

# **Learning mathematics by reading**

## **- a study of students interacting with a text**

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### **Abstract**

This study investigates the situation when students on their own read a new mathematical text, and solve problems relevant to the text. The students worked together in pairs on a given text, about the absolute value of real numbers, with a video camera recording their activity. First, the students were instructed to read and discuss the text without any given tasks. Thereafter, the students were given exercises relevant to the text, and they were allowed to keep the text and use it when working with these exercises. Two pairs of students participated, all of them on their last year on the natural science programme at the Swedish upper secondary school. The observations reveal a variety of different activities among the students, and some questions also arise that would be interesting to examine in more detail.

## Introduction

Reading can be said to appear in mainly two different forms within mathematics education, when comprehending word problems and when reading texts without direct focus on problem solving. And if education in mathematics mainly focuses on problem solving, the reading experience within mathematics tends to be limited to this particular situation. This focus on problem solving is present within research in mathematics education, where much research about *reading mathematics* has been with emphasis on strategies for comprehending word problems, which often causes reading to be treated as a potential obstacle that may interfere with learning (Borasi & Siegel 1990).

This paper will focus on the process of reading mathematical texts for the purpose of learning. Reading will not be limited to ‘extracting a message encoded by the author’, but will be seen as a more interactive process, and will be treated as an opportunity for learning, and not merely as a potential obstacle. This view is close to the so-called *transactional perspective*, which states that “reading always involves each reader actively in the transformation of the text read” and that “the written text is seen as a springboard for generating new meanings, rather than as a template against which a reader’s understanding is measured” (Borasi & Siegel 1990, pp. 9-10).

What in this paper will be labeled *mathematical texts*, are not all sorts of texts with a mathematical content, but perhaps *textbooks* might be a more appropriate word. So when discussing mathematical texts (or sometimes just texts) and textbooks, a text with a mathematical content which have been written for the purpose of being used in educational situations, will be referred to. This includes both texts that are to be used to learn from, and to teach from, although this paper will focus on the situation when learning from texts. Sierpiska (1997) labels these types of texts *didactic texts*, which consist of both a mathematical layer and a didactical layer, where the latter has the purpose of formatting “the behavior of the reader as a learner-user of the textbook and [...] the reader’s interpretations of the mathematical layer” (Sierpiska 1997, p. 5). Different texts can have different quantitative relations between these layers, and can therefore be labeled as strongly or weakly didactic texts.

The use of texts can in general be analyzed in three different ways; *a priori* – analysis of the text as such, with only an indirect connection to the reader, *a tempo* – study of the way students and teachers use texts when learning and teaching, and *a posteriori* – comparing learning results with the text, a sort of evaluation of textbooks. This paper will focus on analysis *a tempo*, but the text used in the experimental situation will also be analyzed *a priori*, although this will mainly serve as a tool for the analysis *a tempo*. More precisely, this study investigates the situation when students on their own read a new mathematical text, and solve problems relevant to the text.

## Reading mathematical texts

As previously mentioned, a mathematical text consists of two different layers, the mathematical and the didactical layer. Thereby, the purpose of the text can also be divided in two parts, one to present a mathematical content for the reader to become

acquainted with, and the other to teach how to relate to the text (or texts in general) and how to relate to the mathematical content. This also points to reasons why reading mathematics could be regarded as an important part of mathematics education. If one wishes for students to develop into more independent mathematical learners, reading can of course play an important role, where especially the combination of the two different types of layers in a text have the potential to guide the reader in this activity. But readers who are not aware of these different aspects of the text might have difficulties comprehending both the purpose and the content of the text. Cowen (1991, p. 50) argues for emphasizing reading within mathematics courses at the university level since “[v]ery few of our students will ever, after leaving our courses, have to give a formal proof of a theorem”, but as he continues, “many of them *will* have to read and understand mathematical writing to apply new ideas to the problems of their jobs.”

Reading can also be viewed as one of several competencies that make up what is meant by ‘mathematical knowledge’, with reading as a part of a communicational aspect of mathematics. KOM-arbejdsgruppen (2002) has characterized what is meant by mathematical knowledge, i.e. what is meant by knowing, understanding, performing and using mathematics. Their result consists of eight competencies, one of which is the competence of communication, including the ability to interpret and understand mathematical texts. These competencies are not isolated, but are closely related to each other. It is argued that all these competencies should be addressed within mathematical education, something that demands great variety in activities.

Textbooks can be used in many different ways depending on several factors, for example the nature of the text and the purpose of using the text. A survey among mathematics teachers in the Netherlands showed that textbooks were used in all sorts of ways, there were for example teachers who only used the textbook as a reference for themselves and others who let their students struggle with their text alone (van Dormolen 1986). But there seems to be little focus on the process of reading within mathematics education, and it is also argued by KOM-arbejdsgruppen (2002) that the use of textbooks in general need to change to make it possible to develop all different competencies.

