two photos of the setup are shown in Figure 4.2. We use a non-collinear PDC source of entangled photon pairs. The non-linear crystal being used is a 2 mm thick BBO (beta barium borate) crystal, appropriately cut ($\theta = 43.5^\circ$) for type-II phase matching of PDC from 390 nm to 780 nm, and properly tilted to achieve the non-collinear case. It is being pumped by UV pulses and the two spatial emission modes $a$ and $b$ at the intersections of the two frequency degenerated (780 nm) down-conversion cones are coupled to single-mode fibers. In front of each of the two fiber couplers, centered on the spatial emission mode, are a HWP and a compensation crystal (BBO) of 1 mm thickness placed to compensate for longitudinal and transversal walk-off effects. Leaving the fibers, the down-conversion light passes through narrow band (FWHM=3 nm) interference filters for spectral filtering and is thereafter split into six spatial modes ($a, b, c, d, e, f$) by ordinary 50/50 beam splitters (BSs). Birefringent optics is used to compensate for phase shifts in the BSs. Each spatial mode ends with a polarisation analysis station and APDs for single photon detection.
Six-photon entanglement

Figure 4.1: Experimental setup for generating and analysing the six-photon polarisation entangled states. The six photons are created in third order PDC processes in a 2 mm thick BBO crystal pumped by UV pulses. After walk-off compensation the intersections of the two cones obtained in non-collinear type-II PDC are coupled to single-mode fibers (SMFs) wound in polarisation controllers. Narrow-band ($\Delta\lambda = 3$ nm) interference filters (Fs) serve to remove spectral distinguishability. The coupled spatial modes are divided into three modes each by 50/50 beam splitters (BSs). Birefringent phase plates (PPs) compensate for phase shifts in the BSs. Each mode can be analysed in an arbitrary basis using half- and quarter wave plates (HWP and QWP) and a polarising beam splitter (PBS). Simultaneous detection of six photons (one in each mode) are being recorded by a 12 channel coincidence counter.
4.1 The experimental setup

Figure 4.2: Photos of the experimental setup. In (a) the PDC, the walk-off compensation and the single-mode fiber couplers are shown and in (b) the part of the setup from one of the fiber outputs to three of the polarisation analysis stations is shown. Compare with Figure 4.1.
4.2 PDC and coupling efficiency

By calculating the ratio of coincidence counts to single counts, an estimate of the total number of down-converted pairs created, as well as an estimate of the total collection efficiencies can be made [50]. Let $E^H_a$ denote the collection, i.e. coupling and detection, efficiency for horizontal photons in mode $a$, and let $N_{dc}$ be the down-conversion rate (the number of down-converted pairs created in one second). The single horizontal counts per second in mode $a$ will now be given by $E^H_a \cdot N_{dc}$ and the coincidence counts between a horizontal photon in mode $a$ and a vertical in mode $d$ will be $E^H_a \cdot E^V_d \cdot N_{dc}$.

If we now take all different single counts together with all coincidence counts between one $H$- and one $V$-photon, we get the following overdetermined system of equations (30 equations with 13 independent variables):

\[
\begin{align*}
C^H_i &= N_{dc}/2 \cdot E^H_i, \quad i = a, b, \ldots, f \\
C^V_i &= N_{dc}/2 \cdot E^V_i, \quad i = a, b, \ldots, f \\
C^{H,V}_{i,j} &= N_{dc}/2 \cdot E^H_i \cdot E^V_j, \quad i = a, b, c; \quad j = d, e, f \\
C^{V,H}_{i,j} &= N_{dc}/2 \cdot E^V_i \cdot E^H_j, \quad i = a, b, c; \quad j = d, e, f
\end{align*}
\]  

(4.1)

This system can readily be solved for all $E^{H/V}_i$, $i = a, b, \ldots, f$ and $N_{dc}$. It is important that the count rates are corrected in such a way that coincidence counts are also counted as single counts in the respective singles channels. For example, for each coincidence event between mode $a$ and mode $d$, the single counts in the channels corresponding to mode $a$ and mode $d$ should be added with one. The results for the down-conversion rate is $N_{dc} = 2.13 \cdot 10^6$ s$^{-1}$ and the results for the efficiencies are given in Table 4.1. For these calculations we have used data measured in $H/V$-basis during around 140 hours. Due to the long measurement time the statistical errors are negligible. Regarding the collection efficiencies, $E^{H/V}_c$ and $E^{H/V}_f$ are almost twice as high as the others, because there is only one beam splitter in the paths to these modes, see Figure 4.1. Also, the consistently higher values for efficiencies belonging to the upper down-conversion arm (modes $a, b, c$), suggests that it has a better coupling to the single-mode fiber than the lower arm (modes $d, e, f$). The total collection efficiencies for each down-conversion arm are given as sums in Table 4.1. Another observation is that horizontal photons are consistently being more efficiently collected than vertical photons. The cause of this is that the emission cone of $H$-photons is smeared out compared to the $V$-cone, see section 2.4.1. After the HWPs in the walk-off compensation, $H$- and $V$-polarisations are interchanged and the blurred cone is now $V$-polarised.
4.2 PDC and coupling efficiency

Table 4.1: Estimated collection efficiencies.

<table>
<thead>
<tr>
<th>mode, ( i )</th>
<th>( E_i^H ) [%]</th>
<th>( E_i^V ) [%]</th>
<th>( \sum )</th>
<th>( \sum )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>5.6</td>
<td>3.9</td>
<td>21.9</td>
<td>16.3</td>
</tr>
<tr>
<td>( b )</td>
<td>5.6</td>
<td>4.6</td>
<td>10.7</td>
<td>7.8</td>
</tr>
<tr>
<td>( c )</td>
<td>10.7</td>
<td>7.8</td>
<td>5.2</td>
<td>3.7</td>
</tr>
<tr>
<td>( d )</td>
<td>4.5</td>
<td>3.0</td>
<td>9.1</td>
<td>6.3</td>
</tr>
</tbody>
</table>

Using the collection efficiencies and the coincidence rates, we have estimated multiphoton emission rates from the crystal up to the third order. The results are shown in Table 4.2. By dividing these rates by the pulse repetition frequency (80 MHz), probabilities are obtained. Let us now recall the theoretical expression for the probability of \( n \):th order non-collinear PDC from section 2.3.2:

\[
P_n(K) = \frac{\tanh^{2n} K}{\cosh^4 K} \cdot (n + 1).
\]

By fitting this function to the estimated emission probabilities, the factor \( K \) has been calculated to \( K = 0.116 \). Another interesting thing we can calculate from our data is the probability for one pump photon to be down-converted in a BBO crystal (\( P_{\text{BBO}} \)). Since the down-conversion rate is proportional to

Table 4.2: Estimated emission rates and probabilities.

<table>
<thead>
<tr>
<th>order</th>
<th>( n )</th>
<th>( m )</th>
<th>photon number state</th>
<th>rate [s(^{-1})]</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(</td>
<td>1H_a, 1V_b, \rangle )</td>
<td>1 062 000</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>(</td>
<td>1V_a, 1H_b, \rangle )</td>
<td>1 027 000</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>(</td>
<td>2H_a, 2V_b, \rangle )</td>
<td>14 900</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>(</td>
<td>1H_a, 1V_a, 1V_b, 1H_b, \rangle )</td>
<td>14 500</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>(</td>
<td>2V_a, 2H_b, \rangle )</td>
<td>13 800</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>(</td>
<td>3H_a, 3V_b, \rangle )</td>
<td>250</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>(</td>
<td>2H_a, 1V_a, 2V_b, 1H_b, \rangle )</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>(</td>
<td>1H_a, 2V_a, 1V_b, 2H_b, \rangle )</td>
<td>190</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>(</td>
<td>3V_a, 3H_b, \rangle )</td>
<td>190</td>
</tr>
</tbody>
</table>
the pump power (for small $K$) this probability is constant. The total flux of pump photons ($F$) entering the BBO per second is given by:

$$F = \frac{\lambda \cdot P}{\hbar \cdot c} \approx \frac{390 \text{ nm} \cdot 1.3 \text{ W}}{\hbar \cdot c} \approx 2.6 \cdot 10^{18} \text{ s}^{-1}, \quad (4.3)$$

where $P$ is the pump power. $P_{\text{BBO}}$ is now given by the ratio of $\bar{N}_{dc}$ to $F$:

$$P_{\text{BBO}} = \frac{\bar{N}_{dc}}{F} \approx 8.4 \cdot 10^{-13}. \quad (4.4)$$

### 4.3 The six-photon entangled states $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$

In this work we use the third order PDC emission of a non-linear crystal, which after proper engineering of the photons may result in one of the two genuinely entangled states $|\Psi^-_6\rangle$ or $|\Psi^+_6\rangle$. Now we will see how these states take form.

In general the state of the emission from non-collinear type-II PDC is given by Eq. (2.12). The third order term, with $p = 3$ in this equation, corresponding to the emission of six photons into two spatial modes in a third order PDC process is proportional to the following superposition of photon number states:

$$|3H_a,3V_b\rangle + e^{i\phi} |2H_a,1V_a,2V_b,1H_b\rangle + e^{2i\phi} |1H_a,2V_a,1V_b,2H_b\rangle + e^{3i\phi} |3V_a,3H_b\rangle,$$ 

where e.g. $3H_a$ means three horizontally polarised photons in mode $a$. By splitting the two modes $a$ and $b$ into six modes ($a, b, c$ and $d, e, f$) and selecting the terms with one photon in each mode, according to section 2.5 and adjusting the phase $\phi$ to either 0 or $\pi$ we arrive at the states $|\Psi^+/_6\rangle$ and $|\Psi^-_6\rangle$.
4.3 The six-photon entangled states $|\Psi_{6}^{-}\rangle$ and $|\Psi_{6}^{+}\rangle$

respectively, given by

$$
|\Psi_{6}^{+/−}\rangle = \frac{1}{2} (|HHHVVV\rangle \pm \frac{1}{6} |HHVHV V\rangle \pm \frac{1}{6} |HVHHVV\rangle \\
\pm \frac{1}{6} |VHHHV V\rangle \pm \frac{1}{6} |HHV VHV\rangle \pm \frac{1}{6} |HVHV HV\rangle \\
\pm \frac{1}{6} |VVH HVH\rangle \pm \frac{1}{6} |HHVV VH\rangle \pm \frac{1}{6} |HVV VH V\rangle \\
\pm \frac{1}{6} |HVHV VH\rangle \pm \frac{1}{6} |HHVVV H\rangle \pm \frac{1}{6} |HVHV V H\rangle \\
\pm \frac{1}{6} |VHVH VH\rangle \pm \frac{1}{6} |VHHV VH\rangle \pm \frac{1}{6} |VHVHV H\rangle \\
\pm \frac{1}{6} |VHVHV V\rangle \pm \frac{1}{6} |VHHHV V\rangle \pm \frac{1}{6} |VVH HHH\rangle \\
\pm \frac{1}{6} |HHVVV H\rangle \pm \frac{1}{6} |HVHVV H\rangle \pm \frac{1}{6} |VVHHV H\rangle \\
\pm \frac{1}{6} |HHVVV H\rangle \pm \frac{1}{6} |HVHVV H\rangle \pm \frac{1}{6} |VHVVHH\rangle \\
\pm \frac{1}{6} |VHVHV H\rangle \pm \frac{1}{6} |VHHVV H\rangle \pm \frac{1}{6} |VHVHV V\rangle \\
\pm \frac{1}{6} |VVHVHV\rangle \pm \frac{1}{6} |VVH HHV\rangle \pm \frac{1}{6} |HVHVHV\rangle \\
\pm \frac{1}{6} |HHVVV H\rangle \pm \frac{1}{6} |HVHVV H\rangle \pm \frac{1}{6} |VVVVHH\rangle \\
\pm \frac{1}{6} |HHVVV H\rangle \pm \frac{1}{6} |HVHVV H\rangle \pm \frac{1}{6} |VVVHH H\rangle ) 
$$

in the computational basis. Even though the two states look very similar, they will in some contexts behave completely differently as will be obvious later on. $|\Psi_{6}^{+/−}\rangle$ can also be expressed in a more compact form as a superposition of a six-qubit Greenberger-Horne-Zeilinger (GHZ) state and two products of three-qubit $W$ states according to

$$
|\Psi_{6}^{-}\rangle = \frac{1}{\sqrt{2}} |GHZ_{6}^{-}\rangle + \frac{1}{2} (|\overline{W}_{3}\rangle |W_{3}\rangle - |W_{3}\rangle |\overline{W}_{3}\rangle ) ,
$$

and

$$
|\Psi_{6}^{+}\rangle = \frac{1}{\sqrt{2}} |GHZ_{6}^{+}\rangle + \frac{1}{2} (|\overline{W}_{3}\rangle |W_{3}\rangle + |W_{3}\rangle |\overline{W}_{3}\rangle ) .
$$

The GHZ states are here defined as

$$
|GHZ_{6}^{+/−}\rangle = \frac{1}{\sqrt{2}} (|HHHVVV\rangle \pm |VVVHHH\rangle ) ,
$$

and the $W$ state is defined as

$$
|W_{3}\rangle = \frac{1}{\sqrt{3}} (|HHV\rangle + |HVV\rangle + |VHH\rangle ) .
$$

$|\overline{W}_{3}\rangle$ is the spin-flipped $|W_{3}\rangle$:

$$
|\overline{W}_{3}\rangle = \frac{1}{\sqrt{3}} (|VHV\rangle + |VHV\rangle + |HVV\rangle ) .
$$

To change the state preparation from one of the states ($|\Psi_{6}^{+/−}\rangle$) to the other, one can slightly tilt one of the compensation BBO crystals. This will introduce a phase shift between $H$ and $V$ in one of the spatial emission modes and consequently $\phi$ in Eq. (4.5) will change.
4.3.1 Measurement data

Let us now look at the measured data of six-photon coincidences for $|\Psi^+_6\rangle$ and $|\Psi^-_6\rangle$. The data of $|\Psi^-_6\rangle$ is depicted in Figure 4.3 and the $|\Psi^+_6\rangle$ data can be viewed in Figure 4.4. The figures show the probabilities to obtain each of the 64 possible sixfold coincidences with one photon detection in each spatial mode, measured in different bases. Figure 4.3(a) and 4.4(a) show the data, where all qubits are measured in $H/V$-basis. The peaks are in very good agreement with theory: half of the detected sixfold coincidences are to be found as $HHHVVV$ and $VVVHHH$, and the other half should be evenly distributed among the remaining events with three $H$ and three $V$ detections. This is a clear effect of the bosonic interference (stimulated emission) in the BBO crystal giving higher probabilities for emission of indistinguishable photons.

