The stack of projective schemes with vanishing second cohomology

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Abstract

In this paper we describe a very general stack parametrizing projective schemes $X$ such that $H^2(X, O_X) = 0$. We show in detail that this stack is algebraic using Artin’s criteria. The main purpose of introducing this stack is the study of the Fulton-MacPherson compactification of the configuration space of points. A secondary purpose is to showcase the technical methods used in verifying Artin’s criteria.

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Introduction

In this paper we define a very general stack $\mathcal{X}_{pv2}$ parametrizing projective schemes with the property that their structure sheaf has vanishing second cohomology. In other words, for each scheme $T$ the category $\mathcal{X}_{pv2}(T)$ consists of maps $X \to T$ where $X$ is an algebraic space and the map is flat, proper and of finite presentation. Furthermore, the geometric fibers $X_t$ for each point $t \in T$ are projective schemes satisfying $H^2(X_t, \mathcal{O}_{X_t}) = 0$. We show in detail using Artin’s criteria that the stack $\mathcal{X}_{pv2}$ is algebraic.

We use the stack $\mathcal{X}_{pv2}$ primarily to study the Fulton-MacPherson compactification of the configuration space of points. For a smooth variety or scheme $X$ the configuration space $F(X,n)$ parametrizes ordered $n$-tuples of distinct points on $X$. The space $F(X,n)$ is easily described as

$$F(X,n) = \underbrace{X \times \cdots \times X}_{n} - \Delta$$

where $\Delta$ is the diagonal locus where two or more points coincide. Fulton and MacPherson [FM94] introduce a smooth compactification $X[n]$ of $F(X,n)$ called the Fulton-MacPherson compactification. The space $X[n]$ is created from the $n$-fold product $X^n$ by a sequence of blowups along the diagonal locus $\Delta$.

The compactification $X[n]$ is not constructed as a moduli space, but the points on the boundary of $X[n]$ can still be interpreted as certain geometric objects called stable $n$-pointed degenerations of $X$. A stable degeneration can be described as follows: suppose that two labelled points $p_1, p_2$ of $X$ come together at some point $x \in X$. Then we blow up $x$, creating an exceptional divisor $E \cong \mathbb{P}^{d-1}$, where $d = \dim(X)$. Next we attach a projective space $\mathbb{P}^d$ to $\text{Bl}_x(X)$, identifying the hyperplane at infinity $H \subseteq \mathbb{P}^d$ with the exceptional divisor $E$. Next we put the two labelled points $p_1, p_2$ on the open set $\mathbb{P}^d - H$ of the attached component $\mathbb{P}^d$. The points on the attached component encodes the directions in which $p_1$ and $p_2$ approach each other when coming together. If we have more points that come together we might need to blow up points on the attached $\mathbb{P}^d$ and continue the process. Stable degenerations are described in more detail in [FM94, p. 194].

It thus natural to try and create a moduli space $SD_{X,n}$ parametrizing stable $n$-pointed degenerations of $X$ and to ask whether $SD_{X,n}$ is isomorphic to $X[n]$. It is not difficult to define the moduli stack $SD_{X,n}$ but the work lies in showing that it is algebraic. The algebraic stack $\mathcal{X}_{pv2}$ defined in this paper may be used to derive algebraicity of a number of related stacks. The most important of these is the stack $\mathcal{X}_{X,n}$ parametrizing tuples $(W, f, p_1, \ldots, p_n)$ where $W$ is a projective scheme with $H^2(W, \mathcal{O}_W) = 0$ together with a map $f: W \to X[n]$. We show that this stack is algebraic.
$f : W \to X$ and $n$ points $p_1, \ldots, p_n$ in the smooth locus of $W$. The hope is to show that if $X = \mathbb{P}^2$ then $SD_{X,n}$ is an open substack of $\mathcal{X}_{X,n}$. Thus we would obtain algebraicity of $SD_{X,n}$ in the case $X = \mathbb{P}^2$.

Another reason for introducing $\mathcal{X}_{\mathbb{P}^2}$ is to demonstrate the various techniques needed to verify Artin’s criteria. Also, some classical moduli spaces can be realized as open substacks of $\mathcal{X}_{\mathbb{P}^2}$ or one of the related stacks and we therefore obtain algebraicity of these classical stacks as well.

1 Preliminary notions on stacks

In this section we recall some preliminary material regarding categories fibered in groupoids and stacks.

**Definition 1.1** (Categories fibered in groupoids). Let $\mathcal{C}$ be a category. A category $\mathcal{X}$ together with a functor $p : \mathcal{X} \to \mathcal{C}$ is called a *category fibered in groupoids* over $\mathcal{C}$, or simply a *CFG* over $\mathcal{C}$ if the following conditions hold:

(i) (Existence of pullbacks.) For each map $\varphi : V \to U$ of $\mathcal{C}$ and each object $x$ of $\mathcal{X}$ such that $p(x) = U$, there is a morphism $f : y \to x$ of $\mathcal{X}$ such that $p(f) = \varphi$.

(ii) (Unique liftings of maps.) For every diagram

\[
\begin{array}{ccc}
z & \xrightarrow{h} & y \\
\downarrow & & \downarrow \\
x & \xleftarrow{f} & y
\end{array}
\]

in $\mathcal{X}$ with image

\[
\begin{array}{ccc}
W & \xrightarrow{x} & V \\
\downarrow & & \downarrow \\
U & \xleftarrow{\varphi} & V
\end{array}
\]

under $p$ and every morphism $\psi : W \to V$ of $\mathcal{C}$ such that $\chi = \varphi \circ \psi$ there is a unique map $g : z \to y$ of $\mathcal{X}$ such that $h = f \circ g$ and $p(g) = \psi$.

Let $U$ be an object of $\mathcal{C}$, and denote by $\mathcal{X}(U)$ the subcategory of $\mathcal{X}$ consisting of objects that map to $U$ via $p$, and morphisms that map to $\text{id}_U$ via $p$. Item (ii) ensures that the category $\mathcal{X}(U)$ is a *groupoid*, i.e. that all morphisms in $\mathcal{X}(U)$ are isomorphisms. Also, if $\varphi : V \to U$ is a map in $\mathcal{C}$ and $x$ is an object of $\mathcal{X}(U)$, then item (i) ensures the existence of an object $y \in \mathcal{X}(V)$ and item (ii) gives the object $y$ up to unique isomorphism.
Thus for each map \( \varphi : V \to U \) we obtain a “pullback” functor \( \varphi^* : \mathcal{X}(U) \to \mathcal{X}(V) \). We may therefore think of the assignment \( U \mapsto \mathcal{X}(U) \) as a contravariant functor from \( \mathcal{C} \) to the category of groupoids. If \( x \) is an object of \( \mathcal{X}(U) \) we may also denote the object \( \varphi^*(x) \) of \( \mathcal{X}(V) \) by \( x|V \), if the map \( \varphi \) is understood.

**Example 1.2.** Let \( \mathcal{C} \) be the category of schemes \( \text{Sch} \) and let \( \mathcal{X} \) be the category whose objects are morphisms \( f : X \to S \) of schemes. We think of \( f \) as a family of schemes over the base \( S \). Morphisms in \( \mathcal{X} \) are cartesian diagrams

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
T & \varphi & \longrightarrow S.
\end{array}
\]

The functor \( p : \mathcal{X} \to \text{Sch} \) sends an object \( f : X \to S \) to the scheme \( S \) and a cartesian diagram as above to the underlying morphism \( \varphi : T \to S \) of schemes. Then \( \mathcal{X} \) is a CFG over \( \text{Sch} \).

For a scheme \( S \), the category \( \mathcal{X}(S) \) is the category of schemes over \( S \) with isomorphisms as maps, and if \( \varphi : T \to S \) is a morphism of schemes the functor \( \varphi^* : \mathcal{X}(S) \to \mathcal{X}(T) \) is defined by taking fiber products \( X \mapsto X \times_S T \).

**Definition 1.3 (Stack).** Let \( S \) be a scheme and let \( \text{Sch}/S \) be the category of schemes over \( S \), i.e. the category whose objects are morphisms of schemes \( U \to S \). Consider a CFG \( \mathcal{X} \) over \( \text{Sch}/S \). We say that \( \mathcal{X} \) is a stack for the \( \acute{e}tale \) topology over \( \text{Sch}/S \) or simply stack over \( \text{Sch}/S \) if the following hold:

(i) (Isom-functors are sheaves.) For every object \( U \) of \( \text{Sch}/S \) and every pair of objects \( x, y \) of \( \mathcal{X}(U) \) the functor

\[\text{Isom}(x, y) : \text{Sch}/U \longrightarrow \text{Sets}\]

that maps a \( U \)-scheme \( V \) to the set \( \text{Hom}_{\mathcal{X}(V)}(x|V, y|V) \) is a sheaf for the \( \acute{e}tale \) topology on \( \text{Sch}/U \).

(ii) (Effectivity of descent data.) Let \( U \) be an object of \( \text{Sch}/S \) and let \( \mathcal{V} = \{ \varphi_i : V_i \to U \}_{i \in I} \) be a family of \( \acute{e}tale \) morphisms whose images cover \( U \). A descent datum for \( \mathcal{V} \) is a set of pairs \( \{(x_i, f_{ij})\}_{(i,j) \in I \times I} \) where \( x_i \) is an object of \( \mathcal{X}(V_i) \) and \( f_{ij} \) is an isomorphism

\[f_{ij} : (x_j|V_{ij}) \longrightarrow (x_i|V_{ij})\]

in \( \mathcal{X}(V_{ij}) \), where \( V_{ij} = V_i \times_U V_j \). The data is required to satisfy the cocycle condition

\[(f_{ik}|V_{ijk}) = (f_{ij}|V_{ijk}) \circ (f_{jk}|V_{ijk})\]

for all \( i, j, k \) in \( I \).
in $\mathcal{X}(V_{ijk})$, where $V_{ijk} = V_i \times_U V_j \times_U V_k$, for every $i, j, k \in I$.

Any such descent datum for $V$ is then effective, i.e. there is an object $x$ of $\mathcal{X}(U)$ and isomorphisms $f_i : (x|V_i) \rightarrow x_i$ for all $i \in I$ such that

$$(f_i|V_{ij}) = f_{ij} \circ (f_j|V_{ij})$$

for all $i, j \in I$.

Condition (i) guarantees that the object $x$ produced in condition (ii) is unique up to unique isomorphism.

**Definition 1.4 (Algebraic stack).** Let $S$ be a scheme and let $\mathcal{X}$ be a stack over the category $\textbf{Sch}/S$. The stack $\mathcal{X}$ is called an *Algebraic stack* or an *Artin stack* if the following two conditions are satisfied:

(i) The diagonal morphism $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is representable, i.e. for each map $U \rightarrow \mathcal{X} \times_S \mathcal{X}$ where $U$ is in $\textbf{Sch}/S$, we have that the fiber product $U \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}$ is an algebraic space.

(ii) There exists a smooth surjective morphism $U \rightarrow \mathcal{X}$ where $U$ is a scheme over $S$. The scheme $U$ is called a *presentation* of $\mathcal{X}$.

If we can choose the surjective morphism in (ii) to be étale, then $\mathcal{X}$ is called a *Deligne-Mumford stack*.

### 2 Artin’s criteria

In this section we give necessary criteria for a stack $\mathcal{X}$ to be an Artin stack. These criteria are known as Artins criteria. The letter $S$ will usually denote a scheme of finite type over an excellent Dedekind ring. We will also identify a ring $A$ with its corresponding affine scheme $\text{Spec}(A)$ and consider it as an object of the category $\textbf{Sch}/S$. Thus when we say for instance that $A$ is of finite type over $S$ we will mean that $\text{Spec}(A)$ is of finite type over $S$. Often we choose $S = \text{Spec}(\mathbb{Z})$ or $S = \text{Spec}(k)$ for a field $k$.

If $A$ is a ring and $\text{Spec}(A)$ is a scheme over $S$, then we will write $\mathcal{X}^e(A)$ instead of $\mathcal{X}^e(\text{Spec}(A))$. If $b \in \mathcal{X}^e(B)$ is an object and $A \rightarrow B$ is a homomorphism of rings, we write $\mathcal{X}^e_b(A)$ for the subcategory of $\mathcal{X}^e(A)$ consisting of objects mapping to $b$ and morphisms mapping to $\text{id}_b$ via the natural functor $\mathcal{X}^e(A) \rightarrow \mathcal{X}^e(B)$.

Following Artin [Art74] we denote the set of isomorphism classes of $\mathcal{X}^e(A)$ by $\mathbb{F}(A)$, and the set of isomorphism classes of $\mathcal{X}^e_b(A)$ is denoted $\mathbb{F}_b(A)$.
Definition 2.1. A surjective map of rings $A' \to A$ with nilpotent kernel is called an \emph{infinitesimal extension}. Let $A' \to A \to A_0$ be a sequence of maps of rings. The sequence is called a \emph{deformation situation} if the maps $A' \to A$ and $A \to A_0$ are infinitesimal extensions and $\text{Ker}(A' \to A) \cdot \text{Ker}(A' \to A_0) = 0$. Then we have in particular that $\text{Ker}(A' \to A)$ is an $A_0$-module.

If $M$ is an $A$-module we use the notation $A + M$ for the $A$-algebra defined as the $A$-module $A \oplus M$ with ring structure given by $(a \oplus m) \cdot (a' \oplus m') = aa' \oplus (am' + a'm)$ for $a, a' \in A$ and $m, m' \in M$.

Definition 2.2 (Deformations and obstructions). Let $S$ be a scheme of finite type over an excellent Dedekind ring. Denote by $\mathcal{X}$ a category fibered in groupoids over the category $\mathbf{Sch}/S$ of schemes over $S$. Let $A_0$ be a ring of finite type over $S$ and consider an object $a_0 \in \mathcal{X}(A_0)$. Then we define the \emph{deformation functor} $D_{a_0}$ from finitely generated $A_0$-modules to sets by

$$D_{a_0}(M) = \mathcal{X}_{a_0}(A_0 + M).$$

Note here that $D_{a_0}(M)$ can be interpreted as the set of isomorphism classes of liftings of $a_0$ from the base $A_0$ to the “thickened” base $A_0 + M$.

We have a canonical map $A_0 \to A_0 + M$ mapping $a$ to $(a, 0)$ and we let $b_0$ denote the image of $a_0$ under the corresponding map $\mathcal{X}(A_0) \to \mathcal{X}(A_0 + M)$. The object $b_0$ of $\mathcal{X}(A_0 + M)$ maps to $a_0$ in $\mathcal{X}(A_0)$ under the map $A_0 + M \to A_0$ and we denote by $\text{Aut}_{a_0}(A_0 + M)$ the group of automorphisms of the element $b_0$ in the category $\mathcal{X}_{a_0}(A_0 + M)$.

By an \emph{obstruction theory} $O$ for $\mathcal{X}$ we mean the following data:

(i) For each infinitesimal extension $A \to A_0$ of rings of finite type over $S$ and each object $a \in \mathcal{X}(A)$ a functor $O_a$ from finitely generated $A_0$-modules to finitely generated $A_0$-modules.

(ii) For each deformation situation $A' \to A \to A_0$ of rings of finite type over $S$ and each $a \in \mathcal{X}(A)$ an element $o_a(A') \in O_a(M)$ where $M = \text{Ker}(A' \to A)$. The element $o_a(A')$ is zero if and only if $\mathcal{X}_a(A') \neq \emptyset$.

Thus the element $o_a(A')$ is the obstruction to lifting the object $a$ over the base $A$ to the infinitesimally thickened base $A'$.

Definition 2.3 (Schlessinger’s conditions). Let $S$ be a scheme of finite type over an excellent Dedekind ring. Consider a deformation situation $A' \to A \to A_0$ of rings of finite type over $S$ with $A_0$ reduced, and let

$$B \downarrow
\begin{array}{c}
A' \\
\rightarrow \\
A
\end{array}$$

6
be a diagram of rings of finite type over $S$ where the composed map $B \to A_0$ is surjective, and the bottom map is the map of the deformation situation.

Let $\mathcal{X}$ be a category fibered in groupoids over the category $\textbf{Sch}/S$ of schemes over $S$. The following conditions on the CFG $\mathcal{X}$ are known as Schlessinger’s conditions:

S1(a) For each object $a \in \mathcal{X}(A)$ the natural map
\[ \mathcal{X}_a(A' \times_A B) \to \mathcal{X}_a(A') \times \mathcal{X}_a(B) \]

is surjective.

S1(b) Consider the special case where $A = A_0$ and $A' = A_0 + M$, where $M$ is a finitely generated $A_0$-module. Let $a_0$ be an object of $\mathcal{X}(A_0)$. Then the canonical map
\[ \mathcal{X}_{a_0}((A_0 + M) \times_{A_0} B) \to \mathcal{X}_{a_0}(A_0 + M) \times \mathcal{X}_{a_0}(B) \]  

is bijective.

S1' For each object $a \in \mathcal{X}(A)$ the natural functor
\[ \mathcal{X}_a(A' \times_A B) \to \mathcal{X}_a(A') \times \mathcal{X}_a(B) \]

is an equivalence of categories.

S1 This condition is that both S1(a) and S1(b) are satisfied. Note that S1' implies S1.

S2 Denote by $M$ a finitely generated $A_0$-module. Let $a_0 \in \mathcal{X}(A_0)$ be an object. Then the deformation set
\[ D_{a_0}(M) = \mathcal{X}_{a_0}(A_0 + M) \]

is a finitely generated $A_0$-module.

Condition S1(b) has the following equivalent form: Let $B \to A_0$ be a surjection of rings of finite type over $S$ with $A_0$ reduced and let $M$ be a finitely generated $A_0$-module. Denote by $a_0 \in \mathcal{X}(A_0)$ the image of an object $b \in \mathcal{X}(B)$ by the canonical functor $\mathcal{X}(B) \to \mathcal{X}(A_0)$. Then the canonical map
\[ \mathcal{X}_b(B + M) \to \mathcal{X}_{a_0}(A_0 + M) \]

is bijective.
When S1(b) is satisfied the deformation set $D_{a_0}(M) = \mathcal{F}_{a_0}(A_0 + M)$ introduced in Definition 2.2 has a natural structure of $A_0$-module. The addition of elements is defined as follows: First consider the canonical map

$$(A_0 + M) \times_{A_0} (A_0 + M) \rightarrow (A_0 + M)$$

that sends a pair $(x_0 + m, x_0 + m')$ to $x_0 + (m + m')$. Applying the functor $\mathcal{F}_{a_0}$ to this map and using the isomorphism (2.3.1) of S1(b) yields a diagram

$$\mathcal{F}_{a_0}((A_0 + M) \times_{A_0} (A_0 + M)) \rightarrow D_{a_0}(M)$$

where the map labelled “+” defines the addition on the set $D_{a_0}(M)$. Multiplication by a scalar $\lambda \in A_0$ is defined by applying the functor $\mathcal{F}_{a_0}$ to the map $A_0 + M \rightarrow A_0 + M$ sending an element $x_0 + m$ to $x_0 + \lambda m$. Schlessinger’s condition S2 is then that the $A_0$-module $D_{a_0}(M)$ is finitely generated when $A_0$ is reduced.

**Remark 2.4.** Let $S$ be a scheme of finite type over an excellent Dedekind ring. Consider a deformation situation $A' \rightarrow A \rightarrow A_0$ of rings of finite type over $S$, and let $M = \text{Ker}(A' \rightarrow A)$. Note that $M$ is a finitely generated $A_0$-module. We have isomorphisms

$$A' \times_A A' \cong A' + M \cong A' \times_{A_0} (A_0 + M)$$

$$(x, y) \mapsto x + (y - x) \mapsto (x, x_0 + (y - x))$$

where $x_0$ is the image of $x$ under the map $A' \rightarrow A_0$.

Suppose that $\mathcal{X}$ is a CFG over $\text{Sch}/S$ such that Schlessinger’s condition S1(b) holds. Let $a_0$ be an object of $\mathcal{X}(A)$. We have a canonical pullback map

$$\mathcal{F}_{a_0}(A' \times_A A') \rightarrow \mathcal{F}_{a_0}(A') \times_{\mathcal{F}_{a_0}(A)} \mathcal{F}_{a_0}(A')$$

and from S1(b) we get an isomorphism

$$\mathcal{F}_{a_0}(A' \times_{A_0} (A_0 + M)) \cong \mathcal{F}_{a_0}(A') \times_{\mathcal{F}_{a_0}(A_0 + M)} \mathcal{F}_{a_0}(A_0 + M) = \mathcal{F}_{a_0}(A') \times D_{a_0}(M).$$

Combining the above two maps along with the isomorphism $A' \times_A A' \cong A' \times_{A_0} (A_0 + M)$ yields a map

$$\mathcal{F}_{a_0}(A') \times D_{a_0}(M) \rightarrow \mathcal{F}_{a_0}(A') \times_{\mathcal{F}_{a_0}(A)} \mathcal{F}_{a_0}(A').$$

(2.4.1)
This map defines for any object \( a \in \mathcal{X}_{a_0}(A) \) an action of the additive group \( D_{a_0}(M) \) on the set \( \mathcal{X}_{a}(A') \), in the case where \( \mathcal{X}_{a}(A') \neq \emptyset \).

When Schlessinger’s condition S1(a) is satisfied the map (2.4.1) is surjective, so the action is then transitive. When condition S1’ is satisfied, the action is free.

**Definition 2.5 (ADO-conditions).** Let \( S \) be a scheme of finite type over an excellent Dedekind ring. Denote by \( \mathcal{X} \) a category fibered in groupoids over the category \( \text{Sch}/S \).