In the transition from upper secondary level to university level, much change within mathematics education, also regarding textbooks, where not only the content and appearance change, but also how texts are being used. At the upper secondary level, textbooks more often seem to be used as a collection of exercises, while at the university level, students are to a larger extent expected to read mathematical texts on their own for learning. But when doing homework and preparing for exams, undergraduate students mostly use the examples in textbooks together with class notes, and material in the text other than examples is rarely used (Stephens & Sloan 1980). Is this simply because the students do not know how to read and use the text? Cowen (1991) argues that undergraduate mathematics education also need to focus on teaching students to read and understand mathematical texts, and that reading also must be part of exams.

There seem to be a widespread agreement among educational researchers that reading a mathematical text is a special kind of activity that needs to be learned (Brunner 1976, Hubbard 1990, Fenwick 2001). But there has also been criticism of this focus on the special language of mathematics for reducing reading to a set of skills to apply to a text so as to ‘extract the information’, which reduces reading within mathematics to an obstacle for learning from texts (Borasi & Siegel 1994). A discussion about these views

will not be carried out here, but a special focus on the language of mathematics when discussing reading does not automatically result in a view of reading as a potential obstacle for learning. Therefore, in this paper the special character of the language of mathematics will be discussed at the same time as reading is seen as an opportunity for learning, and not limited to a potential obstacle.

Several special properties of mathematical texts as compared to ‘ordinary texts’ can be noted (Brunner 1976, Hubbard 1990), for example:

- *Precision in statements*: Words and symbols often have a (defined) exact meaning, a meaning that sometimes is different than in other non-mathematical situations.
- *Concise language*: Compact writing, where much information can be gathered in short statements, which often makes it difficult to infer meaning of unknown words from the context.
- *Sequential structure*: New concepts build on (sequences of) earlier concepts.

One can discuss the necessity of these, and other, properties of mathematical texts, and Hubbard (1992, p. 81) points out

the need for the text to be absolutely mathematically correct and complete so that it cannot be criticized by mathematical colleagues. This requirement results in texts written for mathematicians, not students.

But in this paper, the purpose is not to elaborate different types of texts; instead a ‘typical’ text will be used when observing the process of reading.

If reading a mathematical text demands special abilities, one would need to specify these abilities and give suggestions on how to improve these skills, something that has also been done in many studies (see Turnau 1983 for many references). But these suggestions are often based on “personal experience or opinion, and never on research concerning effectiveness of the proposed methods” (Turnau 1983, p. 172). And perhaps there is not much to gain on this type of search for such *specific methods* for teaching the reading of mathematical texts, especially if one draws parallels to early attempts of finding specific ways of teaching problem solving, which was not that successful (Turnau 1983).

Instead a more general view of reading might be helpful, but where the special characters of mathematical texts are taken into account, for the purpose of learning more about reading within mathematics education. This might involve aspects like ‘levels of constructive activity’ when reading and learning from texts (Desforjes & Bristow 1994), or studies about how epistemological beliefs influence students’ activities when reading (Schommer et al. 1992).

## **Purpose and research questions**

This paper focuses on how students use mathematical texts to learn, in a situation when they are given a new text to discuss. One of the main purposes of this study is to act as guidance for future studies about learning mathematics by reading. Therefore, several different aspects of the observed situations will be analyzed, so as to find more specific

research questions. Some specific questions for this study are formulated in advance, but the possibility to observe and analyze other areas of interest, depending on contents of gathered data, may also be included.

Besides acting as a pilot study, the purpose of this study is to see examples of how students act when reading, discussing and using mathematical texts, together with examples of what kind of knowledge is relevant when trying to learn from a text. But the purpose is not to find specific skills needed to read mathematical texts, instead the situation will be observed in a wider sense, where the goal is to learn more about the process of learning mathematics by reading, and where the special properties of mathematical texts are taken into account. Two main questions will be focused on:

- How do students relate to and use texts in situations without specific tasks to solve?
- How do students use texts when tasks relevant to the text are to be solved?

## Method

To examine how students read and use texts, in this study students get to discuss a text and solve tasks relevant to the text together in pairs in the presence of a video camera without a present observer. The four students who volunteered for this, all came from the same mathematics class on their third (and last) year at the natural science programme at the Swedish upper secondary school, where they at this time were taking the mathematics D course<sup>1</sup>. All students passed with distinction (grade VG) or with special distinction (grade MVG) in all three previous mathematics courses A, B and C.

The study was carried out on location at the students' school, and consisted of three parts, with no given time limits:

*Part 1:* The students were instructed to read and discuss the text, and it was emphasized that they should discuss the whole text.

*Part 2:* The students were given tasks relevant to the text to solve. They were allowed to keep the text, and use it when working with these tasks.

*Part 3:* An interview was carried out with the students, with questions about their recent discussions of the given text and tasks. Their thoughts about mathematics in general and the work in their ordinary mathematics classroom were also discussed.

During their work, the students had access to pen and paper, and their notes were collected afterwards.

The text and tasks that the students worked with were about the absolute value of real numbers, and were created on the basis of texts used in introductory courses at the university level. This was done so that the students would get a 'typical' text used at this level, which dealt with something new to the students. The complete text and the tasks are to be found in Appendix A (in Swedish).