Since we are measuring probabilities the phase information is lost and $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$ will look the same in $H/V$-basis. Nevertheless, by turning to another basis, e.g. the $+/--$ or the $L/R$-basis the two states will differ, see Figure 4.3(b)-(c) and 4.4(b)-(c), respectively. Here, a major difference between the two states is becoming clear: In distinct contrast to $|\Psi^+_6\rangle$, is $|\Psi^-_6\rangle$ invariant (up to a global phase $\tau$) under identical unitary transformations $\hat{U}$ in each mode.

$$\hat{U}^{\otimes 6} |\Psi^-_6\rangle = e^{i\tau} |\Psi^-_6\rangle$$  \hspace{1cm} (4.12)

Experimentally this can be demonstrated by using identical settings of all polarisation analysers. No matter what the setting is, the results should be equal. The results of the measurements of $|\Psi^-_6\rangle$ in the three different bases really makes the invariance of the probabilities, with respect to the joint changes of the measurement basis in all modes, clear. Our implementation of $|\Psi^-_6\rangle$ is also the largest invariant state created so far.

The values of the correlation function, measurement time, number of six-photon coincidences and count rates for the data depicted in Figure 4.3-4.4 are given in Table 4.3 and 4.4.

4.3.2 The $|\Psi^-_6\rangle$ correlation function

Another property of the $|\Psi^+_6/\Psi^-_6\rangle$ states is that they exhibit perfect correlations between measurement results in different modes. For the state $|\Psi^-_6\rangle$ we have calculated the correlation function theoretically as well as performed measurements to experimentally show the correlation function. The correlation function is defined as the expectation value of the product of the six local polarisation observables [38, 39]. By measuring the local observables $\sigma_k(\theta_k) = \sigma_{z,k} \cos \theta_k + \sigma_{x,k} \sin \theta_k$ (with eigenvectors $\sqrt{1/2}(|L_k\rangle + i_k e^{i\theta_k} |R_k\rangle$)
4.3 The six-photon entangled states $|\Psi^{-}_{6}\rangle$ and $|\Psi^{+}_{6}\rangle$

Figure 4.3: Experimental results of the six-photon invariant state $|\Psi^{-}_{6}\rangle$. Six-fold coincidence probabilities corresponding to detections of one photon in each mode in $H/V$-basis (a), $+/\!-\!$-basis (b), and $L/R$-basis (c) are shown. Comparing the three measurement results makes the invariance of the state obvious. For the pure $|\Psi^{-}_{6}\rangle$ state the light blue bars would be zero and in our experiment their amplitudes are all in the order of the noise.
Figure 4.4: Experimental results of the six-photon state $|\Psi^+_6\rangle$. Six-fold coincidence probabilities corresponding to detections of one photon in each mode in $H/V$-basis (a), $+/\!/-$-basis (b), and $L/R$-basis (c) are shown. For the pure $|\Psi^+_6\rangle$ state the light blue bars would be zero and in our experiment their amplitudes are all in the order of the noise.
4.3 The six-photon entangled states $|\Psi_6^-\rangle$ and $|\Psi_6^+\rangle$

Table 4.3: Correlation function, measurement time, number of six-photon coincidences and count rates for $|\Psi_6^-\rangle$.

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/\text{--}$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>$-87.9 \pm 4.5$</td>
<td>$-87.6 \pm 5.0$</td>
<td>$-86.8 \pm 4.3$</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>$-100$</td>
<td>$-100$</td>
<td>$-100$</td>
</tr>
<tr>
<td>Measurement time [h]</td>
<td>143.5</td>
<td>141.5</td>
<td>136.2</td>
</tr>
<tr>
<td>Six-photon coincidences</td>
<td>432</td>
<td>369</td>
<td>488</td>
</tr>
<tr>
<td>Count rate [h$^{-1}$]</td>
<td>3.0</td>
<td>2.6</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Table 4.4: Correlation function, measurement time, number of six-photon coincidences and count rates for $|\Psi_6^+\rangle$.

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/\text{--}$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>$-89.5 \pm 4.9$</td>
<td>$86.3 \pm 6.6$</td>
<td>$82.0 \pm 4.8$</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>$-100$</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Measurement time [hours]</td>
<td>99.7</td>
<td>41.2</td>
<td>140.7</td>
</tr>
<tr>
<td>Six-photon coincidences</td>
<td>368</td>
<td>203</td>
<td>389</td>
</tr>
<tr>
<td>Count rate [h$^{-1}$]</td>
<td>3.7</td>
<td>4.9</td>
<td>2.8</td>
</tr>
</tbody>
</table>

and eigenvalues $l_k = \pm 1$), corresponding to an arbitrary linear polarisation analysis in each spatial mode ($k = a, b, c, d, e, f$), the correlation function on
the prime meridian of the Bloch sphere is obtained:

$$E(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f) = \langle \Psi^-_6 | \sigma_a(\theta_a) \otimes \ldots \otimes \sigma_f(\theta_f) | \Psi^-_6 \rangle =$$

$$= -\frac{1}{2} \cos(\theta_a + \theta_b + \theta_c - \theta_d - \theta_e - \theta_f) - \frac{1}{18} \cos(\theta_a + \theta_b - \theta_c + \theta_d + \theta_e - \theta_f)$$

$$- \frac{1}{18} \cos(\theta_a - \theta_b + \theta_c + \theta_d - \theta_e - \theta_f) - \frac{1}{18} \cos(\theta_a - \theta_b - \theta_c + \theta_d + \theta_e - \theta_f)$$

$$- \frac{1}{18} \cos(\theta_a - \theta_b - \theta_c - \theta_d + \theta_e + \theta_f) - \frac{1}{18} \cos(\theta_a - \theta_b + \theta_c - \theta_d + \theta_e + \theta_f)$$

$$- \frac{1}{18} \cos(\theta_a + \theta_b - \theta_c - \theta_d + \theta_e + \theta_f) - \frac{1}{18} \cos(\theta_a - \theta_b + \theta_c - \theta_d - \theta_e + \theta_f)$$

$$- \frac{1}{18} \cos(\theta_a - \theta_b + \theta_c + \theta_d - \theta_e + \theta_f) - \frac{1}{18} \cos(\theta_a + \theta_b - \theta_c - \theta_d - \theta_e + \theta_f)$$

$$- \frac{1}{18} \cos(\theta_a + \theta_b - \theta_c + \theta_d + \theta_e - \theta_f)$$

(4.13)

Since $|\Psi^-_6\rangle$ is invariant the correlation function around any great circle of the Bloch sphere will look the same for this state.

Let us now see how the different terms in the state contribute to the correlation function. By just comparing the coefficients in Eq. (4.6) and Eq. (4.13), it seems plausible that the GHZ part of the state gives rise to the first term of the correlation function. Indeed, the correlation function around the equator of the Bloch sphere, for the GHZ part only, is exactly $-\frac{1}{2} \cos(\theta_a + \theta_b + \theta_c - \theta_d - \theta_e - \theta_f)$. Going around any other great circle will however give a different result, as the GHZ state is not invariant. All terms of $|\Psi^-_6\rangle$ that are qubit-wise orthogonal in the $H/V$-basis are correlated in all bases in the equatorial plane (every other combination of terms gives zero correlation) and each such pair gives rise to one term in the correlation function. The total invariant correlation function can be obtained by adding these ‘equatorial’ terms coherently. It is only the coherent superposition of all terms together that is invariant and not the individual terms themselves (or pairs, triples, etc. thereof).

Figure 4.5 shows how the correlation function depends on $\theta_b$ with the other analysers fixed at $\theta_a = \theta_c = \theta_d = \theta_e = \theta_f = \pi/2$ corresponding to polarisation analysis in the $+/-$-basis. The correlation function in Eq. (4.13) is then simplified to

$$E\left(\frac{\pi}{2}, \theta_b, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = -\sin \theta_b .$$

(4.14)

The experimental values of the correlation function are obtained from the 64 six-photon coincidence counts $c_{l_{[1, l_{2, l_{3, l_{4, l_{5, l_{6, l_{7}}, l_{8}, l_{9}}, l_{10, l_{11}, l_{12}}}}]}}$ and the eigenvalues of the local observables by first calculating the probabilities for each of the 64 six-photon
4.3 The six-photon entangled states $|\Psi_6^−\rangle$ and $|\Psi_6^+\rangle$

Figure 4.5: Six-photon polarisation correlation function. Modes $a, c, d, e$ and $f$ are analysed in the $+/−$-basis and mode $b$ analysis basis is rotated around the prime meridian of the Bloch sphere ($\sigma_z \cos \theta_b + \sigma_x \sin \theta_b$). The solid line shows a sinusoidal fit to the experimental data with a visibility of $83.8\% \pm 3.0\%$. The average measurement time for each data point was 77 hours, and the average count rate was $2.6 \text{ h}^{-1}$.
Six-photon entanglement

events and then taking the sum over the eigenvalue for each event multiplied by the probability to obtain that particular event:

\[
E(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f) = \sum_{l_a, l_b, l_c, l_d, l_e, l_f} l_a l_b l_c l_d l_e l_f \cdot P_{l_a, l_b, l_c, l_d, l_e, l_f}(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f).
\] (4.15)

where the six-photon probabilities are given by

\[
P(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f) = \frac{c_{l_a, l_b, l_c, l_d, l_e, l_f}(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f)}{\sum_{l_a, l_b, l_c, l_d, l_e, l_f} c_{l_a, l_b, l_c, l_d, l_e, l_f}(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f)}.
\] (4.16)

In order to obtain a value for the visibility of the correlation function a sinusoidal least-square fit was made to the data yielding a visibility, defined as the amplitude of the fit, of \( V = 83.8\% \pm 3.0\% \). The error was calculated as the standard deviation of the absolute values of all extremal points. The visibility of the correlation function can be used as a measure of the quality of the coherence in the state, and the six-photon visibility presented here is the highest achieved world-wide, suggesting that the quality of both our six-photon states is extraordinary.

### 4.3.3 Entanglement witnesses

\(|\Psi_6^{+/-}\rangle\) are both genuine multipartite entangled states, meaning that each of their qubits are entangled with all the remaining qubits of the state. In order to show genuine multipartite entanglement for our experimental states we are using the method of entanglement witness. Entanglement witnesses are observables yielding a negative value only for entangled states. They will not however yield negative values for all entangled states, see Figure 4.6. An entanglement witness is always optimised for a specific entangled state, which will be called the ‘ideal’ state since this state will yield the most negative expectation value. The witness will find entanglement in this ideal state, and also in other entangled states that are close enough to the ideal state. The noise tolerance of a witness is a measure of how much depolarising noise that can be blended with the ideal state, without having a positive expectation value. Thus it will in some sense say how far from the ideal state an entangled state can be, while still being found to be entangled by the witness.
4.3 The six-photon entangled states $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$

Figure 4.6: An entanglement witness ($W$) is an operator with a positive expectation value for all separable states and a negative expectation value for a subset of entangled states. The equation $\text{Tr}(W\rho) = 0$ defines a hyperplane in the convex space of (pure and mixed) quantum states, with all separable states on one side and a subset of entangled states on the other side. For the maximum overlap witness this hyperplane (solid line) is tangent to the subset of separable states, maximising the distance to the ideal state $\rho_{\text{ideal}}$. Any other witness, e.g. a reduced witness (dashed line) will yield a shorter distance, but may have other advantages such as a more convenient physical implementation.
The most common entanglement witness, which is also the best witness regarding noise tolerance, is the maximum overlap witness ($\mathcal{W}$). In general it is given by

$$\mathcal{W} = \zeta 1^{\otimes n} - |\Psi_{\text{ideal}}\rangle \langle \Psi_{\text{ideal}}|$$

and

$$\zeta = \max_{|\phi_{\text{sep}}\rangle \in \mathcal{B}} |\langle \phi_{\text{sep}}|\Psi_{\text{ideal}}\rangle|,$$  

where $\mathcal{B}$ is the set of all biseparable states and $n$ is the number of qubits in $|\Psi_{\text{ideal}}\rangle$ [51, 52]. The maximum overlap witnesses optimised for $|\Psi_{\text{6}}^{+/\text{z}}\rangle$ have the form

$$\mathcal{W}^{+/\text{z}} = \frac{2}{3} 1^{\otimes 6} - |\Psi_{\text{6}}^{+/\text{z}}\rangle \langle \Psi_{\text{6}}^{+/\text{z}}|$$

and have a noise tolerance around 34% [53]. Since these witnesses demand 183 different measurement settings, which would be an experimentally very demanding task to perform, we have developed reduced witnesses that can be implemented using only three measurement settings. The cost is a poorer noise tolerance. Our reduced witness $W^-$, which is optimised for $|\Psi_{\text{6}}^{-}\rangle$, is given by

$$W^- = \frac{181}{576} 1^{\otimes 6} - \frac{1}{64} (\sigma_x^{\otimes 6} + \sigma_y^{\otimes 6} - \sigma_z^{\otimes 6}) - \frac{1}{576} \sum_{i=x,y,z} (3\sigma_i^{\otimes 2} 1^{\otimes 4} + 3\sigma_i 1^{\otimes 4} - 5\sigma_i^{\otimes 3} 1) + \frac{1}{576} (1 \leftrightarrow \sigma_x) + (1 \leftrightarrow \sigma_y) + (1 \leftrightarrow \sigma_z),$$

where $[1 \leftrightarrow \sigma_i]$ denotes the same terms as in the sum, but with $1$ and $\sigma_i$ interchanged. It is obtained from the maximum overlap witness as follows. First the maximum overlap witness is decomposed into direct products of Pauli and identity matrices, next only terms containing one type of Pauli matrices and/or identity matrices are selected. I.e. all terms including products of at least two different Pauli matrices are deleted. Finally, the constant in front of $1^{\otimes 6}$ in the first term of Eq. (4.20) is chosen to be the smallest possible such that all entangled states found by the reduced witness are also found by the maximal overlap witness. See appendix A for more details on the derivation. The reduced witness optimised for $|\Psi_{\text{6}}^{+}\rangle$ ($W^+$) is obtained.
4.3 The six-photon entangled states $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$

by the same techniques and is given by

$$\mathcal{W}^+ = \frac{181}{576} I^\otimes 6 - \frac{1}{64} (\sigma_x^\otimes 6 + \sigma_y^\otimes 6 - \sigma_z^\otimes 6) - \frac{1}{576} \sum_{i=x,y,z} \left( 3\sigma_i^\otimes 2 I^\otimes 4 + 3\sigma_i I\sigma_i I^\otimes 3 \right)$$

$$+ \frac{1}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{1}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3$$

$$+ \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3 + \frac{5}{3} \sigma_i^\otimes 2 I^\otimes 3$$

$$- \frac{1}{576} (\sigma \leftrightarrow \sigma)$$

(4.21)

Our reduced witnesses detects genuine six-partite entanglement of $|\Psi^+_6\rangle$ and $|\Psi^-_6\rangle$ with a noise tolerance of 15.0%.

