Institutionen för systemteknik
Department of Electrical Engineering

Examensarbete

AFS-Assisted Trailer Reversing

Examensarbete utfört i Reglerteknik
vid Tekniska högskolan i Linköping
av

Olof Enqvist

LITH-ISY-EX--06/3752--SE

Linköping 2006

Department of Electrical Engineering
Linköpings universitet
SE-581 83 Linköping, Sweden

Linköpings tekniska högskola
Linköpings universitet
581 83 Linköping

AFS-Assisted Trailer Reversing

Examensarbete utfört i Reglerteknik
vid Tekniska högskolan i Linköping
av

Olof Enqvist

LITH-ISY-EX--06/3752--SE

Handledare: **M.Sc. Christian Lundquist**
ZF Lenksysteme
Lic. Gustaf Hendeby
ISY, Linköpings universitet
Dr. Wolfgang Reinelt
ZF Lenksysteme

Examinator: **Dr. Jacob Roll**
ISY, Linköpings universitet

Linköping, 27 January, 2006

Avdelning, Institution
Division, Department

Division of Automatic Control
Department of Electrical Engineering
Linköpings universitet
S-581 83 Linköping, Sweden

Datum
Date

2006-01-27

Språk
Language

☐ Svenska/Swedish
☒ Engelska/English
☐ ____________

Rapporttyp
Report category

☐ Licentiatavhandling
☒ Examensarbete
☐ C-uppsats
☐ D-uppsats
☐ Övrig rapport
☐ ____________

URL för elektronisk version

`http://www.control.isy.liu.se`
`http://www.ep.liu.se/2006/3752`

ISBN
—

ISRN
LITH-ISY-EX--06/3752--SE

Serietitel och serienummer
Title of series, numbering

ISSN
—

Titel
Title

Aktiv styrning vid backning med släp

AFS-Assisted Trailer Reversing

Författare
Author

Olof Enqvist

Sammanfattning
Abstract

Reversing with a trailer is very difficult and many drivers hesitate to even try it. This thesis examines if active steering, particularly AFS (Active Front Steering), can be used to provide assistance.

For analysis and controller design a simple geometric model of car and trailer is used. The model seems to be accurate enough at the low speeds relevant for trailer reversing. It is shown that the only trailer dependent model parameter can be estimated while driving. This enables use with different trailers.

Different schemes to control the system are tested. The main approach is to use the steering wheel as reference for some appropriate output signal, for example the angle between car and trailer. This makes reversing with a trailer more like reversing without a trailer. To turn left, the driver simply turns the steering wheel left and drives. Test driving, as well as theoretical analysis, shows that the resulting system is stable. Of the eight drivers that have tested this type of control, five found it to be a great advantage while two considered it more confusing than helpful.

A major problem with this control approach has to do with the way AFS is constructed. With AFS, the torque required to turn the front wheels results in a reaction torque in the steering wheel. Together with the reference tracking controllers, this makes the steering wheel unstable. Theoretical analysis implies that this problem has to be solved mechanically. One solution would be to combine AFS with electric power steering.

This thesis also presents a trajectory tracking scheme to autonomously reverse with a trailer. Starting from the current trailer position and the desired trajectory an appropriate turning radius for the trailer is decided. Within certain limits, this will stabilize the car as well. The desired trajectory can be programmed beforehand, but it can also be saved while driving forward. Both variants have been tested with good results.

Nyckelord
Keywords

Automotive Control, Trajectory Tracking, Active Steering

Abstract

Reversing with a trailer is very difficult and many drivers hesitate to even try it. This thesis examines if active steering, particularly AFS (Active Front Steering), can be used to provide assistance.

For analysis and controller design a simple geometric model of car and trailer is used. The model seems to be accurate enough at the low speeds relevant for trailer reversing. It is shown that the only trailer dependent model parameter can be estimated while driving. This enables use with different trailers.

Different schemes to control the system are tested. The main approach is to use the steering wheel as reference for some appropriate output signal, for example the angle between car and trailer. This makes reversing with a trailer more like reversing without a trailer. To turn left, the driver simply turns the steering wheel left and drives. Test driving, as well as theoretical analysis, shows that the resulting system is stable. Of the eight drivers that have tested this type of control, five found it to be a great advantage while two considered it more confusing than helpful.

A major problem with this control approach has to do with the way AFS is constructed. With AFS, the torque required to turn the front wheels results in a reaction torque in the steering wheel. Together with the reference tracking controllers, this makes the steering wheel unstable. Theoretical analysis implies that this problem has to be solved mechanically. One solution would be to combine AFS with electric power steering.

This thesis also presents a trajectory tracking scheme to autonomously reverse with a trailer. Starting from the current trailer position and the desired trajectory an appropriate turning radius for the trailer is decided. Within certain limits, this will stabilize the car as well. The desired trajectory can be programmed beforehand, but it can also be saved while driving forward. Both variants have been tested with good results.

Acknowledgements

A lot of people has helped me with this thesis. Christian Lundquist worked together with me all the way, providing valuable feedback and ideas. Wolfgang Reinelt invited me to do the thesis and shared his experience in control. I want to thank them both. I am also very grateful to Gustaf Hendeby and Jacob Roll, for their patience and for good advice concerning the thesis.

Further, my family and friends, Camorasan, Alexander, Christine, Frank, Gerd, Ralf, Reinhard, Samuel, Schusterle, Thomas and Thorsten are not forgotten... *och Christian, se nu till att du kommer ut och får sladda lite ibland också.*

Contents

Chapter 1

Introduction

When reversing, a car with an attached trailer constitutes an unstable system. Thus, the trailer will tend fold up to the car like a jackknife. To avoid this the driver has to compensate for the movement of the trailer using the steering wheel. This is especially difficult due to another characteristic of the system: To get the trailer more to the left, you need to turn the steering wheel more to the right, which can be quite confusing.

All of this makes trailer reversing very difficult and many drivers hesitate to even try it. This thesis examines if active steering, particularly AFS (Active Front Steering), can be used to provide assistance. We will soon discuss the aim and contents of the thesis more thoroughly, but first a short introduction to active steering might be in place.

1.1 Active Front Steering

Normally, the front wheel angles of a car depend solely on the steering wheel angle. With Active Front Steering (AFS) it is possible to superpose an additional, electronically controlled angle (Figure 1.1). Thus, the steering characteristics can be adjusted according to the driving situation.

The superposition angle is provided by an electric motor connected to a planetary gear. This way, the mechanical connection between steering wheel and front road wheels is maintained. In case of malfunction, the planetary gear will be mechanically locked, allowing the driver to steer normally.

AFS was developed by ZF Lenksysteme. It is optional equipment for some BMW series. There it is used to adjust the steering ratio depending on vehicle speed. At low and medium speeds the steering becomes more direct, requiring less steering effort in, for example, a parking manuevre. At higher speeds, the steering is less direct to enhance directional stability. BMW also uses AFS to stabilize the car in critical driving situations.

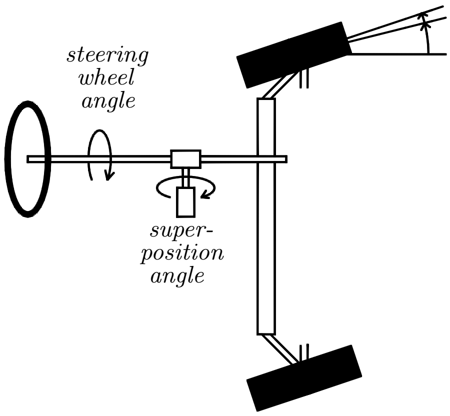

Figure 1.1. With AFS, an angle is superposed to the steering wheel angle.

1.2 This Thesis

The work behind this thesis was conducted at ZF Lenksysteme. They were looking for new applications for AFS. Consequently, the aim of this thesis is to explore the potential of using AFS for trailer reversing. To do so, different assistance functions are developed and evaluated. Though they are primarily intended for presentations, an aim is to make them suitable for serial production as well. Thus, they use existing sensors, with the sole addition of an angle sensor to measure the trailer position. In serial production such a sensor could be placed in the towing hook of the car.

For analysis and controller design a model of the car-trailer system is required. Such a model is derived and validated in Chapter 2. To enable use with different trailers, we need a method to estimate the trailer length while driving. Such a method is presented in Section 2.7. Chapter 3 discusses some important characteristics of the car-trailer system.

The part on control is divided into two parts. Chapter 4 concerns reference tracking controllers, where the driver gives the reference value with the steering wheel. In Chapter 6 a couple of autonomous steering functions are constructed. Using the steering wheel to provide the reference value causes some special problems with AFS. They are analysed in Chapter 5.

Chapter 2

Modelling

The design and analysis of a controller requires an appropriate model of the system that we want to control. In this chapter, a model for the car-trailer system is derived. The modelling is made in a structured manner to enable expansions. Sections 2.1–2.4 discuss the basic assumptions and formulate them as algebraic and differential constraint equations. From these equations a system of differential equations is derived. The derivations can be found in Appendix A and the resulting differential equations in Section 2.5. The validity of the model is examined in Section 2.6 and Section 2.7 presents a method to estimate the trailer length while driving.

2.1 Basics

An important part of modelling, is to choose what kind of model to use. Which effects are essential and which can be disregarded? Choosing the most complex model is not always a good idea. Though such a model is theoretically more accurate, it tends to be more sensitive to variations and it can be difficult to estimate all the parameters. A simpler model is also easier to analyse.

One choice we have to make is whether to include lateral slip in our model. Lateral slip is an effect of cornering. To turn, a car has to be affected by a lateral force. This force is provided by friction when the tyres slip sideways.

A few facts can guide us when we decide whether to include slip in our model. One is that the weight of a specific trailer varies, and thus tyre friction and dynamic properties. If the model depends on these parameters, they would have to be estimated each time the car starts. Moreover, car owners tend to use their car with different trailers. Therefore, we want controllers that can adapt to a new trailer, and thus all trailer dependent parameters have to be estimated while driving.

Since slip is linked to the lateral forces, a model with slip would depend on the dynamic properties of both the car and the trailer. A model without slip, on the other hand, only depends on the geometry of the car and the trailer. Besides, trailer reversing mostly takes place at low speeds, where side forces and lateral slip are small.

It seems a model without slip is more suitable for our purposes. (In cases when slip cannot be disregarded the model in Appendix B might be a starting point.) We will now look a little closer at the geometry of the car. To allow all wheels of the car to roll without lateral slip, the inner front wheel needs to turn more than the outer. The ideal geometry, often called Ackermann steering geometry, is shown in Figure 2.1. Here, all wheels are aligned to move in circles around a common central point, M. The relation between the left front wheel angle, δ_{FL}, and the right, δ_{FR}, is

$$\tan\delta_{FL} - \tan\delta_{FR} = \frac{w_1}{l_1}\,\tan\delta_{FL}\,\tan\delta_{FR},$$

where w_1 and l_1 are the distances between the wheels as can be seen in Figure 2.1. Note that the wheel angles are positive when turning left.

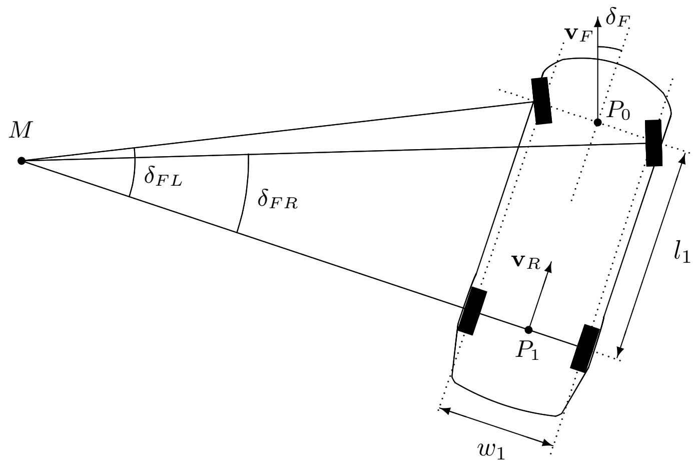

Figure 2.1. The Ackermann steering geometry.

For our model, we will assume that the car has Ackermann steering geometry and that all wheels roll without lateral slip. Thus, all points on the car will move on circles around a common point (Figure 2.1).

We define the points P_0 and P_1 as in Figure 2.1 and $\mathbf{v}_F$ and $\mathbf{v}_R$ as their velocity vectors. The movement of the car can be specified by the signed speed $v_R = \pm|\mathbf{v}_R|$, and the angle δ_F between $\mathbf{v}_F$ and the central axis of the car. In practice δ_F is computed from measurements of the front wheel angles, using

$$\tan\delta_F = \frac{\tan\delta_{FL}}{1 + \frac{w_1}{2\,l_1}\tan\delta_{FL}} = \frac{\tan\delta_{FR}}{1 - \frac{w_1}{2\,l_1}\tan\delta_{FR}}.$$

From now on we will call δ_F the front wheel angle of the car, though that is not entirely true.