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<sup>1</sup> Mathematics at the Swedish upper secondary school consists of courses from A to E, where mathematics A to D are mandatory within the natural science programme, while mathematics E is optional (National agency for education 2001).

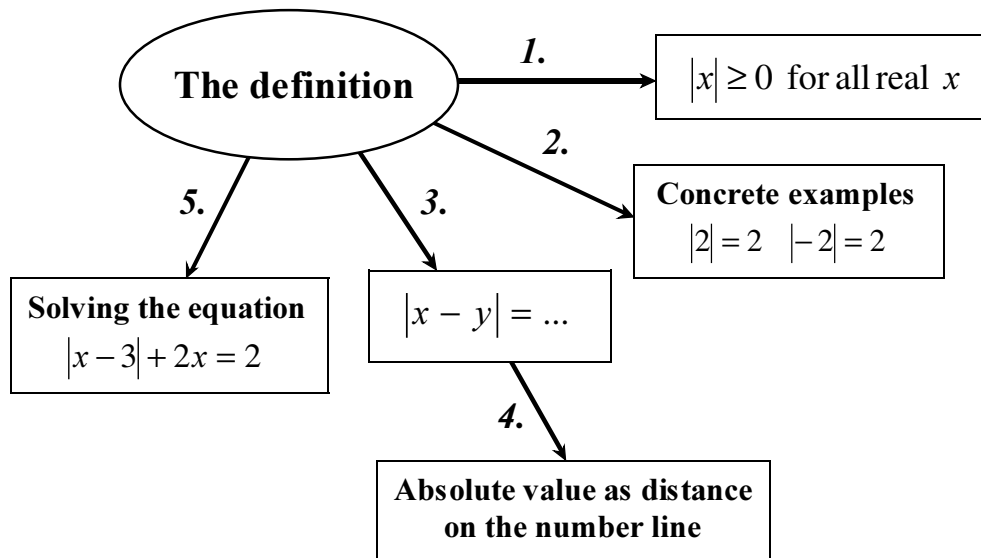


Figure 1. Content and structure of the text about the absolute value of real numbers, which was used in this study.

The content of the text with notions of how different parts of the text relate to each other is illustrated in Figure 1. This shows that everything in the text is connected to the definition, and you could thereby argue that all necessary information is collected within this definition. Thereby it can be concluded that the text also has a didactical layer, which guides the reader on how to interpret and use the mathematical layer.

The five numbered relations illustrated in Figure 1 are described in different ways in the text:

1. The definition implies that...
2. For example...
3. From the definition it also follows that...
4. On a number line this means that...
5. When examining expressions... break into different cases...

In addition to this logical structure of the text there is also a more detailed structure that requires certain understanding, for example an understanding of equalities and inequalities, and knowledge of how to treat these. There are also some mathematical concepts in the text that you need to be familiar with, such as real number, define/definition, number line, distance, and expression.

## Results and analysis

The first two parts of this study will first be analyzed separately for the two participating pairs of students, who hereafter will be referred to as Pair A and Pair B. Thereafter further analysis will be carried out through comparisons of the separate analyses, where

the comparison will be made both between the two pairs of students and between the two parts of the study, so as to find different aspects of the process of reading mathematical texts. The third part of the study will not be analyzed separately, but will be integrated and referred to in all parts of the analysis.

A few general circumstances that can have affected the outcome of the study in an unwanted direction should be mentioned. Firstly, one of the students in Pair A at a few occasions seemed somewhat uncomfortable with the situation when working in front of a video camera, and the other student in this pair acted quite dominantly. Both these issues sometimes seemed to inhibit the discussions, where the dominating student sometimes performed longer monologs. Secondly, the students in Pair B had prior to the study come across the absolute value of real numbers, where they had been informed what function the button 'abs' on the calculator has. But since the text used in this study discusses the absolute value in quite a different and more thorough way, the students in Pair B can still be regarded as reading about something that is new to them.

Since the purpose of this study is not to do any comparisons between the two pairs regarding what and how much the students learn when reading and discussing the text, these mentioned circumstances should not affect the study too negatively. On the contrary, it could be argued that these differences between the pairs could be advantageous since this might increase the possibility to observe different aspects of the activity of reading mathematical texts.

The students' activities in both parts of the study have been divided in smaller episodes, so as to get a better overall picture of the situation. These episodes will be described and used in the following analysis. To separate the individual students, they will be referred to as L and R (*left* and *right* according to their position on the video recording). When giving excerpts from the students' discussions, it should be noted that these are the author's translations from the original discussions in Swedish.

## **Part 1**

In the first part of the study the students were instructed to read and discuss the text, where it was emphasized that they should discuss the whole text.

