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Combinatorial geometries in model theory

Ove Ahlman

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Handledare och examinator: Vera Koponen
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Department of Mathematics
Uppsala University

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Abstract

Model theory and combinatorial pregeometries are closely related through the so called algebraic closure operator on strongly minimal sets. The study of projective and affine pregeometries are especially interesting since they have a close relation to vectorspaces. In this thesis we will see how the relationship occur and how model theory can conclude a very strong classification theorem which divides pregeometries with certain properties into projective, affine and degenerate (trivial) cases.

Abstract

Modellteori är ett ämne som är starkt relaterat till studien av kombinatoriska pregeometrier, detta genom den algebraiska tillslutningsoperatorn som agerar på starkt minimala mängder. Studien av projektiva och affina pregeometrier är speciellt intressant genom dessas relation till vektorrum. I den här uppsatsen kommer vi att se hur denna relation uppstår och hur modellteori kan förklara en väldigt stark klassifikationssats, som delar upp pregeometrier med speciella egenskaper i projektiva, affina och degenererade (triviala) fall.

1 Introduction

This thesis has two major topics woven into each other. The two topics are combinatorial pregeometries (aka matroids) and model theory (in mathematical logic). The understanding of model theory is (as will be shown in this thesis) very closly connected to the study of pregeometries. Many questions in model theory have been answered by the understanding of pregeometries. Conversely, work in model theory has contributed to the understanding of pregeometries (for instance the classification theorem in this essay).

One of the early topics of model theory was to study totally categorical theories, that is, theories with exactly one model up to isomorphism in every infinite cardinality. This class of theories is quite restrictive, but the ideas that where first developed in this context have had great influence on the further development of model theory. A very important result in this area is due to M.Morley, who proved that theories categorical in one uncountable cardinality is categorical in all uncountable cardinalities [8]. Such theories are called uncountably categorical. It was asked if the uncountably categorical theories were finitely axiomatizable. B.Zilber later showed that totaly categorical theories can't have this property [12]. This work by Zilber also gave a lot of other interesting information (necessary for his result) about the structure of the models of uncountable categorical theories. The information did in turn generate, together with the so called "stability theory", also called "classification theory", a subfield inside model theory called "geometric stability theory".

To be able to understand the models of uncountably categorical theories (also called uncountably categorical structures), it is vital to understand how the so called minimal subsets are built up. These are the smallest possible infinite definable subsets of the structure's universe. By using these minimal sets we may prove a lot of important properties of the whole structure. One of the properties of a minimal subset A of a model $\mathfrak{M}$ is that it induces a combinatorical pregeometry, which pretty much rules $\mathfrak{M}$, if $\mathfrak{M}$ is uncountably categorical or totally categorical. Pregeometries obtained from minimal sets were first studied by M.Marh [7], and were later generalized by Baldwin and Lachlan [1].

The pregeometies on minimal sets are one of the connections between combinatorical geometry and model theory. In this thesis we will among other things study how these pregeometries are created from minimal subsets. We will also look at another combinatorical structure, called pseudoplane, and see how it interacts with the pregeometries when defined in a model of an uncountably categorical theory (Trichotomy theorem 7.1). This result will in turn lead us to the Classification theorem 8.1 of infinite homogenous locally finite, pregeometries, which says that any such pregeometry is either degenerated (trivial) or isomorphic to a projective or affine geometry over a finite field. The presentation of the proof of this theorem is based on B.Zilbers work as presented in [12]. The classification theorem was also proved by Cherlin, Harrington and Lachlan [2] (about the same time as Zilber), but using the classification of finite simple groups. It was later proved, using only combinatorical geometry, by Evans [4].

An interesting question, which may arrise while reading about the classification theorem, is if homogenity is a necessary assumption. Is there any infinite locally finite non-homogenous pregeometry? The question was open until the 1990s when E.Hrushovski [6], deviced a method based on amalgamation of finite structures, which made it possible to create this kind of pregeometry (which can be found in [11]).

In this thesis, complete proofs will not be given for all results. Instead we concentrate on explaining the role of the pregeometry obtained by the algebraic closure operator when working on minimal sets. Some very tedious reults in the field of locally projective geometries will also be stated without proofs, but with reference to where they can be found. Zilber's original proof involves some more advanced model theoretic notions like types, Morley-rang, Morley-degree etc. which we will exclude here. Instead we will emphasise how the results fit together and how the geometric and model theoretic results cooperate.

Notation Throughout this whole paper, we will use Gothic letters (like $\mathfrak{M}$) to denote a first order structure for a countable language. The underlying universe will use the corresponding latin letter (in our example M). Note that the letter L will not be used to denote a language. When we need to speak of the structure's language we will instead talk about its signature, which consists of the constant, function and relation symbols of the language of the structure. This set will be denoted by $\Omega_{\mathfrak{M}}$, or if the structure is clear from the context Ω. In the cases where we talk about a formula $\varphi(x_1, ..., x_n)$ it will be a formula created from symbols in Ω, connectives and quantifiers, unless we state differently.

In Example 5.2 and 5.11 we will talk about elimination of quantifiers. When we say that a structure $\mathfrak{M}$ has elimination of quantifiers we mean that there exists a set S which contains only atomic formulas such that for any formula $\varphi(x_1, ..., x_n)$ there exists a formula $\psi(x_1, ..., x_n)$ which is a boolean combination of formulas in S such that

$$\mathfrak{M} \models \forall x_1, ..., x_m(\varphi(x_1, ..., x_m) \leftrightarrow \psi(x_1, ..., x_m)).$$

Quantifier elimination will only be used breifly and we will refer to Hodges [5] who's Lemma 2.7.3. is sufficient when proving this property.

By the notation$\mathfrak{N} \preceq \mathfrak{M}$ we mean that $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$, or equivalently, that $\mathfrak{M}$ is an elementary extension of $\mathfrak{N}$.

The notation $\exists^{=n} x\varphi(x)$ means that there exists exactly n x such that $\varphi(x)$. So $\exists^{=n} x\varphi(x)$ is just an abbreviation of

$$\exists x_1, ..., \exists x_n(\bigwedge_{i<j\leq n} (x_i \neq x_j \wedge \varphi(x_i) \wedge \varphi(x_j)) \wedge \forall y((\bigwedge_{i\leq n} y \neq x_i) \rightarrow \neg\varphi(y)).$$

We may also sometimes write $\exists^{\leq n} x\varphi(x)$ which is the same as

$$\exists^{=0} x\varphi(x) \vee \exists^{=1} x\varphi(x) \vee ... \vee \exists^{=n} x\varphi(x).$$

The symbol $\equiv$ will be used to denote equivalence between formulas. As an example $\varphi(x) \equiv x = x$ say that $\varphi(x)$ denotes (or rather is equivalent to) the formula $x = x$. In order to write out the cardinality of any set A we use the notation $|A|$.

A more detailed introduction to basic model theoretic notions like categoricity etc. may be found in Rothmaler's [10] or Hodges' [5] books.

2 Definability

First some necessary notation.

Definition 2.1 *Let $\mathfrak{M}$ be our structure and $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ a formula from $\Omega_{\mathfrak{M}}$ then*

$$\varphi(a_1, ..., a_n, \mathfrak{M}) = \{(b_1, ..., b_m) \in M^m \mid \mathfrak{M} \models \varphi(a_1, ..., a_n, b_1, ..., b_m)\}.$$

That is, $\varphi(a_1, ..., a_n, \mathfrak{M})$ is the set of all the tuples $(b_1, ..., b_m)$ which fulfill the formula φ together with the elements $a_1, ...a_n$. We call the elements $a_1, ..., a_n$ the parameters of $\varphi(a_1, ..., a_n, x_1, ..., x_m)$.

Examples 2.2

- Let $\varphi(x, y) \equiv x = y$. In every structure $\mathfrak{M}$, $\varphi(\mathfrak{M})$ denotes the set of all pairs of the same element. If the universe of the structure $\mathfrak{M}$ would be $\mathbb{N}$, then this set would be $\varphi(\mathfrak{M}) = \{(0, 0), (1, 1), ...\}$.

- If we have the formula $\varphi(x,y) \equiv y < x$ (free variables y and x) in $\mathfrak{N} = \langle \mathbb{N}, < \rangle$, then choosing x as the parameter 4 we get $\varphi(4,\mathbb{N}) = \{0,1,2,3\}$.
- In any ring $\mathfrak{M} = \langle M, 0, 1, +, * \rangle$ the formula $\varphi(x) \equiv \exists y(x*y = 0 \wedge y \neq 0)$ (where $*$ is interpreted as ring multiplication) produce the set $\varphi(\mathfrak{M})$ of all numbers which multiplied by some non zero number is equal to zero, that is $\varphi(\mathfrak{M})$ consists of all zero divisors. Hence if this ring is an integral domain then $\varphi(\mathfrak{M}) = \{0\}$

Definition 2.3 *Let $\mathfrak{M}$ be a structure and $X \subseteq M$, $A \subset M^k$. We say that A is X-**definable** in $\mathfrak{M}$ if there exists a formula $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ such that for some $b_1, ..., b_n \in X$ we have that*

$$A = \varphi(b_1, ..., b_n, \mathfrak{M})$$

*We say that a subset (or relation) of M is **definable** if it is M-definable. A function $f : M^n \to M$, $n \in \mathbb{N}$, is said to be X-definable if its graph is X-definable.*

Remark 2.4 Most of the times, parametrization like in Definition 2.3 will be written in a more compact way. Instead of writing "for some $b_1, ..., b_n \in X$ and formula $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ we have $\varphi(b_1, ..., b_n, y_1, ...y_m)$" we can write just $\varphi(X, y_1, ..., y_m)$. When this is done the meaning is the same, just not using so much extra space. In the case where it is important exactly which parameters in X that are chosen, we will write them out.