2.2 Coordinates

We introduce a (global) inertial frame R with coordinates x^R and y^R as well as local frames L_i with coordinates x^{L_i} and y^{L_i} (Figure 2.2). The car-fixed frame, L_1, has its origin in the point P_1 between the rear wheels, and its x-axis in the forward direction of the car. Frame L_2 is trailer-fixed with origin between the wheels of the trailer, and x-axis in the forward direction of the trailer. We also define a frame L_0 with origin in P_0, between the front car wheels, and x-axis coinciding with the velocity vector, $\mathbf{v}_F$ (see Section 2.1).

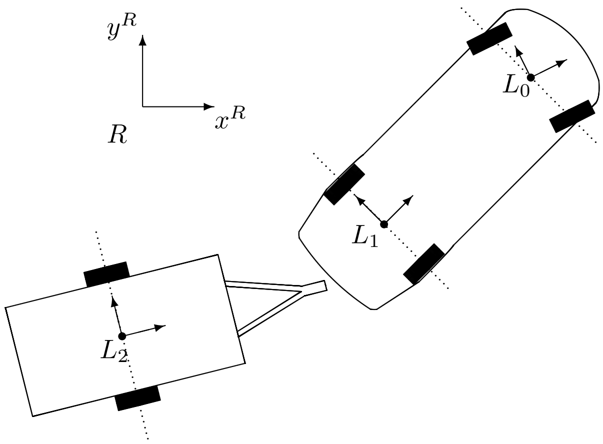

Figure 2.2. The different coordinate frames.

The position of body i in the plane is specified by the global coordinates of the origin P_i of the body-fixed frame L_i, and the orientation of body i is specified by $\psi_i = \psi_{L_iR}$, the angle of rotation of L_i with respect to R. For convenience, we also introduce the displacement vector $\mathbf{r}^R_{P_iO} = (x^R_{P_iO}\,,\, y^R_{P_iO})^T$ (see Figure 2.3).

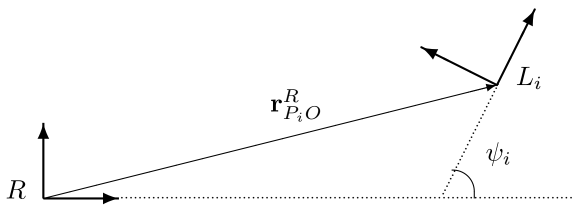

Figure 2.3. Displacement vector and angle of rotation.

2.3 Geometric Constraints

In this section we set up the purely algebraic constraint equations that originate from the geometry of the car-trailer system.

- The angles between the front wheels and the car are controlled by the driver together with the controller. These angles are specified by the single angle δ_F (see Section 2.1). Thus, with ψ_0 and ψ_1 as defined in the previous section,

$$\psi_0 - \psi_1 - \delta_F = 0. \tag{2.1}$$

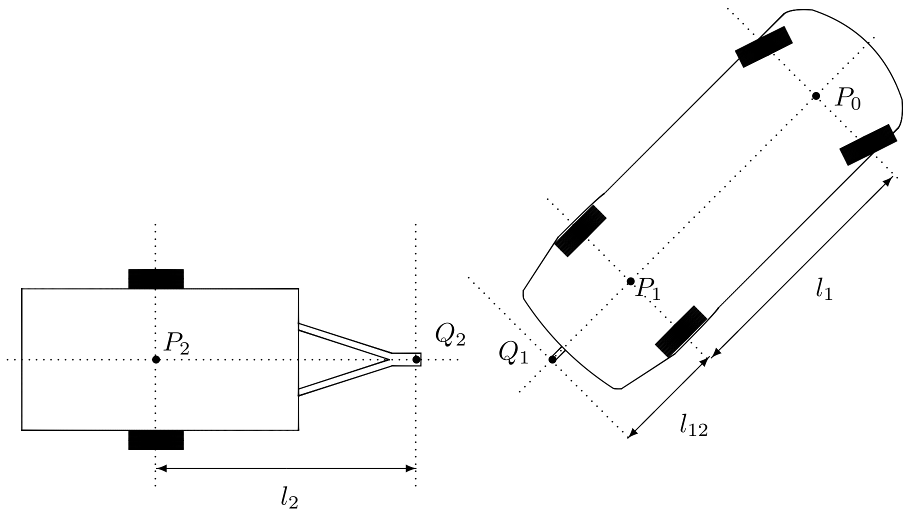

Figure 2.4. Points in the different frames.

- By definition the origin P_0 of frame 0 is fixed in the L_1 frame

$$\mathbf{r}^R_{P_0O} - \mathbf{A}^{RL_1}\,\mathbf{r}^{L_1}_{P_0P_1} - \mathbf{r}^R_{P_1O} = \mathbf{0},$$

where

$$\mathbf{A}^{RL_1} = \begin{pmatrix} \cos\psi_1 & -\sin\psi_1 \\ \sin\psi_1 & \cos\psi_1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}^{L_1}_{P_0P_1} = \begin{pmatrix} l_1 \\ 0 \end{pmatrix}.$$

We get the constraint equations

$$x^R_{P_0O} - l_1\,\cos\psi_1 - x^R_{P_1O} = 0 \tag{2.2a}$$

$$y^R_{P_0O} - l_1\,\sin\psi_1 - y^R_{P_1O} = 0. \tag{2.2b}$$

- We define a point Q_1 for the towing hook of the car and a point Q_2 for the coupling of the trailer (see Figure 2.4). Since the trailer is attached to the car, these points coincide,

$$\mathbf{r}^R_{P_1O} + \mathbf{r}^R_{Q_1P_1} - (\mathbf{r}^R_{P_2O} + \mathbf{r}^R_{Q_2P_2}) = \mathbf{0}.$$

where

$$\mathbf{r}^R_{Q_1P_1} = \mathbf{A}^{RL_1}\begin{pmatrix}-l_{12}\\0\end{pmatrix} \quad \text{and} \quad \mathbf{r}^R_{Q_2P_2} = \mathbf{A}^{RL_2}\begin{pmatrix}l_2\\0\end{pmatrix}.$$

We get

$$x^R_{P_1O} - l_{12}\,\cos\psi_1 - x^R_{P_2O} - l_2\,\cos\psi_2 = 0 \tag{2.3a}$$

$$y^R_{P_1O} - l_{12}\,\sin\psi_1 - y^R_{P_2O} - l_2\,\sin\psi_2 = 0. \tag{2.3b}$$

2.4 Kinematic Constraints

So far, we have only considered the geometry of the car and trailer system. To get further we use the properties discussed in Section 2.1.

- By definition, P_0 moves only along the x-axis of the L_0 frame and as an effect of the no lateral slip assumption P_1 only move along the x-axis of L_1. With a similar argument we assume that the trailer has zero lateral slip. Thus the point P_2 will only move along the x-axis of L_2. We have

$$\mathbf{A}^{L_iR}\,\dot{\mathbf{r}}^R_{P_iO} = \begin{pmatrix}\star\\0\end{pmatrix},\ i = 0, 1, 2$$

which yields the constraint equations

$$-\sin\psi_0\,\dot{x}^R_{P_0O} + \cos\psi_0\,\dot{y}^R_{P_0O} = 0 \tag{2.4a}$$

$$-\sin\psi_1\,\dot{x}^R_{P_1O} + \cos\psi_1\,\dot{y}^R_{P_1O} = 0 \tag{2.4b}$$

$$-\sin\psi_2\,\dot{x}^R_{P_2O} + \cos\psi_2\,\dot{y}^R_{P_2O} = 0. \tag{2.4c}$$

- Finally, the speed of the car can be measured. Let us assume that the speed of the rear wheels is measured. Since there is no slip,

$$\mathbf{A}^{L_1R}\,\dot{\mathbf{r}}^R_{P_1O} = \begin{pmatrix}v_R\\0\end{pmatrix},$$

which yields

$$\cos\psi_1\,\dot{x}^R_{P_1O} + \sin\psi_1\,\dot{y}^R_{P_1O} = v_R. \tag{2.5}$$

Assuming that initial conditions are known we now have enough equations to decide the behaviour of the system as a result of the steering angle, δ_F, and rear wheel speed, v_R.

2.5 Differential Equations

To make our model easier to handle we rewrite it as a system of differential equations. The derivations can be found in Appendix A. Since we have four non-algebraic constraint equations, (2.4) and (2.5), we get four independent variables in our equations. We choose these to be $x_1 = x^R_{P_1O}$, $y_1 = y^R_{P_1O}$, ψ_1 and $\gamma = \psi_1 - \psi_2$. Note that γ is the angle between car and trailer, defined as positive in a left curve. The resulting equations are

$$\dot{x}_1 = v_R \cos\psi_1 \tag{2.6a}$$
$$\dot{y}_1 = v_R \sin\psi_1 \tag{2.6b}$$
$$\dot{\psi}_1 = \frac{v_R}{l_1} \tan\delta_F \tag{2.6c}$$
$$\dot{\gamma} = \left(\frac{v_R}{l_1} + \frac{v_R\, l_{12}}{l_1\, l_2} \cos\gamma\right) \tan\delta_F - \frac{v_R}{l_2} \sin\gamma. \tag{2.6d}$$

For obvious reasons the input δ_F is bounded. We have

$$|\delta_F| \leq \delta_F^{bd}.$$

Sometimes it is more appropriate with a model that uses traveled distance, rather than time, as independent variable. We introduce σ as the distance travelled backwards by the rear wheels. Using the chain rule we get

$$\dot{\gamma}(t) = \gamma'(\sigma)\, \dot{\sigma}(t) = -v_R\, \gamma'(\sigma)$$

and

$$\gamma'(\sigma) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2} \cos\gamma\right) \tan\delta_F + \frac{1}{l_2} \sin\gamma. \tag{2.7}$$

Naturally, the same substitution could be performed in the first three equations, but that will not be necessary.

2.6 Model Validation

To validate the model, we use measurements of rear speed, v_R, front wheel angle, δ_F, and car-trailer angle, γ, from test drives. By solving the model equations with measured inputs v_R and δ_F, we get a simulated output γ. Comparing measured and simulated γ gives an idea of the accuracy of the model. Figure 2.5 shows this comparison for a typical test drive.

It should be mentioned that the agreement between measured and simulated γ is bad when reversing. The reason is that since the system is unstable, a small initial deviation will tend to increase. For this reason, we use measurements from forward driving, trusting that our assumptions are just as valid for reversing.

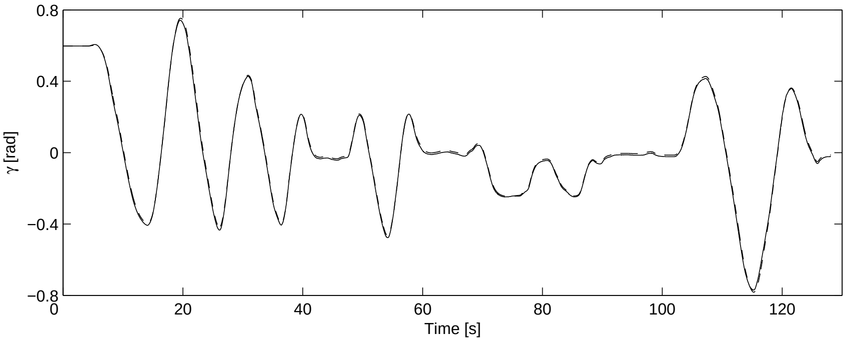

Figure 2.5. Measured (solid) and simulated (dashed) γ, when driving forward at low speed.

2.7 Trailer Length Estimation

To allow driving with trailers of different lengths we need some way to estimate the trailer length while driving. The standard solution is to use a prediction error method, choosing the parameter value that minimizes the quadratic sum of the prediction errors. The basics of parameter estimation can be found in [6].

The basis of our estimation is measurements of γ, δ_F and v_R. We use v_R to get sequences $\gamma(k)$ and $\delta_F(k)$ that are equidistant in space rather than in time. Otherwise errors when driving slowly would get an exaggerated effect.