### *Pair A*

The students in Pair A spent approximately 8 minutes on the first part of the study. Their activities are summarized in Table 1a, which shows that the students had trouble understanding the concept of absolute value during the entire part 1, since they in the last episode did not understand why  $|-4| = 4$ . A couple of times the direct question of what the 'lines' mean is brought up. In episode 2 the question is stated directly after the students have read the first page of the text, but at this time no attempt is made on finding an answer. In episode 6 the question is brought up again and the following dialogue takes place:

*L: I'm still wondering what this is. [Referring to the lines in the absolute value]*

*R: But isn't it like parentheses? No, it's not parentheses, but it's kind of...*

*L: No, because... [Interrupted by R]*

*R: It's like the  $x$  itself, if you have an  $x$ ...*

*L: Oh, yes... [Seems to be agreeing]*

*R: It's probably meant like that. At least I think so.*

From the last comment by student L, the students seem to agree on an interpretation of the absolute value, where  $|x|$  does not have a special meaning but instead is another way of writing  $x$ . This becomes more evident in episode 8, where the students believe that  $|-4|-2$  should equal  $-6$ . It could be noted that their interpretation of absolute value is not justified, neither with some part of the text or in any other way. In the third part of the study the students were asked to describe 'absolute value':

*R: Yes... [Giggling]*

*L: A lot of numbers...*

*R: Yes, exactly, no but... I'm not sure I've understood it correctly, but... I would describe it like you put...  $x$  in an equation, kind of different [cases]... and then you put a definition to it... I'm not really sure... [Giggling]*

Here the absolute value is described as a kind of proposed activity, where you could suspect influence from the example of solving an equation in the text (something that becomes more obvious in part 2). It also becomes obvious in other, more direct ways that the students focus on some parts of the text more than others, where some parts are not

Table 1a. Summary of student activity for Pair A in part 1 of the study

<b>Episode</b>	<b>Length (min:sec)</b>	<b>Description of student activity</b>
1	1:30	Reading first page. Giggling/laughter.
2	0:15	Questions. "What are these?" (referring to the lines in the absolute value) "What are real numbers?"
3	0:45	Student R comments the text from the definition to $ x - y $ , but not in detail. "This is logical" is a common comment.
4	0:45	Frustration? Sighing/laughter. Reading second page.
5	0:20	Similar to episode 3.
6	0:30	The first question from episode 2 is brought up again. Seem to agree on an interpretation of the notion of absolute value.
7	1:00	Reading the example of solving an equation, and student R commenting.
8	2:45	Discussing the last check of the root $x = -1$ . Trouble with understanding why $ -4  = 4$ .
<b>Total</b>	<b>7:50</b>	



mentioned at all by the students, despite that they have not fully understood these parts (remember that the students were encouraged to discuss all parts of the text). For example, the students mentioned in part 3 that they had not understood the part about absolute value as distance on the number line, but in part 1 they have not discussed this part of the text at all.

When the students do discuss the text, a typical example can be given from episode 3, when the parts of the text about the concrete examples and  $|x - y|$  is brought up:

**R:** *But this is quite logical. For example that if, it's the 2... or if x, that is that it should... That's quite logical. If it's negative... negative, negative...*

**L:** *Yes...*

**R:** *...so it becomes plus [positive]. That's kind of logical, we understand that. There too that... I understand that too, that x has to, if x should, if the answer should be greater than zero, then x has to be greater than y. Isn't it so?*

**L:** *Yes...*

**R:** *And the same here y must be greater than x if it should be greater than one, wait, less than one... [Says 'one' instead of 'zero']*

**L:** *Yes... [Modest voice]*

**R:** *It must... Yes, yes, exactly, it's quite logical.*

Here, student R focuses on small parts or fragments of the text, without considering whole statements. In the example  $|-2| = -(-2) = 2$ , only the second equality is considered, which is checked to be correct. In a similar manner, the part about  $|x - y|$  is reduced to the last statement that  $x - y \geq 0$  gives  $x \geq y$ , together with the reversed inequality. It seems like instead of discussing unfamiliar parts of the text, the text is reduced to familiar details that can be checked quite easily. This is also evident in episode 7 when the students discuss the example of solving an equation, where their discussion consists of checking the two potential solutions against the conditions  $x \geq 3$  and  $x \leq 3$  respectively. In part 3 the students cannot explain where these conditions come from, something that was not discussed in part 1.

### *Pair B*

The students in Pair B spent approximately 10 minutes on the first part of the study. Their activities are summarized in Table 1b, which shows that the students, after reading through the whole text, address all parts of the text, one by one. But there are differences in how they discuss the text in episodes 3 and 4 compared to when discussing the example in episode 5. In episode 4, the absolute value as distance on the number line is discussed:

**L:** *It means that this becomes positive all the time, doesn't it? Because...*

**R:** *Yes, it doesn't matter if it is, if the y is on the left side.*

**L:** *No.*

**R:** *It's the distance that, it's just only the distance.*

Table 1b. Summary of student activity for Pair B in part 1 of the study

<b>Episode</b>	<b>Length</b> (min:sec)	<b>Description of student activity</b>
1	3:05	Reading both pages of the text.
2	0:30	Recognizing absolute value from its function on the calculator.
3	0:40	Commenting the concrete example and the definition.
4	0:45	Discussion about $ x - y $ as distance between $x$ and $y$ .
5	3:00	Discussion about all parts of the example of solving an equation.
6	1:05	Reflections. "What is the use of this?" "The top of the iceberg!"
7	0:50	Compares $ x - y  = \dots$ with the example's $ x - 3  = \dots$
<b>Total</b>	<b>9:55</b>	