The theoretical expectation value for both witnesses operating on their ideal states is $\langle \Psi^-_6 | \mathcal{W}^- | \Psi^-_6 \rangle = \langle \Psi^+_6 | \mathcal{W}^+ | \Psi^+_6 \rangle = -1/18 \approx -0.056$ and our experimental results are shown in Table 4.5. Our experimental implementations of $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$ are showing genuine six-qubit entanglement with, respectively, 2.0 and 1.5 standard deviations. As seen in the table the witness $\mathcal{W}^-$ optimised for $|\Psi^-_6\rangle$ does not find entanglement in $|\Psi^-_6\rangle$ and vice versa.

| $|\Psi^-_6\rangle$ | $|\Psi^+_6\rangle$ |
|-----------------|-----------------|
| $\langle \mathcal{W}^- \rangle$ | $-0.023 \pm 0.012$ | $0.233 \pm 0.007$ |
| $\langle \mathcal{W}^+ \rangle$ | $0.224 \pm 0.007$ | $-0.021 \pm 0.014$ |

Table 4.5: Observed witness values.

From the witnesses we can also obtain lower bounds for the state fidelities. For $\mathcal{W}^-$ the spectrum is given by

$$\lambda(\mathcal{W}^-) = \{ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{64} \} ,$$

(4.22)

where $\lambda_1$, corresponding to the eigenstate $|\Psi^-_6\rangle$, is the only negative eigenvalue. This value, together with the second smallest and the largest eigenvalues, corresponding to eigenstates orthogonal to $|\Psi^-_6\rangle$ is given by

$$\lambda_1 = -\frac{1}{18}$$

(4.23)

$$\lambda_2 = \frac{1}{9}$$

(4.24)

$$\lambda_{64} = \frac{13}{36} .$$

(4.25)
The experimental state can be expressed as
\[ \rho_{\text{exp}} = F |\Psi_6^-\rangle \langle \Psi_6^-| + (1 - F) \rho_\perp, \]  \hspace{1cm} (4.26)

where \( \rho_\perp \) is orthogonal to \( |\Psi_6^-\rangle \) and obey \( \text{Tr}(|\Psi_6^-\rangle \langle \Psi_6^-| \rho_\perp) = 0 \), and \( F = \text{Tr}(\rho_{\text{exp}} |\Psi_6^-\rangle \langle \Psi_6^-|) \) is the state fidelity. For the witness expectation value we now arrive at
\[ \lambda_1 \cdot F + \lambda_2 (1 - F) \leq \langle W^- \rangle = \text{Tr}(W^- \rho_{\text{exp}}) \leq \lambda_1 \cdot F + \lambda_{64} (1 - F) \]  \hspace{1cm} (4.27)

\[ \Rightarrow \left( \langle W^- \rangle - \lambda_2 \right) \cdot (\lambda_1 - \lambda_2) \leq F \leq \left( \langle W^- \rangle - \lambda_{64} \right) \cdot (\lambda_1 - \lambda_{64}). \]  \hspace{1cm} (4.28)

Inserting the values for \( \lambda_1, \lambda_2, \lambda_{64} \) and \( \langle W^- \rangle \) from Eq. (4.23)-(4.25) and Table 4.5 we get the following lower and upper bounds of the state fidelity:
\[ 0.81 \pm 0.07 \leq F_{\Psi_6^-} \leq 0.92 \pm 0.03. \]  \hspace{1cm} (4.29)

For \( |\Psi_6^+\rangle \) the lower and upper bounds of the fidelity are similarly calculated to \( 0.79 \pm 0.08 \leq F_{\Psi_6^+} \leq 0.92 \pm 0.03 \).

These lower bounds of the state fidelities are remarkably higher than the bounds for other six-photon entangled states. The first experiment revealing six-photon entanglement in graph states [54], using three consecutive non-collinear sources, as well as two observations of the six-photon Dicke state [55,56], using multiphoton emission from collinear PDC sources, all achieved fidelities below 0.66.

Also note that in the case of implementing a maximum overlap witness (Eq. (4.17)), all non-negative eigenvalues are degenerate \((\lambda_i = \lambda_2 \geq 0, i \geq 2)\) and the lower and upper bounds of Eq. (4.28) will coincide. Consequently, by implementing this kind of witness a value of the fidelity can be obtained, not merely lower and upper bounds.

### 4.3.4 Five-photon entanglement

By taking five-photon subsets of the states from the previous section we can obtain five-photon genuine entangled states. The five-photon subsets can be achieved experimentally by conditioning on a detection of one photon in a specific state, so called projective measurements. This can give information about the entanglement persistency of the multi-photon states against photon loss [57]. For \( |\Psi_5^-\rangle \) we have projected the second qubit (in mode
4.3 The six-photon entangled states $|\Psi^-_6\rangle$ and $|\Psi^+_6\rangle$

\[ |\Psi^-_6\rangle = -\frac{1}{\sqrt{2}} |VVHHH\rangle + \frac{1}{\sqrt{3}} |V_2\rangle |W_3\rangle - \frac{1}{\sqrt{6}} |HH\rangle |W_3\rangle . \]  \hspace{1em} (4.30)

A similar projection onto $|H\rangle$ results in
\[ b \langle H | \Psi^-_6 \rangle = \frac{1}{\sqrt{2}} |HHVVV\rangle - \frac{1}{\sqrt{3}} |V_2\rangle |W_3\rangle + \frac{1}{\sqrt{6}} |VV\rangle |W_3\rangle . \]  \hspace{1em} (4.31)

Figure 4.7 and 4.8 show the measurement results obtained in the $H/V$ and the $+/-$ observation bases for the five-photon states $b \langle V | \Psi^-_6 \rangle$ and $b \langle H | \Psi^-_6 \rangle$. In Figure 4.7(a) and 4.8(a) the results in the computational basis are presented and we clearly see the terms $|HHVVV\rangle$ and $|VVVHHH\rangle$. The results from the measurements in the diagonal basis can be seen in Figure 4.7(b) and 4.8(b). Here the terms $|HHVVV\rangle$ and $|VVVHHH\rangle$ are evident. All these results are in agreement with theoretical predictions.
Figure 4.8: The five-photon state \( \langle b | H | \Psi^- \rangle \) obtained through projective measurement of the \( b \)-qubit onto \( |H\rangle \). The bars represent five-fold coincidence probabilities, conditioned on detecting a \( |H\rangle \)-photon in mode \( b \). In (a) all qubits are measured in the computational basis. The results in (b) correspond to measurements of the five qubits in the diagonal basis. The light blue bars should, according to theory be vanishing.
Chapter 5

Decoherence-free subspaces

In this chapter the concept of decoherence-free (DF) subspaces and DF states will be discussed. DF states of two, four and six qubits will be presented.

5.1 Decoherence

One of the major problems in the field of quantum information is decoherence. This arises because the quantum system of interest (e.g. the qubits of a quantum state) cannot be completely isolated from the surrounding environment. The environment has in general a large number of degrees of freedom beyond our control and due to that, uncontrolled interactions between the quantum system and the environment will lead to loss of phase information (dephasing). A related effect concerning loss of energy is called dissipation, but dissipative processes are usually much slower than dephasing processes. Neglecting dissipation (corresponding to e.g. loss of photons), we can treat the noise causing dephasing as unitary transformations.

One way around the decoherence problem is to process the quantum information much faster than the dephasing time scale. In this case the loss of phase information can be considered slow and can be neglected. The weaker the coupling to the environment is, the longer the dephasing time scale will be. Since photons interact poorly with the environment when traversing through vacuum or air the decoherence will in these cases be small, but when using e.g. communication through long distances of optical fiber, photons will suffer from decoherence. The optical fiber will introduce birefringence due to thermal or mechanical stress variations.

Another way to reduce the effects of decoherence is to encode the information in quantum states of a decoherence-free subspace. If the qubit-
Decoherence-free subspaces

environment interaction exhibits some symmetry, there are states that are invariant under this interaction, making them suitable for protecting quantum information [58, 59]. Such decoherence-free states are hence insensitive to some particular kind of noise connected to the aforementioned symmetry. An interesting example is the so-called collective noise, which arises when the qubit-environment coupling is equal for all qubits of a quantum state. This type of noise is a major cause of decoherence and arises naturally when the spatial (temporal) separation between the qubits is small compared to the correlation length (time) of the environment. An example is when polarisation encoded photons are sent through the same optical fiber. From now on, the term decoherence-free states will be used to describe decoherence-free states immune to collective noise.

5.2 Decoherence-free states

By simulating noise with a random phase shift for each term in the state, we can make an illustrative example showing the effects of the environment interacting with the qubits. A decohering environment acting in the $H/V$-basis will result in the following unitary transformation on a general qubit:

$$c_0 |H\rangle + c_1 |V\rangle \rightarrow e^{i\phi_0} c_0 |H\rangle + e^{i\phi_1} c_1 |V\rangle = e^{i\phi_0} (c_0 |H\rangle + e^{i(\phi_1 - \phi_0)} c_1 |V\rangle) ,$$

(5.1)

where $\phi_0$ and $\phi_1$ are random phases. The global phase is unobservable and can be neglected, but since the phase $(\phi_1 - \phi_0)$ is random the single qubit state is changed.

Consider now the same unitary transformation acting on each qubit of the two qubit states $|\Psi^\pm\rangle = (|HV\rangle \pm |VH\rangle)/\sqrt{2}$:

$$|\Psi^\pm\rangle \rightarrow e^{i\phi_H} |H\rangle e^{i\phi_V} |V\rangle \pm e^{i\phi_V} |V\rangle e^{i\phi_H} |H\rangle = e^{i(\phi_H + \phi_V)} |\Psi^\pm\rangle .$$

(5.2)

The result is, up to a global phase, equal to the initial state. Thus both $|\Psi^+\rangle$ and $|\Psi^-\rangle$ are invariant under collective noise acting in the $H/V$-basis and under this condition they span a two-dimensional decoherence-free subspace. In most cases however, there is no control of the noise and it cannot be guaranteed that the environment acts only in one basis, as the $H/V$-basis. Taking again the states $|\Psi^\pm\rangle$, but now letting the environment act in the
diagonal basis leads to
\[ |\Psi^+\rangle = \frac{|HV\rangle + |VH\rangle}{\sqrt{2}} = \frac{|++\rangle - |--\rangle}{\sqrt{2}} \rightarrow \frac{e^{i2\phi_+}|++\rangle - e^{i2\phi_-}|--\rangle}{\sqrt{2}} \]
\[ (5.3) \]
and
\[ |\Psi^-\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}} = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \rightarrow \frac{e^{i(\phi_+ + \phi_-)}|+-\rangle - e^{i(\phi_- + \phi_+)}|-+\rangle}{\sqrt{2}} = \frac{e^{i(\phi_+ + \phi_-)}|+-\rangle - |-+\rangle}{\sqrt{2}} \]
\[ (5.4) \]
As we can see $|\Psi^+\rangle$ is not invariant under this transformation and is thus not a true decoherence-free state. $|\Psi^-\rangle$ on the other hand remains invariant under any identical unitary transformation on each qubit. For the case of two-qubit states, the singlet state $|\Psi^-\rangle$ is the only DF state. However, since at least two DF states are required to encode the quantum information of one qubit, it is necessary to move to higher dimension Hilbert spaces in order to achieve two or more states to span a DF subspace.

With collective noise all qubits of a state couple equally to the environment and hence they are being subject to the same unknown unitary transformation. In general, an $N$-qubit state is decoherence-free if it is invariant under any such $N$-lateral unitary transformation, i.e. $U^{\otimes N} |\Psi\rangle = |\Psi\rangle$, where $U^{\otimes N} = U \otimes \ldots \otimes U$ denotes the tensor product of $N$ unitary transformations $U$. The number of qubits $N$ also determines the amount of quantum information that can be protected. For even $N$ the dimension $d(N)$ of the decoherence-free subspace is given by
\[ d(N) = \frac{N!}{(N/2)!((N/2) + 1)!}, \]
\[ (5.5) \]
and the number of qubits that can be encoded therein is $\log_2 d(N) [59]$.  

5.3 Building decoherence-free subspaces

In the last section we saw that a single-qubit state can never be decoherence-free, whereas for two-qubit states there is only one truly decoherence-free state. We will now continue to look at four- and six-qubit states spanning DF subspaces of dimension two and five, respectively, according to Eq. (5.5).
5.3.1 The two-dimensional decoherence-free subspace

The two-dimensional DF subspace is the smallest DF subspace that is useful for protecting quantum information. It is spanned by two four-qubit states, of which the first can be chosen to be the double singlet

\[
\left| \bar{0} \right\rangle_{abcd} = \left| \Psi^- \right\rangle_{ab} \otimes \left| \Psi^- \right\rangle_{cd} = \frac{1}{2} \left( \left| 0101 \right\rangle - \left| 0110 \right\rangle - \left| 1001 \right\rangle + \left| 1010 \right\rangle \right)_{abcd} .
\]

(5.6)

The only decoherence-free state that is orthogonal to \( \left| \bar{0} \right\rangle_{abcd} \) is given by

\[
\left| \bar{1} \right\rangle_{abcd} = \left| \Psi_*^- \right\rangle_{abcd} = \frac{1}{2\sqrt{3}} \left( 2 \left| 0011 \right\rangle - \left| 0101 \right\rangle - \left| 0110 \right\rangle - \left| 1001 \right\rangle - \left| 1010 \right\rangle + 2 \left| 1100 \right\rangle \right)_{abcd} .
\]

(5.7)

and it will constitute the second basis state [58]. For \( \left| \bar{0} \right\rangle \) and \( \left| \bar{1} \right\rangle \) collective noise in the computational basis will only result in an equal global phase like \( e^{i(\phi_0 + \phi_1)} \), because all terms consist of the same number of \( \left| 0 \right\rangle - \left| 1 \right\rangle \)-states. Further, since both \( \left| \bar{0} \right\rangle \) and \( \left| \bar{1} \right\rangle \) are invariant under basis changes they will look the same in any basis and thus any phases due to collective noise will be global. The same holds for any superposition of \( \left| \bar{0} \right\rangle \) and \( \left| \bar{1} \right\rangle \) and the two decoherence-free states can now be used to encode an arbitrary qubit as

\[
\left| Q \right\rangle = c_0 \left| \bar{0} \right\rangle + c_1 \left| \bar{1} \right\rangle ,
\]

(5.8)

which is protected against collective noise:

\[
\left| Q \right\rangle \rightarrow \hat{U}^{\otimes 4} \left| Q \right\rangle = c_0 \cdot \hat{U}^{\otimes 4} \left| 0 \right\rangle + c_1 \cdot \hat{U}^{\otimes 4} \left| 1 \right\rangle = c_0 \left| 0 \right\rangle + c_1 \left| 1 \right\rangle = \left| Q \right\rangle .
\]

(5.9)