Performing an Euler step on our differential equation gives us a prediction of $\gamma(k+1)$ given the values of $\gamma(k)$ and $\delta_F(k)$. The prediction is

$$\begin{aligned}\hat{\gamma}(k+1) &= \gamma(k) + h\left[\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma(k)\right)\tan\delta_F(k) - \frac{1}{l_2}\sin\gamma(k)\right] \\ &\triangleq \gamma(k) + g_1(k) + \frac{1}{l_2}\, g_2(k).\end{aligned}$$

We replace the parameter that we want to estimate with $\theta = 1/l_2$ and define

$$\begin{aligned}V_N(\theta) &= \frac{1}{N}\sum_{k=1}^{N} \left(\gamma(k+1) - \hat{\gamma}(k+1|\theta)\right)^2 \\ &= \frac{1}{N}\sum_{k=1}^{N}\left(\gamma(k+1) - \gamma(k) - g_1(k) - \theta\, g_2(k)\ \right)^2.\end{aligned}$$

We get our estimation of l_2 from the θ that minimizes $V_N(\theta)$. Because $\hat{\gamma}$ is linear in θ this minimum can be found easily. First we define

$$f_N = \frac{1}{N}\sum_{k=1}^{N}\Big(\gamma(k+1) - \gamma(k) - g_1(k)\Big)\, g_2(k)$$

and

$$R_N = \frac{1}{N} \sum_{k=1}^{N} g_2(k)^2.$$

If $R_N > 0$

$$\begin{aligned} V_N(\theta) &= \frac{1}{N} \sum_{k=1}^{N} \Big(\gamma(k) - \gamma(k-1) - g_1(k)\Big)^2 + 2\,\theta\, f_N + \theta^2\, R_N \\ &= \frac{1}{N} \sum_{k=1}^{N} g_1(k)^2 - \frac{f_N^2}{R_N} + R_N \left(\theta + \frac{f_N}{R_N}\right)^2. \end{aligned}$$

which is minimal when

$$\theta = -\frac{f_N}{R_N}. \tag{2.8}$$

The condition $R_N > 0$ is worth commenting. For this condition to be false, $g_2(k) = 0$ for all k, which means

$$\frac{l_{12}}{l_1\, l_2} \cos\gamma \tan\delta_F - \frac{1}{l_2} \sin\gamma = 0.$$

There seems to be a whole family of solutions to this equation,

$$\tan\delta_F = \frac{l_1}{l_{12}} \tan\gamma.$$

If the driver chooses the input according to one of these functions, it will be impossible to estimate the trailer length. Fortunately, this seems very unlikely, except perhaps for the special case $\delta_F = \gamma = 0$, which is pretty obvious. We conclude that the estimation method works, except for driving along a straight line.

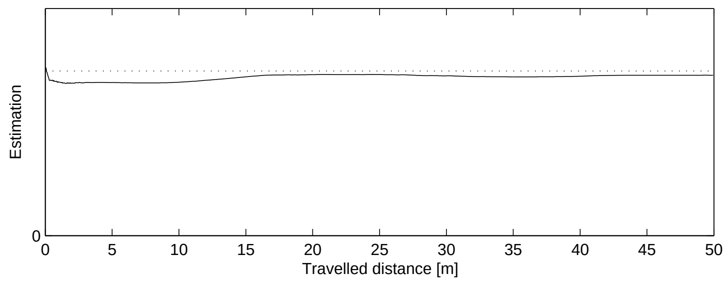

Figure 2.6. Estimation of the trailer length in real-time. The solid line is the estimated trailer length and the dashed line is the actual length.

Figure 2.6 shows estimated and true trailer length from a test drive. The test drive included several curves. As the plot shows, a good estimation of the trailer length was obtained after just a few meters.

Chapter 3

System Characteristics

In the next chapter we will construct controllers for trailer reversing, but first we look at some of the characteristics that make it difficult to reverse without assistance.

In the first section we derive an expression for the equilibria of our model. Then we define the concept of left-right steering. In Section 3.3 we derive an expression for the jackknifing angle and the last section of this chapter concerns stability.

3.1 Equilibria

As in most of this thesis, we will replace the front wheel angle, δ_F, with the more convenient $u = \tan\delta_F$. Not to forget its meaning, we will refer to it as the front wheel tangent. The bounds on the front wheel angle are easily translated. We have

$$|u| < u_{bd} = \tan\delta_F^{bd}.$$

Next, we rewrite (2.7) and define a function for the right hand side,

$$\gamma'(\sigma) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma\right)u + \frac{1}{l_2}\sin\gamma \triangleq f(\gamma, u). \tag{3.1}$$

In order to analyse this equation, we first seek the equilibria. Since we are only interested in realistic car-trailer angles, we can assume $|\gamma| \leq \pi/2$. We put $\gamma' = 0$, which yields

$$u = \frac{l_1 \sin\gamma}{l_2 + l_{12}\cos\gamma}.$$

This equilibrium relation will prove important, so we define the function

$$u_{eq}(\gamma) = \frac{l_1 \sin\gamma}{l_2 + l_{12}\cos\gamma}. \tag{3.2}$$

Note that this function is strictly monotonic on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, for each front wheel tangent, u, one unique car-trailer angle is in equilibrium, and inversely, for each car-trailer angle there is only one front wheel tangent that will put the trailer in equilibrium.

3.2 Left-Right Steering

In the previous section, we found the equilibria of our differential equation. In this section, we study the behaviour around such an equilibrium. Recall from (3.1) that

$$\gamma' = f(\gamma, u),$$

and that $f(\gamma, u_{eq}(\gamma)) = 0$. For $\gamma \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$f'_u(\gamma, u) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma\right) < 0.$$

We conclude that

$$\begin{aligned} u > u_{eq} &\Rightarrow \gamma' = f(\gamma, u) < 0 \\ u < u_{eq} &\Rightarrow \gamma' = f(\gamma, u) > 0. \end{aligned}$$

What this means is that to get the trailer to turn *left*, you have to turn the front wheels of the car to the *right* and inversely. Lacking a better expression, we will call this *left-right* steering. It should be familiar to anyone who has tried reversing with a trailer. For inexperienced drivers it can pose quite a problem.

3.3 Jackknifing

In the previous section we learned that to make γ smaller, we simply make $u > u_{eq}$, but what happens if this is not possible? The problem arises if $u_{eq}(\gamma) > u_{bd}$. Then γ' must be positive and γ will increase. The trailer will fold up to the car like a jackknife (Figure 3.1). The only way for the driver to regain control of the trailer is to stop and drive forward.

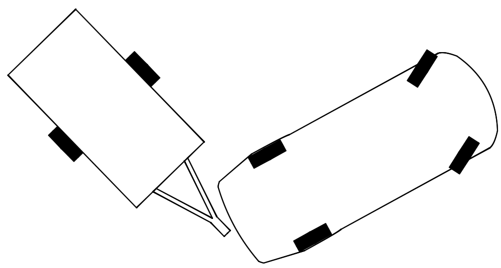

Figure 3.1. Jackknifing.

The critical jackknifing angle γ_{jk} can be found by putting $u_{eq}(\gamma) = u_{bd}$. (Thanks to the symmetry it is sufficient to consider the positive case.) Using

(3.2) yields

$$u_{bd} = u_{eq}(\gamma) = \frac{l_1 \sin\gamma}{l_2 + l_{12}\cos\gamma},$$

and

$$l_1 \sin\gamma - l_{12}\, u_{bd} \cos\gamma = l_2\, u_{bd}.$$

Using the relation $\sin(\alpha+\beta) = \sin\alpha\ \cos\beta + \cos\alpha \sin\beta$ we get an expression for the jackknifing angle

$$\gamma_{jk} = \arcsin\left(\frac{l_2\, u_{bd}}{\sqrt{l_1^2 + l_{12}^2\, u_{bd}^2}}\right) + \arctan\left(\frac{l_{12}\, u_{bd}}{l_1}\right). \tag{3.3}$$

Symmetrically, the trailer will jackknife if $\gamma < -\gamma_{jk}$. From now on we will mainly be interested in angles $\gamma \in (-\gamma_{jk}, \gamma_{jk})$. For them $|u_{eq}(\gamma)| < u_{bd}$ and thus the trailer can be controlled. Notice that at the jackknifing angle itself the system is not controllable.

3.4 Stability

In this section, we examine the stability of our equilibria, and present some theory that will be needed in the next chapter. First we recall the definition of stability.

Consider a scalar autonomous differential equation $z' = f(z)$, $z(0) = z_0$ where f is a continuous function and $f(Z) = 0$.

Definition 3.1 *The equilibrium Z is stable, if for an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that $|z_0 - Z| < \delta$ assures $|z(t) - Z| < \varepsilon$ for all $t > 0$.*

Theorem 3.1 *If $f'(Z) > 0$, then Z is an unstable equilibrium.*

For a proof of this theorem and a good general discussion on stability, see [5].

The definition of stability is valid for autonomous differential equation. A differential equation with an input signal results in an autonomous function if the input signal is kept constant. In our case, assuming $u \equiv U$, yields the equation

$$\gamma' = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma\right) U + \frac{1}{l_2}\sin\gamma = f_U(\gamma).$$

which has only one equilibrium, $\Gamma \in (-\gamma_{jk}, \gamma_{jk})$, such that $U = u_{eq}(\Gamma)$, (see Section 3.1). According to Theorem 3.1, the equilibrium is unstable if $f'(\Gamma) > 0$. Differentiation yields

$$f'(\gamma) = \frac{l_{12}}{l_1\, l_2}\sin\gamma\, U + \frac{1}{l_2}\cos\gamma.$$

Equation (3.2) tells us that Γ and U, will always have the same sign. Thus, $f'(\Gamma)$ is positive and the equilibrium is unstable.

This instability plays an important roll in making trailer reversing difficult. Because of it, the driver has to be active all the time, responding to the movement of the trailer, and thanks to the left-right steering there is a large risk that he or she will respond incorrectly.

Apparently, it should be an aim of any control scheme to stabilize the system. However, normal stability is not always sufficient. In the next chapter we will discuss different reference tracking controllers. Then we expect the current reference value to be a stable equilibrium, not only locally, but for all $\gamma \in (-\gamma_{jk}, \gamma_{jk})$. We also need to be sure that γ never leaves this interval. So what kind of stability do we require? To end this chapter, we define of an *attractive* equilibrium and prove a theorem that will be useful in the next chapter.

Consider a scalar autonomous differential equation $z' = f(z)$, $z(0) = z_0$ *where* f *is a continuous function and* $f(Z) = 0$.

Definition 3.2 *The equilibrium* Z *is attractive in an interval* $\mathcal{I}$, *if* $|z(t) - Z|$ *decreases strictly towards zero for all* $z_0 \in \mathcal{I}$.

Theorem 3.2 *If* $(z - Z)f(z) < 0$ *for* $z \in \mathcal{I} \setminus \{Z\}$, *then* Z *is attractive in* $\mathcal{I}$.

Proof Let $d(t) = \frac{1}{2}(z(t) - Z)^2$ and $d_0 = d(0) = \frac{1}{2}(z_0 - Z)^2$, where $z_0 \in \mathcal{I}$. We note that $d(t) \geq 0$ and that $d'(t) = (z(t) - Z)f(z(t)) \leq 0$. Thus $d(t)$ is decreasing and bounded and consequently, it has a limit, c, satisfying $0 \leq c \leq d_0$. It remains to show that this limit is zero. When $d_0 = 0$ we have $0 \leq c \leq 0$ so we only have to consider the case $d_0 > 0$.

For contradiction, assume $c > 0$. Since d is decreasing this means that d stays in $\Omega = [c, d_0]$ for all t. Since Ω is compact and d' is continuous, d' attain a maximum value in Ω. We call it k. From the condition of the theorem we know that $k < 0$. This means that for all t, $d'(t) \leq k < 0$, implying that $d \to -\infty$. Since $d \geq 0$ this is a contradiction. The conclusion is that $d(t)$ decreases towards zero and since d' is negative except for $d = 0$, the decrease is strict. □

Note that an attractive equilibrium is also asymptotically stable (as defined in [5] or [1]).

Chapter 4

Reference Tracking

Normally, the steering wheel is used to control the front wheels of the car. To control the trailer, the driver has to predict how it will react to different front wheel angles. Because of the factors discussed in the previous chapter, this is quite difficult.

In this chapter we try to make trailer reversing more like reversing without a trailer. To achieve this we break the direct link between steering wheel and front wheels. Instead, the steering wheel will be used to provide a reference signal for some appropriate output. Different outputs are tested. In Section 4.1, two controllers for the car-trailer angle γ are designed, and in Section 4.2, a controller for the trailer turning radius. Experiences from test driving are presented in Section 4.4 and in Section 4.5, some alternatives to reference tracking controllers are discussed.

