Information geometries in black hole physics

Narit Pidokrajt

Department of Physics

Stockholm
University

Thesis for the degree of Doctor of Philosophy in Theoretical Physics
Department of Physics
Stockholm University
Kingdom of Sweden

ISBN: 978-91-7155-916-6

Printed by Universitetsservice US AB, Stockholm

Cover by Isabella Malmnäs

Abstract

In this thesis we aim to develop new perspectives on the statistical mechanics of black holes using an information geometric approach (Ruppeiner and Weinhold geometry). The Ruppeiner metric is defined as a Hessian matrix on a Gibbs surface, and provides a geometric description of thermodynamic systems in equilibrium. This Ruppeiner geometry exhibits physically suggestive features; a flat Ruppeiner metric for systems with no interactions i.e. the ideal gas, and curvature singularities signaling critical behavior(s) of the system. We construct a flatness theorem based on the scaling property of the black holes, which proves to be useful in many cases. Another thermodynamic geometry known as the Weinhold geometry is defined as the Hessian of internal energy and is conformally related to the Ruppeiner metric with the system's temperature as a conformal factor.

We investigate a number of black hole families in various gravity theories. Our findings are briefly summarized as follows: the Reissner-Nordström type, the Einstein-Maxwell-dilaton and BTZ black holes have flat Ruppeiner metrics that can be represented by a unique state space diagram. We conjecture that the state space diagram encodes extremality properties of the black hole solution. The Kerr type black holes have curved Ruppeiner metrics whose curvature singularities are meaningful in five dimensions and higher, signifying the onset of thermodynamic instabilities of the black hole in higher dimensions. All the three-parameter black hole families in our study have non-flat Ruppeiner and Weinhold metrics and their associated curvature singularities occur in the extremal limits. We also study two-dimensional black hole families whose thermodynamic geometries are dependent on parameters that determine the thermodynamics of the black hole in question. The tidal charged black hole which arises in the braneworld gravity is studied. Despite its similarity to the Reissner-Nordström type, its thermodynamic geometries are distinctive.

KEYWORDS: Black holes, Thermodynamics, instability, Hessian, Entropy, Ruppeiner geometry, Weinhold geometry, Information geometry.

To my family, all my physics teachers, and the gentle readers

Vi måste lära känna naturen bättre. Annars kan vi inte lura den.

–Mikael Rode

Abbreviations, conventions and notation

In this thesis we will use natural units where $G = \hbar = c = 1$ unless otherwise stated. The spacetime dimension is denoted by D. The metric signature is $(-, +, \ldots, +)$. The Einstein summation convention is used throughout unless otherwise specified. Greek indices $(\alpha, \beta, \ldots)$ run from 0 to n, where n is the number of spatial dimensions. Abbreviations will be used where appropriate.

Symbol	**Description**
$A_{(\alpha\beta)}$	Symmetrization of $A_{\alpha\beta}$, *i.e.* $\frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$
$A_{[\alpha\beta]}$	Antisymmetrization of $A_{\alpha\beta}$, *i.e.* $\frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$
$\Psi_{,\alpha}$	Partial derivative of Ψ, *i.e.* $\partial_\alpha \Psi$
$A^{\alpha}{}_{;\beta}$	Covariant derivative of A^α, *i.e.* $\nabla_\beta A^\alpha$
$g_{\alpha\beta}$	Metric on manifold $\mathscr{M}$
$\Gamma^{\alpha}{}_{\beta\gamma}$	Christoffel symbol
g	Metric determinant *i.e.* $\det[g_{\alpha\beta}]$
$R_{\alpha\beta\gamma}{}^{\delta}$	Riemann tensor
$R_{\alpha\beta}$	Ricci tensor as constructed from from $R_{\alpha\beta} = R_{\alpha\beta\gamma}{}^{\beta}$
$d\Omega^2$	Line element on unit two-sphere, *i.e.* $d\theta^2 + \sin^2\theta d\varphi^2$
(a)dS	(anti) de Sitter
BH	Black hole
BR	Black ring
D	Number of spacetime dimensions, *i.e.* $D = 1 + n$
GR	General Relativity
KN	Kerr-Newman
RN	Reissner-Nordström
Pl	Planck
QM	Quantum Mechanics
BTZ	Banãdos, Teitelboim and Zanelli (black hole)
Λ	Cosmological constant

Contents

Populärvetenskaplig sammanfattning—a brief summary in Swedish v

Preface vii

Accompanying papers xiii

1 Black hole species—a brief review 1

1.1 Black hole solutions in General Relativity 1
1.1.1 Black hole families in four dimensions 4
1.1.2 Higher-dimensional black holes 6
1.1.3 Kerr black hole *à la* Myers-Perry 9
1.1.4 Black rings 10
1.1.5 Dilaton black holes 10
1.2 Lower-dimensional black holes 12
1.2.1 BTZ black hole 12
1.2.2 Two-dimensional black holes 13
1.3 Tidal charged black holes 14

2 Black hole thermodynamics 17

2.1 The laws of black hole thermodynamics 20
2.2 Black hole in a box 22
2.3 Mass of the black hole 24
2.4 Black hole thermodynamics and scale-invariant gravitation 26
2.5 Derivation of black hole fundamental relations 27
2.5.1 Myers-Perry black hole 27
2.5.2 Dilaton black hole 31
2.5.3 BTZ black hole 31

3 Black hole information geometry 33

3.1 Thermodynamic geometry 35

3.2 Application of information geometry to black hole thermodynamics 40
3.3 Flatness theorem 42
3.3.1 Black hole examples 44

4 Results and discussions **47**
4.1 General relativity black holes 47
4.1.1 Reissner-Nordström black hole 47
4.1.2 Reissner-Nordström—adS black hole 49
4.1.3 Kerr black hole 49
4.1.4 Kerr-Newman black hole 50
4.2 Lower-dimensional black holes 51
4.2.1 BTZ black hole 51
4.2.2 Two-dimensional black holes 51
4.3 Myers-Perry black holes 53
4.3.1 Reissner-Nordström black hole 53
4.3.2 Kerr black hole 55
4.3.3 Kerr black hole with double spins 58
4.4 Dilaton black holes 58
4.4.1 Dilaton black hole with a unit coupling constant 59
4.4.2 Dilaton black hole with arbitrary coupling constant 61
4.5 Tidal charged black holes 62
4.5.1 Thermodynamic geometries of the tidal charged black hole 63
4.5.2 The global structure of the Ruppeiner geometry 64
4.6 State space graphics 67

5 Summary and outlook **69**

Appendices **71**

Appendix A Killing vectors, Killing horizons and surface gravity **71**

Appendix B Dilaton black holes—a transformation between Einstein and string frames **72**

Appendix C Proof of the conformal relation between Ruppeiner and Weinhold metrics **73**

Appendix D Degeneracy of Ruppeiner metric **74**

Index **87**

About the author **89**

Populärvetenskaplig sammanfattning—a brief summary in Swedish

Denna avhandling behandlar frågan om svarta håls fysik på ett teoretiskt sätt. Den sammanfattar mitt arbete under de senaste fem åren. De svarta hålens fysik baserar sig på den allmänna relativitetsteorin som är den mest accepterade teorin för gravitationen. Svarta hål har verifierats observationellt indirekt men det finns fortfarande många problem särskilt av teoretisk art såsom vad de svarta hålens entropi kommer från eller var inom det svarta hålet entropin finns. Svarta håls termodynamik, grundades för cirka 40 år sedan av Stephen W. Hawking med flera och det blev en överraskning för de flesta fysiker att sådana klassiska objekt (enligt relativitetsteorin vilken är en klassisk teori) kunde stråla termiskt där det förväntas bli helt kallt (noll Kelvin). Denna strålning från det svarta hålet kallas "Hawkingstrålning" och denna strålning är ointressant för de astrofysiska svarta hålen men det är av stor betydelse för de små svarta hålen som kanske kommer upptäckas vid världens största partikelfysiklaboratorium (CERN) i Schweiz/Frankrike. Om Hawkingstrålningen kommer att upptäckas då vi vet att *kvantgravitation* är närmare att bli erkänd och det är mycket troligt att Hawking kommer mottaga Nobelpriset i fysik i Stockholm.

Mitt arbete i denna avhandling behandlar inte kvantgravitationen direkt utan att öppna nya perspektiv på samspelet mellan svarta håls termodynamik och svarta håls statistiska mekanik som, om de förstås, möjligtvis kan bana väg till den rätta teorin för statistisk mekanik för svarta hål vilket krävs för vi ska ha en fullständig teori för kvantgravitation. Häri använder vi en geometrisk metod för att utforska svarta håls termodynamik och har fått nya originella resultat, dvs vi utnyttjar *informationsgeometri* (i synnerhet Ruppeiner- och Weinholdgeometrier) för att studera svarta håls termiska egenskaper. Vi har använt den geometriska metoden att utforska svarta hål i olika gravitationsteorier t. ex. de svarta hålen enligt Reissner-Nordström och Kerr i fyra dimensioner samt Myers-Perrys svarta hål som finns när man generaliserar den allmänna relativitetsteorin till högre dimensioner. Vi har också studerat svarta hål i tre dimensioner som kallas BTZ svarta hål, samt dilatoniska svarta hål vilka är intressanta från supersträngteoriernas perspektiv.

Vi sammanfattar våra resultat på följande sätt: Ruppeinermetriken för Kerrs svarta hål är krökt, medan det är platt för Reissner-Nordström i alla rumtidsdimensioner (de tillhör Myers-Perrys klass). Det så kallade BTZ svarta hålet har en platt Ruppeinermetrik vilket är ett förvånande faktum. Det dilatoniska svarta hålet har en platt Ruppeinermetrik men dess termodynamiska egenskaper skiljer sig från Reissner-Nordströms svarta hål. Geometriska mönster kan sammanfattas genom att rita ett diagram för tillståndsrummet. Det tillåter oss att dra slutsatser om de svarta hålens extremalitetsegenskaper. Det dock mest relevanta resultatet vi har fått är att förutse den *termodynamiska instabiliteten* i Myers-Perrys svarta hål från Ruppeinergeometrin. Informationen om sådan instabilitet är *kodad i* krökningsskalären för Ruppeinermetriken.

Vi tror att de resultaten kommer att bli användbara i andra sammanhang av svarthålsfysiken eftersom svarta hål numera utforskas inte bara av gravitationsfysiker utan också av kondenserade materiens fysiker och även av kvantinformationsfysiker. Förhoppningen är att de geometriska mönster vi upptäckt kommer att bli betydande när kvantgravitationen är erkänd.

Preface

I never see what has been done; I only see what remains to be done.

–Buddha

This is a PhD thesis[1] and it contains what I have been doing in the past four and a half years. It has been a rather long period of time given the human's life span but in my mind those years simply vanished. This is certainly due to many things that happened during these years. The PhD program has profoundly influenced my perspectives, visions and thoughts about life as a whole. It has also allowed me to be in new places and environments in which I could learn not only physics but also cultures, peoples and various ways of life. I met many great thinkers[2] who motivated and inspired me in a number of ways. Half way through my PhD program several things happened that made me pause and think about other things I could possibly do in my life than physics but nothing was stronger than the gravitational pull of black holes so I stayed on my trajectory and this manuscript is a result of my increasingly strong will to follow this trajectory even farther.

As I always tell my friends and colleagues, black holes are certainly some of the most exotic entities encountered in physics of the present time. I have always been astonished by their existence. The verbal definition of the black hole is quite straightforward: *it is a region of spacetime surrounded by a boundary known as the event horizon inside which the force of gravity is so strong that not even light can escape, hence it is invisible*. The mathematical definition is not as simple but straightforward as we expect mathematics to be so, and we will discuss and develop mathematics of black holes as we carry on in this manuscript. The history of black hole physics began in 1784 when John Michell, an English clergyman, discussed classical bodies which have escape velocities greater than the speed of light. The first scientist who discussed this problem was Pierre Laplace in 1795 when he derived the gravitational radius using Newtonian gravity. However serious and systematic research in black hole physics might

[1]It is supposed to be advanced enough that lay people may not understand the main part of the thesis. So if you do not know physics at a university level and only wish to know the summarized main ideas and outcomes, please read only this section (you will actually learn a lot about history of black hole physics and this section will contain only one single equation) and skip the rest or just browse as you like.

[2]One of them was Roy P. Kerr who I really had time to chat with. He was born on the same date as me but 45 years earlier.

not have been as it is today had Karl Schwarzschild not been able solve the Einstein field equation in a vacuum for uncharged spherically symmetric systems shortly after Albert Einstein founded the subject of General Relativity in 1915.

An electro-vacuum solution was found later on by H. Reissner and G. Nordström in 1918 which is the solution of an electrically charged spherically symmetric black hole known as the Reissner-Nordström black hole. In 1920 Jørg Tofte Jebsen[3], an unknown Norwegian physicist from Oslo was the first to discover that the Schwarzschild solution is the unique spherically symmetric solution to the Einstein field equation in a vacuum. Later in 1923 George D. Birkhoff established the same theorem (now known as Birkhoff's theorem) which states that the static Schwarzschild metric is the unique solution outside any mass distribution, even when this varies with time as long as the spherical symmetry is maintained[4]. In 1939 the gravitational collapse of a massive star which produces a black hole was first described by Oppenheimer and Snyder.

To give a bit more of history, Schwarzschild himself did not realize that his solution was a black hole solution and this mathematical solution was lurking around in the mind of physicists until 1960s when this topic picked greater attention by both Western and Soviet physicists. It was the late John A. Wheeler who coined the term black hole[5]. Wheeler passed away in 2008 at age 96. The uncharged rotating solution was found in 1963 by Roy P. Kerr, a New Zealander mathematician, which drove black hole physics into a serious research field. It can be said that in the last 40 years or so the field of black hole physics has become a serious business and a large number of physicists make their living on this. There are by now many subfields within black hole physics ranging from gravitational waves research to numerical simulations to black hole thermodynamics to pure mathematical studies of black hole solutions. Furthermore black hole physics is a subject of interest not only to relativists but also cosmologists, astrophysicists, string theorists, mathematicians and even some condensed matter physicists.

Like most areas of physics there remain puzzles and unsolved/open issues in black hole physics. To list a few, the black hole information loss paradox[6], the cosmic censorship conjecture by Penrose and the one related to my research—the origin of the black hole's entropy. It is still far from clear what it is like inside the black hole, let alone the statistical mechanics of this object. We have thermodynamics[7] of black holes thanks to Carter, Bardeen, Bekenstein and Hawking who founded the subject. However it would not be considered valid to the full unless one understands the microscopic pictures of them in a consistent manner. The lack of statistical

[3]Jebsen's publication on the uniqueness proof was indeed the first publication in the field of GR from Sweden. The examiner of his work was C. W. Oseen, a physics professor at Uppsala University who was the pioneer lecturer in GR in Sweden.

[4]In astrophysics, the spherical collapse of the star cannot result in any emission of gravitational radiation.

[5]Black holes were known for some time as dead stars or frozen stars.

[6]which may not be a problem anymore if one takes Hawking's confession in Dublin in 2004 to the full.

[7]Thermodynamics is a macroscopic theory which deals with *e.g.* how energy transfers for a given system. For ordinary systems there are also microscopic pictures which substantiate the thermodynamic pictures, namely the subject known as statistical mechanics.

mechanics of black holes is largely due to the lack of quantum physics[8] of black holes which is, in my view, not going to be solidly established in the near future.

What I have been actively involved in during past five years is the study of black hole thermodynamics using a new approach, namely *information geometry*. More specifically I employ thermodynamic geometry known as *Ruppeiner geometry* (let's denote it by g_{ij}^R) to investigate certain thermal properties of the black holes. The Ruppeiner geometry has a counterpart known as the *Weinhold geometry*, denoted by g_{ij}^W. The two geometries are conformally related with a conformal factor involving the black hole's temperature, T, as follows

$$g_{ij}^R = \frac{1}{T} g_{ij}^W .$$

It has been shown by various groups of scientists that this geometry encodes certain pieces of information. For an ideal gas, the Ruppeiner/Weinhold geometry is flat under some conditions but they are nonflat for systems with underlying statistical mechanical interactions. The curvature singularites (where the curvatures diverge) signals critical phenomena such as phase transitions. I have been studying black hole thermodynamics using this approach and have obtained satisfactory outcomes including a prediction of thermodynamic stability of the Kerr black holes in higher dimensions.

Readers who are not familiar with the concept of higher dimensional physics should not panic as we only study physics in such dimensions theoretically. In short it is when the physical space is higher than 3D which we are used to. Curiously I also go down to a lower-dimensional world in which the physical space is less than three. It may be slightly easier to cope with lower dimensions in terms of doing computations. Simply put, in the flatland (3D) and the lineland (2D) we are able to do several things which cannot be done in the standard (3+1) dimensions.

Since quantum theory of gravity is still in murky water, we hope to test new ideas which may give rise to some new insightful perspectives in black hole physics. The *information geometric approach* is one of the newest ideas in black hole physics.

Stockholm, Kingdom of Sweden
August 20, 2009

[8] This subject is believed to either emerge from string theory or quantum gravity which is still under development. Although attempts to reconcile quantum mechanics and GR began already in 1930s we are still far from a complete framework that combines both theories. The task of combining GR and quantum mechanics remains one of the outstanding problems of theoretical physics.

Acknowledgments

I cannot overemphasize how important it has been for me since I began my PhD program to have received support, contributions and assistance from many people and various organizations. First and foremost, I wish to express my gratitude to my parents my brother teerak for their unconditional love and support. Without my parents's prudent plan I might not have able to come this far. I wish I will be able to give them everything they need by all means.

I acknowledge Professor Ingemar Bengtsson for his work as a supervisor for a number of years. I would like to thank Ingemar for teaching me physics and mathematics needed to do research in black holes physics, for some financial support to GRSweden meetings and some other events in Europe. I would also like to thank him for pointing out some good old books, papers, *a brief history of time* . . . and especially for his intensive proofreading of this thesis.

Professor Jan E. Åman has been a fantastic collaborator, office mate and an indispensable adviser to me in so many situations and events. Without Jan I dare say that I would miss out some good details in the papers and this thesis. He is one of the best navigators roaming the Earth. Jan is well known by relativists worldwide as many people I met at various conferences asked about him. Now they know that he is not only the master of CLASSI but also a black hole physicist. I can safely say that there is no other physicist that knows about trains better than he does. I hope there will be a Maglev connection between Stockholm and Bangkok during your lifetime.

I would like to express "spasibo!" to Professor Aleksandr "Sasha" Zheltukhin for sharing with me his wisdoms, creative ideas, and all the knowledge about the world he has experienced through many episodes. It is better than any history book. Conversations with Sasha are inspiring and at times exciting. I hope to learn more about his SUSY technologies so that I can write a paper with him one day. Also I wish on Sasha's behalf that LHC will detect SUSY in the near future. The SUSY discovery may not be so certain but one certainty is that: *I will certainly miss my office A5:1085*.

It is an honor for me be part of the KoF (Quantum and Field Theory) research group, where I have pretty much grown up into a physicist from just a student. I thank all the group's people from the faculty members to the master students of the present and the past. People on the third floor are very creative and friendly and I really enjoy chatting with all of them on physics, the world affairs as well as the latest technologies (which they know much better than I do). Thanks go to Kate Blanchfield for kindly correcting my English in the thesis and elsewhere. I would also like to extend my acknowledgments to some members in the CoPS group especially Professor Fawad Hassan (my mentor) for sharing his deep physics knowledge and interesting information with me. Fysikum (the Physics Department) has been a wonderful employer for me simply due to the wonderful people it has. I thank Professor Kjell Fransson, the present director of the PhD program, for answering all my tedious questions, and I would like to acknowledge the former director of the graduate program Professor Ove Appelblad (who may never read this thesis) for his helpful guidelines regarding the PhD program at Fysikum. I thank also the present staff members in the administrative corridor especially Mona Holgerstrand for assisting me with the money issues and also for her friendliness and smiles. I thank Elisabet Oppenheimer for her support with the paper work and her clear answers to my annoying questions.

I thank all my collaborators, first and foremost Dr. John Ward of Victoria University for his friendship and our fruitful collaborations. I appreciate his encouragement and all our email correspondences. I wish to thank my other collaborators on the gravity front both in the past and present: Professor László Á. Gergely in Szeged, Dr. Daniel Grumiller in Vienna, Dr. Niko Jokela in Haifa, Dr. Tolga Birkandan in Istanbul, Dr. Stefan Åminneborg in Stockholm, and Dr. Petarpa Boonserm in Wellington. Dr. Burin Gumjudpai of Naresuan University in Thailand is highly appreciated for the sincere friendship and moral support.

I would like to thank Professor Roberto Emparan for stimulating and encouraging discussions, and importantly the kind hospitality of the Departament de Física Fonamental while I was a visitor there for about a month in 2008, and I still recall every moment there vividly. I would also like to thank the local people there for friendliness and support, *i.e.* Alba Gutierrez Pedemonte and Christian Teijon. Hopefully I will be in Barcelona again soon. I would like to thank Professor Michael Bradley of Umeå University for his help with my one-week visit to the University of Cape Town, South Africa. I acknowledge Professor Peter Dunsby for his kind hospitality and for showing me the city of Cape Town.

In the past three years I have been to several physics departments and research institutes, and my visits would not have been so nice and fruitful without kind hospitalities of the following organizations: The Spanish Relativity Society (SEGRE), Nordita, Niels Bohr Institute, Helsinki Institute of Physics, Departament Física Fonamental at Universitat de Barcelona, The Cosmology Group at University of Cape Town, TPTP at Naresuan University, Cosmology Group at Chulalongkorn University, Department of Theoretical Physics at Vienna University of Technology and The Applied Mathematics Department at Linköping University. I would also like to acknowledge financial support from the following foundations: Helge Ax:son Johnsons Stiftelse, K & A Wallenberg Stiftelse, John Söderberg Stiftelse and Liljevach resestipendium. Without their support I might not have been able to have the experience mentioned above.

Last but not least, I would like to thank all my friends. Just to name a few: Diego, Thomas, Surachate, and Isabella. I admire our friendships and every support and comfort you have provided which is certainly important for me. Without all the good friends my life would have been half good.

List of accompanying papers and author's contributions

Papers included in this thesis:

1. Jan E. Åman, Ingemar Bengtsson and Narit Pidokrajt, *Geometry of black hole thermodynamics*, Gen. Rel. Grav. **35** 1733 (2003), [arXiv: gr-qc/0304015]

   The project was suggested by Professor Bengtsson. I found that the Ruppeiner metric of the Reissner-Nordström black hole is flat and the Weinhold metric is nonflat. I was the first to obtain the curvature scalar for the Ruppeiner metric of the Kerr black hole. Professor Bengtsson did the writing of the paper. Professor Åman joined as the project was about half-way done.

2. Jan E. Åman and Narit Pidokrajt, *Geometry of higher-dimensional black hole thermodynamics*, Phys. Rev. **D73** 024017 (2006), [arXiv: hep-th/0510139]

   As I wished to continue with black hole thermodynamic geometry projects, Professor Bengtsson suggested that I compute the Ruppeiner metric for the Reissner-Nordström and Kerr black holes in 5D which I successfully did. Professor Åman became a vital collaborator as we extended the work to the full. We applied this approach to Myers-Perry black holes. Professor Åman was first to obtain the Weinhold metric for MP black holes in arbitrary D. I wrote the entire paper.

3. Jan E. Åman, Ingemar Bengtsson and Narit Pidokrajt, *Flat information geometries in black hole thermodynamics*, Gen. Rel. Grav. **38**, 1305 (2006), [arXiv: gr-qc/060119]

   I had some idea while working on the paper 5 that there were seemingly patterns that flat thermodynamic geometries were not just random. Professor Bengtsson came up with a concrete way to formulate the problem. I double checked the calculations. Professor Bengtsson wrote the paper.

4. Jan E. Åman, James Bedford, Daniel Grumiller, Narit Pidokrajt and John Ward, *Ruppeiner theory of black hole thermodynamics*, J. Phys.: Conf. Ser. **66** 012007 (2007), [arXiv:gr-qc/0611119]

   Dr. Grumiller and I discussed the possibility of applying thermodynamic geometry to black holes in 2D during the MG11 meeting in Berlin. Dr. Grumiller first calculated thermodynamic geometries for the 2D black holes and I double checked the calculations and added an extra section on BTZ. Professor Åman, Dr. Ward, Dr. Bedford joined forces as we attempted to interpret our findings. I wrote most parts of the paper.

5. Jan E. Åman, Narit Pidokrajt and John Ward, *On Geometro-thermodynamics of dilaton black holes*, EAS Publications Series, 30 (2008) 279, [arXiv:0711.2201]

It was initially and originally my idea that we should study thermodynamic geometries of black hole solutions outside Einstein gravity. Dr. Ward who is a string theorist and Professor Åman agreed to tackle the calculations with me. I derived the entropy formula for the dilaton black hole with arbitrary coupling and did find that the state space fills the entire thermodynamic light cone. Professor Bengtsson gave useful suggestions.

6. László Á. Gergely, Narit Pidokrajt and Sergei Winitzki, *Thermodynamics of tidal charged black holes*, (2008), arXiv:0811.1548 [gr-qc]

   Professor Gergely and I met in February 2008 and we decided to pursue the project after stimulating discussions. I did most of the calculations and finalized the paper. We decided to use plots made by Professor Gergely's program but they all agree with mine. I drew the state space plot. Dr. Winitzki joined the project in the very last stage contributing to the proof of Ruppeiner/Weinhold conformal relation in the appendix.

Papers not included in this thesis:

1. Jan E. Åman, Ingemar Bengtsson, Narit Pidokrajt and John Ward, *Thermodynamic geometries of black holes*, Proceedings of the Eleventh Marcel Grossmann Meeting on General Relativity, ed. Kleinert, Jantzen and Ruffini, World Scientific (2008) 1511.