Examples 2.5

- In $\mathfrak{M} = \langle \mathbb{N}, S, *, 0 \rangle$ (where S is the successor function, $*$ is the multiplication function and 0 is a constant interpreted as zero) the set Ev, which consists of all even numbers, is $\{2\}$-definable since if

  $$\psi(z,y) \equiv \exists x(y = z * x) \qquad \text{then} \qquad Ev = \psi(2, \mathfrak{M})$$

  Of course Ev is also $\emptyset$-definable since if

  $$\varphi(y) \equiv \exists x(y = S(S(0)) * x) \qquad \text{then} \qquad Ev = \varphi(\mathfrak{M})$$

  Since 0 is already in $\Omega_{\mathfrak{M}}$ we do not count it as a parameter when saying Ev is $\emptyset$-definable.
- In any group $\mathfrak{M} = \langle M, *, 1 \rangle$ (where the constant 1 is interpreted as the identity element) the formula $\varphi(x) \equiv x * x = 1$ will define the set

$\varphi(\mathfrak{M})$ of all elements of order 2. Hence the elements of order 2 are $\emptyset$-definable. The same principle may be used to define higher orders too for any finite number, just use the formula $x * ... * x = 1$. Hence we may conclude that for every $n = 2, 3, ...$ the property of having order n on elements is $\emptyset$-definable.

The next definition will take the notion of definability even further, doing it for whole structures. That is, we will see what it means that a structure $\mathfrak{N}$ is defined, or at least expressible, inside an other structure $\mathfrak{M}$.

Definition 2.6 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures (not necessarily using the same signature) and $X \subseteq M$. We say that $\mathfrak{N}$ is **X-definable** in $\mathfrak{M}$ if there exists $k \in \mathbb{Z}^+$ and a partial surjective map $\sigma : M^k \to N$ such that:*

1. *The domain of σ is an X-definable subset of M^k.*
2. *The preimage*

   $$\sigma^{-1}(R) = \{(\bar{a}_1, ..., \bar{a}_n) | \bar{a}_1, ..., \bar{a}_n \in M^k, (\sigma(\bar{a}_1), ..., \sigma(\bar{a}_n)) \in R\}$$

   is X-definable in $\mathfrak{M}$ for any relation $R \in \Omega_{\mathfrak{N}}$.
3. *The preimage $\sigma^{-1}(G_f)$ is X-definable in $\mathfrak{M}$ for the graph G_f of any function $f \in \Omega_{\mathfrak{N}}$.*
4. *The preimage $\sigma^{-1}(c)$ is X-definable in $\mathfrak{M}$ for any constant $c \in \Omega_{\mathfrak{N}}$.*

*We say that a model $\mathfrak{N}$ is **definable** in $\mathfrak{M}$ if it is M-definable in $\mathfrak{M}$.*
*The function σ is called an X**-interpretation**.*

Some less formal explanation may be required.

Remark 2.7 Let $\sigma : M^k \to N$ be an X-interpretation of $\mathfrak{N}$ in $\mathfrak{M}$. Then $x \sim y \Leftrightarrow \sigma(x) = \sigma(y)$ is an X-definable equivalence relation (since '$\sim$' is the preimage of '=') on the domain D of σ. If $\bar{a} = (a_1, ..., a_k) \in D$ let $[\bar{a}]$ be the equivalence class which $\bar{a}$ belongs to. Then we denote $D/\sim = \{[\bar{a}] \mid \bar{a} \in D\}$ and define $\bar{\sigma} : D/\sim \to N$ by $\sigma([\bar{a}]) = \sigma(\bar{a})$. This function will be a bijection and by point 2 in the definition, for every relation $R \in \Omega_{\mathfrak{N}}$ there is a formula $\bar{\varphi}_R(x_1, ..., x_n, y_1, ..., y_m)$ and parameters $b_1, ..., b_m \in X$ such that

$$\forall a_1, ..., a_n \in D \qquad \mathfrak{M} \models \bar{\varphi}(a_1, ..., a_n, b_1, ..., b_m) \;\Leftrightarrow\; \mathfrak{N} \models R(\sigma(a_1), ..., \sigma(a_n)).$$

The situation is analogous for functions or constants in $\Omega_{\mathfrak{N}}$. Hence the structure

$$\mathfrak{D} = \langle D/\sim, \{\bar{\varphi}_R | R \in \Omega_{\mathfrak{N}} \text{ or } R = graph(f), f \in \Omega_{\mathfrak{N}} \text{ or } R = c, c \in \Omega_{\mathfrak{N}}\}\rangle$$

is a well defined copy of $\mathfrak{N}$ created from equivalence classes in $\mathfrak{M}$

This might seem a bit abstract, so we will give some more concrete examples.

Examples 2.8 a) Let the abelian group $\mathbb{Z}_5$ be our $\mathfrak{M} = \langle \mathbb{Z}_5, +, 0 \rangle$ and let $\mathfrak{N}$ be the ring $\mathbb{Z}_5$, that is $\mathfrak{N} = \langle \mathbb{Z}_5, +, *, 1, 0 \rangle$. Then $\mathfrak{N}$ will be $\emptyset$-definable in $\mathfrak{M}$ since we can choose the interpretation σ to be the identity mapping, noting that the preimage of multiplication is defined by using addition at most four times. It could also be done in the other way defining $\mathfrak{M}$ in $\mathfrak{N}$, in which case we will just map it straight in since there is no multiplication to take care of in $\mathfrak{M}$

b) Choose $\mathfrak{M} = \langle \mathbb{Z}, +, * \rangle$ and $\mathfrak{N} = \langle \mathbb{Q}, +, * \rangle$. Note that the subset $\{0\}$ is $\emptyset$-definable in $\mathfrak{M}$ and in $\mathfrak{N}$. We want to define $\mathfrak{N}$ inside of $\mathfrak{M}$. The problem here is that in $\mathbb{Q}$ a number may be equal to another number without looking the same ($\frac{3}{3} = 1$ etc.). Hence we have to choose $k = 2$ and so $\sigma : \mathbb{Z}^2 \to \mathbb{Q}$ with the domain of $\sigma = \mathbb{Z}^2 - \{(p, q) | q \neq 0\}$ is defined by $\sigma(p, q) = \frac{p}{q}$. We may on $\mathbb{Z}^2$ define the relation $E((x, y), (u, v)) \Leftrightarrow xv = uy$. Notice how this relation, which is $\emptyset$-definable packs together all pairs which represent the same number in $\mathbb{Q}$. Hence it will be used to define the preimage to equality. The preimage (under σ) of the graph of $+$ is defined by the formula

$$\psi(x, y, a, b, \alpha, \beta) \equiv \beta = y * b \wedge \alpha = x * b + y * a.$$

Since for $\frac{x}{y}, \frac{a}{b}, \frac{\alpha}{\beta} \in \mathbb{Q}$, $\frac{x}{y} + \frac{a}{b} = \frac{\alpha}{\beta} \Leftrightarrow \beta = y * b$ and $\alpha = x * b + y * a$. In a similar way the preimage of the graph of '$*$' can be shown to be $\emptyset$-definable.

3 Minimal sets

Definition 3.1 *In a structure $\mathfrak{M}$, an infinite definable set $X \subseteq M$ is **minimal** if for every other definable set $Y \subseteq M$ we have that*

$$|X \cap Y| < \aleph_0 \qquad \text{or} \qquad |X - Y| < \aleph_0.$$

Note that, since every definable subset of M has a formula which defines it (possibly with parameters from M), we may look at the formulas (with parameters) possible to construct from the signature rather than sets. This is important since if you have a formula, you know that it represents a definable set, but a set in general does not have to be definable.

Sometimes it is necessary to demand something stronger, which is defined here:

Definition 3.2 *An infinite definable set $X \subseteq M$ is called* ***strongly minimal****, if for some $\bar{b} \in M^m$, and formula $\varphi(x_1, ..., x_n, \bar{b})$, $X = \varphi(\mathfrak{M}, b)$ and in every elementary extension $\mathfrak{M} \preceq \mathfrak{N}$ the set $\varphi(\mathfrak{N}, b)$ is minimal.*
A structure $\mathfrak{M}$ is ***strongly minimal*** *if the universe M of the structure is strongly minimal.*

Example 3.3 As examples of strongly minimal structures, you can look at infinite vectorspaces or algebraically closed fields. We will not discuss this further but more information may be found in [10].

We will use the following proposition which is proved in [10].

Proposition 3.4 *Every strongly minimal structure is uncountably categorical.*

4 Geometries

In this section we will talk about geometries as defined in combinatorics (there are other geometries, in algebraic geometry for example, similar but with other definitions).

Definition 4.1 *A* ***geometry*** *$G = \langle \Pi, cl \rangle$ is a set Π together with a closure operation cl which operates on the powerset of Π, and satisfies the following axioms:*
$\forall X \subseteq \Pi$

1. *Reflexivity property:*
   $X \subseteq cl(X)$

2. *Monotonicity property:*
   $\forall Y \subseteq \Pi \ , \ Y \subseteq cl(X) \Rightarrow cl(Y) \subseteq cl(X)$

3. *Exchange property:*
   $\forall a, b \in \Pi \ , \ a \in cl(X \cup \{b\}) - cl(X) \Rightarrow b \in cl(X \cup \{a\})$

4. *Geometry property:*
   $cl(\emptyset) = \emptyset \quad$ *and* $\quad \forall a \in \Pi \ , \ cl(\{a\}) = \{a\}$

5. *Finite Character property:*
   $cl(X) = \bigcup \{cl(X_0) | X_0 \subseteq X \wedge |X_0| < \aleph_0\}$

If all conditions except 4 are fulfilled we call G a ***pregeometry****.*

Examples 4.2

a) The most trivial example of a geometry is obtained if we for some set Π take $G = \langle \Pi, cl \rangle$ and let $cl(X) = X$, $\forall X \subseteq \Pi$. The axioms are satisfied here trivially.

