Calibration of parameters for the Heston model in the high volatility period of market

Master’s Thesis in Financial Mathematics

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Chapter 1

Introduction

The Heston model concerns with the option pricing problems and has achieved great success. We begin our investigation with some words about Black-Scholes model. In the Black-Scholes model for financial equities the volatility is assumed to be constant. This assumption is used in option pricing with the Black-Scholes formula, see for example [3]. The constant volatility is however not consistent with real data as shown in many studies, for example [5], [4]. In the Black-Scholes model, the implied volatility varies with expiring data and with strike price. Also the clustering of data seen in the returns indicates a time varying volatility.

Several different approaches have been used to improve the early financial models. In the 1960’s Mandelbrot suggested a model of the price base on the stable Paretian distribution. In 1973, Clark, [6] proposed the so called Mixture-of-Distribution Hypothesis. In the same year, Merton [7] supposed that volatility was a deterministic function of time. It can explain the different implied volatility levels for different times of maturity, but doesn’t explain the smile shape for different strikes. The work of Heston (1993) led to the development of stochastic volatility models.

The Heston model is one of the most widely used stochastic volatility (SV) models today. In our project we investigate the Heston model and characterize the estimation and calibration problem of this model.

There are many empirical, economic and mathematical reasons for using a model with such a form for investigation the volatility on the market. Empirical studies have shown that an asset’s log-return distribution is non-Gaussian. It is characterized by heavy tails and high peaks. It is also observed that equity returns and empirical volatility are negatively correlated.

Calibration of the stochastic volatility model can be done in some different ways [4], [10]. One of them is to look at a time series of historical data and the corresponding option data. We consider a period of high volatility in exchange market and make calculations using data from such period.
Chapter 1. Introduction

One method of calibration the Heston model is the Indirect Inference method. This method can be described in three steps. At first we consider auxiliary, more simple model and estimate parameters of this model using real date. In our work we use GARCH(1,1) model, and get vector of parameters \((\omega, \alpha, \beta)\). On the second step, we choose the initial values for the parameters for Heston model \((\theta, \kappa, \rho)\). As a result of this step we obtain a vector of option prices and corresponding values of volatility \([S, \upsilon]\). And finally we repeat the first step, but instead of using the real dates we use estimated dates from second step. We obtain the vector with new parameters and compare new vector and last vector.

In this project we study the estimation of the parameters of the Heston model with focus on the period with high-volatility market. Our research question is how this estimation is effected in the period of high turbulence of the financial market.

We use daily option prices from Nordic Market. The data include period from 6.07.2004 to 14.03.2008.

For more effective investigation this problem, we should divide different periods on the Nordic market Therefore we have got a data which include stable period for option prices and high-volatility period. The one of general parts of our work consists problem of change-point detection. We consider base principles of searching this problem.

The Nordic derivatives market (NASDAQ-OMX) is the third largest marketplace for derivatives by volume. It offers exchange-traded options on Danish, Finnish, Icelandic, Norwegian and Swedish equities. The most traded contracts are options on Ericsson, which represent approximately 60% of the total volume in the single stock options.

In our work we use the OMXS30 index. It is a tradable index which consists of the 30 largest capitalized shares at the Nordic Exchange Stockholm. This means that OMXS30 index options are excellent instruments for reducing risk exposure or increasing yields over the Swedish market.
Chapter 2

The Heston model

2.1 The base equations of the Heston model

In this chapter we present information about the Heston model and methods of calibration parameters. Further we describe in detail the influence of each parameter of this model.

We begin by assuming that the spot asset price $S_0$ at time $t$ is determined by a stochastic process:

$$dS(t) = \mu S dt + \sqrt{\nu(t)} S dW_t^{(1)},$$  \hspace{1cm} (2.1)

where $\nu(t)$ is the variance and follows the process:

$$d\nu(t) = \kappa(\theta - \nu(t)) dt + \sigma \sqrt{\nu(t)} dW_t^{(2)},$$  \hspace{1cm} (2.2)

where $W_t^{(1)}$ and $W_t^{(2)}$ are Wiener processes allowed to be correlated with each other by

$$dW_t^{(1)} dW_t^{(2)} = \rho dt.$$  \hspace{1cm} (2.3)

We denote by $(S_t)_{t \geq 0}$ and $(\nu_t)_{t \geq 0}$ the price and volatility processes, respectively. The Brownian motion process are correlated with parameter $\rho$.

$$\rho = corr(dW_t^{(1)}, dW_t^{(2)})$$

Another parameters in the previous equations represent the following:

- $\mu$ is the rate of return of the asset,
• \( \theta \) is a long run average price volatility (long vol),

• \( \kappa \) is the rate of mean reversion to the long term variance,

• \( \sigma \) is the volatility of variance (vol of vol).

On the Figures (2.1)-(2.2) we plot the spot price processes in Heston’s model.

![Figure 2.1: The example of the spot price dynamics in the Heston model.](image)

Figure 2.1: The example of the spot price dynamics in the Heston model.

and on the Figure [2.2] the corresponding volatility process.

To make possible a comparison both trajectories were obtained with the same set of numbers. The initial spot rate \( S_0 = 700 \), yielding a drift of \( \mu = 0.02 \). The volatility in the Heston model is given by a mean reverting process (see Figure [2.2]) with the initial variance \( \nu_0 = 0.02 \), the long term variance \( \theta = 0.04 \), the speed of the mean reversion \( \kappa = 2 \), and the volatility of variance \( \sigma = -0.05 \).