5.3.2 A six-qubit decoherence-free basis

To be able to protect two arbitrary qubits and hence also an entangled state from collective noise, a DF subspace of dimension four or higher is necessary. For this reason the five-dimensional DF subspace spanned by six-qubit states is especially interesting. One choice of basis states in this subspace is the following:

\[
\left| \bar{0} \right\rangle_{abcdef} = \left| \Psi^- \right\rangle_{ab} \otimes \left| \Psi^- \right\rangle_{cd} \otimes \left| \Psi^- \right\rangle_{ef} ,
\]

(5.10)

\[
\left| \bar{1} \right\rangle_{abcdef} = \left| \Psi^- \right\rangle_{ab} \otimes \left| \Psi^- \right\rangle_{cdef} ,
\]

(5.11)

\[
\left| 2 \right\rangle_{abcdef} = \left| \Psi^- \right\rangle_{abcd} \otimes \left| \Psi^- \right\rangle_{ef} ,
\]

(5.12)

\[
\left| 3 \right\rangle_{abcdef} = \left| \Psi^- \right\rangle_{abcdef} .
\]

(5.13)
5.4 Quantum communication without a shared reference frame

As we can see $|\bar{1}\rangle_{abcdef}$ and $|\bar{2}\rangle_{abcdef}$ can be obtained from qubit permutations of the product state of $|\Psi^-\rangle$ from Eq. (1.35) and $|\Psi^-_4\rangle$ from Eq. (5.7). Another state obtained from such permutations is $|\Psi^-\rangle_{cd} \otimes |\Psi^-_4\rangle_{abcdef}$. This state is orthogonal to the states in Eq. (5.10) - (5.12), but not to $|\Psi^-_6\rangle_{abcdef}$. They are however linearly independent. The fifth and last basis state of the five-dimensional DF subspace can be calculated by the Gram-Schmidt method. This state is composed of 18 terms in the computational basis and looks like:

$$
|\bar{3}\rangle_{abcdef} = \frac{1}{6\sqrt{2}} \left( -4|001011\rangle + 2|001101\rangle + 2|001110\rangle + 2|010011\rangle 
- 1|010101\rangle - 1|010110\rangle + 1|011001\rangle + 1|011010\rangle 
- 2|011100\rangle + 2|100011\rangle - 1|100101\rangle - 1|100110\rangle 
+ 1|101001\rangle + 1|101010\rangle - 2|101100\rangle - 2|110001\rangle 
- 2|110010\rangle + 4|110100\rangle \right)_{abcdef}. 
$$

(5.14)

As $|\bar{3}\rangle_{abcdef}$ this state is a genuine six-qubit entangled state. Note that for the single-qubit Hilbert spaces, the $|0\rangle/|1\rangle$-basis can represent any orthonormal basis, since all DF states are invariant [60].

In order to encode arbitrary quantum information in a five-dimensional DF subspace, one could in principle use five sources, one for each DF basis state, and coherently overlap the generated photons. This is however a great technical and experimental challenge. Especially since for the last basis state (Eq. (5.14)) there is yet no description or proposal of how to experimentally prepare the state at all. With the aim to encode two arbitrary qubits in a DF subspace the demands could be slightly eased as four basis states would then be enough.

5.4 Quantum communication without a shared reference frame

Quantum communication without sharing a common reference frame is very much related to the topic of decoherence-free subspaces [60]. In this situation two parties wish to communicate quantum information between each other, but they have no idea how their encoding bases are related. For example when the sender transmits $|H\rangle$, the receiver can obtain any pure state, due to some rotation of the state. An arbitrary rotation of the receiver’s measurement frame relative to the sender’s computational basis frame can be seen as a rotation due to collective noise. To encode one logical qubit
we need four physical qubits and can use the basis states in Eq. (5.6)-(5.7). Similarly, to encode quantum information in a five-dimensional system the basis states in Eq. (5.10)-(5.14) could be used.
Chapter 6

Quantum telecloning

Quantum telecloning is a process combining quantum teleportation and optimal quantum cloning from one input to $M$ outputs. It turns out that both $|\Psi^+_6\rangle$ and $|\Psi^-_6\rangle$ can be used for quantum telecloning, but before we look into that quantum teleportation and optimal quantum cloning will be very briefly discussed.

6.1 Quantum teleportation

Quantum teleportation [11,12] allows an unknown state ($|X\rangle$) of a qubit ($X$) to be perfectly transmitted between two spatially separated parties (Alice and Bob). A necessary condition is that Alice and Bob share a maximally entangled two-qubit state (one of the Bell states), see Figure 6.1. Alice then performs a joint Bell measurement of the unknown qubit $X$ and her qubit of the Bell state. The result of this measurement is sent to Bob as a two-bit message over a classical channel. Depending on the result of the Bell measurement, Bob uses one of the unitary operators $\mathbb{1}$, $\sigma_x$, $\sigma_y$ or $\sigma_z$ to rotate his qubit, see Table 6.1. After the rotation the state of Bob’s qubit is equal to the original state of the $X$-qubit.

6.2 Optimal quantum cloning

Optimal quantum cloning aims for copying quantum information from $N$ identical qubits to $M \geq N$ qubits as efficiently as possible. Here we will only regard universal cloning, where the fidelity of the copies with respect to the original state $|X\rangle$ is independent of $|X\rangle$ [18]. The quantum no-cloning theorem tells us that perfect copying of quantum information is not possible,
Figure 6.1: The quantum teleportation scheme. First Alice and Bob need to share a Bell state (a). Alice then makes a Bell measurement with her half of the Bell state and a qubit in the unknown state $|X\rangle$ (b). After classical communication (CC) of the result, Bob can reconstruct $|X\rangle$ by means of local unitary operations ($U\dagger$) (c).

Table 6.1: Local rotations on teleported qubit. The columns define which Bell state is initially shared between the parties and the rows denote the possible outcomes of the Bell measurement.

| Outcome | $|\Psi^+\rangle$ | $|\Psi^-\rangle$ | $|\Phi^+\rangle$ | $|\Phi^-\rangle$ |
|---------|----------------|----------------|----------------|----------------|
| $|\Psi^+\rangle$ | $I$ | $\sigma_z$ | $\sigma_x$ | $\sigma_y$ |
| $|\Psi^-\rangle$ | $\sigma_z$ | $I$ | $\sigma_y$ | $\sigma_x$ |
| $|\Phi^+\rangle$ | $\sigma_x$ | $\sigma_y$ | $I$ | $\sigma_z$ |
| $|\Phi^-\rangle$ | $\sigma_y$ | $\sigma_x$ | $\sigma_z$ | $I$ |

see section 1.2.3. However, quantum mechanics imposes an upper limit of the fidelity of the copies given by [24]

$$\gamma(N, M) = \frac{M(N + 1) + N}{M(N + 2)}.$$  \hspace{1cm} (6.1)

For the case of one original qubit ($N = 1$), the optimal fidelities for up to three copies are given by Table 6.2.

### 6.3 Quantum telecloning procedure

Both six-photon entangled states $|\Psi_{6/+}^+\rangle$ can be used as so-called telecloning states [18]. I.e. they can be used for implementing optimal quantum telecloning of an unknown single-qubit state $|X\rangle$. In these specific cases (using
6.3 Quantum telecloning procedure

Table 6.2: Optimal cloning fidelities.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$1$</th>
<th>$5/6$</th>
<th>$7/9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma(1,M)$</td>
<td>$1$</td>
<td>$5/6$</td>
<td>$7/9$</td>
</tr>
</tbody>
</table>

$|\Psi_6^{+/−}\rangle$) the number of optimal copies that can be achieved is three ($M = 3$).

Suppose now that Alice has a qubit ($X$) in an unknown state $|X\rangle$, which she wants to share in an optimal way with Bob, Charlie and David. The procedure for achieving this is the following: First they should all share a telecloning state e.g. one of $|\Psi_6^{+/−}\rangle$. Alice will have three of the qubits, which will be called ‘port’ ($P$), ‘ancilla1’ (A1) and ‘ancilla2’ (A2), corresponding to qubits $a$, $b$ and $c$ in $|\Psi_6^{+/−}\rangle$. The remaining three qubits, corresponding to $d$, $e$ and $f$, are called ‘clone1’ (C1), ‘clone2’ (C2) and ‘clone3’ (C3) and will be distributed to Bob, Charlie and David, see Figure 6.2.

Alice now performs a joint Bell measurement on the qubits $X$ and $P$ and obtains one of the four results $|\Psi^{+/−}\rangle_{XP}$, $|\Phi^{+/−}\rangle_{XP}$. She will use classical communication to inform Bob, Charlie and David about her measurement result. Depending on which of the two telecloning states that is shared between the parties and on the result of the Bell projection, it might be necessary for the接收ers to rotate their clone-qubits in order to have optimal copies of $|X\rangle$. The local rotations that they should perform correspond to one of the unitary transformations $\mathbb{1}$, $\sigma_z$, $\sigma_x$ or $\sigma_y$. To implement these rotations one can use a HWP@0° for $\sigma_z$ and a HWP@45° for $\sigma_x$. $\sigma_y$ can be implemented by a HWP@0° and a HWP@45° in series. For the implementation of the identity operator, of course no rotation is needed.

Exactly which rotations are needed to obtain the optimal copies for $|\Psi_6^{+/−}\rangle$ and $|\Psi_6^{−+/−}\rangle$ for the different Bell measurement outcomes are shown in Table 6.3. The fidelities of the clones with respect to the original state $|X\rangle$ will now take the optimal value $7/9$. Note that the ancilla qubits are always traced out.

Using the $|\Psi_6^{+/−}\rangle$ states, telecloning with two receivers ($M = 2$) can be performed. In this case the fidelity of the two optimal copies can reach $5/6$. The procedure is equivalent to the one with $M = 3$ and the local rotations depending on the outcome of the Bell measurement are the same, see Table 6.3 and substitute $|\Psi_6^{+/−}\rangle$ for $|\Psi_6^{−+/−}\rangle$.

By using one of the Bell states as telecloning state, there can only be one receiver. This reduces the scheme to quantum teleportation, which can
Table 6.3: Local rotations on each clone-qubit used in telecloning scheme. The columns define which telecloning state is used and the rows denote the possible outcomes of the Bell measurement.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Shared state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\Psi^+\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\Psi^-\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\Phi^+\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>\Phi^-\rangle$</td>
</tr>
</tbody>
</table>

be seen as a special case of quantum telecloning with $M = 1$. Here, the procedure is equivalent and the local rotations are equal when using $|\Psi^+/\rangle$ instead of $|\Psi^+_6\rangle$, compare Table 6.1 and 6.3.
6.3 Quantum telecloning procedure

Figure 6.2: Quantum telecloning with three optimal clones. By sharing a $|\Psi^+_6\rangle$ or a $|\Psi^-_6\rangle$ state (a), Alice and her partners can make three optimal copies of an unknown state $|X\rangle$ at distant locations. The scheme involves a Bell measurement (b) and classical communication (CC) together with local unitary operations ($\hat{U}^\dagger$) (c). In the end Bob, Charlie and David will each have an optimal copy ($\tilde{X}$) of the original state $|X\rangle$. 
Chapter 7

Excluding non-contextual hidden variable theories

Quantum mechanics can in general only make statistical predictions of measurement outcomes, whereas classical theories often are able predict absolute outcomes. This property of quantum mechanics has lead people to question whether quantum mechanics can be completed, in order to predict absolute measurement results, e.g. by hidden variable (HV) theories [26]. Such theories use so called hidden variables, which should carry information about measurement results for any possible future measurement. I.e. every property of an object should have predefined definite values. Nevertheless, there are two theorems showing that seemingly plausible subsets of HV theories cannot, in fact, make the same statistical predictions of measurement outcomes as quantum mechanics. The two theorems are Bell’s theorem and the Kochen-Specker (KS) theorem.

Bell’s theorem states that no HV theories satisfying locality (i.e. an object cannot be influenced by space-like separated events) can reproduce all predictions of quantum mechanics. Specifically, certain correlations of entangled states of distributed composite systems cannot be properly described by such local hidden variable (LHV) theories. As stated in section 1.3.2 numerous experimental violations of Bell inequalities have lead us to strongly believe (all loopholes have not yet been closed) that LHV theories cannot be used to explain quantum phenomena.

The Kochen-Specker (KS) theorem asserts that non-contextual hidden variable (NCHV) theories cannot reproduce predictions made by quantum mechanics [61–65]. NCHV theories are another class of HV theories that are not excluded by Bell’s theorem. Non-contextuality requires that the
predefined definite values should be independent of which subset of compatible (co-measurable) measurements one might make, i.e. the measurement context. In other words, any NCHV theory should assign the observable \( A \) the same value regardless of whether \( A \) will be measured together with \( B, C, \ldots \), or together with \( a, \alpha, \ldots \), where some of \( B, C, \ldots \) and some of \( a, \alpha, \ldots \) might not be compatible. If we were to measure observable \( A \) first, we could even wait to decide which observables to measure next (\( a, \alpha, \ldots \) or \( B, C, \ldots \)), until we have recorded the outcome of \( A \). Thus the statistics would be equal in both cases and the most plausible HV theory would be a NCHV theory.

The KS theorem is valid for systems of three or more levels. In two-dimensional systems it does not make sense to talk about non-contextuality, since a two-level system only can be measured in one context (any projection operator is orthogonal only to one projection operator). In contrast to Bell’s theorem, the KS theorem does not rely only on correlations of some specific quantum states. As will be shown the KS theorem can be proved for any quantum state, product or superposition, of any system, composite or not, including single individual systems in maximally mixed states [62, 66]. Here it is the measurement correlations of certain quantum mechanical observables acting on the same system that are the key.

\section{Theoretical proof of the KS theorem}

In the following it will be shown that by assuming a NCHV theory for a four-dimensional system, one will arrive at a contradiction, excluding NCHV theories from theories able to predict quantum mechanical phenomena.

Consider the nine dichotomic observables shown in Eq. 7.1, each with eigenvalues \( \pm 1 \) (i.e. the possible measurement outcomes of each observable are \( \pm 1 \)) [63]. In four dimensions it is convenient to write observables as
7.1 Theoretical proof of the KS theorem

tensor products of two Pauli (and/or identity) matrices for spin-$\frac{1}{2}$ particles. Note that this does not imply that one needs to think of the physical four-dimensional system as being composed of two spin-$\frac{1}{2}$ systems, although it might be convenient, both theoretically and experimentally. For now, let $s$ and $p$ denote the first and second qubit, respectively. This notation will be explained below.