2. Jan E. Åman and Narit Pidokrajt, *Ruppeiner Geometry of Black Hole Thermodynamics*, EAS Publications Series, 30 (2008) 269, [arXiv:0801.0016]

3. Jan E. Åman, Stefan Åminneborg, Ingemar Bengtsson and Narit Pidokrajt, *Anti-de Sitter Quotients, Bubbles of Nothing, and Black Holes*, Gen. Rel. Grav. **40**, 2557 (2008), [arXiv:0801.4214]

4. J. E. Åman, N. Pidokrajt and J. Ward, *Information geometry of asymptotically AdS black holes*, AIP Conf. Proc. **1122**, 181 (2009)

Part I:

Background

Chapter 1
Black hole species—a brief review

It is sometimes said that if naked singularities do occur, then this would be disastrous for physics. I do not share this view.

–R. Penrose

In this chapter we will discuss black hole families (solutions) concisely and it is anticipated that our discussions will be a proper background for black hole thermodynamics to be discussed in the next chapter. As promised in the preface we will define a black hole in a more rigorous manner, namely we must define the concept of *event horizon* which is the most important feature of the black hole. An event horizon is a hypersurface separating the spacetime points that are connected to infinity by a timelike path from those that are not. The event horizon constitutes the boundary of the black hole which separates the black hole from the outside universe. We will develop an understanding of this concept as we proceed with black hole solutions in this chapter.

1.1 Black hole solutions in General Relativity

The first and the most familiar black hole solution of the Einstein equation in vacuum is the Schwarzschild solution [1] named after Karl Schwarzschild who discovered it already in 1916. This solution is static and spherically symmetric (for a good review see *e.g.* [3])

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{1.1}$$

where M denotes the asymptotic mass of the black hole, $d\Omega_2^2$ is the metric on the unit two-sphere, *viz.* $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ with the following coordinate ranges: $t \in (-\infty, \infty)$, $r \in (0, \infty)$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. At the origin $r = 0$ there is a curvature singularity as may be verified by computing the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. The black hole's event horizon is located at $r = 2M$ which is also a coordinate singularity (clearly not the black hole's singularity) of the metric but can be shifted away using a (global) coordinate system *e.g.* the *Kruskal-Szekeres coordinates*, which are coordinates that cover the entire spacetime manifold of the maximally extended Schwarzschild solution and are well-behaved everywhere outside the physical singularity. The Schwarzschild black hole is often used as a model for spherical gravitational collapse. By analyzing the radial light rays (θ and ϕ constant and $ds^2 = 0$) we find that $\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)$ tends to zero as it approaches the region $r = 2M$, see Fig. (1.1). Inside the event horizon, where $r < 2M$ the future light cones point inward, toward $r = 0$ which is the black hole's singularity described above. Since particles and photons propagate within or on the light cones, they cannot escape from such a region. For the Earth to become a black hole the collapse would have to reach its Schwarzschild radius of approximately 0.88 cm.

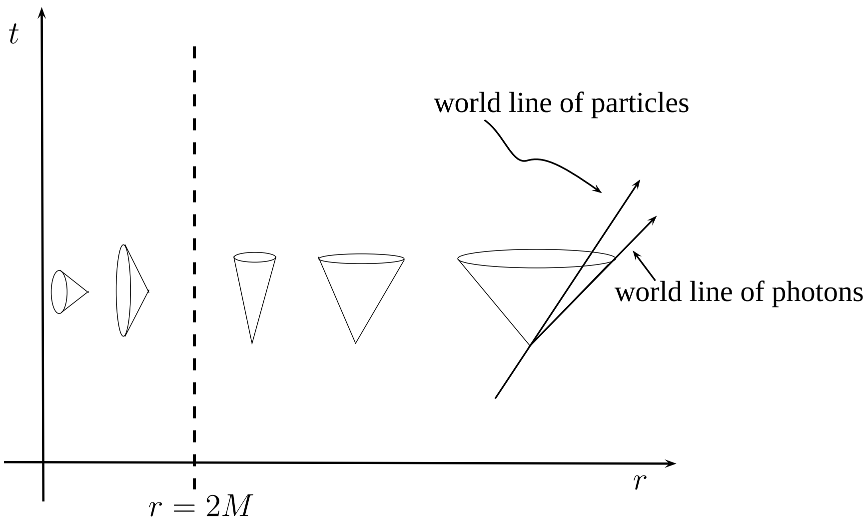

Figure 1.1: *Future light cones in Schwarzschild coordinates outside, near and inside the region $r = 2M$, the event horizon. In this diagram the light cones have slope ± 1 far from the event horizon, but their slope approaches $\pm\infty$ as $r \to 2M$. It is easy to see from the Schwarzschild metric that the t and r coordinates swap character in the region $r < 2M$.*

However the figure above does not capture what happens at infinity, namely in an asymptotic region of the black hole, where the spacetime becomes Minkowskian. Studying the global structure of spacetime—in particular the curved ones—can be very difficult. It is useful to adopt the conformal diagram (often referred to as *Penrose-Carter* or just *Penrose* diagrams) in which infinities are brought to a finite distance, *viz.* the boundary of the diagram, known as *scri* (pronounced "scri" due to the fact that it is written as a script "I"). In the Penrose diagram light rays travel at $45°$. The Penrose diagram of the Minkowski space is depicted in Fig. 1.2, and a few things can be read off as follows: $\mathscr{I}^+$(*future null infinity*) is where the light rays end, and $\mathscr{I}^-$is the *past null infinity*. All timelike geodesics in the Penrose-Carter diagram begin at the point i^- (referred to as *past timelike infinity*) and end at i^+ which is called *future timelike infinity*. Nongeodesic timelike curves that end at null infinity are asymptotically null. The symbol i^0 refers to the *spatial infinity* at which all the spacelike geodesics end.

Now that we have introduced the Penrose diagram, we can refine the definition of the black

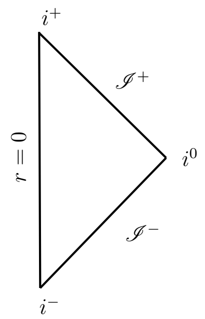

Figure 1.2: *Minkowski space à la Penrose. Note the following symbols: $\mathscr{I}^+$is the future null infinity, $\mathscr{I}^-$is the past null infinity, i^+ is future timelike infinity, i^- is the past timelike infinity and i^0 is the spatial infinity, and they are just points in the diagram.*

hole as follows: A black hole in asymptotically flat spacetime is defined as a region such that no causal signal[1] from it can reach $\mathscr{I}^+$. In the Schwarzschild spacetime, the Killing[2] vector $\xi = \partial_t$ goes from being timelike to spacelike at the event horizon. If a Killing vector field is null along some null hypersurface Σ, then Σ is a Killing horizon of ξ^μ. Note that the Schwarzschild solution is time translation invariant for $r > 2M$. The Schwarzchild black hole at $r = \infty$ has the Minkowskian causal structure. The Schwarzschild solution is a one-parameter family because it is characterized only by its mass M. There are several black hole solutions in GR which are exact solutions (for a comprehensive review of exact solutions to Einstein's equations, see [4]).

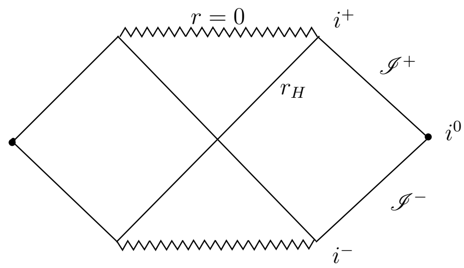

Figure 1.3: *A Penrose diagram for the Schwarzschild black hole spacetime. Zigzag lines ($r = 0$) are the black hole's singularity, r_H denotes the event horizon. Note that in the asymptotical limit it has the same causal structure as the Minkowski spacetime. Note that in the Penrose diagram each point corresponds to a two-sphere.*

---

[1] *i.e.* a signal propagating at velocity not faster than the speed of light.

[2] A vector field on a Riemannian manifold (or pseudo-Riemannian manifold) that preserves the metric is called a Killing vector. Killing fields are the infinitesimal generators of isometries. In a nutshell, if you move along the direction of a Killing vector, then the metric does not change. The Killing vectors satisfy $\nabla_{(\mu}\xi_{\nu)} = 0$. For more discussion, see *e.g.* [2].

1.1.1 Black hole families in four dimensions

The solution of the Einstein equation in electro-vacuum spacetime with imposed spherical symmetry is the black hole solution known as the *Reissner-Nordström* black hole solution [5]. It comes about by solving the Einstein equations coupled to the source-free Maxwell's equations, *i.e.*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \tag{1.2}$$

where the energy-momentum tensor of the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda}, \tag{1.3}$$

with $\nabla_\mu F^{\mu\nu} = 0$ and $\partial_{[\mu} F_{\nu\lambda]} = 0$. The RN black hole metric is given by

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2, \tag{1.4}$$

where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. M and Q are ADM mass and charge of the RN black hole respectively. The RN metric has a curvature singularity at $r = 0$. Note that the Ricci scalar of this spacetime is vanishing, *i.e.* $R = g_{\mu\nu} R^{\mu\nu} = 0$. This is due to the fact that $T_{\mu\nu}$ is traceless. The electromagnetic fields associated with this solution are given by

$$E_r = F_{rt} = \frac{Q}{4\pi r^2}, \tag{1.5}$$

which comes from solving the Maxwell equations in vacuum. It is obvious from (1.5) that Q is the ADM charge.[3] We can verify it by showing that the integration of the electric field over the two-sphere at infinity gives rise to an electric charge, *i.e.*

$$Q = -\lim_{r\to\infty} \int_{S^2} d\theta \, d\varphi \, r^2 \, \sin\theta \, E^r. \tag{1.6}$$

The event horizon of the RN black hole is obtained by solving for r from the equation $f(r) = 0$ whose solution is given by the charge of the black hole.

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{1.7}$$

which is where the Killing vector becomes null. There are two roots r_+ which is the outer horizon and $r = r_-$ which is a hypersurface known as the Cauchy horizon [7]. The function $f(r)$ might have one, two or no real zeroes depending on the relative values of M and Q as follows:

[3] It is worth pointing out that the ADM (Arnowitt-Deser-Misner) conserved charge [6] is a surface integral evaluted at spatial infinity, which is used when one considers asymptotically flat spacetime. In other words, the ADM mass is a component of the four-momentum of asymptotically flat manifolds. The ADM energy is a component of the ADM four-momentum. If the black hole is static, the ADM mass is identical to the ADM energy. The ADM charge is also given by a surface integral.

- $\underline{M^2 < Q^2}$ This is when we have a naked singularity [8] which is unphysical from the cosmic censorship conjecture [9]. The RN black hole in this case would not be formed by any gravitational collapse.

- $\underline{M^2 = Q^2}$ We have the so-called extremal RN black hole. For astrophysical or macroscopic black holes this seems to be an unlikely situation. However the extremal RN solution is interesting because it has a supersymmetry.

- $\underline{M^2 > Q^2}$ This is a case expected in realistic gravitational collapse but the charge Q would be very small on an astrophysical scale. The surfaces defined by $r_\pm$ are null and they are outer (r_+) and inner (r_-) horizons. However the singularity at $r = 0$ is timelike, not spacelike surface as in the Schwarzschild solution.

Black holes can also rotate. The uncharged rotating black hole is known as the *Kerr* black hole [10]. The black hole that is electrically charged and rotating is named the *Kerr-Newman* black hole [11, 12]. The metrics describing the Kerr and Kerr-Newman black holes are more complicated than the RN and Schwarzschild metrics. The gravitational field of the Kerr black hole with nonzero angular momentum is described by an axisymmetric solution which, in Boyer-Lindquist coordinates, takes the form

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2} dt\, d\varphi \\ &\quad + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \end{aligned} \tag{1.8}$$

where $a = J/M$ is angular momentum per unit mass and

$$\Delta = r^2 - 2Mr + a^2, \tag{1.9}$$

$$\rho^2 = r^2 + a^2\cos^2\theta. \tag{1.10}$$

Its event horizons are (assuming that $a^2 < M^2$)

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{1.11}$$

The Kerr metric or Kerr solution is stationary (but not static) and axially symmetric and has two horizons *i.e.* outer and inner horizons. In between the event horizon and the static limit[4] lies the so-called *ergosphere* inside which nothing can remain stationary. The area of the event horizon of the Kerr black hole is given by $A = 4\pi r_+^2$. The Kerr black hole family is of most relevance to the real world as it has been confirmed that there are some black holes *e.g.* in the center of the Milky way which are near-extremal Kerr black holes. Note also that the Kerr black

[4]The static limit is the surface of the ergosphere where $\xi^\mu(t)$ becomes null.

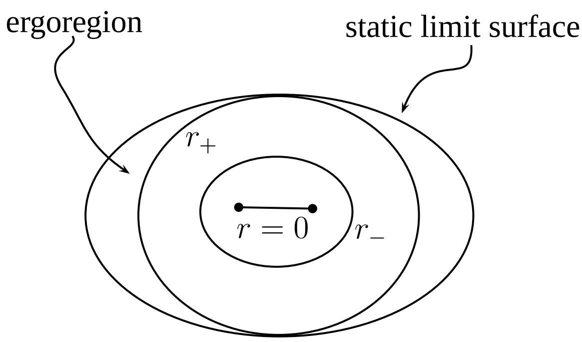

Figure 1.4: *Horizon structure around the Kerr solution (side view). Note that $r = 0$ is not a point in space but a disk and it is sometimes called ring singularity because it spreads out over the ring.*

hole solution is meaningful when $J \leqslant M^2$. The solution becomes extremal when $J = M^2$ and such a black hole solution is called an extremal Kerr black hole. It is readily seen that when $a = 0$ we recover the Schwarzschild metric.

The black holes we have discussed so far are asymptotically flat black holes in GR (assuming the cosmological constant, Λ, is zero). However they also exist in the background where $\Lambda \neq 0$. Black hole solutions in the background in which $\Lambda < 0$ are called *Anti de Sitter* (AdS) black holes which are relevant to string theory[5], whereas the black hole solutions in the background in which $\Lambda > 0$ are called *de Sitter* (dS) black holes, which are relevant to cosmology.

1.1.2 Higher-dimensional black holes

The idea of higher dimensions dates back to the work of Kaluza [15] and Klein [16] in which the tiny extra dimensions are compactified and can be probed only with very high energy. However it was not until 1963 that black holes were studied in higher dimensions, *i.e.* when Tangherlini [17] was able to obtain a vacuum solution of the Einstein equation in higher dimensions, essentially the Schwarzschild black hole in arbitrary dimension. In 1986 Myers and Perry [18] found asymptotically flat black hole solutions in an arbitrary number of spacetime dimensions which we from now on refer to as the *Myers-Perry black hole*. In recent years the black hole solutions have been studied extensively and some new discoveries were made, an important one being the *black ring* solution. We can safely say that GR in more than four spacetime dimensions has been the subject of constantly increasing attention. There are various reasons why

[5] The context in which the AdS space is very interesting to string theorists is known as the *AdS/CFT correspondence* [14] (Anti-de-Sitter space/Conformal Field Theory correspondence), *a.k.a. Maldacena duality*. This is the conjectured equivalence between a string theory defined on one space (5D gravity), and a quantum field theory without gravity (say, supersymmetric Yang-Mills theory) defined on the conformal boundary of this space, whose dimension is lower by one or more. This conjecture comes out of the holographic principle of string theory in that the Yang-Mills theory can be thought of as a hologram on the boundary of the 5D space where gravity takes place. and that the quantum field theory is a conformal field theory (CFT).

we should study Einstein's gravity theory in higher dimensions, and in particular its black hole solutions. I would like to quote Emparan and Reall [20]:

1. *String theory contains gravity and requires more than four dimensions. In fact, the first successful statistical counting of black hole entropy in string theory was performed for a five-dimensional black hole. This example provides the best laboratory for the microscopic string theory of black holes.*

2. *The AdS/CFT correspondence relates the properties of a d-dimensional black hole with those of a quantum field theory in $(d-1)$ dimensions.*

3. *The production of higher-dimensional black holes in future colliders becomes a conceivable possibility in scenarios involving large extra dimensions and TeV-scale gravity*[6].

4. *As mathematical objects, black hole spacetimes are among the most important Lorentzian Ricci-flat manifolds in any dimension.*

As a matter of fact the higher-dimensional black holes are richer than their counterparts in 4D due to more rotational dynamics and the appearance of extended black objects. It is also worth mentioning that gravity is more difficult in $D > 4$ due to the larger number of degrees of freedom, and there are issues of black hole instabilities which are absent for 4D black holes. In brief, the physics of higher-dimensional black holes can be uniquely different and richer than in four dimensions. A number of reviews on the subject are available, see *e.g.* [20–22]. The electrically charged non-rotating black hole solution in arbitrary dimension in a background with a generalized cosmological constant is obtained by solving for the field equation from the Einstein-Maxwell action

$$S = -\frac{1}{16\pi G}\int_{\mathscr{M}} d^D x\sqrt{-g}\left(R - 2\lambda - F^2\right), \tag{1.12}$$

where we have defined

$$\lambda = \frac{(D-1)(D-2)\Lambda}{6}. \tag{1.13}$$

G is Newton's universal gravitational constant, g a determinant of the metric tensor and the negative sign under the square root is there to prevent it from being imaginary due to the metric signature. R is the Ricci scalar whereas F^2 represents the modulus of the Maxwell field-strength tensor $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, where A_ν is the electromagnetic vector potential. The field equation derived from (1.12) gives the Einstein-Maxwell black hole solution with cosmological constant in the following form

$$G_{\mu\nu} + \frac{(D-1)(D-2)\Lambda}{6}g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.14}$$

[6]At the moment there is no experimental data available at the TeV scale but facilities are being set up at CERN (European Organization for Nuclear Research). Updated and detailed information can be acquired from `www.cern.ch`.

The field equation for the Maxwell field takes the same form as (1.3). However in arbitrary dimension the energy-momentum tensor is not traceless. The contraction of $T_{\mu\nu}$ with the contravariant metric tensor gives

$$T = -(D-4)F^2, \tag{1.15}$$

which obviously vanishes in the ordinary spacetime dimensions. In this thesis we will only deal with the higher-dimensional black holes in a background without Λ, in which the most general static solution with spherical symmetry is given by

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{(D-2)}, \tag{1.16}$$

where r is a radial coordinate, $d\Omega^2_{(D-2)}$ is the line element of a unit $(D-2)$-sphere whose area is given by

$$\Omega_{(D-2)} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}, \tag{1.17}$$

where Γ is the Gamma function [19]. This solution is indeed the RN black hole solution in higher dimensional spacetime. The function $f(r)$ is given by

$$f(r) = 1 - \frac{\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}. \tag{1.18}$$

The mass parameter μ and the charge parameter q are the ADM mass and charge respectively. They are related to the mass and electric charge of the RN black hole as follows:

$$\mu = \frac{16\pi GM}{(D-2)\Omega_{(D-2)}}, \tag{1.19}$$

$$q = \sqrt{\frac{8\pi G}{(D-2)(D-3)}}Q. \tag{1.20}$$

An event horizon of the RN black hole is where $f(r) = 0$ which can be solved analytically in any dimension.

$$r_\pm = \left(\frac{\mu}{2} \pm \frac{\mu}{2}\sqrt{1 - \frac{4q^2}{\mu^2}}\right)^{1/(D-3)}. \tag{1.21}$$

We use $G = \Omega^2_{(D-2)}/16\pi$ in order to eliminate all the π's under the root in (1.19) for the sake of simplicity in further calculations. Note that r_+ stands for the RN black hole's outer horizon whereas r_- refers to the Cauchy horizon. It is worth noticing that

$$r_+^{D-3} + r_-^{D-3} = \mu \quad \text{and} \quad r_+^{D-3} r_-^{D-3} = q^2. \tag{1.22}$$

This equation can be expressed in terms of the ADM mass, charge and dimension as follows

$$r_+^{D-3} = \frac{M\Omega_{(D-2)}}{2(D-2)}\left(1+\sqrt{1-\frac{D-2}{2(D-3)}\frac{Q^2}{M^2}}\right). \tag{1.23}$$

The RN black hole becomes extremal when

$$\frac{Q^2}{M^2} = \frac{2(D-3)}{D-2}. \tag{1.24}$$

We can readily see that in $D = 4$ the extremal limit is $Q^2 = M^2$ as we already know.

1.1.3 Kerr black hole *à la* Myers-Perry

Owing to the fact that there is the possibility of rotation in several independent rotation planes [18], the spinning (Kerr) black hole in higher dimension has more than one rotation plane. We refer to the Kerr black hole in higher dimensions with more than one angular momentum as the *multiple-spin Kerr black hole*. The metric of this black hole in Boyer-Lindquist coordinates for odd D is given by

$$ds^2 = -d\bar{t}^2 + (r^2+a_i^2)(d\mu_i^2 + \mu_i^2 d\bar{\phi}_i^2) + \frac{\mu r^2}{\Pi F}(d\bar{t} + a_i\mu_i^2 d\bar{\phi}_i)^2 + \frac{\Pi F}{\Pi - \mu r^2}dr^2, \tag{1.25}$$

where

$$d\bar{t} = dt - \frac{\mu r^2}{\Pi - \mu r^2}dr, \tag{1.26}$$

$$d\bar{\phi}_i = d\phi_i + \frac{\Pi}{\Pi - \mu r^2}\frac{a_i}{r^2+a_i^2}dr, \tag{1.27}$$

with the constraint

$$\mu_i^2 = 1. \tag{1.28}$$

The functions Π and F are defined as follows:

$$\begin{aligned} \Pi &= \prod_{i=1}^{(D-1)/2}(r^2+a_i^2), \\ F &= 1 - \frac{a_i^2\mu_i^2}{r^2+a_i^2}. \end{aligned} \tag{1.29}$$

The metric is slightly modified for even D [18]. The event horizons in the Boyer-Linquist coordinates will occur where $g^{rr} = 1/g_{rr}$ vanishes. They are the largest roots of

$$\Pi - \mu r = 0 \qquad \text{even D}, \tag{1.30}$$

$$\Pi - \mu r^2 = 0 \qquad \text{odd D}. \tag{1.31}$$

For an arbitrary D the position of the horizon r cannot be solved analytically. Myers and Perry showed that the properties of the Kerr black hole in D dimensions are similar to that in 4D. Note that for $D \geqslant 6$, black holes can exist with arbitrary large angular momentum for a fixed mass. This is owing to the fact that there is no Kerr bound in $D \geqslant 6$ in the same way as in 4D.

1.1.4 Black rings

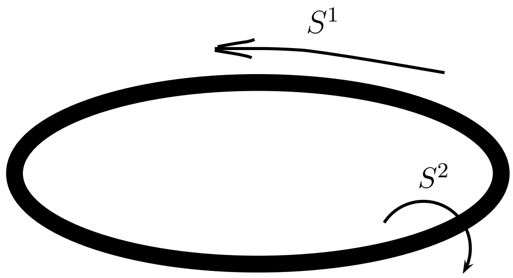

Figure 1.5: *Black ring in 5D with the horizon topology* $S^1 \times S^2$

The 5D MP Kerr black hole has an event horizon with S^3 topology. However we can also have the so-called black ring solution [23–25] due to the fact that there is the possibility of rotation in several independent rotation planes [18]. Discovered by Emparan and Reall in 2001 this solution is the asymptotically flat[7] black hole solution with the event horizon's topology of $S^1 \times S^2$. The S^1 describes a contractible circle, not stabilized by topology but by centrifugal force due to rotation. The solution is regular on and outside the event horizon provided that it has angular momentum along the S^1 direction. This construction can also be, heuristically, understood as: take a piece of black string, with $S^2 \times \mathbb{R}$ horizon, and curve it to form a black ring with horizon topology $S^2 \times S^1$. Since the black string has a tension, then the S^1, being contractible, will tend to collapse. But we may try to set the ring into rotation and in this way provide a centrifugal repulsion that balances the tension. This turns out to be possible in any $D \geqslant 5$, so we expect that non-spherical horizon topologies are a generic feature of higher-dimensional GR.

1.1.5 Dilaton black holes

The dilaton black hole solutions can be obtained from the low-energy action of string theory [27] by dropping all the fields except the metric $g_{\mu\nu}$, the dilaton[8] scalar field ϕ and a Maxwell field

[7] Recently the black ring solution in AdS background has been found using approximate methods [26]

[8] It is a hypothetical elementary particle having zero mass and zero spin, which is introduced in constructing a scale invariant theory involving massive particles. In string theory this particle arises naturally in the low-energy spectrum. So far it has never been observationally verified.

$F_{\mu\nu}$. The main results on the dilaton black hole are given in [28]. The black hole solution coupled to a massive dilaton was also obtained by Horne and Horowitz in the early 1990s [29]. Dilaton black hole solutions in dS and AdS spacetimes are studied in [30–32]. The rotating dilaton black hole solution was found by Horne and Horowitz [33] in 1992, in which they discussed how a small amount of angular momentum can qualitatively change the properties of extremal charged black holes coupled to a dilaton. This was also further discussed in [34], where explicit solutions were obtained when the coupling parameter took either of the special solutions $a = 0$ or $a = \sqrt{3}$. It was argued that these values correspond to minimal couplings of the theory with $a = 0$ corresponding to no dilatonic coupling as in the RN solution, and $a = \sqrt{3}$ corresponding to the minimal coupling of a five-dimensional solution. Further evidence for the special nature of $a = \sqrt{3}$ also appeared in [35]. The more general case of dilaton black holes in higher spacetime dimensions is discussed *e.g.* in [36, 37].

The case of $a = 1$ corresponds directly to a solution obtained from the low energy limit of string theory coupled to an Abelian gauge field. This coupling arises when we transform from the string frame to the Einstein frame via a conformal rescaling of the metric. The dilaton arises naturally in this context as the zero mode of the closed string and uniquely determines the string coupling through the relation $g_s = e^{\langle\phi\rangle}$, where $\langle\phi\rangle$ denotes the vacuum expectation value of the field. The action for the 4D dilaton gravity is slightly modified from that of pure Einstein-Maxwell gravity in that it has an electromagnetic field coupled to the dilaton scalar, hence there is an additional kinetic term to consider. It can be seen to take the following form in the Einstein frame (see Appendix B for a transformation between Einstein and string frames) [38]

$$S = \int d^4x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right] \tag{1.32}$$

with a being the dilaton coupling constant, and $F^2 = F_{\mu\nu}F^{\mu\nu}$. We will also assume that $a \geqslant 0$ as there is a $\mathbb{Z}_2$ symmetry for the dilaton allowing us to exchange $\phi \to -\phi$. It should be noted that the only known values of a arising from supergravity theories are $a = 0, \frac{1}{\sqrt{3}}, 1$ and $\sqrt{3}$, with the second example arising from black string solutions. Interestingly it is known that when we take $a \gg 1$ the corresponding extremal black hole solution can be interpreted as an elementary particle [29].