*L: Yes, exactly. Since you only calculate with... you do not bother about plus and minus... so then it is just like so.*

*R: Yes.*

Here, the text is referred to in quite general terms and no details are discussed. For example, they do not discuss how this part of the text follows from the definition, or in detail how  $|x - y|$  can represent the distance between  $x$  and  $y$ . Instead, their discussion starts with the *meaning* of the text, and does not deal with how or why the presented result is possible to obtain. In episode 3 the students comment the definition briefly, and also the text thereafter:

*L: But that is quite clear. [Pointing at the definition]*

*R: Yes it is.*

*L: And yes, that... means that... that, here, it just means that it is positive numbers. [Pointing at  $|x| \geq 0$ ]*

*R: Yes.*

*L: Strange way describing it.*

Whether the students have understood the definition in the same way is not discussed. And again, their following discussion starts with the meaning of the text and no details are discussed, for example about how this part relates to the definition.

But when the students in episode 5 discuss the example, this is done in a more detailed manner:

*R: Yes here you can place, if  $x$  is greater than three...*

*L: Yes.*

*R: ...you can use the top one,  $x$  minus three.*

*L: Yes.*

*R: But then it fails, because then you get  $x$  equals five [thirds]. So...*

*L: Yes, but it's just to put in then... yes... exactly. What did they do next, do they test the other? [Turns to page 2]*

*R: Yes, and then it was correct. Then they put in that  $x$  was less than three.*

*L: Then they can use that other. [Turns to page 1] That. [Points on page 1]*

*R: That. [Points on page 1, then back to page 2]*

*L: And then it becomes... Yes, exactly.*

*R: Yes.*

In contrast to their previous discussions, here more details of the text are discussed. This focus on the example also appears in the third part of the study when the students explain what they thought of the text:

*L: I use this example quite a lot, to kind of understand what... it is about, because... this with  $x$  and  $y$  and such, it doesn't say that much really.*

*R: But since, yes since they have mixed, they have mixed with that then they have mixed with example... then it gets easier to understand.*

The students', or at least student L's, view of and opinion about the text appears a few more times. For example, earlier in part 3, student L gives spontaneous comments about the text:

*L: Yes it describes more mathematically than... than like you think. This first thing, that one [the definition], it really just describes that you remove the minus sign, on all of it, but... well, it doesn't say that.*

So in addition to discussions about the content of the text, the students also reflect on the text, and in episode 7 they also discuss possible uses of the absolute value. They do not manage to find any concrete useful applications, and ask about this at the end of part 3.

## **Part 2**

In the second part of the study the students were given tasks relevant to the text to solve, and they were allowed to keep the text when working with these tasks. From the students' activities much could be said about their ability to solve mathematical tasks in general and to conduct logical reasoning. But this study focuses on how the students use the text when trying to solve the tasks, regardless if they manage to solve these in an acceptable way.

### *Pair A*

The students in Pair A spent approximately 26 minutes on the second part of the study. Their activities are summarized in Table 2a.

When the students try to solve the tasks, they (of course) use their interpretation of the absolute value (see part 1), an interpretation that seems to have strong influences from the example of solving an equation in the text. This focus on the example here becomes obvious for several reasons:

- When they discuss task 1, they talk about “calculate two cases... and see which one is correct” and that they get several/different ‘solutions’.

- At two occasions the students ask themselves what kind of equation they have in task 1.
- In episode 3 they jump to task 2 since “it looks much easier, because then it’s just to do the same as in the example”.

In their attempts to solve task 1, the students use different parts of the text. But problems arise since they, as in part 1, focus on parts of statements in the text; parts that they also have difficulties to handle correctly. For example, in the definition the students interpret  $x \geq 0$  and  $x \leq 0$  as if the “answer” should be greater and less than zero respectively. Thereby, it is for the students not the same  $x$  on all places in the definition. When they with the help of the definition get a result they are not satisfied with (“nothing works”), an attempt is made to solve the task with the help of another part of the text ( $|x - y| = \dots$ ). In this way, the students conduct a sort of search for ‘clues’ or ‘methods’ described in the text that can be applied to the task. In episode 4 it is also noted by the students that “they have put this condition here also, maybe you should think of this”. It is not reflected upon where this condition comes from, but it is only established that it is stated in the text, and that you thereby probably should use it. It thus seems like the students do not expect the text to *explain* anything, for example (referring to part 1) the students do not seem to expect that the text should explain the meaning of  $|x|$ , instead they present theories of what they *believe* it could mean without any direct connection to anything in the text. Thus, the students seem to see the text primarily as a collection of (sometimes isolated) statements or methods that do not necessarily need to have a logical structure.