b) A more applied example of a closure operator is the operation working on fields making them algebraically closed. Using an algebraically closed field $\mathbb{F}$ as our universe we define $G = \langle \mathbb{F}, cl \rangle$ and for $X \subseteq \mathbb{F}$

$$cl(X) = \text{The smallest algebraically closed subfield which includes X.}$$

Reflexivity is fullfilled since the extension of the field which contain X into an algebraic closed field will also include X. Monotonicity is fullfilled since an algebraically closed field is closed by definition and hence the closure of a subset of it can not create something which is not already there. Exchange property may be proved through some algebraic work (see Rothmaler [10] for a proof). It is not a geometry but just a pregeometry since $cl(\emptyset) \neq \emptyset$, because $\emptyset$ is not even a field. The finite character property is fullfilled since if $a \in cl(X)$ then there is an irreducible polynomial $P(x)$ with coeficients $b_1, ..., b_n \in X$ and with a as one, among finitely many, roots. Hence $a \in cl(b_1, ..., b_n)$.

c) Use a vector space V as the universe of the geometry $G = \langle V, cl \rangle$. As our closure on $X \subseteq V$ we take the points in the smallest subspace containing all the points in X. Using basic results about vector spaces, it is straightforward to verify that properties $1-3$ and 5 holds. Notice that this is not a geometry, but rather only a pregeometry since we have that $cl(\emptyset) = \{0\}$ and especially the closure of any point not zero is the line through origin containing that point.

d) Another way to define cl on a vectorspace is this for $X \subseteq V$. Let $W \subseteq V$ be a subspace of minimal dimension such that for some $v_0 \in X$ we have that $X \subseteq v_0 + W = \{v_0 + w \mid w \in W\}$; let $cl(X) = v_0 + W$ for this W and v_0. Let $cl(\emptyset) = \emptyset$.
Or in other words, we get all the points which lie in the smallest "non origin centralized" subspace, so we may get all points in a plane which does not necessarily contain the origin. The definition is unambiguous, which can be seen as follows. Suppose that $v_0, v_0' \in X$, $X \subseteq v_0 + W$ and $X \subseteq v_0' + W'$, where W and W' are subspaces of V. Then $\{x - v_0 | x \in X\} \subseteq W$ and $\{x - v_0' | x \in X\} \subseteq W'$ and in particular $v_0' - v_0 \in W$.

It follows that for every $x \in X$, $x - v_0' = x - v_0 + (v_0 - v_0') \in W$, because $x - v_0, v_0 - v_0' \in W$. Hence $\{x - v_0' | x \in X\} \subseteq W, W'$ and since both W and W' has minimal dimension, both of them must be the linear span of $\{x - v_0' | x \in X\}$. So then $W = W'$.
It does not matter which vector you choose to represent the plane. Since if $v_0, v_1 \in X$ and v_0 is used to describe the closure. Then $v_1 = w + v_0$ for some $w \in W$. Choosing v_1 to describe the closure instead of v_0 would create $v_1 + W = v_0 + w + W = v_0 + W$ (since $w \in W$). Notice that the big difference here compared to Example c) is that we do not handle the origin as a special point, but rather become free in our geometry. This is also what saves us from becoming just a pregeometry, since in this space $cl(\emptyset)$ is empty and $cl(\{a\}) = a$ for a single point $a \in V$ since then we will take $\{0\}$ as our subspace W and hence $cl(\{a\}) = a + \{0\} = \{a\}$.

Remark 4.3 Instead of writing $cl(\{a\})$ or $cl(\{a, b, c\})$ whenever we speak of the closure of single elements $a, b, c \in \Pi$, we will write $cl(a)$ or $cl(a, b, c)$ which will mean the same, just without having to write the set parentheses all the time. Whenever we use set operations inside the closure we will of course write it out: for example, $cl(\{a, b, c\} - \{a\})$.

The closure operation is actually closing the sets. We will show that in this lemma:

Lemma 4.4 *Let $G = \langle \Gamma, cl \rangle$ be a pregeometry and $X \subseteq \Gamma$. Then $cl(cl(X)) = cl(X)$.*

Proof. $cl(X) \subseteq cl(cl(X))$ follows from the reflexivity property. The monotonicity property say that $Y \subseteq cl(X) \Rightarrow cl(Y) \subseteq cl(X)$ for any $Y \subseteq \Gamma$. Choose $Y = cl(X) \subseteq \Gamma$. Then since $Y = cl(X) \subseteq cl(X)$ we get that $cl(Y) = cl(cl(X)) \subseteq cl(X)$. Hence $cl(cl(X)) = cl(X)$. $\square$

Remark 4.5 In example 4.2 c) we see that sometimes when we try to get a geometry we only get a pregeometry. This may always be fixed by defining a new geometry, using the old pregeometry. Take a pregeometry $P = \langle \Gamma, cl \rangle$. Then define our new geometry $\hat{P} = \langle \Pi, kl \rangle$ as follows:

$\Pi = \{cl(x) | x \in \Gamma - cl(\emptyset)\}$

Observe that every $Y \subseteq \Pi$ has the form $Y = \{cl(x) | x \in X\}$ for some $X \subseteq \Gamma$. In the other direction, if $X \subseteq \Gamma$ then $\hat{X} \subseteq \Pi$ is defined as
$\hat{X} = \{cl(x) | x \in X - cl(\emptyset)\}$. Now define $kl(\hat{X}) = \{cl(x) | x \in cl(X) - cl(\emptyset)\}$. Then $\hat{P} = \langle \Pi, kl \rangle$ is a geometry which we will call the associated geometry of P.

Example 4.6 If we take the vector space $\mathbb{R}^2$ and create the geometry $G = \langle \mathbb{R}^2, cl \rangle$ such that for $X \subseteq \mathbb{R}^2$, $cl(X)$ is the least subspace which contains all points in X, then the associated geometry $\hat{G}$ consists of all the lines (not the points on the lines but this time the actual lines) which go through the origin.

Remark 4.7 Define our geometry over a vectorspace as in Example 4.2 d), that is, $G = \langle V, cl \rangle$ and if $X \subseteq V$ then $cl(X)$ is the smallest "non origin centralized" subspace containing all points in X. We can convert this geometry defined on a vector space into the other type which we defined on a vector space in Example 4.2 c), by redefining our closure operation. Fix any element $a \in V$ and for every $X \subseteq V$ define $cl_a(X) = cl(X \cup \{a\})$. Now we will suddenly have a center in the pregeometry $G_a = \langle V, cl_a \rangle$, of course it is not a geometry anymore but only a pregeometry since $cl_a(\emptyset) = \{a\}$. Notice that this pregeometry is isomorphic to the pregeometry in example 4.2 c) (see definition below of isomorphisms between geometries).

Definition 4.8 *When we talk about* ***isomorphisms*** *between pregeometries* $G = \langle \Gamma, cl \rangle$ *and* $G' = \langle \Gamma', cl' \rangle$ *we mean a map* $\varphi : \Gamma \to \Gamma'$ *such that* φ *is bijective and for all* $X \subseteq \Gamma$, $\varphi(cl(X)) = cl'(\varphi(X))$. *A pregeometry is said to be isomorphic to another pregeometry if there exists an isomorphism between them. If* $G' = G$ *that is* φ *has the same range as domain, we call* φ *an* ***automorphism****.*

In our previous examples and remarks we have spotted some really interesting properties. Important enough to generalize and hence here are a few definitions.

Definitions 4.9 *Take a pregeometry* $G = \langle \Gamma, cl \rangle$*. Then for every* $X \subseteq \Gamma$*:*

Localization *The localization of* G *with respect to* $Y \subseteq \Gamma$ *is the geometry* $\widehat{G}_Y$ *where* $G_Y = \langle \Gamma, cl_Y \rangle$ *and* $cl_Y(X) = cl(X \cup Y)$.

G is called:

Degenerate *if* $cl(X) = X$.

Locally finite *if* $|X| < \aleph_0 \Rightarrow |cl(X)| < \aleph_0$.

Projective *if it is isomorphic to a geometry* $\hat{H}$*, where* $\hat{H}$ *is defined as in Remark 4.5, where* $H = \langle V, kl \rangle$*,* V *is a vector space and* $kl(X)$ *the points in the smallest subspace which contains all points in* X *(as in Example 4.2 c)).*

Affine *if it is isomorphic to a geometry* $\hat{H}$ *such that* $H = \langle V, kl \rangle$ *where* V *is a vector space and the closure* kl *is defined as follows:*
Let W *be a subspace of minimal dimension of* V *such that for some* $v_0 \in X$, $X \subseteq v_0 + W = \{v_0 + w | w \in W\}$*; then* $kl(X) = v_0 + W$ *for this* W *and* v_0*. Let* $cl(\emptyset) = \emptyset$ *(as in Example 4.2 d)).*

Locally projective *if the localization* $\widehat{G_{\{a\}}}$ *is projective for all* $a \in \Gamma$ *(that is for every* $a \in \Gamma$, $\widehat{G_{\{a\}}}$ *is isomorphic to the associated geometry of a vector space.*

Homogeneous *if for all* $y_1, y_2 \in \Gamma - cl(X)$ *there exists an automorphism* φ *of* G *such that* $\varphi(y_1) = y_2$ *and* $\varphi(y_2) = y_1$ *and for all* $x \in X$ *we have that* $\varphi(x) = x$.

All examples in 4.2 are homogeneous. This can be proved by using basic results about linear independence and linear mappings in vector spaces.

Example 4.10 Choose $G = \langle \mathbb{R}^3, cl \rangle$ where $cl(X)$ is the smallest subspace containing $X \subseteq \mathbb{R}^3$, as in Example 4.2 c). Then if $cl(X)$ is a plane, choosing any two points y_1, y_2 outside that plane will induce an automorphism. The automorphism can be constructed as a linear mapping by choosing the plane $cl(X)$ as an eigenspace for the map, and then by rotation, and possibly enlargement/shrinking, mapping y_1 and y_2 to each other. This may be done in a similar way if $cl(X)$ is a line, and trivially if $cl(X) = \{0\}$ or $cl(X) = \mathbb{R}^3$. Hence the geometry G is homogeneous.