In the stochastic volatility model the value function of a general contingent claim \( U(S,\nu,t) \) is dependent on the randomness of the asset \( \{S_t\}_{t\leq0} \) and the randomness associated with the volatility of the asset’s return \( \{\nu_t\}_{t\leq0} \).

In frame of the Heston model the value of any option must satisfy the following partial differential equation,

\[
\frac{1}{2} \nu S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma \nu S \frac{\partial^2 U}{\partial S \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + r S \frac{\partial U}{\partial S} + \left\{ \kappa(\theta - \nu) - \lambda(S, \nu, t) \right\} \frac{\partial U}{\partial \nu} - r U + \frac{\partial U}{\partial t} = 0. \tag{2.4}
\]
The function $\lambda(S, \nu, t)$ is called the market price of the volatility risk. Without loss of generality its functional form can be reduced to $\lambda(S, \nu, t) = \lambda \nu$ (see paper [4]). A European option with the strike price $K$ and time to maturity $T$ satisfies the PDE (2.4) subject to the following boundary conditions:

\[
\begin{align*}
U(S, \nu, t) &= \max(0, S - K), \quad (2.5) \\
U(0, \nu, t) &= 0, \quad (2.6) \\
\frac{\partial U}{\partial S}(\infty, \nu, t) &= 1, \quad (2.7) \\
rS \frac{\partial U}{\partial S}(S, 0, t) + \kappa \theta \frac{\partial U}{\partial \nu}(S, 0, t) - rU(S, 0, t) + U_t(S, 0, t) &= 0, \quad (2.8) \\
U(S, \infty, t) &= S. \quad (2.9)
\end{align*}
\]

The solutions have the following form:

\[
C(S, \nu, t) = SP_1 - KP(t, T)P_2, \quad (2.10)
\]

where the first term is the present value of the spot asset price, and the second term is the value of the strike-price payment. Both of them must satisfy the original PDE (2.4).

If option price satisfy the conditions in equations (2.5)-(2.9), the function

\[
P_f(S, \nu, T; \ln |K|) = I_{\{x \leq \ln |K|\}} \quad (2.11)
\]
may be interpreted as "risk-neutralized" probabilities (Cox and Ross (1976)). We can immediately get the probabilities in the closed form. In the Heston’s work (1993) we can find that the characteristic function in form
\[ f(S, \nu, t; \phi) = e^{C(T-t;\phi)+D(T-t;\nu+i\phi)S}, \] (2.12)
where
\[ C(\tau; \phi) = r\phi i\tau + \frac{\alpha}{\sigma^2} \left\{ (b_j - \rho \sigma \phi i + d)\tau - 2\ln \left[ \frac{1 - ge^{d\tau}}{1 - g} \right] \right\}, \] (2.13)
\[ D(\tau; \phi) = \frac{b_j - \rho \sigma \phi \tau + d}{\sigma^2} \left[ 1 - e^{d\tau} \right], \] (2.14)
and
\[ g = \frac{b_j - \rho \sigma \phi \tau + d}{b_j - \rho \sigma \phi \tau - d} \] (2.15)
\[ d = \sqrt{(\rho \sigma \phi \tau - b_j)^2 - \sigma^2(2u_j \phi i - \phi^2)}, \quad \text{for } j = 1, 2, ... \] (2.16)
To get the required probabilities one can invert the characteristic functions:
\[ P_f(S, \nu, T; \ln [K]) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{e^{-i\phi\ln[K]}f_j(S, \nu, T; \phi)}{i\phi} \right] d\phi. \] (2.17)
Equations (2.4), (2.10), and (2.12) give the solution to the European call options.

2.2 A selection of parameters

The goal of our work is to investigate the behavior of the spot price process of the Heston model according to the parameters. For example, the description of the spot price process can be various in the period of high volatility and in the low volatility market. The effective utilisation of the stochastic volatility model depends on the initial parameters and calibration parameters on such type of market. The calibration can be done in different ways. For example, one method is to look at a time series of the historical data. The method of estimation such as Generalized or Efficient Methods of Moments have been applied earlier (Chernov and Ghysels (2000)). Unfortunately, the attempts to use empirical distribution for this goal have one common flaw - they do not allow to estimate the market price of volatility risk \( \lambda(S, \nu, t) \). Instead of the historical approach we calibrate the model by derivative prices. At first we have to understand how each parameters influence to the option price and chose the most important.
As a preliminary step, we will retrieve the strikes since the smile in exchange markets is specified as a function of the deltas. On the next step we will fit five parameters: initial variance $\nu_0$, volatility of variance $\sigma$, long-run variance $\theta$, mean reversion $\kappa$, and correlation $\rho$. In all plots obtained for $\nu_0 = 0.01$, $\sigma = 0.25$, $\theta = 0.015$, and $\rho = 0.05$.

First, let us look at the volatility change of variance (vol of vol) on the shape of the smile.

![Figure 2.3: The effect of changing the variance volatility. Dashed curve reflect a smile with $\sigma = 0$, the continuous line with circles corresponds to $\sigma = 0.25$ and the continuous line shows $\sigma = 0.6$.](image)

Setting $\sigma$ equal to zero produces a deterministic process of the variance and consequently volatility which does not admit any smile. On the other hand, increasing the volatility of variance increases the convexity of the fit, see Figure 2.3.