Suppose that \( \mathcal{X} \) satisfies Schlessinger’s conditions S1 and S2, and assume that there is an obstruction theory \( O \) for \( \mathcal{X} \). Then the **ADO-conditions** are the following conditions on the automorphisms, deformation and obstruction functors defined in Definition 2.2:

1. **(Compatibility with étale maps.)** Let \( A \to B \) and \( A_0 \to B_0 \) be étale maps of rings of finite type over \( S \). Consider infinitesimal extensions \( A \to A_0 \) and \( B \to B_0 \) such that the diagram

   \[
   \begin{array}{ccc}
   A & \longrightarrow & B \\
   \downarrow & & \downarrow \\
   A_0 & \longrightarrow & B_0
   \end{array}
   \]

   is commutative. Let \( a_0 \in \mathcal{X}(A_0) \) and \( a \in \mathcal{X}(A) \) be objects and denote by \( b_0 \in \mathcal{X}(B_0) \) and \( b \in \mathcal{X}(B) \) their respective images via the given étale maps. Then for each finitely generated \( A_0 \)-module \( M \) we have canonical isomorphisms

   (A1) \( \text{Aut}_{b_0}(B_0 + (M \otimes A_0 B_0)) \cong \text{Aut}_{a_0}(A_0 + M) \otimes A_0 B_0, \)

   (D1) \( D_{b_0}(M \otimes A_0 B_0) \cong D_{a_0}(M) \otimes A_0 B_0, \)

   (O1) \( O_a(M \otimes A_0 B_0) \cong O_{a_0}(M) \otimes A_0 B_0. \)

2. **(Compatibility with completion.)** Let \( A_0 \) be a ring of finite type over \( S \) and \( \mathfrak{m} \) a maximal ideal of \( A_0 \). Denote by \( \hat{A}_0 \) the completion \( \lim \leftarrow \mathfrak{m}^n A_0 \).

   Then for each object \( a_0 \in \mathcal{X}(A_0) \) and for each finitely generated \( A_0 \)-module \( M \) we have isomorphisms

   (A2) \( \text{Aut}_{a_0}(A_0 + M) \otimes A_0 \hat{A}_0 \cong \lim \text{Aut}_{a_0}(A_0 + (M/\mathfrak{m}^n M)) \),

   (D2) \( D_{a_0}(M) \otimes A_0 \hat{A}_0 \cong \lim D_{a_0}(M/\mathfrak{m}^n M) \).

3. **(Constructibility.)** Let \( A_0 \) be a reduced ring of finite type over \( S \) and \( a_0 \in \mathcal{X}(A_0) \) an object. Fix a finitely generated \( A_0 \)-module \( M \). There is an open dense set of points \( p \in \text{Spec}(A_0) \) of finite type such that there are isomorphisms
(A3) $\text{Aut}_{a_0}(A_0 + M) \otimes_{A_0} k(p) \cong \text{Aut}_{a_0}(A_0 + (M \otimes_{A_0} k(p)))$,

(D3) $D_{a_0}(M) \otimes_{A_0} k(p) \cong D_{a_0}(M \otimes_{A_0} k(p))$.

Also, for each infinitesimal extension $A \to A_0$ and each object $a$ of $\mathcal{X}(A)$ there is an open dense set of points $p \in \text{Spec}(A_0)$ of finite type such that there is an injection

(O3) $O_a(M) \otimes_{A_0} k(p) \hookrightarrow O_a(M \otimes_{A_0} k(p))$.

**Definition 2.6.** Let $S$ be a scheme and let $\mathcal{X}$ be a CFG over the category $\text{Sch}/S$. Then $\mathcal{X}$ is said to be limit preserving or locally of finite presentation if for each filtered direct system of rings $\{A_\lambda\}_{\lambda \in \Lambda}$ over $S$ we have that the canonical functor

$$\lim \mathcal{X}(A_\lambda) \to \mathcal{X}(\lim A_\lambda)$$

is an equivalence of categories.

**Proposition 2.7** (Artin’s Criteria, [Art74, Cor. 5.2]). Let $S$ be a scheme of finite type over an excellent Dedekind ring. Consider a limit preserving stack $\mathcal{X}$ over the category $\text{Sch}/S$, and assume that $\mathcal{X}$ has an obstruction theory $O$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $S$ if the following hold:

1. The diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable.
2. Schlessinger’s conditions S1 and S2 of Definition 2.3 are satisfied.
3. If $\hat{A}$ is a complete local ring over $S$ with maximal ideal $m$ and with residue field of finite type over $S$, then $\mathcal{X}(\hat{A}) \to \lim \mathcal{X}(\hat{A}/m^n)$ has a dense image, i.e. the composed map $\mathcal{X}(\hat{A}) \to \mathcal{X}(\hat{A}/m^n)$ is surjective for each $n$.
4. The conditions D1-D3 and O1,O3 of Definition 2.5 are satisfied.

**Remark 2.8.** With the notation of Proposition 2.7 let $T$ be an $S$-scheme of finite type and consider a map $T \to \mathcal{X} \times \mathcal{X}$ corresponding to an object $(a_1, a_2)$ in $\mathcal{X} \times \mathcal{X}$. We have the following cartesian diagram

$$\begin{CD}
\text{Isom}_T(a_1, a_2) @>>> T \\
@VVV @VVV \\
\mathcal{X} @>{\Delta}>> \mathcal{X} \times \mathcal{X}
\end{CD}$$

where $\text{Isom}_T(a_1, a_2)$ is the functor $\text{Sch}/T \to \text{Sets}$ defied by mapping a $T$-scheme $U$ to the set of isomorphisms $\phi : a_1|U \to a_2|U$. Condition (1) of
Proposition 2.7 is then the assertion that the functor \( \text{Isom}_T(a_1, a_2) \) is represented by an algebraic space.

To prove the representability of \( \text{Isom}_T(a_1, a_2) \) one may use Proposition 2.7 again with \( \text{Isom}_T(a_1, a_2) \) instead of \( \mathcal{X} \). This leads to a slightly condensed version of Artin’s criteria shown below.

**Theorem 2.9** (Artins Criteria, [Art74, Thm. 5.3]). Let \( S \) be a scheme of finite type over an excellent Dedekind ring. Consider a limit preserving stack \( \mathcal{X} \) over the category \( \text{Sch}/S \), and assume that \( \mathcal{X} \) has an obstruction theory \( O \). Then \( \mathcal{X} \) is an algebraic stack locally of finite type over \( S \) if the following hold:

1. Schlessinger’s conditions \( S1 \) and \( S2 \) of Definition 2.3 are satisfied. Moreover, if \( A_0 \) is a ring of finite type over \( S \) and \( M \) is a finitely generated \( A_0 \)-module, then \( \text{Aut}_{a_0}(A_0 + M) \) is a finitely generated \( A_0 \)-module for any object \( a_0 \) of \( \mathcal{X}(A_0) \).

2. If \( \hat{A} \) is a complete local ring over \( S \) with maximal ideal \( m \) and with residue field of finite type over \( S \), then \( \mathcal{X}(\hat{A}) \to \lim \mathcal{X}(\hat{A}/m^n) \) is faithful and has a dense image, i.e. the composed map \( \mathcal{X}(\hat{A}) \to \mathcal{X}(\hat{A}/m^n) \) is surjective for each \( n \).

3. The conditions \( A1-A3, D1-D3 \) and \( O1,O3 \) of Definition 2.5 are satisfied.

4. Suppose that \( A_0 \) is a ring of finite type over \( S \) and that \( a_0 \) is an object of \( \mathcal{X}(A_0) \). Let \( \phi \) be an automorphism of \( a_0 \) that induces the identity in \( \mathcal{X}(k(p)) \) for a dense set of points \( p \in \text{Spec}(A_0) \) of finite type over \( S \). Then \( \phi \) induces the identity in \( \mathcal{X}(U) \) where \( U \) is a non-empty open subset of \( \text{Spec}(A_0) \).

### 3 Schlessinger’s conditions

In this section we show that Schlessinger’s condition \( S1' \) holds for a number of CFG:s. This will be used in Section 7 to verify Artin’s Conditions for the stack \( \mathcal{X}_{pv2} \).

**Proposition 3.1.** Let \( S \) be a scheme of finite type over an excellent Dedekind ring. Let \( A \) be the category of flat affine morphisms of schemes fibered over the category \( \text{Sch}/S \). In other words, for an \( S \)-scheme \( U \) the objects of the category \( A(U) \) are \( U \)-schemes such that the map \( X \to U \) is flat and affine.
For each ring \( A \) over \( S \) we will identify the category \( \mathcal{A}(A) \) with the opposite category of flat \( A \)-algebras, and for each morphism \( A \to B \) of rings over \( S \) the associated functor \( \mathcal{A}(A) \to \mathcal{A}(B) \) sends an \( A \)-algebra \( R \) to the \( B \)-algebra \( R \otimes_A B \). Then \( \mathcal{A} \) satisfies Schlessinger’s condition \( S1' \) of Definition 2.3.

Proof. Consider a diagram

\[
\begin{array}{ccc}
A' \times_A B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A
\end{array}
\]

of rings over \( S \), with \( A' \to A \) an infinitesimal extension. Let \( R \) be an object of \( \mathcal{A}(A) \). We wish to show that the natural pullback functor

\[
\Phi : \mathcal{A}_R(A' \times_A B) \longrightarrow \mathcal{A}_R(A') \times \mathcal{A}_R(B)
\]

is an equivalence of categories.

Let \((R',S)\) be an object of the category \( \mathcal{A}_R(A') \times \mathcal{A}_R(B) \). Define \( B' = A' \times_A B \) and let \( S' = R' \times_R S \). Then \( S' \) is a \( B' \)-algebra, which is flat over \( B' \) and satisfying \( S' \otimes_{B'} A = R \) by [Sch68, Lem. 3.4]. This is illustrated by the diagram below:

\[
\begin{array}{ccc}
S' & \longrightarrow & S \\
\downarrow & & \downarrow \\
R' & \longrightarrow & R \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A
\end{array}
\]

Let \((R'_1,S_1)\) and \((R'_2,S_2)\) be two objects of \( \mathcal{A}_R(A') \times \mathcal{A}_R(B) \). A morphism between \((R'_1,S_1)\) and \((R'_2,S_2)\) is a pair of maps \( f : R'_1 \to R'_2 \) and \( g : S_1 \to S_2 \) such that the \( A \)-algebra homomorphisms \( f \otimes_{A'} \text{id}_A \) and \( g \otimes_B \text{id}_A \) are the identity on \( R \).

The maps \( f \) and \( g \) give rise to a map \( R'_1 \times_R S_1 \to R'_2 \times_R S_2 \) from the universal property of the fiber product. This map is a morphism in the category \( \mathcal{A}_R(A' \times_A B) \) and we have thus shown that the assignment

\[
(R',S) \longmapsto R' \times_R S
\]

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gives a functor

\[ \Psi : \mathcal{A}_R(A') \times \mathcal{A}_R(B) \longrightarrow \mathcal{A}_R(A' \times_A B). \]

By [Sch68, Lem. 3.4] we have that if \((R', S)\) is an object of \(\mathcal{A}_R(A') \times \mathcal{A}_R(B)\) then the canonical projection maps

\[ (R' \times_R S) \otimes_{B'} A' \longrightarrow R', \quad (R' \times_R S) \otimes_{B'} B \longrightarrow S \]

are isomorphisms. This shows that we have an isomorphism from \(\Phi \circ \Psi\) to the identity functor on \(\mathcal{A}_R(A') \times \mathcal{A}_R(B)\).

Moreover, let \(C\) be an object of \(\mathcal{A}_R(B')\), where \(B' = A' \times_A B\). This means in particular that \(C \otimes_{B'} A = R\). Then by [Sch68, Cor. 3.6] we have that the canonical map

\[ C \longrightarrow (C \otimes_{B'} A') \times_{(C \otimes_{B'} A)} (C \otimes_{B'} B) \]

sending an element \(x\) to \((x \otimes 1, x \otimes 1)\) is an isomorphism. This shows that we have an isomorphism from the identity functor on \(\mathcal{A}_R(B')\) to \(\Psi \circ \Phi\), and so \(\Phi\) is an equivalence of categories. \(\square\)

**Remark 3.2.** With the notation and assumptions of Proposition 3.1, consider a diagram

\[ \begin{array}{ccc}
A' & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A
\end{array} \]

of rings of finite type over the base scheme \(S\), with \(A' \to A\) an infinitesimal extension. Let \(X = \text{Spec}(R)\) be an affine scheme, flat over \(A\). By Proposition 3.1 the functor

\[ \Phi : \mathcal{A}_X(A' \times_A B) \longrightarrow \mathcal{A}_X(A') \times \mathcal{A}_X(B). \]

has an inverse \(\Psi\) given by the fiber product of rings. If \((X', Y)\) is an object of \(\mathcal{A}_X(A') \times \mathcal{A}_X(B)\) with \(X' = \text{Spec}(R')\) and \(Y = \text{Spec}(S)\) we will use the notation

\[ \Psi(X', Y) = X' \cup_X Y = \text{Spec}(R' \times_R S) \]

for the corresponding object of \(\mathcal{A}_X(A' \times_A B)\). Note that since \(R' \times_R S\) is a pullback, or inverse limit, of the diagram

\[ \begin{array}{ccc}
S & \longrightarrow & R \\
\downarrow & & \downarrow \\
R' & \longrightarrow & R
\end{array} \]
in the category of rings, we have that $X' \cup_X Y$ is a pushout, or direct limit, of the diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \\
\uparrow & & \uparrow \\
X' & \longleftarrow & X
\end{array}
$$

in the category of affine schemes.

**Proposition 3.3.** Let $S$ be a scheme of finite type over an excellent Dedekind ring and consider the category $\mathcal{S}$ of flat families of schemes, fibered over the category $\text{Sch}/S$ of schemes over $S$. Then $\mathcal{S}$ satisfies Schlessinger’s condition $S_1'$ of Definition 2.3.

**Proof.** Consider a diagram

$$
\begin{array}{ccc}
A' \times_A B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A
\end{array}
$$

of rings over $S$, with $A' \to A$ an infinitesimal extension. Let $X$ be a scheme, flat over $A$. We wish to construct an inverse to the pullback functor

$$
\Phi : S_X(A' \times_A B) \longrightarrow S_X(A') \times S_X(B).
$$

Let $X'$ and $Y$ be flat schemes over $A'$ and $B$ respectively, such that $X = X' \otimes_{A'} A$ and $X = Y \otimes_B A$. Let $\{V_i = \text{Spec } S_i\}_{i \in I}$ be an open affine cover of $Y$. This cover pulls back to an open affine cover $\{U_i = \text{Spec } R_i\}_{i \in I}$ of $X$.

Since $A' \to A$ is an infinitesimal extension, we have that $X'$ and $X$ share the same underlying topological space. We may thus consider the open cover $\{U'_i\}_{i \in I}$ of $X'$, where $U'_i = U_i$ as topological spaces. As schemes we have that $U_i \cong U'_i \otimes_{A'} A$ so by [EGA1, Prop. 5.1.9] we have that each $U'_i$ is affine, say $U'_i = \text{Spec } R'_i$.

Consider the affine schemes $V'_i = U'_i \cup_{U_i} V_i = \text{Spec}(R'_i \times_{R_i} S_i)$ over $B' = A' \times_A B$ for $i \in I$. These affines can be glued together to a scheme $Y'$ over $B'$. This is possible because if $V_{ij}$ is an open affine subscheme of $V_i \cap V_j$ and $U_{ij}$ is its pullback to $X$, then we may find a unique lifting of $U_{ij}$ to an open affine $U'_{ij}$ of $X'$ and the scheme $U'_{ij} \cup_{U_{ij}} V_{ij}$ is then an open subscheme of both $U'_i \cup_{U_i} V_i$ and $U'_j \cup_{U_j} V_j$ which we can use for the gluing.

We will use the notation $Y' = X' \cup_X Y$. By Proposition 3.1 we have that each $V'_i$ is flat over $B'$ for all $i \in I$, and also that we have canonical isomorphisms $U_i \cong V'_i \otimes_{B'} A'$ and $V_i \cong V'_i \otimes_{B'} B$. It follows that $X' \cup_X Y$ is
flat over $B'$ and that we have canonical isomorphisms $X' \cong Y' \otimes_{B'} A'$ and $Y \cong Y' \otimes_{B'} B$.

Now let $(X_1', Y_1)$ and $(X_2', Y_2)$ be objects of $S_X(A') \times S_X(B)$. A morphism between these objects is a pair of morphisms $f : X_1' \to X_2'$ and $g : Y_1 \to Y_2$ that pull back to the identity on $X$. Let $V_1$ and $V_2$ be open affines of $Y_1$ and $Y_2$ respectively such that $g(V_1) \subseteq V_2$. Since the map $g$ pulls back to the identity on $X$ the map of affines $V_1 \to V_2$ pulls back to an inclusion $U_1 \subseteq U_2$ of open affine subschemes of $X$. Let $U_1' \subseteq X_1'$ and $U_2' \subseteq X_2'$ be the corresponding open affine subschemes of $X_1'$ and $X_2'$. Since the map $f$ pulls back to the identity on $X$ we have that $f(U_1') \subseteq U_2'$.

By the universal property of the fiber product of rings we obtain a unique map $U_1' \cup U_1 \to U_2' \cup U_2$, and by gluing together these kinds of maps we get a morphism $X_1' \cup_X Y_1 \to X_2' \cup_X Y_2$. Thus we have obtained a functor

$$\Phi : S_X(A') \to S_X(A' \times_A B)$$

by the assignment $(X', Y) \mapsto X' \cup_X Y$. Since we have canonical isomorphisms $X' \cong (X' \cup_X Y) \otimes_{B'} A'$ and $Y \cong (X' \cup_X Y) \otimes_{B'} B$ it follows that $\Phi \circ \Psi$ is isomorphic to the identity functor on $S_X(A') \times S_X(B)$.

Let $Z$ be an object of $S_X(A' \times_A B)$, and let $\{W_i\}_{i \in I}$ be an open affine cover of $Z$. Then by Proposition 3.1 we have for each $i \in I$ a canonical isomorphism

$$(W_i \otimes_{B'} A') \cup_{W_i \otimes_{B'} A} (W_i \otimes_{B'} B) \cong W_i$$

which gives a canonical isomorphism

$$(Z \otimes_{B'} A') \cup_{Z \otimes_{B'} A} (Z \otimes_{B'} B) \cong Z.$$

This gives an isomorphism from $\Psi \circ \Phi$ to the identity functor on $S_X(A' \times_A B)$, and so $\Phi$ is an equivalence of categories. \qed

**Remark 3.4.** With the notation and assumptions of Proposition 3.3, consider a diagram

$$
\begin{array}{ccc}
A' \times_A B & \to & B \\
\downarrow & & \downarrow \\
A' & \to & A
\end{array}
$$

of rings of finite type over the base scheme $S$, with $A' \to A$ an infinitesimal extension. Let $X$ be a scheme, flat over $A$. By Proposition 3.3 the pullback functor

$$\Phi : S_X(A' \times_A B) \to S_X(A') \times S_X(B).$$
has an inverse $\Psi$, and if $(X', Y)$ is an object of $\mathcal{S}_X(A') \times \mathcal{S}_X(B)$ we will use the notation $\Psi(X', Y) = X' \cup_X Y'$ for the corresponding object of $\mathcal{S}_X(A' \times_A B)$. The scheme $X' \cup_X Y$ will be a pushout, or direct limit, of the diagram

\[
\begin{array}{ccc}
Y & \rightarrow & X' \\
\downarrow & & \downarrow \\
X' & \rightarrow & X
\end{array}
\]

in the category of schemes.

We now need some preliminary results regarding closed immersions defined by nilpotent ideals. These results will be used in the proof of Proposition 3.11.

**Lemma 3.5.** Let $A$ be a ring, $I \subseteq A$ a nilpotent ideal and $f : M \rightarrow N$ a map of $A$-modules. Suppose that the induced map $M/IM \rightarrow N/IN$ is surjective. Then $f$ is surjective.

**Proof.** Let $K$ be the cokernel of the map $f : M \rightarrow N$. Tensoring the right exact sequence

\[M \rightarrow N \rightarrow K \rightarrow 0\]

with $A/I$ gives $K/IK = 0$. Hence $K = IK$ and so $K = 0$ since $I$ is nilpotent. $\Box$

**Proposition 3.6.** Let $f : X \rightarrow Y$ be a representable morphism of algebraic spaces and assume that there is a closed immersion $Y_0 \rightarrow Y$ defined by a nilpotent ideal such that $f_0 : X_0 = X \times_Y Y_0 \rightarrow Y_0$ is a closed immersion. Then $f$ is a closed immersion.

**Proof.** Let $Z$ be a scheme and let $Z \rightarrow Y$ be a morphism. Pulling back along the map $f : X \rightarrow Y$ gives a morphism of schemes $g : W = X \times_Y Z \rightarrow Z$. We wish to show that $g$ is a closed immersion. This is true if $g^{-1}(U) \rightarrow U$ is a closed immersion for each affine open $U$ of $Z$. Thus we may assume that $Z$ is affine. Pulling back the map $g$ along the closed immersions $Y_0 \rightarrow Y$ and $X_0 \rightarrow X$ gives a cartesian diagram

\[
\begin{array}{ccc}
W_0 & \rightarrow & Z_0 \\
\downarrow & & \downarrow \\
W & \rightarrow & Z
\end{array}
\]

where the vertical maps are closed immersions defined by nilpotent ideals, and the top horizontal map is a closed immersion. It follows that the schemes
Z₀ and W₀ are affine. Thus W₀ → W is a closed immersion of schemes defined by a nilpotent ideal, with W₀ affine. Thus by [EGA, Prop. 5.1.9] we have that W is affine. Thus g is a closed immersion by Lemma 3.5.