<p>| | | |</p>
<table>
<thead>
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<tbody>
<tr>
<td>$A$</td>
<td>$B$</td>
<td>$C$</td>
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<tr>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$\gamma$</td>
</tr>
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</table>

$$A = \sigma^s_z \hspace{1cm} B = \sigma^p_z \hspace{1cm} C = \sigma^s_z \otimes \sigma^p_z$$
$$a = \sigma^p_x \hspace{1cm} b = \sigma^s_x \hspace{1cm} c = \sigma^s_x \otimes \sigma^p_x$$
$$\alpha = \sigma^s_z \otimes \sigma^p_z \hspace{1cm} \beta = \sigma^s_x \otimes \sigma^p_z \hspace{1cm} \gamma = \sigma^s_y \otimes \sigma^p_y$$

(7.1)

Also note that the observables in each of the three rows, as well as in each of the three columns are mutually commuting. Moreover, the products of the three observables in each row, and in the left and middle columns are $+\mathbb{1}$. The product of the observables in the last column is $-\mathbb{1}$. Hence, the product of the values assigned to the three observables in each row must be $+1$. This follows from the fact that the values of mutually commuting observables must obey any functional identity that is satisfied by the observables themselves [63]. I.e.

$$f(A, B, C, \ldots) = 0 \Rightarrow f(v(A), v(B), v(C), \ldots) = 0 ,$$

(7.2)

where $v(A)$ is the value of observable $A$, etc. That the row products equal $+1$, further implies that each row must contain either zero or two negative values ($-1$) and thus in total there must be an even number of negative values.

In a similar manner, the product of the values assigned to the three observables in the first and middle columns must also be $+1$, and the product of the values in the last column must be $-1$. This conveys that the number of negative values must be either zero or two in each of the first two columns, but one or three in the last column. Thus the total number of negative values must be odd, which contradicts the requirements of the row identities. This contradiction implies the impossibility to assign predefined values to all nine observables.

7.1.1 The 18 vector proof

Another proof of the KS theorem in four dimensions, using a more general construction relating to colouring of vectors, is the 18 vector proof. This kind of proof can also be used to prove the KS theorem for three dimensions, but then at least 31 vectors are needed [67]. Here we have 18 vectors, each
representing the projection operator onto the corresponding normalised vector, see Table 7.1. As an example “1-111” corresponds to the projection onto the vector \((1, -1, 1, 1)/2\). Each column contains four mutually orthogonal vectors and hence the four corresponding projectors add up to the identity. In any NCHV theory, each column must contain one and only one vector that is assigned the value ’1’, the others should be ’0’. This is a consequence of the sum rule of quantum mechanics for orthogonal resolutions of the identity (i.e., if the sum of a subset of mutually orthogonal projection operators is the identity, one and only one of the corresponding answers ought to be yes) [62, 65]. It is common to translate this into a colouring problem, where one vector in each column must be coloured white, while the others are black. Such an assignment is however impossible since in Table 7.1 each vector appears in two columns. That is, there will always be an even number of white entries (’1’s) in the table. Nevertheless the number of columns is odd (nine) and there can only be one white entry in each column. Hence we have a contradiction.

7.2 A state-independent KS inequality

The contradictive proofs above can be rephrased into an experimentally testable inequality with an upper bound for NCHV theories, that is lower than the quantum mechanical bound. This was recently shown by Cabrero [64] and his inequality reads

\[
\chi \equiv \langle ABC \rangle + \langle abc \rangle + \langle \alpha\beta\gamma \rangle + \langle Aaa \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle \leq 4 ,
\]

(7.3)

with the observables defined as in Eq. (7.1). The “classical” bound is simply found by generating all the \(2^9\) possible values of \(\chi\), with \(A, B, C, a, b, c, \alpha, \beta, \gamma = \pm 1\), and taking the maximum value which is 4.

Nevertheless, by using the row and column identities \((ABC = abc = \alpha\beta\gamma = Aaa = Bb\beta = -Cc\gamma = 1)\) we find that, according to quantum
mechanics, the left-hand side of Eq. (7.3) must be 6. Hence, there is a significant contrast between the prediction of quantum mechanics and the classical bound. Thus an experimental violation of the inequality should be feasible and would prove the KS theorem. Furthermore, since every term in the inequality simplifies to the identity operator, the value of $\chi$ should be 6 for any state. I.e. in stark contrast to any Bell inequality, the inequality (7.3) is state-independent. This inequality has recently been violated experimentally using a composite system of two trapped $^{40}$Ca$^+$ ions [68]. In the following, the first experimental violation with single systems (using single photons) will be described.

7.3 A state-independent KS experiment with single photons

Here we want to test the quantum prediction of inequality (7.3) on different quantum states of a four-level single-particle system, and thus experimentally prove the KS theorem for four-dimensional systems. An appropriate choice of physical system for encoding the four-level system is the polarisation and path degrees of freedom of single photons. In this way we can readily encode a four-level system in a single particle. In particular this encoding provides a natural way of decomposing the four-level system into two qubits (two-level systems), making the observables as stated in Eq. (7.1) more comprehensible. In our case the first qubit is encoded in the spatial path ($s$) of the photon and the second qubit in its polarisation ($p$). The quantum states $|t\rangle_s$ and $|r\rangle_s$, where $t$ and $r$ denote the transmitted and reflected paths of the photon, respectively, provide a basis for describing any quantum state of the photon’s spatial path. Similarly, $|H\rangle_p$ and $|V\rangle_p$, where $H$ and $V$ denote horizontal and vertical polarisation, respectively, provide a basis for describing any quantum state of the photon’s polarisation. The basis states in the computational basis for the complete system can now be defined as

$$
|t\rangle |H\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |t\rangle |V\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
$$

$$
|r\rangle |H\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |r\rangle |V\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
$$

(7.4)
7.3.1 State encoding

The first part of the experiment serves to encode any quantum state in the four-level system (Figure 7.2). It is composed of an attenuated laser, working as a quasi-single photon source, followed by a half wave plate (HWP) and a polarising beam splitter (PBS). This arrangement lets us adjust the absolute amplitudes in the two paths $t$ and $r$. The phase between the paths is set by the wedge ($W$) placed in the $r$-arm. The two outputs of the PBS can, independently, be rotated to any polarisation using a HWP and a quarter wave plate (QWP) in each spatial path. The laser we used was a narrow bandwidth diode laser emitting at 780 nm and offering a long coherence length ($>1$ m). The laser was attenuated so that the two-photon coincidences were negligible. The mean photon number per detection time window was 0.058.

![Figure 7.2: State preparation. This arrangement allows for encoding of arbitrary states (in a four-dimensional Hilbert space) using the path and polarisation degrees of freedom of single photons. The symbols are defined at the bottom of Figure 7.4.](image)

7.3.2 Measurement devices

A fundamental point of great importance in the experiment is that the actual measurements of the nine observables in Eq. (7.1) are context-independent, in the sense that every measurement device should look the same regardless of in which context it is used. Thus the experimental apparatus for measuring an observable, e.g. $c$, must be the same in every context within which $c$ is being measured, i.e. the setup for measuring $c$ should be the same when $c$ is being measured together with $a$ and $b$, and when it is being measured together with $C$ and $\gamma$ [69]. The devices for measuring the nine observables are shown in Figure 7.3. The $A$-measurement, which is the simplest one, corresponds to distinguishing the two paths $t$ and $r$. The measurements of $B$ and $a$ are polarisation measurements in the $H/V$- and $+/-$-bases, respectively. The reason for these devices having only one input path is that they only are used at points where the path qubit not will be measured any more.
7.3 A state-independent KS experiment with single photons

Figure 7.3: Measurement devices for the nine observables. Measurement results can be obtained by placing a detector in each output path. When further processing is desired, the corresponding eigenstate must be prepared in each output path.
80 Excluding non-contextual hidden variable theories

To measure observable $b$, its eigenstates $(|t\rangle \pm |r\rangle) / \sqrt{2}$ should be mapped to the output paths $t$ and $r$, respectively. This is accomplished by phase sensitive interference, controlled with a wedge, in a 50/50 beam splitter (BS). The remaining five observables $C$, $c$, $\alpha$, $\beta$ and $\gamma$ are products of one path observable and one polarisation observable, $\sigma_s^i \otimes \sigma_p^j$. The corresponding devices must have only two outputs: one for the case in which the product observable yields the value $+1$, and the other for the case in which it yields $-1$, without making any distinction between the different eigenstates of $\sigma_s^i$ and $\sigma_p^j$. If, to obtain the result of $\sigma_s^i \otimes \sigma_p^j$, one simply measured $\sigma_s^i$ and $\sigma_p^j$, then a subsequent measurement of $\sigma_s^j \otimes \sigma_p^i$ would not be compatible.

For the observables to be compatible they need to share a common set of eigenstates. The four sets of observables in the first two rows and the first two columns in Figure 7.3, each share eigenstates corresponding to products of eigenstates from their non-composite observables. As an example the observables $A$, $B$ and $C$ share the eigenstates $|t\rangle |H\rangle$, $|t\rangle |V\rangle$, $|r\rangle |H\rangle$, $|r\rangle |V\rangle$. The last column observables ($C$, $c$, $\gamma$) have the four Bell states, given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|t\rangle |H\rangle + |r\rangle |V\rangle), \quad (7.5)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|t\rangle |H\rangle - |r\rangle |V\rangle), \quad (7.6)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|t\rangle |V\rangle + |r\rangle |H\rangle), \quad (7.7)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|t\rangle |V\rangle - |r\rangle |H\rangle), \quad (7.8)$$

as their common eigenstates. This implies that these observables can be implemented as Bell measurements, with different distributions of the Bell states in the two output paths. Similarly $\alpha$ and $\beta$ can be implemented as Bell measurements preceded by polarisation rotations. These two observables share, together with $\gamma$, the following eigenstates:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\Phi^-\rangle + |\Psi^+\rangle) = \frac{1}{2}(|t\rangle |H\rangle + |t\rangle |V\rangle + |r\rangle |H\rangle - |r\rangle |V\rangle), \quad (7.9)$$
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Psi^-\rangle) = \frac{1}{2}(|t\rangle |H\rangle - |t\rangle |V\rangle + |r\rangle |H\rangle + |r\rangle |V\rangle), \quad (7.10)$$
$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(|\Phi^-\rangle - |\Psi^+\rangle) = \frac{1}{2}(|t\rangle |H\rangle - |t\rangle |V\rangle - |r\rangle |H\rangle - |r\rangle |V\rangle), \quad (7.11)$$
$$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Psi^-\rangle) = \frac{1}{2}(|t\rangle |H\rangle - |t\rangle |V\rangle + |r\rangle |H\rangle + |r\rangle |V\rangle). \quad (7.12)$$
7.3 A state-independent KS experiment with single photons

Figure 7.4: Experimental setups for measuring the six sets of observables in inequality (7.3). In each setup there are seven boxes, corresponding to the measurement devices in Figure 7.3. In some cases additional components have been added in order to transform a device’s output to the correct eigenstate. The state encoding setup in Figure 7.2 is the first part in each experimental setup, but has been omitted in these schematics.
7.3.3 Sequential measurements

As each term in the inequality (7.3) corresponds to a different experimental context, six different experimental setups are actually needed, one setup for each term in the inequality. We have measured each set of compatible observables, e.g. $A$, $B$, $C$, using sequential measurements in tree configurations. After the state preparation the photon enters the first measurement device ($A$) through its input and interacts with the device. This interaction makes the photon exit the device in one of two possible output paths, representing different measurement values ($\pm 1$). By placing detectors in the outputs we could obtain this value, but we would be unable to perform subsequent measurements. Instead we have constructed two identical $B$-devices, one at each of $A$’s outputs, measuring the second observable ($B$). Similarly, we have four $C$-devices, one at each of the $B$-devices’ outputs. Finally we have placed a detector after each of the eight outputs. A single photon passing through the whole arrangement causes a single detection in one of the eight detectors. This detection reveals the measurement results of the three observables $A$, $B$ and $C$. As single-photon detectors we used Silicon avalanche photodiodes. All single counts were registered using an 8 channel coincidence logic with a time window of 1.7 ns. The overall detection efficiency

Figure 7.5: Photo of the $c\gamma C$ setup. Compare with Figure 7.4.
7.3 A state-independent KS experiment with single photons

of the experiment, defined as the ratio of detected to prepared photons, was $\eta = 0.50$. This value was obtained considering that the detection efficiency of the single-photon detectors is 55% and the fiber coupling is 90%. Therefore, the fair sampling assumption (i.e., the assumption that detected photons are an unbiased subensemble of the prepared photons) is needed to conclude a violation of the inequality.

The predictions of both NCHV theories and quantum mechanics do not depend on the order of the compatible measurements. Therefore we have chosen the most convenient order for each set of observables in the experiment (e.g., we measured $CAB$ instead of $ABC$). This was usually the configuration which minimised the number of required interferometers. We first constructed the $CAB$ setup and measured its expectation value for 16 different pure states. Then we constructed the $\alpha Aa$ setup and measured its expectation value for the same 16 states, and so on for the $cba$, $\beta bB$, $\beta \gamma \alpha$ and $c\gamma C$ setups. Schematics for all six setups are shown in Figure 7.4 and a photo of the $c\gamma C$ setup is shown in Figure 7.5.

Note that some interferometers in the setups are phase insensitive in the sense that the relative phase between the two spatial paths ($t$ and $r$) is of no significance. This phase difference is usually much more unstable than the phase difference between $H$- and $V$-polarisation in a single spatial mode, which is very stable in isotropic media. An example of a phase insensitive interferometer is the interferometer including the state preparation and the $C$-device in the $CAB$ setup (Figure 7.4). Here, the two PBS outputs of the $C$-device are going directly to two other PBSs. Hence, any phase difference between $H$ and $V$ in any of the $C$-outputs, originating from the phase between the incoming $t$ and $r$ paths, is of no relevance (the two output paths will never meet again).

Figure 7.6: A Mach-Zehnder interferometer (a), and two displaced Sagnac interferometers with one (b) or two (c) beam splitters (BSs).
All the interferometers in the experimental setup are based on free space displaced Sagnac interferometers [34, 70]. These interferometers are more stable than e.g. Mach-Zehnder interferometers due to their, to some extent, intrinsically stable design, see Figure 7.6. Since all mirrors are reflecting both beam paths, small vibrations in one mirror will cause similar path length fluctuations in both paths. Furthermore, when possible we have built interferometers with a single PBS, instead of two independent PBSs, increasing the stability further. Using these techniques we have reached a visibility above 99% for phase insensitive overlaps, and a visibility ranging between 90% and 95% for phase sensitive interferometers. The phases were set using tilted glass plates.

7.3.4 Experimental violation

We have tested inequality (7.3) with 20 different states with different purity, ranging from completely pure states to a maximally mixed state, see Table 7.2. For each of the 16 pure states we have calculated the average value of each set of compatible observables for about $1.7 \times 10^7$ detected photons. The results for the mixed states were obtained by suitably combining pure state data. Figure 7.7 shows the expectation value of $\chi$ for each of the 20 states. It is clear that all tested states violate the inequality with a comfortable margin, and hence we undoubtedly see a state-independent violation of non-contextual hidden variable theories. The average value of $\chi$ for all pure states is 5.455. Due to experimental imperfections our measured values do not reach the quantum mechanical prediction of $\chi = 6$. The main systematic errors were due to imperfect beam overlaps, phase drifts between the time of calibration and the measurement, and imperfect polarisation optics. The statistical errors were deduced from propagated Poissonian counting statistics of the raw detection events.
7.3 A state-independent KS experiment with single photons

Table 7.2: Experimental value of $\langle CAB \rangle + \langle cba \rangle + \langle \beta \gamma \alpha \rangle + \langle \alpha Aa \rangle + \langle \beta bB \rangle - \langle c\gamma C \rangle$ for 20 quantum states. The average value is $5.4550 \pm 0.0006$ and on average we violate the inequality with 655 standard deviations (stds).