It could be argued that this approach is misguided, that it will change the steering characteristics too much and confuse the driver. Even if this is true, it is interesting to test reference tracking systems. The reason is that they let us examine, more generally, the possibility of controlling a car-trailer system. The experiences can then be used for more intricate control schemes or, as in Chapter 6, for autonomous steering.

4.1 Controlling γ

Since the car-trailer angle, γ, is measured, that is perhaps the most natural choice of output. To turn left, the driver steers left, indicating that he wants a positive angle between car and trailer.

In Chapter 2 we found the differential equation, describing how the trailer angle depends on the steering input. Although this equation is nonlinear, a linear controller would probably work. However, because of the nonlinearity, the effectiveness of such a controller could depend on the trailer length. It seems more satisfactory to use nonlinear controllers. In the following sections we try input-output linearization and optimal feedback.

4.1.1 Input-Output Linearization

As discussed in Chapter 3, for each car-trailer angle γ there is a balancing front wheel tangent $u_{eq}(\gamma)$. To increase γ we need to make $u < u_{eq}(\gamma)$ and opposite to decrease γ. Thus, any controller for γ should work around this equilibrium. One way to obtain such a controller is input-output linearization (described in [1]).

Input-output linearization is based on the fact that linear systems are rather easy to control. To control a nonlinear system like ours

$$\gamma' = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2} \cos\gamma\right) u + \frac{1}{l_2} \sin\gamma,$$

we first fake a linear system by adding a kind of filter. Choosing

$$u = \frac{-w + \frac{1}{l_2} \sin\gamma}{\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2} \cos\gamma}, \tag{4.1}$$

yields

$$\gamma' = w \text{ as long as } |u| \leq u_{bd}$$

which is, within the bounds, a linear system from the new input signal w to the output γ. This linear system can be controlled with a normal P-controller, $w = K_1(r_\gamma - \gamma)$, where r_γ is the reference value for γ. All combined, we get the controller

$$u = \frac{l_1 \sin\gamma}{l_2 + l_{12} \cos\gamma} + \frac{K_1'(\gamma - r_\gamma)}{l_2 + l_{12} \cos\gamma}. \tag{4.2}$$

To examine the stability of the controlled system, we assume that the reference value is kept constant $r_\gamma = R \in (-\gamma_{jk}, \gamma_{jk})$. This yields an autonomous differential equation

$$\gamma' = K_1(R - \gamma),\ |u| \leq u_{bd}, \tag{4.3}$$

with one equilibrium $\gamma = R$.

Proposition 4.1 *The equilibrium $\gamma = R$ is attractive (and asymptotically stable) in $(-\gamma_{jk}, \gamma_{jk})$.*

Theorem 3.2 states that the system is attractive if γ' switches sign at the equilibrium (and has no other zeros). It is pretty obvious that this is the case for (4.3).

4.1.2 Optimal Feedback

Another approach to nonlinear control is optimal control. In optimal control we seek a control law that minimizes some criterion. Choosing this criterion is no exact science, but adjusting the criterion can often be more intuitive than adjusting the parameters of a controller. In this section we try a certain kind of optimal control called optimal feedback. The theory can be found in [1].

The system we want to control is described by

$$\gamma'(\sigma) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma(\sigma)\right) u(\sigma) + \frac{1}{l_2}\sin\gamma(\sigma) = f(\gamma, u).$$

The criterion we want to minimize is a special case of the criterion in (18.68) in [1].

$$\min \int_{\tau}^{\infty} L(u(\sigma), \gamma(\sigma))\, d\sigma \tag{4.4a}$$

$$\dot{\gamma} = f(\gamma, u) \tag{4.4b}$$

$$u(s) \in U = [-u_{bd}, u_{bd}] \tag{4.4c}$$

$$\gamma(0) = z \tag{4.4d}$$

Getting a good controller is a question of choosing $L(u, \gamma)$. Clearly, γ should track its reference value, r_γ and hence we put a term $(\gamma - r_\gamma)^2$ in L.

$$L(u, \gamma) = (\gamma - r_\gamma)^2 + \dots$$

Furthermore, we want the steering to feel smooth. To ensure this we add a term $(u - u_{eq}(\gamma))^2$, where u_{eq} refers to the current equilibrium. Recall the definition from Section 3.1,

$$u_{eq}(\gamma) = \frac{l_1 \sin\gamma}{l_2 + l_{12}\cos\gamma}.$$

Further, we need some way to adjust the trade-off between reference tracking and smoothness, so we also introduce a design parameter $K_2 > 0$,

$$L(u, \gamma) = \frac{K_2^2}{2}(\gamma - r_\gamma)^2 + \frac{1}{2}(u - u_{eq}(\gamma))^2. \tag{4.5}$$

We now seek a control law that is optimal for this criterion. Again, the theory can be found in [1]. A central roll is played by the optimal return function

$$V(\tau, z) = \int_{\tau}^{\infty} L(u^*, \gamma^*) d\sigma,$$

where u^* and γ^* refer to functions satisfying (4.4).

Since the interval is infinite and all functions are position invariant the optimal return function must also be independent of the starting position τ. This means that

$$V'_\tau \equiv 0 \quad \text{and} \quad V(\tau, z) = V(z).$$

We get the Hamilton-Jacobi equation

$$0 = \min_u V'_z(\gamma)\, f(\gamma) + L(u, \gamma) =$$

$$= \min_u V'_z(\gamma)\left(-\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma\right) u + \frac{1}{l_2}\sin\gamma\right) + L(u, \gamma).$$

The minimum can be found by completing the square. It is attained for

$$u = \frac{l_1 \sin\gamma}{l_2 + l_{12}\,\cos\gamma} + V_z'(\gamma)\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\,\cos\gamma\right).$$

Note that the bounds on the input could easily have been included here. If the minimizing u is too large, it is optimal to choose the biggest possible $u = \pm u_{bd}$. Reinserting the new expression yields the equation

$$-\frac{1}{2}\,V_z'(\gamma)^2\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\cos\gamma\right)^2 + \frac{K_2^2}{2}(\gamma - r_\gamma)^2 = 0$$

with the solutions

$$V_z'(\gamma) = \pm K_2 \frac{\gamma - r_\gamma}{\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\cos\gamma}.$$

The definition of $V(\gamma)$, excludes the negative sign, which gives us the optimal controller

$$u = \frac{l_1\,\sin\gamma}{l_2 + l_{12}\,\cos\gamma} + K_2(\gamma - r_\gamma). \tag{4.6}$$

4.1.3 Comparing the γ-controllers

The controllers from the previous sections, (4.2) and (4.6), are very similar. This inspires a closer look at the optimality of linearizing controllers.

By definition, input-output linearization creates a linear system from output signal to a new input signal. In our case we got

$$\gamma'(\sigma) = w.$$

To control this new system, we used a (linear) P-controller. According to the theory of linear quadratic control (compare Chapter 9 in [1]), such a controller is optimal for a linear quadratic criterion

$$\min_w \int_0^\infty \left(\frac{K_1^2}{2}(\gamma - r_\gamma)^2 + \frac{1}{2}\,w^2\right) d\sigma.$$

Solving (4.1) for w we get

$$w = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\,\cos\gamma\right) u + \frac{1}{l_2}\,\sin\gamma = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\,\cos\gamma\right)(u - u_{eq}(\gamma))$$

and the criterion

$$\min_u \int_0^\infty \left(\frac{K_1^2}{2}(\gamma - r_\gamma)^2 + \frac{1}{2}\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\,\cos\gamma\right)^2 (u - u_{eq})^2\right) d\sigma,$$

which is almost the same criterion as the one we chose for the optimal feedback. (Note that to some extent this discussion could be repeated in a multi-variable case.)

Since the difference between the controllers is so small, the work we did on optimal feedback can be seen mainly as a motivation for the linearizing controller.

4.2 Controlling the Turning Radius

A drawback of controlling γ is that the transient behaviour of the trailer can be somewhat unpredictable. The reason is that the actual movement of the trailer is not regarded. Sometimes a controller for the turning radius of the trailer might be better. In this section we construct such a controller.

In Appendix A.2 an expression for the turning radius, ρ_T, is derived, or rather an expression for the curvature

$$\kappa = \pm\frac{1}{\rho_T} = \frac{l_1\,\sin\gamma - l_{12}\,u\,\cos\gamma}{l_1\,l_2\,\cos\gamma + l_{12}\,l_2\,u\,\sin\gamma}. \tag{4.7}$$

Since the curvature is not measured, it cannot be controlled using standard methods. Instead we try a kind of predictive control choosing the control signal that, according to the model, should give the desired output.

Let r_κ be the reference value for κ. To find the value of u that will give the desired curvature, we put $\kappa = r_\kappa$, in (4.7) and solve for u. This gives us the control law

$$u = g(r_k, \gamma) = \frac{l_1}{l_{12}}\,\frac{\tan\gamma - l_2\,r_\kappa}{1 + l_2\,r_\kappa\,\tan\gamma}. \tag{4.8}$$

Note that the controller has a discontinuity. At $l_2\,r_\kappa\,\tan\gamma = -1$ the denominator turns zero and u switches sign. Remembering that $u = \tan\delta_F$, we understand that the controller switches between $\delta_F = \pi/2$ and $\delta_F = -\pi/2$, that is, the wheels are supposed to turn from full right to full left. Though this is mathematically correct, it is naturally undesirable in practice. Moreover, the front wheel angle can never become larger than δ_F^{bd}, corresponding to $u = u_{bd} = \tan\delta_F^{bd}$. This maximum is attained for

$$r_\kappa = \frac{l_1\tan\gamma - l_{12}\,u_{bd}}{l_1\,l_2 + l_2\,l_{12}\,u_{bd}\,\tan\gamma}.$$

We modify the controller accordingly. The modified controller is $u = h(r_\kappa, \gamma)$, with

$$h(r_\kappa, \gamma) = u_{bd} \text{ when } r_\kappa \le \frac{l_1\tan\gamma - l_{12}\,u_{bd}}{l_1\,l_2 + l_2\,l_{12}\,u_{bd}\,\tan\gamma}, \tag{4.9a}$$

$$h(r_\kappa, \gamma) = -u_{bd} \text{ when } r_\kappa \ge \frac{l_1\tan\gamma + l_{12}\,u_{bd}}{l_1\,l_2 - l_2\,l_{12}\,u_{bd}\,\tan\gamma} \tag{4.9b}$$

and otherwise

$$h(r_\kappa, \gamma) = g(r_\kappa, \gamma) = \frac{l_1}{l_{12}}\,\frac{\tan\gamma - l_2\,r_\kappa}{1 + l_2\,r_\kappa\,\tan\gamma}. \tag{4.9c}$$

Now we would like to check if this controller will actually stabilize the trailer. First we need to find the equilibria of the system. Since we are only interested in equilibria $\gamma \in (-\gamma_{jk}, \gamma_{jk})$, where $|u_{eq}(\gamma)| < u_{bd}$, the saturation in (4.9) is of no interest. Thus, we insert (4.8) in the differential equation (3.1),

$$\gamma' = -\left(\frac{1}{l_{12}} + \frac{1}{l_2}\,\cos\gamma\right)\,\frac{\tan\gamma - l_2\,r_\kappa}{1 + l_2\,r_\kappa\,\tan\gamma} + \frac{1}{l_2}\,\sin\gamma.$$

In equilibrium $\gamma' = 0$, so

$$\frac{1}{l_{12}}(\sin\gamma - l_2\, r_\kappa\, \cos\gamma) - r_\kappa\, \cos^2\gamma = r_\kappa\, \sin^2\gamma$$

and

$$r_\kappa = \frac{\sin\gamma}{l_{12} + l_2\, \cos\gamma}. \tag{4.10}$$

This relation shows us that if the reference value r_κ gets too large, the controller will steer towards an equilibrium $\gamma > \gamma_{jk}$, and cause the trailer to jackknife. To prevent this the reference value has to be kept within the bound

$$|r_\kappa| < r_\kappa^{bd} = \frac{\sin\gamma_{jk}}{l_{12} + l_2\, \cos\gamma_{jk}}.$$

Now that we have found a relation that describes the equilibria of the controlled system, we go on to check the stability of these equilibria. As commented in Section 3.4, stability is defined for autonomous differential equations. As in that section, we get an autonomous equation by assuming a constant input signal. In this case we assume that the reference signal is kept constant $r_\kappa = R$, with $|R| < r_\kappa^{bd}$. We write the resulting differential equation

$$\gamma' = f_R(\gamma) \tag{4.11}$$

where the R indicates that we get a different equation for each value of R. The autonomous equation has only one equilibrium $\Gamma \in (-\gamma_{jk}, \gamma_{jk})$.