**Corollary 3.7.** Let X be an algebraic space and let X₀ → X be a closed immersion defined by a nilpotent ideal. Then X is separated if and only if X₀ is separated.

**Proof.** Apply Proposition 3.6 to the diagonal map Δ : X → X × X and the immersion X₀ × X₀ → X × X. □

**Proposition 3.8.** Let X be an algebraic space and let X₀ → X be a closed immersion defined by a nilpotent ideal of Oₓ. Then X is an affine scheme if and only if X₀ is an affine scheme.

**Proof.** The “only if” part is clear. We will use the Weak Serre Criterion [Knu71, Thm. III.2.3] to show that if X₀ is an affine scheme, then X is also affine.

Suppose therefore that X₀ is affine, defined by a nilpotent ideal I of Oₓ. By considering the nested closed subspaces Xₙ defined by the ideals Iₙ₊₁ for n ≥ 0 we may reduce to the case where I² = 0.

Since X₀ and X share the same underlying topological space and X₀ is quasicompact we conclude that X is also quasicompact. Moreover, the space X is separated by Corollary 3.7.

To use the Weak Serre Criterion [Knu71, Thm. III.2.3] we need to show that the global sections functor Γ(X, −) from quasicoherent sheaves on X to abelian groups is exact and faithful. Thus for each quasicoherent sheaf F on X we need to show that Hⁱ(X, F) = 0 for i > 0, and that if Γ(X, F) = 0 then F = 0. Let I be the ideal defining X₀, and recall that I² = 0. Consider the exact sequence of Oₓ-modules

\[ 0 \rightarrow IF \rightarrow F \rightarrow F/IF \rightarrow 0. \]

Taking global sections, we obtain a long exact sequence of cohomology:

\[ 0 \rightarrow Γ(X, IF) \rightarrow Γ(X, F) \rightarrow Γ(X, F/IF) \rightarrow \cdots. \]

(3.8.1)

Since IF and F/IF are Oₓ/I-modules we have that these modules are defined on the affine scheme X₀ so by [Knu71, Cor. II.4.15] we obtain

\[ Hⁱ(X, IF) = Hⁱ(X₀, IF) = 0, \quad Hⁱ(X, F/IF) = Hⁱ(X₀, F/IF) = 0. \]
for \(i > 0\). Thus \(H^i(X, \mathcal{F}) = 0\) for \(i > 0\) and so \(\Gamma(X, -)\) is exact. Moreover, if \(\Gamma(X, \mathcal{F}) = 0\) then by the exact sequence (3.8.1) we have the relation \(\Gamma(X, \mathcal{F}/\mathcal{I}\mathcal{F}) = \Gamma(X_0, \mathcal{F}/\mathcal{I}\mathcal{F}) = 0\). Hence by faithfulness of \(\Gamma(X_0, -)\) we have that \(\mathcal{F}/\mathcal{I}\mathcal{F} = 0\) and so \(\mathcal{F} = \mathcal{I}\mathcal{F} = \mathcal{I}^2\mathcal{F} = 0\). Thus \(\Gamma(X, -)\) is faithful and by the Weak Serre Criterion we have that \(X\) is affine.

\section*{Corollary 3.9}

Let \(X\) be an algebraic space and let \(X_0 \to X\) be a closed immersion defined by an nilpotent ideal of \(\mathcal{O}_X\). Then \(X\) is a scheme if and only if \(X_0\) is a scheme.

\begin{proof}
The “only if” part is clear, so assume that \(X_0\) is a scheme. Since the immersion \(X_0 \to X\) is defined by a nilpotent ideal we have that \(X_0\) and \(X\) share the same underlying topological space. We have by [Kmu71, Prop. 6.10] that there is a one-to-one correspondence between open subspaces of \(X\) and open sets in the Zariski topology on \(X\). Let \(U \subseteq X\) be an open subspace such that the corresponding open subscheme \(U_0\) of \(X_0\) is affine. Then \(U_0\) is a closed subscheme of \(U\) defined by a nilpotent ideal so by Proposition 3.8 \(U\) is an affine scheme. Thus \(X\) has an open cover of affine schemes, so \(X\) is a scheme.
\end{proof}

The following proposition is a version of [EGAIV, Thm. 18.1.2] for algebraic spaces:

\section*{Proposition 3.10}

Let \(S\) be an algebraic space and let \(S_0\) be a closed subspace defined by a nilpotent ideal of \(\mathcal{O}_S\). Then the functor

\[ X \mapsto X \times_S S_0 \]

from the category of schemes étale over \(S\) to the category of schemes étale over \(S_0\) is an equivalence of categories.

\begin{proof}
Let \(U \to S\) be an étale presentation of \(S\) and let \(R = U \times_S U\). Consider the pullback \(U_0 = U \times_S S_0 \to S_0\) which is an étale presentation of \(S_0\). Let \(R_0 = U_0 \times_{S_0} U_0\). Then \(U_0\) is a closed subscheme of \(U\) with the same underlying topological space, and \(R_0\) is a closed subscheme of \(R\) with the same underlying topological space.

Let \(p_1, p_2\) denote the two projections \(R \to U\) and let \(p_{01}, p_{02}\) denote the two projections \(R_0 \to U_0\). Consider an algebraic space \(X_0\), étale over \(S_0\). The space \(X_0\) pulls back to a scheme \(X'_0\) over \(U_0\) together with an isomorphism of schemes \(\phi_0 : p_{01}^*X'_0 \cong p_{02}^*X'_0\) satisfying a cocycle condition. The data \((X'_0, \phi_0)\) is a descent datum for the covering \(U_0 \to S_0\) and determines the algebraic space \(X_0\) up to unique isomorphism by taking the quotient of the equivalence relation \(p_{01}^*X'_0 \Rightarrow X'_0\).
By [EGAIV, Thm. 18.1.2] we can find a scheme $X'$ over $U$, unique up to unique isomorphism, such that $X'_0 = X' \times_U U_0$ along with a unique isomorphism $\phi : p_1^*X' \cong p_2^*X'$ such that the pullback of $\phi$ is $\phi_0$. Let $X'' = p_1^*X'$. Then we obtain an étale equivalence relation $X'' \rightrightarrows X'$ whose quotient is an algebraic space $X$, étale over $S$ such that $X_0 = X \times_S S_0$.

Again by [EGAIV, Thm. 18.1.2] the pullback $X' \mapsto X' \times_U U_0$ is an equivalence of categories between the category of schemes étale over $U$ and the category of schemes étale over $U_0$. Likewise we get an equivalence of categories between the category of descent data for the covering $U \to X$ and the category of descent data for the covering $U_0 \to X_0$. Since any descent datum gives rise to an algebraic space, unique up to unique isomorphism, we have that the pullback $X \mapsto X \times_S S_0$ gives an equivalence of categories between algebraic spaces étale over $S$ and algebraic spaces étale over $S_0$. By Corollary 3.9 an algebraic space $X$ over $S$ is a scheme if and only if $X \times_S S_0$ is a scheme. We thus obtain the desired equivalence of categories between schemes étale over $S$ and schemes étale over $S_0$.

**Proposition 3.11.** Let $S$ be a scheme of finite type over an excellent Dedekind ring. Let $\mathcal{AS}$ be the fibered category of flat families of algebraic spaces, fibered over the category $\textbf{Sch}/S$ of schemes over $S$. Then $\mathcal{AS}$ satisfies Schlessinger’s condition $S1'$ of Definition 2.3.

**Proof.** Consider a diagram

\[
\begin{array}{ccc}
A' \times_A B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A
\end{array}
\]

of rings over $S$, with $A' \to A$ an infinitesimal extension. Let $X$ be an algebraic space, flat over $A$. We wish to construct an inverse to the functor

\[\Phi : \mathcal{AS}_X(A' \times_A B) \longrightarrow \mathcal{AS}_X(A') \times \mathcal{AS}_X(B).\]

Let $X'$ and $Y$ be algebraic spaces that are flat over $A'$ and $B$ respectively, such that $X \cong X' \otimes_{A'} A$ and $X \cong Y \otimes_B A$. Denote by $V \to Y$ an étale presentation of $Y$ and let $U = V \times_Y X \to X$ be the étale presentation of $X$ pulled back from $V$.

By Proposition 3.10 there is an étale presentation $U' \to X'$ of $X'$, unique up to unique isomorphism, such that $U \cong U' \times_{X'} X$. Define the schemes

\[R = U \times_X U, \quad R' = U' \times_{X'} U', \quad S = V \times_Y V.\]
These define étale equivalence relations $R \Rightarrow U$, $R' \Rightarrow U'$ and $S \Rightarrow V$ over $A$, $A'$ and $B$ respectively. We can then by Proposition 3.3 define an étale equivalence relation $R' \cup R S \Rightarrow U' \cup U V$ over $B' = A' \times_A B$ which gives an algebraic space $Y'$ that is flat over $B'$, along with canonical isomorphisms

$$R' \cong (R' \cup R S) \otimes_{B'} A', \quad S \cong (R' \cup R S) \otimes_{B'} B$$

and

$$U' \cong (U' \cup U V) \otimes_{B'} A', \quad V \cong (U' \cup U V) \otimes_{B'} B.$$

It follows that we obtain canonical isomorphisms

$$X' \cong Y' \otimes_{B'} A', \quad Y \cong Y' \otimes_{B'} B.$$

We use the notation $Y' = X' \cup_X Y$.

Let $(X'_1, Y_1)$ and $(X'_2, Y_2)$ be objects of $\mathcal{A}S_X(A') \times \mathcal{A}S_X(B)$. A morphism between these objects is a pair of maps $f : X'_1 \to X'_2$ and $g : Y_1 \to Y_2$ pulling back to the identity on $X$. Suppose that $Y_i$ is the quotient of an étale equivalence relation $S_i \Rightarrow V_i$ for $i = 1, 2$ such that we have a commutative diagram of equivalence relations

$$
\begin{array}{c}
S_1 \rightarrow V_1 \rightarrow Y_1 \\
\downarrow \quad \downarrow \quad \downarrow f \\
S_2 \rightarrow V_2 \rightarrow Y_2
\end{array}
$$

over $B$. The above diagram pulls back to a commutative diagram

$$
\begin{array}{c}
R_1 \rightarrow U_1 \rightarrow X \\
\downarrow \quad \downarrow \quad \downarrow \text{id}_X \\
R_2 \rightarrow U_2 \rightarrow X
\end{array}
$$

(3.11.1)

of equivalence relations over $A$. Since $X$ is a closed subspace of $X'_1$ and $X'_2$ defined by nilpotent ideals we can use Proposition 3.10 and find a diagram

$$
\begin{array}{c}
R'_1 \rightarrow U'_1 \rightarrow X'_1 \\
\downarrow \quad \downarrow \quad \downarrow g \\
R'_2 \rightarrow U'_2 \rightarrow X'_2
\end{array}
$$

of equivalence relations over $A'$, unique up to unique isomorphism, pulling back to the diagram (3.11.1). By the universal property of pushouts of
schemes described in Remark 3.4 we obtain a diagram

\[
\begin{array}{ccc}
R'_1 \cup R_1 S_1 & \xrightarrow{=} & U'_1 \cup U_1 V_1 \\
\downarrow & & \downarrow \\
R'_2 \cup R_2 S_2 & \xrightarrow{=} & U'_2 \cup U_2 V_2 \\
\end{array}
\xrightarrow{h} \begin{array}{ccc}
X'_1 \cup X_1 Y_1 & \rightarrow & X'_2 \cup X_2 Y_2 \\
\downarrow & & \downarrow \\
X'_2 \cup X_2 Y_2 & \rightarrow & X'_2 \cup X_2 Y_2
\end{array}
\]

of equivalence relations over \( B' = A' \times_A B \). Then \( h : X'_1 \cup X_1 Y_1 \rightarrow X'_2 \cup X_2 Y_2 \) is our desired morphism of algebraic spaces, showing that the assignment \((X', Y) \mapsto X' \cup X Y\) gives a functor

\[ \Psi : \mathcal{AS}_X(A') \times \mathcal{AS}_X(B) \rightarrow \mathcal{AS}_X(A' \times_A B). \]

We have already shown that there are canonical isomorphisms

\[ X' \cong (X' \cup X Y) \otimes_{B'} A', \quad Y \cong (X' \cup X Y) \otimes_{B'} B \]
which means that we have an isomorphism from the identity functor on \( \mathcal{AS}_X(A') \times \mathcal{AS}_X(B) \) to \( \Phi \circ \Psi \).

On the other hand, let \( Z \) be an object of \( \mathcal{AS}_X(A' \times_A B) \), given as the quotient of an étale equivalence relation \( T \rightrightarrows W \). Then by Proposition 3.3 we have canonical isomorphisms

\[ (T \otimes_{B'} A') \cup_{T \otimes_{B'} A} (T \otimes_{B'} B) \xrightarrow{=} T \]

and

\[ (W \otimes_{B'} A') \cup_{W \otimes_{B'} A} (W \otimes_{B'} B) \xrightarrow{=} W \]

respecting the equivalence relation, which gives a canonical isomorphism

\[ (Z \otimes_{B'} A') \cup_{Z \otimes_{B'} A} (Z \otimes_{B'} B) \xrightarrow{=} Z. \]

This provides an isomorphism from the composed functor \( \Psi \circ \Phi \) to the identity functor on \( \mathcal{AS}_X(A' \times_A B) \), and so \( \Phi \) is an equivalence of categories. \( \square \)

**Lemma 3.12.** Let \( A \) be a ring, \( I \subseteq A \) a nilpotent ideal, and \( B \) an \( A \)-algebra. Suppose that \( B/I B \) is an \( A/I \)-algebra of finite type. Then \( B \) is an \( A \)-algebra of finite type.

Moreover, if \( B \) is flat as an \( A \)-algebra and \( B/I B \) is an \( A/I \)-algebra of finite presentation, then \( B \) is an \( A \)-algebra of finite presentation.

**Proof.** First let \( B \) be an arbitrary \( A \)-algebra. Let \( A_0 = A/I \) and \( B_0 = B/I B \). Choose a presentation

\[ 0 \rightarrow J_0 \rightarrow A_0[x_1, \ldots, x_n] \rightarrow B_0 \rightarrow 0 \quad \text{(3.12.1)} \]
of \( B_0 \) as an \( A_0 \)-algebra. Let \( b_{01}, \ldots, b_{0n} \) be the images of \( x_1, \ldots, x_n \) in \( B_0 \). Since \( B \to B_0 \) is surjective we can lift these images to elements \( b_1, \ldots, b_n \) of \( B \). Consider the map \( A[x_1, \ldots, x_n] \to B \) that maps \( x_i \) to \( b_i \) for \( 1 \leq i \leq n \). This map is surjective after tensoring with \( A_0 \) so it is surjective by Lemma 3.5. Thus \( B \) is an \( A \)-algebra of finite type.

Now assume that \( B \) is flat as an \( A \)-algebra, and that \( B_0 \) is an \( A_0 \)-algebra of finite presentation. Let \( J \) denote the kernel of the map \( A[x_1, \ldots, x_n] \to B \) defined above. Since \( B \) is flat we may tensor the exact sequence

\[
0 \to J \to A[x_1, \ldots, x_n] \to B \to 0
\]

by \( A_0 \) and obtain the exact sequence (3.12.1). Let \( \{f_{01}, \ldots, f_{0m}\} \) be generators of the ideal \( J_0 \) and lift these to elements \( \{f_1, \ldots, f_m\} \) of the ideal \( J \) via the surjective map \( J \to J_0 \). Denote by \( \tilde{J} \) the ideal of \( A[x_1, \ldots, x_n] \) generated by the elements \( \{f_1, \ldots, f_m\} \). Then we have an inclusion \( i : \tilde{J} \subseteq J \) that is surjective after tensoring with \( A_0 \), and so \( i \) is surjective by Lemma 3.5. Thus \( \tilde{J} = J \) so \( B \) is an \( A \)-algebra of finite presentation.

**Proposition 3.13.** Let \( A \) be a ring and \( f : X \to \text{Spec}(A) \) a morphism where \( X \) is an algebraic space. Let \( A \to A_0 \) be a ring homomorphism with nilpotent kernel. Let \( X_0 = X \otimes_A A_0 \) and consider the cartesian diagram

\[
\begin{array}{ccc}
X_0 & \to & X \\
\downarrow f_0 & & \downarrow f \\
\text{Spec}(A_0) & \to & \text{Spec}(A).
\end{array}
\]

Suppose that \( f_0 \) has any of the following properties:

(i) Quasicompact,

(ii) Quasiseparated, i.e. the diagonal \( X_0 \to X_0 \times_{A_0} X_0 \) is quasicompact,

(iii) Universally closed,

(iv) Separated,

(v) Locally of finite type,

(vi) Proper.

Then \( f \) has the corresponding property. Moreover, suppose that the morphism \( f \) is flat. Then if \( f_0 \) has the property

(vii) Locally of finite presentation

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so does the map $f$.

Proof. The algebraic spaces $X_0$ and $X$ share the same underlying topological space, as do the schemes $\text{Spec}(A_0)$ and $\text{Spec}(A)$. The properties (i), (ii) and (iii) depend only on the underlying topological spaces and so these are clear. Property (iv) follows from Corollary 3.7. The properties (v) and (vii) follow from Lemma 3.12. Finally, property (vi) is simply properties (i), (iii) and (v).

\[\square\]

\section{Derived categories and derived functors}

In this section we discuss the basics of derived categories and functors. This material will be used in Section 5 to derive some results about Ext-groups. These results will then be used in Section 7 to verify that Theorem 2.9(3) holds for the stack $\mathcal{F}_{pv2}$. Theorem 2.9(3) is the part of Artin's criteria that deals with automorphisms, deformations and obstructions.

\textbf{Definition 4.1} (The derived category). Let $\mathcal{A}$ be an abelian category. We define $\mathbb{C}(\mathcal{A})$ to be the category of cochain complexes

$$A^\bullet = \ldots \rightarrow A^{-1} \rightarrow A^0 \rightarrow A^1 \rightarrow \ldots$$

in $\mathcal{A}$ with maps of cochain complexes as morphisms.

Next define $\mathbb{K}(\mathcal{A})$ to be the category whose objects are the objects of $\mathbb{C}(\mathcal{A})$ and whose morphisms are morphisms of cochain complexes modulo homotopy. Recall that a map $A^\bullet \rightarrow B^\bullet$ of cochain complexes is called a quasi-isomorphism if the corresponding map of cohomology $H^i(A^\bullet) \rightarrow H^i(B^\bullet)$ is an isomorphism for every integer $i$. The condition of being a quasi-isomorphism is invariant under homotopy so we may speak of quasi-isomorphisms also in the category $\mathbb{K}(\mathcal{A})$.

Finally we define the \textit{derived category} $\mathbb{D}(\mathcal{A})$ as the category $\mathbb{K}(\mathcal{A})$ localized in the class $Q$ of quasi-isomorphisms. This means that $\mathbb{D}(\mathcal{A})$ is the category whose objects are the objects of $\mathbb{K}(\mathcal{A})$ but where we have “formally inverted” quasi-isomorphisms much like we may formally invert elements of a ring to produce a localization of a ring. Formally a morphism $A^\bullet \rightarrow B^\bullet$ in $\mathbb{D}(\mathcal{A})$ is a commutative triangle

\[
\begin{array}{ccc}
\text{C}^\bullet & \rightarrow & \text{B}^\bullet \\
\downarrow f & & \downarrow g \\
A^\bullet & \rightarrow & B^\bullet \\
\end{array}
\]
in \( K(\mathcal{A}) \) where \( f \) is a quasi-isomorphism. One should think of the map \( h \) as \( g \circ f^{-1} \). We write \( D(\mathcal{A}) = Q^{-1}K(\mathcal{A}) \). There is a canonical localization functor \( q : K(\mathcal{A}) \to D(\mathcal{A}) \). For details and further discussion see [Wei94, Ch. 10.1-10.4].

We also denote by \( K^b(\mathcal{A}), K^+(\mathcal{A}) \) and \( K^-(\mathcal{A}) \) the full subcategories of \( K(\mathcal{A}) \) defined by bounded, bounded below and bounded above complexes respectively. Their subsequent localizations \( D^b(\mathcal{A}), D^+(\mathcal{A}) \) and \( D^-(\mathcal{A}) \) in the class of quasi-isomorphisms are then full subcategories of \( D(\mathcal{A}) \).

Suppose that \( X \) is a ringed space, scheme or algebraic space, and let \( \mathcal{A} \) denote the category of \( \mathcal{O}_X \)-modules. Then the derived category \( D(\mathcal{A}) \) exists and we denote this category \( D(X) \). We will also use the notation \( K(X) \) for the homotopy category, as well as the notation \( K^+(X), K^-(X), K^b(X) \). We let \( D_{qc}(X) \) denote the subcategory of complexes with quasi-coherent cohomology and we let \( D_c(X) \) denote the subcategory of complexes with coherent cohomology.