The initial variance has a different influence on the smile. Changing allows adjustments in the height of the smile curve rather than the shape, see Figure 2.4.

The next parameter is the long-run variance.

We can notice that effects of changing the long-run variance are similar to effects of changing the initial variance. This requires some attention in the calibration process. It seems we can choose the initial variance a priori and only look at the long-run variance. In particular, a different initial variance for different maturities would be incompatible. In figure 2.5 the effect of changing long-run variance is shown.
Chapter 2. The Heston model

Figure 2.4: The effect of changing the initial variance. The dashed curve reflects a smile with $\nu_0 = 0.008$, and the dashed-circles curve corresponds to $\nu_0 = 0.012$.

Figure 2.5: The effect of changing the long-run variance. The dashed-circles curve reflects a smile with $\theta = 0.01$, the continuous line shows $\theta = 0.015$ and the dashed curve corresponds to $\theta = 0.02$.

The effect of speed changing of mean reversion on the shape of the smile is displayed on the Figure 2.6.
Figure 2.6: The illustration of the spread changing the of the mean reversion. The dashed curve reflect a smile with $\kappa = 0.01$, the continuous line shows $\kappa = 1.4$, and the dashed-circles curve corresponds to $\kappa = 3$.

The changing of the mean reversion is an evident impact, the increasing of the mean reversion lifts the center. Further, the influence of mean reversion can be compensated by a stronger volatility of variance.

Finally, let us look at the influence of correlation.

The uncorrelated case give a quite symmetric smile curve centered at-the-money. However, it is not exactly symmetric. The changing such parameter changes the degree of symmetry, see Figure 2.7. In particular, positive correlation makes calls more expensive, negative correlation makes puts more expensive.
Figure 2.7: The effect of the correlation parameter changing. The continuous line reflect a smile with $\rho = 0$, the continuous line with circles shows $\rho = -0.15$, and the dashed-circles curve corresponds to $\rho = -0.5$. 
Chapter 3

The Indirect Inference method

In this chapter we present the Indirect inference method and its application to our case. This method was first introduced by Smith (1990, 1993) and later generalized in important ways evolved by Gourieroux, Monfort, Renault (1993), Gallant and Tauchen (1996), [11].

Indirect inference is a simulation-based method employed for the estimating the parameters of economic models and it has many interesting applications, mainly in finance, macroeconomics, because of its flexibility. Its utilization in our work is related to the estimation of the parameters of the Heston stochastic volatility model. The principle of this method is the use an auxiliary model to capture aspects of the data which we use for the estimation.

3.1 The basic idea of the method

To compute option prices from the Heston model, we need input parameters that are not observable from the market data. We attempt to estimate the parameters from the time series return data and from the corresponding option data.

Assume that $\mu = 0.02$. The discrete time approximation of equations (2.1) and (2.2) is

$$R_t = \sqrt{\nu_t \tau} \epsilon_{1t},$$

$$\nu_t = \kappa \theta \tau + (1 - \kappa \tau) \nu_{t-\tau} + \sigma \sqrt{\nu_{t-\tau} \tau \epsilon_{2t}},$$

where $R_t$ is the return of two consecutive stock prices, and $\epsilon_{1t}$ and $\epsilon_{2t}$ are two correlated standard normal random numbers.

For example we can take a GARCH(1,1) model, as an auxiliary model in our indirect method. GARCH(1,1) process defined by following

$$R_t = \sqrt{h_t} \epsilon_t,$$
$h_t = \omega + \alpha \epsilon_{t-1}^2 + \beta h_{t-1}, \quad (3.4)$

where $\epsilon_t$ is a normal random number with mean 0 and variance $h_t$. Due to the restriction from the auxiliary model, we can estimate four parameters $(\sigma, \theta, \kappa, \rho)$. Given a set of parameters $(\sigma, \theta, \kappa, \rho)$ we simulate the return data at one-day interval, remarking that we have 252 trading days.

The general idea of the indirect inference method is to match the moments of the auxiliary model with market data with those of the simulated data. For any set of simulated series, the structural parameter set $(\sigma, \theta, \kappa, \rho)$ is known. Application the auxiliary model to our real data from Nordic market yield an optimal set of parameters $Q_R = (\omega, \alpha, \beta, \sigma)$. For the calibration our parameters we use, at first, central moments (mean and variance) and normalised central moments (skewness and kurtosis) and on the second step we use the GARCH(1,1) like auxiliary model. Therefore in the next two sections we should include the base theoretical information about these models.

### 3.2 Moments

There are four parameters that can be interesting for us: mean, variance, skewness and kurtosis.

If $F$ is a cumulative probability distribution function which may have a density function, then the $n$-th moment of the probability distribution is given by the following form

$$
\mu_n = E[(X - \mu)^n] = \int_{-\infty}^{+\infty} (x - \mu)dF(x). \quad (3.5)
$$

The first moment around zero, if it exists, is the expectation of $X$.

$$
\mu = E[X].
$$

But in the case when

$$
E(|X^n|) \text{ is not finite.} \quad (3.6)
$$

the moment is said not to exist.