Proposition 4.2 *The equilibrium Γ is attractive (and asymptotically stable) in $(-\gamma_{jk}, \gamma_{jk})$.*

We will divide the proof of this assertion in three steps. First recall the definition of g in (4.8) and the defintion of h in (4.9).

Lemma 4.1 *For two equilibria R_1, Γ_1 and R_2, Γ_2 with $\Gamma_1 < \Gamma_2$ and $\Gamma_1, \Gamma_2 \in (-\gamma_{jk}, \gamma_{jk})$, it holds that*

$$\text{(i) } h(R_1, \Gamma_2) > h(R_2, \Gamma_2)$$

$$\text{(ii) } h(R_2, \Gamma_1) < h(R_1, \Gamma_1)$$

Proof (of (ii)) First, note that $h(r_\kappa, \gamma) = g(r_\kappa, \gamma)$ or $h(r_k, \gamma) = \pm u_{bd}$. Since

$$g_r'(r_\kappa, \gamma) = -\frac{l_1}{l_{12}}\, \frac{1 + \tan^2\gamma}{(1 + l_2\, r_\kappa\, \tan\gamma)^2} < 0,$$

we know that $h_r' \leq 0$.

Further, for an equilibrium $\Gamma_1 \in (-\gamma_{jk}, \gamma_{jk})$, the corresponding input $u = h(R_1, \Gamma_1)$ will be smaller than the maximum input $|u| < u_{bd}$. Compare with Section 3.3. Thus the equilibrium lies in an interval where h is strictly decreasing.

Now, (4.10) shows that $R_2 > R_1$. Since h is decreasing everywhere and strictly decreasing around R_1, $h(R_2, \Gamma_1) < h(R_1, \Gamma_1)$. □

Next we define

$$f(\gamma, u) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\cos\gamma\right)u + \frac{1}{l_2}\sin\gamma$$

noting that (4.11) can be written

$$\gamma' = f_R(\gamma) = f(\gamma, h(R, \gamma))$$

Lemma 4.2 *For the same* R_1, Γ_1 *and* R_2, Γ_2 *as earlier,*

$$\text{(i) } f(\Gamma_2, h(R_1, \Gamma_2)) < 0$$

$$\text{(ii) } f(\Gamma_1, h(R_2, \Gamma_1)) > 0$$

Proof (of (ii)) First note that, since R_1, Γ_1 is an equilibrium

$$f(\Gamma_1, h(R_1, \Gamma_1)) = 0.$$

Further, we note that

$$f_u'(\gamma, u) = -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\,l_2}\cos\gamma\right) < 0.$$

From Lemma 4.2 we know that $h(R_2, \Gamma_1) < h(R_1, \Gamma_1)$ and consequently,

$$f(\Gamma_1, h(R_2, \Gamma_1)) > f(\Gamma_1, h(R_2, \Gamma_1)) = 0$$

□

Proof (Proposition) To use Theorem 3.2, we need to prove that for $\gamma \in (-\gamma_{jk}, \gamma_{jk})$

$$\gamma > \Gamma \Rightarrow f_R(\gamma) < 0$$

$$\gamma < \Gamma \Rightarrow f_R(\gamma) > 0.$$

This follows from Lemma 4.2 after noticing that each $\gamma \in (-\gamma_{jk}, \gamma_{jk})$ corresponds to an equilibrium with

$$r_\kappa = \frac{\sin\gamma}{l_{12} + l_2\,\cos\gamma}.$$

□

4.3 Anti-Jackknifing

As discussed in Chapter 3, one of the disturbing characteristics of a car-trailer system, is its tendency to jackknife. A strength of the reference tracking controllers is that jackknifing can be prevented. All we need to do is bound the reference values. For the γ-controllers, the obvious bound is

$$|r_\gamma| < \gamma_{jk}.$$

In case of the turning radius controller the bound has already been mentioned, but we repeat it here

$$|r_\kappa| < r_\kappa^{bd} = \frac{\sin \gamma_{jk}}{l_{12} + l_2 \cos \gamma_{jk}}.$$

In practice it would, of course, be necessary to have a safety margin, staying well below these limits, but even so jackknifing could occur. The reason is the bounds on the angular speed of the AFS electric motor. At higher vehicle speeds the motor might not be able to provide the superposition angles needed to avoid jackknifing. In a commercial system, it could therefore be necessary to limit the vehicle speed when the system is active.

4.4 Test Driving

To examine the characteristics of the controllers designed earlier in this chapter, they were tested in a prototype car with AFS. Of the two γ-controllers, only the linearizing controller from Section 4.1.1 was tested.

First a few words on the subject of the next chapter are necessary. The testing of the reference tracking controllers was made difficult by the torque feedback in the steering wheel. The feedback causes the steering wheel to feel unstable and it gets hard to steer. Though this strange steering wheel feel was hard to ignore completely, it was possible to get a good idea of how well the reference tracking controllers work.

In the previous sections, we proved stability for our controllers. In practice this was tested by driving with a constant reference value (steering wheel angle). The clear result was that both the controllers do stabilize the trailer, at least for speeds up to 25–30 km/h. (Higher speeds were not tested.) The stability was very useful when making a turn. You only had to turn the steering wheel once and then hold it instead of the customary turning back and forth.

Limiting the reference value (as discussed in the previous section), was effective for avoiding jackknifing. The limits to the speed of the electric motor seem to have little effect in normal driving situation, that is, at normal speeds.

Though these results are fine, the important question remains. Do the controllers make it easier to reverse with a trailer? At least there are advantages, no instability, no left-right steering and no risk of jackknifing, but there are problems too. Apart from the torque feedback, some drivers were confused by the fact that they now control the trailer rather than the car. Not counting the author, eight persons tested these controllers (some only a short time). Of these five were decidedly positive, two negative and one not quite certain.

To conclude this section, we present two plots from test drives. Figure 4.1 shows how the car-trailer angle tracks its reference value when using the linearizing controller. Figure 4.2 shows slalom driving with and without control. It should be mentioned that the driver had been using the controller for about five minutes.

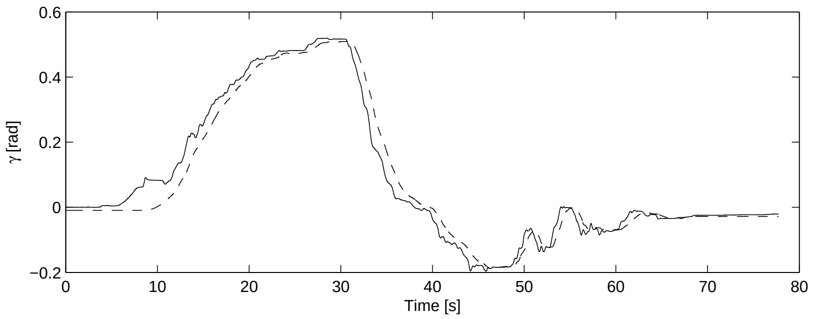

Figure 4.1. Steering wheel reference signal (solid) and car-trailer angle γ (dashed), when using the linearizing γ-controller.

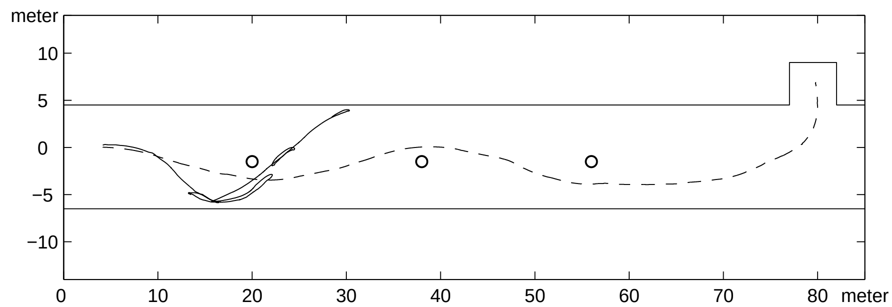

Figure 4.2. Backwards slalom driving with (dashed) and without controller.

4.5 Alternatives

In this section, we discuss a few alternatives to the reference tracking controllers that we covered earlier in this chapter.

4.5.1 Error Correction

A rather common approach to automotive control is to intervene only when the driver has done something wrong. A similar idea is used, successfully, in systems like ESP (Electronic Stability Program). There, the rather safe assumption is that most drivers do not want to lose control of their car. Unfortunately, it seems harder to make any assumptions concerning trailer reversing. In the right situation, any action (even jackknifing) can be desired. Estimating if a certain action was desired or a mistake seems very difficult. This approach is not discussed in this thesis.

4.5.2 Modified Steering Characteristics

A problem with the reference tracking controllers is that they dramatically change the steering characteristics of the car. In this section a more conservative approach is presented.

The approach is inspired by the fact that a long trailer is easier to reverse (though naturally demanding more space). The reason is that the trailer angle changes more slowly for a long trailer, giving the driver more time to think. It would be possible to obtain the same effect using active steering. The principle is quite simply to make the steering gain less direct around the stabilizing angle. Figure 4.3 shows the steering ratio with and without such a modification. The result is that small deviations around the stabilizing angle have less dramatic effects.

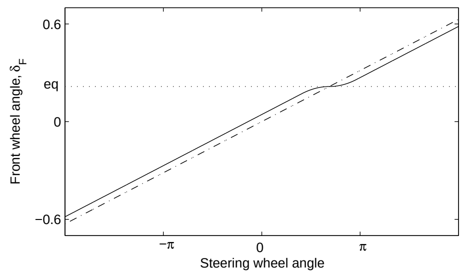

Figure 4.3. Steering ratio with (solid) and without control.

This approach was also implemented and tested. It was most effective when making a turn to, for example, park. When trying to drive straight backwards it provided little or no help. A variant is to use a γ-controller for reversing straight, and modified characteristics in all other driving situations.

4.5.3 Assisting Steering Wheel Torque

If active steering is combined with electric (rather than hydraulic) power steering, it is possible to simultaneously control the front wheel angles and the steering wheel torque. This opens for new assistance methods. A nice approach might be to keep the normal relation between steering wheel and front road wheel, but use the steering wheel torque to gently guide the driver to the current equilibrium. The idea is that the driver shall know where the equilibrium is.

One way to obtain this effect is to feed a γ-controller with the current car-trailer angle as reference value. The output of the controller will be the balancing front wheel angle. A drawback is that since the balancing angle cannot be calculated exactly, γ might change slowly. To avoid this some kind of memory could be used when deciding the input to the γ-controller. The aim is that if the driver lets go of the steering wheel the controller should stabilize the trailer autonomously.

Chapter 5

Steering Wheel as Reference

Our assistance functions are meant to assist drivers not used to reversing with a trailer. The aim was therefore to make it as similar as possible to reversing without trailer. That means that turning the steering wheel to the left should cause the trailer to go left. In Section 3.2 we called this left-left steering. This turned out to work poorly with AFS.

With AFS the torque needed to turn the front wheels will cause an opposing torque in the steering wheel. Normally, turning the steering wheel to the left will cause the wheels to go left and the opposing torque in the steering wheel will tend to decelerate the steering wheel movement.

However, when reversing with a trailer the front wheels initially have to turn right for the trailer to move left. The torque in the steering wheel will therefore tend to accelerate the steering wheel movement. This makes it very hard to steer.

In this chapter, we analyse the torque feedback problem using linear models of the involved systems. In Section 5.1 we show that left-left steering will cause the steering to become unstable, while left-right steering creates a stable system. The theoretical possibility of solving this problem, is explored in Section 5.2. Finally, we look at a few methods to make the instability less disturbing (Section 5.3).

5.1 Linear Analysis

In this section we examine the effect of using a normal P-controller to control the trailer. Linear models of both the steering and the car-trailer system are used. We show that though the car-trailer system can be stabilized with a P-controller, left-left steering will make the complete system unstable.

We start by linearizing our nonlinear car-trailer model around $(\gamma, \delta_F) = (0, 0)$,

$$\begin{aligned} \gamma' &= -\left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\cos\gamma\right)\tan\delta_F + \frac{1}{l_2}\sin\gamma \\ &\approx \left(\frac{1}{l_1} + \frac{l_{12}}{l_1\, l_2}\right)\delta_F - \frac{v_R}{l_2}\,\gamma \triangleq -b\,\delta_F + a\,\gamma. \end{aligned}$$

Laplace transformation gives us the transfer function between δ_F and γ

$$G(s) = -\frac{b}{s-a}.$$

Now assume that an independent reference value r_γ is provided somehow, and that we use a P-controller $\delta_F = K(r_\gamma + \gamma)$. Then the transfer function for the closed loop system,

$$G_{c,0}(s) = \frac{K\,b}{s-a-K\,b}$$

is stable. Thus it is possible to stabilize the trailer using P-control, but what happens when we also include the steering. To try this we need a model from the input torque of the driver, T, to the steering wheel angle.