The metric for the dilaton black hole is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + R^2(r)d\Omega_2^2, \tag{1.33}$$

where we have defined

$$f(r) = \frac{(r - r_-)(r - r_+)}{R^2} \tag{1.34}$$

and

$$R(r) = r\left(1 - \frac{r_-}{r}\right)^{a^2/(1+a^2)}. \tag{1.35}$$

The horizon is located at $r = r_+$ and there is a singularity at $r = r_-$ for $a \neq 0$. The extremal limit of this black hole occurs at $r_+ = r_-$. Note however that the extremal limit of the dilaton black hole does not admit the $AdS_2 \times S^2$ geometry unlike its counterpart with $a = 0$, *i.e.* the RN black hole. In fact the extremal limit of the dilaton black hole has a timelike naked singularity. The ADM mass and charge of the solution are

$$M = \frac{r_+}{2} + \frac{1-a^2}{1+a^2}\frac{r_-}{2}, \tag{1.36}$$

$$Q = \sqrt{\frac{r_+ r_-}{1+a^2}}. \tag{1.37}$$

Solving these expressions for r_+ and r_- we obtain the following

$$r_+ = M + M\sqrt{1-(1-a^2)\frac{Q^2}{M^2}}, \tag{1.38}$$

$$r_- = \frac{(1+a^2)Q^2}{r_+}. \tag{1.39}$$

The condition $r_+ = r_-$ can be written in terms of M and Q as

$$\frac{Q^2}{M^2} = 1 + a^2. \tag{1.40}$$

1.2 Lower-dimensional black holes

1.2.1 BTZ black hole

When the cosmological constant is zero, a vacuum solution of (2+1)-dimensional gravity is necessarily flat and one expects no black hole solutions. However black hole solutions were shown to exist for a negative cosmological constant in 1992 by Bañados, Teitelboim, and Zanelli [39], hence the name. The BTZ black hole is remarkably similar to the (3+1)-dimensional black hole. Much like the Kerr black hole it contains an inner and an outer horizon. It has "no hairs" and is fully characterized by ADM mass, angular momentum and charge. However the BTZ black hole is asymptotically adS whereas the 4D Kerr solution can be asymptotically flat. For a comprehensive review on BTZ black holes, see *e.g.* [40, 41]. The line element of the BTZ black hole can be written as

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2(N^\varphi dt + d\varphi)^2, \tag{1.41}$$

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\varphi = -\frac{J}{2r^2}, \tag{1.42}$$

with $t \in (-\infty, \infty)$, $r \in (0, \infty)$ and $\varphi \in [0, 2\pi]$. $N^2(r)$ and N^φ are called the squared lapse and angular shift respectively. The event horizons can be obtained by solving $N^2(r) = 0$ and take the form:

$$r_\pm = l\sqrt{\frac{M}{2}(1 \pm \Delta)} \tag{1.43}$$

where

$$\Delta = \sqrt{1 - \left(\frac{J}{Ml}\right)^2} \tag{1.44}$$

with imposed conditions that

$$M > 0 \text{ and } |J| \leqslant Ml. \tag{1.45}$$

In the extremal case $J = |Ml|$, the two event horizons coincide. Note that l is the radius of curvature which provides the length scale in order to have dimensionless mass. The BTZ black hole is similar to its (3+1) counterpart, the Kerr solution. The BTZ black hole has an ergosphere, namely $r_{\text{erg}} = l\sqrt{M}$ and an upper bound in angular momentum for any given mass. The spacetime geometry of the black hole is one of constant negative curvature, so it is locally that of adS space. The BTZ black hole can only differ from the adS space in its global properties [39].

1.2.2 Two-dimensional black holes

In 2D the Einstein action is topologically invariant, and it has no dynamical content. To add dynamics to the system we include the dilaton fields hence it is sometimes known as *2D dilaton black hole*[9]. The action reads

$$I_{2DG} = \frac{1}{4\pi}\int d^2x\sqrt{-g}\left(XR + U(X)(\nabla X)^2 - 2V(X,q)\right) , \tag{1.46}$$

where X is a scalar field (dilaton), U, V are arbitrary functions thereof defining the model and R is the Ricci scalar associated with the 2D metric $g_{\mu\nu}$. The function V additionally depends on a parameter q which may be interpreted as charge.[10] In this way charged black hole solutions can be described, including the RN black hole. We shall employ the definitions

$$Q(X) = \int^X U(z)dz\,, \qquad w(X,q) = \int^X e^{Q(z)}V(z,q)dz\,. \tag{1.47}$$

The quantity $w(X, q)$ is invariant under dilaton dependent conformal transformations. In terms of these functions it can be shown that the solution for the line-element in Eddington-Finkelstein gauge reads

$$ds^2 = 2e^{Q(X)}du\,(dX - (w(X,q) + M)du) , \tag{1.48}$$

[9] For a comprehensive review cf. *e.g.* [42, 43].

[10] Such a dependence on q emerges for instance if one introduces in 2D an abelian Maxwell-term and integrates it out exactly. Its only remnant is the conserved $U(1)$ charge q which enters the potential V.

where M is a constant of motion corresponding to the mass. Killing horizons emerge for

$$w(X,q) + M = 0\,. \tag{1.49}$$

The solution of this equation for the outermost horizon is denoted by $X = X_h$.

1.3 Tidal charged black holes

Black hole solutions arise also in brane-world[11] gravity models [44]. There are many brane-world scenarios, but in the simplest gravity model evolves in a curved 5D space-time (the bulk), which contains a temporal 4D hypersurface (the brane), on which all the fields of the standard model are localized. Gravitational dynamics on the brane is governed by an effective Einstein equation [45, 46]. The most well-known brane black hole is the spherically symmetric vacuum *tidal charged* black hole, derived in [47]:

$$ds^2 = -f\left(r\right)dt^2 + f^{-1}\left(r\right)dr^2 + r^2 d\Omega_2^2. \tag{1.50}$$

The metric function f is given as

$$f\left(r\right) = 1 - \frac{2M}{r} + \frac{q}{r^2}\,. \tag{1.51}$$

Such black holes are characterized by two parameters: their mass M and tidal charge q. The latter arises from the Weyl curvature of the 5D space-time into which the brane is embedded (more exactly, from its "electric" part as computed with respect to the brane normal).

Formally the metric (4.93) agrees with the RN solution of a spherically symmetric Einstein-Maxwell system in GR, provided we replace the tidal charge q by the square of the electric charge Q. Thus $q = Q^2$ is always positive, when the metric (4.93) describes the spherically symmetric exterior of an electrically charged object in GR. By contrast, in brane-world theories the metric (4.93) allows for any sign of q. A positive tidal charge weakens the gravitational field of the black hole in precisely the same way the electric charge of the RN black hole does. A negative tidal charge, however, strengthens the gravitational field, contributing to the localization of gravity on the brane.

The structure of the tidal charged black hole in the case $q > 0$ is in full analogy with the general relativistic RN solution[12]. For $q \in (0, M^2)$ it describes tidal charged black holes with two horizons, located at $r_\pm = M \pm \sqrt{M^2 - q}$, both below the Schwarzschild radius. For $q = M^2$ the two horizons coincide at $r_e = M$ (this is the analogue of the extremal RN black hole). Finally there is a new possibility (but unphysical in GR) due to physical considerations on

[11] According to string theory, we may be confined in a braneworld, which is a sub-universe embedded in the higher dimensional bulk universe.

[12] In making analogy with the RN black hole one can also consider the Born-Infeld black holes, which is a nonlinear generalization of the RN black hole [48]

the smallness of the electric charge. This is when $q > M^2$ for which the metric (4.93) describes a naked singularity. Such a situation can arise whenever the mass M of the brane object is small enough, compared to the effect of the Weyl curvature expressed as a tidal charge. If we assume that the tidal charge q is a more or less global property of the brane, then the latter can contain many black holes of mass $M \geqslant \sqrt{q}$.

For any $q < 0$ there is only one horizon, at $r_+ = M + \sqrt{M^2 + |q|}$. For these black holes, gravity is increased on the brane by the presence of the tidal charge. This again contributes towards the localization of gravity on the brane.

Work on the tidal charged black hole includes the matching with an interior stellar solution, a procedure requiring a negative q [49], the study of weak deflection of light to second order in both parameters [50], a confrontation with solar system tests [51], and the evolution of thin accretion disks in this geometry [52].

Chapter 2
Black hole thermodynamics

> *I believe that in order to gain a better understanding of the degrees of freedom responsible for black hole entropy, it will be necessary to achieve a deeper understanding of the notion of entropy itself.*
>
> –R. M. Wald

This chapter concerns the subject of black hole thermodynamics, which has been studied extensively during the past four decades. This is a field that has been generating surprises since its first emergence. There have been strong hints that there are very deep and fundamental relationships between gravity, thermodynamics and quantum theory (for a review on black hole thermodynamics, see *e.g.* [53]).

The history of black hole physics reached a climax when Hawking discovered that black holes were actually not black, *i.e.* a black hole radiates as if it were a black body. This phenomenon was well described in his world renowned book 'A brief history of time' [54]. Hawking himself had a hard time believing the truth he uncovered just like the community then. Hawking was even told by the chairman of the session at the conference where he first presented his calculations that his results were all nonsense.

As a matter of fact, before Hawking's startling discovery there were already pieces of information that suggested that black holes could be thought of as thermodynamic systems, namely J. Bekenstein—then a research student at Princeton—suggested that the area of the event horizon was a measure of the entropy of the black hole [55]. We would not enjoy the rest of the chapter as much if we conceal the truth that Hawking was indeed irritated by Bekenstein's claim even though Hawking had shown before that in fact the area of an event horizon can never decrease under quite general assumptions [57][1]. As written clearly in his very famous book, he felt Bekenstein misused his findings to claim that the surface area of the black hole was related to its entropy. One year after Bekenstein, Hawking together with Bardeen and Carter wrote a paper on black hole mechanics [59] discussing also similarities between entropy and the area of the event horizon motivated partly by his irritation with Bekenstein. The contribution of Bekenstein was later acknowledged by Hawking after the startling discovery.

[1]Hawking proved that if there are no naked singularities, the cross sectional area of a future event horizon cannot be decreasing anywhere.

In a compact (mathematical) language we will recast the history described above as follows: Bekenstein claimed that

$$S_{\rm BH} \propto A. \tag{2.1}$$

where A is the surface area of black hole which is directly related to the geometry of the event horizon. The event horizon is characterized by a quantity, κ, known as the surface gravity[2]. The surface gravity is uniformly constant over the event horizon. The black hole's surface gravity seemingly has *temperature-like* properties in that it has absolute zero, arbitrary scale and is defined in equilibrium. We can thus suspect that the black hole's temperature is proportional to its surface gravity

$$T_{\rm BH} \propto \kappa. \tag{2.2}$$

Now the great contribution of Hawking to black hole physics—despite all the surprises and initial incredulity—is that he convincingly and systematically derived the proportionality constants [60] for both Eqs. (2.1) and (2.2) by the method of quantum field theory on a black hole background. The temperature[3] of the black hole (*a.k.a. Bekenstein-Hawking* or just *Hawking temperature*) is therefore given by

$$T_{\rm BH} = \frac{\hbar\kappa}{2\pi c k_B}, \tag{2.3}$$

where k_B is the Boltzmann's constant and the Bekenstein entropy (now known as Bekenstein-Hawking entropy)

$$S_{\rm BH} = \frac{c^3 k_B A}{4G\hbar}, \tag{2.4}$$

where A refers to the area of the black hole's horizon, G the Newton's gravitational constant, and $\hbar$ the Planck's constant. As a quick and simple exercise (we use only natural units here), we derive temperature and entropy of the Schwarzschild black hole. Since the surface area of the Schwarzschild black hole is given by

$$A = 4\pi r^2 = 16\pi M^2, \tag{2.5}$$

hence the entropy can be easily calculated to be

$$S = 4M^2, \tag{2.6}$$

using $k_B = 1/\pi$. Next the temperature of the Schwarszchild black hole is given by

$$T = \frac{\partial M}{\partial S} = \left(\frac{\partial S}{\partial M}\right)^{-1} = \frac{1}{8M}. \tag{2.7}$$

[2]For more detailed information, see Appendix A.

[3]There exists an analogous effect in flat spacetime, known as *Unruh* effect in which an observer moving at an acceleration in a flat spacetime will record a temperature proportional to the magnitude of his acceleration. See *e.g.* [58]. This result is also based on quantum field theory.

One can clearly see that the larger the mass the lower the temperature. Should we be able to measure the black hole's temperature it would be from a rather small black hole instead of supermassive black holes. Another way of computing the temperature is to work out the surface gravity κ (see detailed discussion in Appendix A) and insert it in Eq. (2.3) in order to obtain the black hole's temperature. The Schwarzschild black hole has

$$\kappa = \frac{1}{4M}, \tag{2.8}$$

thus we see that the resulting temperature coincides with the one calculated in Eq. (2.7).

For the sake of completeness we will compute the Hawking temperature of an astrophysical black hole as follows:

$$T_{\mathrm{H}} = \frac{\hbar}{8\pi G k_{\mathrm{B}} M} \approx 6.2 \times 10^{-8} \frac{M_{\odot}}{M}\ \mathrm{K}, \tag{2.9}$$

where $M_{\odot} = 1.98892 \times 10^{30}$ kg is the solar mass. This is utterly negligible for large black holes—the black hole absorbs much more from the microwave background radiation than it radiates itself. In the case of the rotating "Kerr" black hole, the Hawking temperature is reduced by the rotation, explicitly [13]

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}} = 2\left(1 + \frac{M}{\sqrt{M^2 - a^2}}\right)^{-1} \frac{\hbar}{8\pi M k_{\mathrm{B}}} < \frac{\hbar}{8\pi M k_{\mathrm{B}}}, \tag{2.10}$$

where $a = J/M$. For the RN black hole, one has

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}} = \left(1 - \frac{Q^4}{r_+^4}\right) \frac{\hbar}{8\pi M k_{\mathrm{B}}} < \frac{\hbar}{8\pi M k_{\mathrm{B}}}. \tag{2.11}$$

Thus, electric charge also reduces the Hawking temperature. As a conclusive remark one can safely say that the Hawking radiation plays no role in the case of large-sized black holes. The only type of black hole where one can hope to observe this radiation is the so-called mini black hole, which may have formed in the primordial stage of the Universe. It has been recently discussed how to observe the Hawking radiation from black hole analogs *e.g.* from acoustic black holes in atomic Bose-Einstein condensates [61].

It is important to note that black hole thermodynamics rests on this *no-hair theorem* [62] which states that the final state of a gravitational collapse is a stationary state characterized by a small number[4] of parameters. In other words the stationary black hole is described by a geometry specified merely by the macroscopic parameters such as M, J and Q. Based on the fact that the black hole is stationary (equilibrium) we can define the black hole's temperature since the surface gravity can be defined only when we have the Killing horizon where the norm of the Killing vector goes null in a spacetime [63].

[4]much like a given thermodynamical system in equilibrium which is characterized by a small number of parameters only.

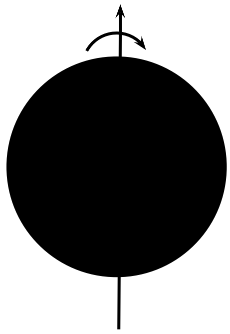

Figure 2.1: *A final stage of GR black hole in four dimensions in equilibrium is governed only by a few control parameters. Its entropy is simply* $S(M, Q, J)$.

2.1 The laws of black hole thermodynamics

The laws of black hole thermodynamics were written down as we compare the laws of black hole mechanics in GR with the laws of thermodynamics. Due to the black hole *no-hair theorems* (see, *e.g.* [64]) we can establish that *mechanically conserved parameters* of stationary black holes (black holes in equilibrium) are analogous to the *state parameters* of ordinary thermodynamics. In the corresponding laws, the role of energy, E, is played by the mass, M, of the black hole; the role of temperature, T, is played by a constant times the surface gravity, κ, of the black hole; and the role of entropy, S, is played by a constant times the area, A, of the black hole. The fact that E and M represent the same physical quantity provides a strong hint that the mathematical analogy between the laws of black hole mechanics and the laws of thermodynamics might be of physical significance. The constants aforementioned are fixed thanks to Hawking's startling discovery [60].

The four laws of black hole thermodynamics

We can write down the four laws of black hole thermodynamics as follows:

1. Zeroth law

   The black hole's physical temperature (we now denote it by T_H) is given by

$$T_{\rm H} = \frac{\kappa}{2\pi} \tag{2.12}$$

   where κ is the black hole's surface gravity. This formula coincides with $T = \frac{\partial M}{\partial S}$ provided one has the fundamental relation.

2. First law

This law is concerned with the mass (energy) change, dM when a black hole switches from one stationary state to another.

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \text{"work terms"}. \tag{2.13}$$

or

$$dM = T_{\rm H}\, dS_{\rm BH} + \text{"work terms"}. \tag{2.14}$$

It is readily seen that the above equations are analogous to the first law of thermodynamics, *i.e.*

$$dE = TdS + \text{"work terms"}. \tag{2.15}$$

And the entropy of the black hole is thus represented by a quarter of the area of the event horizon

$$S_{\rm BH} = \frac{A}{4}. \tag{2.16}$$

The "work terms" are given differently depending on the type of the black holes. For the Kerr-Newman black hole family, the first law would be

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \Omega\, dJ + \Phi\, dQ. \tag{2.17}$$

where Ω is the angular velocity of the hole and Φ is the electric potential which are defined at the horizon by

$$\Omega = \frac{\partial M}{\partial J}, \tag{2.18}$$

$$\Phi = \frac{\partial M}{\partial Q}. \tag{2.19}$$

3. Second law

In any classical process, the area of the event horizon does not decrease

$$\delta A \geqslant 0, \tag{2.20}$$

nor does the black hole's entropy, $S_{\rm BH}$. The second law relies on the *weak energy condition* given by

$$T_{\mu\nu} v^\mu v^\nu \geqslant 0, \tag{2.21}$$

where v^μ is any timelike vector. Note that the second law of black hole thermodynamics (mechanics) can be violated if we take into account quantum effects, *i.e.* the *Hawking radiation*. This is because the area theorem proven by Hawking rests on the energy condition. Gedanken (thought) experiments show that since there is black hole radiation in nature, there must be a rise in entropy in the surrounding region. In order not to violate the second law of thermodynamics, Bekenstein [55, 56] introduced the so-called *generalized entropy*, S' to account for the entropy of this sort and it is defined as

$$S' = S_{BH} + S_m, \tag{2.22}$$

where S_m is the entropy of the surrounding matter. The statement is known as the *Generalized Second Law* (GSL)

$$\delta S' \geqslant 0. \tag{2.23}$$

The ordinary second law seems to fail when the matter is dropped into a black hole because according to classical GR, the matter will disappear into a spacetime singularity, in this manner the total entropy of the universe decreases as there is no compensation for the lost entropy. The virtue of the GSL keeps the law of entropy valid as the total entropy of the universe still increases when that matter is dropped into the black hole.

4. Third law

The third law of thermodynamics also has an analog in black hole physics, namely the surface gravity of the horizon cannot be reduced to zero in a finite number of steps. There is a Planck-Nernst form of the third law of thermodynamics, which states that $S \to 0$ as $T \to 0$. The analog of this law fails in black hole mechanics since there exist extremal black holes (*i.e.* black holes with $\kappa = 0$) with finite A. However, there is good reason to believe that the Planck-Nernst theorem should not be viewed as a fundamental law of thermodynamics [65] but rather as a property of the density of states near the ground state in the thermodynamic limit, which happens to be valid for commonly studied materials. Indeed, examples can be given of ordinary quantum systems that violate the Planck-Nernst form of the third law in a manner very similar to the violations of the analog of this law that occur for black holes [66]. Other examples are frustrated spin systems systems which violate the Planck-Nernst version of the third law [67].

2.2 Black hole in a box

Let's consider a Schwarzschild black hole in an unspecified thermal bath emitting black body radiation at temperature $T_{\rm rad}$. As long as the Hawking temperature of the black hole $T_{\rm BH} = T_{\rm rad}$ then we have an equilibrium. However if the size of the thermal bath is large then the equilibrium is unstable. The situation is as follows: if we, say, let[5] a black hole absorb less energy than it has radiated away then its mass diminishes slightly but its temperature $T_{\rm BH}$ will increase resulting in a further increase of the radiation rate and a further reduction of the black

[5] One can imagine that this can happen randomly.

hole's mass. However a random fluctuation that increases the mass of the black hole reduces its temperature (and the rate of Hawking radiation) which means that the radiation accreted onto the black hole becomes the dominating process. So we have two possible situations for the Schwarzschild black hole in an unspecified thermal bath: either complete evaporation of the black hole or an unlimited growth of its size. This characteristic is due to the black hole's *negative specific heat*. The formula is well-known

$$C = \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T} = T\left(\frac{\partial T}{\partial S}\right)^{-1} = T\left(\frac{\partial^2 M}{\partial S^2}\right)^{-1}. \tag{2.24}$$

For the Schwarzschild black hole it is easy to see that $C \propto -M^2$. This property is characteristic for systems with long-range attractive forces *e.g.* gravitating bodies. Black holes are self-gravitating systems and like most such systems they exhibit negative specific heats, in which case they must be treated in microcanonical ensemble. There is however some exception *e.g.* the BTZ black hole whose specific heat is positive. Note also that systems with long-range interactions are not extensive in nature, *i.e.*

$$S(\lambda U, \lambda V) \neq \lambda S(U, V), \tag{2.25}$$

where λ is a scaling variable.

Now we can render the situation differently if we put the *black hole in a box*, *viz.* we allow the black hole to be a part of a finite-size thermodynamic system. If the total energy is fixed within this box, it is shown that a stable equilibrium configuration can exist [74]. In this case we have the black body radiation and a black hole, both a temerpature T, then the energy and entropy due to the radiation are given by

$$E_{\rm rad} = \sigma V T^4, \tag{2.26}$$

$$S_{\rm rad} = \frac{4}{3}\sigma V T^3, \tag{2.27}$$

where σ is the Stefan's constant, and V the volume of the box. The condition of stable equilibrium is achieved by maximazing the generalized entropy

$$S_{\rm tot} = S_{\rm BH} + S_{\rm rad} = 4M^2 + \frac{4}{3}\sigma V T^3 \tag{2.28}$$

for a fixed value of total energy

$$E_{\rm tot} = M_{\rm BH} + E_{\rm rad} = M + \sigma V T^4. \tag{2.29}$$

There are two scenarios to consider, *i.e.* one where there is only a black body radiation and the other one in which there is a stable configuration where the black hole is in equilibruim with the black body radiation. In the latter the radiation and black hole temperatures coincide, *i.e.* $8MT = 1$ which is obviously the temperature of the Schwarzschild black hole. The equilibrium is stable if $\frac{d^2S}{dT^2} < 1$. However since we cannot "naturally" fix the volume of the box, it is suggestive that black holes in nature live in canonical ensemble.

It is also worth mentioning that a black hole in AdS background can have positive specific heat when it is large compared to the AdS space's radius, whilst its specific heat is negative when it is small. There is a critical temperature in which this occurs and this transition is known as the *Hawking-Page* (HP) phase transition [75]. The AdS space can be understood as some sort of box for the AdS black hole, and hence we can imagine that the BTZ black hole lives in an AdS box. If one considers the specific heat plot of the black hole in question, the HP transition is where the specific heat changes from negative infinity to positive infinity at the minimum temperature. In fact the research in the discipline of black hole thermodynamics has developed significantly since the work of HP. In particular, the phase transition of the AdS black hole in 5D inspired by string theory has generated renewed attention because it relates to confining-deconfining phase transition on the gauge theory side through the AdS/CFT duality [76].

2.3 Mass of the black hole

In our research program we need the fundamental relation in explicit form, which satisfies the first law of black hole thermodynamics. However computing the black hole's mass can be complicated for certain spacetimes. As discussed earlier we showed that electric charge is given by a surface integral at infinity. In GR we normally use the ADM formula [68] to compute mass, which is the surface integral at infinity. For the result to be consistent one always has to check that it satisfies the first law of thermodynamics. In 1972 L. Smarr [69] was able to obtain the mass of the Kerr black hole (later the generalized formula for the mass of black hole is known as the *Smarr mass* formula). There have been papers dedicated to studying masses of various black holes. In general, working out the black hole's mass can be a complicated task. In 4D there is a generalized Smarr formula that includes the negative cosmological constant (Kerr-Newman black hole in AdS Space). Fortunately, the mass formula for the KN AdS black hole can be expressed in a fairly compact form given by Marco Caldarelli, G. Cognola and D. Klemm [70]. They use the formulas given in [71] and obtain[6]

$$M = \frac{\sqrt{S}}{2}\sqrt{\left(1 - \frac{\Lambda S}{3} + \frac{Q^2}{S}\right)^2 + \frac{4J^2}{S^2}\left(1 - \frac{\Lambda S}{3}\right)} \tag{2.30}$$

This is the main *fundamental relation* for our study in this thesis. Having the mass function expressed explicitly in terms of other control parameters one can invert it to obtain the entropy formula. We can also use the Bekenstein-Hawking entropy knowing the outer horizon radius of

[6] The mass formula in Eq. (2.30) is not as presented in the paper [70] but a more compact form we discovered.