The second central moment is the variance. If the random variable $X$ has expected value (mean) $\mu = E(x)$, then the variance $\text{Var}(X)$, is given by

$$
\sigma^2 = \text{Var}(X) = E\left((X - \mu)^2\right). \quad (3.7)
$$
To describe the kurtosis and skewness we should also introduce the notion of a standardized moment. It is $\frac{\mu_n}{\sigma^n}$, where $\mu_n$ is the n-th moment and $\sigma$ is the standard deviation.

The third central moment is a measure of the degree of asymmetry of a frequency distribution. Any distribution will have a third central moment. The normalized third central moment is called the skewness, it can be defined as

$$\beta = \frac{\mu_3}{\sigma^3},$$ \hspace{1cm} (3.8)

where $\mu_3$ is the third moment about the mean and $\sigma^3$ is the standard deviation, as it was in definition of standardized moments. A distribution that is skewed to the left (the tail of the distribution is heavier on the left) will have a negative skewness. A distribution that is skewed to the right (the tail of the distribution is heavier on the right), will have a positive skewness, see Figure 3.1.

The fourth standardized moment is a measure of the flatness (versus peakedness) of the distribution. The kurtosis is illustrate in the Figure 3.2.

The kurtosis is more commonly defined as the fourth cumulant divided by the square of the variance of the probability distribution,

$$\gamma = \frac{\mu_4}{(\sigma^2)^2} - 3.$$ \hspace{1cm} (3.9)
Figure 3.2: A distribution with a high kurtosis has a sharper peak (left panel), while a low kurtosis distribution has a more rounded peak (right panel).

### 3.3 The GARCH(p,q) model

In this section we introduce some theoretical information about the GARCH(p,q) model. For the description of the qualitative changes of option prices $h = (h_n)_{n\leq 1}$, with

$$h_n = \ln \frac{S_n}{S_{n-1}}, \quad (3.10)$$

is used, so called, conditionally-gaussian model, the ARCH(p) model, where

$$h_n = \sigma_n \epsilon_n, \quad (3.11)$$

and the ”volatility” $\sigma_n$ is determined like

$$\sigma_n^2 = \omega_0 + \sum_{i=1}^{\rho} a_i \sigma_{n-i}^2. \quad (3.12)$$

For the calibration of parameters for the Heston model we use Generalized Autoregressive Conditional Hederoskedastic model with parameters p and q (GARCH(p,q)). It was introduced by T.Bollerslev (1986). In this model

$$h_n = \sigma_n \epsilon_n, \quad \epsilon \in N(0, 1).$$

The volatility

$$\sigma_n^2 = \omega_0 + \sum_{i=1}^{\rho} a_i h_{n-i}^2 + \sum_{j=1}^{q} \beta_i \sigma_{n-j}^2, \quad (3.13)$$
where $\omega_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$, if all $\beta_j$ is equal zero, then we obtain the ARCH(p) model. The GARCH(1,1) model equations have the following form:

\[ h_n = \sigma_n \epsilon_n, \]  
\[ \sigma_n^2 = \omega_0 + \alpha_1 h_{n-1}^2 + \beta_1 \sigma_{n-1}^2. \]  

(3.14)  
(3.15)

Remember that $\epsilon \in N [0, 1]$, then

\[ Eh_n^2 = \omega + (\alpha + \beta) E\sigma_{n-1}^2. \]

If the process is stationary, it satisfy the following condition

\[ Cov(X_t, X_s) = f (t - s), \]

then

\[ Eh_n^2 = \omega + (\alpha + \beta) Eh_n^2. \]

Correspondingly

\[ Eh_n^2 = \frac{\omega}{1 - \alpha - \beta}. \]  

(3.16)

The stationary value exists if $\alpha + \beta < 1$. 

Chapter 4

An analysis of real data

4.1 The historical review of real data

The empirical data used in this project consists of the OMX 30 index from the NASDAQ OMX Nordic Exchange Market. The NASDAQ (National Association of Securities Dealers Automated Quotations) is an American stock exchange market. It is the largest electronic screen-based equity securities trading market in the United States. The OMX (Optionsmaklarna/Helsinki Stock Exchange) is a financial services company, which in 2006 comprised stock exchanges in all Nordic and Baltic states. It has two divisions, the OMX Exchanges, which operates eight stock exchanges in the Nordic and Baltic countries, and the OMX Technology, which develops and markets systems for financial transactions used by the OMX Exchanges, as well as by other stock exchanges. The company’s stock market activities are categorized into three divisions:

- Nordic Market (Copenhagen, Stockholm, Helsinki, Iceland),

- Baltic Market (Tallinn, Riga, Vilnius),

- First North (alternative exchange).

On May 25, 2007 NASDAQ concluded a treaty of acquiring with OMX financial company. The acquisition was completed in Dec. 2007 and since then NASDAQ is included into the European market. After this fusion the one third of the International Exchange St Petersburg (IXSP) in St Petersburg is also include into NASDAQ company because IXSP was founded with the assistance of OMX. The newly merged company was renamed the NASDAQ OMX Group.
In our project we use the data of the value of OMXS30 Index. OMX Stockholm 30 is OMX Nordic Exchange Stockholm’s leading share index. The index consists of the 30 most actively traded stocks on the OMX Nordic Exchange Stockholm, see [14]. The limited number of constituents guarantees that all the underlying shares of the index have excellent liquidity. The composition of the OMXS30 index is revised twice a year. The OMXS30 Index is a market weighted price index. The index consists a portfolio of the largest and most traded shares, representing the majority sectors of economy. The base date for the OMX Stockholm 30 Index is September 30, 1986, with a base value of 125. The indexes serve as an indicator of the overall trend in its market and are intended to offer a cost effective index that an investor can fully replicate.