The feedback torque, T_f, in the steering wheel is approximately proportional to the angular velocity of the front wheels, $\dot{\delta}_F$. Thus,

$$T_f = -k\,\dot{\delta}_F \quad \Rightarrow \quad E(s) = -k\,s.$$

Assuming that friction in the steering shaft can be ignored, the steering wheel angle, δ_S, follows

$$\ddot{\delta}_S = J\,(T + T_f) \quad \Rightarrow \quad H(s) = \frac{J}{s^2}.$$

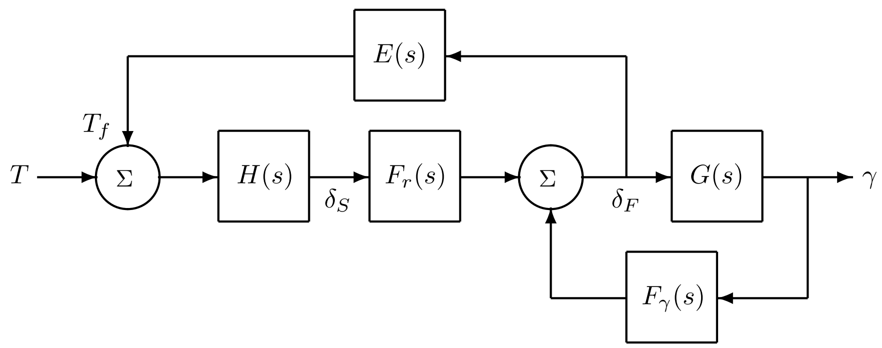

Figure 5.1. Block diagram showing the linear model used to analyse the torque feedback. The input is the driver's steering wheel torque, T, and the output is the car-trailer angle, γ. Note the undesired loop created by $E(s)$.

Now, we have a model from driver input to trailer angle (see Figure 5.1). To examine stability we calculate the transfer function, $G_C(s)$, from driver torque, T, to trailer angle, γ,

$$G_C(s) = \frac{F_r\,G\,\frac{J}{s^2}}{1 - F_r\,\frac{J}{s^2}\,k\,s - F_\gamma\,G}.$$

From test drives we know the nice properties of left-right steering. Are they also reflected in this linearized model? First look at proportional control, $F_r = K_r > 0$,

$F_\gamma = K_\gamma > 0$. Note that a positive steering wheel angle will give a negative trailer angle. We get the transfer function

$$G_C(s) = \frac{-K_r\, b\, J}{s^2 + (K_\gamma\, b - K_r\, J\, k - a)\, s + K_r\, Jkb}\, \frac{1}{s}$$

and the system poles (not counting the integration)

$$s = -\frac{1}{2}(K_\gamma\, b - K_r\, Jk - a) \pm \sqrt{\frac{1}{4}(K_\gamma\, b - K_r\, Jk - a)^2 - K_r\, Jkb}.$$

For small positive K_r, the system is stable (apart from the integration), but if K_r gets too large the system will become oscillating or even unstable.

To get left-left-steering instead, we use the controller $F_r = -K_r$. We then get the transfer function

$$G_C(s) = \frac{K_r\, b\, J}{s^2 + (K_\gamma\, b + K_r\, Jk - a)\, s - K_r\, Jkb}\, \frac{1}{s}$$

and the poles

$$s = -\frac{1}{2}(K_\gamma\, b + K_r\, Jk - a) \pm \sqrt{\frac{1}{4}(K_\gamma\, b + K_r\, Jk - a)^2 + K_r\, Jkb}.$$

Now, the square root is always greater than the first term, and thus there will always be at least one unstable pole.

5.2 Stabilizing the Steering Wheel

In the last section we found that P-control combined with left-left steering will make the steering unstable. A natural question is if it is possible to construct a controller that, while keeping left-left steering, creates a stable system. The aim of this section is to examine this possibility.

The front wheels has to react fairly quickly to the driver's actions, at least a lot quicker than the trailer. Therefore it is not unreasonable to assume constant trailer angle in these discussions.

5.2.1 No Steering Shaft Friction

The task is thus to construct a controller from steering wheel angle to front wheel angle. From Figure 5.1 we get

$$\begin{aligned}\delta_F(s) &= F_r(s)\, H(s)\, (T + T_f) + F_\gamma(s)\, \gamma(s)\\ &= F_r(s)\, H(s)\, T + F_r(s)\, H(s)\, E(s)\, \delta_F(s) + F_\gamma(s)\, \gamma(s).\end{aligned}$$

As discussed earlier the feedback through γ is disregarded. The system we want to stabilize can thus be described by the transfer function

$$G_S(s) = \frac{F_r(s)\, H(s)}{1 - F_r(s)\, H(s)\, E(s)}.$$

Consider a proper linear controller

$$F_r(s) = \frac{P(s)}{Q(s)}.$$

With functions $H(s)$ and $E(s)$ from the last section, the transfer function for the steering is

$$G_S(s) = \frac{P(s)\,J}{s\,Q(s) + Jk\,P(s)}\,\frac{1}{s}$$

and the poles are given by

$$R(s) = s\,Q(s) + Jk\,P(s) = 0.$$

We can always choose to have $Q(0) > 0$. If we want the controller to be stable in itself, $Q(s)$ cannot have any zeros in the right half plane, which means that $Q(s) \to \infty$ when $s \to \infty$. To get left-left-steering, the steady state gain must be negative. That is, when steering left, the front wheels turn right. Since

$$0 > F_r(0) = \frac{P(0)}{Q(0)}$$

we get $P(0) < 0$. This means

$$R(0) = 0 \cdot Q(0) + Jk\,P(0) < 0$$

and

$$R(s) \to \infty \text{ when } s \to \infty.$$

Since $R(s)$ is continuous, this implies that $R(s)$ has a real positive zero. The conclusion is that there is no stable linear controller that can locally stabilize the system.

5.2.2 With Steering Shaft Friction

In a real car, there will be some friction in the steering shaft, though not very much. In this case

$$H(s) = \frac{J}{s^2 + s\mu}$$

and the transfer function for the steering

$$G_S(s) = \frac{P(s)\,J}{(s+\mu)\,Q(s) + Jk\,P(s)}\,\frac{1}{s}.$$

With the same argument as in the previous section, we find that for any stable, linear controller it must hold

$$\mu\,Q(0) + Jk\,P(0) \geq 0$$

which means

$$F_r(0) = \frac{P(0)}{Q(0)} \geq -\frac{\mu}{Jk}. \tag{5.1}$$

Since there is very little friction in the steering shaft, μ is small and thus this is a hard restraint. The only ways to get stability is left-right steering ($F_r(0) < 0$), or with a very indirect steering ratio. The latter means that the driver will have to turn the steering wheel a lot to get the desired reaction.

5.3 Limiting Instability

Though the analysis in the previous section, uses a very simplified linear model and only concerns linear controllers, its result are probably valid. It is probably very difficult or impossible to get accaptable steering characteristics with left-left steering. This does not mean that the reference tracking controllers are useless though. We have already mentioned that if AFS is combined with electric power steering it is possible to control the front wheel angle and steering wheel torque simultaneously.

Until then we try making the instability less disturbing. There are a few approaches.

- The most straightforward solution is probably to decrease the steering ratio. If it is made low enough the steering should become stable, but then the steering ratio is unacceptably low.
- Another approach is to limit how fast the wheels turn, thereby limiting the steering wheel torque. This could be done by low pass filtering. The experience from test drives is that the controller will feel very slow long before the steering feels stable.
- The nice properties of the left-right steering inspired a compromise. Allowing locally inverted steering it should be possible to stabilize the steering wheel while keeping the left-left-steering. An example is given by the following relation between steering wheel angle, and the reference signal, r (Figure 5.2).

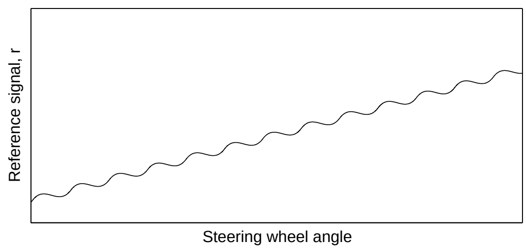

Figure 5.2. Relation between reference signal and steering wheel angle.

Practically, this means that the front wheels move back and forth. The major drawback is that the steering feels uneven.

Chapter 6

Autonomous Steering

In this chapter we look at the concept of autonomous steering. The aim is to design a trajectory tracking controller. This controller can then be used with different trajectories. Two examples are given in Section 6.4. We also look at the problem of saving a trajectory when driving forward, in order to track it backwards (Section 6.3).

6.1 Tracking

Let us assume that we know the current position of car and trailer, as well as the desired trajectory. How do we track it? Most trajectory tracking schemes in the literature seem to involve some kind of prediction control. The path is predicted for a couple of different input values and the different paths are evaluated. In our case, this would mean solving our nonlinear differential equation in each step. Since we have very limited computing capacity in the car, this would not work. We need a simpler control scheme.

When driving forward with a trailer, you mostly only consider the movement of the car, expecting the trailer to follow. The principle is illustrated in Figure 6.1. The driver indicates with the steering wheel how much he wants the car to turn. Since the system from car movement (turning radius) to trailer angle is stable, the trailer will follow the car in a nice way.

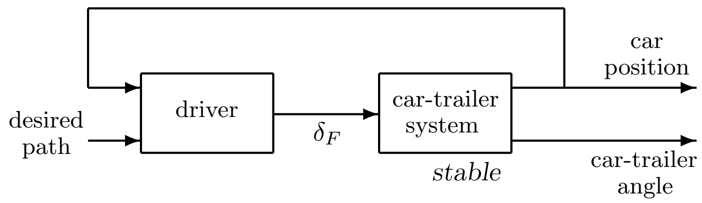

Figure 6.1. Driving forward with a trailer.

In Section 4.2, we showed that controlling the turning radius of the trailer would also stabilize the trailer angle, γ. This result inspires the trajectory tracking scheme in Figure 6.2. Comparing the current trailer position with the desired path, an autonomous steering function decides an appropriate turning radius for the trailer. To obtain this turning radius, the radius controller from Section 4.2 is used. Since this also stabilizes the car-trailer angle, the car position can normally be disregarded.

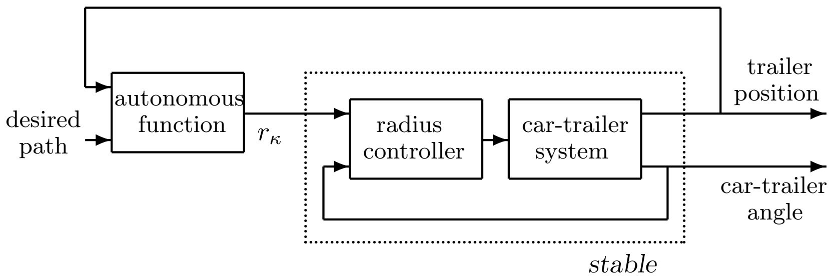

Figure 6.2. Trajectory tracking scheme.

Next, we need a way to decide the appropriate turning radius, given the desired path and the current trailer position. It seems natural to consider the point on the trajectory that is closest to the current position as our desired position. We define d_{pos} as the deviation from the desired position, and d_ψ as the deviation from the desired orientation of the trailer (as an angle). Ideally d_{pos} and d_ψ are both zero. The appropriate turning radius for the trailer is then given by the curvature of the trajectory,

$$r_\kappa = \kappa_{traj}.$$

If the deviations are not zero the turning radius should be adjusted accordingly,

$$r_\kappa = \kappa_{traj} \pm K_1\, d_{pos} + K_2\, d_\psi,$$

where the sign in front of $K_1\, d_{pos}$ depends on whether the current position is to the left or to the right of the desired path.