<table>
<thead>
<tr>
<th>State</th>
<th>Expectation value</th>
<th># std</th>
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<tbody>
<tr>
<td>$</td>
<td>\psi_1\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_2\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_3\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_4\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
</tr>
<tr>
<td>$\rho_5 = \frac{13}{16}</td>
<td>\psi_1\rangle \langle \psi_1</td>
<td>+ \frac{1}{16} \sum_{j=2}^{4}</td>
</tr>
<tr>
<td>$\rho_6 = \frac{5}{8}</td>
<td>\psi_1\rangle \langle \psi_1</td>
<td>+ \frac{1}{8} \sum_{j=2}^{4}</td>
</tr>
<tr>
<td>$\rho_7 = \frac{11}{16}</td>
<td>\psi_1\rangle \langle \psi_1</td>
<td>+ \frac{5}{16} \sum_{j=2}^{4}</td>
</tr>
<tr>
<td>$</td>
<td>\psi_8\rangle =</td>
<td>t\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_9\rangle =</td>
<td>t\rangle</td>
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<tr>
<td>$</td>
<td>\psi_{10}\rangle =</td>
<td>r\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{11}\rangle =</td>
<td>r\rangle</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{12}\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
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<tr>
<td>$</td>
<td>\psi_{13}\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle</td>
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<tr>
<td>$</td>
<td>\psi_{14}\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle +</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{15}\rangle = \frac{1}{\sqrt{2}}(</td>
<td>t\rangle + i</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{16}\rangle = \frac{1}{2}(</td>
<td>t\rangle +</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{17}\rangle = \frac{1}{2}(</td>
<td>t\rangle + i</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{18}\rangle = \frac{1}{2}(</td>
<td>t\rangle +</td>
</tr>
<tr>
<td>$</td>
<td>\psi_{19}\rangle = \frac{1}{2}(</td>
<td>t\rangle + i</td>
</tr>
<tr>
<td>$\rho_{20} = \frac{1}{4} \sum_{j=1}^{4}</td>
<td>\psi_j\rangle \langle \psi_j</td>
<td>$</td>
</tr>
</tbody>
</table>
Figure 7.7: Expectation values of $\chi$ for 20 different states. The red solid line indicates the bound of inequality (7.3) and the blue dashed line indicates the average value ($5.4550 \pm 0.0006$) of the 16 pure states. All 20 states clearly violate the inequality. The quantum states are presented in Table 7.2. The statistical errors are negligible on the presented scale.
Chapter 8

Conclusions

In this thesis experimental tests of foundations of quantum mechanics have been presented. These tests include multi-photon correlations of entangled photons and genuine multi-qubit entanglement detection with entanglement witnesses. Also pure quantum mechanical properties such as rotational invariance of multi-photon states have been shown and a few applications of multi-photon entanglement have been briefly described. Finally we have presented results of a novel state-independent inequality testing whether non-contextual hidden variable theories can complete quantum mechanics.

8.1 Multi-photon entanglement

In the work on multi-photon entanglement we have experimentally generated and characterised two six-photon entangled states ($|\Psi^+_{6}/^-_{6}\rangle$) of very high quality. For the experimental generation of the entangled states we have used the technique of parametric down-conversion (PDC), linear optics and conditional detection. We chose to entangle our states in the polarisation degree of freedom of single photons and use the non-collinear configuration of type-II PDC in a BBO crystal. By suitable processing and filtering we were able to collect the third order emission of the PDC. In order to achieve the desired states we have utilised bosonic emission enhancement due to indistinguishability of multiple down-conversion pairs. To take advantage of this and to produce states of high quality we needed to accurately control the following properties of the PDC-emission:

- Spatial modes – First we had to consider the transversal walk-off effect for light of extraordinary polarisation compared to that of ordinary
polarisation. This effect is caused by birefringence in the BBO crystal and was optimally compensated for by compensation crystals of half the thickness of the PDC-BBO. Single spatial modes were then defined by coupling the light to single-mode fibers.

- **Spectral distribution** – The spectral distribution will directly influence the coherence length of the photons. Therefore the spectral distribution needed to be narrow enough to have a coherence length covering all photons, and yielding a coherent superposition of high quality. On the other hand with a narrower spectrum, the number of transmitted photons would decrease. Hence there is a trade-off here between the quality of the states and the measurement time needed to have sufficient counting statistics. In the experiments we have used 3 nm (FWHM) filters.

- **Timing** – To achieve up to six simultaneous photons, we first need to define what we should mean by ‘simultaneous’. Here the important factor is that the photons should be coherent, i.e. by ‘simultaneous’ we mean “within the coherence time”. Since the coherence time given by a bandwidth of 3 nm lies in the order of 300 femtoseconds, we were using ultra short pump pulses in the femtosecond regime. There were also longitudinal walk-off effects, separating photons of different polarisation, and hence diminishing the coherence. These effects were optimally compensated for by the same compensation technique as for the transversal walk-off.

- **Polarisation** – Since the qubits were encoded in the polarisation of single photons we needed to maintain the polarisation of the light when traversing the fibers. This was done with passive polarisation controllers.

- **The phase** – By changing the phase between horizontal and vertical polarisation we could adjust which state to produce. The phase was altered by slightly tilting one of the compensation crystals.

After this processing the photons were divided into six spatial modes by 50/50 beam splitters. Each qubit could then be analysed in an arbitrary basis and the right state was found by conditional detection of one qubit in each mode. A multi-channel coincidence unit was developed to keep track of all detection data. This unit is able to record all possible single and coincidence counts between any of the 12 detectors. In addition it was used for adjusting the delay for all signals from the single photon detectors.
8.1 Multi-photon entanglement

With this setup we have not only looked at six-photon entanglement, but also four- and two-photon entangled states ($|\Psi^+/\rangle$ and $|\Psi^+/\rangle$) have been studied. One class of states that we have generated and which are of special importance are the invariant states $|\Psi^-\rangle$, $|\Psi^-\rangle$ and $|\Psi^-\rangle$. These states remain unchanged under simultaneous identical unitary transformations of all their qubits.

All our experimentally observed states showed characteristic properties of the ideal states with high quality. The fidelities of the six-photon states were in the ranges $0.81 \leq F_{\Psi^-} \leq 0.92$ and $0.79 \leq F_{\Psi^+} \leq 0.92$. The four-photon states had lower bounds of their fidelities at 0.92 and the fidelities of the two-photon Bell states were 0.97. Also projections of the states onto different bases had results in very good agreement with theoretical predictions. The rotational invariance is unquestionable for our implementation of $|\Psi^-\rangle$, and also for $|\Psi^+\rangle$ the high coherence of the state is obvious. The absolute values of the correlation function of measurements in three different bases were all well above 80% and most of them above 86%. Classically the correlation function would be zero for other bases than the preparation basis.

To verify that we actually were dealing with genuine entanglement we have derived a reduced witness, capable of affirming genuine six-partite entanglement with only three measurement settings. With this type of witness we showed genuine entanglement in our experimental implementations of $|\Psi^-\rangle$ and $|\Psi^+\rangle$ with 2.0 and 1.5 standard deviations, respectively.

One of many possible applications for our generated states is the quantum telecloning protocol, combining optimal quantum cloning and quantum teleportation. With the $|\Psi^+/\rangle$ states quantum telecloning with three receivers and an optimum fidelity of $7/9$ could be performed. Using the $|\Psi^+/\rangle$ states the scheme with two receivers could be implemented with fidelities up to $5/6$. When using the $|\Psi^+/\rangle$ states the scheme reduces to the quantum teleportation scheme with theoretically perfect fidelity.

Another application of the rotationally invariant states are as basis states in a decoherence-free subspace, making decoherence-free encoding possible. A decoherence-free subspace spanned by six-qubit states would be five-dimensional, and thereby big enough to accomplish decoherence-free encoding of arbitrary two-qubit states (including entangled states). Other possible applications of the $|\Psi^-\rangle$ state include secret sharing, communication without a common reference frame and remote state preparation.

One could imagine going to eight-photon entanglement using a similar setup. This could be possible provided higher quantum efficiency of single-photon detectors, or better coupling to single-mode fibers (e.g. by using
Another interesting approach is the continuing development of periodically poled crystals and waveguides. These crystals have very high non-linearities and therefore don’t require high pump powers. Also their down-conversion spectra seems to be narrower than for bulky crystals [71]. Further advantages include single spatial mode emission, using waveguides. So far this kind of crystals has shown great results for two-photon entanglement [72], but higher-order entanglement has not yet been shown.

8.2 Excluding non-contextual hidden variable theories

In chapter 7 the work on an experimental test of a Kochen-Specker inequality was described. This inequality is valid for four-level systems and is fulfilled for any non-contextual hidden variable theory [64]. According to quantum theory it is remarkably violated by any quantum state, with an equal margin of violation, and is hence state-independent.

We used the spatial path and the polarisation degrees of freedom of single photons to encode the four-level system. The state-independent inequality was for the first time violated shortly before we finished our work using a composite system of two ions [68]. Nevertheless, the violation described here is the first using single systems. Our results here show that hidden variable theories based on a non-contextuality assumption cannot explain experimentally observed outcomes of simple measurements on single photons. The inequality, which is fulfilled for any non-contextual hidden variable theory, was violated for all 20 states that were tested. These states included both product states, superposition states (with and without entanglement between the internal degrees of freedom) and even a maximally mixed state, as well as states of different mixtures. Hence entanglement is not the only key element distinguishing quantum mechanics from classical physics.
Appendix A

Derivation of the reduced witness

We start to define the maximum overlap witness \[51, 52\] given by

\[
\tilde{W}_n^{+/−} = ζ \mathbb{1}^\otimes n - \rho_{\Psi_n^{+/−}} = ζ \mathbb{1}^\otimes n - |\Psi_n^{+/−}\rangle \langle \Psi_n^{+/−}| ,
\]  

(A.1)

where \(n\) is the number of qubits in the state and \(ζ\) is the maximum overlap of \(\rho_{\Psi_n^{+/−}}\) with any \(n\)-qubit biseparable state \(|\Psi_{sep}\rangle\):

\[
ζ = \max_{\Psi_{sep}} \{⟨\Psi_{sep}|\rho_{\Psi_n^{+/−}}|\Psi_{sep}\rangle\} .
\]  

(A.2)

The value of \(ζ\) can conveniently be found using the overlapb command in the QUBIT4MATLAB package [53].

For the reduced witness we use the non-mixed Pauli operators of the density operator of the state \(\rho_{\Psi_n^{+/−}}\). By non-mixed Pauli operators we mean operators that can be written as a direct product of only one type of Pauli operators (\(σ_x, σ_y\) or \(σ_z\)) and identity operators (\(\mathbb{1}\)), e.g. \(σ_x^\otimes 2 \otimes \mathbb{1}^\otimes n−2\). Let us now define the operator \(\text{NMP}(\rho)\) as all non-mixed Pauli operators in decomposition of \(\rho\), except the \(n\)-qubit identity operator \(\mathbb{1}^\otimes n\). The reduced witness can now be expressed as

\[
W_n^{+/−} = κ \mathbb{1}^\otimes n - \text{NMP}(\rho_{\Psi_n^{+/−}}) ,
\]  

(A.3)

where the coefficient \(κ\) is dictated by the inequality

\[
⟨W_n^{+/−}\rangle = \text{Tr}(W_n^{+/−} \rho) \geq ξ \cdot \text{Tr}(\tilde{W}_n^{+/−} \rho) = ξ \cdot ⟨\tilde{W}_n^{+/−}\rangle \quad \forall \quad \rho .
\]  

(A.4)

\(ξ\) is a real-valued positive constant (\(ξ > 0\)). Eq. (A.4) makes sure that the reduced witness only may be negative for states yielding negative expectation
values for the maximum overlap witness. I.e. the reduced witness will be able to detect genuine multi-qubit entanglement [73]. $|\Psi_n^{+/−}\rangle$ is an eigenstate to both $\tilde{W}_n^{+/−}$ and $W_n^{+/−}$ and in both cases it corresponds to the minimum eigenvalue, $\lambda_1$ and $\tilde{\lambda}_1$, respectively:

$$\tilde{W}_n^{+/−} |\Psi_n^{+/−}\rangle = \tilde{\lambda}_1 |\Psi_n^{+/−}\rangle,$$

$$W_n^{+/−} |\kappa=0\rangle |\Psi_n^{+/−}\rangle = \lambda_1 |\Psi_n^{+/−}\rangle.$$ (A.5) (A.6)

The inequality in Eq. (A.4) with the expectation values taken over $|\Psi_n^{+/−}\rangle$ now gives

$$\langle \Psi_n^{+/−} | (W_n^{+/−} - \xi \cdot \tilde{W}_n^{+/−}) |\Psi_n^{+/−}\rangle = \kappa + \lambda_1 - \xi \tilde{\lambda}_1 \geq 0.$$ (A.7)

All other eigenvalues of $\tilde{W}_n^{+/−}$, except for $\tilde{\lambda}_1$, are degenerate and thus have the same value, denoted $\tilde{\lambda}_2$. The non-minimum eigenvalues of $W_n^{+/−}$ are not all degenerate, but in addition to its minimum eigenvalue only its second smallest eigenvalue, denoted $\lambda_2$, will be needed in this derivation.