Definition 4.11 The definition of a geometry also gives us a natural way of talking about span, basis and dimension (like in a vector space). A set X is **spanned** by elements $a_1, ..., a_n$ if $cl(a_1, ..., a_n) = X$. It is **independent** if $\forall a \in X(a \notin cl(X - \{a\}))$. An element a is called **independent of** (we may sometimes also say 'in') X if $a \notin cl(X - \{a\})$. A set S (or single element) is said to be **independent over** X if every element $\alpha \in S$ is independent of $X \cup S$. An element a is called **dependent** of X if $a \in cl(X - \{a\})$. A **basis** is a spanning independent set. The size (that is, cardinality) of a basis is called the **dimension**. Sets of dimension 1 are called points, of dimension 2 lines, of dimension 3 planes etc.

Example 4.12 Again take the pregeometry $G = \langle \Gamma, cl \rangle$ over the vector space $\Gamma = \mathbb{R}^2$, where $cl(X)$ is the smallest subspace containing X. Let $X = \{(1,1), (-1,-1), (5,5)\}$. Then $(1,0) \notin cl(X)$ and hence $(1,0)$ is independent of X. Then X is not an independent set, since whatever point we choose to remove, the other elements will still span the same space. The set $Y = \{(5,5)\}$ is independent, since $(5,5) \notin cl(\emptyset)$.

We will now start to list and prove some propositions regarding geometries which will be needed later in the more advanced chapters.

Proposition 4.13 *Let $G = \langle \Gamma, cl \rangle$ be a pregeometry and $X \subseteq \Gamma$, $a \in X$. If a is dependent of X then $cl(X) = cl(X - \{a\})$.*

Proof. That a is dependent in X means that $a \in cl(X - \{a\})$. Having this in mind and using the reflexivity property we get $(X - \{a\}) \cup \{a\} \subseteq cl(X - \{a\})$. Now using the monotonicty property we get that $cl(X) \subseteq cl(X - \{a\})$.
The other direction follows directly from the monotonicity and reflexivity property without the need of dependency. By reflexivity, $X - \{a\} \subseteq X \subseteq cl(X)$,and now by monotonicity, $cl(X - \{a\}) \subseteq cl(X)$. □

The previous proposition will sometimes be used without extra notice.

Remark 4.14 The finite character property ($cl(X) = \bigcup\{cl(X_0) | X_0 \subseteq X \wedge |X_0| < \aleph_0\}$) will be very important in proofs using geometries. The property implies that in order to prove an attribute, it is often sufficient to prove that this attribute holds for any finite sets.

The following proposition will show that dependency works as we expect it to, that is, as in vector spaces.

Proposition 4.15 *Let $G = \langle \Gamma, cl \rangle$ be a pregeometry and let $X \subseteq \Gamma$. If there exists $c \in X - cl(\emptyset)$ such that c is dependent of $X - \{c\}$ ($c \in cl(X - \{c\})$) then there exists a different element $a \in X$ which is dependent of $X - \{a\}$.*

Proof. Assuming that X is finite (by Remark 4.14) we use induction to prove this.

Base: $X = \{c, x_0\}$. Assume that $c \in cl(x_0)$ and $c \notin cl(\emptyset)$. The exchange property of cl gives that $x_0 \in cl(c)$.

Induction Step: Suppose that $|X| = n$ say $X = \{c\} \cup Z$ where $Z = \{z_1, ..., z_{n-1}\}$ and the lemma holds for Z in place of X. If $c \in cl(Z)$ then we have the following scenario. Either $c \in cl(Z - \{z_1\})$ and hence by the induction hypothesis we are done (using the set $\{c\} \cup \{z_2, ..., z_{n-1}\}$ as X). Otherwise $c \notin cl(Z - \{z_1\})$ and hence $c \in cl(Z) - cl(Z - \{z_1\})$. By the exchange property we get that $z_1 \in cl(\{c\} \cup \{z_2, ..., z_{n-1}\})$ and hence we have found another dependent element. □

Being locally projective implies a couple of properties which will be used later.

Proposition 4.16 *For a pregeometry $G = \langle A, cl \rangle$ (or a geometry, since a geometry is also a pregeometry), $\hat{G}$ is locally projective iff for every $c \in A - cl(\emptyset)$:*

1. *if $x, y \in A$ and x, y, c are all independent, then $cl(x, y, c) \neq cl(x, c) \cup cl(y, c)$.*

2. *if $y \in A$, $Z \subseteq A$, $x \in cl(\{y, c\} \cup Z)$ then $\exists z \in cl(c, Z)$ such that $x \in cl(y, z, c)$.*

Some explanation for this is required, but no real proof will be given. The hard part is to show that the conjunction of 1 and 2 implies that G is locally projective. Details may be found in [3].

Explanation. First, remember that in the case of just creating the localization of $\hat{G}$ we will need to remake the pregeometry A by defining $kl(X) = cl(X \cup \{c\})$ as the localization of G and then identifying the same closures with eachother. Because of this we can see that the closure of a single point in the geometry is $cl(x, c)$. Hence property 1 say that points in the union of two lines through c is not equal to the span of these two lines; a property which seems pretty obvious when considering that the localization is to be isomorphic with the vector space geometry (Example 4.2 c)).

The second property, in view of our explanation of property 1, states that in the localized geometry around c of the geometry $\hat{G}$, if x is in the span of y and some set Z, then Z on its own can span a single point z which is the only thing needed together with y, in order to get x. Or in more casual words, instead of finding x inside of a (possibly) huge dimensional space spanned by y and Z, we may take something which Z spans, and together with it and y, create a line, which is the only thing needed in order to find x. This property is also obviously true in the vector space geometry.□

Proposition 4.17 *If $G = \langle \Gamma, cl \rangle$ is a pregeometry, $X, Y \subseteq \Gamma$ then $cl(X) \cap cl(Y) = cl(cl(X) \cap cl(Y))$. Or in other words, the intersection between two spans is still a span (hence closed).*

Proof. Let $Z = cl(X) \cap cl(Y)$. Now since $Z \subseteq cl(X)$ we get from the monotonicity property that $cl(Z) \subseteq cl(X)$.

Same thing other way, we get $cl(Z) \subseteq cl(Y)$ from the monotonicity property since $Z \subseteq cl(Y)$. Hence $cl(Z) \subseteq cl(Y) \cap cl(X)$.
The other direction: By the reflexivity property we get that $cl(X) \cap cl(Y) \subseteq cl(cl(X) \cap cl(Y))$. By definition we have that $cl(cl(X) \cap cl(Y)) = cl(Z)$. Put these two together and we get $cl(X) \cap cl(Y) \subseteq cl(Z)$. $\square$

5 Algebraic closure

The algebraic closure is, in model theory, a very important tool and it will play an important part in the later sections of this thesis.

Definition 5.1 *Let $X \subseteq M$. The* ***algebraic closure*** *of X in $\mathfrak{M}$, denoted $acl_{\mathfrak{M}}(X)$, is the set of all $q \in M$ such that for some formula $\varphi(x, y_1, ..., y_k)$ (buildt from $\Omega_{\mathfrak{M}}$) and some $b_1, ...b_k \in X$, $q \in \varphi(\mathfrak{M}, b_1, ..., b_k)$ and $|\varphi(\mathfrak{M}, b_1, ..., b_k)| < \aleph_0$. We call $acl_{\mathfrak{M}}$ the algebraic closure operator in $\mathfrak{M}$. When $\mathfrak{M}$ is clear from the context we may omit the subscript $\mathfrak{M}$ and write $acl(X)$ instead.*

Put into words, the algebraic closure of a set X contains all elements a such that there exists a formula with parameters in X, which is only satisfied by a finite number of elements and a (that is less than n elements for some $n \in \mathbb{N}$).

Examples 5.2 In the first, third and fourth examples, below, $\mathfrak{M}$ has elimination of quantifiers, which is necessary to reason the way we do there. To see that they have quantifier elimination one can use Lemma 2.7.3 in [5] (Hodges).

- Let $\mathfrak{M} = \langle M, m \rangle$ be the structure that is only consisting of the infinite universe M and the interpretation m of a constant symbol. Then, for $X \subseteq M$, $acl(X) = X \cup \{m\}$ since M is finite and the only atomic formula, with one variable and parameters from X we can create is $x = a$, $a \in X \cup \{m\}$. In particular $acl(\emptyset) = \{m\}$.

- Let $\mathfrak{M} = \langle \mathbb{N}, < \rangle$. Here $acl(\emptyset) = \mathbb{N}$ since the formula $\varphi_0(x) \equiv \neg\exists y(y < x)$ is only satisfied by the element zero. Using the formula for zero we can create the formula $\varphi_1(x) \equiv \forall y(y < x \leftrightarrow \varphi_0(y))$ which is only satisfied by the element one. In this way, we may inductively define the formulas
$$\varphi_{n+1}(x) \equiv \forall y(y < x \leftrightarrow \bigvee_{i=0}^{n} \varphi_i(y))$$

which are all defined without parameters, and only satisfied by finitely number of elements. Hence $acl(\emptyset) = \mathbb{N}$.

- Let $\mathfrak{M} = \langle \mathbb{Q}, < \rangle$. This set is not as easy as $\mathbb{N}$ since the set is dense with respect to '$<$'. Any formula speaking about some interval will speak about an infinite amount of numbers, since we have no more information to use than the $<$ relation (and '$=$' as usual). Hence $acl(X) = X$, for all $X \subseteq \mathbb{Q}$.

- Let $\mathfrak{M} = \langle G, I \rangle$ where G is a graph which consists of infinite many disjoint copies of the complete graph with exactly 3 vertices, denoted K_3, such that nodes in different copies are never connected. Let I be the relation $I(a, b) \Leftrightarrow$ "a is a neighbour to b". Then $acl(\emptyset) = \emptyset$ since we are not able to see the difference between the copies of K_3. But $acl(X)$ is the set of all nodes which lie in the same complete 3 graph as any node in X, since these can be found with the formula $x_0 I y$ where $x_0 \in X$.