At present (reshuffle of 2008-05-24) the index is composed of the following list- ings: Electrolux, Ericsson, Hennes Mauritz (HM), Nokia, SCANDIA, SEB, Swed- bank, Tele2, Volvo Group and other. The data we use is values of the OMX-index, from January 2, 2004 to May 20, 2008. Plots of the index evolution during this period is shown on Figure (4.1) - (4.3):

![Figure 4.1: The OMXS30 Index during the period: 2004.01.02 - 2008.05.20](image)
4.2 A background of the change-point detection problem

In the previous chapter we gave some historical description of market, indexes and influence of events to our real data. But for the further analysis we need to determine the change-moments of market’s conditions. It is important for the exact conclusions about the Heston model and it’s application in practice. The problem of the change-point detection was consider in many papers, for example [12], [13]. Therefore, there are different methods which are allowed to find required
points.

We should say that there are some different types of such problem. In one case we have the changes of the regression coefficients. So, if the data presented serially and the parameters change in process we have an on-line (prospective) detection, but if we identify stationary intervals in advance and the data presented en-mass we have off-line (retrospective) detection. Also one should separate the change point problem for single and multiple changepoints. We interested in retrospective detection because we already have a data during the fixed period, and we can analyse all set of presented option prices. In this chapter we give the essential principles of the off-line detection and describe base steps of the off-line statistical test. Finally we consider a Bayesian point of view on the retrospective analysis and give some illustrative examples with application to our real data and with some conclusions.

Let’s consider hypothesis testing, test assertions about general parameters of a process (e.g., mean, variance, covariance). So, let

\begin{itemize}
  \item $H_0$ (Null hypothesis): normal situation,
  \item $H_1$ (Alternative hypothesis): abnormal situation.
\end{itemize}

This statistic test use the real data

$$T(y) = f(y_1, \ldots, y_N).$$

We also introduce a decision function of $d(T)$ which classifies values of $y = (y_1, \ldots, y_N)$

$$d(T) \in \{0, 1\}$$

It determines if the test statistic is within an acceptable range

\begin{itemize}
  \item if $d(T) = 0$: we have normal situation,
  \item if $d(T) = 1$: this moment can be the changepoint from normal to abnormal situation.
\end{itemize}

For example, we can represent $T(y)$ in the following form

$$T(y) = \sum_i (y_i - \bar{y})^2.$$

The next step is to chose critical values (upper or lower limits). The decision function $d(T)$ is equal to
• 1, if $T < T^L_C$ or $T > T^H_C$,

• 0, otherwise.

We have raise an alarm if $d(T) = 1$. Denote by $C$ the set of values for which $H_0$ is rejected. It is reasonably to introduce probabilities of switching from the normal situation to the abnormal one, and the probability of the inverse change.

• $P(\text{false positive}) = \alpha = P(y \in C|H_0),$

• $P(\text{false negative}) = \beta = P(y \notin C|H_1).$

To better understanding we can make a Figure 4.4 with our real data, a specially for an unstable period.

![Figure 4.4: The high-volatility period of Nordic market. Two lines illustrate a switching between a normal situation and an abnormal situation.](image)

Denote $\pi(\theta) = P(\phi \text{ rejects } H_0|\theta),$

The components, which we introduced are allowed to design statistical test which provide a good way to find points of change. It is reasonable to select test $\phi$ which minimize $\beta_\phi$ and also $\alpha_\phi$ should be not too large.

Denote $\pi(\theta)$
and

\[ \pi(\theta) = \alpha_\phi, \text{ if } \theta \in H_0, \]
\[ \pi(\theta) = 1 - \beta_\phi, \text{ if } \theta \in H_1. \]

The test will be ideal if the system after a normal situation change to a normal position but after an abnormal situation doesn’t follow normal situation.

\[ \pi(\theta) = 0, \text{ if } \theta \in H_0, \]
\[ \pi(\theta) = 1, \text{ if } \theta \in H_1. \]

Note that off-line test use the appearance like the effect of clustering, and we choose points such that the variance within a cluster is smaller than the variance between clusters.
4.3 The Bayesian analysis of change-points for our data

There we consider the problem of change-point detection under the Bayesian viewpoint. Let \( y^t = (y_1, ..., y_n) \) be a vector of observable data. The distribution \( f(y|\theta), \theta \in \Theta \). Parameter \( p, 1 \leq p \leq n - 1 \), denote the number of changepoints. \( r_p = (r_1, ..., r_p) \) the positions at which the changes occur. And \( S_{r_p} = (y_1^t, ..., y_{p+1}^t) \) denote the partition of the vector of data \( y \). The generic partition is following

\[
S_{r_p} = (y_1^t, ..., y_{p+1}^t) = \{ (y_1, ..., y_{r_1}), (y_{r_1+1}, ..., y_{r_2}), ..., (y_{r_p+1}, ..., y_n) \}.
\]

All models can be classified into parts (boxes) \( \{\mathcal{I}_0, ..., \mathcal{I}_{n-1}\} \), where box \( \mathcal{I}_0 \) contains the model with no changes, box \( \mathcal{I}_1 \) contains the model with just one change.