**Definition 4.2 (Exact triangles).** Let \( \mathcal{A} \) be an abelian category. Suppose we have three maps \( u : A^\bullet \to B^\bullet, v : B^\bullet \to C^\bullet \) and \( w : C^\bullet \to A^\bullet [1] \) in \( K(\mathcal{A}) \), where \( A^\bullet [1] \) denotes the shifted complex defined by \( A^p[1] = A^{p+1} \). We say that \((u, v, w)\) is an exact triangle if there is a map \( u' : (A')^\bullet \to (B')^\bullet \) in \( K(\mathcal{A}) \) and a commutative diagram

\[
\begin{array}{cccccc}
A^\bullet & \overset{u}{\rightarrow} & B^\bullet & \overset{v}{\rightarrow} & C^\bullet & \overset{w}{\rightarrow} & A^\bullet [1] \\
\downarrow{f} & & \downarrow{g} & & \downarrow{h} & & \downarrow{f[1]} \\
(A')^\bullet & \overset{u'}{\rightarrow} & (B')^\bullet & \rightarrow & \mathrm{Cone}(u') & \rightarrow & (A')^\bullet [1]
\end{array}
\]

in \( K(\mathcal{A}) \), where \( \mathrm{Cone}(u') \) denotes the mapping cone [Wei94, 1.5.1] and the maps \( f, g, h \) are isomorphisms in \( K(\mathcal{A}) \). Thus one can think of \((u, v, w)\) as being an exact triangle if it is isomorphic to the natural triangle defined by the mapping cone of some map. We may also define exact triangles in the category \( D(\mathcal{A}) \) but it is a bit more complicated, see [Wei94, Prop. 10.4.1]. For more information on general triangles and triangulated categories, see [Wei94, Ch. 10.2]. Note however that in the book [Wei94] the author uses the convention \( A[i]^p = A^{p-i} \) so one must then use \( A^\bullet [-1] \) instead of \( A^\bullet [1] \) in the diagrams.

**Definition 4.3 (Derived functors).** Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and let \( F : K^+(\mathcal{A}) \to K(\mathcal{B}) \) be a morphism of triangulated categories. This means that \( F \) is an additive covariant functor commuting with shifting of complexes and taking exact triangles to exact triangles. Then a right derived
functor of $F$ is a morphism of triangulated categories $R F : \mathcal{D}^+(A) \rightarrow \mathcal{D}(B)$ together with a natural transformation $\xi$ from the composition

$$K^+(A) \xrightarrow{F} K(B) \xrightarrow{q} D(B)$$

to the composition

$$K^+(A) \xrightarrow{q} D^+(A) \xrightarrow{RF} D(B)$$

where $q$ denotes the natural localization functor. The natural transformation $\xi$ has the universal property that if $G : \mathcal{D}^+(A) \rightarrow \mathcal{D}(B)$ is another functor with a natural transformation $\zeta : q \circ F \rightarrow G \circ q$ then there is a unique natural transformation $\eta : RF \rightarrow G$ such that $\zeta_{A^*} = \eta_{q(A^*)} \circ \xi_{A^*}$ for every object $A^*$ of $\mathcal{D}^+(A)$. This universal property makes $RF$ unique up to natural isomorphism if it exists. One should think of $RF$ as the functor from $\mathcal{D}^+(A) \rightarrow \mathcal{D}(B)$ that best approximates the functor $F : K^+(A) \rightarrow K(B)$.

Dually we may consider a morphism $F : K^-(A) \rightarrow K(B)$ of triangulated categories. A left derived functor of $F$ is then a morphism of triangulated categories $LF : \mathcal{D}^-(A) \rightarrow \mathcal{D}(B)$ together with a natural transformation $\xi : LF \circ q \rightarrow q \circ F$ satisfying the dual universal property.

**Remark 4.4.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \rightarrow \mathcal{B}$ is a covariant additive functor. Then $F$ may be extended to a functor $C(A) \rightarrow C(B)$ which commutes with shifting of complexes and preserves mapping cones and exact triangles. Thus $F$ induces a morphism of triangulated categories $K(A) \rightarrow K(B)$. If the right derived functor of this morphism exists we denote it by $RF$, and if the left derived functor exists we denote it by $LF$.

**Theorem 4.5.** Let $\mathcal{A}$ be an abelian category and consider a morphism of triangulated categories $F : K^+(A) \rightarrow K(B)$. If $\mathcal{A}$ has enough injectives the right derived functor $RF : \mathcal{D}^+(A) \rightarrow \mathcal{D}(B)$ exists and if $I^*$ is a bounded below complex of injective objects we have

$$RF(I^*) \cong q(F(I^*))$$

where $q : K(B) \rightarrow D(B)$ is the canonical localization functor.

Dually, if $F : K^-(A) \rightarrow K(B)$ is a morphism of triangulated categories and $\mathcal{A}$ has enough projectives, then we have the existence of a left derived functor $LF : \mathcal{D}^-(A) \rightarrow \mathcal{D}(B)$ and if $P^*$ is a bounded above complex of projective objects, then

$$LF(P^*) \cong q(F(P^*))$$

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Proof. This is [Wei94, Thm. 10.5.6].

Remark 4.6. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. Then the derived functor $RF$ might exist even though $\mathcal{A}$ does not have enough injectives.

Suppose that there is a triangulated subcategory $K' \subseteq K^+(\mathcal{A})$ such that each object of $K^+(\mathcal{A})$ admits a quasi-isomorphism to an object of $K'$. Furthermore, assume that for any acyclic object $I^\bullet$ of $K'$ we have that $F(I^\bullet)$ is an acyclic object of $K(\mathcal{B})$. Then the right derived functor $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ exists and for each object $I^\bullet$ of $K'$ we have $RF(I^\bullet) \cong q(F(I^\bullet))$.

Dually, assume that there is a triangulated subcategory $K' \subseteq K^-(\mathcal{A})$ such that each object of $K^-(\mathcal{A})$ admits a quasi-isomorphism to an object of $K'$. Then the left derived functor $LF : D^-(\mathcal{A}) \to D(\mathcal{B})$ exists and for each object $P^\bullet$ of $K'$ we have $LH^i(F(P^\bullet)) \cong q(F(P^\bullet))$.

Remark 4.7. Consider abelian categories $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A}$ has enough injectives. Then if $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ is a right derived functor we may compute $RF(A^\bullet)$ for any object $A^\bullet$ of $D^+(\mathcal{A})$. To do this, let $I^\bullet$ be a bounded below complex of injectives together with a quasi-isomorphism $A^\bullet \to I^\bullet$. The complex $I^\bullet$ exists by [Wei94, Lem. 5.7.2]. Now $A^\bullet \to I^\bullet$ is an isomorphism in $D^+(\mathcal{A})$ and so

$$RF(A^\bullet) \cong RF(I^\bullet) \cong q(F(I^\bullet))$$

by Theorem 4.5. Dually, we may use projective resolutions to compute left derived functors.

Remark 4.8. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact covariant additive functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, then one defines the functors $RF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ as $RF(A^\bullet) = H^i(RF(A))$ where we view $A$ as a complex concentrated in degree zero. The condition that $F$ be left exact is not strictly necessary, but this condition leads to $RF(A^\bullet) \cong F(A^\bullet)$ which is a very useful property.

Definition 4.9 (Derived Hom). Let $\mathcal{A}$ be an abelian category with enough injectives. Suppose that $A^\bullet$ is a bounded above complex and $B^\bullet$ is a bounded below complex. Then we may form the bounded below complex $\text{Hom}_A^\bullet(A^\bullet, B^\bullet)$ by

$$\text{Hom}_A^i(A^\bullet, B^\bullet) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_A(A^p, B^{p+i})$$

with differential

$$d^i : \text{Hom}_A^i(A^\bullet, B^\bullet) \to \text{Hom}_A^{i+1}(A^\bullet, B^\bullet)$$

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given by

\[ (f^p : A^p \to B^{p+1})_{p \in \mathbb{Z}} \mapsto (d_B \circ f^p + (-1)^i f^{p+1} \circ d_A : A^p \to B^{p+i+1})_{p \in \mathbb{Z}} \]

where \( d_A \) and \( d_B \) denote the differentials in the complexes \( A^\bullet \) and \( B^\bullet \) respectively.

Let \( \text{Ab} \) denote the category of abelian groups. Then for a fixed complex \( A^\bullet \) in \( \text{K}^- (\mathcal{A}) \) the functor

\[ \text{Hom}_A^\bullet (A^\bullet, -) : \text{K}^+ (\mathcal{A}) \to \text{K}(\text{Ab}) \]

is a morphism of triangulated categories and we may thus form the derived functor

\[ \mathcal{R}\text{Hom}_A^\bullet (A^\bullet, -) : \text{D}^+ (\mathcal{A}) \to \text{D}(\text{Ab}) \]

The assignment \( (A^\bullet, B^\bullet) \mapsto \mathcal{R}\text{Hom}_A^\bullet (A^\bullet, B^\bullet) \) is a bifunctor

\[ \mathcal{R}\text{Hom}_A^\bullet (-, -) : \text{D}^- (\mathcal{A})^{\text{op}} \times \text{D}^+ (\mathcal{A}) \to \text{D}(\text{Ab}) \]

and if \( \mathcal{A} \) has enough projectives we may define \( \mathcal{R}\text{Hom}_A^\bullet (-, B^\bullet) \) as a right derived functor in the first variable. For complexes \( A^\bullet, B^\bullet \) and for \( i \geq 0 \) we define

\[ \text{Ext}^i_{A \mathcal{O}X} (A^\bullet, B^\bullet) = H^i(\mathcal{R}\text{Hom}_{A \mathcal{O}X}^\bullet (A^\bullet, B^\bullet)) \]

If \( X \) is a ringed topological space, scheme or algebraic space we may consider the analogous complex \( \mathcal{H}\text{om}_{\mathcal{O}X}^\bullet (\mathcal{F}^\bullet, \mathcal{G}^\bullet) \) with \( \mathcal{F}^\bullet \) being a bounded above complex of \( \mathcal{O}_X \)-modules and \( \mathcal{G}^\bullet \) being a bounded below complex of \( \mathcal{O}_X \)-modules. Then the functor

\[ \mathcal{H}\text{om}_{\mathcal{O}X}^\bullet (\mathcal{F}^\bullet, -) : \text{K}^+ (X) \to \text{K}(X) \]

is a morphism of triangulated categories and the corresponding derived functor is denoted \( \mathcal{R}\text{Hom}_{\mathcal{O}X}^\bullet (\mathcal{F}^\bullet, -) \). We define the local Ext groups for \( i \geq 0 \) as

\[ \mathcal{E}xt_{\mathcal{O}X}^i (\mathcal{F}^\bullet, \mathcal{G}^\bullet) = H^i(\mathcal{R}\text{Hom}_{\mathcal{O}X}^\bullet (\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \]

For more information, see [LH09, Ch. 2.4] and [Har66, Ch. II.3].

**Definition 4.10 (Derived tensor product).** Let \( X \) be a ringed space, scheme or algebraic space and let \( R \) be a commutative ring with identity. Let \( \mathcal{A} \) denote the category of \( R \)-modules or the category of \( \mathcal{O}_X \)-modules. Then if \( A^\bullet \) and \( B^\bullet \) are two complexes of \( \mathcal{A} \) we define the tensor product \( A^\bullet \otimes B^\bullet \) to be the complex given by

\[ (A^\bullet \otimes B^\bullet)^i = \bigoplus_{p+q=i} A^p \otimes B^q \]
with differential
\[ d^i : (A^\bullet \otimes B^\bullet)^i \rightarrow (A^\bullet \otimes B^\bullet)^{i+1} \]
defined by
\[ d^i(a^p \otimes b^q) = d^p_A(a^p) \otimes b^q + (-1)^i a^p \otimes d^q_B(b^q). \]
for \( a^p \otimes b^q \in A^p \otimes B^q \). Then for a fixed bounded below complex \( A^\bullet \) the functor
\[ (A^\bullet \otimes -) : K^-(A) \rightarrow K(A) \]
is a morphism of triangulated categories. Also, the category of \( \mathcal{O}_X \)-modules has enough flat modules and the functor \((A^\bullet \otimes -)\) takes acyclic complexes of flat modules to acyclic complexes [Har66, Lem. II.4.1]. Thus the functor \((A^\bullet \otimes -)\) satisfies the criteria of Remark 4.6 so the left derived functor
\[ L(A^\bullet \otimes -) : D^- (A) \rightarrow D(A) \]
extists. We denote this derived functor \((A^\bullet \otimes L^- \otimes -)\). If \( X \) is a scheme or an algebraic space the derived tensor product can be extended to a functor
\[ \otimes^L : D(X) \times D(X) \rightarrow D(X). \]
For more information, see [LH09, Ch. 2.5] and [Har66, Ch. II.4].

**Definition 4.11** (Derived push-forward and pullback). Let \( X, Y \) be ringed spaces, schemes or algebraic spaces and consider a morphism \( f : X \rightarrow Y \). Denote by \( \mathcal{O}_X\)-\textbf{Mod} the category of \( \mathcal{O}_X \)-modules and by \( \mathcal{O}_Y\)-\textbf{Mod} the category of \( \mathcal{O}_Y \)-modules. Then by Remark 4.4 the functor \( f_* \) has a right derived functor
\[ Rf_* : D^+(X) \rightarrow D^+(Y) \]
and from the fact that the category \( \mathcal{O}_Y\)-\textbf{Mod} has enough flat modules we obtain a left derived functor
\[Lf^* : D^-(Y) \rightarrow D^-(X). \]
See [LH09, Ch. 3.1] and [Har66, Ch. II.2] for more information.

5 Hom complexes and Ext groups

In this section we prove a number of statements about Ext-groups. These statements are related to Schlessinger’s conditions of Definition 2.3 and the ADO-conditions of Definition 2.5. Specifically, Proposition 5.1 is connected to Schlessinger’s condition S2 and Propositions 5.2, 5.6 and 5.10 are related to ADO-conditions (1), (2) and (3).
**Proposition 5.1.** Let $A$ be a noetherian ring and let $X$ be an algebraic space with a proper morphism $f : X \to \text{Spec}(A)$. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules such that the cohomology $H^i(\mathcal{F}^\bullet)$ is coherent for each $i$. Let 

$$
\mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0
$$

be a bounded above complex of $\mathcal{O}_X$-modules such that the cohomology $H^i(\mathcal{L}^\bullet)$ is coherent for each $i$. Then the $\mathcal{O}_X$-modules $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet)$ are coherent and the $A$-modules $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet)$ are finitely generated for each integer $i$.

**Proof.** Let $Y = \text{Spec}(A)$. We view $\mathcal{L}^\bullet$ as an object of the derived category $D_{-c}(X)$ and $\mathcal{F}^\bullet$ as an object of the derived category $D_{+c}(X)$. Then by [Har66, Prop. II.3.3] we have that $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet)$ is in $D_{+c}(X)$. This shows that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) = H^i(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet))$ is coherent for each $i$. We have an isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \cong Rf_*(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet))$$

in $D^+(Y)$ by [Wei94, Ex. 10.8.6]. Furthermore, the derived functor $Rf_*$ maps $D_+^c(X)$ to $D_+^c(Y)$ by [Har66, Prop. II.2.2] and [Knu71, Thm. IV.4.1]. Thus $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet)$ is in $D_+^c(Y)$ which means that

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) = H^i(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet))$$

is a finitely generated $A$-module. 

**Proposition 5.2.** Let $A$ be a noetherian ring, $X$ an algebraic space and $f : X \to \text{Spec}(A)$ a separated morphism of finite type. Consider a flat morphism $A \to B$. Defining $Y = X \otimes_A B$ we obtain a cartesian diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & X \\
\downarrow{g} & & \downarrow{f} \\
\text{Spec}(B) & \xrightarrow{u} & \text{Spec}(A).
\end{array}
$$

Let 

$$
\mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0
$$

be a complex of $\mathcal{O}_X$-modules bounded above such that the cohomology $H^i(\mathcal{L}^\bullet)$ is coherent for all $i$ and let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules bounded below such that the cohomology $H^i(\mathcal{F}^\bullet)$ is quasi-coherent for all $i$. 

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Then for each integer \(i\) there are canonical isomorphisms

\[
\phi^* \operatorname{Ext}^i_{\mathscr{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \sim \operatorname{Ext}^i_{\mathcal{O}_X}(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet)
\]

and

\[
\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \otimes_A B \sim \operatorname{Ext}^i_{\mathcal{O}_X}(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet).
\]

**Proof.** We view \(\mathcal{L}^\bullet\) as an object of \(\mathbf{D}^{-}(X)\) and \(\mathcal{F}^\bullet\) as an object of \(\mathbf{D}^{+}_{qc}(X)\). Then by [Har66, Prop. II.5.8] we have a canonical isomorphism

\[
\phi^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \sim \mathcal{R} \operatorname{Hom}_{\mathcal{O}_Y}^\bullet(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet) \tag{5.2.1}
\]

in \(\mathbf{D}^+(Y)\). Since \(\phi^*\) is an exact functor we have that \(\phi^*\) commutes with cohomology. Thus we obtain an isomorphism

\[
\phi^* \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) = \mathcal{H}^i(\phi^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet)) \cong \\
\cong \mathcal{H}^i(\mathcal{R} \operatorname{Hom}_{\mathcal{O}_Y}^\bullet(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet)) = \operatorname{Ext}^i_{\mathcal{O}_Y}(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet).
\]

Furthermore, we have by [Har66, Prop. II.5.12] a canonical isomorphism

\[
u^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \sim \mathcal{R} \operatorname{g}_s(\phi^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet)) \tag{5.2.2}
\]

Since \(\nu\) is flat the functor \(\nu^*\) is exact and thus commutes with cohomology. This fact together with the isomorphisms (5.2.2) and (5.2.1) gives us the desired result

\[
\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}^\bullet) \otimes_A B = \nu^* H^i(\mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet)) \cong \\
\cong H^i(\nu^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet))) \cong H^i(\mathcal{R} \operatorname{g}_s(\phi^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{L}^\bullet, \mathcal{F}^\bullet))) \cong \\
\cong H^i(\mathcal{R} \operatorname{g}_s(\phi^* \mathcal{R} \operatorname{Hom}_{\mathcal{O}_Y}^\bullet(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet))) = \operatorname{Ext}^i_{\mathcal{O}_Y}(\phi^* \mathcal{L}^\bullet, \phi^* \mathcal{F}^\bullet).
\]

\[\square\]

Next we show some results regarding gradings of \(\operatorname{Ext}\)-modules. These results will be used in the proof of Proposition 5.6 to show that certain graded modules are finitely generated.

**Proposition 5.3.** Let \(A\) be a noetherian ring, \(X\) an algebraic space and \(f : X \to \operatorname{Spec}(A)\) a morphism of finite type. Moreover, let \(m\) be a maximal ideal of \(A\) and consider the Rees algebra \(S = \oplus_{k \geq 0} m^k\) with \(m^0 = A\) and multiplication inherited from the multiplication of \(A\). Let \(\mathcal{S}' = f^*(S)\) and suppose that \(\mathcal{M}\) is a quasi-coherent \(\mathcal{O}_X\)-module that is also a finitely generated graded \(\mathcal{S}'\)-module. Let

\[
\mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0
\]

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be a bounded above complex of $\mathcal{O}_X$-modules such that the cohomology $\mathcal{H}^i(\mathcal{L}^\bullet)$ is coherent for all $i$. Then the modules

$$\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M})$$

are finitely generated graded $S'$-modules for all $p \geq 0$.

**Proof.** Since the question is local on $X$ we can reduce to the case where $X$ is affine, say $X = \text{Spec}(B)$. We then have a graded $B$-algebra $S' = \oplus_{k \geq 0} m^k B$ corresponding to the module $S'$, a finitely generated $S'$-module $M$ corresponding to the module $\mathcal{M}$ and a complex

$$L^\bullet = \ldots \to L^{-2} \to L^{-1} \to L^0 \to 0$$

of finitely generated $B$-modules corresponding to the complex $\mathcal{L}^\bullet$. Note that the ring $S'$ is noetherian by [Knu71, Prop. V.2.15]. We wish to show that the $B$-module

$$\mathcal{E}xt^p_B(L^\bullet, M)$$

is finitely generated as an $S'$-module. Denote by $F$ the contravariant functor

$$G \mapsto \text{Hom}_B(G, M)$$

from $B$-modules to $B$-modules. We may also view $F$ as a covariant functor from $(\text{B-Mod})^{\text{op}}$ to $\text{B-Mod}$. Then we have

$$\mathcal{E}xt^p_B(L^\bullet, M) = R^p F(L^\bullet)$$

where $R^p F$ denotes the hyperderived functor of $F$. By [EGAIII, 0.11.4.3] we have a convergent spectral sequence of hypercohomology

$$\mathcal{E}xt^p_B(H^{-q}(L^\bullet), M) \implies \mathcal{E}xt^{p+q}_B(L^\bullet, M).$$

From this sequence we conclude that if $\mathcal{E}xt^p_B(G, M)$ is a finitely generated $S'$-module for all finitely generated $B$-modules $G$ and all $p \geq 0$, then $\mathcal{E}xt^p_B(L^\bullet, M)$ is also a finitely generated $S'$-module for all $p \geq 0$.