In these examples (except the second one) we only use structures with at most a single relation symbol (besides '$=$') and elimination of quantifiers. This is of course not necessary in order to take the algebraic closure, but if you have more relations and maybe some function symbols in the signature, we can create more complex formulas, so it may be very hard to find out which elements are and which are not in the algebraic closure.

Notice how the algebraic closure in Examples 5.2 have some features in common with the closure operator of a pregeometry. We will start proving some lemmas which show that most of the properties for pregeometries hold for the *acl* operator on any structure $\mathfrak{M}$. In the following lemmas assume that $X \subset M$ for some structure $\mathfrak{M}$.

Lemma 5.3 $X \subseteq acl(X)$.

Proof. For all $a \in X$ we may choose the formula $a = x$. This formula is only finitely satisfied (only satisfied by a). □

Lemma 5.4 $acl(X) = \bigcup \{acl(X_0) | X_0 \subseteq X \wedge |X_0| < \aleph_0\}$.

Proof. For each $a \in acl(X)$ there is a formula φ which is only finitely satisfied using some parametrization from X. But this formula is finite (since all formulas are), and hence it may only use a finite $X_0 \subseteq X$ to parametrize the formula used to pick out a. Hence we have proved that

$acl(X) \subseteq \bigcup\{acl(X_0) | X_0 \subseteq X \wedge |X_0| < \infty\}$. The other direction follows from Lemma 5.5 a), which does not use this result in the proof. For every finite subset X_0 of X we get that $acl(X_0) \subseteq acl(X)$, hence $\bigcup acl(X_0) \subseteq acl(X)$. $\square$

Notice that the formula $\mathfrak{M} \models \varphi(a, X) \wedge \exists^{\leq n} x \varphi(x, X)$ is the same thing as saying $a \in acl_{\mathfrak{M}}(X)$, under the restriction that we use the same parameters in when parametrising with X. In the upcoming proof of Lemma 5.5 we will assume that the formulas $\mathfrak{M} \models \varphi(a, X)$ and $\exists^{\leq n} x \varphi(x, X)$ use the same parameters from X.

This next lemma is needed in order to prove the monotonicity property.

Lemma 5.5

a) $Y \subseteq X \Rightarrow acl(Y) \subseteq acl(X)$.

b) $acl(acl(X)) = acl(X)$.

Proof. a) For each $a \in acl(Y)$ we have a formula φ with parameters $\vec{y} \in Y^m$ such that

$$\mathfrak{M} \models \varphi(\vec{y}, q) \wedge \exists^{=n} z \varphi(\vec{y}, z). \qquad (*)$$

But since $Y \subseteq X$ we have that $\vec{y} \in X$ too, so $(*)$ is a parametrization from X too and hence $a \in acl(X)$, which leads to $acl(Y) \subseteq acl(X)$.

b) We see that $acl(acl(X)) \supseteq acl(X)$ is satisfied by lemma 5.3.
Now for the other direction. For all $a \in acl(acl(X))$ there is a formula $\varphi(y, x_1, ..., x_m)$ such that $\varphi(a, acl(X))$ is finitely satisfied, which means that we may choose $b_1, ..., b_m \in acl(X)$ such that

$$\mathfrak{M} \models \varphi(a, b_1, ..., b_m) \wedge \exists^{=n} y \varphi(y, b_1, ..., b_m). \qquad (**)$$

But since $b_i \in acl(X)$ for $i = 1, ..., m$ we can also for $b_1, ..., b_m$ find formulas $\psi_1, ..., \psi_m$ and $n' \in \mathbb{N}$ such that

$$\mathfrak{M} \models \psi_1(b_1, X) \wedge \exists^{\leq n'} y \psi_1(y, X), ..., \mathfrak{M} \models \psi_m(b_m, X) \wedge \exists^{\leq n'} y \psi_m(y, X).$$

and hence

$$\mathfrak{M} \models \psi_1(b_1, X) \wedge \exists^{\leq n'} y \psi_1(y, X) \wedge ... \wedge \psi_m(b_m, X) \wedge \exists^{\leq n'} y \psi_m(y, X).$$

Put this together with our formula $(**)$ and we get:

$$\mathfrak{M} \models \bigwedge_{k=1}^{m} \big(\psi_k(b_k, X) \wedge \exists^{\leq n'} y \psi_k(y, X)\big) \wedge \varphi(a, b_1, ..., b_m) \wedge \exists^{=n} y \varphi(y, b_1, ..., b_m).$$

By existence introduction we get:

$$\mathfrak{M} \models \exists x_1, ..., x_m \big(\bigwedge_{k=1}^{m} \psi_k(x_k, X) \wedge \exists^{\leq n'} y \psi_k(y, X)$$

$$\wedge \varphi(a, x_1, ..., x_m) \wedge \exists^{=n} y \varphi(y, x_1, ..., x_m) \big).$$

Since for each finite tuple $(x_1, ..., x_m)$ there are at most n a's satisfying this big formula, this whole formula is only finitely satisfied. But this formula only contain parameters in X and is satisfied by a and hence $a \in acl(X)$, so $acl(acl(X)) \subseteq acl(X)$. $\square$

Lemma 5.6 $Y \subseteq acl(X) \Rightarrow acl(Y) \subseteq acl(X)$.

Proof. $Y \subseteq acl(X) \Rightarrow acl(Y) \subseteq acl(acl(X))$ by lemma 5.5 a).
But $acl(acl(X)) = acl(X)$ by lemma 5.5 b). Hence $acl(Y) \subseteq acl(X)$. $\square$

Notice how this proof is done only from the geometric properties proved in Lemma 5.5, and not anything extra. We now state the proposition which we were looking for, but now with the extra minimality demand, which is not necessary to get the previously proved properties.

Proposition 5.7 *Let $\mathfrak{M}$ be a structure, and $A \subseteq M$ strongly minimal. Then A together with the algebraic closure operation, restricted to A, forms a homogeneous pregeometry $G = \langle A, acl_A \rangle$, where $acl_A(X) = acl_{\mathfrak{M}}(X) \cap A$ for any $X \subseteq A$.*

Remark 5.8 When we talk about the geometry of a strongly minimal set A, we mean the geometry $\hat{A}$ formed as in Remark 4.5. In the case where A already is a geometry $\hat{A} = A$, so there is no need to make any exceptions. If $A \subseteq M$ for some strucure $\mathfrak{M}$ and we say $acl(X)$ for some $X \subseteq A$ we mean the algebraic closure of X which is inside of A, that is $acl_A(X)$, and not in the whole structure $\mathfrak{M}$. It would be possible to write $acl_A(X) = acl(X) \cap A$ every time to be very exact, but since this will be the case every time when we have a strongly minimal set A we use the shorter notation $acl(X)$. We will never speak of a geometry using acl on anything else but on strongly minimal sets.

Remark 5.9 As will be shown later, we will only need minimality of A in order to get a pregeometry, using the acl operator. The assumption of strong minimality is made in order to derive that the pregeometry has the homogenity property.

To prove Proposition 5.7 we will need to show that this G satisfies all the five properties in Definition 4.1 except property 4, since we only claim it is a pregeometry. We have already proved the monotonicity, finite character and reflexivity properties. The proof of the homogenity property will be left out, since the proof demands using the notion of types, Zorn's lemma and other things we do not go into too deep here. A proof of the homogenity property may be found in Rothmaler's book [10].
Hence we have the exchange property left to prove. In the following lemma we will assume that $\mathfrak{M}$ is a structure, $A \subseteq M$ is strongly minimal and $X \subseteq A$.

Lemma 5.10 $\forall a, b \in A,\ a \in acl(X \cup \{b\}) - acl(X) \Rightarrow b \in acl(X \cup \{a\})$.

Proof. Since $a \in acl(X \cup \{b\})$ we have a formula $\varphi(x, y, X)$ such that $\mathfrak{M} \models \varphi(a, b, X) \wedge \exists^{\leq m} x \varphi(x, b, X)$ for some $m \in \mathbb{N}$. Let $\psi(y, X)$ denote the formula $\exists^{\leq m} x \varphi(x, y, X)$.

If this formula would be satisfied by a finite amount of elements, then (since $\psi(y, X)$ is a formula only using parameters from X) $b \in acl(X)$ and hence $acl(X \cup \{b\}) = acl(X)$ by Lemma 5.5 b). But then $a \in acl(X) - acl(X) = \emptyset$. Hence this formula must be satisfied by infinite amount of elements, call this set $R = \psi(\mathfrak{M}, X)$.

Let $T = \varphi(a, \mathfrak{M}, X)$. If $|T| < \aleph_0$ then (because $\mathfrak{M} \models \varphi(a, b, X)$ and $\varphi(a, y, X)$ is a formula with only parameters in $X \cup \{a\}$) $b \in acl(X \cup \{a\})$ and hence we are done.

Instead assume that $|T| \geq \aleph_0$. Then since A is minimal, $|A - T| = n$ for some $n \in \mathbb{N}$. Denote the formula $\exists^{=n} y \neg \varphi(x, y, X)$ by $\chi(x, X)$, and let $E = \chi(\mathfrak{M}, X)$. Because $\mathfrak{M} \models \chi(a, X)$ we get that $|E| \geq \aleph_0$, since if it would not be infinite then (since $\chi(x, X)$ has parameters only from X) $a \in acl(X)$.

Choose distinct $a_0, ..., a_m \in E$. Note that by the choice of a_i the set $\varphi(a_i, \mathfrak{M}, X)$ will be infinite and contain all but n elements from A. Hence the set $\bigcap_{0 \leq i \leq m} \varphi(a_i, \mathfrak{M}, X)$ containts all but a finite number of elements from A. Since R is infinite, we get that the set $R \cap \bigcap_{0 \leq i \leq m} \varphi(a_i, \mathfrak{M}, X)$ is also infinite. Pick any $c \in R \cap \bigcap_{0 \leq i \leq m} \varphi(a_i, \mathfrak{M}, X)$. From the property of c we get that $\mathfrak{M} \models \bigwedge_{0 \leq i \leq m} \varphi(a_i, c, X)$. But this contradicts that $c \in R$ because this implies that $\exists^{\leq m} x \varphi(x, c, X)$. Hence $|T| < \aleph_0$ and we are done. □

All properties mentioned in Proposition 5.7, have now been proved except the homogeneity which, as said before, we will not prove here.