6.2 Positioning

Our only way to decide the position of the trailer is to calculate from measurements of rear wheel speed, v_R, front wheel angle, δ_F and car-trailer angle, γ. For this purpose we will use the model from Section 2.5. We define the initial position and orientation of the car as zero.

$$\hat{x}_1(0) = \hat{y}_1(0) = \hat{\psi}_1(0) = 0,$$

In each sample we calculate a new position using

$$\begin{aligned}
\hat{x}_1(k+1) &= \hat{x}_1(k) + h\, v_R(k)\, \cos\hat{\psi}_1(k) \\
\hat{y}_1(k+1) &= \hat{y}_1(k) + h\, v_R(k)\, \sin\hat{\psi}_1(k) \\
\hat{\psi}_1(k+1) &= \hat{\psi}_1(k) + h\, \frac{v_R(k)}{l_1}\, \tan\delta_F(k),
\end{aligned}$$

where h is the sample time. The trailer position can then be decided using the geometric relations from Section 2.3.

$$\begin{aligned}
\hat{x}_2 &= \hat{x}_1 - l_{12}\, \cos\hat{\psi}_1 - l_2\, \cos\hat{\psi}_2 \\
\hat{y}_2 &= \hat{y}_1 - l_{12}\, \sin\hat{\psi}_1 - l_2\, \sin\hat{\psi}_2 \\
\hat{\psi}_2 &= \hat{\psi}_1 - \gamma.
\end{aligned}$$

6.3 Saving a Trajectory

We now have a controller to track a given trajectory, but we have yet to decide how to represent and store such a trajectory. The choice of representation is especially important when the trajectory is being saved while driving.

A nice choice of representation would be to use some kind of local polynomials, like splines. That way curvature, direction and position of the trajectory would fit together. However, finding appropriate polynomials given the sampled position of the trailer seems difficult. For example, spline interpolation involves solving large matrices in real-time. Again, this might be a bit too much for the computing unit of the car.

Instead, we choose to save curvature, orientation angle and position at certain points. Between these points we use linear interpolation to get a trajectory. This makes computations easy and all signals continuous.

More accurately, we save the trailer position and angle in points with equal distances, T_i. With distance we here mean distance in space, not in time. In those same points the current curvature of the trailer is calculated from measurements using (4.7). When tracking the trajectory, two points T_i and T_{i+1} are active at each moment in time. The trajectory is represented (locally) by the line

$$\mathbf{r}(\tau) = (1-\tau)\, \mathbf{r}^R_{T_i O} + \tau\, \mathbf{r}^R_{T_{i+1} T_i}.$$

The point on this line that is closest to the current trailer position is considered our desired position. With the corresponding value of τ, we calculate desired values for ψ and κ.

This method seems to work well in practice. Figure 6.3 compares the saved forward trajectory to the trajectory chosen by the controller. Note that the trailer positions in the plot are estimated. Errors in the estimation are not regarded, but other tests imply that they are quite small.

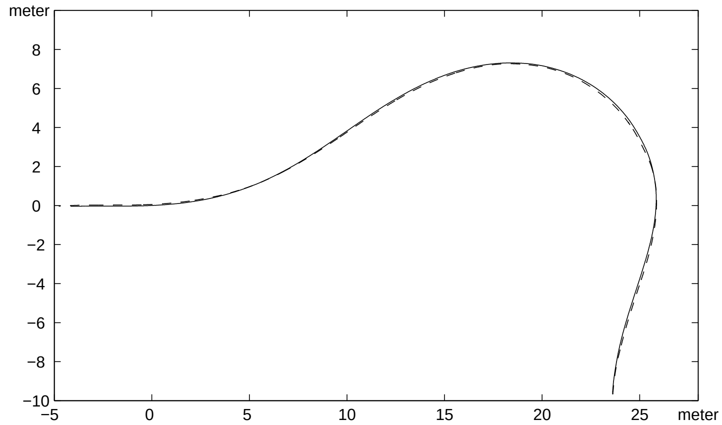

Figure 6.3. Tracking a saved trajectory. The solid line is the trajectory saved when driving forward, and the dashed line shows how the controller tracks this trajectory when reversing.

6.4 Special Trajectories

Except for tracking a saved trajectory, the tracking controller can be used with some constant standard trajectories. Figure 6.4 shows the tracking of a parking trajectory. This example shows one of the strengths of the control approach. An initial oversteer is noticed and compensated by the controller. The result is that the trailer is parked nicely in the correct position. The trajectory tracking controller can also be used for reversing along a straight line.

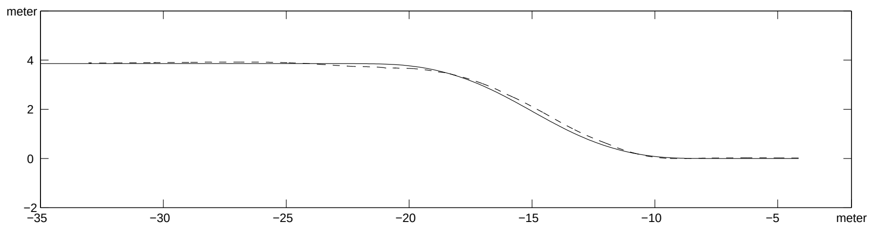

Figure 6.4. Tracking a parking trajectory.

Chapter 7

Conclusions and Future Work

7.1 Modelling

In Chapter 2 we derived a simple model for car and trailer. Data from test drives shows that the model works well for low speeds. We can also note that the model reflects several of the system characteristics that we know from practice. If a model for higher speeds is required, the one in Appendix C can be used, but first it has to be validated and the parameters in it have to be estimated.

To enable use with different trailers, a method to estimate the trailer length was presented in Section 2.7. Initial tests imply that it is effective, but more tests are desirable. It would also be interesting to see how sensitive the controllers are to errors in the trailer length. Perhaps it would be possible to start the controllers with a guessed trailer length and then switch to the estimated value when the estimation is done.

7.2 Reference Tracking Controllers

In Chapter 4, a few reference tracking controllers were designed. It was shown theoretically and in test drives that the controllers could stabilize the trailer. It was also possible to get left-left steering and to avoid the risk of jackknifing.

Due to the torque feedback discussed in Chapter 5, the reference tracking controllers are not appropriate for use with AFS as it works today. However, the problem with the feedback can be solved by adding a brake or electric power steering. It would be interesting to test the controllers with such a solution. The question is how many would prefer the controlled system to driving without assistance.

With electric power steering, it would also be possible to test the potential of an assisting steering wheel torque (see Section 4.5.3).

7.3 Trajectory Tracking

In Chapter 6 methods to save and track a trajectory was discussed. The basis for the trajectory tracking was the turning radius controller from Chapter 4. In test drives we saw that the trajectory tracking seems to work nicely. With small adjustments it could be used for presentational purposes.

It would be worth to work more on the parking function. Using sensors to measure the distance to surrounding cars, it would be possible to adapt the trajectory accordingly. That would create a truly useful assistance function.

Appendix A

Derivations

A.1 Deriving the Differential Equations

The purpose of this section is to derive differential equations describing the car-trailer system, from the constraint equations in Sections 2.3 and 2.4. From (2.4b) and (2.5) we get the relations

$$\dot{x}_1 = v_R \cos\psi_1$$

$$\dot{y}_1 = v_R \sin\psi_1$$

Differentiating (2.2a), (2.2b) and combining them with (2.4a) we get

$$\begin{aligned} & l_1\,\dot{\psi}_1 \sin\psi_0 \sin\psi_1 + l_1\,\dot{\psi}_1 \cos\psi_0 \cos\psi_1 - \sin\psi_0\,\dot{x}_1 + \cos\psi_0\,\dot{y}_1 \\ & \quad = l_1\,\dot{\psi}_1 \cos(\psi_0 - \psi_1) - \dot{x}_1 \sin\psi_0 + \dot{y}_1 \cos\psi_0 = 0 \end{aligned}$$

Using (2.1) and the newly found expressions for $\dot{x}_1$ and $\dot{y}_1$ we can rewrite this

$$l_1\,\dot{\psi}_1 \cos\delta_F = v_R \sin\delta_F.$$

For $|\delta_F| < \pi/2$ this can be rewritten

$$\dot{\psi}_1 = \frac{v_R}{l_1} \tan\delta_F.$$

(A model that can handle $\delta_F = \pi/2$, should be based on the speed of the front wheels rather than that of the rear wheels.) Differentiating (2.3a) and (2.3b) and combining them with (2.4c) we get

$$\dot{\psi}_2 = -\frac{v_R\,l_{12}}{l_1\,l_2} \cos\gamma \tan\delta_F + \frac{v_R}{l_2} \sin\gamma.$$

The derivative of the trailer angle γ is simply

$$\dot{\gamma} = \dot{\psi}_1 - \dot{\psi}_2 = \left(\frac{v_R}{l_1} + \frac{v_R\,l_{12}}{l_1\,l_2} \cos\gamma\right) \tan\delta_F - \frac{v_R}{l_2} \sin\gamma.$$

A.2 Deriving the Trailer Velocity

For the design of the controller in Section 4.2 we needed an expression for the turning radius of the trailer. Since the angular velocity is already known we only need the forward velocity, that is

$$\dot{x}^{L_2}_{P_2O} = \cos\psi_2 \cdot \dot{x}^{R}_{P_2O} + \sin\psi_2 \cdot \dot{y}^{R}_{P_2O}.$$

Using (2.3a) and (2.3b) we find

$$\dot{x}^{L_2}_{P_2O} = \dot{x}_1 \cdot \cos\psi_2 + \dot{y}_1 \sin\psi_2 + l_{12}\,\dot{\psi}_1 \sin\gamma.$$

Using the equations from the previous section this can be rewritten

$$\dot{x}^{L_2}_{P_2O} = v_R \cos\gamma + \frac{v_R\, l_{12}}{l_1}\ \tan\delta_F \sin\gamma.$$

A reasonable measure for the turning radius, ρ_T, of the trailer is given by the curvature, κ

$$\kappa = \frac{\dot{\psi}_2}{\dot{x}^{L_2}_{P_2O}} = \pm\frac{1}{\rho_T} = \frac{l_1\ \sin\gamma - l_{12}\ \tan\delta_F\ \cos\gamma}{l_1\, l_2\ \cos\gamma + l_2\, l_{12}\ \tan\delta_F\ \sin\gamma}.$$

Notice that the curvature is positive when turning left and negative when turning right.

Appendix B

Model with Lateral Slip

In this section we derive a model that includes lateral wheel slip. The mechanical laws that are needed can be found in any textbook on mechanics, for example [4]. We use the coordinate frames from Section 2.2, but also introduce some new. To get easier equations we build the model around the coordinates of the car hook $(x, y)^T$. We also introduce the longitudinal and lateral velocities of the towing hook

$$v_x = \dot{x}\, \cos\psi_1 + \dot{y}\, \sin\psi_1$$

$$v_y = -\dot{x}\, \sin\psi_1 + \dot{y}\, \cos\psi_1,$$

and the accelerations

$$a_x = \ddot{x}\, \cos\psi_1 + \ddot{y}\, \sin\psi_1$$

$$a_y = -\ddot{x}\, \sin\psi_1 + \ddot{y}\, \cos\psi_1,$$

noting that

$$\dot{v}_x = a_x + \omega_1\, v_y$$

$$\dot{v}_y = a_y - \omega_1\, v_x.$$

Geometric constants l for the car, and λ for the trailer are defined as shown Figure B.1. Furthermore, we define $l_C = l_R + l_{CR}$, $l_F = l_C + l_C F$ and $\lambda_R = \lambda_C + \lambda_{CR}$.

Our aim is to create a state space model, with inputs v_x, $\dot{v}_x$, δ_F and state space variables x, y, ψ_1, ψ_2, v_y, ω_1 and ω_2.

B.1 Slip Angles

Lateral slip is an effect of cornering. To turn, a car needs to be affected by lateral forces. These are provided by the friction when the wheels slip.