Thus let $G$ be a finitely generated $B$-module and let $F_\bullet$ be a finitely generated free resolution of $G$, with $F_i = B^{n_i}$ for $i \geq 0$. From the exact sequence

$$\text{Hom}_B(F_{p-1}, M) \xrightarrow{d^{p-1}} \text{Hom}_B(F_p, M) \xrightarrow{d^p} \text{Hom}_B(F_{p+1}, M)$$

we calculate $\mathcal{E}xt^p_B(G, M)$ as $\text{Ker}(d^p)/\text{Im}(d^{p-1})$. As $\text{Hom}_B(F_p, M) = M^{n_p}$ is a finitely generated graded $S'$-module and $S'$ is a noetherian ring we conclude that $\mathcal{E}xt^p_B(G, M)$ is a finitely generated graded $S'$-module as well. This concludes the proof. \qed
Proposition 5.4. Let $A$ be a noetherian ring, $X$ an algebraic space and $f : X \to \text{Spec}(A)$ a proper morphism. Moreover, let $\mathfrak{m}$ be a maximal ideal of $A$ and consider the Rees algebra $S = \bigoplus_{k \geq 0} \mathfrak{m}^k$ with $\mathfrak{m}^0 = A$ and multiplication inherited from the multiplication of $A$. Let $S' = f^*(S)$ and suppose that $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module that is also a finitely generated graded $S'$-module. Let

$$L^\bullet = \ldots \to L^{-2} \to L^{-1} \to L^0 \to 0$$

be a bounded above complex of $\mathcal{O}_X$-modules such that the cohomology $\mathcal{H}^i(L^\bullet)$ is coherent for all $i$. Then the modules

$$\text{Ext}^p_{\mathcal{O}_X}(L^\bullet, \mathcal{M})$$

are finitely generated $S$-modules for all $p \geq 0$.

Proof. By [Wei94, Cor. 10.8.3] we have a local-to-global convergent spectral sequence

$$H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(L^\bullet, \mathcal{M})) \Longrightarrow \text{Ext}^{p+q}_{\mathcal{O}_X}(L^\bullet, \mathcal{M})$$

so we are done if we can show that each of the modules $H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(L^\bullet, \mathcal{M}))$ is finitely generated as an $S$-module. By Proposition 5.3 we have that $\mathcal{E}xt^p_{\mathcal{O}_X}(L^\bullet, \mathcal{M})$ is a finitely generated graded module over the $\mathcal{O}_X$-algebra $f^*(S)$. The result [EGAIII, Prop. 3.3.1] holds in the case of algebraic spaces since the main part of the proof is that the higher direct image of a coherent sheaf under a proper map is coherent [Knu71, Thm. IV.4.1]. By [EGAIII, Prop. 3.3.1] we may then conclude that the groups

$$H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(L^\bullet, \mathcal{M}))$$

are finitely generated graded $S$-modules. Thus the module $\text{Ext}^p_{\mathcal{O}_X}(L^\bullet, \mathcal{M})$ is also a finitely generated and graded $S$-module. \hfill \square

Lemma 5.5. Let $A$ be a noetherian ring, $X$ an algebraic space and consider a proper morphism $f : X \to \text{Spec}(A)$. Suppose that

$$L^\bullet = \ldots \to L^{-2} \to L^{-1} \to L^0 \to 0$$

is a complex of $\mathcal{O}_X$-modules bounded above such that the cohomology $\mathcal{H}^i(L^\bullet)$ is coherent for all $i$ and let $\{\mathcal{M}_k\}_{k \geq 0}$ be an collection of $\mathcal{O}_X$-modules where each $\mathcal{M}_k$ is quasicoherent. Define $\mathcal{M} = \bigoplus_{k \geq 0} \mathcal{M}_k$.

Then for each $n \geq 0$ there is a canonical isomorphism

$$\bigoplus_{k \geq 0} \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, \mathcal{M}_k) \isom \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, \mathcal{M}).$$
Proof. We show first that there is an isomorphism

$$\bigoplus_{k \geq 0} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, M_k) \xrightarrow{\sim} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, M)$$

for each $n \geq 0$. By [EGAIII, 0.11.4.3] there is a convergent spectral sequence

$$\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{L}^\bullet), M_k) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{L}^\bullet, M_k)$$

for each $k \geq 0$ and thus a convergent spectral sequence

$$\bigoplus_{k \geq 0} \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{L}^\bullet), M_k) \Rightarrow \bigoplus_{k \geq 0} \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{L}^\bullet, M_k).$$

Similarly there is a convergent spectral sequence

$$\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{L}^\bullet), M) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{L}^\bullet, M).$$

Thus if we can find canonical isomorphisms

$$\bigoplus_{k \geq 0} \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{L}^\bullet), M_k) \xrightarrow{\sim} \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{H}^{-q}(\mathcal{L}^\bullet), M)$$

(5.5.1)

for all $p, q$ we will by [Wei94, Thm. 5.2.12] obtain canonical isomorphisms

$$\bigoplus_{k \geq 0} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, M_k) \xrightarrow{\sim} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, M)$$

for all $n \geq 0$.

Thus let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. We wish to show that the canonical map

$$\bigoplus_{k \geq 0} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{G}, M_k) \xrightarrow{\sim} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{G}, M)$$

is an isomorphism. For each $k \geq 0$ let $M_k \to \mathcal{G}^\bullet_k$ be an injective resolution of $M_k$. We can use the result of [Har66, Cor. II.7.9] since $X$ is noetherian and we thus conclude that $\mathcal{M} \to \bigoplus_{k \geq 0} \mathcal{G}^\bullet_k$ is an injective resolution of $\mathcal{M}$. We have for each $n \geq 0$ that

$$\bigoplus_{k \geq 0} \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{G}, M_k) = \bigoplus_{k \geq 0} \mathcal{H}^n(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G}^\bullet_k)) \cong \mathcal{H}^n(\bigoplus_{k \geq 0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G}^\bullet_k)).$$
Since each module \( \mathcal{G} \) is coherent we have by [Lan02, Ex. III.26] an isomorphism
\[
\bigoplus_{k \geq 0} \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}_k^\bullet) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \bigoplus_{k \geq 0} \mathcal{I}_k^\bullet).
\]

Thus we have
\[
\mathcal{H}^n\left( \bigoplus_{k \geq 0} \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}_k^\bullet) \right) \cong \mathcal{H}^n\left( \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \bigoplus_{k \geq 0} \mathcal{I}_k^\bullet) \right) = \text{Ext}^n_{\mathcal{O}_X}(\mathcal{G}, \mathcal{M}).
\]

Thus we have shown the existence of the canonical isomorphisms (5.5.1) so we obtain the canonical isomorphisms
\[
\bigoplus_{k \geq 0} \text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k) \cong \text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M})
\]
for each \( n \geq 0 \). By [Wei94, Cor. 10.8.3] we have local-to-global convergent spectral sequences
\[
H^p(X, \text{Ext}^q_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k)) \Rightarrow \text{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k)
\]
for each \( k \geq 0 \) which gives a spectral sequence
\[
\bigoplus_{k \geq 0} H^p(X, \text{Ext}^q_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k)) \Rightarrow \bigoplus_{k \geq 0} \text{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k).
\]

Thus the canonical isomorphisms
\[
\bigoplus_{k \geq 0} H^p(X, \text{Ext}^q_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k)) \cong H^p(X, \bigoplus_{k \geq 0} \text{Ext}^q_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k)) \cong H^p(X, \text{Ext}^q_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M})).
\]
for each \( p, q \) give by [Wei94, Thm. 5.2.12] the desired canonical isomorphism
\[
\bigoplus_{k \geq 0} \text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M}_k) \cong \text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{M})
\]
for each \( n \geq 0 \).

\[\square\]

**Proposition 5.6.** Let \( A \) be a noetherian ring, \( X \) an algebraic space and \( f : X \to \text{Spec}(A) \) a proper, flat morphism. Consider a maximal ideal \( \mathfrak{m} \) of \( A \) and denote by \( \hat{A} \) the \( \mathfrak{m} \)-adic completion of \( A \). Let
\[
\mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0
\]
be a complex of \( \mathcal{O}_X \)-modules bounded above such that \( \mathcal{H}^i(\mathcal{L}^\bullet) \) is coherent for each \( i \) and let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module.

Then for each \( n \geq 0 \) there is a canonical isomorphism
\[
\text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_A \hat{A} \cong \varprojlim \text{Ext}^n_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}/\mathfrak{m}^{k+1}\mathcal{F}).
\]
Proof. We will follow closely the proof of [Knu71, Thm. V.3.1]. Fix an integer \( n \) and consider for \( k \geq 0 \) the exact sequence
\[
0 \to m^{k+1} \mathcal{F} \to \mathcal{F} \to \mathcal{F}_k \to 0
\]
where \( \mathcal{F}_k = \mathcal{F}/m^{k+1} \mathcal{F} \). The long exact sequence of Ext-groups gives us an exact sequence
\[
\text{Ext}^n_{O_X}(\mathcal{L} \cdot, m^{k+1} \mathcal{F}) \to \text{Ext}^n_{O_X}(\mathcal{L} \cdot, \mathcal{F}) \xrightarrow{\varphi_{n,k}} \text{Ext}^n_{O_X}(\mathcal{L} \cdot, \mathcal{F}_k) \xrightarrow{\delta_k} \text{Ext}^{n+1}_{O_X}(\mathcal{L} \cdot, m^{k+1} \mathcal{F}).
\]
(5.6.1)

For a fixed \( n \) we introduce the notation
\[
E = \text{Ext}^n_{O_X}(\mathcal{L} \cdot, \mathcal{F}), \quad E_k = \text{Ext}^n_{O_X}(\mathcal{L} \cdot, \mathcal{F}_k), \quad R_k = \text{Ker}(\varphi_{n,k}), \quad Q_k = \text{Im}(\delta_k)
\]
so that we have exact sequences
\[
0 \to R_k \to E \xrightarrow{\varphi_{n,k}} E_k \xrightarrow{\delta_k} Q_k \to 0
\]
for each \( k \geq 0 \).

Consider integers \( k, m \geq 0 \) and the commutative diagram
\[
\begin{array}{c}
0 \to m^{k+1} \mathcal{F} \to \mathcal{F} \to \mathcal{F}_k \to 0 \\
\downarrow \\
0 \to m^{k+m+1} \mathcal{F} \to \mathcal{F} \to \mathcal{F}_{k+m} \to 0
\end{array}
\]
where the leftmost vertical map is the canonical injection, the middle vertical map is the identity map and the rightmost vertical map is the canonical projection. By taking the long exact sequence of Ext-groups we obtain a commutative diagram
\[
\begin{array}{c}
0 \to R_k \to E \xrightarrow{id_E} E_k \xrightarrow{\nu_{k,m}} Q_k \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to R_{k+m} \to E \xrightarrow{id_E} E_{k+m} \xrightarrow{\nu_{k,m}} Q_{k+m} \to 0
\end{array}
\]
(5.6.2)

with exact rows. From this diagram we obtain an exact sequence of projective systems
\[
0 \to \{E/R_k\} \to \{E_k\} \to \{Q_k\} \to 0.
\]
Since every map \( E/R_{k+m} \to E/R_k \) is surjective we may use the result of [AM69, Prop. 10.2] and obtain an exact sequence
\[
0 \to \lim E/R_k \to \lim E_k \to \lim Q_k \to 0.
\]
(5.6.3)
Now our goal is to prove two things. The first thing we will show is that the modules \( \{ R_k \} \) define a stable \( m \)-filtration on \( E \) (c.f. [AM69, Ch. 10 p. 105]). Since \( E \) is a finitely generated \( \hat{A} \)-module by Proposition 5.1 we may then conclude that \( E \otimes_A \hat{A} = \hat{E} = \lim \leftarrow E/R_k \).

The next thing to prove is that \( \lim \leftarrow Q_k = 0 \). By (5.6.3) we then obtain the desired isomorphism

\[
E \otimes_A \hat{A} = \lim \leftarrow E/R_k \longrightarrow \lim \leftarrow E_k.
\]

Now we begin showing that the submodules \( \{ R_k \} \) of \( E \) define a stable \( m \)-filtration on \( E \). Let \( m \geq 0 \) and choose an element \( x \in m^m \). Multiplication by \( x \) in \( m^k \mathcal{F} \) defines a homomorphism \( m^k \mathcal{F} \rightarrow m^{k+m} \mathcal{F} \) and thus induces a homomorphism

\[
\mu_{x,k,m} : \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, m^k \mathcal{F}) \longrightarrow \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, m^{k+m} \mathcal{F}).
\]

Let \( S = \bigoplus_{k \geq 0} m^k \) with \( m^0 = A \) considered as a graded \( A \)-algebra. Then \( S \) is noetherian by [Bou61, III.2.10, Cor. 4 to Thm. 2] and the module

\[
F = \bigoplus_{k \geq 0} \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, m^k \mathcal{F})
\]

is a finitely generated graded \( S \)-module by Lemma 5.5 and Proposition 5.4 with multiplication by \( x \in S_m \) defined by the functions \( \mu_{x,k,m} \) defined above.

To show that the modules \( R_k \) define a stable \( m \)-filtration it is enough by [Bou61, III.3.1, Thm. 1] to show that \( mR_k \subseteq R_{k+1} \) for all \( k \geq 0 \) and that \( R = \bigoplus_{k \geq 0} R_k \) is a finitely generated \( S \)-module. To show that \( mR_k \subseteq R_{k+1} \) choose an element \( x \in m^m \) and consider the commutative diagram

\[
m^{k+1} \mathcal{F} \xrightarrow{x} m^{k+m+1} \mathcal{F} \xrightarrow{x} \mathcal{F}
\]

where the horizontal arrows are multiplication by \( x \) and the vertical arrows are the natural injections. The diagram

\[
\text{Ext}^n_{\mathcal{O}_X}(L^\bullet, m^{k+1} \mathcal{F}) \xrightarrow{\mu_{x,k+1,m}} \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, m^{k+m+1} \mathcal{F}) \]

\[
\text{Ext}^n_{\mathcal{O}_X}(L^\bullet, \mathcal{F}) \xrightarrow{\mu_{x,0}} \text{Ext}^n_{\mathcal{O}_X}(L^\bullet, \mathcal{F})
\]
is then also commutative. In view of the exact sequence (5.6.1) we may view $R_k$ as the image of the map

$$\text{Ext}^n_{O_X}(L^\bullet, m^{k+1}F) \rightarrow \text{Ext}^n_{O_X}(L^\bullet, F).$$  \quad (5.6.6)

The commutativity of (5.6.5) now implies that $m^m R_k \subseteq R_{k+m}$. From (5.6.5) and the fact that $R_k$ can be viewed as the image of (5.6.6) we also have that the graded $S$-module $R = \oplus_{k \geq 0} R_k$ is a quotient of the submodule

$$\bigoplus_{k \geq 0} \text{Ext}^n_{O_X}(L^\bullet, m^{k+1}F)$$

of the $S$-module $F$. Thus $R$ is a finitely generated graded $S$-module and so the modules $R_k$ define a stable $m$-filtration on $E$. Thus we have

$$E \otimes_A \hat{A} = \hat{E} = \varprojlim E/R_k.$$ \quad (5.6.7)

Next we will consider the modules $Q_k$. Define the graded $S$-module

$$N = \bigoplus_{k \geq 0} \text{Ext}^{n+1}_{O_X}(L^\bullet, m^{k+1}F).$$

The module $N$ is a finitely generated graded $S$-module by Lemma 5.5 and Proposition 5.4, and we have that $Q_k \subseteq N_k$ for all $k \geq 0$. Choose $x \in m^m$ and consider the commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & m^{k+1}F & \rightarrow & F & \rightarrow & F_k & \rightarrow & 0 \\
\downarrow{x} & & \downarrow{x} & & \downarrow{x} & & \downarrow{x} & & \downarrow{x} \\
0 & \rightarrow & m^{k+m+1}F & \rightarrow & F_k & \rightarrow & F_{k+m} & \rightarrow & 0 \\
\end{array}$$

with exact rows, where vertical arrows denote multiplication by $x$. Using the long exact sequence of Ext-groups we obtain a commutative diagram

$$\begin{array}{cccccc}
\text{Ext}^n_{O_X}(L^\bullet, F_k) & \rightarrow & \text{Ext}^{n+1}_{O_X}(L^\bullet, m^{k+1}F) \\
\downarrow{\delta_k} & & \downarrow{\delta_{k+m}} & & \downarrow{\delta_{k+m}} \\
\text{Ext}^n_{O_X}(L^\bullet, F_{k+m}) & \rightarrow & \text{Ext}^{n+1}_{O_X}(L^\bullet, m^{k+m+1}F) \\
\end{array}$$

where the rightmost vertical map represents multiplication by $x \in S_m$ in the graded $S$-module $N$. Since $Q_k = \text{Im}(\delta_k)$ we conclude that $S_m Q_k \subseteq Q_{k+m}$ and so $Q = \oplus_{k \geq 0} Q_k$ is a sub-$S$-module of $N$ and hence finitely generated.
For \( m \geq 0 \) let \( \alpha_m \) denote the injection \( m^m \to A \) which we may think of as a map \( S_m \to S_0 \). Since \( m^{k+1}F_k = 0 \) we have that the \( A \)-module \( \text{Ext}_{\mathcal{O}_X}^n(L^\bullet, F_k) \) is annihilated by \( m^{k+1} \). Since \( Q_k \) is the image of the map
\[
\delta_k : \text{Ext}_{\mathcal{O}_X}^n(L^\bullet, F_k) \to \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^{k+1}F)
\]
we have that \( Q_k \), as an \( A \)-module, is also annihilated by \( m^{k+1} \). Thus in the \( S \)-module \( Q_k \) we have
\[
\alpha_{k+1}(S_{k+1})Q_k = 0.
\]
Since \( \alpha_m(S_m) \subseteq \alpha_{k+1}(S_{k+1}) \) for \( m > k \) we conclude that
\[
\alpha_m(S_m)Q_k = 0 \text{ for all } m > k. \tag{5.6.8}
\]
Since \( Q \) is a finitely generated \( S \)-module we have by [EGAII, Lem. 2.1.6(ii)] the existence of integers \( h \) and \( k_0 \) such that
\[
Q_{h+k} = S_hQ_k \text{ for all } k \geq k_0. \tag{5.6.9}
\]
We will now show that
\[
\alpha_{h+k_0}(S_{h+k_0})Q = 0. \tag{5.6.10}
\]
We have by (5.6.8) that \( \alpha_{h+k_0}(S_{h+k_0})Q_m = 0 \) when \( m < h + k_0 \). If on the other hand \( m \geq h + k_0 \) there is an integer \( a \) such that
\[
m = ah + k, \text{ with } k_0 \leq k < h + k_0.
\]
In this case we have by (5.6.9) and (5.6.8) that
\[
\alpha_{h+k_0}(S_{h+k_0})Q_m = \alpha_{h+k_0}(S_{h+k_0})Q_{ah+k} = S_{ah}\alpha_{h+k_0}(S_{h+k_0})Q_k = 0.
\]
Thus we have \( \alpha_{h+k_0}(S_{h+k_0})Q = 0 \).

Next we consider for \( k, m \geq 0 \) the canonical \( A \)-module homomorphism
\[
\nu_{k,m} : \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^{k+m}F) \longrightarrow \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^kF)
\]
obtained from the injection \( m^{k+m}F \to m^kF \). For each \( x \in m^m \) we have a factorization
\[
\text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^kF) \xrightarrow{\mu_{x,m}} \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^{k+m}F) \xrightarrow{\nu_{k,m}} \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^kF)
\]
of the map \( \mu_{x,k,0} \). Since \( Q_k \) is a submodule of \( \text{Ext}_{\mathcal{O}_X}^{n+1}(L^\bullet, m^kF) \) we then have in the \( S \)-module \( N \) that
\[
\nu_{k,m}(S_mQ_k) = \alpha_m(S_m)Q_k. \tag{5.6.11}
\]
Now we wish to prove that there is an integer $m > 0$ such that $\nu_{k,m}(Q_{k+m}) = 0$ for $k \geq k_0$. We prove this by choosing $m > h + k_0$ such that $m$ is a multiple of $h$. Thus $Q_{k+m} = S_m Q_k$ for $k \geq k_0$ and we may use (5.6.11) and (5.6.10) to conclude that

$$\nu_{k,m}(Q_{k+m}) = \alpha_m(S_m)Q_k \subseteq \alpha_{h+k_0}(S_{h+k_0})Q_k = 0.$$  

The projective system $\{Q_k\}$ is defined by the maps $\nu_{k,m}$ and we therefore conclude that $\lim Q_k = 0$.

Thus by (5.6.3) and (5.6.7) we have obtained the desired isomorphism

$$E \otimes_A \hat{A} = \hat{E} = \lim E/R_k \sim \lim E_k.$$

We now need a few results on derived categories of sheaves in order to prove Proposition 5.10.

**Proposition 5.7** (Projection formula). Let $X, Y$ be separated, noetherian algebraic spaces of finite Krull dimension and let $f : X \to Y$ be a morphism. Then we have an isomorphism in $D(Y)$

$$(Rf_*\mathcal{F}^*) \otimes^L_{O_Y} \mathcal{G}^* \sim Rf_*(\mathcal{F}^* \otimes^L_{O_X} Lf^*\mathcal{G}^*)$$

for $\mathcal{F}^* \in D(X)$ and $\mathcal{G}^* \in D_{qc}(Y)$.