Here is an example showing that without the minimality, we may not get the exchange property and hence not a pregeometry.

Example 5.11 Let $\mathfrak{M} = \langle M, R, P \rangle$ where R is a binary relation and P is a unary relation. Define these relations on $\mathfrak{M}$ as follows:

1. P and $M - P$ are both infinite (and hence $\mathfrak{M}$ is not minimal).

2. $\forall a, b \in M,\ R(a, b) \rightarrow a \in P$ and $b \in M - P$.

3. $\forall a \in P,$ there is an infinite amount of elements b such that $\mathfrak{M} \models R(a, b)$.

4. $\forall b \in M - P,\ \exists^{=1} a\ R(a, b)$.

Using Lemma 2.7.3 in [5] Hodges, we see that $\mathfrak{M}$ has quantifier elimination. Since all $a \in P$ satisfy the same quantifier free formulas, without parameters, and all $b \in M - P$ satisfy the same quantifier free formulas we get that $acl(\emptyset) = \emptyset$. Let $b \in M - P$. By property 4 of $\mathfrak{M}$ we get that there exists exactly one $a \in M$ such that $\mathfrak{M} \models R(a, b)$, which gives us that $a \in acl(b) = acl(b) - acl(\emptyset)$. If $\mathfrak{M}$ would have the exchange property, we would get that $b \in acl(a)$. But from property 3 of $\mathfrak{M}$ and quantifier elimination we get that $b \notin acl(a)$ and hence the exchange property is not satisfied by $\mathfrak{M}$

The next proposition shows what we talked about in the introduction, that the structure of a minimal structure does depend on how it's geometry behaves. It is also necessary for further results.

Proposition 5.12 *Let $\mathfrak{M}$ be a strongly minimal structure. Assume that $A = \{a_1, a_2, a_3\}, B = \{b_1, b_2, b_3\} \subseteq M$ are both independent sets, and that there is $c \in acl(a_1, a_2, a_3)$ such that:*
$c \notin acl(a_i, a_j)\ \ \forall i, j \in \{1, 2, 3\}$.
Then there is $d \in acl(b_1, b_2, b_3)$ such that:
$d \notin acl(b_i, b_j)$ *for all* $i, j \in \{1, 2, 3\}$.

In order to prove this proposition we will need a lemma which we will not prove here but may be found in [10].

Lemma 5.13 *If $\{a_1, ..., a_n\}$ is independent and $\{b_1, ..., b_n\}$ is independent, then for every formula $\varphi(x_1, ..., x_n)$ we have that*

$$\mathfrak{M} \models \varphi(a_1, ..., a_n) \Leftrightarrow \mathfrak{M} \models \varphi(n_1, ..., b_n)$$

Proof Proposition 5.12 Since $c \in acl(a_1, a_2, a_3)$ we have that there is a formula $\varphi(x_0, x_1, x_2, x_3)$ and $n \in \mathbb{N}$ such that

$$\mathfrak{M} \models \varphi(c, a_1, a_2, a_3) \wedge \exists^{\leq n} x \varphi(x, a_1, a_2, a_3). \qquad (*)$$

From the other property on c we have that for any formula $\psi(x, y_1, y_2)$ we have that

$$\mathfrak{M} \models \psi(c, a_i, a_j) \wedge \neg\exists^{\leq n} x \psi(x, a_i, a_j) \qquad (**)$$

for any $i, j \in \{1, 2, 3\}$ and $n \in \mathbb{N}$. Hence by Lemma 5.13 and $(*)$ we have that:

$$\mathfrak{M} \models \exists^{\leq n} x \varphi(x, b_1, b_2, b_3).$$

Choose one of those x satisfying that formula, call it d, and choose it such that $d \notin acl(b_i, b_j)$ where $i, j \in \{1, 2, 3\}$. This can be done because of $(**)$ and Lemma 5.13. $\square$

6 Incidence systems

Incidence systems have an important role in restricting what possibilities the pregeometries of strongly minimal structures have. This is shown in Theorem 7.1 which in turn is used in the Classification theorem 8.1.

Definition 6.1 *An incidence system is a triple $A = \langle P, L, I \rangle$ where P is a set of elements called points, L is a set of elements called lines and $I \subseteq P \times L$ is a so-called incidence relation.*

Since there are not any axioms for the incidence systems, any structure with 2 sets and a relation will do as an example.

Examples 6.2 a) Let the incidence system be $\mathbb{R}^3$ and P the points, L the lines and I the relation "$a \in P$ lies on the line $b \in L$".

b) A more absurd example: X is any set, $P = X, L = \wp(X)$ (powerset of X) and I is the relation $aIB \Leftrightarrow a \in B$.

Any incidence system may be interpreted as a first-order structure by using the set $P \cup L$ as the universe. There is nothing which stop this structure from having a nonempty cut between L and P, that is a line could also be a point. The signature of this structure has a binary relation which is noted by I. Seeing this, we can concider an insidence system like any other structure and so we will use the same conventions that we have already created for structures, for incidence systems too.

Definition 6.3 *An incidence system $A = \langle P, L, I \rangle$ satisfying the following properties is called a* ***pseudo plane:***

1. *$\forall p \in P$ we have that $|I(p, A)| \geq \aleph_0$.*

2. *$\forall l \in L$ we have that $|I(A, l)| \geq \aleph_0$.*

3. *$\forall p_1, p_2 \in P$ with $p_1 \neq p_2$ we have that $|I(p_1, A) \cap I(p_2, A)| < \aleph_0$.*

4. *$\forall l_1, l_2 \in L$ with $l_1 \neq l_2$ we have that $|I(A, l_1) \cap I(A, l_2)| < \aleph_0$.*

Example 6.4 The Euclidean geometry forms a pseudoplane. Consider $\mathbb{R}^2$, with the set of points as P, set of lines as L and pIl if a point p is on a line l. Any point is on an infinite amount of lines, and every line consists of infinite amount of points. Any two not equal points do only have one line in common (hence finite). Any two lines only cross at one single point (hence finite). So we have verified that it is a pseudoplane.

7 The Trichotomy Theorem

This section will present a very important result which classifies structures into structures with pseudoplanes or structures with special kinds of geometries. The proof for this theorem will only be partially done since it has some fairly advanced parts (which are presented in [12]).

Theorem 7.1 *Let $\mathfrak{M}$ be a totaly categorical structure. Then one and only one of the following conditions hold:*

1. *There is a pseudoplane definable in $\mathfrak{M}$.*

2. *The geometry of any strongly minimal structure $\mathfrak{U}$ definable in $\mathfrak{M}$ is locally projective.*

3. *The geometry of any strongly minimal structure $\mathfrak{U}$ definable in $\mathfrak{M}$ is degenerate.*

Remark 7.2 A straigthforward consequence of definability is that if a pseudoplane is definable in a structure $\mathfrak{U}$ and $\mathfrak{U}$ is definable in $\mathfrak{M}$, then there is a pseudoplane definable in $\mathfrak{M}$ too (transitivity of definability you could call it). Hence, if we in the trichotonomy theorem 7.1 are in case 2 or 3, we won't be able to define a pseudoplane in any of those structures $\mathfrak{U}$.

The following proposition 7.3, will together with 7.8 and 7.9 prove the Trichotomy theorem. This section will mostly be devoted to proving Proposition 7.3. This is because the proof deals almost only with pregeometries and does not require advanced model theoretic notions, such as Morley rank and degree of types.

Until the end of this chapter let $\mathfrak{U}$ be a strongly minimal structure definable in $\mathfrak{M}$ and assume that no pseudoplane is definable in $\mathfrak{M}$ and hence not in $\mathfrak{U}$. Let A be the universe of $\mathfrak{U}$.

Proposition 7.3 *The geometry $\hat{A}$ is locally projective or degenerate.*

Notice first how we do not put any emphasis on separating the two statements locally projective and degenerate as in Theorem 7.1. This is because a geometry can not be both degenerate and locally projective.
In order to prove this proposition we will need a lemma. The proof of it uses the notions of Morley rank and degree and results about these notions (see Zilber [12] Lemma 2.2.4)

Lemma 7.4 *For any five elements $a_1, a_2, b_1, b_2, c \in A$ there exists four of them which are algebraically dependent, $c \in acl(a_1, a_2, b_1, b_1)$ or*

$$acl(a_1, a_2, c) \cap acl(b_1, b_2, c) \supsetneq acl(c).$$

We will at one point add constants from a set Z_0 to the signature and consider the localization $acl_{Z_0}(X) = acl(X \cup Z_0)$ in order to use this lemma. This may be done without loss of generality, since if no pseudoplane is definable in $\mathfrak{U}$, then no pseudoplane is definable in the expansion of $\mathfrak{U}$ by constants from Z_0.

From now on assume that $c \in A - acl(\emptyset)$. In Proposition 4.16 the two properties needed to be shown in order to conclude local projectiveness of $\hat{A}$ are stated. The two properties are the following:

1. If $y \in A$, $Z \subseteq A$, $x \in acl(\{y, c\} \cup Z)$, then $\exists z \in acl(c, Z)$ such that $x \in acl(y, z, c)$.
2. If $x, y \in A$ and $\{x, y, c\}$ is an independent set, then $acl(x, y, c) \neq acl(x, c) \cup acl(y, c)$.

First we will show that property 1 is true for A; in fact it is true for every strongly minimal structure definable in $\mathfrak{M}$ (because we assume no pseudoplane is definable in $\mathfrak{M}$). The second property will after that be what makes a difference between a locally projective and a degenerate geometry. We will show that if the second property does not hold, then the geometry $\hat{A}$ is degenerate.
The following lemma gives us property 1.