The main interest in our settings is making inferences on three magnitudes. At first on the number of changepoints \( p \), second, is the configuration \( r \) conditionally on \( p \). Third, value of \( r \) on the whole set of models. Then we need to compute \( \pi(p|y), \pi(r|y, p) \) and \( \pi(r|y) \).

Let \( m(y|M_0) \) be the marginal of the set of data \( y \)

\[
m(y|M_r) = m(y|r) = \int f(y|\theta, r, p)\pi(\theta|r)d\theta,
\]

and \( m(y|M_0) \) the marginal under the no change model \( M_0 \)

\[
m(y|M_0) = m(y|0) = \int f(y|\theta_0, 0, p)\pi(\theta_0|0)d\theta.
\]

The value

\[
\frac{m(y|M_r)}{m(y|M_0)},
\]

denotes the Bayes factor for comparing model \( M_r \) and \( M_0 \).

The base equations for the computing the posterior probabilities are following

\[
\pi(p|y) = \frac{\pi(p)\sum_{s \in \mathcal{I}_p} \pi(s|p)B_{sn}(y)}{\sum_{q=0}^{n-1} \pi(q)\sum_{s \in \mathcal{I}_q} \pi(s|q)B_{sn}(y)}, \text{ for } p \in \{0, 1, ..., n - 1\},
\]

\[
P(M_r|y, p) = \frac{\pi(r|p)B_{rn}(y)}{\sum_{s \in \mathcal{I}_p} \pi(s|q)B_{sn}(y)}, \text{ for } M_r \in \mathcal{I}_p,
\]

\[
P(M_r|y) = \frac{\pi(p)\pi(r|p)B_{rn}(y)}{\sum_{q=0}^{n-1} \pi(q)\sum_{s \in \mathcal{I}_q} \pi(s|q)B_{sn}(y)}, \text{ for } M_r \in \mathcal{I}_p.
\]
Chapter 4. An analysis of real data

Let’s consider our real data. The period of time which we used from market is long and really difficult to analyse. The results from the factor analysis and conditions being up to five change-points, i.e., \( p \leq 5 \). As seen from posterior probabilities (Table 4.5) the number of change-points, indicate the existence of a double change-points. Five of all probable models displayed in Table 4.6 in decreasing order.

| \( p \) | \( \pi(p | y) \) |
|-------|----------------|
| 0     | 0.024          |
| 1     | 0.066          |
| 2     | 0.562          |
| 3     | 0.220          |
| 4     | 0.052          |
| 5     | 0.076          |

Figure 4.5: The posterior probabilities of the change-points number \( p \).

<table>
<thead>
<tr>
<th>Models ( r )</th>
<th>600</th>
<th>1042</th>
<th>924</th>
<th>964</th>
<th>602</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(M_r</td>
<td>y) )</td>
<td>0.448</td>
<td>0.412</td>
<td>0.115</td>
<td>0.038</td>
</tr>
</tbody>
</table>

Figure 4.6: The posterior probabilities of the most probable models.

We can also look at the plot, when we use two lines to divide our data according to analysis of change-points (see Figure 4.7).

Figure 4.7: The OMXS30 Index during the period 2004.01.02-2008.05.20 with change-points.

Conclusion: there are two change-points at the position 600 and position 1042. In the context of our goal we interested in the change-point which have position...
600. It can be more reasonable to get just this point and divide our data in to two different periods. So, option prices from first to 600th position we associated with the stable market, and for the data from 600th position to the end of table we use for analysis of the Heston model on the high volatility period of market.
CHAPTER 4. AN ANALYSIS OF REAL DATA
Chapter 5

Programme realization

We described how we done calibration of our parameters by the Indirect method. This method include the utilization of the auxiliary model.

In one case we can use GARCH(1,1) model like the auxiliary model. Also it is possible to obtain good results by using general moments, in particular second moment, skewness and kurtosis. Further, we describe in detail the structure of the operations.

In the case of general moments we use real data from Nordic Exchange market to find four parameters: the first moment $\mu$, the second moments $\sigma^2$, the skewness $\beta$ and the kurtosis $\gamma$ (or $\omega, \alpha, \beta$ if we use the GARCH(1,1) model). Then we save the vector with this parameters $Q_R = (\mu_R, \sigma_R, \beta_R, \gamma_R)$ (R means parameters for the real data). On the next step we use the Heston simulation. We choose initial parameters for the Heston model $H = (\sigma_0, \theta_0, \kappa_0, \rho_0)$, put into our program, and as a result we obtain a vector of the stock prices (the number of the real data and the simulated data should be the same). The third step is similar to the first step, but now we use the simulated data. We find four parameters: the first moment, the second moment, the skewness and the kurtosis. So, we have another vector $Q_S = (\mu_S, \sigma_S, \beta_S, \gamma_S)$ (S means parameters for the simulated data). Finally we compare two vectors with parameters, therefore we construct the function

$$f(\mu, \sigma, \beta, \gamma) = \sqrt{((\mu_R - \mu_S)^2 + (\sigma_R - \sigma_S)^2 + (\beta_R - \beta_S)^2 + (\gamma_R - \gamma_S)^2)} \quad (5.1)$$

The goal is to minimize this function by changing initial parameters for the Heston simulation step. There we can use one of the optimization methods, for example the gradient method. The algorithm of the program is shown on the Picture [5]. Some parts of program can be found in appendix.