Table 2a. Summary of student activity for Pair A in part 2 of the study

<b>Episode</b>	<b>Length (min:sec)</b>	<b>Description of student activity</b>
1	4:30	Task 1a. First using the definition, then $ x - y  = \dots$ . Final conclusion: “No solution is ok”
2	3:20	Task 1b. Alternating $+/-$ on the two absolute values, which gives four ‘solutions’.
3	2:40	Task 2a. Breaking into two cases, but no conditions on $x$ .
4	1:10	Task 2b. Solving case one: plus on both absolute values. Notices the conditions on $x$ in the text’s example.
5	2:00	Task 2a. Adding the conditions on $x$ , removing one solution. Thinking about how to do the same on tasks 2b and 1.
6	2:00	Task 1. Four solutions to task 1c (same as 1b). Unsure about 1a.
7a	2:20	Task 2b. Alternating $+/-$ on the two absolute values, solving each equation.
7b	5:10	Trying to place different conditions on the four different cases, removing one solution (not correct).
8	3:00	Reading and checking their solutions, agreeing on task 2b.
<b>Total</b>	<b>26:10</b>	

On the other hand, the students often state what they *believe* and *feel* about the text and tasks without any explanations of their thoughts, for example at the end of episode 1 when the students seem puzzled by their ‘solutions’:

**R:** *What kind of answer should we get anyway?*

**L:** *We should calculate it.*

**R:** *Yes...*

**L:** *[Laughing]*

**R:** *If we try it this way instead, we put this as  $x$ ... And we should calculate... but it doesn't work anyway.*

**L:** *But then we have done it right or... because it is like that.*

**R:** *It should be correct.*

**L:** *I think so.*

In other situations other comments of the same kind occur, for example that “it is probably this way it should be done”, without further comments on why, or more exactly what is referred to; and that “it feels like we got it right on 2a at least”.

#### *Pair B*

The students in Pair B spent approximately 15 minutes on the second part of the study. Their activities are summarized in Table 2b.

A first example of the students’ discussions can be given from episode 3, when they

Table 2b. Summary of student activity for Pair B in part 2 of the study

<b>Episode</b>	<b>Length (min:sec)</b>	<b>Description of student activity</b>
1	1:35	Task 1a. Calculating 2–3 first, comparing with last check in example.
2	0:35	Solving tasks 1b and 1c.
3	3:40	Task 2a. Comparing with beginning of example and the definition. Solving two cases, and discussing whether $x = 1$ is a solution.
4	6:35	Task 2b. Discussing the difference between $ x - 5 $ and $ x + 5 $ . Removing the notation of the absolute values, and discussing when allowed to do that. Breaking into three cases, solving two of them and discussing for which $x$ they are valid.
5	0:35	Checking task 2a and the two first cases in 2b by inserting the solutions in the original equations.
6	1:00	Solving last case of task 2b, checking the solution both by the condition on $x$ and by inserting in the original equation.
7	1:20	Reading their solutions and the text.
<b>Total</b>	<b>15:20</b>	

have solved the two cases in task 2a and gotten  $x = 1$  in the second case:

**R:** *But this can't, this one isn't ok is it, that one [pointing to the solution  $x = 1$ ].*

**L:** *Yes... what?*

**R:** *This one is not correct, because then it becomes one equals... minus one [if you put  $x = 1$  into the original equation].*

**L:** *Yeeaaaaahhh...*

**R:** *And this one should, this should also be... if  $x$  is less than zero. And  $x$  is greater than zero then, so... [Pointing on the definition in the text]*

**L:** *Yes, exactly.*

**R:** *It's only this one that is... [Referring to the solution from the first case]*

**L:** *Yes, that one.*

**R:** *...that is correct.*

**L:** *We can't use that one if  $x$  is one.*

**R:** *No.*

**L:** *So then... this one disappears. [Sigh] You're right.*

In this discussion, where the students at the beginning do not agree, the definition in the text has a central role when student R motivates why  $x = 1$  cannot be a solution. The students often use the text in this way, as argumentation in situations when they do not agree, or as a tool when one student is explaining something to the other. But even when both students seem to agree and understand what to do, the text is used in their joint work with the tasks, as can be seen in episode 4 when the notation of the absolute values are removed in task 2b:

**L:** *Should we write... just write like this and without the absolute signs. [Pointing at task 2b]*

**R:** *Without absolute signs... [Writing]*

**L:** *Yes*

**R:** *Multiply the two here... [Writing. 'The two': in front of the absolute value]*

**L:** *Yes, then it will be parenthesis around that. Yes, hmm. Oh, how did we do this, now we have... if we just do like this... what could  $x$  be then.*

**R:** *Then  $x$  will...*

**L:** *Here it becomes like that if  $x$  was greater than three, so  $x$  should be greater than minus five, and greater than... two. [Pointing in the example in the text]*

**R:** *Yes.*

**L:** *For this to be correct.*

**R:** *Yes.*

**L:** *So if we calculate that one and hope for... it to be greater than two.*

In many situations, which the above excerpt is an example of, the text is used partly as help with some details in solving the tasks and partly to check that their suggestions on

how to solve the tasks agree with the text, for example in episode 1, where the text confirms for the students that it is possible to start with calculating the '2-3' within the absolute value.