Lemma 7.5 *If $y \in A$, $Z \subseteq A$, $x \in acl(\{y, c\} \cup Z)$, then $\exists z \in acl(c, Z)$ such that $x \in acl(y, z, c)$.*

Proof. Assuming that Z is finite (in accordance to Remark 4.14), we prove this property using induction over the size of Z.

Base: $Z = \{z_0\}$.

Because of the first property of acl we know that $z_0 \in acl(c, Z)$ and hence $x \in acl(\{y, c\} \cup Z) = acl(y, z_0, c)$ and we are done.

Induction Step: Let $Z = \{z_1, z_2\} \cup Z_0$, $x \in acl(\{y, c, z_1, z_2\} \cup Z_0)$. Also assume that the lemma olds for Z' in the place of Z if $|Z'| < |Z|$, as our induction assumption.

Claim We may assume the set $\{z_1, z_2, y, c\}$ to be independent over Z_0.

If it is not independent over Z_0 then there is an $\alpha \in \{z_1, z_2, y, c\}$ such that $\alpha \in acl((\{z_1, z_2, y, c\} - \{\alpha\}) \cup Z_0)$. Hence we may exclude one element and get the same closure. If z_1 or z_2 are depending on any of the others over Z_0, we are done by the induction hypothesis (since then $|Z - \{z_i\}| < |Z|$ and $acl(\{y, c\} \cup Z) = acl(\{y, c\} \cup (Z - \{z_i\}), i \in \{1, 2\})$. If y is not independent of $Z_0 \cup \{c, z_1, z_2\}$, then $x \in acl(\{y, c\} \cup Z) = acl(\{c\} \cup Z)$, and hence we may choose x as our z, since $x \in acl(y, x, c)$ by the reflexivity property of acl. If c is dependent of $Z_0 \cup \{y, z_1, z_2\}$ we have, by Proposition 4.15, another element $a \in Z_0 \cup \{y, z_1, z_2\}$ which is dependent. We have in proving this claim already concluded that any element $a \in \{y, z_1, z_2\}$ may be assumed to be indenpendent. Hence $a \in Z_0$, but then by Proposition 4.13

$$acl(\{y, c, z_1, z_2\} \cup Z_0) = acl(\{y, c, z_1, z_2 \cup (Z_0 - \{a\})\}) = acl(\{y, c\} \cup (Z - \{a\}).$$

Since $|Z - \{a\}| < |Z|$ we use the induction assumption and conclude that we are done. So we may assume that y, z_1, z_2, c are all independent over Z_0, hence we have proved the claim.

We add Z_0 to the signature in order to use Lemma 7.4. The first case of the lemma is that there exist four elements among x, y, z_1, z_2, c which are algebraically dependent over Z_0.

Claim We may assume the set $\{x, y, z_1, z_2\}$ to be dependent over Z_0.

To start off, x must be one of those elements since the other four were previously concluded independent. If one of z_1 or z_2 would not belong to the set of those four elements then we would be able to conclude $x \in acl(\{c, y, z_i\} \cup Z_0)$ and hence by the induction assumption we are done. Thus z_1 and z_2 belong to the subset with 4 dependent elements of $\{x, y, c, z_1, z_2\}$. If y would not be among those four elements, we would have $x \in acl(c, Z)$. Hence choosing $z = x$ we see that $x \in acl(y, x, c)$ and we would be done as well. So we have now proved the claim. Because of this:

$$x \in acl(\{y\} \cup Z).$$

Now $z_1 \in A - acl(\emptyset)$ and $|\{z_2\} \cup Z_0| < |Z|$ gives us by the induction hypothesis (using z_1 instead of c) that $\exists z \in acl(Z)$ such that

$$x \in acl(y, z, z_1).$$

Using monotonicity we get that $x \in acl(y, z, z_1) \subseteq acl(\{y, c\} \cup \{z_1, z\})$. Since $|\{z_1, z\}| < |Z|$ we use the induction hypothesis, under the condition of a basis of size more than 2, and get that $\exists z' \in acl(c, z, z_1)$ such that $x \in acl(y, z', c)$. This z' works as the element we are looking for since $z \in acl(Z)$ implies by monotonicity that $z' \in acl(c, z, z_1) \subseteq acl(c, Z)$. Therefore we would now be done with the first case if we had that the lemma works for a basis of size 2, which is what we now are going to prove. If $x \in acl(y, z, c)$ or $x \in acl(y, z_1, c)$ we are done, so assume that this is not the case.

Claim We may assume that $\{y, z_1, z, c\}$ is an independent set.

If $z \in acl(z_1, y, c)$ then $x \in acl(z_1, y, z) \subseteq acl(z, y, c, z_1) = acl(z_1, y, c)$ but this contradicts the assumption that $x \notin acl(z_1, y, c)$. Hence z must be independent of $\{z, y, c\}$. The exactly same argument works to show that z_1 is independent of the others. Assume $c \in acl(y, z_1, z)$, we have from before that $z \in acl(z_1, z_2)$, hence either $z \in acl(z_2)$ or $z \notin acl(z_2)$ in which case $z_2 \in acl(z, z_1)$. In either case we get that $c \in acl(y, z_1, z) \subseteq acl(y, z_1, z_2)$ but this contradicts the first claim we proved, hence c must be independent. If y wold be dependent of the others then there would exist some other element in $\{y, z_1, z, c\}$ which would be dependent of the other three elements, by Proposition 4.15. Hence we have proved the claim.
We may assume that

$$x \notin acl(U) \ \text{ where } U \subsetneq \{y, z_1, z\}. \qquad (+)$$

since $x \in acl(z_1, y) \subseteq acl(c, z_1, y)$ or $x \in acl(z, y) \subseteq acl(c, z_1, y)$ means we are done. If $x \in acl(z_1, z)$ then we may choose x as the z' we are suposed to find in the lemma.
Using (+), the previously proved claim, Proposition 5.12 and that $x \in$

$acl(y, z, z_1)$ we get that there exists $z_1' \in acl(z_1, z, c)$ such that $z_1' \notin acl(U)$ if $U \subsetneq \{z, z_1, c\}$. Also from the exchange property of $x \in acl(y, z, z_1)$ and (+) we get that $z_1 \in acl(x, y, z)$ and hence $acl(x, y, z) = acl(x, y, z, z_1) = acl(y, z, z_1)$.

Claim We may assume that any four out of $\{x, y, z_1', z_1, c\}$ are independent.

We get four cases that we need to check. The proof is done by assuming the opposite and showing either that it becomes trivial or that we have a contradiction.

- $\{x, y, z_1', c\}$ dependent. Then $x \in acl(y, z_1', c)$, since we may assume that $\{y, z_1', c\}$ is independent. Hence we are done.
- $\{x, y, z_1, c\}$ dependent. Then $x \in acl(y, z_1, c)$, since we may assume that $\{y, z_1, c\}$ is independent. Hence we are done.
- $\{x, y, z_1, z_1'\}$ dependent. Then $z_1' \in acl(x, y, z_1) = acl(y, z, z_1)$. We know that $z_1' \notin acl(z, z_1)$ and hence by exchange property $y \in acl(z_1', z, z_1) \subseteq acl(z_1', z, z_1, c) = acl(z_1, z, c)$ which contradicts the previous claim.
- $\{y, z, z_1', c\}$ dependent. Then $y \in acl(z_1', z_1, c) \subseteq acl(z, z_1, c, z_1') = acl(z, z_1, c)$ which contradicts the previous claim.

Hence we have shown that all cases are contradictions or trivial cases and so we have proved the claim.
Now using Lemma 7.4 again, we see that $acl(z_1', z_1, c) \cap acl(x, y, c) - acl(c)$ is not empty and hence we can find an element z' in there. Notice that $z' \in acl(\{c\} \cup Z)$ and since $z' \in acl(x, y, c) - acl(c)$ we have by the exchange property that $x \in acl(y, z', c)$. Altough in order to use the exchange property we need to know that $z' \notin acl(y, c)$. But if $z' \in acl(y, c) - acl(c)$ then $y \in acl(z', c) \subseteq acl(z_1', z_1, c)$ which contradicts the claim. Hence $z' \notin acl(y, c)$ Hence we have proved the lemma for $|Z| = 2$ and so we are done with the first case.

The second case we have is that

$$acl(\{c\} \cup Z) \cap acl(\{x, y, c\} \cup Z_0) \supsetneq acl(\{c\} \cup Z_0).$$

When taking the intersection between two spans, that is algebraically closed sets, we get a new span by Proposition 4.17. This together with the proper subset gives us that if we add an element from the left hand side which is not already in the right hand span, we will still get a span which is included in the left hand span. Hence $\exists z \in A - acl(\{c\} \cup Z_0)$ such that

$$acl(\{c\} \cup Z) \cap acl(\{x, y, c\} \cup Z_0) \supseteq acl(\{z, c\} \cup Z_0). \qquad (*)$$

Using the reflexivity property on the righthand side we see that z is in both of the lefthand side sets, hence we have this information:

$$z \in acl(\{c\} \cup Z) \qquad \text{and} \qquad z \in acl(\{x, y, c\} \cup Z_0). \qquad (**)$$

Claim $z \notin acl(\{y, c\} \cup Z_0)$.

Assume that $z \in acl(\{y, c\} \cup Z_0)$. Then we conclude that $z \in acl(\{y, c\} \cup Z_0) - acl(\{c\} \cup Z_0)$ and hence by the exchange property, $y \in acl(\{z, c\} \cup Z_0)$. By the first part of $(**)$, $acl(\{z, c\} \cup Z_0) \subseteq acl(\{c\} \cup Z)$ and hence $y \in acl(\{c\} \cup Z)$. This is impossible since we assumed (by the first claim) that y, z_1, z_2, c are independent over Z_0. Hence we can conclude that

$$z \notin acl(\{y, c\} \cup Z_0).$$

This claim together with the second result in $(**)$ gives us that:

$$z \in acl(\{x, y, c\} \cup Z_0) - acl(\{y, c\} \cup Z_0).$$

Which, by the exchange property of geometries (property 3) implies that

$$x \in acl(\{z, y, c\} \cup Z_0).$$

Now $|\{z\} \cup Z_0| < |Z|$ and so we can use the induction hypothesis to get that $\exists z' \in acl(\{z, c\} \cup Z_0) \subseteq acl(\{c\} \cup Z)$ such that $x \in acl(y, z', c)$. Hence we are done with the second case and so we are done $\square$.