For any state $|\Psi_{⊥}^{+/−}\rangle$, that is orthogonal to $|\Psi_n^{+/−}\rangle$, now applies that

$$\langle \Psi_{⊥}^{+/−} | (W_n^{+/−} - \xi \cdot \tilde{W}_n^{+/−}) |\Psi_{⊥}^{+/−}\rangle \geq \kappa + \lambda_2 - \xi \tilde{\lambda}_2 \geq 0,$$ (A.8)

where the last inequality makes sure that the inequality in Eq. (A.4) holds. By combining Eq. (A.7) and the last inequality in Eq. (A.8), we arrive at the following system of equations, where we have set the equations equal to zero as we want $\kappa$ as small as possible:

$$\begin{cases} \kappa + \lambda_1 - \xi \tilde{\lambda}_1 = 0 \\ \kappa + \lambda_2 - \xi \lambda_2 = 0 \end{cases}.$$ (A.9)

By solving for $\xi$ and $\kappa$ we obtain the results presented in Table A.1 for $n = 2, 4, 6$. The value of $\xi$ can be seen as measure of how close the reduced witness is to the corresponding optimal (maximum overlap) witness. With $n = 2$ the value of $\xi = 1$, which means that the reduced witness in this case is the same as the maximum overlap witness.
Table A.1: $\xi$ and $\kappa$ in the reduced witness for $|\Psi_n^{+/−}\rangle$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\xi$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>$1/4 = 0.25$</td>
</tr>
<tr>
<td>4</td>
<td>$5/12 \approx 0.42$</td>
<td>$11/24 \approx 0.46$</td>
</tr>
<tr>
<td>6</td>
<td>$1/6 \approx 0.17$</td>
<td>$181/576 \approx 0.31$</td>
</tr>
</tbody>
</table>
Appendix B

The four-photon entangled states $|\Psi_{-4}^{+}\rangle$ and $|\Psi_{+4}^{+}\rangle$

By selecting two of the three spatial modes $a, b, c$ combined with two of $d, e, f$ and looking at the four-photon coincidences our experimental setup will give us one of the four-photon states $|\Psi_{+4}^{+/-}\rangle$ originating from second order PDC emission. It is important that the modes are selected such that exactly two spatial modes are chosen from each PDC emission arm, see fig 4.1.

Taking the second order term ($p = 2$) of Eq. (2.12) leads to the photon-number state:

$$|2H_a, 2V_b\rangle + e^{i\phi}|1H_a, 1V_a, 1H_b, 1V_b\rangle + e^{2i\phi}|2V_a, 2H_b\rangle .$$

(B.1)

After beam splitting and selection of the terms with one photon in each spatial mode and adjustment of the phase $\phi$ to 0 or $\pi$ we arrive at the $|\Psi_{+/-4}^{+/-}\rangle$ states:

$$|\Psi_{+/-4}^{+/-}\rangle = \sqrt{\frac{1}{3}}|HHVV\rangle + \sqrt{\frac{1}{3}}|VVHH\rangle \pm \frac{1}{2\sqrt{3}}|HVHV\rangle$$

$$= \pm \frac{1}{2\sqrt{3}}|HVHV\rangle \pm \frac{1}{2\sqrt{3}}|VHHV\rangle \pm \frac{1}{2\sqrt{3}}|VHVH\rangle .$$

(B.2)

These two states can also be expressed as superpositions of a four-photon GHZ state and a product of two Bell states

$$|\Psi_{+/-4}^{+/-}\rangle = \sqrt{\frac{2}{3}}|GHZ_4\rangle \pm \sqrt{\frac{1}{3}}|\Psi^{+}\rangle |\Psi^{+}\rangle .$$

(B.3)
The four-photon entangled states $|\Psi^-_4\rangle$ and $|\Psi^+_4\rangle$

The GHZ state is here equal to

$$|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|HHVV\rangle + |VVHH\rangle)$$  \hspace{1cm} (B.4)

and the Bell state is given by

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle).$$  \hspace{1cm} (B.5)

$|\Psi^-_4\rangle$ is invariant under identical unitary transformations $\hat{U}$ on each qubit according to

$$\hat{U}^{\otimes 4} |\Psi^-_4\rangle = e^{i\tau} |\Psi^-_4\rangle.$$  \hspace{1cm} (B.6)

The four-fold coincidence probabilities corresponding to projective measurements in the three usual bases are shown in Figure B.1 ($|\Psi^-_4\rangle$) and in Figure B.2 ($|\Psi^+_4\rangle$). The corresponding values of the correlation function as well as measurement time and total four-photon coincidence counts are given in Table B.1 and B.2.

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/-$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>93.4 ± 0.4</td>
<td>91.9 ± 0.4</td>
<td>90.2 ± 0.2</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Measurement time [h]</td>
<td>143.5</td>
<td>141.5</td>
<td>136.2</td>
</tr>
<tr>
<td>Four-photon coincidences</td>
<td>354 226</td>
<td>325 783</td>
<td>407 439</td>
</tr>
<tr>
<td>Count rate [s$^{-1}$]</td>
<td>0.69</td>
<td>0.64</td>
<td>0.83</td>
</tr>
</tbody>
</table>

### B.1 The $|\Psi^-_4\rangle$ correlation function

As for $|\Psi^-_6\rangle$ in section 4.3.2, we will now calculate the $|\Psi^-_4\rangle$ correlation function on the prime meridian. It can be expressed significantly more compact than its six-photon counterpart and looks like

$$E(\theta_a, \theta_b, \theta_c, \theta_d) = \langle \Psi^-_4 | \sigma_a(\theta_a) \otimes \ldots \otimes \sigma_d(\theta_d) | \Psi^-_4 \rangle =$$

$$= \frac{2}{3} \cos(\theta_a + \theta_b - \theta_c - \theta_d) + \frac{1}{3} \cos(\theta_a - \theta_b) \cos(\theta_c - \theta_d).$$  \hspace{1cm} (B.7)
B.1 The $|\Psi^-_4\rangle$ correlation function

Figure B.1: Experimental results of the four-photon invariant state $|\Psi^-_4\rangle$ in modes $a$, $b$, $d$ and $e$. Four-fold coincidence probabilities corresponding to detections of one photon in each mode in $H/V$-basis (a), $+/\sim$-basis (b), and $L/R$-basis (c) are shown. Comparing the three measurement results makes the invariance of the state obvious. Due to the long measurement time the statistical errors are negligible.
The four-photon entangled states $|\Psi^+_4\rangle$ and $|\Psi^-_4\rangle$

Figure B.2: Experimental results of the four-photon state $|\Psi^+_4\rangle$ in modes $a$, $b$, $d$ and $e$. Four-fold coincidence probabilities corresponding to detections of one photon in each mode in $H/V$-basis (a), $+/-$-basis (b), and $L/R$-basis (c) are shown. Due to the long measurement time the statistical errors are negligible.
B.1 The $|\Psi^-_4\rangle$ correlation function

Table B.2: Correlation function, measurement time, number of four-photon coincidences and count rates for $|\Psi^+_4\rangle$ in modes $a$, $b$, $d$ and $e$.

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/-$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>93.2 ± 0.2</td>
<td>91.5 ± 0.3</td>
<td>90.3 ± 0.2</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Measurement time [h]</td>
<td>99.7</td>
<td>41.2</td>
<td>140.7</td>
</tr>
<tr>
<td>Four-photon coincidences</td>
<td>287 757</td>
<td>142 562</td>
<td>330 569</td>
</tr>
<tr>
<td>Count rate [s$^{-1}$]</td>
<td>0.80</td>
<td>0.96</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Figure B.3(a)-(f) shows the correlation functions dependence on the angle $\theta_b$ of the local observable $\sigma_b(\theta_b) = \sigma_{z,b} \cos \theta_b + \sigma_{x,b} \sin \theta_b$ in mode $b$. The other polarisation analysers are held fixed at the $\{+/−\}$ polarisation basis, simplifying Eq. (B.7) to

$$E\left(\frac{\pi}{2}, \theta_b, \frac{\pi}{2}, \frac{\pi}{2}\right) = \sin \theta_b.$$  \hfill (B.8)

The state is implemented in several combinations of spatial modes and in the figure we include all implementations comprising mode $b$, whose analysis basis have been varied. The visibilities of the correlation functions in Figure B.3 as well as the average count rates for each set of modes are presented in Table B.3.

Table B.3: Visibilities of $|\Psi^-_4\rangle$ correlation function.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Figure B.3</th>
<th>Visibility [%]</th>
<th>Count rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b, d, e$</td>
<td>(a)</td>
<td>91.9 ± 1.4</td>
<td>0.6 s$^{-1}$</td>
</tr>
<tr>
<td>$a, b, d, f$</td>
<td>(b)</td>
<td>91.8 ± 1.1</td>
<td>1.3 s$^{-1}$</td>
</tr>
<tr>
<td>$a, b, e, f$</td>
<td>(c)</td>
<td>91.9 ± 1.4</td>
<td>1.1 s$^{-1}$</td>
</tr>
<tr>
<td>$b, c, d, e$</td>
<td>(d)</td>
<td>92.0 ± 1.2</td>
<td>1.2 s$^{-1}$</td>
</tr>
<tr>
<td>$b, c, d, f$</td>
<td>(e)</td>
<td>91.8 ± 0.9</td>
<td>2.6 s$^{-1}$</td>
</tr>
<tr>
<td>$b, c, e, f$</td>
<td>(f)</td>
<td>92.0 ± 1.2</td>
<td>2.1 s$^{-1}$</td>
</tr>
</tbody>
</table>
The four-photon entangled states $|\Psi^-_4\rangle$ and $|\Psi^+_4\rangle$

Figure B.3: Four-photon polarisation correlation functions of $|\Psi^-_4\rangle$. Modes $a, c, d, e$ and $f$ are analysed in the $+/-$-basis and mode $b$ analysis basis is rotated around the prime meridian of the Bloch sphere ($\sigma_z \cos \theta_b + \sigma_x \sin \theta_b$). The figures correspond to different implementations of the state, using different modes ($abde$, $abdf$, $abef$, $bcde$, $bcdf$ and $bcef$). The solid lines show sinusoidal fits to the experimental data with an average visibility of $V_4 = 91.9\% \pm 0.5\%$. The average measurement time for each data point was 77 hours; the statistical errors are negligible on the presented scale.
B.2 Reduced witnesses

The reduced witnesses derived for $|\Psi_4^{+/-}\rangle$ are given by

$$
W_4^- = \frac{11}{24} I^4 - \frac{1}{48} \sigma_x \sigma_x \mathbb{1} - \frac{1}{48} \sigma_y \sigma_y \mathbb{1} - \frac{1}{48} \sigma_z \sigma_z \mathbb{1} + \frac{1}{24} \sigma_x \mathbb{1} \sigma_x \\
+ \frac{1}{24} \sigma_y \mathbb{1} \sigma_y + \frac{1}{24} \sigma_z \mathbb{1} \sigma_z + \frac{1}{24} \mathbb{1} \sigma_x \sigma_x + \frac{1}{24} \mathbb{1} \sigma_y \sigma_y + \frac{1}{24} \mathbb{1} \sigma_z \sigma_z \\
+ \frac{1}{24} \sigma_x \mathbb{1} \sigma_x + \frac{1}{24} \sigma_y \mathbb{1} \sigma_y + \frac{1}{24} \sigma_z \mathbb{1} \sigma_z + \frac{1}{24} \mathbb{1} \sigma_x \sigma_x + \frac{1}{24} \mathbb{1} \sigma_y \sigma_y + \frac{1}{24} \mathbb{1} \sigma_z \sigma_z \\
+ \frac{1}{24} \sigma_x \mathbb{1} \sigma_x - \frac{1}{48} \mathbb{1} \sigma_x \sigma_x - \frac{1}{48} \mathbb{1} \sigma_y \sigma_y - \frac{1}{48} \mathbb{1} \sigma_z \sigma_z \\
- \frac{1}{16} \sigma_x \sigma_x \sigma_x \sigma_x - \frac{1}{16} \sigma_y \sigma_y \sigma_y \sigma_y - \frac{1}{16} \sigma_z \sigma_z \sigma_z \sigma_z
$$

(B.9)

and

$$
W_4^+ = \frac{11}{24} I^4 - \frac{1}{48} \sigma_x \sigma_x \mathbb{1} - \frac{1}{48} \sigma_y \sigma_y \mathbb{1} - \frac{1}{48} \sigma_z \sigma_z \mathbb{1} - \frac{1}{24} \sigma_x \mathbb{1} \sigma_x \\
- \frac{1}{24} \sigma_y \mathbb{1} \sigma_y + \frac{1}{24} \sigma_z \mathbb{1} \sigma_z - \frac{1}{24} \mathbb{1} \sigma_x \sigma_x - \frac{1}{24} \mathbb{1} \sigma_y \sigma_y + \frac{1}{24} \mathbb{1} \sigma_z \sigma_z \\
- \frac{1}{24} \sigma_x \mathbb{1} \sigma_x - \frac{1}{24} \sigma_y \mathbb{1} \sigma_y + \frac{1}{24} \sigma_z \mathbb{1} \sigma_z - \frac{1}{24} \mathbb{1} \sigma_x \sigma_x - \frac{1}{24} \mathbb{1} \sigma_y \sigma_y + \frac{1}{24} \mathbb{1} \sigma_z \sigma_z \\
+ \frac{1}{24} \mathbb{1} \sigma_x \mathbb{1} \sigma_x - \frac{1}{48} \mathbb{1} \sigma_x \sigma_x - \frac{1}{48} \mathbb{1} \sigma_y \sigma_y - \frac{1}{48} \mathbb{1} \sigma_z \sigma_z \\
- \frac{1}{16} \sigma_x \sigma_x \sigma_x \sigma_x - \frac{1}{16} \sigma_y \sigma_y \sigma_y \sigma_y + \frac{1}{16} \sigma_z \sigma_z \sigma_z \sigma_z
$$

(B.10)

The noise tolerance for these witnesses is 18.5% (cf. 26.7% for the maximum overlap witness counterpart) and their expectation values for their ideal states are $\langle \Psi_4^- | W_4^- | \Psi_4^- \rangle = \langle \Psi_4^+ | W_4^+ | \Psi_4^+ \rangle = -5/48 \approx -0.104$. The experimentally observed witness values are presented in Table B.4-B.5. Our experimental implementations of $|\Psi_4^-\rangle$ and $|\Psi_4^+\rangle$ are showing genuine four-qubit entanglement with well above 100 standard deviations, and the average lower bound fidelities are 0.919 ± 0.002 and 0.919 ± 0.003, respectively.

B.3 Three-qubit entanglement by projective measurements

Similarly to achieving five-photon entangled states by projective measurements of one qubit in $|\Psi_5^-\rangle$, we can also create three-photon entangled states by doing the same thing with $|\Psi_4^\pm\rangle$. Here we project qubit $b$ onto the $H/V$-basis and analyse the remaining three-qubit states in the $H/V$- as well as
The four-photon entangled states $|\Psi^-\rangle$ and $|\Psi^+\rangle$

Table B.4: Observed witness values, witness violation in number of standard deviations, and lower and upper bounds of the fidelity for $|\Psi^-\rangle$.