It could be noted that the students succeed in solving the tasks quite easily, and that this possibly has limited their need to use the text. In practically all situations, at least one of the students has relatively quickly presented a possible solution, at least to a part of the task, and they have at no time had any major *common* problems, something that could have shown other examples of how these students use texts when solving tasks.

## Comparisons

In the previous sections several examples of students' view and usage of mathematical texts have been observed. To reveal more aspects of the observed situation when students read, discuss and use the text, comparisons will be made between the two parts of the study and also between the two pairs of students. Some direct comparisons can of course be done between earlier described activities, but here other aspects will be discussed, which have not already come up in a natural way.

The students in Pair A seemed quite unused with the situation in the first part of the study, where the students read and discussed the text freely (i.e. without further instructions). For example, after having read both pages of the text, the following dialogue takes place:

*R: Shall we go through this?*

*L: Go through?*

*R: Yes, what we can discuss about this.*

*L: Hmm... Were we supposed to do that now?*

*R: Yes.*

*L: Okay.*

However, the students in Pair B seem more comfortable in this situation, and in the third part of the study student L comments the situation:

*It's a good thing that we're working together also, if you should sit alone and just think about this, it can be different, then you don't come up with all thoughts yourself. But, yes, it's in principle like this we work on the lessons also.*

It could here be remembered that all four students come from the same mathematics class. But the students in Pair A seem more comfortable in the second part of the study (compared to the first), where they had specific tasks to solve. And in part three, after commenting several parts of the text as "quite logical", student R continues "then how to use it is of course another thing". This shows the student's own view of the difference between the two parts of this study.

There seems to be a significant difference between the two pairs' purpose or goal with their discussions and activities. The students in Pair B seem to try to reach a kind of *understanding* of the content of the text, where they for example sometimes start with a discussion about the meaning of the text, without direct reference to any details. The

students in Pair A on the other hand have the details of the text as their starting point, where focus seems to be on *doing* something, and where ‘doing something’ seems to be a goal by itself. For example at the end of the first part they do not understand the final check by inserting  $x = -1$  in the original equation, in the example of solving an equation. After a while, they instead try to insert the solution in the equation present earlier in the text, where the notation of the absolute value has been removed. And after checking that  $x = -1$  indeed is a solution to this equation, the students seem satisfied.

A similar difference between the pairs is also present in part two of the study, where the students in Pair B reflect on what they are doing and why, for example by checking that their suggested ways of solving the tasks agree with the text, while the students in Pair A seem focused on finding a method described in the text that they feel gives a reasonable result.

## Discussion and conclusions

The two main questions focused on in this paper<sup>2</sup> are about the students’ concrete activities in the observed situation when reading and discussing a mathematical text, but are also about their view of the text, for example concerning what the students consider possible to learn from mathematical texts and their view of how to use a text of this kind. The two pairs of students who were observed acted very differently, something that of course was advantageous for this study since this made it possible to observe different examples of how students act when they read, discuss and use a mathematical text. But by this it can also be concluded that students on their final year at the upper secondary school can have very different views of and abilities to use mathematical texts, even if they have similar grades.

Two kinds of activities, that certainly seem important for understanding and learning from the text, seem to be absent in the work of the first pair of students: Although they were aware of parts of the text they did not seem to understand, they did not try to make sense of it with the help of a discussion. Since everything in the text is connected to another part of the text, this may cause great problems in trying to understand the text and learn from it. At a more detailed level, these students often focused on small parts or fragments of the text, without considering its role in a larger context. This includes cases when focusing on and analyzing a fragment of a statement, without considering the whole statement.

The process of reading can sometimes be divided into two main steps; in the first step “there is a superficial understanding of words and sentences”, and in the second step “there may be the construction of a mental model, that is, a representation of what the text is about rather than of the text itself” (Newton & Merrell 1994, p. 457). From this you could say that the students in the first pair remain in the first step. But it can also be noted that their focus is on this first step when discussing the text, unlike the students in

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<sup>2</sup> See previous section ‘Purpose and research questions’ on page 4



the second pair who in a sense sometimes skip the first step<sup>3</sup>, and go directly to step two with discussions about the meaning of the text.

In the given text, both pairs of students focused on the example of solving an equation, both in their free discussions about the text and also when solving the tasks. This focus on examples in mathematical texts has also been observed among students at university level (Stephens & Sloan 1980). But the example, and the text in general, was used in very different ways by the two pairs, where the students in the first pair tried to find clues in the text on how to deal with the exercises, a search that was carried out in a non-systematical way, while the students in the second pair used the text mainly to check that their ways of solving the tasks agreed with what was said in the text.

The observations in this study show that the two pairs seem to have fundamentally different goals with their activities, where the students in the first pair focus on details in the text and on *doing something* with the text without any thorough reflections on their activities, while the students in the second pair try to *reach an understanding* of the content of the text and of the solutions of the tasks.

One of the main purposes of this study is to act as a guidance for future studies about learning mathematics by reading, and from the described observations some issues arise that would be interesting to examine in more detail.

The situation observed in this study is of course rather special for several reasons, such as having to work in front of a video camera. An interesting question is if the students would act (or do act) in the same way in their normal mathematical studies when using mathematical texts. And if the text was to be part of a larger perspective and context, as is the case in ordinary textbooks, would this result in any different activities among the students?