**Proof.** See [LH09, Prop. 3.9.4].

**Corollary 5.8.** Let $S$ be a scheme of finite type over an excellent Dedekind ring and let $A$ be a noetherian ring of finite type over $S$. Consider an algebraic space $X$ and a proper morphism $f : X \to \text{Spec}(A)$. Let

$$\mathcal{L}^* = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0$$

be a complex of coherent $\mathcal{O}_X$-modules concentrated in negative degree and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then for each finitely generated $A$-module $M$ there is an isomorphism

$$R\text{Hom}_{\mathcal{O}_X}^*(\mathcal{L}^*, \mathcal{F}) \otimes^L_A M \sim R\text{Hom}_{\mathcal{O}_X}^*(\mathcal{L}^*, \mathcal{F} \otimes^L_{\mathcal{O}_X} Lf^*(M))$$

in the derived category $D(A)$.
Proof. By [Har66, Prop. 5.14] we have an isomorphism
\[ R\mathcal{H}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_{\mathcal{O}_X}^L Lf^*(M) \xrightarrow{\sim} R\mathcal{H}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X}^L Lf^*(M)) \]
in the category $\mathcal{D}(X)$. Combining this isomorphism with Proposition 5.7 we obtain a sequence of isomorphisms
\[ (Rf_! R\mathcal{H}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F})) \otimes_X M \xrightarrow{\sim} Rf_!(R\mathcal{H}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_{\mathcal{O}_X}^L Lf^*(M)) \]
\[ \xrightarrow{\sim} Rf_! R\mathcal{H}_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X}^L Lf^*(M)). \]
The proposition follows from the canonical isomorphism
\[ Rf_! \circ R\mathcal{H}_{\mathcal{O}_X}(-, -) \xrightarrow{\sim} R\mathcal{H}_{\mathcal{O}_X}(-, -) \]
of [Wei94, Ex. 10.8.6]. \qed

**Proposition 5.9.** Let $A$ be a regular noetherian ring of finite Krull dimension. Consider an algebraic space $X$ and let $f : X \to \text{Spec}(A)$ be a proper, flat morphism. Let
\[ \mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0 \]
be a complex of $\mathcal{O}_X$-modules bounded above such that $\mathcal{H}^i(\mathcal{L}^\bullet)$ is coherent for each $i$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over $A$.

Consider an integer $N > 0$ and define for $0 \leq i < N$ the functor
\[ T^i(M) = \text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X} f^*(M)) \]
from finitely generated $A$-modules to finitely generated $A$-modules. Then there is a bounded complex
\[ F^\bullet = 0 \to F^0 \to F^1 \to F^2 \to \ldots \to F^N \to 0 \]
of finitely generated free $A$-modules such that we have a functorial isomorphism
\[ T^i(M) \cong H^i(F^\bullet \otimes_A M) \]
for $0 \leq i < N$ and for each finitely generated $A$-module $M$. 40
Proof. Let $M$ be a finitely generated $A$-module. Since $\mathcal{F}$ is a flat $\mathcal{O}_X$-module we have an isomorphism $\mathcal{F} \otimes_{\mathcal{O}_X} f^*(M) \cong \mathcal{F} \otimes_{\mathcal{O}_X}^L f^*(M)$ in the derived category $D(X)$. Thus by the projection formula of Corollary 5.8 we have

$$R\text{Hom}_{\mathcal{O}_X}(L^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X} f^*(M)) \cong R\text{Hom}_{\mathcal{O}_X}(L^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X}^L f^*(M)) \cong R\text{Hom}_{\mathcal{O}_X}(L^\bullet, \mathcal{F}) \otimes_A M.$$ (5.9.1)

Denote by $R^\bullet$ the element $R\text{Hom}_{\mathcal{O}_X}(L^\bullet, \mathcal{F})$ of $D(A)$. We will first show that we may truncate $R^\bullet$ in high degrees while retaining the cohomology groups $H^i(R^\bullet \otimes^L_A M)$ for $i < N$.

Thus let $K$ be the projective dimension of $A$. This coincides with the Krull dimension of $A$ since $A$ is regular. Denote by $\tilde{R}^\bullet$ the object of $D(A)$ corresponding to the complex defined by $\tilde{R}^p = R^p$ for $p \leq N + K$ and $\tilde{R}^p = 0$ for $p > N + K$.

Let $P^\bullet = \ldots \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0$ be a projective resolution of $M$ and note that $P^{-k} = 0$ when $k > K$. Then

$$R^\bullet \otimes^L_A M \cong R^\bullet \otimes_A P^\bullet$$

in $D(A)$ and for $i \leq N$ we have

$$(R^\bullet \otimes_A P^\bullet)^i = \bigoplus_{r \geq 0} R^r \otimes_A P^{i-r}.$$

Since $P^{i-r} = 0$ when $i - r < -K$ we have that $R^i \otimes_A P^{i-r}$ is independent of $R^r$ when $r > N + K > i + K$. We thus conclude that

$$H^i(R^\bullet \otimes^L_A M) \cong H^i(\tilde{R}^\bullet \otimes^L_A M)$$ (5.9.2)

for $i < N$.

From the isomorphisms (5.9.1) and (5.9.2) we have for $i < N$ a canonical isomorphism

$$\text{Ext}^i_{\mathcal{O}_X}(L^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X} f^*(M)) = H^i(R\text{Hom}_{\mathcal{O}_X}(L^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X} f^*(M))) \cong H^i(\tilde{R}^\bullet \otimes^L_A M).$$

By [Har77, Lem. III.12.3] there is a quasi-isomorphism $F^\bullet \simeq \tilde{R}^\bullet$ where $F^\bullet$ is a bounded complex of finitely generated free $A$-modules. Since the modules $F^p$ are projective we have an isomorphism in the derived category

$$F^\bullet \otimes_A M \simeq \tilde{R}^\bullet \otimes^L_A M.$$

Thus we obtain the desired canonical isomorphism

$$\text{Ext}^i_{\mathcal{O}_X}(L^\bullet, \mathcal{F} \otimes_{\mathcal{O}_X} f^*(M)) \simeq H^i(F^\bullet \otimes_A M)$$

for $i < N$. 

$\square$
Proposition 5.10. Let $S$ be a scheme of finite type over an excellent Dedekind ring and let $A$ be a noetherian reduced ring of finite type over $S$. Consider an algebraic space $X$ and a proper, flat morphism $f : X \to \text{Spec}(A)$. Let

$$\mathcal{L}^\bullet = \ldots \to \mathcal{L}^{-2} \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to 0$$

be a complex of $\mathcal{O}_X$-modules bounded above such that $\mathcal{H}^i(\mathcal{L}^\bullet)$ is coherent for each $i$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over $A$.

Then for each integer $N > 0$ there is a dense open set $U \subseteq \text{Spec}(A)$ such that for each $y \in U$ such that $k(y)$ is of finite type over $A$ and for each $0 \leq i < N$ there is a canonical isomorphism

$$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_A k(y) \xrightarrow{\sim} \text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_A k(y)).$$

Proof. Since $A$ is of finite type over an excellent ring we have by the result [EGA IV, 7.8.3] that $A$ is excellent and that the set

$$\text{Reg}(A) = \{ p \in \text{Spec}(A) : A_p \text{ is regular} \}$$

is open in $\text{Spec}(A)$. Thus there is an open immersion $\text{Spec}(A') \to \text{Spec}(A)$ with $A'$ regular. Let $X' = X \otimes_A A'$ and let $(\mathcal{L'})^\bullet$ and $\mathcal{F}'$ be the pullbacks of $\mathcal{L}^\bullet$ and $\mathcal{F}$ to $X'$ respectively. Since the map $A \to A'$ is flat we have by Proposition 5.2 that

$$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_A A' \cong \text{Ext}^i_{\mathcal{O}_{X'}}((\mathcal{L'})^\bullet, \mathcal{F}')$$

Thus for each $y \in \text{Spec}(A')$ we have

$$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F}) \otimes_A k(y) \cong \text{Ext}^i_{\mathcal{O}_{X'}}((\mathcal{L'})^\bullet, \mathcal{F}') \otimes_{A'} k(y)$$

and since $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_A k(y))$ is an $A'$-module we have

$$\text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_A k(y)) \cong \text{Ext}^i_{\mathcal{O}_{X'}}((\mathcal{L'})^\bullet, \mathcal{F}' \otimes_{A'} k(y)).$$

Thus we may reduce to the case where the ring $A$ is regular. Then the scheme $\text{Spec}(A)$ is a disjoint union of regular and irreducible schemes. We may then study each of the components of $\text{Spec}(A)$ and reduce to the case where $A$ is regular and irreducible.

Let $0 \leq i < N$ be an integer. We will study the functor $T^i$ from finitely generated $A$-modules to finitely generated $A$-modules defined by

$$T^i(M) = \text{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{F} \otimes_A f^*(M)).$$
By Proposition 5.9 there is a complex
\[ F^\bullet = 0 \to F^0 \to F^1 \to \ldots \to F^N \to 0 \]
of finitely generated free \( A \)-modules which is bounded above, and a functorial isomorphism
\[ T^i(M) \cong H^i(F^\bullet \otimes_A M). \]
Consider the function \( \phi^i \) on \( \text{Spec}(A) \) defined by
\[ \phi^i(y) = \dim_{k(y)} T^i(k(y)) \]
for \( y \in \text{Spec}(A) \). Then \( \phi \) is upper semicontinuous by \([\text{EGA} \text{IV}, \text{Thm. 7.6.9}]\). Thus if \( \eta \in \text{Spec}(A) \) is the generic point there is an open set \( U_i \ni \eta \) such that \( \phi^i(y) \leq \phi^i(\eta) \) for all \( y \in U_i \). Moreover, if \( y \in U_i \) is any point, then there is an open set \( V \ni y \) such that \( \phi^i(y') \leq \phi^i(y) \) for all \( y' \in V \). Since \( \eta \in V \) we thus have \( \phi^i(\eta) \leq \phi^i(y) \) and so \( \phi^i(\eta) = \phi^i(y) \). Thus \( U_i \) is an open set on which \( \phi^i \) is constant.

Let \( \text{Spec}(A') \neq \emptyset \) be an open subscheme of \( \text{Spec}(A) \) contained in the intersection of the sets \( U_i \) for \( 0 \leq i < N \). Then the functions \( \phi^i \) are constant on \( \text{Spec}(A') \) for \( 0 \leq i < N \). We may then restrict our attention to the open set \( \text{Spec}(A') \) and thus reduce to the case where the functions \( \phi^i \) are constant on all of \( \text{Spec}(A) \) for \( 0 \leq i < N \).

For each \( 0 \leq i < N \) we have by \([\text{EGA} \text{IV}, \text{Thm. 7.6.9}]\) that the functor \( T^i \) is exact since \( \phi^i \) is constant. By \([\text{EGA} \text{IV}, \text{Prop. 7.3.3}]\) this means that there is an canonical isomorphism
\[ T^i(A) \otimes_A M \xrightarrow{\sim} T^i(M) \]
for each finitely generated \( A \)-module \( M \). Especially we have for each point \( y \in \text{Spec}(A) \) of finite type over \( A \) a canonical isomorphism
\[ T^i(A) \otimes_A k(y) \xrightarrow{\sim} T^i(k(y)) \]
which proves the proposition.

\[ \square \]

6 The cotangent complex

In this section we define the cotangent complex and relate this complex to the study of deformations and obstructions. We show in Propositions 6.3 and 6.4 that automorphisms, deformations and obstructions are related to the Ext-groups of the cotangent complex.

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Definition 6.1 (Cotangent complex). Let $A \to B$ be a ring homomorphism. We will define $A$-algebras $P_n$ for $n \geq 0$ starting with

$$P_0 = A[B] = A\left[x_b^0 : b \in B\right].$$

This notation means that $P_0$ is a polynomial ring over $A$ in variables $x_b^0$ for $b \in B$. We iterate this construction and define for $n \geq 0$

$$P_n = A[P_{n-1}] = A\left[x_p^n : p \in P_{n-1}\right].$$

where we define $P_{-1} = B$. We have natural ring maps $d^i_n : P_n \to P_{n-1}$ for $0 \leq i \leq n$ defined iteratively starting with the canonical map

$$d^0_0 : A[B] \longrightarrow B$$

mapping $x_b^0$ to $b$. For $n > 0$ we may then define

$$d^n_0 : P_n \longrightarrow P_{n-1}$$

by $d^n_0(x^n_p) = p$ and for $0 < i \leq n$ we define

$$d^n_i : P_n \longrightarrow P_{n-1}$$

by $d^n_i(x^n_p) = x^{n-1}_{p'}$ where $p' = d^{i-1}_{n-1}(p) \in P_{n-2}$.

Note that the collection $(P_n, d^n_i)$ of rings and maps defines a simplicial $A$-algebra [Ill71, I.1.1]. This means that the maps $d^n_i$ satisfy the relation $d^{i-1}_{n-1}d^n_i = d^{i-1}_{n-1}d^n_i$ whenever $i < j$. A consequence of this relation is that any composition of maps

$$d^n_0 \circ d^n_1 \circ \cdots \circ d^n_i : P_n \longrightarrow B$$

will be equal to the composition

$$d^n_0 \circ d^n_1 \circ \cdots \circ d^n_i : P_n \longrightarrow B.$$

Thus for every $n \geq 0$ the ring $B$ has a canonical structure of $P_n$-algebra.

We now define a complex $L_{B/A}^\bullet$ of $B$-modules called the cotangent complex of the morphism $A \to B$. For $n > 0$ we define $L^n_{B/A} = 0$ and for $n \geq 0$ we let

$$L^{-n}_{B/A} = \Omega^1_{P_n/A} \otimes_{P_n} B.$$

The differentials

$$d^{-n} : L^{-n}_{B/A} \longrightarrow L^{-n+1}_{B/A}$$

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are defined by
\[ d^{-n} = \sum_{i=0}^{n} (-1)^i f_n^i \otimes P_n \text{id}_B \]
where the maps \( f_n^i : \Omega^1_{P_n/A} \to \Omega^1_{P_{n-1}/A} \) are the maps induced from the ring homomorphisms \( d_n^i : P_n \to P_{n-1} \).

We may also globalize the construction of the cotangent complex. Thus let \( X \) be a ringed space, scheme or algebraic space and let \( A \to B \) be a morphism of sheaves of rings on \( X \). Then we can mirror the construction of the cotangent complex in the local case. We may define sheaves \( P_0 = A[\mathcal{B}] \) and \( P_n = A[P_{n-1}] \) for \( n > 0 \). The maps \( d_n^i \) can also be defined globally, and we may thus define a complex of \( B \)-modules \( L^\bullet_{\mathcal{B}/A} \) on \( X \) called the cotangent complex of \( A \to B \).

If \( f : X \to Y \) is a map of ringed spaces, schemes or algebraic spaces, we define the \textit{cotangent complex of} \( f \) to be the complex of \( O_X \)-modules given by \( \mathcal{L}_{X/Y} \). If \( Y = \text{Spec}(A) \) is an affine scheme we write \( \mathcal{L}^\bullet_{X/A} \) instead of \( \mathcal{L}_{X/Y} \).

For more information on the construction and basic properties of the cotangent complex, see [Ill71, Ch. II].

**Lemma 6.2.** Let \( A \) be a noetherian ring and let \( f : X \to \text{Spec}(A) \) be a morphism of finite type, where \( X \) is an algebraic space. Then the cotangent complex \( \mathcal{L}^\bullet_{X/A} \) is in \( D^-_c(X) \), i.e. the cohomology \( H^i(\mathcal{L}^\bullet_{X/A}) \) is a coherent \( O_X \)-module for all \( i \).

**Proof.** This is [Ill71, Cor. II.2.3.7].

**Proposition 6.3.** Let \( S \) be a scheme of finite type over an excellent Dedekind ring. Denote by \( \mathcal{X}^\bullet_{\text{fl,pr,fp}} \) the CFG over \( \text{Sch}/S \) as follows: For each \( S \)-scheme \( T \) the category \( \mathcal{X}^\bullet_{\text{fl,pr,fp}}(T) \) has objects which are families \( f : X \to T \) where \( X \) is an algebraic space and \( f \) is flat, proper and of finite presentation.

The following data is then an obstruction theory \( O \) for \( \mathcal{X} \): Consider an infinitesimal extension \( \varphi : A \to A_0 \) of rings of finite type over \( S \), and an object \( f : X \to \text{Spec}(A) \) of \( \mathcal{X}(A) \). Then define \( O_X \) as the following functor from \( A_0 \)-modules to \( A \)-modules:

\[ M \mapsto \text{Ext}^2_{\mathcal{O}_X}(\mathcal{L}^\bullet_{X/A}, f^*M_{[\varphi]}). \]

Here the notation \( M_{[\varphi]} \) means restriction of scalars and \( \mathcal{L}^\bullet_{X/A} \) denotes the cotangent complex.

**Proof.** The functor in question is a functor from finitely generated \( A_0 \)-modules to \( A \)-modules. Since the module \( M_{[\varphi]} \) is annihilated by elements of \( \text{Ker}(\varphi) \) it
follows that $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{L}^\bullet_{X/A}, f^*M[\varphi])$ is also annihilated by elements of $\text{Ker}(\varphi)$. Consequently the functor $O_X$ maps $A_0$-modules to $A_0$-modules. Furthermore, we have that $f^*M[\varphi]$ is a coherent $\mathcal{O}_X$-module so by Lemma 6.2 and Proposition 5.1 the functor takes values in finitely generated $A_0$-modules.

That the group $\text{Ext}^2_{\mathcal{O}_X}(\mathcal{L}^\bullet_{X/A}, f^*M[\varphi])$ contains the obstructions to lifting flat deformations follows from [Ill71, Thm. III.2.1.7]. That any such flat lifting is proper and of finite presentation follows from Proposition 3.13.

Proposition 6.4. Let $S$ be a scheme of finite type over an excellent Dedekind ring. Denote by $\mathcal{X}_{\text{fl}, \text{pr}, \text{fp}}$ the CFG over $\text{Sch}/S$ as follows: For each $S$-scheme $T$ the category $\mathcal{X}_{\text{fl}, \text{pr}, \text{fp}}(T)$ has objects which are families $f : X \to T$ where $X$ is an algebraic space and $f$ is flat, proper and of finite presentation.

Let $A_0$ be a noetherian ring and $M$ a finitely generated $A_0$-module. Consider an object $f^0 : X_0 \to \text{Spec}(A_0)$ of $\mathcal{X}_{\text{fl}, \text{pr}, \text{fp}}(A_0)$ and define $A = A_0 + M$, with $\varphi : A \to A_0$ the natural map.

Then the group $\text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ is isomorphic to the group of isomorphism classes of flat, proper, finitely presented liftings of $X_0/A_0$ to $A$. In other words, with the notation of Definition 2.2 we have

$$D_{X_0}(M) \cong \text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M).$$

Suppose next that $X/A$ is a lifting of $X_0/A_0$ to $A$. We then have that the group $\text{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ is isomorphic to the group of automorphisms of $X/A$ in the category $(\mathcal{X}_{\text{fl}, \text{pr}, \text{fp}})_X(A)$ of liftings of $X_0/A_0$ to $A$. With the notation of Definition 2.2 we have

$$\text{Aut}_{X_0}(A_0 + M) \cong \text{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M).$$

Furthermore, the groups $\text{Ext}^i_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ are finitely generated $A_0$-modules for $i \in \{0, 1\}$.

Proof. That $\text{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ is isomorphic to the group of isomorphism classes of flat liftings of $X_0/A_0$ to $A$ is the statement of [Ill71, Thm. III.2.1.7]. That any such flat lifting is proper and of finite presentation follows from Proposition 3.13.

The fact that $\text{Ext}^i_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ is finitely generated for $i \in \{0, 1\}$ follows from Lemma 6.2 and Proposition 5.1.

Finally, that $\text{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{L}^\bullet_{X_0/A_0}, f_0^*M)$ is isomorphic to the group of automorphisms of a particular flat lifting follows from [Ill71, Thm. III.2.1.7].
7 The stack of projective schemes with vanishing second cohomology

In this section we define the stack $\mathcal{X}_{pv2}$ parametrizing projective schemes whose structure sheaf has vanishing second cohomology. The main theorem of the paper is Theorem 7.6 where we use Artin’s criteria of Theorem 2.9 to show that $\mathcal{X}_{pv2}$ is an algebraic stack.

**Definition 7.1.** Let $S$ be a scheme of finite type over an excellent Dedekind ring. Let $\mathcal{X}_{pv2}$ denote the CFG over $\textbf{Sch}/S$ where $\mathcal{X}_{pv2}(T)$ is the category consisting of morphisms $f : X \to T$ where $X$ is an algebraic space, the map $f$ is flat, proper and of finite presentation such that for all $t \in T$ the geometric fibers $X_t/k(t)$ are projective schemes such that $H^2(X_t, \mathcal{O}_{X_t}) = 0$.

**Lemma 7.2.** Let $T$ be an integral noetherian scheme and let $f : X \to T$ be a morphism of algebraic spaces which is flat and proper. Let $E \subseteq T$ be the set of points $t \in T$ such that the fiber $f_t : X_t \to \text{Spec}(k(t))$ is an object of $\mathcal{X}_{pv2}(k(t))$. Assume that the generic point of $T$ is in $E$. Then $E$ is open in $T$.

*Proof.* The question is local on $T$ so we may reduce to the case where $T$ is affine, say $T = \text{Spec}(A)$.

That the set $U$ of points $t \in T$ such that the fibers $X_t$ satisfy $H^2(X_t, \mathcal{O}_{X_t}) = 0$ is open follows from [Har77, Thm. III.12.8]. This theorem is applicable in our situation since the proof only uses the fact that the functor

$$M \mapsto H^2(X, \mathcal{O}_X \otimes_A M)$$

from $A$-modules to $A$-modules is isomorphic to a functor of the form

$$M \mapsto H^2(L^\bullet \otimes_A M)$$

where $L^\bullet$ is a bounded above complex of finitely generated free $A$-modules. The proof of this fact is [Har77, Prop. III.12.2] which is applicable in our case since the proof only relies on the fact that we may compute cohomology using the Čech complex.