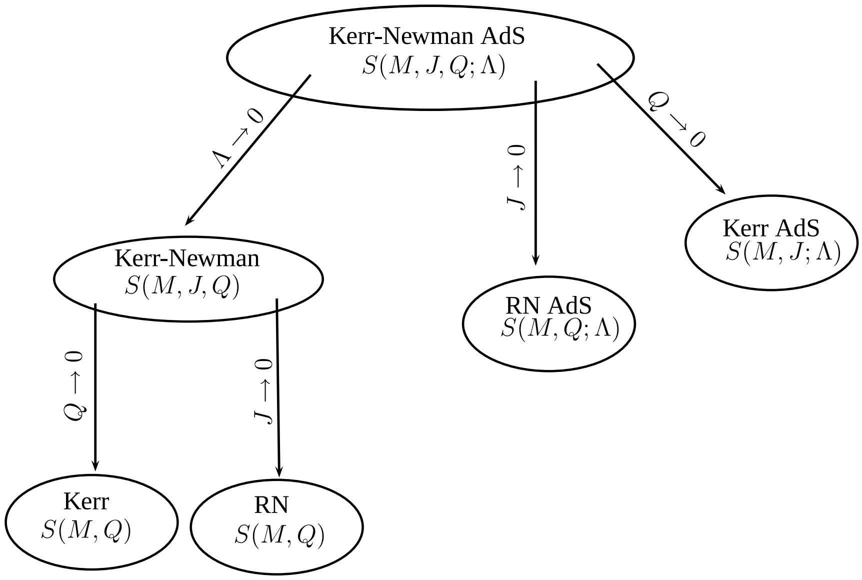

Figure 2.2: *A diagram displaying black hole families in 4D that we investigate in this thesis.*

the black hole, *e.g.* for the KN black hole we have

$$r_+ = M + M\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}}. \tag{2.31}$$

Thus the area of the event horizon is given by

$$A = 4\pi r_+^2, \tag{2.32}$$

Using the entropy-area formula we obtain the entropy for the KN black hole as follows:

$$S = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}}, \tag{2.33}$$

where we have used $k_B = 1/\pi$. Inverting the entropy equation (2.33) we obtain the mass formula for the KN black hole as

$$M = \sqrt{\frac{S}{4} + \frac{1}{S}\left(J^2 + \frac{Q^4}{4}\right) + \frac{Q^2}{2}}, \tag{2.34}$$

where J and Q are the hole's spin and the electric charge respectively. This formula agrees perfectly with Eq. (2.30) when one takes the limit $\Lambda \to 0$. Since this mass function satisfies the first law of thermodynamics, we can straightforwardly derive intensive parameters *i.e.* temperature, angular velocity and electric potential from it.

2.4 Black hole thermodynamics and scale-invariant gravitation

Black hole thermodynamic fundamental relation (2.25) is not extensive in nature since black hole is a gravitating body but in some cases a somewhat similar property holds. The extensitivity of the system is related to the scale-invariant property of the theory which can be understood as follows. Physical theory is understood to be invariant under a constant change of units. Having scale invariance it is possible to extend our measurement to arbitrary spacetime-dependent transformations of units [72,73]. To be more precise, a scale transformation means that we allow the length to scale as $L \to \Omega(x^i)L$ where Ω is some arbitrary function and $\Omega \in (0, \infty)$. Under such a transformation, mass with the dimension $\frac{1}{[\mathrm{L}]}$ transforms as $m \to \Omega^{-1}(x^i)m$. The spacetime metric transforms as $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. In this section we discuss how the Einstein-Maxwell action is affected by the presence of the cosmological constant which breaks the scale-invariant properties of their construction. The action of the Maxwell field coupled to gravity in arbitrary dimension in the cosmological background can be written as

$$S = \int d^D x \sqrt{-g} \left(g^{\alpha\beta} R_{\alpha\beta} - 2\Lambda - g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right). \tag{2.35}$$

Now let ξ be a constant scaling parameter and let the metric tensor together with the field-strength tensor transform as

$$g_{\alpha\beta} \to g'_{\alpha\beta} = \xi^2 g_{\alpha\beta}, \tag{2.36}$$

$$F_{\alpha\beta} \to F'_{\alpha\beta} = \xi F_{\alpha\beta}. \tag{2.37}$$

We can write the Einstein-Maxwell action fully as

$$S[g^{\alpha\beta}, F^{\alpha\beta}] = \int d^D x \left[\sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}(g^{\alpha\beta}) - 2\Lambda\sqrt{-g} - \sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right]. \tag{2.38}$$

The question to ask is the following: *if* $(g_{\alpha\beta}, F_{\alpha\beta})$ *is a solution, is* $(g'_{\alpha\beta}, F'_{\alpha\beta})$ *also a solution?* We need to see how each component of the action transforms with this scaling parameter. First we notice that the Christoffel symbols and the Ricci tensors are invariant under the transformation, whereas the inverse metric tensor and the determinant of the metric tensor transform as follows:

$$g'^{\alpha\beta} = \frac{1}{\xi^2} g^{\alpha\beta}, \tag{2.39}$$

and

$$\det g' = \xi^{2D} \det g. \tag{2.40}$$

Substituting these transformed variables in the action we wish to transform, it becomes

$$S[g^{'\alpha\beta}, F^{'\alpha\beta}] = \int d^D x \big[\xi^D \sqrt{-g}\frac{1}{\xi^2} g^{\alpha\beta} R_{\alpha\beta} \\ - \Lambda \xi^D \sqrt{-g} - \xi^D \sqrt{-g}\frac{1}{\xi^2}\frac{1}{\xi^2}\xi\xi g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}\big]. \tag{2.41}$$

Clearly, when $\Lambda = 0$ we obtain

$$S[g^{'\alpha\beta}, F^{'\alpha\beta}] = \xi^{D-2} \int d^D x \left[\sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}(g^{\alpha\beta}) - \sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta}\right]. \tag{2.42}$$

Therefore we can conclude that in the absence of the cosmological constant the action scales with ξ, namely

$$S[g^{'\alpha\beta}, F^{'\alpha\beta}] = \xi^{D-2} S[g^{\alpha\beta}, F^{\alpha\beta}]. \tag{2.43}$$

The Einstein equations derived from action without the cosmological constant will be scale-invariant, and thus thermodynamic properties of the black holes as they can be scaled in the unit of length. We will elaborate this issue in Chapter 3 in the section on the flatness theorem.

2.5 Derivation of black hole fundamental relations

In this section we will discuss how one obtains explicit entropic/energetic fundamental relations (entropy/mass functions) for certain black holes as we will eventually have to use them. So far we have shown one explicit black hole's entropy, *i.e.* that of the Schwarzschild black hole which is trivial but there are certainly cases where it is very tricky to obtain the entropy function analytically, and one will have to resort to its counterpart—the mass function. The fundamental relations for GR black holes are functions of mass, M, electric charge, Q and angular momentum J. Selectively, we will discuss only three families of black holes, *i.e.* the Myers-Perry black hole, BTZ and the dilaton black hole.

2.5.1 Myers-Perry black hole

There is no need to work out the entropic/energic function of the Reissner-Nordström and Kerr black holes in 4D once we have worked out the fundamental relation for the MP black hole because the MP solutions work in all $D \geqslant 4$.

MP Reissner-Nordström black hole

The Reissner-Nordström black hole in arbitrary spacetime has the metric described by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2_{(D-2)}, \tag{2.44}$$

with

$$f(r) = 1 - \frac{\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}. \tag{2.45}$$

The ADM mass and charge are defined as in (1.19) and (1.20) respectively. The area of the event horizon of the RN is given by

$$A = \Omega_{(D-2)} r_+^{(D-2)}, \tag{2.46}$$

where r_+ is given by (1.21). The entropy of the black hole can be calculated using

$$S = \frac{k_B}{4G\hbar}\Omega_{(D-2)} r_+^{(D-2)}. \tag{2.47}$$

We found a way to simplify the computations by introducing the Boltzmann's constant as

$$k_B = \frac{[2(D-2)]^{\frac{D-2}{D-3}}}{4\pi\Omega_{(D-2)}^{1/(D-3)}}. \tag{2.48}$$

Hence the entropy of the black hole becomes

$$S = r_+^{(D-2)} = \left(r_+^{D-3}\right)^{\frac{D-2}{D-3}}. \tag{2.49}$$

Explicitly we can write out the entropy in terms of the black hole's mass and electric charge as

$$S = \left(M + M\sqrt{1 - \frac{D-2}{2(D-3)}\frac{Q^2}{M^2}}\right)^{\frac{D-2}{D-3}}. \tag{2.50}$$

Inverting the equation we obtain the mass of the MP Reissner-Nordström black hole as

$$M = \frac{S^{\frac{D-3}{D-2}}}{2} + \frac{D-2}{2(D-3)}\frac{Q^2}{S^{\frac{D-3}{D-2}}}. \tag{2.51}$$

MP Kerr black hole

We have two cases (i) MP Kerr black hole with a single nonzero spin (ii) MP Kerr black hole with a multiple nonzero spins but we will only discuss the single-spin case in arbitary D and two-spin case in $D = 5$.

The important quantity of concern to us is the event horizon which is obtained by solving the horizon equation in Section 1.1.3

$$r_+^2 - a^2 - \frac{\mu}{r_+^{D-5}} = 0. \tag{2.52}$$

We can simplify the calculations by introducing Newton's gravitational constant as

$$G = \frac{\Omega_{(D-2)}}{4\pi}. \tag{2.53}$$

The area of the horizon becomes

$$A = \Omega_{(D-2)} r^{(D-4)} (r_+^2 + a^2). \tag{2.54}$$

The ADM mass and charge of the black hole are given by

$$\mu = \frac{4M}{D-2}, \tag{2.55}$$

$$a = \frac{D-2}{2}\frac{J}{M}. \tag{2.56}$$

The entropy can now be written as

$$S = r_+^{(D-4)} (r_+^2 + a^2) = r_+ \mu, \tag{2.57}$$

which is a useful formula for our work. Unfortunately, this (2.57) cannot be solved analytically in arbitrary dimension. The mass function of the Kerr black hole in D dimension is given by

$$M = \frac{D-2}{4} S^{\frac{D-3}{D-2}} \left(1 + \frac{4J^2}{S^2}\right)^{1/(D-2)}. \tag{2.58}$$

One can obtain the temperature of this black hole by differentiating the above mass function with respect to the entropy, *i.e.*

$$T = \frac{(D-3)\left(1 + 4\frac{D-5}{D-3}\frac{J^2}{S^2}\right)}{4S^{\frac{1}{D-2}}\left(1 + 4\frac{J^2}{S^2}\right)^{\frac{D-3}{D-2}}}. \tag{2.59}$$

MP Kerr black hole with double spins

In arbitrary dimension, areas of the event horizon of higher dimensional Kerr black holes are given by

$$A = \frac{\Omega_{(d-2)}}{r_+} \prod_i (r_+^2 + a_i^2) \qquad \text{odd dimension}, \tag{2.60}$$

$$A = \Omega_{(d-2)} \prod_i (r_+^2 + a_i^2) \qquad \text{even dimension}. \tag{2.61}$$

In 5D there can be only two angular momenta associated with the Kerr black hole, thus the area of the event horizon reads

$$A = \frac{2\pi^2}{r_+} (r_+^2 + a_1^2)(r_+^2 + a_2^2). \tag{2.62}$$

The temperature of the 5D Kerr black hole with two spins is the Hawking temperature $T = \kappa/2\pi$ where the surface gravity κ is given by

$$\kappa = r_+ \left(\frac{1}{r_+^2 + a_1^2} + \frac{1}{r_+^2 + a_2^2} \right) - \frac{1}{r_+}. \tag{2.63}$$

Since there are two angular momenta, there are two angular velocities associated with this black hole,

$$\Omega_{a_1} = \frac{a_1}{r_+^2 + a_1^2}, \qquad \Omega_{a_2} = \frac{a_2}{r_+^2 + a_2^2}. \tag{2.64}$$

The first law of thermodynamics for this black hole takes the form [77]

$$dM = TdS + \Omega_{a_1} dJ_{a_1} + \Omega_{a_2} dJ_{a_2}. \tag{2.65}$$

The entropy of the 5D Kerr black hole with double spins is given by

$$S = \frac{k_B A}{4G} = \frac{k_B}{4G} \frac{2\pi^2}{r_+} (r_+^2 + a_1^2)(r_+^2 + a_2^2). \tag{2.66}$$

We can choose k_B and G such that S simplifies as

$$S = \frac{1}{r_+} (r_+^2 + a_1^2)(r_+^2 + a_2^2), \tag{2.67}$$

where r_+ is the largest root of

$$(r^2 + a_1^2)(r^2 + a_2^2) - \mu r^2 = 0, \tag{2.68}$$

where μ is the ADM mass defined in (2.55) with 5D and $a_i = 3J_i/2M$. The temperature of the 5D double-spin Kerr black hole reaches zero in the extremal limit which is given by

$$a_1 + a_2 = \sqrt{\mu} \tag{2.69}$$

or explicitly in terms of mass and the two spins as

$$J_1 + J_2 = \frac{4M^{3/2}}{3\sqrt{3}}. \tag{2.70}$$

Since solving for the entropy function directly is rather complicated, we thus use the same procedure as in the case of the single-spin Kerr black hole and obtain the mass as a function of entropy and two angular momenta as

$$M = \frac{3S^{2/3}}{4} \left(1 + \frac{4J_1^2}{S^2} \right)^{\frac{1}{3}} \left(1 + \frac{4J_2^2}{S^2} \right)^{\frac{1}{3}}. \tag{2.71}$$

2.5.2 Dilaton black hole

In this case the area of the event horizon of a black hole is given by

$$A = 4\pi R^2(r_+) \tag{2.72}$$

where $R^2(r_+)$ appears in Eq. (1.35). A straightforward calculation shows that the entropy of the dilaton black hole takes the form

$$\begin{aligned} S &= \frac{k_B c^3 A}{4G\hbar} = R^2(r_+) \\ &= M^2 \left(1 + \sqrt{1 - (1-a^2)\frac{Q^2}{M^2}}\right)^2 \left(1 - \frac{(1+a^2)Q^2}{M^2\left(1 + \sqrt{1-(1-a^2)\frac{Q^2}{M^2}}\right)^2}\right)^{\frac{2a^2}{1+a^2}} \end{aligned} \tag{2.73}$$

where we are using $k_B = 1/\pi$. Clearly the entropy of the system vanishes when we take the extremal limit (1.40) implying that the area of the horizon has shunk to zero size.

2.5.3 BTZ black hole

This is a case of the (2+1) dimensional black hole—the BTZ black hole. Even though it is just a toy model, it is important for studies in gravitational physics as it is a counterpart of Kerr black hole in 4D. It also possesses thermodynamical properties analogous to the (3+1)-dimensional black hole, *e.g.* its entropy is captured by a law directly analogous to the Bekenstein bound in (3+1)-dimensions, essentially with the surface area replaced by the BTZ black holes circumference. One interesting fact about this black hole is that its heat capacity is always positive, yet it is a *bona fide* black hole. This is related to the presence of the AdS background which renders the specific heat of the RNAdS black hole positive above the Hawking-Page phase transition. We investigate the information geometry of this black hole by starting with the Weinhold metric as it is simpler, *i.e.*

$$M = S^2 + \frac{J^2}{4S^2}. \tag{2.74}$$

The hole's temperature is given by

$$T = 2S - \frac{J^2}{2S^3}, \tag{2.75}$$

whereas its angular velocity takes a rather simple form

$$\Omega = \frac{J}{2S^2}. \tag{2.76}$$

The heat capacity is evaluated as

$$C = \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T} = T\left(\frac{\partial T}{\partial S}\right)^{-1} = T\left(\frac{\partial^2 M}{\partial S^2}\right)^{-1} = \frac{S(4S^4 - J^2)}{4S^4 + 3J^2}. \tag{2.77}$$

We can readily seen that the heat capacity of the BTZ black hole is positive.

Chapter 3
Black hole information geometry

If you haven't found something strange during the day, it hasn't been much of a day.

–J. A. Wheeler

In this chapter we highlight many attempts to understand the (unsettled) underlying statistical mechanics of black holes, and introduce black hole information geometry—the main research program for this thesis. Getting to the point, we all know that the thermodynamic theory which is a macroscopic theory ought to have a microscopic counterpart, namely a statistical mechanics which is a microscopic theory. In classical gravity, a black hole is nothing but empty space with a very strong gravitational field. It is therefore a highly nontrivial question whether the similarity between black holes and the ordinary thermodynamical systems goes so far as to include the possibility of a statistical mechanical foundation of black hole thermodynamics. One easy way to see this nontriviality is to notice the Bekenstein-Hawking entropy, $S = \frac{A}{4}$, and write down the microscopic entropy $S = k_B \ln \Omega$ where Ω is the number of accessible microstates. Comparing them we readily see that the $\Omega = e^{A/4k_B}$. However A, the black hole's surface area that we have so far is a mere function of macroscopic parameters. Since the entropy of ordinary matter is understood to arise from the number of quantum states accessible to the matter at given values of the energy and other state parameters, it is then natural to demand a proper understanding of why the surface area of the black hole represents the entropy of a black hole in GR. In order to identify the quantum dynamical degrees of freedom of a black hole, we will need to go beyond the classical and semiclassical theories and consider black holes within a fully quantum theory of gravity—a theory we apparently do not have. Furthermore there are important questions one should address when one tries to study black hole statistical mechanics *e.g.* (i) what are the microscopic degrees of freedom of black holes? (ii) where are they located (if at all)? (iii) what happens to the black hole entropy after the black hole has evaporated?

Attempts to solve the black-hole-and-entropy problems date back to the 1970s when Bekenstein used information theoretic approach to describe the black hole entropy [56]. In 1985 t'Hooft proposed the so-called brick wall model which allows one to relate the black hole entropy to the entropy of thermal radiation at the Hawking temperature located outside the black hole with the mirror-like boundary [78]. His calculations led to the entropy of the black hole being proportional to its surface area but the proportionality constant is $1/4$ only at the Planck scale. Furthermore it does not address how thermal properties outside the black hole are connected with the loss of information concerning the states in the black hole interior. In the same year Zurek and Thorne suggested that the entropy of a black hole can be interpreted as the logarithm of the number of quantum-mechanical distinct ways that the hole could have been made [79]. In 1986 Bombelli, Koul, Lee and Sorkin [80] tried to explain the origin of black hole entropy by using entanglement entropy. In 1993 Susskind speculated that the classical entropy of a black hole arose from configurations of strings with ends which are frozen on the horizon. He also suggested that quantum corrections to this entropy are finite unlike the case in quantum field theory, and he also thought that all black holes are single string states [81]. In 1996 Strominger and Vafa derived the Bekenstein-Hawking entropy for a class of five-dimensional extremal black holes in string theory by counting the degeneracy of BPS bound states [82]. This technique has been very popular and the Strominger-Vafa paper has been cited over a thousand times. Attempts to locate the degrees of freedom of the black hole entropy were discussed *e.g.* in [83] Horowitz and Marolf show that in string theory many modes of the gravitational field exist only inside the horizon of an extremal black hole. In 1997 Ashtekar, Baez, Corichi and Krasnov [84] studied black hole entropy in loop quantum gravity (LQG). They quantized the classical phase space of the exterior of a black hole in vacuum GR, and were able to show that the entropy of a large non-rotating black hole is proportional to its horizon area. The constant of proportionality depends upon the hand-picked *Barbero-Immirzi* or just *Immirzi* parameter, which fixes the spectrum of the area operator in loop quantum gravity. That the black hole entropy could be derived in LQG (even though the proportionality constant does not emerge naturally) has been advocated by some physicists as one of the most important achievements of LQG [85]. It is worth mentioning that R. Sorkin has presented his philosophical viewpoint on black hole entropy in an article titled "Ten theses on black hole entropy" [86].

Due to the lack of a consensus and complete understanding of black hole statistical mechanics we are motivated to resort to a new idea/method in the hope of opening up a new perspective in the subject. Herein we resort to an idea of *thermodynamic geometry*, a subclass of information geometry, which is a subject in the realm of mathematical statistics. Information geometry is one of the newest ideas in attempts to understand how black hole thermodynamics is related to its statistical mechanical description. Succinctly, information geometry is the study of probability and information by way of differential geometry [87]. Specifically we resort to the idea of thermodynamic geometry which began with Gibbs's reformulation of the theory in terms of equilibrium states rather than processes. The surface of the set of equilibrium states was Gibbs's primary object of study and foreshadowed much of the modern differential geometric theory of manifolds [88]. The work of Gibbs [89]—which was followed by Caratheodory [90], Hermann [91] and later by Mrugała [92, 93]—concerns a differential geometric approach based upon the contact structure[1] of the thermodynamic phase space generated by a one-form (*a.k.a.*

[1]Contact geometry is the study of a geometric structure on smooth manifolds specified by a one-form, for more information see *e.g.* [94].

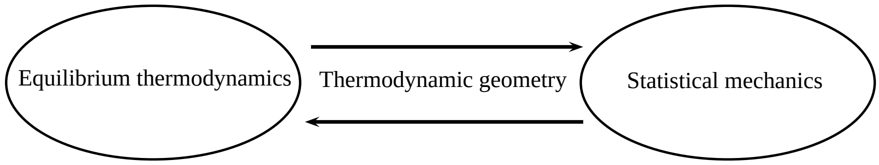

Figure 3.1: *Underlying statistical mechanics is encoded in thermodynamic geometry.*

a Gibbs form) $\mathcal{G}$. This space is $(2n+1)-$dimensional and is coordinatized by n extensive variables E^a and n intensive variables I^a, together with the thermodynamic potential Φ. The first law of thermodynamics is then incorporated into this approach naturally through differential forms. A particular subspace of $\mathcal{G}$ is the space of thermodynamic equilibrium states $\mathcal{S}$. On $\mathcal{S}$, the laws of thermodynamics are valid and thermodynamic systems are specified by means of a fundamental equation.

With the idea that thermodynamics can be geometrized, Weinhold [95] and Ruppeiner [96] proposed *thermodynamic metrics* on the space of equilibrium states. The idea of Weinhold and Ruppeiner is that we can describe thermodynamic systems in terms of a metric whose components are given as the Hessian of the internal thermodynamic energy (Weinhold) or entropy (Ruppeiner). This approach has been widely used to study properties of thermodynamic space generated by the Weinhold and Ruppeiner metrics [97–99], the thermodynamic length [100–102], the chemical and physical properties of various two-dimensional thermodynamic systems [103–107], and the associated Riemannian structure [108–110]. Information geometry in dimension larger than two was studied in [111].

3.1 Thermodynamic geometry

We refer to both the Weinhold and Ruppeiner geometries as thermodynamic geometry. The *Weinhold metric*—a precursor to the Ruppeiner metric—is orginally defined as the Hessian of the energy/mass, M, as a function of energy, S, and other mechanically conserved charges,[2] N^a
.

$$g_{ij}^{W} = \partial_i\partial_j M(S, N^a). \tag{3.1}$$

The *Ruppeiner metric* is defined as the negative of the Hessian of the entropy function with respect to the thermodynamic system's mechanically conserved quantities, *i.e.*

$$g_{ij}^{R} = -\partial_i\partial_j S(M, N^a), \tag{3.2}$$

The infinitesimal distance on the thermodynamic state space is defined as

$$ds^2 = g_{ij}^{R} dx^i dx^j. \tag{3.3}$$

[2]They are parameters that are additive in magnitude such as mass, entropy, electric charge, volume, etc.

It is clearly assumed that the coordinates x^i form some preferred affine coordinates, and that is the reason why we can think of the Hessian of the energy/entropy as a metric on thermodynamic space (in question). The functions on Gibbssian surface allows for affine parametrization. An affine transformation (affine map) between two vector spaces is given by

$$x \mapsto Ax + b. \tag{3.4}$$

What extensive thermodynamic systems, black hole thermodynamics, and mathematical statistics have in common is that there is a preferred set of variables (extensive quantities, additive conserved charges, probability distributions). These things are arbitrary to some extent, for instance you might want to change the zero point (add constants to them), but if you subject them to a coordinate transformation that is more general than an affine one they are no longer extensive (additive conserved charges, probabilities). In this sense affine coordinate transformation have a special status. Then it makes sense to use a definition of the metric which is invariant under affine transformations only.

The Hessian matrix of the entropy function which we call the Ruppeiner metric transforms as a metric, provided we restrict ourselves to only affine coordinate transformations. In thermodynamics the coordinates x^i represent *e.g.* M, J and Q for the Kerr-Newman black hole. It is worth noting that the thermodynamic metric is similar to the Kähler metric which is a metric defined on a complex manifold $\mathscr{M}$, *i.e.*

$$g_{a\bar{a}} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^{\bar{a}}}$$

where $K = K(z, \bar{z})$ is the Kähler potential. The Kähler metric preserves its form under transformation $z' = z'(z)$ and $\bar{z}' = \bar{z}'(\bar{z})$ which are the transformations that preserves the complex structure.