It is unclear why the students in the first pair do not discuss parts of the text that are not fully understood. Is it due to a didactical contract, where the students act as they believe they are expected to act; with focus on solving problems, and not (necessarily) on understanding all of the text? If so, is this contract caused by the special experimental situation, or would the students act the same way in a more 'normal' situation? Or is this phenomenon caused by some properties of the text itself or perhaps by the students' epistemological beliefs about mathematical texts or mathematics in general?

There are also several other aspects of the observed situation that seem to have a strong connection to more general aspects of mathematics than just specifically to reading mathematical texts, for example that the students sometimes are satisfied with asking each other *whether* they have understood a certain part of the text without dealing with *how* they have understood this part, as if there is only one way of understanding something.

A comparison with the conclusions from Lithner (2000), who focused on students' mathematical reasoning in task solving among university students, shows some similarities between the student activities in that study and the activities observed in this study. For example Lithner observes that the students often try to understand each

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<sup>3</sup> At least they seem to skip the first step in their common discussions, but since it could be argued that the first step is a prerequisite for the second step, step one can be assumed to be carried out individually when reading the text.

separate step of a solution procedure, but only *locally*, which resembles the first pair of students' focus on checking details in the text without considering the detail's position in a wider context.

But what relationship exists between the more general aspects and the more specific about reading mathematical texts, and how could you 'isolate' the interesting aspects of the specific about the situation when reading mathematical texts?

## Absolutbelopp

För varje reellt tal  $x$  betecknar  $|x|$  det s.k. *absolutbeloppet* av  $x$ , som definieras genom att

$$|x| = \begin{cases} x & \text{om } x \geq 0 \\ -x & \text{om } x \leq 0 \end{cases}$$

Observera att  $-x \geq 0$  om  $x \leq 0$  och att definitionen därför medför att

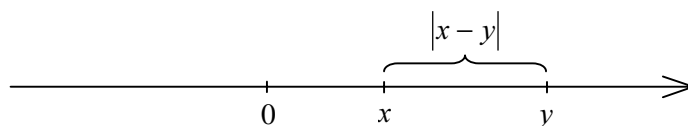
$$|x| \geq 0 \quad \text{för alla reella tal } x.$$

T.ex. är  $|2| = 2$  och  $|-2| = -(-2) = 2$ .

Av definitionen följer också att

$$|x - y| = \begin{cases} x - y & \text{om } x - y \geq 0 \text{ dvs om } x \geq y \\ y - x & \text{om } x - y \leq 0 \text{ dvs om } x \leq y \end{cases}$$

där  $x$  och  $y$  är reella tal. På en tallinje betyder det att  $|x - y|$  är *avståndet* mellan punkterna  $x$  och  $y$  oberoende av hur de ligger i förhållande till varandra. Detta illustreras i figuren nedan. Speciellt är då  $|x| = |x - 0|$  avståndet mellan punkterna  $x$  och  $0$  på tallinjen.



När vi skall undersöka ett uttryck som innehåller absolutbelopp måste vi med hjälp av definitionen dela in i olika fall som i följande exempel:

**Exempel.** Vi skall bestämma alla lösningar  $x$  till ekvationen

$$|x - 3| + 2x = 2$$

Här är

$$|x - 3| = \begin{cases} x - 3 & \text{om } x - 3 \geq 0 \text{ dvs om } x \geq 3 \\ -(x - 3) & \text{om } x - 3 \leq 0 \text{ dvs om } x \leq 3 \end{cases}$$

Vi får därför två fall:

Då  $x \geq 3$  ser vi att ekvationen  $|x - 3| + 2x = 2$  kan skrivas  $x - 3 + 2x = 2$  och sedan lösas

$$x - 3 + 2x = 2$$

$$3x = 5$$

$$x = \frac{5}{3}$$

Men  $x = \frac{5}{3}$  uppfyller inte villkoret  $x \geq 3$  och är därför inte en lösning till ekvationen.

Då  $x \leq 3$  ser vi att ekvationen  $|x - 3| + 2x = 2$  kan skrivas  $-(x - 3) + 2x = 2$  och sedan lösas

$$\begin{aligned}-(x - 3) + 2x &= 2 \\x + 3 &= 2 \\x &= -1\end{aligned}$$

som uppfyller villkoret  $x \leq 3$  och alltså är en lösning till ekvationen.

Den enda lösningen till ekvationen är alltså

$$x = -1.$$

Kontroll av  $x = -1$  i den givna ekvationen:

$$\text{Vänsterled} = |x - 3| + 2x = |-1 - 3| + 2 \cdot (-1) = |-4| - 2 = 4 - 2 = 2 = \text{Högerled.}$$

## Övningar

1. Räkna ut

(a)  $|2 - 3|$

(b)  $|2| + |-3|$

(c)  $|2| - |3|$

2. Bestäm alla lösningar till följande ekvationer:

(a)  $|x| = 3x - 4$

(b)  $|x + 5| = 2|x - 2| + 1$

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