<table>
<thead>
<tr>
<th>Modes</th>
<th>$\langle W^- \rangle$</th>
<th># std</th>
<th>$F_{\text{min}}$</th>
<th>$F_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b, d, e$</td>
<td>$-0.0714 \pm 0.0005$</td>
<td>138</td>
<td>$0.9214 \pm 0.0012$</td>
<td>$0.9563 \pm 0.0007$</td>
</tr>
<tr>
<td>$a, b, d, f$</td>
<td>$-0.0699 \pm 0.0004$</td>
<td>195</td>
<td>$0.9179 \pm 0.0009$</td>
<td>$0.9544 \pm 0.0005$</td>
</tr>
<tr>
<td>$a, b, e, f$</td>
<td>$-0.0693 \pm 0.0004$</td>
<td>176</td>
<td>$0.9164 \pm 0.0009$</td>
<td>$0.9535 \pm 0.0005$</td>
</tr>
<tr>
<td>$a, c, d, f$</td>
<td>$-0.0704 \pm 0.0003$</td>
<td>262</td>
<td>$0.9190 \pm 0.0006$</td>
<td>$0.9550 \pm 0.0004$</td>
</tr>
<tr>
<td>$b, c, d, e$</td>
<td>$-0.0705 \pm 0.0004$</td>
<td>188</td>
<td>$0.9193 \pm 0.0009$</td>
<td>$0.9551 \pm 0.0005$</td>
</tr>
<tr>
<td>$b, c, d, f$</td>
<td>$-0.0704 \pm 0.0003$</td>
<td>271</td>
<td>$0.9190 \pm 0.0006$</td>
<td>$0.9550 \pm 0.0003$</td>
</tr>
<tr>
<td>$b, c, e, f$</td>
<td>$-0.0700 \pm 0.0003$</td>
<td>246</td>
<td>$0.9181 \pm 0.0007$</td>
<td>$0.9545 \pm 0.0004$</td>
</tr>
</tbody>
</table>

the $+/−$-basis. In the computational basis the projection of the $b$-qubit onto $|V\rangle$ leads to

$$b\langle V|\Psi^-\rangle = \sqrt{\frac{2}{3}} |VHH\rangle - \frac{1}{\sqrt{6}} |HHV\rangle - \frac{1}{\sqrt{6}} |HVH\rangle.$$  \hspace{1cm} (B.11)

A similar projection onto $|H\rangle$ results in

$$b\langle H|\Psi^-\rangle = \sqrt{\frac{2}{3}} |HVV\rangle - \frac{1}{\sqrt{6}} |VHV\rangle - \frac{1}{\sqrt{6}} |VVH\rangle.$$  \hspace{1cm} (B.12)

Figure B.4 shows the experimental results from measurements in $H/V$- and $+/−$-basis for the two three-qubit states.
B.3 Three-qubit entanglement by projective measurements

Figure B.4: The three-photon states $\langle V | \Psi^-_4 \rangle_{abde}$ (a)-(b) and $\langle H | \Psi^-_4 \rangle_{abde}$ (c)-(d) obtained through projective measurement of the $b$-qubit onto $|V\rangle$ and $|H\rangle$, respectively. The bars represent three-fold coincidence probabilities, conditioned on detecting a $|V\rangle$- or $|H\rangle$-photon in mode $b$. In (a) and (c) all qubits are measured in the computational basis. The results in (b) and (d) correspond to measurements of the three qubits in the diagonal basis. The statistical errors are negligible on the presented scale.
The four-photon entangled states $|\Psi_4^-\rangle$ and $|\Psi_4^+\rangle$

Table B.5: Observed witness values, witness violation in number of standard deviations, and lower and upper bounds of the fidelity for $|\Psi_4^+\rangle$.

<table>
<thead>
<tr>
<th>Modes</th>
<th>$\langle W^+_4 \rangle$</th>
<th># std</th>
<th>$F_{\text{min}}$</th>
<th>$F_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b, d, e$</td>
<td>$-0.0699 \pm 0.0007$</td>
<td>106</td>
<td>$0.9178 \pm 0.0016$</td>
<td>$0.9543 \pm 0.0009$</td>
</tr>
<tr>
<td>$a, b, d, f$</td>
<td>$-0.0710 \pm 0.0005$</td>
<td>155</td>
<td>$0.9205 \pm 0.0011$</td>
<td>$0.9558 \pm 0.0006$</td>
</tr>
<tr>
<td>$a, b, e, f$</td>
<td>$-0.0721 \pm 0.0005$</td>
<td>144</td>
<td>$0.9230 \pm 0.0012$</td>
<td>$0.9572 \pm 0.0007$</td>
</tr>
<tr>
<td>$a, c, d, f$</td>
<td>$-0.0702 \pm 0.0003$</td>
<td>205</td>
<td>$0.9184 \pm 0.0008$</td>
<td>$0.9547 \pm 0.0005$</td>
</tr>
<tr>
<td>$b, c, d, e$</td>
<td>$-0.0689 \pm 0.0005$</td>
<td>145</td>
<td>$0.9153 \pm 0.0011$</td>
<td>$0.9529 \pm 0.0006$</td>
</tr>
<tr>
<td>$b, c, d, f$</td>
<td>$-0.0691 \pm 0.0003$</td>
<td>211</td>
<td>$0.9159 \pm 0.0008$</td>
<td>$0.9533 \pm 0.0004$</td>
</tr>
<tr>
<td>$b, c, e, f$</td>
<td>$-0.0710 \pm 0.0004$</td>
<td>198</td>
<td>$0.9204 \pm 0.0009$</td>
<td>$0.9558 \pm 0.0005$</td>
</tr>
</tbody>
</table>
Appendix C

The two-photon entangled states $|\Psi^-\rangle$ and $|\Psi^+\rangle$

By selecting only one spatial mode from each PDC emission arm ($a$, $b$ or $c$ together with $d$, $e$ or $f$) and looking for two-photon coincidences, the two-photon Bell states $|\Psi^+/\rangle$ can be obtained.

$$|\Psi^+/\rangle = \frac{1}{\sqrt{2}} (|HV\rangle \pm |VH\rangle)$$

(C.1)

$|\Psi^-\rangle$ is invariant under identical unitary transformations $\hat{U}$ on both qubits according to

$$\hat{U} \otimes^2 |\Psi^-\rangle = e^{i\tau} |\Psi^-\rangle.$$  (C.2)

The experimental results of $|\Psi^+/\rangle$ in the spatial modes $a$ and $d$ are presented in Figure C.1 and in Table C.1-C.2.

Table C.1: Correlation function, measurement time, number of two-photon coincidences and count rate for $|\Psi^-\rangle$ in modes $a$ and $d.$

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/-$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>$-97.5 \pm 0.2$</td>
<td>$-96.4 \pm 0.2$</td>
<td>$-95.6 \pm 0.1$</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>$-100$</td>
<td>$-100$</td>
<td>$-100$</td>
</tr>
<tr>
<td>Measurement time [h]</td>
<td>143.5</td>
<td>141.5</td>
<td>136.2</td>
</tr>
<tr>
<td>Two-photon coincidences</td>
<td>$2.27 \cdot 10^9$</td>
<td>$2.18 \cdot 10^9$</td>
<td>$2.42 \cdot 10^9$</td>
</tr>
<tr>
<td>Count rate [s$^{-1}$]</td>
<td>4400</td>
<td>4280</td>
<td>4930</td>
</tr>
</tbody>
</table>
The two-photon entangled states $|\Psi^-\rangle$ and $|\Psi^+\rangle$

Figure C.1: Experimental results of the two-photon invariant Bell states $|\Psi^-\rangle$ (a)-(c) and $|\Psi^+\rangle$ (d)-(f) in modes $a$ and $d$. Two-fold coincidence probabilities corresponding to detections of one photon in each mode in $H/V$-basis (a), (d), $+/-$-basis (b), (e), and $L/R$-basis (c), (f) are shown. Comparing the three measurement results (a)-(c) makes the invariance of $|\Psi^-\rangle$ obvious. Due to the long measurement time the statistical errors are negligible.
Table C.2: Correlation function, measurement time, number of two-photon coincidences and count rate for $|\Psi^+\rangle$ in modes $a$ and $d$.

<table>
<thead>
<tr>
<th>Analysis basis</th>
<th>$H/V$</th>
<th>$+/−$</th>
<th>$L/R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr. function (exp.) [%]</td>
<td>−97.1 ± 0.1</td>
<td>96.2 ± 0.1</td>
<td>95.8 ± 0.1</td>
</tr>
<tr>
<td>Corr. function (theo.) [%]</td>
<td>−100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Measurement time [h]</td>
<td>99.7</td>
<td>41.2</td>
<td>140.7</td>
</tr>
<tr>
<td>Two-photon coincidences</td>
<td>$1.72 \cdot 10^9$</td>
<td>$7.8 \cdot 10^8$</td>
<td>$2.20 \cdot 10^9$</td>
</tr>
<tr>
<td>Count rate [$s^{-1}$]</td>
<td>4790</td>
<td>5240</td>
<td>4340</td>
</tr>
</tbody>
</table>

C.1 The $|\Psi^-\rangle$ correlation function

The $|\Psi^-\rangle$ correlation function on the prime meridian is given by

$$E(\theta_a, \theta_b) = \langle \Psi^- | \sigma_a(\theta_a) \otimes \sigma_b(\theta_b) | \Psi^- \rangle = −\cos(\theta_a - \theta_b). \quad (C.3)$$

Figure C.2 shows the correlation functions dependence on the angle $\theta_b$ of the local observable $\sigma_b(\theta_b) = \sigma_z \cos \theta_b + \sigma_x \sin \theta_b$ in mode $b$. The other polarisation analyser is held fixed at the $\{+/−\}$ polarisation basis, simplifying Eq. (C.3) to

$$E\left(\frac{\pi}{2}, \theta_b\right) = −\sin \theta_b. \quad (C.4)$$

The state is implemented in several combinations of spatial modes and in the figure we include all implementations comprising mode $b$, whose analysis basis have been varied. The visibilities of the correlation functions in Figure C.2 as well as the average count rates for each set of modes are presented in Table C.3.

Table C.3: Visibilities of $|\Psi^-\rangle$ correlation function.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Figure C.2</th>
<th>Visibility [%]</th>
<th>Count rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b, d$ (a)</td>
<td>96.2 ± 0.4</td>
<td>4604 $s^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$b, e$ (b)</td>
<td>96.3 ± 0.6</td>
<td>3814 $s^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$b, f$ (c)</td>
<td>96.2 ± 0.4</td>
<td>7916 $s^{-1}$</td>
<td></td>
</tr>
</tbody>
</table>
The two-photon entangled states $\lvert \Psi^- \rangle$ and $\lvert \Psi^+ \rangle$

Figure C.2: Two-photon polarisation correlation functions of $\lvert \Psi^- \rangle$. Modes $d$, $e$ and $f$ are analysed in the $+/-$-basis and mode $b$ analysis basis is rotated around the prime meridian of the Bloch sphere ($\sigma_z \cos \theta_b + \sigma_x \sin \theta_b$). The figures correspond to different implementations of the state, using different modes ($bd$, $be$ and $bf$). The solid lines show sinusoidal fits to the experimental data with an average visibility of $V_2 = 96.2\% \pm 0.3\%$. The average measurement time for each data point was 77 hours; the statistical errors are negligible on the presented scale.
C.2 Entanglement witnesses for $|\Psi^\pm\rangle$ and $|\Psi^\mp\rangle$

For the two-qubit states $|\Psi^+/\rangle$, are the maximum overlap witnesses decomposable in non-mixed Pauli and identity operators and hence we do not need to calculate reduced witnesses. The maximum overlap witnesses, optimised for $|\Psi^+/\rangle$, are given by

$$\tilde{W}_2^- = \frac{1}{2} \mathbb{1} - |\Psi^-\rangle \langle \Psi^-| = \frac{1}{4} (\mathbb{1} + \sigma_x \sigma_x + \sigma_y \sigma_y + \sigma_z \sigma_z)$$

and

$$\tilde{W}_2^+ = \frac{1}{2} \mathbb{1} - |\Psi^+\rangle \langle \Psi^+| = \frac{1}{4} (\mathbb{1} - \sigma_x \sigma_x - \sigma_y \sigma_y + \sigma_z \sigma_z).$$

The noise tolerance for these witnesses is $2/3 \approx 66.7\%$ and their expectation values for their ideal states are $\langle \Psi^- | \tilde{W}_2^- | \Psi^- \rangle = \langle \Psi^+ | \tilde{W}_2^+ | \Psi^+ \rangle = -1/2 = -0.5$. The experimentally observed witness values are presented in Table C.4-C.5. Our experimental implementations of $|\Psi^-\rangle$ and $|\Psi^+\rangle$ are showing entanglement with well above 50 000 standard deviations.

For maximum overlap witnesses all non-negative eigenvalues are degenerate and hence the lower and upper bounds of the fidelity, as derived in section 4.3.3, will coincide. Thus, using such witnesses we obtain a value for the fidelity directly, instead of just lower and upper bounds. Fidelities of five implementations of $|\Psi^+/\rangle$ are given in Table C.4-C.5.

Table C.4: Observed witness values, witness violation in number of standard deviations, and fidelities for $|\Psi^-\rangle$.

<table>
<thead>
<tr>
<th>Modes</th>
<th>$\langle W^- \rangle$</th>
<th># std</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b, d$</td>
<td>-0.473669 ± 0.000006</td>
<td>75 405</td>
<td>0.973669 ± 0.000006</td>
</tr>
<tr>
<td>$b, e$</td>
<td>-0.472787 ± 0.000007</td>
<td>68 302</td>
<td>0.972787 ± 0.000007</td>
</tr>
<tr>
<td>$b, f$</td>
<td>-0.471278 ± 0.000005</td>
<td>98 237</td>
<td>0.971278 ± 0.000005</td>
</tr>
<tr>
<td>$a, d$</td>
<td>-0.473591 ± 0.000007</td>
<td>72 690</td>
<td>0.973591 ± 0.000007</td>
</tr>
<tr>
<td>$c, f$</td>
<td>-0.471397 ± 0.000004</td>
<td>131 309</td>
<td>0.971397 ± 0.000004</td>
</tr>
</tbody>
</table>
The two-photon entangled states $|\Psi^-\rangle$ and $|\Psi^+\rangle$

Table C.5: Observed witness values, witness violation in number of standard deviations, and fidelities for $|\Psi^+\rangle$.

<table>
<thead>
<tr>
<th>Modes</th>
<th>$\langle W^+ \rangle$</th>
<th># std</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b, d$</td>
<td>$-0.470938 \pm 0.000008$</td>
<td>56 498</td>
<td>$0.970938 \pm 0.000008$</td>
</tr>
<tr>
<td>$b, e$</td>
<td>$-0.473377 \pm 0.000009$</td>
<td>52 078</td>
<td>$0.973377 \pm 0.000009$</td>
</tr>
<tr>
<td>$b, f$</td>
<td>$-0.474413 \pm 0.000006$</td>
<td>75 355</td>
<td>$0.974413 \pm 0.000006$</td>
</tr>
<tr>
<td>$a, d$</td>
<td>$-0.472946 \pm 0.000009$</td>
<td>54 458</td>
<td>$0.972946 \pm 0.000009$</td>
</tr>
<tr>
<td>$c, f$</td>
<td>$-0.470981 \pm 0.000005$</td>
<td>99 689</td>
<td>$0.970981 \pm 0.000005$</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
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</table>


Part II

Scientific papers