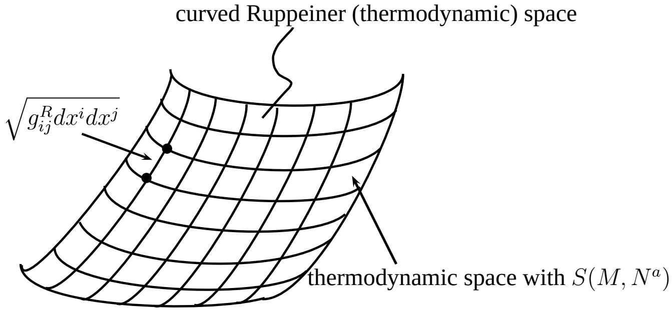

Figure 3.2: *A visualized curved Ruppeiner geometry on thermodynamic state space characterized by entropy S and other mechanically conserved parameters such as mass, M, charge Q and angular momentum J. The squared distance on this curved geometry is $ds^2 = g^R_{ij} dx^i dx^j$.*

The original idea of Ruppeiner is that the Riemannian curvature in some sense measures the complexity of the underlying statistical mechanical model. The Ruppeiner geometry was first constructed in the context of thermodynamic fluctuation theory [99]. We have found that even though thermodynamic fluctuations may not be associated with black hole systems, some essential information can still be extracted from the Ruppeiner geometry. Let's try to remind ourselves how Ruppeiner formed his idea in relation to classical thermodynamic fluctuation theory: let Ω be the number of (equiprobable) microstates consistent with a given macroscopic state. Boltzmann argued that the macroscopic entropy is given by

$$S = k_B \ln \Omega \ . \tag{3.5}$$

Einstein rewrote this equation as

$$P \propto e^{S/k_B} \ , \tag{3.6}$$

where P is the probability that the given macrostate will be realized. We can Taylor expand the entropy around an equilibrium state, taking into account that the entropy has a maximum there, and introduce the Hessian matrix

$$g_{ij} \equiv -\partial_i \partial_j S(x) \ . \tag{3.7}$$

Here x stands for the n extensive variables shifted so that they take the value zero at equilibrium. The matrix is positive definite if the entropy is concave. If we normalize the resulting probability distribution (using $k_B = 1$) we arrive at

$$P(x) = \frac{\sqrt{g}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} g_{ij} x^i x^j} \tag{3.8}$$

as the probability distribution governing fluctuations around the equilibrium state. The pair correlation functions are then given by the contravariant metric tensor,

$$\langle x^i x^j \rangle = g^{ij} \ . \tag{3.9}$$

This derivation is valid as we assume that the fluctuations are small [112]. We should pause and note here that the physical situation here is a system described by the canonical (or grand canonical) ensemble, plus the fact that one extensive parameter (volume) has been set aside and used to give an appropriate physical dimension to g_{ij}. Ruppeiner argues that the Riemannian geometry of the metric tensor g_{ij} carries information about the underlying statistical mechanical model of the system. In particular he argues that the metric is flat if and only if the statistical mechanical system is noninteracting, while curvature singularities are a signal of critical behavior—more precisely of divergent correlation lengths. This viewpoint has been confirmed in a number of soluble models, see *e.g.* [113, 114]. The construction of the Ruppeiner metric used in thermodynamics is related to the Fisher–Rao metric that is used in mathematical statistics.

The Ruppeiner geometry is conformally related to the Weinhold geometry via

$$ds^2 = g^R_{ij} dM^i dM^j = \frac{1}{T} g^W_{ij} dS^i dS^j \ , \tag{3.10}$$

where T is the system's temperature. The proof of this conformal relation is given in Appendix C using the first law of thermodynamics in differential two-forms. In any regard *Eq. (3.10) is one of the most important equations in this thesis* as it turns out to be very useful in most of our calculations when they are not easily done in Ruppeiner coordinates (to be discussed and elaborated on in Chapter 4). In addition, we state that the Weinhold geometry does not have the same physical meaning as the Ruppeiner geometry.

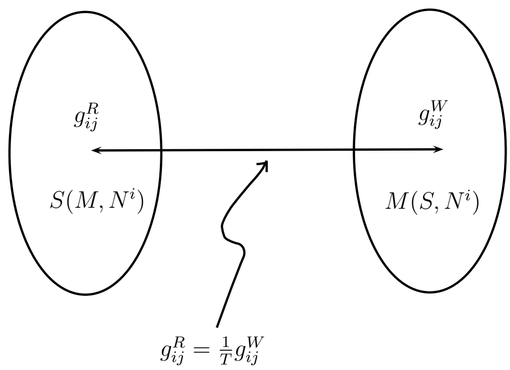

Figure 3.3: *A correspondence between Ruppeiner and Weinhold manifolds.*

It is worth stating that when $S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$ the thermodynamic metric will be degenerate[3]. So far, from the known models the Ruppeiner geometry is flat *if and only if* the underlying statistics is noninteracting, *i.e.* that of the classical ideal gas [96, 99]. The ideal gas has the fundamental relation (this equation is known as *Sackur-Tetrode* equation)

$$S = N \ln \left[\frac{V}{N} \left(\frac{U}{N} \right)^c \right] + k_1 N. \tag{3.11}$$

where c is the ratio of specific heats, k_1 is constant. Consider the ideal gas at fixed volume we obtain the Ruppeiner metric as follows

$$ds^2 = \frac{C_V}{T^2} dT^2 + \frac{V}{T\rho^2 K_T} d\rho^2, \tag{3.12}$$

where C_V is the heat capacity at constant volume and K_T is the isothermal compressibility of the system. We are in the coordinates (T, ρ) where $\rho \equiv N/V$ is the density. The equation of state of the ideal gas has the form $P = \rho T$ and $C_V = N\kappa$ where P is the pressure and κ is a positive definite constant. To see whether the metric in (3.12) is flat or not we can either

[3]This follows from the degree-one homogeneity property of S and the Euler's theorem that follows. See Appendix D for more details.

compute the Riemann curvature scalar (the Ricci scalar) of the metric or perform the coordinate transformation. Using the equation of state and the C_V equation we get

$$ds^2 = V\rho\left(\frac{\kappa}{T^2}dT^2 + \frac{1}{\rho^2}d\rho^2\right). \tag{3.13}$$

We can further transform by using

$$\begin{aligned} x_1 &= \sqrt{2V\rho}\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right), \\ x_2 &= \sqrt{2V\rho}\left(\cos\frac{t}{2} - \sin\frac{t}{2}\right), \end{aligned} \tag{3.14}$$

where

$$t = \int_{T_0}^{T}\sqrt{\frac{\kappa}{T^2}}\,dT, \tag{3.15}$$

with T_0 being an arbitrary positive constant. The metric now reads

$$ds^2 = dx_1^2 + dx_2^2. \tag{3.16}$$

This is evidenly a flat metric and the state space covers an infinite two-dimensional plane. According to Ruppeiner the vanishing Ruppeiner curvature corresponds to the absence of statistical interactions of the ideal gas. Interestingly the ideal gas at constant volume also has a flat Weinhold metric, which takes the form

$$ds_W^2 = N\frac{dT^2}{T} + T\frac{dN^2}{N}, \tag{3.17}$$

where $N = \rho V$ and $T = P/\rho$, and we assume that $N, T \in \mathbb{R}$. By direct calculation of the Riemann curvature the metric in Eq. (3.17) is flat, but we nevertheless wish to bring it into a manifestly flat form. By mathematical wisdom we can rewrite Eq. (3.17) as

$$ds^2 = NT\left(\frac{dT^2}{T^2} + \frac{dN^2}{N^2}\right). \tag{3.18}$$

We can use $x = \ln T$ and $y = \ln N$ (where $x, y \in \mathbb{R}$) to turn the metric above into

$$ds^2 = e^{x+y}(dx^2 + dy^2). \tag{3.19}$$

This form is not yet a manifestly flat form so we transform further using

$$u = \frac{1}{\sqrt{2}}(x+y) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(x-y), \tag{3.20}$$

and we arrive at

$$ds^2 = e^{\sqrt{2}u}(du^2 + dv^2). \tag{3.21}$$

Next we use

$$\alpha = \sqrt{2}e^{u/\sqrt{2}} \quad \text{and} \quad \beta = \frac{v}{\sqrt{2}}, \tag{3.22}$$

where $\alpha \in \mathbb{R}^+$ whilst $\beta \in \mathbb{R}$, and we finally arrive the sought-after manifestly flat metric

$$ds^2 = d\alpha^2 + \alpha^2 d\beta^2. \tag{3.23}$$

The state space of this Weinhold metric is recognizable as an infinite covering of the plane, *a.k.a.* a barber pole spiral.

3.2 Application of information geometry to black hole thermodynamics

Don Page was likely the first who thought of applying geometrized thermodynamics to black holes as he wrote about his idea together with simple calculations in the letter to Physics Today in January 1977 [115]. However he was negatively responded to by F. Weinhold. Approximately two decades after thermodynamic geometry was established, G. W. Gibbons together with S. Ferrara and R. Kallosh suggested the use of Ruppeiner geometry in black hole physics but they did not explicitly compute any geometrical quantities out of the Ruppeiner and Weinhold metrics [116]. Since then thermodynamic geometries have been computed for a number of black hole families ranging from lower-dimensional black holes to GR black holes to dilaton black holes to Myers-Perry black holes to black rings to black holes in unified theories. The number of articles on this topic has been growing including both agreeable and conflicting results, see *e.g.* [117, 120, 121, 123–131, 144, 154, 155]. Our approach since 2003 has been mainly to use this approach to uncover black hole thermodynamics's geometrical patterns and interpret encoded pieces of information relevant to black hole physics. The most satisfactory outcome of this research program is the *prediction of the onset of thermodynamics of the Myers-Perry black hole in* $D > 5$. Detailed calculations and methodologies used will be discussed in Chapter 4. Recently thermodynamic geometry of hot QCD system has been evaluated [132].

The Quevedo approach

Since 2007 H. Quevedo and his team in Mexico City having been investigating black hole thermodynamics using the proposed modified thermodynamic geometry formalism named *Geometrothermodynamics* or GTD in short [133, 134] . In the GTD formalism thermodynamic geometry is made to incorporate invariance under Legendre transformation. Their statement is

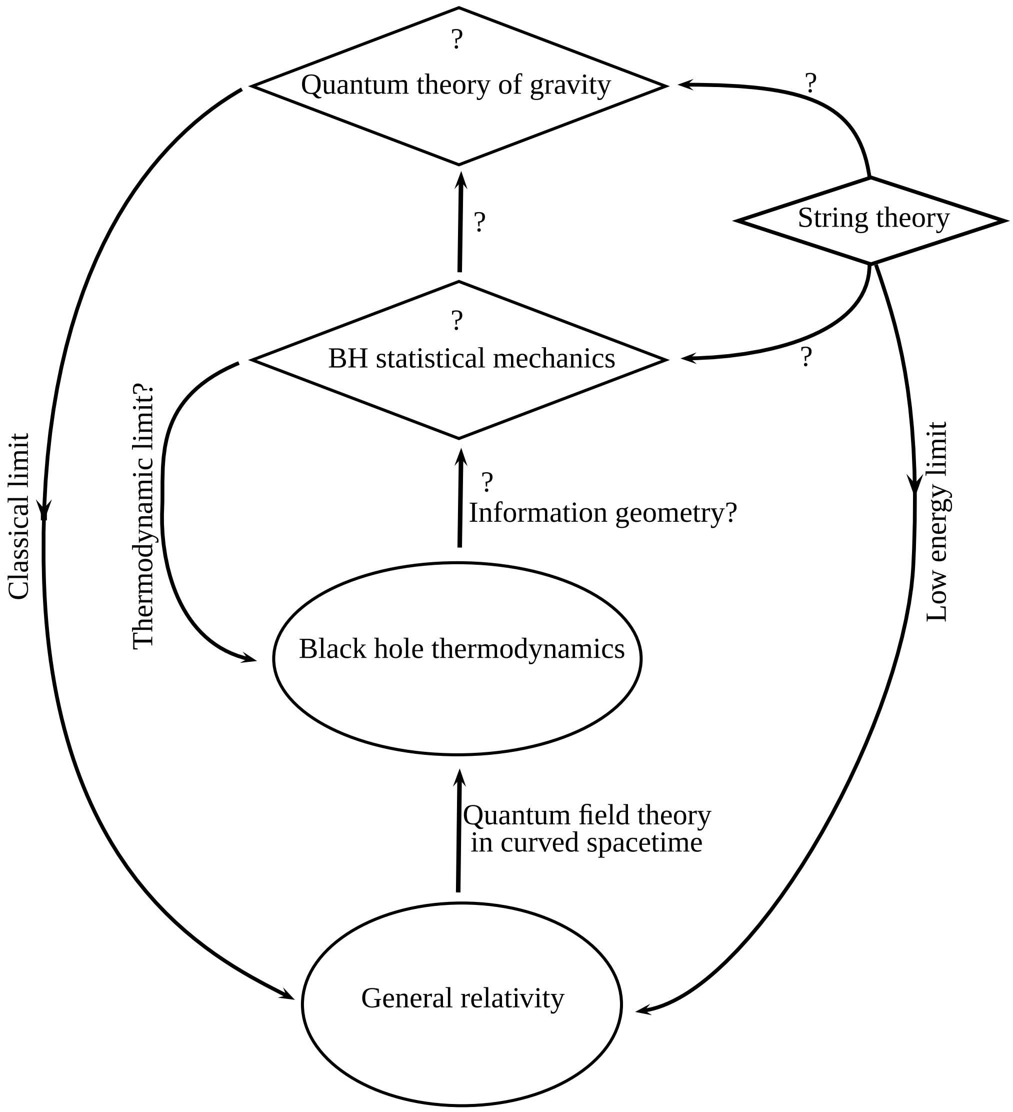

Figure 3.4: *A diagram showing how black hole thermodynamics is related to the broader structure of the overall theory of gravity. An ellipse denotes an acceptable theory to date whereas a rhombus denotes a theory to be established and "?" means unsettled/unfinished. We might understand black hole statistical mechanics through the new perspective(s) opened up by information geometry.*

that GTD allows one to derive Legendre invariant metrics in the space of equilibrium states. They have applied this approach to black hole thermodynamics yielding different results from ours. For example the modified thermodynamic metrics of the RN black hole are more complicated than our results, and in the case of the Kerr black hole both generalized geometries are flat and, they claim, cannot reproduce its thermodynamic behavior and should be considered as a negative result for the use of geometry in black hole thermodynamics. The Quevedo team have also worked out their modified geometry for black holes in two dimensions [137], BTZ black holes [136] and black holes in asymptotically AdS space [138].

3.3 Flatness theorem

It is observable that there are seemingly geometrical patterns of thermodynamic geometries for black hole families. It is then natural to investigate why some thermodynamic geometries are flat, whilst the others are not. In information geometry we can define a metric in some preferred affine coordinate system by

$$g_{ij} = \partial_i \partial_j \psi, \tag{3.24}$$

where ψ is any reasonable function. In mathematical statistics we have an example with the choice of potential of the form

$$\psi = \sum_{i=1}^{N} x^i \ln x^i, \qquad x^i > 0 \tag{3.25}$$

where ψ is minus the *Shannon entropy*. The Hessian matrix of ψ is known as the *Fisher information matrix*[4]. It is a flat metric on the positive cone and it is round on the probability simplex defined by $\sum_i p_i = 1$ [139]. In our context, ψ is either the entropy with a negative sign in front or the energy function. If

$$\psi = -S(M, N^a), \tag{3.26}$$

the corresponding metric is the Ruppeiner metric or it is the Weinhold metric if we have

$$\psi = M(S, N^a). \tag{3.27}$$

Now, the main question is *when is an information metric flat?* One possibility is that

$$\psi = \sum_{i=1}^{N} f_i(x^i). \tag{3.28}$$

[4]In statistical geometry the metric tensor has a potential, *i.e.* there is a convex function ψ. The Hessian function of ψ is called the *Fisher-Rao metric*.

Observations tell us that the Fisher metric on the positive cone comes from such a potential. As well, the Ruppeiner metric of the ideal gas at fixed particle number is of this type, whereas the black hole information metric is not. It turns out that if we assume the potential ψ to have the quasi-homogeneity property

$$\lambda^{a_3}\psi(x,y) = \psi(\lambda^{a_1}x, \lambda^{a_2}y), \tag{3.29}$$

assuming that $x > 0$ and $\psi > 0$, then we can rewrite

$$\psi(x,y) = x^a f(x^b y), \tag{3.30}$$

where f is some function, a, b being some exponents. If we choose $\lambda^{a_1}x = 1$ we will find that $a = a_3/a_1, b = -a_2/a_1$. We have found that if $\psi(x,y) = x^a f(x^{-1}y)$ then the information metric is flat. However the converse is not true. So we state a sufficient but not necessary condition theorem and provide a proof as follows:

Theorem: *The Ruppeiner metric defined through* $g^R_{ij} = -\partial_i\partial_j\Theta$ *is a flat metric in any dimension if* $\Theta = x^a f(x^b y)$ *with* $b = -1$ *and* $a \neq 1$, x *and* y *are coordinates on the state space and* f *is some smooth function.*

Proof: We change coordinates on state space $(x,y) \to (\psi,\sigma)$

$$\psi = x^a f(x^b y) \quad \text{and} \quad \sigma = x^b y. \tag{3.31}$$

A calculation shows that the metric formed by the Hessian of ψ is given by

$$\begin{aligned} ds^2 = &\left(\frac{a-1}{a} - \frac{b(b+1)}{a^2}\frac{\sigma f'}{f}\right)\frac{d\psi^2}{\psi} + 2(b+1)\left(\frac{f'}{af} + \frac{b\sigma f'^2}{a^2 f^2}\right)d\psi d\sigma \\ &+ \psi\left(\frac{f''}{f} - \frac{2b+a+1}{a}\frac{f'^2}{f^2} - \frac{b(b+1)}{a^2}\frac{\sigma f'^3}{f^3}\right)d\sigma^2. \end{aligned} \tag{3.32}$$

This metric is diagonal provided $b = -1$. If we introduce the new coordinate $r = \sqrt{\psi}$ it is a manifestly flat metric, and it covers a wedge shaped region. Hence the theorem is proved. ■

As a matter of fact we found that by using x and σ as coordinates we obtain the metric comparable to the Weinhold metric as

$$ds^2 = x^{a-2}\left[(a(a-1)f - b(b+1)\sigma f')dx^2 + 2(a+b)xf'dxd\sigma + x^2 f'' d\sigma^2\right]. \tag{3.33}$$

If $a + b = 0$ this metric is diagonal, which takes the form

$$ds^2 = \psi_{,x}\left((a-1)\frac{dx^2}{x} + \frac{x}{a}\frac{f''}{f - \sigma f'}d\sigma^2\right), \tag{3.34}$$

where $\psi_{,x}$ is the derivative with respect to x of $\psi(x,y)$. This is the metric on a flat wedge multiplied with the conformal factor $\psi_{,x}$. It is worth noting that for $a=1$ the Ruppeiner metric is degenerate and this corresponds to the fact that the system's entropy is extensive.

Alternatively, we can prove this by means of computing the Riemannian curvature scalar. First it is instructive to look at the Riemannian curvature tensor. In the preferred coordinate system the Christoffel symbols (with one index lowered using the metric) are given by

$$\Gamma_{ijk} = \frac{1}{2}\partial_i\partial_j\partial_k\psi. \tag{3.35}$$

The expression for the Riemannian curvature tensor now reduces to

$$R_{ijkl} = \Gamma_{ikm}g^{mn}\Gamma_{njl} - \Gamma_{ilm}g^{mn}\Gamma_{njk}. \tag{3.36}$$

The scalar curvature takes a fairly complicated form [155]

$$\begin{aligned} R =& \frac{(b+1)x^{3a+4b-4}}{2g^2}\Big[a(a-1)(a+b)ff'f'' - 2a(a-1)(a+2b)ff''^2 - ab(a-1)\sigma ff''f''' \\ &+ (a+b)^2(a+b-1)f'^2f'' + b(a+b)(2a+b-1)\sigma f'^2f''' \\ &+ b(2b-a^2-3ab)\sigma f'f''^2 + b^2(b+1)\sigma^2(f'f''f''' - f''^2)\Big], \end{aligned} \tag{3.37}$$

where g is the determinant of the metric having the form

$$g = x^{2(a+b-1)}\left[a(a-1)ff'' - (a+b)^2f'^2 - b(b+1)\sigma f'f''\right]. \tag{3.38}$$

Obviously the metric is flat for $b=-1$ for it sends the curvature scalar in (3.37) to zero, regardless of the form of the function f. However this is not a one-to-one statement. Note also that for $a+b=0$ the Weinhold metric is flat.

3.3.1 Black hole examples

The fundamental relation for the entropy of black hole relates the area of the event horizon to the ADM charges of the black hole as in (2.16). In the two-parameter family we occasionally encounter the entropy function of the form

$$S = M^a f(M^bQ) \tag{3.39}$$

where M and Q are conserved quantities of the black hole system. We focus on the case where the cosmological constant vanishes, *i.e.* when the Einstein-Maxwell equations are scale invariant. This enables us to perform the dimensional analysis; using length as the only basic unit we can write down the black hole parameters as

$$[S] = L^{D-2}, \ [M] = L^{D-3}, \ [Q] = L^{D-3}, \ [J] = L^{D-2}, \tag{3.40}$$

where D is the spacetime dimension. The scale invariance of the Einstein-Maxwell equation gives the entropy relation quasi-homogeneity properties, with definite exponents:

$$L^{D-2}S(M, Q, J) = S(L^{D-3}M, L^{D-3}Q, L^{D-2}J). \tag{3.41}$$

Hence we can see that in the case of two parameters (RN and Kerr black hole respectively) the black hole entropies will be of the form

$$S = M^{\frac{D-2}{D-3}} f\left(\frac{Q}{M}\right) \ \text{and} \ S = M^{\frac{D-2}{D-3}} f\left(\frac{J}{M^{\frac{D-2}{D-3}}}\right). \tag{3.42}$$

We can readily see that the Ruppeiner geometry of the RN black hole will be flat in any dimension. Similarly, the Weinhold metric of the Kerr black hole is flat in any dimension. This will be also true for black rings in five dimensions. Explicitly we can express the entropy function of the RN black hole (for $D \geqslant 4$) as follows

$$S = M^c \left(1 + \sqrt{1 - \frac{c}{2}\frac{Q^2}{M^2}}\right)^c, \ \ c \equiv \frac{D-2}{D-3}. \tag{3.43}$$

The Kerr black hole in arbitrary spacetime D has the fundamental relation

$$M = \frac{D-2}{4} S^{\frac{D-3}{D-2}} \left(1 + \frac{4J^2}{S^2}\right)^{1/(D-2)}. \tag{3.44}$$

Our scale invariance argument works for both Kerr and RN black holes, however it fails in the case of the BTZ black hole which is the 2+1 *bona fide* black hole in the presence of a cosmological constant. In this case the fundamental relation has the form

$$S = M^a f\left(\frac{J}{M}\right). \tag{3.45}$$

The Ruppeiner metric of this black hole's state space is a flat metric which translates into a flat wedge in an Euclidean space. For exotic black hole examples such as the black ring the fundamental formula can be found in [120].

It is instructive to compare the black hole examples to the ideal gas. Inverting the Eq. (3.11) we obtain the internal energy as

$$U = k_2 \frac{N^{(c+1)/c}}{V^{1/c}} e^{\frac{S}{cN}}, \tag{3.46}$$

where c is the ratio of specific heats, k_2 is constant. It is worth stating again that when $S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N)$ the thermodynamic metric will be degenerate, but if we set V fixed then it belongs to the the class $\psi(x, y) = x^a f(x^b y)$ which is our flatness theorem.

To conclude this section, we found that our small theorem gives a sufficient but not necessary condition for flatness and it shows that the ideal gas is even more special than our black hole examples. This theorem will be useful in a number of black hole cases *e.g.* the dilaton black hole to be discussed in the next chapter.

Chapter 4
Results and discussions

I have noticed even people who claim everything is predestined, and that we can do nothing to change it, look before they cross the road.

–S. W. Hawking

In this chapter we summarize results we have obtained which are presented in the selected publications as shown on page xiii. For conciseness we do not repeat discussions on the black holes's metrics, rather we begin with the thermodynamic metrics, and discuss calculations done which in some case lead to very satisfactory interpretations.

4.1 General relativity black holes

In ordinary spacetime we deal with three families of black holes, namely the Reissner-Nordström black hole, Kerr and the Kerr-Newman black hole. The order here is based on the complexity of the problem. The results in this section are based on [144].

4.1.1 Reissner-Nordström black hole

The Gibbs surface of the RN black hole is defined by

$$S(M,Q) = M^2 - Q^2 + M^2\sqrt{1 - \frac{Q^2}{M^2}}. \tag{4.1}$$

As we have found in [144] it is simpler to start by working out the Weinhold metric. Thus we invert (4.1) and obtain

$$M = \frac{\sqrt{S}}{2}\left(1 + \frac{Q^2}{S}\right). \tag{4.2}$$

The Weinhold metric can then be obtained and it takes the form

$$ds^2_W = \frac{1}{8S^{\frac{3}{2}}}\left[-\left(1 - \frac{3Q^2}{S}\right)dS^2 - 8QdSdQ + 8SdQ^2\right]. \tag{4.3}$$

After some coordinate transformations it simplifies to

$$ds_W^2 = \frac{1}{8S^{\frac{3}{2}}}\left[-(1-u^2)dS^2 + 8S^2du^2\right], \tag{4.4}$$

where we have used

$$u = \frac{Q}{\sqrt{S}}; \qquad u \in (-1, 1). \tag{4.5}$$

Surprisingly the variable u has the same value as the electric potential, $\Phi = \frac{\partial M}{\partial Q}$, of the RN black hole. The Hawking tempeature is given by

$$T = \frac{\partial M}{\partial S} = \frac{1}{4\sqrt{S}}\left(1 - \frac{Q^2}{S}\right), \tag{4.6}$$

which vanishes in the extremal limit of the RN black hole. By using the conformal transformation, we obtain the Ruppeiner metric of the RN black hole in Weinhold coordinates as

$$ds_R^2 = -\frac{dS^2}{2S} + 4S\frac{du^2}{1-u^2}. \tag{4.7}$$

This is a flat metric. With some insight, we can turn the Ruppeiner metric above into Rindler coordinates as follows

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2, \tag{4.8}$$

by using

$$\tau = \sqrt{2S} \quad \text{and} \quad u = \sin\frac{\sigma}{\sqrt{2}}. \tag{4.9}$$

Turning this into Minkowski coordinates (t, x) we obtain

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2 = -dt^2 + dx^2, \tag{4.10}$$

where we have used

$$\begin{aligned} t &= \tau\cosh\sigma. \\ x &= \tau\sinh\sigma. \end{aligned} \tag{4.11}$$

Thus the line element (4.8) is a timelike wedge in Minkowski space. Note that the range of σ is $[-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}]$. We can also express the entropy function in terms of x and t as

$$S = \frac{1}{2}(t^2 - x^2). \tag{4.12}$$

It is readily seen that the black hole entropy vanishes on the thermodynamic light cone. We will revisit the RN black hole again in Sec. (4.3.1)

4.1.2 Reissner-Nordström—adS black hole

The formula of the RNadS is given by

$$M = \frac{\sqrt{S}}{2}\left(1 + \frac{S}{l^2} + \frac{Q^2}{S}\right). \tag{4.13}$$

The extremal limit of this black hole is where

$$\frac{Q^2}{S} = 1 + \frac{3S}{l^2}, \tag{4.14}$$

which is consistent with the extremal limit of the ordinary RN black hole when the cosmological constant is switched off ($\Lambda = 0$ *i.e.* $l \to \infty$). The Hawking temperature is found to be

$$T = \frac{1}{4\sqrt{S}}\left(1 + \frac{3S}{l^2} - \frac{Q^2}{S}\right). \tag{4.15}$$

This temperature vanishes in the extremal limit as anticipated. The Weinhold metric of this black hole is

$$ds_W^2 = \frac{1}{8S^{3/2}}\left[-\left(1 - \frac{3S}{l^2} - \frac{3Q^2}{S}\right)dS^2 - 8QdSdQ + 8SdQ^2\right]. \tag{4.16}$$

This is a curved metric. By using the conformal relation we obtain the Ruppeiner metric in the form

$$ds^2 = \frac{1}{1 + \frac{3\tau^2}{2l^2} - u^2}\left[-\left(1 - \frac{3\tau^2}{2l^2} - u^2\right)d\tau^2 + 2\tau^2 du^2\right]. \tag{4.17}$$

We can observe that the metric above changes its properties as the signature of the metric changes. It is related to the thermodynamic stability of the black hole. It is well known that for sufficiently large black holes, the entropy function becomes concave [75]. The quantity of concern to us in this context is of course the curvature scalar, which is found to be

$$R = \frac{9}{l^2}\frac{\left(\frac{3S}{l^2} + \frac{Q^2}{S}\right)\left(1 - \frac{S}{l^2} - \frac{Q^2}{S}\right)}{\left(1 - \frac{3S}{l^2} - \frac{Q^2}{S}\right)^2\left(1 + \frac{3S}{l^2} - \frac{Q^2}{S}\right)}. \tag{4.18}$$

It is readily observable that the curvature scalar diverges both in the extremal limit and along the curve where the metric changes signature. A quick glance tells us that as $l \to 0$; R diverges but this is not a physical situation because the limit $\Lambda \to \infty$ does not exist.