The fact that the set $V$ of points $t \in T$ such that $X_t$ is projective is open follows from [EGAIV, Prop. 9.6.2]. Now we have $E = U \cap V$ which is open in $T$. \hfill $\square$

**Remark 7.3.** The result of Lemma 7.2 is that the property of being in $\mathcal{X}_{pv2}$ is an *ind-constructible* property [EGAIV, Def. 9.2.1].
Lemma 7.4. Let $S$ and $\mathcal{X}_{pv2}$ be as in Definition 7.1. Then $\mathcal{X}_{pv2}$ is limit-preserving.

Proof. Let $\mathcal{X} = \mathcal{X}_{pv2}$ denote the CFG in question. We need to check that for any filtered direct system of rings $\{A_\lambda\}_{\lambda \in \Lambda}$ over $S$ we have that the canonical functor

$$\Phi : \lim_{\rightarrow} \mathcal{X}(A_\lambda) \longrightarrow \mathcal{X}(\lim_{\rightarrow} A_\lambda)$$

is an equivalence of categories.

We introduce the notation

$A = \lim_{\rightarrow} A_\lambda$,  $T = \text{Spec}(A)$,  $T_\lambda = \text{Spec}(A_\lambda)$ for $\lambda \in \Lambda$.

To show that $\Phi$ is essentially surjective we let $f : X \rightarrow T$ be an object of $\mathcal{X}(T)$. Let $U \rightarrow X$ be an étale presentation and let $R = U \times_X U$. Then $R \Rightarrow U$ defines an étale equivalence relation in the category of schemes over $T$. By [EGA IV, Thm. 8.8.2] we can find a $\lambda \in \Lambda$ and an equivalence relation $R_\lambda \Rightarrow U_\lambda$ in the category of schemes over $T_\lambda$. The schemes $R_\lambda$ and $U_\lambda$ are of finite presentation over $T_\alpha$ and $R \Rightarrow U$ is the pullback of the equivalence relation $R_\lambda \Rightarrow U_\lambda$ by the canonical map $T \rightarrow T_\lambda$. The two maps in $R_\lambda \Rightarrow U_\lambda$ can be chosen to be étale by [EGA IV, Prop. 17.7.8], and hence $R_\lambda \Rightarrow U_\lambda$ is an étale equivalence relation. Furthermore, by [EGA IV, Thm. 11.2.6] and [EGA IV, Thm. 10.8.5(iv)] we can choose $\lambda$ such that $U_\lambda$ is flat over $T_\lambda$ and such that the canonical map $R_\lambda \rightarrow U_\lambda \times_{T_\lambda} U_\lambda$ is a closed immersion. Thus $R_\lambda \Rightarrow U_\lambda$ defines an algebraic space $X_\lambda$ which is flat, separated and of finite presentation over $T_\lambda$ together with an isomorphism $X \cong X_\lambda \times_{T_\lambda} T$.

From Chow’s Lemma [Knu71, Thm. IV.3.1] we have a commutative diagram

$$
\begin{array}{ccc}
X'_\lambda & \xrightarrow{j_\lambda} & P_\lambda \\
\downarrow g_\lambda & & \downarrow p_\lambda \\
X_\lambda & \xrightarrow{f_\lambda} & T_\lambda
\end{array}
$$

(7.4.1)

where $p_\lambda$ is projective, the map $g_\lambda$ is projective and surjective and $j_\lambda$ is an open immersion. The diagram (7.4.1) pulls back to a diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{j} & P \\
\downarrow g & & \downarrow p \\
X & \xrightarrow{f} & T
\end{array}
$$

where $p$ is projective, the map $g$ is a projective and surjective and $f$ is proper. Thus $j$ is a proper immersion of schemes and hence a closed immersion. We
may thus by [EGAIV, Thm. 8.10.5(iv)] find an element $\mu \geq \lambda$ such that the diagram (7.4.1) pulls back to a diagram

$$
\begin{array}{ccc}
X_\mu' & \to & P_\mu \\
\downarrow g_\mu & & \downarrow p_\mu \\
X_\mu & \to & T_\mu
\end{array}
$$

where $j_\mu$ is a closed immersion. Thus $f_\mu \circ g_\mu = p_\mu \circ j_\mu$ is proper. We have that $f_\mu$ is separated and of finite type and the property of being universally closed is purely topological. Thus it follows from [EGAII, Cor. 5.4.3] that $f_\mu$ is proper. Thus $f_\mu : X_\mu \to T_\mu$ is flat, proper and of finite presentation and pulls back to $f : X \to T$. By Lemma 7.2 and [EGAIV, Prop. 9.3.3] we may also choose $f_\mu$ so that the fibers are projective with vanishing second cohomology. Thus the functor $\Phi$ is essentially surjective.

Next we show that $\Phi$ is fully faithful. First of all, let $X \to T$ and $Y \to T$ be objects of $\mathcal{X}(T)$ and consider a $T$-morphism $f : X \to Y$. We may find étale presentations $U \to X$ and $V \to Y$ of $X$ and $Y$ respectively, such that $f$ is induced by morphisms $g, h$ in a diagram

$$
\begin{array}{ccc}
R & \to & U \\
\downarrow h & & \downarrow g \\
S & \to & V
\end{array}
\to
\begin{array}{ccc}
 & & X \\
 & & \downarrow f \\
 & & \\
 & & Y
\end{array}
$$

where $R = U \times_X U$ and $S = V \times_Y V$. By [EGAIV, Thm. 8.8.2] we may find $\lambda \in \Lambda$ and equivalence relations $R_\lambda \to U_\lambda$ and $S_\lambda \to V_\lambda$ together with maps $g_\lambda, h_\lambda$ making a commutative diagram

$$
\begin{array}{ccc}
R_\lambda & \to & U_\lambda \\
\downarrow h_\lambda & & \downarrow g_\lambda \\
S_\lambda & \to & V_\lambda
\end{array}
$$

The equivalence relations $R_\lambda \to U_\lambda$ and $S_\lambda \to V_\lambda$ define algebraic spaces $X_\lambda$ and $Y_\lambda$ and the maps $g_\lambda$ and $h_\lambda$ give a map $f_\lambda : X_\lambda \to Y_\lambda$ that pull back to the morphism $f$ via the map $T \to T_\lambda$. Thus the map on Hom-sets is surjective.

Next we show that the map on Hom-sets is injective. Thus choose $\alpha \in \Lambda$ and let $f'_\alpha$ and $f''_\alpha$ be morphisms $X_\alpha \to Y_\alpha$ where $X_\alpha$ and $Y_\alpha$ are objects of $\mathcal{X}(T_\alpha)$. Assume furthermore that the pullbacks $f', f'' : X \to Y$ of the morphisms via the map $T \to T_\alpha$ coincide. Then we need to show that
there is an element $\mu \geq \alpha$ such that $f'_\mu = f''_{\mu}$. Let $U_\alpha \to X_\alpha$ and $V_\alpha \to Y_\alpha$ be étale presentations such that the morphisms $f'_\alpha, f''_\alpha$ are induced by morphisms $g'_\alpha, g''_\alpha : U_\alpha \to V_\alpha$ and morphisms $h'_\alpha, h''_\alpha : U_\alpha \times_{X_\alpha} U_\alpha \to V_\alpha \times_{Y_\alpha} V_\alpha$ compatible with the projection maps.

The diagram

\[
\begin{array}{ccc}
U_\alpha \times_{X_\alpha} U_\alpha & \longrightarrow & U_\alpha \\
\downarrow h'_\alpha & & \downarrow f'_\alpha \\
V_\alpha \times_{Y_\alpha} V_\alpha & \longrightarrow & V_\alpha
\end{array}
\]

pulls back to a diagram

\[
\begin{array}{ccc}
U \times_{X} U & \longrightarrow & U \\
\downarrow h' & & \downarrow f' \\
V \times_{Y} V & \longrightarrow & V
\end{array}
\]

and similarly the pair of maps $(g''_\alpha, h''_\alpha)$ pull back to a pair of maps $(g', h')$ inducing the map $f'' : X \to Y$.

Consider now for each $\lambda \in \Lambda$ the category $\mathcal{S}(T_\lambda)$ of schemes separated and of finite presentation over $T_\lambda$. We define similarly the category $\mathcal{S}(T)$. Then by [EGAIV, Thm. 8.8.2] we have that the canonical map

\[
\lim \mathcal{S}(T_\lambda) \longrightarrow \mathcal{S}(T)
\]

defined by pullback is an equivalence of categories. The morphisms $f'$ and $f''$ are completely determined by the pairs $(g', h')$ and $(g'', h'')$ respectively. Since the pairs $(g', h')$ and $(g'', h'')$ of morphisms in $\mathcal{S}(T)$ induce the same map $f' = f''$ we thus have some $\mu \geq \alpha$ such that the pairs $(g''_\alpha, h''_\alpha)$ and $(g''_\mu, h''_\mu)$ induce the same map $f'_\mu = f''_\mu$. This shows that the functor $\Phi$ is fully faithful.

**Lemma 7.5.** Let $S$ and $\mathcal{X}_{pv2}$ be as in Definition 7.1. Then $\mathcal{X}_{pv2}$ is a stack.

**Proof.** Let $T$ be a scheme over $S$. Any family $\{ \varphi_i : T_i' \to T \}_{i \in I}$ of étale maps whose images cover $T$ can be replaced with the single étale surjective map $\varphi : T' = \coprod_{i \in I} T_i' \to T$. Also, the data of giving an algebraic space $X'_i$ over $T_i'$ for each $i \in I$ is equivalent to giving the algebraic space $X' = \coprod_{i \in I} X_i'$ over the scheme $T'$. Thus when discussing descent data for the stack $\mathcal{X}_{pv2}$ it is enough to consider étale surjective maps $f' : T'' \to T$ whose descent data is of the form $(X', \phi)$ where $X'$ is an algebraic space over $T'$ and $\phi$ is an isomorphism $p_1^* X' \to p_2^* X'$ where $p_1$ and $p_2$ denote the projection maps $T' \times_T T' \to T'$.  

50
Thus let $f' : T' \to T$ be an étale surjective map and as above we denote the maps $T'' \times_T T' \to T'$ by $p_1, p_2$. To show that a descent datum $(X', \phi)$ for $f'$ is effective we let $X'' = p_1^* X'$. This gives us an étale equivalence relation $X'' \rightrightarrows X'$ where one of the maps is the canonical map $p_1^* X' \to X'$ and the other map is the composition of $\phi$ and the map $p_2^* X' \to X'$. Then by [Knu71, Prop. II.3.14] the categorical quotient $X$ of $X'' \rightrightarrows X'$ exists in the category of algebraic spaces, and by the universal property of the quotient we obtain a unique map $f : X \to T$ as in the diagram below:

\[
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow & & \downarrow f' \\
T' \times_T T' & \longrightarrow & T'.
\end{array}
\]

Also, the rightmost square in the diagram is cartesian by [Knu71, Prop. I.5.8]. By [Knu71, Def. II.3.2, Prop. II.3.4, Prop. II.7.2] we conclude that since $f'$ is flat, proper and of finite presentation, the map $f$ has these properties as well.

Also, if $\text{Spec}(k) \to T$ is a geometric point we can always find a factorization $\text{Spec}(k) \to T' \to T$ and so the geometric fibers of $X \to T$ will have the desired property of being projective with vanishing second cohomology of the structure sheaf.

Next let $X, Y$ be objects of $\mathcal{X}_{pv2}(T)$, and denote by $X'$ and $Y'$ the pullbacks of $X$ and $Y$ via the map $f' : T' \to T$. Also, let $X''$ and $Y''$ denote the pullbacks to $T'' = T' \times_T T'$. Thus $X$ and $Y$ are quotients of the equivalence relations $X'' \rightrightarrows X'$ and $Y'' \rightrightarrows Y'$ respectively. Any $T$-morphism $X \to Y$ is then uniquely determined by the corresponding morphisms

\[
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y'' & \longrightarrow & Y'.
\end{array}
\]

This shows that the Hom-functor of morphisms from $X$ to $Y$ is a sheaf in the étale topology. Consequently the CFG $\mathcal{X}_{pv2}$ is a stack.

**Theorem 7.6.** Let $S$ and $\mathcal{X}_{pv2}$ be as in Definition 7.1. Then the limit preserving stack $\mathcal{X}_{pv2}$ satisfies Artins criteria of Theorem 2.9. Consequently $\mathcal{X}_{pv2}$ is an algebraic stack.

**Proof.** We go through each of Artin’s conditions in Theorem 2.9.

**Step 1.** We begin by showing that Schlessingers condition $S1'$ of Definition 2.3 holds. Thus consider a deformation situation $A' \to A \to A_0$ of rings
of finite type over $S$ with $A_0$ reduced. Moreover, let

$$
\begin{array}{ccc}
B & \to & A \\
\downarrow & & \downarrow \\
A' & \to & A
\end{array}
$$

be a diagram of rings of finite type over $S$ where the composed map $B \to A_0$ is surjective, and the bottom map is the morphism in the deformation situation. We then need to show that for each object $X/A$ of $\mathcal{X}_{\text{pv}2}(A)$ the natural functor

$$
\Phi: (\mathcal{X}_{\text{pv}2})_X(A' \times_A B) \to (\mathcal{X}_{\text{pv}2})_X(A') \times (\mathcal{X}_{\text{pv}2})_X(B)
$$

is an equivalence of categories. That $\Phi$ is fully faithful follows from Proposition 3.11. We need to show that $\Phi$ is essentially surjective. Thus let $(X',Y)$ be an object of $(\mathcal{X}_{\text{pv}2})_X(A') \times (\mathcal{X}_{\text{pv}2})_X(B)$. By Proposition 3.11 there is an algebraic space $Y'$, flat over $B' = A' \times_A B$ such that $X' \cong Y' \otimes_{B'} A'$ and $Y \cong Y' \otimes_{B'} B$. The fact that $Y'$ is proper and of finite presentation over $B'$ follows from Proposition 3.13. Moreover, the map $B' \to B$ is a nilpotent immersion, so any geometric point $\text{Spec}(k) \to \text{Spec}(B')$ will factor through the map $\text{Spec}(B) \to \text{Spec}(B')$ thus it follows that the geometric fibers of the map $Y' \to B'$ are projective with vanishing second cohomology of the structure sheaf. Thus $Y'$ is an object of $(\mathcal{X}_{\text{pv}2})_X(B')$ so the functor $\Phi$ is essentially surjective.

That Schlessinger’s Condition S2 and the corresponding statement for automorphisms are satisfied follows from Proposition 6.4 and Proposition 5.1.

**Step 2.** Let $\hat{A}$ be a complete local ring over $S$ with maximal ideal $m$ with residue field $k$ of finite type over $S$. For each $n \geq 0$ define $\hat{A}_n = \hat{A}/m^{n+1}$ and let $\{ f_n : X_n \to \text{Spec}(\hat{A}_n) \}_{n \geq 0}$ be an object of $\lim\left\downarrow \right\mathcal{X}_{\text{pv}2}(\hat{A}_n)$. In other words, the sequence $\{ f_n \}$ is a formal deformation in the sense that for each $m < n$ the map $f_m$ is the pullback of the map $f_n$ under the natural map $\text{Spec}(\hat{A}_m) \to \text{Spec}(\hat{A}_n)$. The algebraic space $X_0$ is a projective scheme over $\text{Spec}(k)$ and so by Corollary 3.9 the algebraic spaces $X_n$ are schemes for each $n \geq 0$. It follows by [Vis97, Prop. 6.3] that there is a scheme $X$ over $\text{Spec}(\hat{A})$ that pull back to $X_n$ via the canonical map $\text{Spec}(\hat{A}_n) \to \text{Spec}(\hat{A})$ for each $n \geq 0$. Note that this is the only step of the proof that require the fibers to be projective with vanishing second cohomology of the structure sheaf. We have thus shown that the canonical functor

$$
\Phi: \mathcal{X}_{\text{pv}2}(\hat{A}) \to \lim\left\downarrow \right\mathcal{X}_{\text{pv}2}(\hat{A}_n)
$$

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is essentially surjective. We have by the Grothendieck Existence Theorem [Knu71, Thm. V.6.3] and [Vis97, Prop. 6.3] that if $X/A$ is any object of $\mathcal{X}_{pv2}(\hat{A})$ then in fact $X$ is a scheme. Thus the fact that the canonical functor
$$\Phi : \mathcal{X}_{pv2}(\hat{A}) \longrightarrow \lim_{\leftarrow} \mathcal{X}_{pv2}(\hat{A}_n)$$
is fully faithful follows from [EGAIII, Thm. 5.4.1].

**Step 3.** We need to check the conditions A1-A3, D1-D3 and O1,O3 of Definition 2.5. Thus let $A_0$ be a ring of finite type over $S$ and consider an object $f_0 : X_0 \to \text{Spec}(A_0)$ of $\mathcal{X}_{pv2}(A_0)$. Furthermore, let $M$ be a finitely generated $A_0$-module and let $\varphi : A \to A_0$ be an infinitesimal deformation. Suppose that $f : X \to \text{Spec}(A)$ is a lifting of $X_0$ to $A$ and that $X$ is flat, proper and of finite presentation over $A$. Then $X$ is in $\mathcal{X}_{pv2}(A)$, since any geometric point $\text{Spec}(k) \to \text{Spec}(A)$ must factor via the map $\text{Spec}(A_0) \to \text{Spec}(A)$. We thus have by Proposition 6.4 the automorphism and deformation modules
$$\text{Aut}_{X_0}(A_0 + M) = \text{Ext}^0_{\mathcal{X}_0}(\mathcal{L}_{X_0/A_0}^\bullet, f_0^*M)$$
$$D_{X_0}(M) = \text{Ext}^1_{\mathcal{X}_0}(\mathcal{L}_{X_0/A_0}^\bullet, f_0^*M)$$
for the stack $\mathcal{X}_{pv2}$. By Proposition 6.3 we have the obstruction functor
$$O_X(M) = \text{Ext}^2_{\mathcal{X}/A}(\mathcal{L}_{X/A}^\bullet, f^*M_{[\varphi]})$$
for the stack $\mathcal{X}_{pv2}$.

Next suppose that $B_0$ is a ring of finite type over $S$ and let $A_0 \to B_0$ be an étale map of rings over $S$. Let $Y_0 = X_0 \otimes_{A_0} B_0$ so that we have a diagram
$$
\begin{array}{ccc}
Y_0 & \xrightarrow{\phi} & X_0 \\
\downarrow & & \downarrow \\
\text{Spec}(B_0) & \longrightarrow & \text{Spec}(A_0)
\end{array}
$$
which is cartesian. Then by [Ill71, Eq. 1.2.3.5] we have that
$$\mathcal{L}_{Y_0/B_0}^\bullet \cong \phi^* \mathcal{L}_{X_0/A_0}^\bullet$$
where $\mathcal{L}_{X_0/A_0}^\bullet$ and $\mathcal{L}_{Y_0/B_0}^\bullet$ denote the cotangent complex of $X_0/A_0$ and $Y_0/B_0$ respectively.

Now conditions A1-A3, D1-D3 and O1,O3 follow from Propositions 5.2, 5.6 and 5.10 which give the corresponding conditions for the Ext-groups.

**Step 4.** Let $A_0$ be ring of finite type over $S$, and let $X_0$ be an element of $\mathcal{X}_{pv2}(A_0)$. Choose an automorphism $\phi$ of $X_0$ and suppose that $\phi$ induces the
identity map in $X_{pv2}(k(p))$ for a dense set of points $p \in \text{Spec}(A_0)$ of finite type over $S$. We may consider the closed subspace $Z_0 = \text{Ker}(\phi, \text{id}_{X_0})$ of $X_0$ defined by the cartesian diagram

$$
\begin{array}{c}
\text{Ker}(\phi, \text{id}_{X_0}) \\
\downarrow \\
X_0 \xrightarrow{\Delta} X_0 \times X_0.
\end{array}
$$

Set-theoretically $Z_0$ is the set of points of $X_0$ where $\phi$ is equal to the identity map. Denote by $U_0$ the complement of the closed subspace $Z_0$. The set $W_0$ of points $p \in \text{Spec}(A_0)$ such that $\phi$ induces the identity map in $X_{pv2}(k(p))$ is contained in the closed set $V_0 = \text{Spec}(A_0) - f_0(U_0)$ of $\text{Spec}(A_0)$. Since the set $W_0$ is dense it follows that $V_0$ is dense and therefore equal to $\text{Spec}(A_0)$. We conclude that $f_0(U_0) = \emptyset$ and so $U_0 = \emptyset$. Thus $\phi = \text{id}_{X_0}$ and so the last of Artin’s Conditions is fulfilled.

\section{Extensions and applications}

In this section we derive algebraicity of some stacks related to the stack $X_{pv2}$. These include the stack $\mathcal{X}_n$ parametrizing families of objects of $X_{pv2}$ together with $n$ smooth distinct sections. Also for a scheme $X$ smooth and proper over the base scheme $S$ we consider the stack $\mathcal{X}_{X,n}$ parametrizing objects of $\mathcal{X}_n$ together with a morphism to $X$.

We also give some applications showing the algebraicity of the classical moduli spaces $M_{g,n}$ of smooth, $n$-pointed curves of genus $g$. Finally we discuss the Fulton-MacPherson compactification $X[n]$ and the stack $SD_{X,n}$ of stable $n$-pointed degenerations.