We model each wheel pair with a single wheel placed on the central axis of the car (or trailer). The slip angle, β_i, is defined as the angle between the central axis of the wheel and the actual direction of the wheel movement. For reasonably small

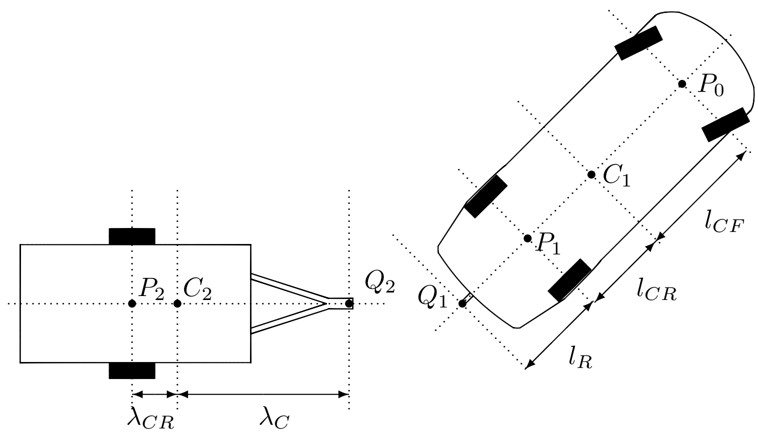

Figure B.1. Points in the different frames.

slip angles it is a good approximation to assume that the lateral friction force in the tyre is proportional to the slip angle,

$$F_i = k_i \beta_i. \tag{B.1}$$

A more thorough discussion of wheel slip and of tyre properties in general can be found in [3]. Assuming that the wheels moved without lateral slip, gave us (2.4). With a slip angle β_i, similar derivations yield

$$-\sin(\psi_i + \beta_i)\, \dot{x}^R_{P_iO} + \cos(\psi_i + \beta_i)\, \dot{y}^R_{P_iO} = 0.$$

Using (2.2a) and (2.2b) from Section 2.3 and the definitions of v_x and v_y we get

$$\omega_1 = \frac{v_x}{l_F} \tan(\delta_F + \beta_0) - \frac{v_y}{l_F}$$

and

$$v_y + l_R\, \omega_1 = v_x \, \tan \beta_1.$$

With these equations we can express β_0 and β_1 in terms of the state space variables,

$$\beta_0(\mathbf{z}) = \arctan\left(\frac{l_F\, \omega_1 + v_y}{v_x}\right) - \delta_F \tag{B.2}$$

$$\beta_1(\mathbf{z}) = \arctan\left(\frac{v_y + l_R\, \omega_1}{v_x}\right). \tag{B.3}$$

A similar expression can be derived for the trailer

$$\beta_2(\mathbf{z}) = \arctan\left(\frac{v_x \, \sin\gamma + v_y \, \cos\gamma + \lambda_R\, \omega_2}{v_x \, \cos\gamma + v_y \, \sin\gamma}\right) \tag{B.4}$$

From now on these expressions are intended when we write β_0, β_1 and β_2.

B.2 Dynamics

In this section we use the basic laws of mechanics to derive the equations of motion for car and trailer. For the coordinates of the car center of gravity, $(x^R_{C_1O}, y^R_{C_1O})$, it holds that

$$x^R_{C_1O} - l_C \cos\psi_1 - x = 0$$

$$y^R_{C_1O} - l_C \sin\psi_1 - y = 0$$

and for the trailer

$$x - \lambda_C \cos\psi_2 - x^R_{C_2O} = 0$$

$$y - \lambda_C \sin\psi_2 - y^R_{C_2O} = 0.$$

Differentiating gives the corresponding relation of the accelerations. For the car

$$a_{C_1,y} = a_y + l_C\,\dot\omega_1$$

and for the trailer

$$\ddot x \cos\psi_2 + \ddot y \sin\psi_2 + \lambda_C\,\omega_2^2 = a_{C_2,x}$$

$$-\ddot x \sin\psi_2 + \ddot y \cos\psi_2 - \lambda_C\,\dot\omega_2 = a_{C_2,y}.$$

Together with the definitions of a_x and a_y, this yields

$$a_{C_2,x} = a_x \cos\gamma - a_y \sin\gamma + \lambda_C\,\omega_2^2$$

$$a_{C_2,y} = a_x \sin\gamma + a_y \cos\gamma - \lambda_C\,\dot\omega_2.$$

For the centers of gravity, we can use Newton second law of motion, $F = ma$. For the car, we only look at the lateral axis, since longitudinal movement is considered a measured input. The external forces in this case are the slip forces from the wheels and the lateral force that the trailer causes F_{y_1}. This gives us

$$F_{y_1} = m_1\,a_{C_1,y} = m_1\,a_y - k_0\,\beta_0 \cos\delta_F - k_1\,\beta_1 + m_1\,l_C\,\dot\omega_1$$

For the trailer the external forces are the slip force (along the lateral axis) and the car pulling force. We define F_{x_2} as the longitudinal pulling force and F_{y_2} as the lateral, noting that

$$F_{x_2} \sin\gamma - F_{y_2} \cos\gamma = F_{y_1}. \tag{B.5}$$

For the trailer Newton's law yields

$$F_{x_2} = m_2\,a_{C_2,x} = m_2\,a_x \cos\gamma - m_2\,a_y \sin\gamma + m_2\,\lambda_C\,\omega_2^2$$

$$F_{y_2} = m_2\,a_{C_2,y} = m_2\,a_x \sin\gamma + m_2\,a_y \cos\gamma - m_2\,\lambda_C\,\dot\omega_2 - k_2\,\beta_2.$$

Euler's equation $J\,\dot\omega = \sum M_i$ gives us relations for the angular accelerations

$$J_1\,\dot\omega_1 = l_{CF}\,k_0\,\beta_0 \cos\delta_F - l_{CR}\,k_1\,\beta_1 - l_C\,F_{y_1}$$

$$J_2\,\ddot\psi_2 = \lambda_C\,F_{y_2} - \lambda_{CR}\,k_2\,\beta_2.$$

Also using (B.5), we get

$$
\begin{aligned}
(J_1 + m_1\, l_C^2)\, \dot{\omega}_1 &= (l_{CF} + l_C)\, k_0\, \beta_0 \cos\delta_F - (l_{CR} + l_C)\, k_1\, \beta_1 - m_1\, l_C\, a_y \\
(J_2 + m_2\, \lambda_C^2)\, \dot{\omega}_2 &= m_2\, \lambda_C\, a_x\, \sin\gamma + m_2\, \lambda_C\, a_y\, \cos\gamma - (\lambda_{CR} + \lambda_C)\, k_2\, \beta_2 \\
F_{y_1} &= -m_2\, a_y + \sin\gamma(m_2\, \lambda_C\, \omega_2^2) + \cos\gamma(m_2\, \lambda_C\, \dot{\omega}_2 + k_2\, \beta_2).
\end{aligned}
$$

Replacing F_{y_1} yields

$$
\begin{aligned}
(m_1 + m_2)\, a_y =& k_0\, \beta_0 \cos\delta_F + k_1\, \beta_1 + k_2\, \beta_2\, \cos\gamma \\
& - m_1\, l_C\, \dot{\omega}_1 + m_2\, \lambda_C\, \dot{\omega}_2\, \cos\gamma + m_2\, \lambda_C\, \omega_2^2\, \sin\gamma. \qquad \text{(B.6)}
\end{aligned}
$$

It remains to solve the following system

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{l_{CF} + l_C}{J_1 + m_1\, l_C^2}\, k_0\, \beta_0 \cos\delta_F - \frac{l_{CR} + l_C}{J_1 + m_1\, l_C^2}\, k_1\, \beta_1 - \frac{m_1\, l_C}{J_1 + m_1\, l_C^2}\, a_y \\
\dot{\omega}_2 &= \frac{m_2\, \lambda_C}{J_2 + m_2\, \lambda_C^2}\, a_x\, \sin\gamma + \frac{m_2\, \lambda_C}{J_2 + m_2\, \lambda_C^2}\, a_y\, \cos\gamma - \frac{\lambda_{CR} + \lambda_C}{J_2 + m_2\, \lambda_C^2}\, k_2\, \beta_2 \\
(m_1 + m_2)\, a_y =& k_0\, \beta_0 \cos\delta_F + k_1\, \beta_1 + k_2\, \beta_2\, \cos\gamma \\
& - m_1\, l_C\, \dot{\omega}_1 + m_2\, \lambda_C\, \dot{\omega}_2\, \cos\gamma + m_2\, \lambda_C\, \omega_2^2\, \sin\gamma.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\left(\frac{m_1\, J_1}{J_1 + m_1\, l_C^2} + \frac{m_2\, J_2}{J_2 + m_2\, \lambda_C^2} + \frac{m_2^2\, \lambda_C^2}{J_2 + m_2\, \lambda_C^2}\, \sin^2\gamma \right) a_y = \\
&= \frac{J_1 - m_1\, l_C\, l_{CF}}{J_1 + m_1\, l_C^2}\, k_0\, \beta_0 \cos\delta_F + \frac{J_1 + m_1\, l_C\, l_{CR} + 2\, m_1\, l_C^2}{J_1 + m_1\, l_C^2}\, k_1\, \beta_1 + \\
&+ \frac{J_2 - m_2\, \lambda_C\, \lambda_{CR}}{J_2 + m_2\, \lambda_C^2}\, k_2\, \beta_2\, \cos\gamma + \frac{m_2^2\, \lambda_C^2}{J_2 + m_2\, \lambda_C^2}\, a_x\, \sin\gamma\, \cos\gamma + m_2\, \lambda_C\, \omega_2^2\, \sin\gamma.
\end{aligned}
$$

From this we define $f(\mathbf{z})$ so that

$$a_y = f(\mathbf{z})$$

where $\mathbf{z}$ refers to the the state space variables. This gives us a state space model of the system

$$
\begin{aligned}
\dot{x} &= v_x\, \cos\psi_1 - v_y\, sinx \\
\dot{y} &= v_x\, \sin\psi_1 + v_y\, \cos x \\
\dot{\psi}_1 &= \omega_1 \\
\dot{\psi}_2 &= \omega_2 \\
\dot{v}_y &= f(\mathbf{z}) - \omega_1\, v_x \\
\dot{\omega}_1 &= \frac{l_{CF} + l_C}{J_1 + m_1\, l_C^2}\, k_0\, \beta_0(\mathbf{z}) \cos\delta_F - \frac{l_{CR} + l_C}{J_1 + m_1\, l_C^2}\, k_1\, \beta_1(\mathbf{z}) - \frac{m_1\, l_C}{J_1 + m_1\, l_C^2}\, f(\mathbf{z}) \\
\dot{\omega}_2 &= \frac{m_2\, \lambda_C}{J_2 + m_2\, \lambda_C^2}\, a_x\, \sin\gamma + \frac{m_2\, \lambda_C}{J_2 + m_2\, \lambda_C^2}\, f(\mathbf{z})\, \cos\gamma - \frac{\lambda_{CR} + \lambda_C}{J_2 + m_2\, \lambda_C^2}\, k_2\, \beta_2(\mathbf{z}).
\end{aligned}
$$

Note that everything in these equations is either a constant or a function of the state space variables.

Bibliography

[1] Glad, Torkel and Ljung, Lennart: *Control Theory – Multivariable and Nonlinear Methods*, Taylor & Francis, 2000.

[2] Glad, Torkel and Ljung, Lennart: *Modellbygge och simulering*, Studentlitteratur, 2004.

[3] Wong, Jo Yung: *Theory of Ground Vehicles*, John Wiley & Sons, 2001.

[4] Hahn, Hubert: *Rigid Body Dynamics of Mechanics – 1 Theoretical Basis*, Springer, 2002.

[5] Hansson, Kurt: *Föreläsningar i ordinära differentialekvationer, 2005*, URL: `http://www.mai.liu.se/~kuhan/kurser/NMAC09/`, 2005.

[6] Reinelt, Wolfgang; Klier, Willy; Reimann, Gerd; Schuster, Wolfgang and Grossheim, Reinhard: *Active Front Steering: Safety and Functionality, SAE Paper 2004-01-1101*, SAE World Congress, 2004.

Upphovsrätt

Detta dokument hålls tillgängligt på Internet — eller dess framtida ersättare — under 25 år från publiceringsdatum under förutsättning att inga extraordinära omständigheter uppstår.

Tillgång till dokumentet innebär tillstånd för var och en att läsa, ladda ner, skriva ut enstaka kopior för enskilt bruk och att använda det oförändrat för ickekommersiell forskning och för undervisning. Överföring av upphovsrätten vid en senare tidpunkt kan inte upphäva detta tillstånd. All annan användning av dokumentet kräver upphovsmannens medgivande. För att garantera äktheten, säkerheten och tillgängligheten finns det lösningar av teknisk och administrativ art.

Upphovsmannens ideella rätt innefattar rätt att bli nämnd som upphovsman i den omfattning som god sed kräver vid användning av dokumentet på ovan beskrivna sätt samt skydd mot att dokumentet ändras eller presenteras i sådan form eller i sådant sammanhang som är kränkande för upphovsmannens litterära eller konstnärliga anseende eller egenart.

För ytterligare information om Linköping University Electronic Press se förlagets hemsida `http://www.ep.liu.se/`

Copyright

The publishers will keep this document online on the Internet — or its possible replacement — for a period of 25 years from the date of publication barring exceptional circumstances.

The online availability of the document implies a permanent permission for anyone to read, to download, to print out single copies for his/her own use and to use it unchanged for any non-commercial research and educational purpose. Subsequent transfers of copyright cannot revoke this permission. All other uses of the document are conditional on the consent of the copyright owner. The publisher has taken technical and administrative measures to assure authenticity, security and accessibility.

According to intellectual property law the author has the right to be mentioned when his/her work is accessed as described above and to be protected against infringement.

For additional information about the Linköping University Electronic Press and its procedures for publication and for assurance of document integrity, please refer to its www home page: `http://www.ep.liu.se/`

© Olof Enqvist