4.1.3 Kerr black hole

From the fundamental relation (2.30) we obtain the entropy of the Kerr black hole in the form

$$S = 2M^2 + 2M^2\sqrt{1 - \frac{J^2}{M^4}}. \tag{4.19}$$

The extremal limit of the Kerr black hole occurs when

$$\frac{J}{M^2} = \pm 1. \tag{4.20}$$

The Ruppeiner metric of the Kerr black hole is found to be

$$ds_R^2 = \frac{2}{\left(1 - \frac{J^2}{M^4}\right)} \left\{ -2 \left[\left(1 - \frac{J^2}{M^4}\right)^{3/2} + 1 - \frac{3J^2}{M^4} \right] dM^2 - \frac{4J}{M^3} dMdJ + \frac{dJ^2}{M^2} \right\}, \tag{4.21}$$

which can be diagonalized using

$$v = \frac{J}{M^2}; \qquad v \in (-1, 1). \tag{4.22}$$

The diagonalized metric reads

$$ds_R^2 = -2\left(1 + \frac{2}{\sqrt{1 - v^2}}\right) dM^2 + \frac{2M^2}{(1 - v^2)^{3/2}} dv^2. \tag{4.23}$$

The Ruppeiner geometry is curved and its curvature scalar is given by

$$R = \frac{1}{4M^2} \frac{\sqrt{1 - \frac{J^2}{M^4}} - 2}{\sqrt{1 - \frac{J^2}{M^4}}}. \tag{4.24}$$

This curvature scalar diverges in the extremal limit. Since the entropy function is not concave the thermodynamic theory *à la* Ruppeiner could not be used. Furthermore we have learned from [120] that unstable modes do not appear[1] in the extremal limit of the Kerr black hole in 4D. However we have a different story in dimension higher than five as we will see in Sec. (4.3.2)

4.1.4 Kerr-Newman black hole

The fundamental relation for the KN black hole is given by (2.30). The first law of thermodynamics for this black hole takes the form

$$dM = TdS + \Omega dJ + \Phi dQ, \tag{4.25}$$

which enables us to compute Ω (angular velocity), Φ (electric potential) and T (temperature) by means of partial differentiations.

[1] meaning that there is no change of stability despite the presence of a vertical slope in the extremal limit in the conjugacy diagram (*i.e.* a plot of conjugate thermodynamic quantities such as $\beta = 1/T$ versus mass M), Fig. 3(a) in [120]. Analyzing the conjugacy plot is part of the so-called Poincaré (turning point) method which is a standard method for stability analysis. This method is in contrary to the analysis based on the sign of the specific heat. In effect, the Poincaré method does not predict any instability in 4D Kerr spacetimes.

Unfortunately the Ruppeiner metric of the KN black hole is far too complicated to be presented here. We also could not deal with it by hand, thus some computer programs such as CLASSI [145] and GRTENSOR were employed for performing the computations. The results we obtained [144] were not surprising; both the Ruppeiner and Weinhold metrics are curved. The Ruppeiner curvature diverges in the extremal limit of the KN black hole. Furthermore we found that the Ruppeiner geometry of this black hole is not conformally flat.

4.2 Lower-dimensional black holes

4.2.1 BTZ black hole

The Weinhold metric for the BTZ black hole takes the form

$$ds_W^2 = \left(2 + \frac{3J^2}{2S^4}\right) dS^2 - \left(\frac{2J}{S^3}\right) dSdJ + \left(\frac{1}{2S^2}\right) dJ^2, \tag{4.26}$$

which is a non-flat metric. Using the conformal relation we obtain the Ruppeiner metric, which after diagonalization, is given by

$$ds_R^2 = \frac{1}{S} dS^2 + \left(\frac{S}{1-u^2}\right) du^2, \tag{4.27}$$

where we have used

$$\lambda = \frac{J}{2S^2}; \qquad u \in (-1, 1) \tag{4.28}$$

The Ruppeiner metric in [144] is a flat metric, in other words the space of its thermodynamic state is a flat space. After an obvious and final coordinate transformation to polar coordinates we find that this is a wedge in an Euclidean flat space. Note also that the metric signature of the Ruppeiner metric for the BTZ black hole is Euclidean which corresponds to the positive definiteness of its specific heat. The RNAdS black hole has positive specific heat above the Hawking-Page phase transition.

4.2.2 Two-dimensional black holes

This section is a summary of paper [146]. In 2D the Hawking-Unruh temperature (as derived *e.g.* from surface gravity [147]) is given by

$$T = |w'(X, q)|_{X=X_h} \,. \tag{4.29}$$

Prime denotes differentiation with respect to X. The Bekenstein-Hawking entropy (as derived e.g. from Wald's Noether charge technique [148]) is given by

$$S = X_h \,. \tag{4.30}$$

We are now able to derive the Weinhold and Ruppeiner metrics. Because of (1.49)-(4.30) both metrics depend on the conformally invariant function $w(X,q)$ only. So we are free to choose $Q=0$ to simplify the calculations. Putting together all definitions yields the Weinhold metric

$$ds_W^2 = -w''(S,q)dS^2 - 2\dot{w}'(S,q)dSdq - \ddot{w}(S,q)dq^2\,, \tag{4.31}$$

where dot denotes differentiation with respect to q. The Ruppeiner metric follows as

$$ds_R^2 = \frac{1}{|w'(S,q)|}ds_W^2\,. \tag{4.32}$$

The conformal factor between these two metrics never vanishes unless the horizon degenerates. We discuss briefly two important classes of examples.

Reissner-Nordström like black holes The family of models ($b \neq -1 \neq c$)

$$w = -\frac{A}{b+1}X^{b+1} - \frac{B}{2(c+1)}X^{c+1}q^2 \tag{4.33}$$

is simple and interesting, as it contains the spherically reduced RN black hole from D dimensions $b=-1/(D-2)$, as well as charged versions of the Witten black hole $b=0$ [149] and of the Jackiw-Teitelboim model $b=1$ [150]. With the coordinate redefinition $u=qS^{c+1}$ the Weinhold metric simplifies to diagonal form,

$$ds_W^2 = (bAS^{b-1} - (\frac{c}{2}+1)Bu^2S^{-c-3})dS^2 + \frac{B}{c+1}S^{-c-1}du^2\,. \tag{4.34}$$

It is flat for $b=0$ or $c=b-2$. Similarly, the Ruppeiner metric turns out as

$$ds_R^2 = \frac{1}{S(AS^b + \frac{B}{2}u^2S^{-c-2})}\left[b(AS^b - \frac{\frac{c}{2}+1}{b}Bu^2S^{-c-2})dS^2 + \frac{B}{c+1}S^{-c}du^2\right]\,. \tag{4.35}$$

The Ruppeiner metric (4.35) is not flat in general. However, if the condition $c=-b-2$ holds, then (4.35) simplifies considerably,

$$ds_R^2 = b\frac{dS^2}{S} + 2S\frac{1}{(b+1)}\frac{du^2}{(-2\frac{A}{B}-u^2)}\,. \tag{4.36}$$

The Ruppeiner metric (4.36) is flat and has Lorentzian or Euclidean signature, depending on b and the sign of u^2+2A/B. The particular subclass

$$U = -\frac{b+1}{X}\,, \qquad V = -AX^{2b+1} - \frac{B}{2}\frac{q^2}{X} \tag{4.37}$$

describes the spherically reduced RN black hole from D dimensions with $b=-1/(D-2)$. It fulfills the condition $c=-b-2$, and thus all corresponding Ruppeiner metrics are flat. This agrees with the results in section 4.1.1 below: the line-element (4.36) essentially coincides with the line-element (4.45) upon rescaling u and choosing B appropriately.

Chern-Simons like black holes In some cases, like the Kaluza-Klein reduced gravitational Chern-Simons term [151] or the toroidally reduced BTZ black hole [152], the charge q does not enter quadratically in the potential but only linearly. Therefore, we consider here the class of models defined by

$$w = -\frac{A}{b+1}X^{b+1} - \frac{B}{c+1}X^{c+1}q\,. \tag{4.38}$$

We obtain the flat[2] Weinhold metric

$$ds_W^2 = (bAS^{b-1} + cBS^{c-1}q)dS^2 + 2BS^c dS dq\,, \tag{4.39}$$

and the Ruppeiner metric

$$ds_R^2 = \frac{1}{AS^b + BqS^c}\left[\frac{b}{S}(AS^b + \frac{c}{b}BqS^c)dS^2 + 2BS^c dS dq\right] \tag{4.40}$$

The Ruppeiner metric (4.40) is not flat in general. However, if the condition $c = b$ holds, then (4.40) simplifies considerably,

$$ds_R^2 = b\frac{dS^2}{S} + B\frac{2dSdq}{A+Bq}\,. \tag{4.41}$$

The Ruppeiner metric (4.41) is flat and has Lorentzian signature.

We conclude this section with a remark on a duality found in [153]. It connects two different models leading to the same classical solutions for the line-element (1.48) and therefore to the same surface gravity (4.29), but the respective entropies differ in general. It would be interesting to study the behavior of the Weinhold and Ruppeiner metrics under this duality.

4.3 Myers-Perry black holes

In [154] we study thermodynamic geometries of Myers-Perry black holes (let us remind the reader that we deal with higher spacetime dimensions in this section) and obtain the following results:

4.3.1 Reissner-Nordström black hole

From [144, 154] it is found that for the RN black hole, it is simpler to work in Weinhold coordinates. Hence we start with the mass function given in (2.51)

$$M = \frac{S^{\frac{D-3}{D-2}}}{2} + \frac{D-2}{2(D-3)}\frac{Q^2}{S^{\frac{D-3}{D-2}}}. \tag{4.42}$$

[2] The line element (4.39) describes a flat (Rindler type) geometry because the coordinate q appears only linearly.

In $D = 4$ the mass formula becomes that of the ordinary RN black hole as in (4.2). The Weinhold metric of RN black hole in arbitrary dimension can be diagonalized by choosing the new coordinate

$$u = \sqrt{\frac{D-2}{2(D-3)}}\frac{Q}{S^{\frac{D-3}{D-2}}}; \qquad u \in (-1, 1). \tag{4.43}$$

The Weinhold metric now becomes

$$ds_W^2 = S^{\frac{D-1}{D-2}} \left(-\frac{1}{2}\frac{D-3}{(D-2)^2}(1-u^2)dS^2 + S^2 du^2 \right), \tag{4.44}$$

which is a curved Lorentzian metric. The Ruppeiner metric can be obtained by using the conformal relation, thus

$$ds_R^2 = \frac{-1}{D-2}\frac{dS^2}{S} + 2S\frac{D-3}{D-2}\frac{du^2}{1-u^2}. \tag{4.45}$$

This is a flat metric. The black hole's temperature is found to be

$$T = \frac{D-3}{2(D-2)}\frac{1-u^2}{S^{\frac{1}{D-2}}}. \tag{4.46}$$

We have found that the Ruppeiner metric (4.45) can be written in Rindler coordinates as

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2, \tag{4.47}$$

by using

$$\tau = 2\sqrt{\frac{S}{D-2}} \quad \text{and} \quad u = \sin\frac{\sigma\sqrt{2(D-3)}}{D-2}. \tag{4.48}$$

The angle σ then lies within the following interval

$$-\frac{(D-2)\pi}{2\sqrt{2(D-3)}} \leqslant \sigma \leqslant \frac{(D-2)\pi}{2\sqrt{2(D-3)}}. \tag{4.49}$$

Turning this into Minkowski coordinates (t, x) we obtain

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2 = -dt^2 + dx^2, \tag{4.50}$$

Using the new parameters defined in terms of mass and charge, we can represent the entropy of the RN black hole in Minkowskian coordinates as follow:

$$S = \frac{1}{4}(D-2)(t^2 - x^2). \tag{4.51}$$

The Ruppeiner metric can be presented as a Rindler wedge as shown in Fig 4.1. Note that curves of constant S are segments of hyperbolas and the opening angle of the wedge grows as $D \to \infty$.

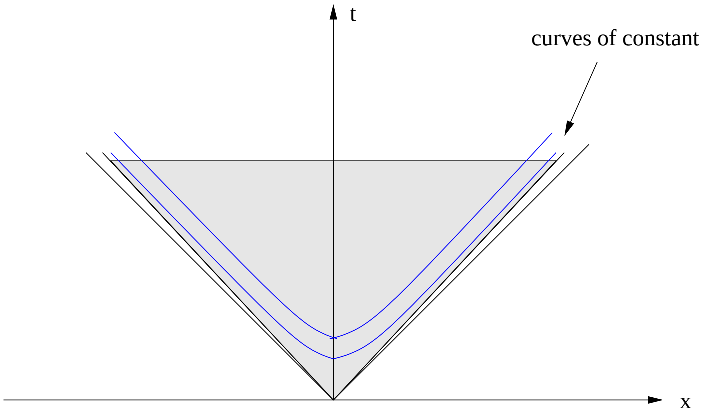

Figure 4.1: *The state space of the 4D RN black holes shown as a wedge in a flat Minkowski space.*

4.3.2 Kerr black hole

We will start with the Weinhold metric as it turns out to be simpler than the Ruppeiner metric. The mass function of the Kerr black hole in D dimension is given by

$$M = \frac{D-2}{4} S^{\frac{D-3}{D-2}} \left(1 + \frac{4J^2}{S^2}\right)^{1/(D-2)}. \tag{4.52}$$

One can obtain the temperature of this black hole by differentiating the above mass function with respect to the entropy, *i.e.*

$$T = \frac{(D-3)\left(1 + 4\frac{D-5}{D-3}\frac{J^2}{S^2}\right)}{4S^{\frac{1}{D-2}}\left(1 + 4\frac{J^2}{S^2}\right)^{\frac{D-3}{D-2}}}. \tag{4.53}$$

The Weinhold metric then takes the form

$$\begin{aligned} ds^2_W = \lambda\Big(& \left[-48(D-5)J^4 + 24S^2J^2 - (D-3)S^4\right] dS^2 \\ & + \left[64(D-5)J^3S - 16(D-1)JS^3\right] dSdJ \\ & + \left[-32(D-4)J^2S^2 + 8(D-2)S^4\right] dJ^2 \Big). \end{aligned} \tag{4.54}$$

The factor λ is given by

$$\lambda = \frac{1}{4(D-2)(S^2+4J^2)^{\frac{2D-5}{D-2}} S^{\frac{D+1}{D-2}}}. \tag{4.55}$$

We can diagonalize this metric by using

$$u = \frac{J}{S}, \tag{4.56}$$

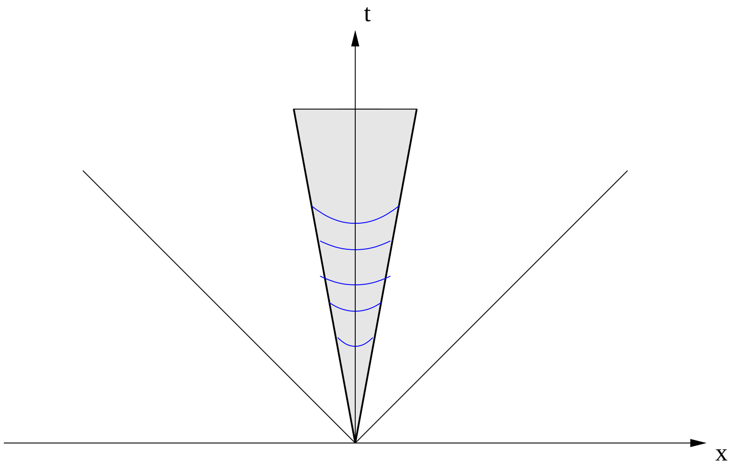

Figure 4.2: *The state space of the 4D Kerr black holes shown as a wedge in a flat Minkowski space. Curves of constant entropy gives causal structure to the state space of the black hole.*

and

$$\tau = \sqrt{\frac{D-2}{D-3}}\, S^{\frac{D-3}{2(D-2)}} (1+4u^2)^{\frac{1}{2(D-2)}}. \tag{4.57}$$

Hence the Weinhold metric in a diagonal form reads

$$ds_W^2 = -d\tau^2 + \frac{2(D-3)}{(D-2)} \frac{(1 - 4\frac{D-5}{D-3}u^2)}{(1+4u^2)^2} \tau^2 du^2. \tag{4.58}$$

This metric is flat. In $D = 4$ we can write it in Rindler coordinates as

$$ds_W^2 = -d\tau^2 + \tau^2 d\sigma^2 \tag{4.59}$$

using

$$u = \frac{1}{2} \sinh 2\sigma. \tag{4.60}$$

Note that in 4D and 5D we have extremal Kerr black holes whereas in 6D and above there is no extremal limit in the solution. In 4D we have $J = M^2$ as the extremal limit, hence u is bounded by

$$|u| \leqslant \frac{1}{2}, \tag{4.61}$$

which translates into

$$-\frac{1}{2}\sinh^{-1} 1 \leqslant \sigma \leqslant \frac{1}{2}\sinh^{-1} 1. \tag{4.62}$$

Numerically it is

$$|\sigma| \leqslant 0.4406. \tag{4.63}$$

By using (4.11) we obtain the wedge of the state space of the Kerr black hole in 4D in a flat Minkowski whose edge is bounded by

$$-\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \leqslant \frac{x}{t} \leqslant \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}. \tag{4.64}$$

In 5D we have the extremal limit

$$\frac{J^2}{M^3} = \frac{16}{27}, \tag{4.65}$$

and u is bounded as $u \in (-\infty, \infty)$ where

$$u = \frac{1}{2} \tan \sqrt{3}\sigma. \tag{4.66}$$

The angle of opening of this wedge is then given by

$$|\sigma| \leqslant \frac{1}{\sqrt{3}} \arctan \infty = \frac{\pi}{2\sqrt{3}} \approx 0.9069. \tag{4.67}$$

Notably the opening angle of the wedge in 5D is wider than than of 4D. For 6D and higher we have the wedge that fills the entire lightcone as in those dimensions there are no extremal limits for the Kerr black holes.

The causal structure of state space is determined by curves of constant entropy rather than by the lightcone itself. For the 4D Kerr black hole the curves of constant entropy are given by

$$S = \frac{(t^2 - x^2)}{4(t^2 + x^2)^2}. \tag{4.68}$$

The Ruppeiner geometry of the arbitrary-dimensional Kerr black hole is curved with the curvature scalar in the following form

$$R = -\frac{1}{S} \frac{1 - 12\dfrac{D-5}{D-3}\dfrac{J^2}{S^2}}{\left(1 - 4\dfrac{D-5}{D-3}\dfrac{J^2}{S^2}\right)\left(1 + 4\dfrac{D-5}{D-3}\dfrac{J^2}{S^2}\right)}. \tag{4.69}$$

In 4D the curvature scalar diverges along the curve $4J^2 = S^2$ which is consistent with the previous result [144]. The curvature scalar (4.69) is valid in any dimension higher than three. In 5D it is reduced to

$$R = -\frac{1}{S}, \tag{4.70}$$

which diverges in the extremal limit of the 5D Kerr black hole. Remarkably, in 6D we have a curvature divergence but not in the limit of extremality, rather at

$$4J^2 = \frac{D-3}{D-5} S^2. \tag{4.71}$$

This is where Emparan and Myers [157] suggest that the Kerr black hole becomes unstable and changes its behavior to be like a black membrane. This is also where the temperature of the higher-dimensional ($D \geqslant 6$) Kerr black hole reaches its minimum. We would like to add that issues of instabilities of Kerr black holes in higher dimensions have recently gained attention, see *e.g.* [158], [159] and [160].

4.3.3 Kerr black hole with double spins

We start with the mass function in order to compute the Weinhold metric, *i.e.* we use Eq. (2.71)

$$M = \frac{3S^{2/3}}{4}\left(1+\frac{4J_1^2}{S^2}\right)^{\frac{1}{3}}\left(1+\frac{4J_2^2}{S^2}\right)^{\frac{1}{3}}. \tag{4.72}$$

The Hessian of M with respect to the entropy and two angular momenta yields the Weinhold metric, which is found to be curved. The curvature scalar of the Weinhold metric takes the form

$$R = \frac{16}{3}\frac{S^{\frac{2}{3}}(S^8+3S^6J_1^2+3S^6J_2^2+4S^4J_1^2J_2^2+64J_1^4J_2^4)}{(S^2+4J_1^2)^{\frac{1}{3}}(S^2+4J_2^2)^{\frac{1}{3}}(S^2-4J_1J_2)^2(S^2+4J_1J_2)^2}. \tag{4.73}$$

We next transform it into the Ruppeiner metric via the conformal relation with an inverse temperature as a conformal factor. The temperature of the double-spin Kerr black hole in five dimensions is given by

$$T = \frac{1}{2S^{5/3}}\frac{(S^2+4J_1J_2)(S^2-4J_1J_2)}{(S^2+4J_1^2)^{2/3}(S^2+4J_2^2)^{2/3}}. \tag{4.74}$$

The Ruppeiner curvature scalar of the double-spin Kerr black hole in five dimensions reads

$$R = -\frac{S^8+20S^6J_1^2+20S^6J_2^2+256S^4J_1^2J_2^2+192J_1^4J_2^2S^2+192J_1^2J_2^4S^2-256J_1^4J_2^4}{2S(S^2+4J_1^2)(S^2+4J_2^2)(S^2-4J_1J_2)(S^2+4J_1J_2)}. \tag{4.75}$$

Note that both the Weinhold and Ruppeiner curvature scalars are divergent at

$$J_1J_2 = \frac{S^2}{4}, \tag{4.76}$$

which is the extremal limit of the 5D double-spin Kerr black hole. Note also that this curvature scalar does not vanish either in the limit of $J_1 = 0$ or $J_2 = 0$.

4.4 Dilaton black holes

By calculating the surface gravity, κ, of the dilaton black hole the temperature [161] is found to be

$$T = \frac{\kappa}{2} = \frac{1}{4r_+}\left(1-\frac{r_-}{r_+}\right)^{\frac{1-a^2}{1+a^2}}, \tag{4.77}$$

which becomes zero when $r_- = r_+$ (for $a = 0$ (RN black hole) and $0 < a < 1$), whilst it remains finite in the case of $a = 1$ and diverges for $a > 1$. Alternatively the temperature of the black hole can also be found by a simple formula, *i.e.* $T = (\partial M/\partial S)_Q$. The entropy in terms of r_- and r_+ reads [161]

$$S = r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}}. \tag{4.78}$$

The Gibbs free energy is given by

$$G = TS = \frac{1}{4}(r_+ - r_-). \tag{4.79}$$

4.4.1 Dilaton black hole with a unit coupling constant

It is a useful exercise to explicitly check these results obtained from the flatness theorem, as we can explicitly read off the physics from the Ruppeiner metric. It is natural to start with the case $a = 1$, as we expect the case of arbitrary a to be complicated. Interestingly from the higher dimensional black p-brane perspective we see that $a = 1$ corresponds to the dimensional reduction of an infinite dimensional object [37], and so this is actually the maximal coupling one could envisage. The fundamental thermodynamic relation of this black hole in terms of mass and charge is given by

$$S = 4M^2 \left(1 - \frac{Q^2}{2M^2}\right). \tag{4.80}$$

Notice that the entropy function vanishes in the extremal limit $Q^2 = 2M^2$, and the temperature of the solution can be written as

$$T = \frac{1}{8M}. \tag{4.81}$$

The corresponding Ruppeiner metric is unsurprisingly simple and takes the following Lorentzian form

$$ds^2 = -8dM^2 + 4dQ^2, \tag{4.82}$$

which is conformally Minkowski if we are able to scale the charge such that $Q \to \sqrt{2}Q$. To better understand the state space let us introduce two new variables

$$t = \sqrt{8}M \quad \text{and} \quad x = 2Q, \tag{4.83}$$

which are constrained to fall in the ranges $x^2/t^2 \leqslant 1$ with $t \in [0, \infty)$ and $x \in (-\infty, \infty)$ [3], and thus we obtain the Ruppeiner metric in Minkowski space as follows

$$ds^2 = -dt^2 + dx^2. \tag{4.84}$$

[3] This may plausibly give a minimal cutoff for the black hole's mass.