\textbf{Definition 8.1.} Let $n > 0$ be an integer and $S$ a scheme of finite type over an excellent Dedekind ring. Consider the CFG $\mathcal{X}_n$ over $\text{Sch}/S$ defined as follows: For each $S$-scheme $T$ the category $\mathcal{X}_n(T)$ has objects $(W/T, t_1, \ldots, t_n)$ where $W$ is an object of $X_{pv2}(T)$ and $t_i : T \to W$ are sections for $1 \leq i \leq n$. A morphism between two objects $(W/T, t_1, \ldots, t_n)$ and $(W'/T, t'_1, \ldots, t'_n)$ is a commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{t_i} & W' \\
\downarrow & & \downarrow \\
T & \xleftarrow{t'_i} & T
\end{array}
$$

for each $i$. It is clear that $\mathcal{X}_n$ is a stack and we have a canonical morphism of stacks $\Phi : \mathcal{X}_n \to X_{pv2}$ forgetting the sections.
Proposition 8.2. Let $\widetilde{\mathcal{X}}_n$ and $S$ be as in Definition 8.1. Then the morphism $\Phi : \widetilde{\mathcal{X}}_n \to \mathcal{X}_{pv2}$ is representable and hence $\widetilde{\mathcal{X}}_n$ is an algebraic stack.

Proof. Let $T$ be an $S$-scheme and consider a morphism $T \to \mathcal{X}_{pv2}$ corresponding to an object $W/T$ of $\mathcal{X}_{pv2}(T)$. Then the diagram

$$
\begin{array}{ccc}
W^n & \rightarrow & T \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{X}}_n & \rightarrow & \mathcal{X}_{pv2}
\end{array}
$$

is 2-cartesian, where $W^n$ denotes the $n$-fold product $W \times_T \ldots \times_T W$ and the morphism $W^n \to \widetilde{\mathcal{X}}_n$ corresponds to the element $(\Delta : W \to W^n, p_1, \ldots, p_n)$ of $\widetilde{\mathcal{X}}_n$, where $p_i$ denotes the $i$:th projection. Thus $\Phi$ is representable and so $\widetilde{\mathcal{X}}_n$ is algebraic by [LMB00, Prop. 4.5(ii)].

Definition 8.3. Let $n > 0$ be an integer and $S$ a scheme of finite type over an excellent Dedekind ring. Consider the CFG $\mathcal{X}_n$ over $\text{Sch}/S$ defined as follows: For each $S$-scheme $T$ the category $\mathcal{X}_n(T)$ has objects $(W/T, t_1, \ldots, t_n)$ of $X_{pv2}(T)$ where $t_i : T \to W$ are sections for $1 \leq i \leq n$ that are distinct and such that the closed subspace $t_i(T)$ is in the smooth locus of $W$ for each $i$. We have that $\mathcal{X}_n$ is a stack and we have a canonical inclusion of stacks $\Psi : \mathcal{X}_n \to \widetilde{\mathcal{X}}_n$.

Proposition 8.4. Let $\mathcal{X}_n$ and $S$ be as in Definition 8.3. Then the morphism $\Psi : \mathcal{X}_n \to \widetilde{\mathcal{X}}_n$ is representable and hence $\mathcal{X}_n$ is an algebraic stack.

Proof. Let $T$ be an $S$-scheme and consider the object $\eta$ of $\widetilde{\mathcal{X}}_n(T)$ given by $\eta = (f : W \to T; t_1, \ldots, t_n)$. Let $Z$ denote the closed subspace of $W$ defined by

$$
Z = (W_{\text{sing}} \cap (t_1(T) \cup \ldots \cup t_n(T))) \cup (t_1(T) \cap \ldots \cap t_n(T)).
$$

Then let $T' = T - f(Z)$ which is an open subspace of $T$ since $f$ is proper. The pullback of $\eta$ to $T'$ is then by construction an object of $\mathcal{X}_n(T')$ and if $U \to T$ is another map such that the pullback of $\eta$ to $U$ is in $\mathcal{X}_n(U)$, then $U \to T'$ factors via $T' \to T$.

Hence if $T \to \widetilde{\mathcal{X}}_n$ is the morphism corresponding to the object $\eta$ of $\widetilde{\mathcal{X}}_n(T)$, then the diagram

$$
\begin{array}{ccc}
T' & \rightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{X}_n & \rightarrow & \widetilde{\mathcal{X}}_n
\end{array}
$$

is 2-cartesian. Thus the morphism $\Psi$ is representable and so by the result [LMB00, Prop. 4.5(ii)] and Proposition 8.2 the stack $\mathcal{X}_n$ is algebraic. \qed
Example 8.5. Let $S$ and $\mathcal{X}_n$ be as in Definition 8.3. Choose integers $g, n \geq 0$ and consider the stack $\mathcal{M}_{g,n}$ over $\textbf{Sch}/S$ defined as follows: For each $S$-scheme $T$ the objects of $\mathcal{M}_{g,n}(T)$ are tuples $(f : W \to T, t_1, \ldots, t_n)$ where $f$ is a smooth, proper morphism such that the geometric fibers are projective schemes of dimension 1 with genus $g$, and $t_1, \ldots, t_n$ are disjoint sections. There is an inclusion of stacks $\Phi : \mathcal{M}_{g,n} \to \mathcal{X}_n$ and we wish to show that this inclusion is an open immersion.

Thus let $T$ be a noetherian scheme and consider a morphism $T \to \mathcal{X}_n$ corresponding to an object $(f : W \to T, t_1, \ldots, t_n)$ of $\mathcal{X}_n(T)$. For each $T$-scheme $T$ the objects of $\mathcal{M}_{g,n}(T)$ are tuples $(f : W \to T, t_1, \ldots, t_n)$ where $f$ is a smooth, proper morphism such that the geometric fibers are projective schemes of dimension 1 with genus $g$, and $t_1, \ldots, t_n$ are disjoint sections.

There is an inclusion of stacks $\Phi : \mathcal{M}_{g,n} \to \mathcal{X}_n$ and we wish to show that this inclusion is an open immersion.

Thus let $T$ be a noetherian scheme and consider a morphism $T \to \mathcal{X}_n$ corresponding to an object $(f : W \to T, t_1, \ldots, t_n)$ of $\mathcal{X}_n(T)$. There is an open subspace $U \subseteq W$ defining the locus where the map $f$ is smooth. Thus the open subscheme $T' = T - f(W - U)$ is the locus of points $y \in T$ such that $W_y \to \text{Spec}(k(y))$ is smooth. Next let $W' \to W$ be an étale presentation of $W$. Let $E$ be the locus of $T$ consisting of points $y \in T$ such that the fibers $W'_y \to \text{Spec}(k(y))$ are one-dimensional. By [EGAIV, Cor. 14.2.5] we have that $E$ is an open subscheme of $T$ and so $E$ is the open subscheme such that the fibers $W_y \to \text{Spec}(k(y))$ are one-dimensional. Let $E' = T' \cap E$.

Furthermore, let $\text{Spec}(A)$ be an open affine subscheme of $E'$ and define $W_A = W \times_T \text{Spec}(A)$. Consider the functors $T^i : M \mapsto H^i(W_A, f^*(M))$ from $A$-modules to $A$-modules for $i \geq 0$. The result of [Har77, Prop. III.12.2] is valid for algebraic spaces so there is a bounded complex $L^\bullet$ of finitely generated free $A$-modules such that $T^i(M) \cong H^i(L^\bullet \otimes_A M)$ for any $A$-module $M$ and any $i \geq 0$. By [EGAIII, Cor. 7.9.3] we then have that the function

$$y \mapsto \sum_{i \geq 0} \dim_{k(y)} T^i(k(y))$$

is constant on $\text{Spec}(A)$. Thus the Euler characteristic of the structure sheaf of the fibers, defined as

$$y \mapsto \dim_{k(y)} H^0(W_y, \mathcal{O}_{W_y}) - \dim_{k(y)} H^1(W_y, \mathcal{O}_{W_y})$$

is locally constant on $\text{Spec}(A)$. Since $\dim_{k(y)} H^0(W_y, \mathcal{O}_{W_y}) = 1$ for all points $y \in \text{Spec}(A)$ we conclude that also the arithmetic genus $\dim_{k(y)} H^1(W_y, \mathcal{O}_{W_y})$ is locally constant on $\text{Spec}(A)$, and thus locally constant on $E'$. Thus let $E''$ be the open subscheme of points $y \in E'$ such that $g = \dim_{k(y)} H^1(W_y, \mathcal{O}_{W_y})$. Then $E''$ is the open set of points $y \in T$ where the fiber $W_y \to \text{Spec}(k(y))$ is a smooth, projective curve of genus $g$. Thus we obtain a 2-cartesian diagram

$$\begin{array}{ccc}
E'' & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,n} & \Phi & \mathcal{X}_n
\end{array}$$
where $E'' \to T$ is the canonical open immersion. This shows that $\Phi$ is an open immersion and so $\mathcal{M}_{g,n}$ is an algebraic stack.

We will define some stacks parametrizing morphisms in Definition 8.10, but first we need some preliminary results regarding representability of Hom-functors.

**Lemma 8.6.** Let $f : X \to Y$ be a morphism of algebraic spaces which is quasifinite, proper, and locally of finite presentation. Then $f$ is finite.

**Proof.** The statement is local on $Y$ so we may assume that $Y$ is an affine scheme. The map $f$ is locally quasifinite, locally of finite presentation and separated so by [Knu71, Cor. II.6.16] we have that $X$ is a scheme. Thus the result follows from the corresponding result [EGAIV, Cor. 18.12.4] for schemes.

**Proposition 8.7.** Let $X \to S$ and $Y \to S$ be flat morphisms of finite type, where $X,Y$ are algebraic spaces and $S$ is a locally noetherian scheme. Consider a proper $S$-morphism $f : X \to Y$. Suppose that the morphism $f_s : X_s \to Y_s$ of fibers is an isomorphism for every $s \in S$. Then $f$ is an isomorphism.

**Proof.** The question is local on $Y$ so we may assume that $Y$ is an affine scheme, say $Y = \text{Spec}(A)$. The morphism $f$ is quasi-finite since all morphisms $f_s$ are isomorphisms. The map $f$ is also proper and so $f$ is finite by Lemma 8.6. Thus $X$ is an affine scheme, say $X = \text{Spec}(B)$.

Since each morphism $f_s$ is an isomorphism we have that $f_s$ is in particular flat. By the “fiberwise criterion of flatness” [EGAIV, Cor. 11.3.11] we thus have that $f$ is flat.

The corresponding map of rings $\varphi : A \to B$ makes $B$ into a finite, flat $A$-algebra. In particular, the localization $B \otimes_A A_p$ is a flat $A_p$-algebra for every $p \in \text{Spec}(A)$ and hence $B \otimes_A A_p$ is free as an $A_p$-module since $A$ is noetherian.

Thus $B \otimes_A A_p \cong A_p^n$ for some integer $n$, and we have by assumption that $B \otimes_A k(p) = k(p)$. This shows that $n = 1$, and so the map $\varphi_p : A_p \to B \otimes_A A_p$ makes $B \otimes_A A_p$ into a free $A_p$-module of rank 1. Thus $\varphi_p$ is an isomorphism for all $p \in \text{Spec}(A)$ and so $\varphi$ is an isomorphism.

**Proposition 8.8.** Let $X$, $Y$ be algebraic spaces that are flat and proper over a noetherian base scheme $S$ and let $f : X \to Y$ be an $S$-morphism. Then there is an open subscheme $S' \subseteq S$ such that for every map $T \to S$ we have that the pullback map $f_T : X \times_ST \to Y \times_ST$ is an isomorphism if and only if the map $T \to S$ factors through $S'$.
Proof. Denote by \( \pi_X : X \to S \) and \( \pi_Y : Y \to S \) the structure maps. Let \( V \to Y \) and \( U \to X \times Y \) be étale presentations of \( Y \) and \( X \times Y \) respectively. Consider the induced map of schemes \( f : U \to V \). By Chevalley’s theorem on fiber dimension [EGAIV, Thm. 13.1.3] the set

\[
U' = \{ u \in U : \dim(f^{-1}(f(u))) = 0 \}
\]

is an open subscheme of \( U \). Denote the image of \( U' \) in \( X \) by \( X' \), which is then an open subspace of \( X \) since \( U \to X \) is étale.

Consider the closed subspace \( Z = f(X - X') \) of \( Y \) and let \( S_1 = S - \pi_Y(Z) \). Then \( S_1 \) is open in \( S \). By the construction of \( S_1 \) we have that for any map \( T \to S \) the pullback \( f_T : X \times_S T \to Y \times_S T \) is quasifinite (hence finite by Lemma 8.6) if and only if \( T \to S \) factors via \( S_1 \). Hence we have reduced to the case where the map \( f : X \to Y \) is finite.

Since \( f : X \to Y \) is finite there exists a coherent sheaf \( \mathcal{A} \) of \( \mathcal{O}_Y \)-modules such that \( X = \text{Spec}(\mathcal{A}) \). Choose an étale presentation \( p : V \to Y \) of \( Y \). Then \( p^* \mathcal{A} \) is a coherent sheaf on \( V \) and \( X \times_Y V = \text{Spec}(i^* \mathcal{A}) \). Let \( V' \subseteq V \) be the open subscheme where \( p^* \mathcal{A} \) is locally free of rank 1, and let \( Y' = p(V') \). Note that \( p' : V' \to Y' \) is an étale presentation of \( Y' \) and \( Y' \) is thus the open subspace of \( Y \) where the sheaf \( \mathcal{A} \) is locally free of rank 1. This implies that \( Y' \) is the open subspace of \( Y \) where the map \( f \) is an isomorphism.

Define the open subscheme \( S' = S - \pi_Y(Y - Y') \) of \( S \). By construction of \( S' \) we have that if \( s \in S \) is a point, then the map of fibers \( f_s : X_s \to Y_s \) is an isomorphism if and only if \( s \in S' \). As fiberwise isomorphisms coincide with isomorphism by Proposition 8.7 we have that \( S' \) satisfies the desired universal property.

Proposition 8.9. Let \( X \) and \( Y \) be algebraic spaces flat and proper over a noetherian base scheme \( S \). Then the functor \( \text{Hom}_S(X, Y) \) is an open subfunctor of the Hilbert functor \( \text{Hilb}_S(X \times_S Y) \). Consequently the functor \( \text{Hom}_S(X, Y) \) is representable by an algebraic space. The open immersion \( \text{Hom}_S(X, Y) \to \text{Hilb}_S(X \times_S Y) \) is defined by mapping a morphism to its graph.

Proof. We construct a map of functors

\[
\Phi : \text{Hom}_S(X, Y) \to \text{Hilb}_S(X \times_S Y)
\]

as follows: For any \( S \)-scheme \( T \), let \( f : X_T \to Y_T \) be a morphism. Then \( \Phi(T) \) maps this morphism to the graph \( \Gamma(f) \), which is a closed subspace of \( (X \times_S Y) \times_S T = X_T \times_T Y_T \), since \( f \) is separated. Furthermore, if we let
$Z \subseteq X_T \times_T Y_T$ be a closed subspace, then $Z$ is in the image of $\Phi(T)$ if and only if the composed map

$$Z \rightarrow X_T \times_T Y_T \rightarrow X_T$$

is an isomorphism, where the rightmost map is the projection.

To show that $\Phi$ is an open immersion, let $T \rightarrow \text{Hilb}_S(X \times_S Y)$ be a map from a scheme $T$, corresponding to a closed subspace $Z \subseteq X_T \times_T Y_T$, flat over $T$. Let $g$ denote the composed map $Z \rightarrow X_T$.

Since $Z$ is proper and flat over $T$ we have by Proposition 8.8 that there is an open subscheme $T' \subseteq T$ with the property that for any map $U \rightarrow T$ of schemes, the pullback map $g_U : Z_U \rightarrow X_U$ is an isomorphism if and only if the map $U \rightarrow T$ factors via $T'$.

What this means in terms of functors is that we have a cartesian diagram

$$
\begin{array}{ccc}
T' & \rightarrow & T \\
\downarrow & & \downarrow \\
\text{Hom}_S(X, Y) & \overset{\Phi}{\rightarrow} & \text{Hilb}_S(X \times_S Y)
\end{array}
$$

where the above arrow is the open immersion $T' \subseteq T$. This shows that $\Phi$ is an open immersion. Since $\text{Hilb}_S(X \times_S Y)$ is representable by [Art69, Thm. 6.1] it follows that $\text{Hom}_S(X, Y)$ is representable by an algebraic space. \hfill \Box

**Definition 8.10.** Let $n > 0$ be an integer and $S$ a scheme of finite type over an excellent Dedekind ring. Consider a scheme $X$ which is flat and proper over $S$ and define the CFG $\mathcal{X}_{X,n}$ as follows: For each $S$-scheme $T$ the objects of $\mathcal{X}_{X,n}(T)$ are tuples $(W/T, t_1, \ldots, t_n, \rho)$ where $(W/T, t_1, \ldots, t_n)$ is an object of $\mathcal{X}_n(T)$ and $\rho : W \rightarrow X \times_S T$ is a $T$-morphism. A morphism between the objects $(W/T, t_1, \ldots, t_n, \rho)$ and $(W'/T, t'_1, \ldots, t'_n, \rho')$ is a commutative diagram

$$
\begin{array}{ccc}
W & \overset{\rho}{\rightarrow} & W' \\
\downarrow & & \downarrow \\
X \times_S T & \rightarrow & X \times_S T \\
\downarrow & & \downarrow \\
T & \rightarrow & T
\end{array}
$$

for each $i$. The CFG $\mathcal{X}_{X,n}$ is a stack and there is a canonical morphism of stacks $\mathcal{X}_{X,n} \rightarrow \mathcal{X}_n$ forgetting the map to $X$.

**Proposition 8.11.** Let the schemes $S$ and $X$ and the stack $\mathcal{X}_{X,n}$ be as in Definition 8.10. Then the canonical morphism $\mathcal{X}_{X,n} \rightarrow \mathcal{X}_n$ is representable and so $\mathcal{X}_{X,n}$ is an algebraic stack.
Proof. Let $T$ be a noetherian scheme and let $T \to \mathcal{Z}_n$ be a morphism corresponding to an object $(W/T, t_1, \ldots, t_n)$ of $\mathcal{Z}_n(T)$. Then the diagram

\[
\begin{array}{ccc}
\text{Hom}_T(W, X \times_S T) & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{Z}_{X,n} & \longrightarrow & \mathcal{Z}_n
\end{array}
\]

is 2-cartesian, where $\text{Hom}_T(W, X \times_S T)$ denotes the étale sheaf on $T$ defined by $\text{Hom}_T(W, X \times_S T)(U) = \text{Hom}_U(W \times_T U, X \times_S U)$ for each étale map $U \to T$. The morphism $\text{Hom}_T(W, X \times_S T) \to \mathcal{Z}_{X,n}$ is defined for each $U \to T$ as the functor

\[
\text{Hom}_T(W, X \times_S T)(U) \longrightarrow \mathcal{Z}_{X,n}(U)
\]

that maps $\rho : W \times_T U \to X \times_S U$ to the object $(W \times_T U, u_1, \ldots, u_n, \rho)$ of $\mathcal{Z}_{X,n}(U)$, where $u_i$ is the pullback of $t_i$ to $U$ for all $i$. The homomorphism sheaf $\text{Hom}_T(W, X \times_S T)$ is representable by an algebraic space by Proposition 8.9 and so the morphism $\mathcal{Z}_{X,n} \to \mathcal{Z}_n$ is representable. Hence $\mathcal{Z}_{X,n}$ is algebraic by [LMB00, Prop. 4.5(ii)]. □

Definition 8.12 (The stack of stable degenerations). Let $k$ be a field and let $S = \text{Spec}(k)$. Consider a scheme $X$ which is smooth and separated over $S$. We define $\mathcal{SD}_{X,n}$ to be the CFG over $\text{Sch}/S$ defined as follows: For a scheme $T$ over $S$, an object of $\mathcal{SD}_{X,n}(T)$ is a tuple $(W/T, t_1, \ldots, t_n, \rho)$ making a commutative diagram

\[
\begin{array}{ccc}
W & \longrightarrow & X \times_k T \\
\downarrow t_i & & \downarrow \text{pr}_2 \\
T & \ phantom &
\end{array}
\]

where the maps $t_i$ are sections, the map $\rho$ is proper and $W$ is an algebraic space which is flat and of finite presentation over $T$. Also, for each point $y \in T$ the geometric fiber $(W_y/k(y), t_{y,1}, \ldots, t_{y,n}, \rho_y)$ is an $n$-pointed stable degeneration of $X_y = (X \times_k T) \times_T \text{Spec}(k(y)) = X \otimes_k k(y)$ in the sense of [FM94, p. 194]. The map $\rho_y : W_y \to X_y$ is the natural map contracting all attached components of the stable degeneration $W_y$. Note that if $X$ is projective over $k$, then the algebraic space $W$ is proper over $T$.

Remark 8.13. Note that if $X$ is projective and has the property that every stable degeneration $W$ of $X$ satisfies $H^2(W, \mathcal{O}_W) = 0$, then the CFG $\mathcal{SD}_{X,n}$ is a substack of $\mathcal{Z}_{X,n}$. We would then want to show that $\mathcal{SD}_{X,n} \to \mathcal{Z}_{X,n}$ is a representable morphism to obtain algebraicity of $\mathcal{SD}_{X,n}$.

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References