Note, that we check parameters for the high-volatility period of market and for the stable period. Therefore we don’t use this algorithm once, it is interesting to see more than 10 values of each parameter. It is allowed us to calculate mean and standart deviation.

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Chapter 5. Programme realization

Figure 5.1: The algorithm of program
Chapter 6

Results and conclusions

Stochastic volatility models are increasingly important in practical derivatives pricing applications. At first, we make some graphical interpretation of the Heston simulation. The following plots show the curves of the real data and the Heston simulation of two periods: period of a low volatility (Figure 6.1), and the period of the high volatility (Figure 6.2).
Figure 6.1: The Heston simulation data and the real data from the stable period of the Nordic market. The graphic of the real data increase with the small volatility, the simulated data doesn’t increase and also has the small volatility.

Figure 6.2: The Heston simulation data and the real data from the high-volatility period of the Nordic market. Values of the option prices are less than the simulated prices for the first part of 150 values and for the last of 100 values. In the middle of such period the real option prices are larger than the simulated prices.

The main idea of our work was the calibration of the parameters for the Heston stochastic volatility model. We make our calculations more than 10 times and now we have a possibility to compare the mean and the standard deviation for each set.
of parameters. Let us first take a look to the result for the stable period from our data.

<table>
<thead>
<tr>
<th>No</th>
<th>sigma</th>
<th>theta</th>
<th>rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.456</td>
<td>0.0055</td>
<td>-0.0131</td>
</tr>
<tr>
<td>2</td>
<td>0.364</td>
<td>0.0111</td>
<td>-0.0104</td>
</tr>
<tr>
<td>3</td>
<td>0.113</td>
<td>0.0189</td>
<td>-0.0211</td>
</tr>
<tr>
<td>4</td>
<td>0.064</td>
<td>0.0211</td>
<td>-0.0162</td>
</tr>
<tr>
<td>5</td>
<td>0.164</td>
<td>0.0201</td>
<td>-0.0112</td>
</tr>
<tr>
<td>6</td>
<td>0.186</td>
<td>0.0130</td>
<td>-0.0238</td>
</tr>
<tr>
<td>7</td>
<td>0.172</td>
<td>0.0361</td>
<td>-0.0096</td>
</tr>
<tr>
<td>8</td>
<td>0.226</td>
<td>0.0276</td>
<td>-0.0076</td>
</tr>
<tr>
<td>9</td>
<td>0.144</td>
<td>0.0192</td>
<td>-0.0105</td>
</tr>
<tr>
<td>10</td>
<td>0.23</td>
<td>0.0296</td>
<td>-0.0229</td>
</tr>
<tr>
<td>11</td>
<td>0.234</td>
<td>0.0419</td>
<td>-0.0134</td>
</tr>
<tr>
<td>12</td>
<td>0.146</td>
<td>0.0385</td>
<td>-0.0001</td>
</tr>
<tr>
<td>13</td>
<td>0.144</td>
<td>0.0331</td>
<td>-0.0176</td>
</tr>
<tr>
<td>14</td>
<td>0.212</td>
<td>0.0364</td>
<td>-0.0192</td>
</tr>
<tr>
<td>15</td>
<td>0.186</td>
<td>0.0421</td>
<td>-0.0190</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>0.2054</strong></td>
<td><strong>0.026967</strong></td>
<td><strong>-0.01615</strong></td>
</tr>
<tr>
<td><strong>StdDev</strong></td>
<td><strong>0.102552</strong></td>
<td><strong>0.010973</strong></td>
<td><strong>0.010693</strong></td>
</tr>
</tbody>
</table>

Figure 6.3: The results for the parameters of the Heston model for the stable period of the market.

The average of volatility of variance ($\sigma$) is 0.2054, the average for the long-run variance ($\theta$) is 0.26967, and the average of the correlation parameter ($\rho$) is -0.01615.

<table>
<thead>
<tr>
<th>No</th>
<th>sigma</th>
<th>theta</th>
<th>rho</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.415</td>
<td>0.0255</td>
<td>-0.0452</td>
</tr>
<tr>
<td>2</td>
<td>0.484</td>
<td>0.0254</td>
<td>-0.0304</td>
</tr>
<tr>
<td>3</td>
<td>0.344</td>
<td>0.0319</td>
<td>-0.0211</td>
</tr>
<tr>
<td>4</td>
<td>0.45</td>
<td>0.0165</td>
<td>-0.0430</td>
</tr>
<tr>
<td>5</td>
<td>0.344</td>
<td>0.0256</td>
<td>-0.0022</td>
</tr>
<tr>
<td>6</td>
<td>0.145</td>
<td>0.0388</td>
<td>-0.0415</td>
</tr>
<tr>
<td>7</td>
<td>0.72</td>
<td>0.0656</td>
<td>-0.0246</td>
</tr>
<tr>
<td>8</td>
<td>0.326</td>
<td>0.0414</td>
<td>-0.0005</td>
</tr>
<tr>
<td>9</td>
<td>0.244</td>
<td>0.0165</td>
<td>-0.0302</td>
</tr>
<tr>
<td>10</td>
<td>0.162</td>
<td>0.0375</td>
<td>-0.0062</td>
</tr>
<tr>
<td>11</td>
<td>0.72</td>
<td>0.0734</td>
<td>-0.0302</td>
</tr>
<tr>
<td>12</td>
<td>0.415</td>
<td>0.0165</td>
<td>-0.0606</td>
</tr>
<tr>
<td>13</td>
<td>0.344</td>
<td>0.0256</td>
<td>-0.0324</td>
</tr>
<tr>
<td>14</td>
<td>0.45</td>
<td>0.043</td>
<td>-0.0522</td>
</tr>
<tr>
<td>15</td>
<td>0.266</td>
<td>0.0526</td>
<td>-0.0832</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>0.3924</strong></td>
<td><strong>0.038687</strong></td>
<td><strong>0.002799</strong></td>
</tr>
<tr>
<td><strong>StdDev</strong></td>
<td><strong>0.1714</strong></td>
<td><strong>0.022141</strong></td>
<td><strong>0.027571</strong></td>
</tr>
</tbody>
</table>