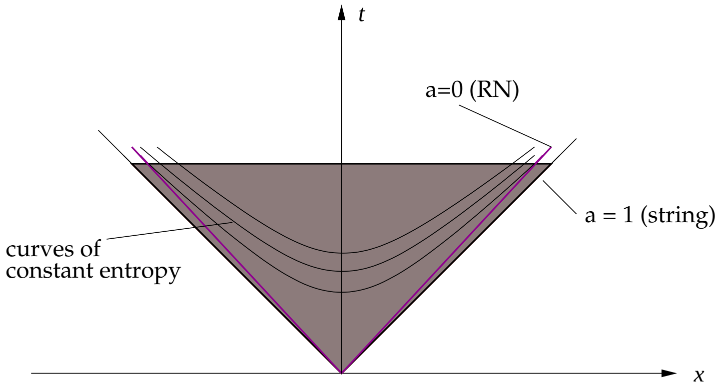

Figure 4.3: *The state space of the dilaton black hole with a unit coupling constant* $a = 1$ *is a wedge on the null cone. This can be compared with the case of Reissner-Nordström black hole in which* $a = 0$*. Curves of constant entropy give a causal structure to this plot.*

The state space of this black hole can thus be presented as a wedge in Minkowski space and one can clearly see that the wedge of the state space lies on the lightcone since (see Fig. 4.3)

$$\frac{x}{t} = 1. \tag{4.85}$$

We can rewrite the black hole's entropy in terms of the re-defined Minkowskian coordinates as

$$S = \frac{1}{2}(t^2 - x^2), \tag{4.86}$$

and it is worth noting that the entropy of the black hole vanishes on the null cone. Out of curiosity we can calculate the Weinhold metric for this black hole in Ruppeiner coordinates and obtain

$$ds_W^2 = -\frac{1}{M}(dM^2 + \frac{1}{2}dQ^2). \tag{4.87}$$

This is a curved metric as expected *a priori*, a result consistent with our previous works which indicated that for two-parameter black hole solutions when one metric is curved, the other is typically flat. The curvature scalar takes the following simple form

$$R_{\text{Weinhold}} = \frac{1}{M}. \tag{4.88}$$

Clearly we see there is no curvature singularity, however the metric remains curved unless we take $M \to \infty$ which reduces the number of physical degrees of freedom. However in the extremal limit of the Weinhold picture we find that the scalar curvature invariant is inversely proportional to the electric charge. Interestingly the above expression is the same as the temperature of the black hole, up to a factor of $\mathcal{O}(1)$.

4.4.2 Dilaton black hole with arbitrary coupling constant

We choose to present the Ruppeiner metric of the dilaton black hole with arbitrary a in a compact form by using Eq. (11) in Ref. [155] for $a = 2$ and $b = -1$ where we take $\psi = -S$. This clearly simplifies to become

$$ds_R^2 = -\frac{1}{2S}dS^2 + S\left(\frac{1}{2}\frac{f'^2}{f^2} - \frac{f''}{f}\right)du^2 \tag{4.89}$$

where $u = Q/M$ and $f = f(u)$ is taken from Eq. (2.73). Our wish is to see how this metric can be written in a manifestly flat form like those we studied earlier in [144, 154]. It is anticipated that the wedge of the state space should fill the entire null cone, as in the case of $a = 1$.

Ideally we would like to transform Eq. (4.89) into coordinates that are functions of S and u which demonstrate the manifest flatness of the metric, *i.e.* we wish to transform Eq. (4.89) into Rindler coordinates ($ds^2 = -d\tau^2 + \tau^2 d\sigma^2$) by choosing

$$\tau = \sqrt{2S}, \tag{4.90}$$

and σ to be some transcendental function of u (*e.g.* $\arcsin$ in the case of RN black hole). Having the metric in Rindler coordinates we can then transform it into the Minkowski metric using

$$t = \tau \cosh\sigma \quad \text{and} \quad x = \tau \sinh\sigma. \tag{4.91}$$

The metric fills the future null cone when $x/t = \tanh\sigma = 1$, which occurs when σ tends to infinity. Put simply we wish to see the range of σ from the transformed metric component.

Unfortunately it is impractical to perform any direct coordinate transformation of the second term of Eq. (4.89) into any transcendental function, since the function f and its derivatives combined are quite intractable. Thus we have to analyse the range of σ knowing the behaviour of the function f, namely that it goes to zero in the extremal limit $u = \sqrt{1 + a^2}$. We can state our goal for the analysis as follows:

Let $g = \frac{1}{2}\frac{f'^2}{f^2} - \frac{f''}{f}$ and let σ be a function of u, therefore the integral of $g(u)$ will represent the full range of σ when integrating it up to the extremal limit,

$$\sigma = \int^{u_{ext}} g(u)du. \tag{4.92}$$

Again this integral is not analytically tractable, therefore we resort to the standard method of power-series expansion about the extremal solution. The expansion is performed about the variable x where $x = \sqrt{1 + a^2} - u$. By definition the extremal limit occurs when $x = 0$. We have done the power-series expansion of g up to $\mathcal{O}(x^2)$ and found that the integral diverges for any $a > 0$, meaning that σ is infinite. Thus we conclude that the wedge of state space for the dilaton black hole with arbitrary coupling constant fills the entire future null cone as anticipated.

That the state space wedge for the dilaton black holes fills the entire thermodynamic light cone shows that thermodynamic geometry can distinguish black hole solutions with genuine extremal limits from those that do not have. We discussed the difference between genuine extremal limits and those that are not in Section 1.1.5.

4.5 Tidal charged black holes

We study thermodynamics of the tidal charged black hole mainly using the geometric methods provided by the thermodynamic metrics. We start by deriving the entropy of the tidal charged black hole and analyzing its mass and tidal charge dependences. These thermodynamic considerations could be useful not only per se, but also for the analysis of the possible 5D extensions of the tidal charged black hole. We summarize our findings in brief as follows:

The tidal charged black hole has the metric

$$ds^2 = -f\left(r\right)dt^2 + f^{-1}\left(r\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{4.93}$$

The metric function f is given as

$$f\left(r\right) = 1 - \frac{2M}{r} + \frac{q}{r^2} \, . \tag{4.94}$$

Such black holes are characterized by two parameters: their mass M and tidal charge q. The exterior horizon for $q > 0$ or the only horizon for $q < 0$ are both given by

$$r_+ = M + \Theta \, , \tag{4.95}$$

where we have introduced the shorthand notation $\Theta = \sqrt{M^2 - q}$, real for any $q \leqslant M^2$. The black hole's entropy can be calculated using the celebrated Bekenstein-Hawking formula (2.4) with geometrized units and $k_B = 1/\pi$ chosen for convenience,

$$S = \frac{A}{4\pi} = r_+^2 = \left(M + \Theta\right)^2 \ . \tag{4.96}$$

By the first law of thermodynamics, the temperature of the black hole is given by

$$T(M,q) = \partial_S M = \frac{1}{\partial_M S} = \frac{\Theta}{2\left(M + \Theta\right)^2} \, . \tag{4.97}$$

The same value $T(M,q)$ is found by computing the temperature of the Hawking radiation if one uses the well-known formula for the surface gravity of a spherically symmetric Killing horizon (see e.g. [163]).

The temperature $T(M,q)$ increases with q for $q < 0$ up to the maximal value $T = 1/(8M)$ at $q = 0$, then decreases with increasing $q > 0$ down to $T = 0$

$$C_q = \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T} = T\left(\frac{\partial T}{\partial S}\right)^{-1} = T\left(\frac{\partial^2 M}{\partial S^2}\right)^{-1} = \frac{-2S(S-q)}{S-3q} \, . \tag{4.98}$$

In the domain of its negative values the heat capacity reaches a local maximum at the Schwarzschild configuration ($\Theta = M$). The thermodynamical interpretation of negative heat capacity is that such a black hole cannot be in a stable equilibrium with an infinite heat reservoir held at $T = T_{BH}(M, q)$. For instance, a small thermal fluctuation may transfer some heat to the black hole and make the black hole colder, thus making heat transfer even more efficient. This is the typical behavior of Schwarzschild black holes, which are unstable with respect to emission of Hawking radiation in empty space and can be stable only in thermal contact with a finite-volume reservoir.

Since the Universe may be considered as an infinite heat reservoir having the temperature of the cosmic background radiation, these considerations may be relevant to the cosmological stability of primordial or near-extremal black holes that have very low temperature. A near-extreme black hole with tidal charge $q > 3M^2/4$ has a positive heat capacity and thus can remain in a stable equilibrium with an infinite heat reservoir at $T = T_{BH}$.

4.5.1 Thermodynamic geometries of the tidal charged black hole

The Ruppeiner metric

The geometry of the tidal charged black hole depends on two parameters: M and q. From the generic definition (3.2) we find the corresponding Ruppeiner metric as

$$ds_R^2 = \frac{1}{\Theta^3}\left[2\left(M - 2\Theta\right)\left(M + \Theta\right)^2 dM^2 - 2(M^2 - \Theta^2)dMdq + \frac{M}{2}dq^2\right] . \qquad (4.99)$$

The Ruppeiner curvature scalar is

$$R = \frac{1}{2\Theta(M + \Theta)}. \qquad (4.100)$$

It is readily seen that the curvature scalar diverges in the extremal limit for $q > 0$, but stays regular for any $q < 0$.

The Weinhold metric

By passing to coordinates $(M,\ \Theta)$ in the Ruppeiner metric and using the conformal relation (3.10) as well as the expression for the temperature, we obtain the Weinhold metric explicitly as

$$ds_W^2 = T ds_R^2 = \frac{-\left(M + 2\Theta\right) dM^2 - 2\Theta dM d\Theta + M d\Theta^2}{\left(M + \Theta\right)^2} . \qquad (4.101)$$

This can be further simplified by introducing the new coordinate r_+ replacing Θ:

$$ds_W^2 = \frac{dr_+}{r_+}\left(M\frac{dr_+}{r_+} - 2dM\right) , \qquad (4.102)$$

then passing to $(Z = \log r_+,\ W = \log(r_+/M^2))$ we find

$$ds_W^2 = M dZ dW\ , \tag{4.103}$$

with

$$M = \exp\left(\frac{Z-W}{2}\right)\ . \tag{4.104}$$

In the coordinates $(U_+ = 2\exp(Z/2)\,,\ U_- = 2\exp(-W/2))$ the Weinhold metric becomes manifestly flat, $ds_W^2 = -dU_+ dU_-$. One can also introduce Minkowskian coordinates as $U_\pm = X \pm Y$, finding $ds_W^2 = -dX^2 + dY^2$. The sequence of coordinate transformations leading to this result can be summarized as

$$\begin{aligned} X &= \sqrt{r_+} + \frac{M}{\sqrt{r_+}}\ , \\ Y &= \sqrt{r_+} - \frac{M}{\sqrt{r_+}}\ . \end{aligned} \tag{4.105}$$

The inverse transformation is

$$\begin{aligned} 4r_+ &= (X+Y)^2\ , \\ 4M &= X^2 - Y^2\ . \end{aligned} \tag{4.106}$$

4.5.2 The global structure of the Ruppeiner geometry

The expression of the temperature in the $(X,\ Y)$ coordinates is

$$T = \frac{r_+ - M}{2r_+^2} = \frac{4Y}{(X+Y)^3}\ , \tag{4.107}$$

which leads to the manifestly conformally flat form of the Ruppeiner metric:

$$ds_R^2 = \frac{(X+Y)^3}{4Y}\left(-dX^2 + dY^2\right)\ . \tag{4.108}$$

Note that the domain of the original Ruppeiner coordinates is

$$M \in (0,\ \infty)\,,\ q \in \left(-\infty,\ M^2\right)\ . \tag{4.109}$$

The corresponding ranges of the variables Θ, r_+ are $\Theta \geqslant 0,\ r_+ \geqslant 0$; the Minkowskian coordinates defined by (4.105) have the range $X > Y \geqslant 0$. Thus the state space is equivalent to the right half of the interior of the future light cone of a Minkowski plane, with the vertical boundary included but the light-like boundary excluded. (The light cone describes $m = 0$ states as can be seen from $4M = X^2 - Y^2$, which for $q \geqslant 0$ does not correspond to black hole metrics.) The extremal states are located at $\left(X = 2\sqrt{M} > 0,\ Y = 0\right)$, i.e. on the positive half of the

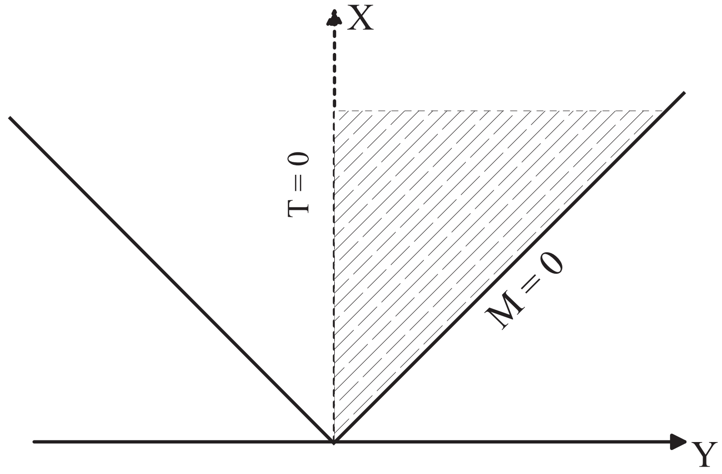

Figure 4.4: A Ruppeiner state space plot of the tidal charged black hole embedded in the flat Minkowskian parameter space. Note that the thermodynamic light cone (TLC) describes $M = 0$ and the wedge fills the right half of the TLC with the light cone excluded. The vertical axis represents the extremal limit in which $T = 0$.

time-like coordinate axis (the vertical boundary). This can be also seen by writing the curvature scalar (4.100) of the Ruppeiner metric in the (X, Y) coordinates:

$$R = \frac{1}{2\,(r_+ - M)\,r_+} = \frac{4}{Y\,(X+Y)^3}\ . \tag{4.110}$$

We also remark that passing to the $(X,\ Y)$ coordinates by the transformation (4.106) induces a degeneracy. For each pair of coordinates $(M,\ q)$, as well as for (M, Θ) or $(M,\ r_+)$, we can associate any of the combinations $(\pm X,\ \pm Y)$, with $X,\ Y$ defined by Eq. (4.105). Therefore the light cone of the Minkowski plane provides a four-fold coverage of the original state space. This is similar to the introduction of the well-known Kruskal coordinates for the Schwarzschild geometry: Kruskal coordinates cover four patches in the Kruskal-Szekeres diagram, while the original coordinates cover only one patch. We end our discussions on the tidal charged black hole by the following remarks:

(A) The induced thermodynamic (Ruppeiner) geometry can have a physical singularity only for $q > 0$, in the extremal mass limit. It is worth noticing that while the Ruppeiner geometry of the tidal charged black hole is non-flat, the Ruppeiner metric for the Reissner-Nordström black hole (when we promote $q = Q^2$ in the metric) is flat [144]. This is a sharp difference emerging in spite of the similarities in the RN and tidal charged black holes.

(B) While the state space of the RN black hole is a Rindler wedge embedded in a Minkowski parameter space, we have found that the state space for the tidal charged black hole is the right half of the interior of the future light cone of the Minkowski plane, with the vertical boundary included but the light-like boundary excluded. The light cone of the Minkowski plane provides a four-fold coverage of this state space, similarly to the four-fold covering of the curvature coordinates for a Schwarzschild black hole by Kruskal coordinates.

(C) With regard to the divergence found in the heat capacity, even tough the heat capacity diverges, the energy (mass) function is regular in the respective point $\Theta = M/2$. In fact we can see that the Ruppeiner metric (4.101) becomes degenerate at that point (the coefficient of dM^2 vanishes) but according to Ruppeiner that is not a sign of phase transition. A contradictory opinion is expressed by Davies [141], according to which a singularity of the heat capacity appears when the black hole undergoes phase transition. In this controversial context we stress again that a singularity in the heat capacity also emerged for RN black holes [144], in the same point where the metric became degenerate, and it was not accompanied by a phase transition. As nothing special happens with the tidal charged black hole at the respective parameter values, we expect the same here.

4.6 State space graphics

In this section we present state space plots in a unified picture. This plot is for flat thermodynamic geometries that can be brought into a Minkowskian form. The main structure of the state space plot consists of a Thermodynamic Light Cone (TLC), which is where the entropies of the black holes vanish. We present conjectures on the wedge structure for higher dimensional black holes based on the established outcomes.

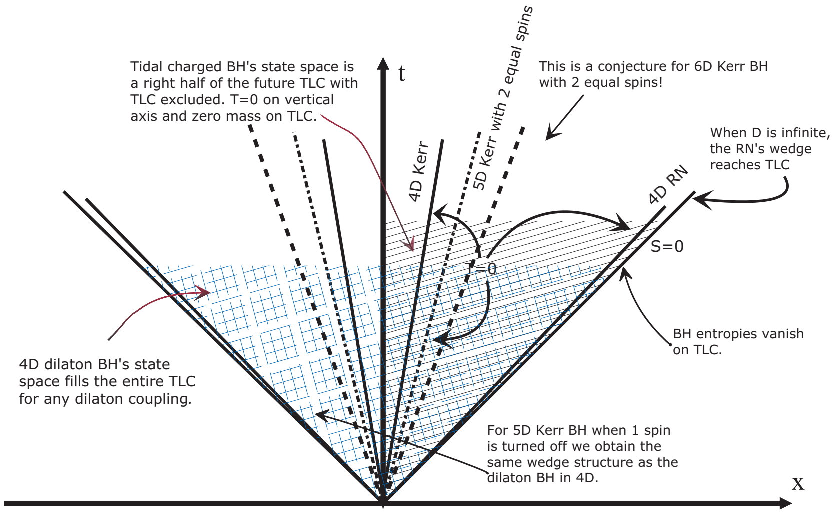

Figure 4.5: A state space plot of Einstein-Maxwell black holes.

Chapter 5
Summary and outlook

Our imagination is stretched to the utmost, not, as in fiction, to imagine things which are not really there, but just to comprehend those things which are there.

–R. P. Feynman

The information geometric theory applied to thermodynamics has generated intruiging results. It provides an alternative and elegant route to obtain insight into thermodynamics through Riemannian geometry. Its power is due to the fact that the Ruppeiner metric together with its associated curvature and signature encodes many aspects of thermodynamics consistent with the known results in systems whose statistical mechanics are known. Since the underlying statistical mechanics of black holes is still unsettled, the application of the Ruppeiner theory to black hole thermodynamics then gives a new perspective on this subject. Although we currently have few results that are physically significant, the outcomes of this project should give a clue for further explorations in this field. We believe that the uncovered geometrical patterns may play an important role in the future, when quantum gravity is better understood.

As a matter of fact, the Ruppeiner theory of black hole thermodynamics can be applied to every class of black holes as long as their fundamental relations are well-defined. The difficulty consists in finding reasonable interpretation(s) of the calculated Ruppeiner geometries. We think that prediction of instabilities of ultraspinning Myers-Perry black holes using our method is probably the most valuable outcome of this research program.

Future works of this research program includes 1) generalization of the flatness theorem to three dimensions 2) understanding the AdS/CFT correspondence in terms of thermodynamic geometries for certain systems, and 3) developing more clever computer programs for computing[1] all the relevant quantities where the only input needed is the fundamental relation.

Last but not least, we end this chapter by presenting the outcomes of this research program as shown in Table 5.1.

[1] this includes plots and stability analysis (based on curvature singularities of BH thermodynamic geometries).

Spacetime dimension	**Black hole family**	**Ruppeiner**	**Weinhold**
$d = 2$	(1+1) RN like BH (generic)	Curved	Curved
	(1+1) reduced RN BH	Flat	Curved
	(1+1) CS like BH (generic)	Curved	Flat
$d = 3$	(2+1) BTZ	Flat	Curved
	(2+1) BTZ (Chern-Simons)	Flat	Curved
	(2+1) BTZ (Log corrections)	Curved	Curved
$d = 4$	RN	Flat	Curved
	Kerr	Curved	Flat
	Kerr-Newman	Curved	Curved
	Braneworld (tidal charged)	Curved	Flat
	Dilaton	Flat	Curved
$d = 5$	Kerr	Curved	Flat
	double-spin Kerr	Curved	Curved
	RN	Flat	Curved
	Black ring	Curved	Flat
any d	Kerr	Curved	Flat
	RN	Flat	Curved

Table 5.1: GEOMETRY OF BLACK HOLE THERMODYNAMICS.

Appendices

Appendix A: Killing vectors, Killing horizons and surface gravity

Normally one would think of the surface gravity (of *e.g.* a planet) as the gravitational acceleration experienced at its surface. However this concept is subtler in the context of black hole physics we discuss in this thesis. Fortunately we have a notion that captures this concept but it is limited to black holes whose event horizons are Killing horizons[1]. The Killing horizon is a null surface to which the Killing vector field $\boldsymbol{\xi}$ is normal. That the norm of $\boldsymbol{\xi}$ is zero on the horizon is a necessary but not a sufficient condition for the Killing horizon[2]. We now state the definition of surface gravity in a mathematical language as follows: if ξ^μ is a suitably normalized Killing vector then the surface gravity, κ, can be defined by

$$\nabla_\mu(\xi^\nu \xi_\nu) = -2\kappa\xi_\mu, \tag{A-1}$$

This equation is to be evaluated at the event horizon, and it implies that $\xi^2 = 0$ which defines a null hypersurface. An important feature of surface gravity is that it globally constant all over the event horizon owing to the following facts:

(a) κ is constant along null generators of the Killing horizon, namely an invariance of Eq. (A-1) under the isometries generated by $\boldsymbol{\xi}$ implies that κ is constant along each null generator, $\xi^\mu \nabla_\mu \kappa = 0$.

(b) κ does not vary from one generator to another. In order to prove that κ does not vary from generator to generator, one uses the fact[3] that for $\kappa \neq 0$ there exists a 2D bifurcation surface, $\mathcal{S}$ on which $\boldsymbol{\xi} = 0$.

[1]In [63] Hawking and Ellis prove that the event horizon of a stationary black hole is a Killing horizon.

[2]For example, in the Kerr spacetime the norm of $t^\mu \equiv \partial_t^\mu$ is zero on the static limit which is not a null hypersurface, but the norm of the Killing vector $\xi^\mu \equiv \partial_t^\mu + \Omega_H \partial_\varphi^\mu$ is zero on the event horizon and there it defines a null hypersurface.

[3]In [59] the constancy of κ is proved without assuming at $\kappa \neq 0$ but requires the use of the Einstein equations with matter obeying the dominant energy condition.

Another way of verifying that κ is constant on the Killing horizon is by showing that $\nabla_{[\alpha}\omega_{\beta]} = 0$ where $\omega_\alpha = \epsilon_{\alpha\beta\gamma\delta}\xi^\beta\nabla^\gamma\xi^\delta$, known as a twist of the $\boldsymbol{\xi}$ field. This proof is given by Racz and Wald in [143].

Eq. (A-1) can also be expressed as

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu. \tag{A-2}$$

The surface gravity also admits the following representation

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu). \tag{A-3}$$

The surface gravity (for static black holes) can be viewed as a force required by an observer at infinity to hold a test particle (by means of an infinitely long, massless string) at the event horizon. For a spherically symmetric spacetime whose Killing vector is $\xi_{(t)} = \partial_t$ the surface gravity is calculated to be

$$\kappa = \frac{1}{2} f'(r_+) \tag{A-4}$$

where $f(r)$ appears in

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2. \tag{A-5}$$

For the Schwarzschild black hole with mass M the surface gravity is given by $\kappa = \frac{1}{4M}$. The Kerr-Newman black hole has a more complicated surface gravity

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} \tag{A-6}$$

where

$$r_+ = M + \sqrt{M^2 - a^2 - Q^2}. \tag{A-7}$$

It is readily seen that in the limit of $J = Q = 0$ we recover the surface gravity of the Schwarzschild black hole.

Appendix B: Dilaton black holes—a transformation between Einstein and string frames

The dilaton black hole in string frame can be transformed into Einstein frame[1] in the following way: We use the action given in [27] with spacelike metric signature as

$$S = \int d^4x \sqrt{-g}\, e^{-2\phi} \left(R + 4(\nabla\phi)^2 - F^2 \right) \tag{B-1}$$

[1] *i.e.* a frame in which there is no function multiplying the Ricci scalar.

This is the low-energy action of string theory in string frame. We want to transform the action in (B-1) into Einstein frame whose spacetime metric is related to the metric of the string frame as:

$$g^E_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta} \Leftrightarrow g_{\alpha\beta} = e^{2\phi} g^E_{\alpha\beta} \tag{B-2}$$

So we need to know how other quantities in (B-1) transform under the conformal transformation in (B-2). There are four quantities that tranform and they do so in the following manner:

$$\text{Metric determinant:} \quad \sqrt{-g} = e^{4\phi}\sqrt{-g^E} \tag{B-3}$$

$$\text{A kinetic term:} \quad (\nabla\phi)^2 = g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi = e^{-2\phi}(\nabla_E\phi)^2 \tag{B-4}$$

$$\text{Field strength term:} \quad F^{\alpha\beta}F_{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} = e^{-4\phi}F^2 \tag{B-5}$$

$$\text{Ricci scalar:} \quad R = e^{-2\phi}R_E - 6e^{-3\phi}\Box e^{\phi} \tag{B-6}$$

Rewriting (B-1) with transformed quantities in the four equations above:

$$S_E = \int d^4x\sqrt{-g}\; e^{4\phi}e^{-2\phi}\left(e^{-2\phi}R_E - 6e^{-3\phi}\Box e^{\phi} + 4e^{-2\phi}(\nabla_E\phi)^2 - e^{-4\phi}F^2\right) \tag{B-7}$$

After integration by parts, this gives the action for dilaton gravity in the Einstein frame as

$$S_E = \int d^4x\,\sqrt{-g^E}\left(R_E - 2(\nabla_E\phi)^2 - e^{-2\phi}F^2\right) \tag{B-8}$$

Note the sign difference in front of the kinetic term.