We have now proved the first property which we needed to prove. In order to prove the second property we will use two lemmas.

Lemma 7.6 *If $x_0, x_1, x_2 \in A$, $x_0 \notin acl(x_1, x_2)$ and $acl(x_0, x_1) \cup acl(x_0, x_2) = acl(x_0, x_1, x_2)$ then $acl(x_1, x_2) = acl(x_1) \cup acl(x_2)$.*

Proof. We get that $acl(x_1, x_2) \supseteq acl(x_1) \cup acl(x_2)$ because of the reflexivity and monotonicity property of acl, $X \subseteq acl(X)$. Now for the other way, assume that $acl(x_1, x_2) - (acl(x_1) \cup acl(x_2))$ is not empty and we can pick $y \in acl(x_1, x_2) - (acl(x_1) \cup acl(x_2))$. Since $y \in acl(x_1, x_2) \subseteq acl(x_0, x_1, x_2)$ we get by assumption that $y \in acl(x_0, x_1) \cup acl(x_0, x_2)$. Hence either $y \in acl(x_0, x_1)$ or $y \in acl(x_0, x_2)$. We use the first case and conclude that the second case is symmetric to the first. By definition $y \in acl(x_0, x_1) - acl(x_1)$ and the exchange property implies that $x_0 \in acl(y, x_1)$. Now since $y \in acl(x_1, x_2)$ we get that (by the monotonicity property) $x_0 \in acl(y, x_1) \subseteq acl(x_1, x_2)$ which is impossible by our assumption on x_0. Hence the assumption that there exists such an y is false and so $acl(x_1, x_2) - (acl(x_1) \cup acl(x_2)) = \emptyset$ and so $acl(x_1, x_2) \subseteq acl(x_1) \cup acl(x_2)$. □

Lemma 7.7 *If for all $a_0, a_1, a_2 \in A$ such that $a_0 \notin acl(\emptyset)$ we have $acl(a_0, a_1) \cup acl(a_0, a_2) = acl(a_0, a_1, a_2)$ then for every nonempty $X \subseteq A$ and $x_0 \in X$, $acl(X) = \bigcup_{x \in X} acl(x_0, x)$.*

Proof. Let X be finite (by remark 4.14), so $X = \{x_0, ..., x_n\}$. Obviously $acl(x_0, ..., x_{n-1}) \supseteq acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1})$ because of the reflexivity property of acl. The other direction is proved by induction on the size of X.

Base: $|X| = 1$, $acl(x) = acl(x)$ where $\{x\} = X$.

Induction hypothesis: If $|X| = n-1$ then $acl(x_0, ..., x_{n-1}) \subseteq acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1})$

Induction Step: For any $y \in acl(x_0, ..., x_n)$ we use Lemma 7.5 to see that there exists $x \in acl(x_0, ..., x_{n-1})$ such that $y \in acl(x, x_n, x_0) = acl(x_0, x_n) \cup acl(x_0, x)$ with equality by our assumption for the lemma. But $x \in acl(x_0, ..., x_{n-1}) = acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1})$ by induction hypothesis also $x_0 \in acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1})$ because of the reflexivity property. Hence $acl(x_0, x) \subseteq acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1})$ because of the monotonicity property (choose $Y = \{x\} \cup \{x_0\}$) and so $y \in acl(x_0, x_n) \cup acl(x_0, x) \subseteq acl(x_0, x_1) \cup ... \cup acl(x_0, x_{n-1}) \cup acl(x_0, x_n)$, hence we have proved the induction step. □

Proof proposition 7.3. In order to get local projectivity we need to prove the following by 4.16:

1. if $y \in A$,$Z \subseteq A$, $x \in acl(\{y, c\} \cup Z)$ then $\exists z \in acl(c, Z)$ such that $x \in acl(y, z, c)$.
2. if $x, y \in A$ and x, y, c all independent, then $acl(x, y, c) \neq acl(x, c) \cup acl(y, c)$.

By Lemma 7.5, we always have property one if no pseudoplane is definable. If property two is fullfilled, the geometry is locally projective.
If property two is not fullfilled then then there are independent $x, y, c \in A$

such that $acl(x,y,c) = acl(x,c) \cup acl(y,c)$. By prop 5.12 this must be true for all independent x, y, c. Now Lemma 7.6 and 7.7 will together imply that for any independent $x_1, ..., x_n$

$$acl(x_0, ..., x_n) = acl(x_0, x_1) \cup ... \cup acl(x_0, x_n) = acl(x_0) \cup ... \cup acl(x_n)$$

Hence $\hat{A}$ is degenerate, which is equivalent with that A is degenerate. □

In order to complete the proof of Theorem 7.1 we need two more results which are not proved here.

Proposition 7.8 *Let $\mathfrak{M}$ be totaly cathegorical. If a pseudoplane is definable in $\mathfrak{M}$, then for every strongly minimal $\mathfrak{U}$ definable in $\mathfrak{M}$, its geometry is neither degenerate nor locally projective.*

This proposition is stated in slightly other words by Zilber [12] (Proposition 2.3.1).
Using yet another, picked out from the air (or rather proved by Zilber [12]) proposition we can at last prove the Trichotomy theorem 7.1.
The following proposition is stated as Proposition 3.4 in Zilber [12].

Proposition 7.9 *Let $\mathfrak{U}$ and $\mathfrak{U}'$ be strongly minimal structures with universes A and A', definable in an uncountably categorical structure $\mathfrak{M}$. Then the geometry $\widehat{A}$ is degenerate iff $\widehat{A}'$ is degenerate.*

Proof of Theorem 7.1 Let $\mathfrak{M}$ be a totaly categorical structure. If a pseudoplane is definable in $\mathfrak{M}$ then Lemma 7.8 says that no degenerate or locally projective geometry is definable. If there is not any pseudoplane definable in $\mathfrak{M}$ then Proposition 7.3 tells us that either the geometry on some strongly minimal $\mathfrak{N}$ definable in $\mathfrak{M}$ is locally projective or it is degenerate. Then Proposition 7.9 tells us that it is degenerate iff all other strongly minimal substructures geometries are degenerate. Hence either all are degenerate and if not then none are and hence they must all be locally projective by Proposition 7.3. □

8 Classification theorem

The following theorem is the main purpose of this thesis. Apart from classifying certain pregeometries it is important for the understanding of countably categorical ω-stable structures, as shown in [2].

Theorem 8.1 (Classification theorem) *A nondegenerate infinite, locally finite homogeneous geometry is isomorphic to an affine or projective geometry of infinite dimension over a finite field.*

We prove the theorem with the help of the following propositions and lemmas. The next proposition will not be proved here but is stated as Proposition 3.5.1 in [12].

Proposition 8.2 *Let $\mathfrak{M}$ be a totaly categorical structure. If a strongly minimal set whose geometry is not degenerate is definable in $\mathfrak{M}$, then in some enrichment $\mathfrak{M}'$ of $\mathfrak{M}$ by constants, there is a $\emptyset$-definable infinite vector space V over a finite field. On this vectorspace the associated geometry of the 'acl' operator will be projective.*

The following lemma is proved in [3].

Lemma 8.3 *Every locally projective infinite, locally finite homogeneous geometry is an affine or projective geometry over a finite field.*

Geometries are not actually real first order strucutres, since the *cl* operator acts on sets and not on elements. The following proposition shows how to create a first-order structure from a geometry. The construction is possible even without the structure being infinite, locally finite or homogenous, but then we will not be able to conclude that the resulting structure has so many nice properties.

Proposition 8.4 *Let $G = \langle A, cl \rangle$ be an infinite homogenous locally finite pregeometry. Consider the structure $\mathfrak{U} = \langle A, C^0, ..., C^n, ... \rangle$ where $C^n(x_0, x_1, ..., x_n)$ is the predicate meaning that $x_0 \in cl(\{x_1, ..., x_n\})$. Then $\mathfrak{U}$ is countably categorical and strongly minimal, and the operator cl in G coincides with the operator acl in $\mathfrak{U}$.*

Sketchy proof. That $\mathfrak{U}$ is strongly minimal follows from the assumptions on G. If $\varphi(x, \bar{a})$ is a formula $\bar{a} \in G$, then $acl(\bar{a})$ will be finite. If there exists $b \in A - acl(\bar{a})$ such that $\mathfrak{U} \models \varphi(b, \bar{a})$ then $|\varphi(\mathfrak{U}, \bar{b})| \geq \aleph_0$. This is because of the homogenity on G and that automorphies preserve truth of formulas. To see that $\mathfrak{U}$ is uncountably categorical is quite mutch harder, hence we refer to Rothmaler [10]. □

Proof of the Classification theorem. Let $G = \langle A, cl \rangle$ be our infinite locally finite homogenous geometry. From this G use Proposition 8.4 to create the new stucture $\mathfrak{M} = \langle A, C^0, ... \rangle$ which is countably categorical and strongly minimal (so totaly categorical by Proposition 3.4) and where the operators *acl* and *cl* coincide. Either the geometry defined on $\mathfrak{M}$ is degenerate, and since it coincides with G we get that G is degenerate. Else we use our $\mathfrak{M}$ and apply Proposition 8.2, to see that for some $X \subseteq A$ there is an infinite vector space V which is X-definable in $\mathfrak{M}$. Since the geometry of this vector space is projective, it will also be locally projective. The Trichotomy Theorem 7.1 gives us that any geometry definable in $\mathfrak{M}$ is locally projective. Hence, also the geometry using *acl* on $\mathfrak{M}$ is locally projective and since it coincides with G we get that G is locally projective. From Lemma 8.3 we get that either G is projective or G is affine, and hence we are done. □

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