Figure 6.4: The results for the parameter calibration for the Heston model for the high-volatility period of market.

If we compare each parameter for different periods we will see that the variance volatility and long-run variance for unstable period larger than for the stable
period. But for the correlation parameters we have inverse situation. Of cause the result of utilization the Heston stochastic volatility model during the stable period will be better for the price prediction. As you see from Table 6.3 standard deviation for each parameter which was compute of for the stable period is less than the standard deviation for the same parameter (Table 6.4) for the unstable period on the Nordic market. We can suppose that the exactness of the option prices will be higher.

On the other hand the standard deviation and the parameters error for the stable period is less then for the unstable period, but the difference is not so big. Before we make analysis of the parameters for the Heston model we use methods of the off-line deviation detection to separate our data into different parts. There are also some methods for the on-line deviation detection. This methods can give information about conditions on the market. Our question was about the estimation parameters of Heston model on the unstable period on the market. The division of the data into the small periods is allowed to obtain more exact results. With the proper choice of parameters, the Heston stochastic volatility model appears to be flexible. Thus, this model can be applied to option prices valuation by different conditions on the market.
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[12] F. Javier Giron, Elias Moreno, George Casella  


Appendix

This part of program read OMXS30 Index data from file:

```r
z <- read.table("H:\data.csv",sep="", skip=461)
y <- z[,5]
RealData=c(length(y),NA)
for (i in 1:length(y))
  RealData[i]=y[length(y)-i+1]
DifRealData<-diff(log(RealData))
```

The following function calculate the general moments for the real data:

```r
Qreal<-c(4,NA);
Qreal[1]<-mean(DifRealData);
Qreal[2]<-var(DifRealData);
Qreal[3]<-skewness(DifRealData);
Qreal[4]<-kurtosis(DifRealData);

print("Moments for Real Data:")
print(Qreal);
```

The function which we use for the Heston simulation:

```r
heston<-function(S0,v0,mu,sigma,kappa,theta,rho,N)
{
  S<-matrix(nrow=1,ncol=N);
v<-matrix(nrow=1,ncol=N);
S[1]=S0;
v[1]=v0;
dt=1/N;
x<-rnorm(N,0,1);
}```
\[
y \leftarrow \text{rnorm}(N,0,1);
B1 \leftarrow x;
B2 \leftarrow \rho x + \sqrt{\text{abs}(1-\rho^2)} y;
\]

for (l in c(1:N))
{
  \text{S}[l+1] = \text{S}[l] + \text{S}[l] \left( \mu dt + \sqrt{\text{abs}(\text{v}[l])} \times \sqrt{dt} \times B1[l] \right);
  \text{v}[l+1] = \text{v}[l] + \kappa \times (\theta - \text{v}[l]) \times dt + \sigma \times \sqrt{\text{abs}(\text{v}[l])} \times \sqrt{dt} \times B2[l];
}
\]

To make a comparison between two vectors of additional parameters we minimize the following function:

\[
\text{func} \leftarrow \text{function}(\sigma, \kappaappa, \theta, \rho) \{
\text{InitParam} \leftarrow \text{c}(4, \text{NA});
\text{InitParam}[1] = \sigma;
\text{InitParam}[2] = \theta;
\text{InitParam}[3] = \kappaappa;
\text{InitParam}[4] = \rho;
\text{SimData} \leftarrow \text{heston}(700, 0.2, 0.02, \sigma, \kappaappa, \theta, \rho, 900);
\text{DifSimData} \leftarrow \text{diff}(\log(\text{SimData}))
\text{Qsim} \leftarrow \text{c}(4, \text{NA});
\text{Qsim}[1] \leftarrow \text{mean}(\text{DifSimData});
\text{Qsim}[2] \leftarrow \text{var}(\text{DifSimData});
\text{Qsim}[3] \leftarrow \text{skewness}(\text{DifSimData});
\text{Qsim}[4] \leftarrow \text{kurtosis}(\text{DifSimData});
\]

\[
f = \sqrt{(Qreal[1]-Qsim[1])^2 + (Qreal[2]-Qsim[2])^2 + (Qreal[3]-Qsim[3])^2 + (Qreal[4]-Qsim[4])^2};
\text{print("f"});
\text{print(f);}
f;
\}