Appendix C: Proof of the conformal relation between Ruppeiner and Weinhold metrics

The first law of thermodynamics can be written in differential forms as

$$D^2U = dTdS - dpdV + d\mu_i dN_i \tag{C-1}$$

where U, T, S, p, V, μ and N are internal energy, temperature, entropy pressure, chemical potential and particle number respectively. Similarly we have

$$D^2S = d\left(\frac{1}{T}\right)dU + d\left(\frac{p}{T}\right)dV - d\left(\frac{\mu_i}{T}\right)dN_i. \tag{C-2}$$

Invoking the first law of thermodynamics in its original form

$$dU = TdS - pdV + \mu_i dN_i. \tag{C-3}$$

Inserting Eq. (C-3) into the RHS of Eq. (C-2) and we obtain

$$\begin{aligned}
D^2S &= -\frac{1}{T^2}dT(TdS - pdV + \mu_i dN_i) + \frac{TdP - pdT}{T^2}dV - \frac{Td\mu_i - \mu_i dT}{T^2}dN_i \\
&= -\frac{1}{T}dTdS + P\cancel{\frac{dVdT}{T^2}} + \mu_i\cancel{\frac{dNdT}{T^2}} + \frac{dPdV}{T} - P\cancel{\frac{dTdV}{T^2}} - \frac{d\mu_i dN_i}{T} - \mu_i\cancel{\frac{dNdT}{T^2}} \\
&= -\frac{1}{T}(dTdS - dpdV + d\mu_i dN_i).
\end{aligned}$$

Thus

$$D^2S = -\frac{1}{T}D^2U. \tag{C-4}$$

This proves that the Ruppeiner[1] and Weinhold metrics are conformally related

$$g_{ij}^R(U, x^i) = \frac{1}{T}g_{ij}^W(S, x^i). \tag{C-5}$$

where x^i are the conserved parameters.

Appendix D: Degeneracy of Ruppeiner metric

In this appendix we show how extensitivity of the entropic fundamental relation leads to a degenerate Ruppeiner metric. The entropy function of an extensive system *e.g.* for the ideal gas takes the form

$$U(\lambda S, \lambda V, \lambda N) = \lambda U(S, V, N). \tag{D-1}$$

Since we will prove this in general, namely we write

$$U(\lambda S, \lambda x_i) = \lambda U(S, x_i), \tag{D-2}$$

where x_i are extensive parameters. In mathematics Eqs.(D-1) and (D-2) are homogeneous functions of degree one (because the exponent of λ is unity). A subsequent theorem of the homogeneous function is the Euler theorem which states that Eq. (D-2) implies that

$$(S\partial_S + x_i\partial_{x_i})\,U = U. \tag{D-3}$$

[1]The minus sign is absorbed in the definition of the Ruppeiner metric, *i.e.* $g_{ij}^R = -\partial_i\partial_j S$.

Using thermodynamic relations we can rewrite the equation above as

$$ST + x_i y_i = U \tag{D-4}$$

where T is temperature, and $y_i \equiv \partial_{x_i} U$ are intensive parameters. Taking the differential of Eq. (D-4) gives

$$SdT + TdS + x_i dy_i + y_i dx_i = dU. \tag{D-5}$$

Grouping the equation above we can see that (using the recognizable form of the first law of thermodynamics)

$$\cancel{(TdS + y_i dx_i)} + SdT + x_i dy_i = \cancel{dU}, \tag{D-6}$$

thus we are left with

$$SdT + x_i dy_i = 0 \tag{D-7}$$

known as the *Gibbs-Duhem relation*. More elegantly we rewrite it as

$$x^a dy_a = 0, \qquad a = 1, \ldots, n. \tag{D-8}$$

We can express

$$dy_a = \frac{dy_a}{dx^b} dx^b, \tag{D-9}$$

and because

$$y_a = \partial_a U, \tag{D-10}$$

hence

$$dy_a = \frac{\partial^2 U}{\partial x^a \partial x^b} dx^b = g^W_{ab} dx^b \tag{D-11}$$

where we have used the definition of Weinhold metric. We now have

$$x^a dy_a = x^a g^W_{ab} dx^b = 0, \tag{D-12}$$

meaning that

$$x^a g^W_{ab} = 0. \tag{D-13}$$

This implies that x^a are null eigenvectors, therefore the Ruppeiner metric is degenerate.

Bibliography

[1] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **1**, 189 (1916).

[2] R. M. Wald, "General Relativity", The University of Chicago Press, Chicago and London (1984).

[3] S. Carroll, "Spacetime and Geometry: An Introduction to General Relativity", Addison Wesley (2004).

[4] H. Stephani, *et al.* (2003), "Exact solutions of Einstein's field equations", Cambridge, UK, Cambridge University Press (2003).

[5] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einstein'schen Theorie". Annalen der Physik **355**, 106 (1916).
Nordström, G. "On the Energy of the Gravitational Field in Einstein's Theory". Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam **26**: 1201 (1918).

[6] R. L. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **117**, 1595 (1960).

[7] S. Chandrasekhar, "The Mathematical Theory of Black Holes", Oxford University Press, USA (1983)

[8] R. Penrose, Annals N. Y. Acad. Sci. **224**, 125 (1973).

[9] R. Penrose, Rivista del Nuovo Cimento **1**, 252 (1969).

[10] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[11] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, J. Math. Phys. **6**, 918 (1965).

[12] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).

[13] H. Falcke and F. W. Hehl, "The galactic black hole", IOP Publishing Ltd, United Kingdom (2003)

[14] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[15] T. Kaluza, Sitz. Preuss. Akad. Wiss K., **1**, 966 (1921).

[16] O. Klein, Z. Phys. **37**, 895 (1926).

[17] F. R. Tangherlini, Nuovo Cim. **27**, 636 (1963).

[18] R. C. Myers and M. J. Perry, Annals Phys. **172**, 304 (1986).

[19] G. Arfken and H. Weber, "Mathematical Methods for Physicists", Harcourt/Academic Press, (2000).

[20] R. Emparan and H. S. Reall, Living Rev. Rel. **11**, 6 (2008) [arXiv:0801.3471 [hep-th]].

[21] V. P. and I. D. Novikov, "Black Hole Physics", Kluwer Academic Publishers, The Netherlands (1980).

[22] N. A. Obers, Lect. Notes Phys. **769**, 211 (2009) [arXiv:0802.0519 [hep-th]].

[23] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002) [arXiv:hep-th/0110260].

[24] R. Emparan, JHEP **0403**, 064 (2004) [arXiv:hep-th/0402149].

[25] R. Emparan and H. S. Reall, Class. Quant. Grav. **23**, R169 (2006) [arXiv:hep-th/0608012].

[26] M. M. Caldarelli, R. Emparan and M. J. Rodriguez, JHEP **0811**, 011 (2008) [arXiv:0806.1954 [hep-th]].

[27] C. G. . Callan, E. J. Martinec, M. J. Perry and D. Friedan, Nucl. Phys. B **262**, 593 (1985).

[28] G. W. Gibbons and K.-i. Maeda, Nucl. Phys. B **298**, 741 (1988).

[29] J. H. Horne and G. T. Horowitz, Nucl. Phys. B **399**, 169 (1993).

[30] S. J. Poletti, J. Twamley and D. L. Wiltshire, Class. Quant. Grav. **12**, 1753 (1995) [Erratum-ibid. **12**, 2355 (1995)] [arXiv:hep-th/9502054].

[31] C. J. Gao and S. N. Zhang, Phys. Rev. D **70** 124019 (2004).

[32] C. J. Gao and S. N. Zhang, Phys. Lett. B **612**, 127 (2005).

[33] J. H. Horne and G. T. Horowitz, Phys. Rev. D **46**, 1340 (1992).

[34] B. Jensen and U. Lindström, Phys. Rev. D **52** 3543 (1995).

[35] V. Frolov, A. Zelnikov and U. Bleyer, Ann. Phys. (Leipzig) **44** 371 (1987).
B. Jensen and U. Lindström, Class. Quant. Grav. **11** 2435 (1994).

[36] G. W. Gibbons, Nucl. Phys. B **298**, 741 (1988).

[37] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quant. Grav. **12**, 297 (1995).

[38] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D **43**, 3140 (1991).

[39] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].

[40] S. Carlip, Class. Quant. Grav. **22**, R85 (2005) [arXiv:gr-qc/0503022].

[41] S. Carlip, "Quantum Gravity in 2+1 Dimensions: The Case of a Closed Universe", Living Rev. Relativity 8, (2005), 1. URL (cited on 11 April 2009): http://www.livingreviews.org/lrr-2005-1

[42] D. Grumiller, W. Kummer and D. V. Vassilevich, Phys. Rept. **369**, 327 (2002) [arXiv:hep-th/0204253].

[43] D. Grumiller and R. Meyer, Turk. J. Phys. **30**, 349 (2006) [arXiv:hep-th/0604049].

[44] R. Maartens, "Brane-World Gravity", Living Rev. Relativity **7**, (2004), 7.

URL (cited on 20 April 2009): http://www.livingreviews.org/lrr-2004-7

[45] T. Shiromizu, K.-i. Maeda, M. Sasaki, Phys. Rev. D **62,** 024012 (2000).

[46] L. Á. Gergely, Phys. Rev. D **68**, 124011 (2003).

[47] N. Dadhich, R. Maartens, P. Papadopoulos, V. Rezania, Phys. Lett. B **487**, 1 (2000).

[48] Y. S. Myung, Y. W. Kim and Y. J. Park, Phys. Rev. D **78**, 084002 (2008) [arXiv:0805.0187 [gr-qc]].

[49] C. Germani, R. Maartens, Phys. Rev. D **64**, 124010 (2001).

[50] L. Á. Gergely, B. Darázs, Publ. Astron. Dept. Eötvös Univ. (PADEU) **17**, 213 (2006); [arXiV:astro-ph/0602427].

[51] C. G. Boehmer, T. Harko, F. S. N. Lobo, Class. Quant. Grav. **25**, 045015 (2008).

[52] C. S. J. Pun, Z. Kovacs and T. Harko, Phys. Rev. D **78**, 084015 (2008) [arXiv:0809.1284 [gr-qc]].

[53] R. M. Wald, "The Thermodynamics of Black Holes", Living Rev. Relativity **4** 2001

URL: http://www.livingreviews.org/lrr-2001-6 (cited on 15 April 2009)

[54] S. W. Hawking, "A brief history of time", A Bantam Book, United States of America (1996)

[55] J. D. Bekenstein, Lett. Nuovo Cim. **4**, 737 (1972).

[56] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[57] S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971)

[58] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[59] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).

[60] S. W. Hawking, Nature **248**, 30 (1974).

[61] I. Carusotto *et al.* , New J. Phys. **10**, 103001, (2008).

[62] R. Ruffini and J. A. Wheeler, Physics Today **24**, 30 (1971).

[63] G. F. Ellis and S. W. Hawking, "The Large-Scale Structure of Spacetime", Cambridge University Press, (1973).

[64] M. Heusler, "Black Hole Uniqueness Theorems", Cambridge University Press, Cambridge; New York, (1996).

[65] M. Aizenman and E. H. Lieb, J. Stat. Phys. **24**, 279, (1981).

[66] R. M. Wald, Phys. Rev. D **56**, 6467 (1997) [arXiv:gr-qc/9704008].

[67] S. Isakov, "Correlations in Frustrated Magnets: Classical and Quantum Aspects", PhD Thesis, Stockholm University, Sweden, (2004)

[68] R. L. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **122**, 997 (1961).

[69] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973) [Erratum ibid **30**, 521 (1973)].

[70] M. M. Caldarelli, G. Cognola and D. Klemm, Class. Quant. Grav. **17**, 399 (2000)

[71] S. W. Hawking, C. J. Hunter and M. Taylor, Phys. Rev. D **59**, 064005 (1999) [arXiv:hep-th/9811056].

[72] F. Hoyle and J. V. Narlikar, "Action at a Distance in Physics and Cosmology", Freeman, San Francisco, (1974)

[73] A. K. Kembhavi and M. D. Pollock, Mon. Not. R. astr. Soc. **197**, 1087, (1981)

[74] G. W. Gibbons and M. J. Perry, Proc. R. Soc. London A **358**, 467, (1978)

[75] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87** (1983) 577.

[76] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 505 [arXiv:hep-th/9803131].

[77] G. W. Gibbons, M. J. Perry and C. N. Pope, Class. Quant. Grav. **22**, 1503 (2005) [arXiv:hep-th/0408217].

[78] G. 't Hooft, Nucl. Phys. B **256**, 727 (1985).

[79] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54**, 2171 (1985).

[80] L. Bombelli, R. K. Koul, J. H. Lee and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).

[81] L. Susskind, arXiv:hep-th/9309145.

[82] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996) [arXiv:hep-th/9601029].

[83] G. T. Horowitz and D. Marolf, Phys. Rev. D **55**, 3654 (1997) [arXiv:hep-th/9610171].

[84] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998) [arXiv:gr-qc/9710007].

[85] J. F. B. G. and E. J. S. Villasenor, Class. Quant. Grav. **26**, 035017 (2009) [arXiv:0810.1599 [gr-qc]].

[86] R. D. Sorkin, Stud. Hist. Philos. Mod. Phys. **36**, 291 (2005) [arXiv:hep-th/0504037].

[87] Shun'ichi Amari, Hiroshi Nagaoka, "Methods of information geometry", Transactions of mathematical monographs; v. 191, American Mathematical Society, (2000).

[88] P. Salamon, "Thermodynamic Geometry", URL (cited on 15 April 2009): www-rohan.sdsu.edu/∼psalamon/GeomThermo/UdineNotes.pdf

[89] J. Gibbs, Vol. 1, Thermodynamics, Yale University Press, 1948.

[90] C. Charatheodory, Gesammelte Mathematische Werke, Band 2, Munich, 1995.

[91] R. Hermann, "Geometry, physics and systems", Marcel Dekker, New York, 1973.

[92] R. Mrugała, Rep. Math. Phys. **14**, 419 (1978).

[93] R. Mrugała, Rep. Math. Phys. **21**, 197 (1985).

[94] H. Geiges, "Contact Geometry", arXiv:math/0307242, (2003).

[95] F. Weinhold, J. Chem. Phys. **63**, 2479 (1975).

[96] G. Ruppeiner, Phys. Rev. A **20**, 1608 (1979)

[97] R. Gilmore, J. Chem. Phys. **75**, 5964 (1981).

[98] T. Feldman, B. Andersen, A. Qi and P. Salamon, Chem. Phys. **83**, 5849 (1985).

[99] G. Ruppeiner, Rev. Mod. Phys. **67**, 605 (1995).

[100] P. Salamon, B. Andersen, P. D. Gait and R. S. Berry, J. Chem. Phys. **73**, 1001 (1980).

[101] P. Salamon, J. Nulton and E. Ihrig, J. Chem. Phys. **80**, 436 (1984).

[102] P. Salamon, J. Nulton and J. D. Berry, J. Chem. Phys. **82**, 2433 (1985).

[103] J. Nulton and P. Salamon, Phys. Rev. A **31**, 2520 (1985).

[104] M. Santoro, J. Chem. Phys. **121**, 2932 (2004).

[105] M. Santoro, Chem. Phys. **310**, 269 (2005).

[106] M. Santoro, Chem. Phys. **313**, 331 (2005).

[107] M. Santoro and Serge Preston, "Curvature of the Weinhold metric for thermodynamical systems with 2 degrees of freedom", arXiv:math-ph/0505010, (2005)

[108] G. F. Torres del Castillo and M. Montesinos-Velasquez, Rev. Mex. Fís. **39**, 194 (1993).

[109] G. Hernández and E. A. Lacomba, Diff. Geom. and Appl. **8**, 205 (1998).

[110] W. Janke, D. A. Johnston, and R. Kenna, Physica A **336**, 181 (2004).

[111] D.A. Johnston, W. Janke, and R. Kenna, Acta Phys. Polon. B **34**, 4923 (2003).

[112] L. Landau and E. M. Lifshitz, "Statistical Physics", Pergamon Press, London (1980).

[113] R. Mrugała, Physica **125A**, 631 (1984).

[114] D. C. Brody and A. Ritz, "Geometric phase transitions", arXiv:cond-mat/9903168, (1999).

[115] D. Page, a letter to Physics Today, pp 12-14, January (1977)

[116] S. Ferrara, G. W. Gibbons and R. Kallosh, Nucl. Phys. B **500**, 75 (1997) [arXiv:hep-th/9702103].

[117] R. G. Cai and J. H. Cho, Phys. Rev. D **60**, 067502 (1999) [arXiv:hep-th/9803261].

[118] J. E. Åman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **35** 1733 (2003).

[119] J. E. Åman and N. Pidokrajt, Phys. Rev. D **73** 024017 (2006).

[120] G. Arcioni and E. Lozano-Tellechea, Phys. Rev. D **72**, 104021 (2005) [arXiv:hep-th/0412118].

[121] J. y. Shen, R. G. Cai, B. Wang and R. K. Su, Int. J. Mod. Phys. A **22**, 11 (2007) [arXiv:gr-qc/0512035].

[122] J. E. Åman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **38**, 1305 (2006) [arXiv:gr-qc/0601119].

[123] T. Sarkar, G. Sengupta and B. Nath Tiwari, JHEP **0611**, 015 (2006) [arXiv:hep-th/0606084].

[124] B. Mirza and M. Zamani-Nasab, JHEP **0706**, 059 (2007) [arXiv:0706.3450 [hep-th]].

[125] A. J. M. Medved, Mod. Phys. Lett. A **23**, 2149 (2008) [arXiv:0801.3497 [gr-qc]].

[126] B. N. Tiwari, arXiv:0801.4087 [hep-th].

[127] G. Ruppeiner, Phys. Rev. D **78**, 024016 (2008) [arXiv:0802.1326 [gr-qc]].

[128] Y. S. Myung, Y. W. Kim and Y. J. Park, Phys. Lett. B **663**, 342 (2008) [arXiv:0802.2152 [hep-th]].

[129] T. Sarkar, G. Sengupta and B. N. Tiwari, JHEP **0810**, 076 (2008) [arXiv:0806.3513 [hep-th]].

[130] S. Bellucci and B. N. Tiwari, arXiv:0808.3921 [hep-th].

[131] L. A. Gergely, N. Pidokrajt and S. Winitzki, arXiv:0811.1548 [gr-qc].

[132] S. Bellucci, V. Chandra and B. N. Tiwari, arXiv:0812.3792 [hep-th].

[133] H. Quevedo, Gen. Rel. Grav. **40**, 971 (2008) [arXiv:0704.3102 [gr-qc]].

[134] H. Quevedo and A. Vazquez, AIP Conf. Proc. **977**, 165 (2008) [arXiv:0712.0868 [math-ph]].

[135] J. L. Alvarez, H. Quevedo and A. Sanchez, Phys. Rev. D **77**, 084004 (2008) [arXiv:0801.2279 [gr-qc]].

[136] H. Quevedo and A. Sanchez, Phys. Rev. D **79**, 024012 (2009) [arXiv:0811.2524 [gr-qc]].

[137] H. Quevedo and A. Sanchez, arXiv:0902.4488 [gr-qc].

[138] H. Quevedo and A. Sanchez, JHEP **0809**, 034 (2008) [arXiv:0805.3003 [hep-th]].

[139] I. Bengtsson and K. Życzkowski, "Geometry of Quantum States", Cambridge University Press (2006).

[140] S. Ferrara, G. W. Gibbons and R. Kallosh, Nucl. Phys. B **500**, 75 (1997) .

[141] P. C. W. Davies, Proc. R. Soc. Lond. A **353**, 499 (1977) .

[142] H. Elvang and R. Emparan, JHEP **0311**, 035 (2003) [arXiv:hep-th/0310008].

[143] I. Racz and R. M. Wald, Class. Quant. Grav. **13**, 539 (1996) [arXiv:gr-qc/9507055].

[144] J. E. Åman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **35**, 1733 (2003) [arXiv:gr-qc/0304015].

[145] J. E. Åman, "Manual for CLASSI: Classification Programs for Geometries in General Relativity", ITP, Stockholm University, Technical Report, 2002. Provisional edition. Distributed with the sources for SHEEP and CLASSI.

[146] J. E. Åman, J. Bedford, D. Grumiller, N. Pidokrajt and J. Ward, J. Phys. Conf. Ser. **66**, 012007 (2007) [arXiv:gr-qc/0611119].

[147] W. Kummer and D. V. Vassilevich, Annalen Phys. **8**, 801 (1999) [arXiv:gr-qc/9907041].

[148] J. Gegenberg, G. Kunstatter and D. Louis-Martinez, Phys. Rev. D **51**, 1781 (1995) [arXiv:gr-qc/9408015].

[149] E. Witten, Phys. Rev. D **44**, 314 (1991).

C. G. Callan, *et al.* , Phys. Rev. D **45**, 1005 (1992).

[150] R. Jackiw and C. Teitelboim, "Quantum Theory Of Gravity", ed Christensen S (Bristol: Adam Hilger) (1984).

[151] G. Guralnik, A. Iorio, R. Jackiw and S. Y. Pi, Annals Phys. **308**, 222 (2003).

D. Grumiller and W. Kummer, Annals Phys. **308**, 211 (2003).

L. Bergamin, D. Grumiller, A. Iorio and C. Nuñez, JHEP **0411**, 021 (2004).

[152] A. Achúcarro and M. E. Ortiz, Phys. Rev. D **48**, 3600 (1993).

[153] D. Grumiller and R. Jackiw, Phys. Lett. B **642**, 530 (2006) [arXiv:hep-th/0609197].

[154] J. E. Åman and N. Pidokrajt, Phys. Rev. D **73**, 024017 (2006) [arXiv:hep-th/0510139].

[155] J. E. Åman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. **38**, 1305 (2006) [arXiv:gr-qc/0601119].

[156] J. E. Åman, N. Pidokrajt and J. Ward, EAS Publ. Ser. **30**, 279 (2008) [arXiv:0711.2201 [hep-th]].

[157] R. Emparan and R. C. Myers, JHEP **0309**, 025 (2003) [arXiv:hep-th/0308056].

[158] R. Monteiro, M. J. Perry and J. E. Santos, arXiv:0903.3256 [gr-qc].

[159] H. Kodama, R. A. Konoplya and A. Zhidenko, arXiv:0904.2154 [gr-qc].

[160] O. J. C. Dias, P. Figueras, R. Monteiro, J. E. Santos and R. Emparan, arXiv:0907.2248 [hep-th].

[161] G. W. Gibbons, "Metaphysics", unpublished

[162] L. A. Gergely, N. Pidokrajt and S. Winitzki, arXiv:0811.1548 [gr-qc].

[163] T. Padmanabhan, Phys. Rept. **406**, 49 (2005) [arXiv:gr-qc/0311036].

Index

ADM mass, 4
AdS, *see* Anti de Sitter
AdS/CFT correspondence, 6, 7
Anti de Sitter space, 13

Barbero-Immirzi parameter, 34
Bekenstein-Hawking entropy, 22
black hole
  acoustic, 19
  analogs, 19
  braneworld, 14
  BTZ, 12, 51
  dilaton, 10
  extremal BTZ, 13
  extremal Kerr, 6
  higher-dimensional, 6
  Kerr, 5
  Kerr-Newman, 5
  mini, 19
  multiple-spin Kerr, 9
  Myers-Perry, 6
  physical temperature, 18, 20
  Tangherlini, 6
  tidal charged, 14
  two-dimensional, 13
  uniqueness theorems, 20
black ring, 6
Boyer-Lindquist coordinates, 5, 9

canonical ensemble, 24
Carter-Penrose diagram, 3
Cauchy horizon, 4, 8
CLASSI, 51
conformal transformation, 73
contact geometry, 35
cosmic censorship conjecture, 5
cosmological constant, 6

degeneracy of the Ruppeiner metric, 74

Einstein frame, 72
Einstein-Maxwell action, 7, 26
ergosphere, 6
event horizon, 13

Fisher information matrix, 42
Fisher-Rao metric, 42
fundamental relation, 24

generalized second law, 22
geometrothermodynamics, 42
Gibbs-Duhem relation, 75
gravitational collapse, 5
GRTensor, 51
GSL, 22

Hawking radiation, 19, 21
Hawking temperature, 18, 19
Hawking-Page phase transition, 24

Immirzi parameter, 34

Kretschmann scalar, 1

loop quantum gravity, 34

Maxwell field-strength tensor, 7
metric
  Kerr, 5

microcanonical ensemble, 24

naked singularity, 5
negative specific heat, 23, 24
no hair theorem, 19
null hypersurface, 3

Penrose diagram, 3
positive heat capacity, 51

quasi-homogeneity, 43

Rindler wedge, 54
ring singularity, 6

Sackur-Tetrode equation, 38
scale invariance, 26
self-gravitating systems, 23
Shannon entropy, 42
singularity, 22
state parameters, 20
string frame, 72
surface gravity, 18, 20

TeV-scale gravity, 7
thermodynamic light cone, 65
thermodynamic geometry, 38

Unruh effect, 18

weak energy condition, 21

About the author

Narit Pidokrajt was born in May 1979 in Songkhla, Kingdom of Thailand. He obtained his primary and secondary education in the Southern cities of Songkhla and Hat Yai respectively. He did his undergraduate studies in physics between 1997-1999 at Chiang Mai University in Northern Thailand. He obtained his MSc in physics from Lund University, Sweden in 2003 before coming to Stockholm. Narit began his PhD studies in theoretical physics in October 2004 with Professor Ingemar Bengtsson as a supervisor and Professor Jan E. Åman as an adviser. His field of specialization is black hole physics, in particular the *geometric* approach for studying an interface between thermodynamics and statistical mechanics of black holes, which opens up new perspectives on the subject. He was the first to obtain the thermodynamic (Ruppeiner) scalar for the spinning "Kerr" black hole in four dimensions.

Narit has given seminars, presentations and talks on his PhD research in a number of European countries and recently in South Africa. He visited several research institutes and universities in the past 4 years for research collaborations, with the longest visit of one month at Universitat de Barcelona, Spain where he was a visitor of Professor Roberto Emparan. Narit has worked as a teaching assistant for statistical mechanics and computational physics courses at Stockholm University during his PhD studies. Apart from the scientific work he has also been involved in other activities, *e.g.* he was a full member of the PhD student council for 3 years, and a webmaster of the KoF research group, "`www.kof.physto.se`" from 2005 to 2009.

Part II:

Papers