Comment on ‘Monte Carlo simulation study of the two-stage percolation transition in enhanced binary trees’

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Abstract. The enhanced binary tree (EBT) is a nontransitive graph which has two percolation thresholds \( p_{c1} \) and \( p_{c2} \) with \( p_{c1} < p_{c2} \). Our Monte Carlo study implies that the second threshold \( p_{c2} \) is significantly lower than a recent claim by Nogawa and Hasegawa (J. Phys. A: Math. Theor. 42 (2009) 145001). This means that \( p_{c2} \) for the EBT does not obey the duality relation for the thresholds of dual graphs \( p_{c2} + \overline{p}_{c1} = 1 \) which is a property of a transitive, nonamenable, planar graph with one end. As in regular hyperbolic lattices, this relation instead becomes an inequality \( p_{c2} + \overline{p}_{c1} < 1 \). We also find that the critical behavior is well described by the scaling form previously found for regular hyperbolic lattices.

PACS numbers: 64.60.ah, 02.40.Ky, 05.70.Fh
Recently, Nogawa and Hasegawa [1] reported the two-stage percolation transition on a nonamenable graph which they called the enhanced binary tree (EBT). While the first transition had little ambiguity, they mentioned that the behavior at the second threshold did not look like a usual continuous phase transition.

A quantity of interest was the mass of the root cluster, denoted as $s_0$, where the root cluster was defined as the one including the root node of the EBT. Using this observable, we briefly check the first transition point, $p_{c_1}$, where an unbounded cluster begins to form. As in [2], we have used the Newman-Ziff algorithm [3, 4] and taken averages over $10^6$ samples throughout this work. The number of generations, $L$, of the EBT defines a typical length scale of the system, and [1] showed the finite-size scaling of $s_0$ as

$$s_0/L \propto \tilde{f}_1[(p-p_{c_1})L^{1/\nu}],$$

with $\nu = 1$. Figure 1(a) confirms both of the percolation threshold $p_{c_1}$ and the scaling form, equation (1). Equivalently, one can measure $b$, the number of boundary points connected to the root node, which becomes finite above $p_{c_1}$ as shown in figure 1(b). It also scales as

$$b \propto \tilde{f}_2[(p-p_{c_1})L^{1/\nu}],$$

with the same exponent $\nu$. Comparing this with [2], we see that the percolation transition in the EBT at $p = p_{c_1}$ belongs to the same universality class as that of regular hyperbolic lattices. One may argue that this scaling form actually corresponds to the case of Cayley trees [2]. The convincing results in figure 1 imply that the estimation in [1] for the dual of the EBT, $\overline{p}_{c_1} = 0.436$, is also correct.

On the other hand, the second percolation transition at $p = p_{c_2}$ indicates uniqueness of the unbounded cluster. We have thus employed a direct observable to detect this transition, i.e., the ratio between the first and second largest cluster masses [2]. The idea is that even the second largest cluster would become negligible if there can exist...
only one unique unbounded cluster. Measuring \( s_2/s_1 \) in the EBT, where \( s_i \) means the \( i \)th largest cluster mass, we have found the second transition at \( p_{c,2} \approx 0.48 \) (figure 2(a)), certainly lower than Nogawa and Hasegawa’s estimation, \( p = 0.564 \).

As an alternative quantity for \( p_{c,2} \), we divide \( b \) by the number of all the boundary points, \( B \). This fraction \( b/B \) is supposed to become finite above \( p_{c,2} \). Based on the Cayley tree result [2], we have assumed that as the system size \( N \) varies, one can write down the following asymptotic form:

\[
b/B \sim c_1 N^{\phi-1} + c_2,
\]

with some constants \( c_1 \) and \( c_2 \) and an exponent \( \phi \). From the finite-size data, we extrapolate the large-system limit by equation (3), which suggests \( p_{c,2} \approx 0.49 \) (figure 2(b)). This is very close to the estimation above from \( s_2/s_1 \). Moreover, in accordance with equation (3), we have suggested the following scaling hypothesis to describe the critical behavior at this transition point [2]:

\[
b/B \propto N^{\phi-1} f_3 [(p - p_{c,2}) N^{1/\bar{\nu}}],
\]

with an exponent \( \bar{\nu} \). Applying this hypothesis to EBT data, we see that \( \phi = 0.84 \) and \( 1/\bar{\nu} = 0.12 \) give a good fit (figure 2(c)) with the same value of \( p_{c,2} = 0.48 \), where the numeric values of the scaling exponents are again consistent with [2].

To make a direct comparison to the observation in [1], we have also calculated the mass fraction of the root cluster, \( s_0/N \), as a function of \( p \). As above, performing
extrapolation to the large-system limit, we see that this quantity becomes positive finite at \( \rho \gtrsim 0.49 \) (figure 3).

All of these observations suggest that the predicted value of \( \rho_{c2} \) in [1] is too high, and it seems that this overestimation led them to consider ‘discontinuity’ since \( s_0/N \) became already so large at that point as shown in figure 3.

Finally, even though our estimation suggests such a different \( \rho_{c2} \) that \( \rho_{c2} + \overline{\rho}_{c1} < 1 \), we note that it does not violate the duality relation proved in [5] for a transitive, nonamenable, planar graph with one end: As Nogawa and Hasegawa correctly pointed out [1], the EBT does not possess transitivity. The inequality \( \rho_{c2} + \overline{\rho}_{c1} < 1 \) was explicitly verified for a pair of hyperbolic dual lattices \( \{7,3\} \) and \( \{3,7\} \) in [2]. This inequality means the existence of a narrow region of \( \rho \) between \( \rho_{c2} \) and \( 1 - \overline{\rho}_{c1} \), where one would find a unique unbounded cluster in a given graph whereas infinitely many unbounded clusters in its dual graph. Such a region does not exist for a transitive case [5]. A typical state in this region is illustrated in figure 4, which shows a situation with many unbounded clusters of radii comparable to \( L \) at the same time as a single unbounded cluster occupies the dominant part of the dual graph.

Acknowledgments

SKB and PM acknowledge the support from the Swedish Research Council with the Grant No. 621-2002-4135. BJK was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) with Grant No. KRF-2007-313-C00282. This research was conducted using the resources of High Performance Computing Center North (HPC2N).

References

Figure 4. Visualization of a triangular hyperbolic lattice projected on the Poincaré disk, where the maximum length from the origin is chosen to be $L = 4$. Bonds are randomly occupied with probability of $p = 0.42$, which are colored red, while only the rest of them appear as occupied in the dual lattice, as colored green, so that the dual probability corresponds to $\overline{p} = 1 - p = 0.58$. Note that $p$ lies between $p_{c2}$ and $1 - p_{c1}$, since this structure has $p_{c2} \approx 0.37$ and its dual has $\overline{p}_{c1} \approx 0.53$, according to [2]. While most clusters have been already absorbed into the largest red cluster, many of green clusters still have radii comparable to $L$ since $p_{c1} < \overline{p} < \overline{p}_{c2} \approx 0.72$. 