Population growth — analysis of an age structure population model

Nina Håkansson

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Examensarbete: 20p
Level: D
Examiner: Vladimir Kozlov
Department of mathematics
Applied mathematics
Linköpings universitet
Supervisors: Vladimir Kozlov and Bengt Ove Turesson
Department of mathematics
Applied mathematics
Linköpings universitet
Uno Wennergren
Department of biology
Theory and Modeling
Linköpings universitet

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This report presents an analysis of a partial differential equation, resulting from population model with age structure. The existence and uniqueness of a solution to the equation are proved. We look at stability of the solution. The asymptotic behaviour of the solution is treated. The report also contains a section about the connection between the solution to the age structure population model and a simple model without age structure.

Nyckelord
Age structure, population model, partial differential equation, asymptotics, stability.
Abstract

This report presents an analysis of a partial differential equation, resulting from population model with age structure. The existence and uniqueness of a solution to the equation are proved. We look at stability of the solution. The asymptotic behaviour of the solution is treated. The report also contains a section about the connection between the solution to the age structure population model and a simple model without age structure.
# Contents

1 Introduction ................................................. 9  
1.1 Population models ........................................ 9  
1.1.1 Population models without age structure .......... 9  
1.1.2 Population models with age structure — time-independent case ........................................ 9  
1.1.3 Population models with age structure — time-dependent case ........................................ 10  
1.1.4 Other population models with age structure ...... 10  
1.2 Presentation of the model to study ..................... 11  
1.2.1 What to study ......................................... 12  

2 Existence and uniqueness of solutions .................... 13  
2.1 The integral equation ..................................... 13  
2.2 Existence and uniqueness of the solution .............. 15  

3 Stability of the model ........................................ 17  
3.1 Stability, considering m ................................... 17  

4 Asymptotics of $N$ — the time-independent case ........ 18  
4.1 Representation of $n(0, \cdot)$ ............................. 19  
4.1.1 Domain of convergence ................................ 19  
4.1.2 The Laplace transform of $n(0, \cdot)$ ............... 20  
4.1.3 Asymptotic behaviour of $Ln(0, \cdot)$ ............... 22  
4.1.4 The representation of $n(0, \cdot)$ .................... 22  
4.2 Asymptotics of $N$ ........................................ 24  

5 Asymptotics of $N$ — the time-dependent case .......... 26  
5.1 Upper bounds for $n(0, \cdot)$ ............................. 26  
5.2 Lower bounds for $n(0, \cdot)$ ............................. 29  
5.3 Upper and lower bounds for $N$ ........................... 31  
5.4 Comparison with the time-independent case .......... 32  

6 Comparison with the model without age structure ...... 33  
6.1 Biological analysis of the requirement $\sigma_0 > -1$ ........ 34  

7 Conclusions .................................................. 35  

A Banach theory ................................................ 37  
A.1 Definitions ................................................ 37  
A.2 The Banach fixed point theorem ......................... 37
1 Introduction

This section presents population models of different types. In the following sections we will analyse the age structure population models.

1.1 Population models

The number of individuals \( N \) in a population at time \( t \), from a given start time \( t = 0 \), can be described with differential equations of varying complexity and accuracy. The following models are presented in [1].

1.1.1 Population models without age structure

The change of the population can be described by the conservation equation:

\[
\frac{dN(t)}{dt} = \text{births} - \text{deaths} + \text{migration}.
\]

The simplest model occurs when there is no migration and the death and birth rates are proportional to \( N \). This model is frequently used in physics, chemistry and biology:

\[
\frac{dN(t)}{dt} = bN(t) - dN(t) \quad \text{with solution} \quad N(t) = N_0e^{(b-d)t},
\]

where \( N_0 = N(0) \) is the initial population, \( b \) the constant birth rate and \( d \) the constant death rate. This model predicts that \( N \) will increase as \( t \to \infty \) if \( b > d \).

It is reasonable to assume that a population will not increase forever; the environment is probably limiting \( N \). At a certain population size there will, for example, be insufficient resources to support a larger population. The logistic growth (suggested by Verhulst 1836) includes such a limiting part:

\[
\frac{dN(t)}{dt} = rN(t) \left( 1 - \frac{N(t)}{K} \right).
\]

Assume \( r > 0 \). If \( N_0 < K \) the population will increase, but it will never exceed \( N = K \). If \( N_0 > K \) the population will instead decrease. The constant \( K \) is called the carrying capacity of the environment and is the largest \( N \) that the environment can support. This model predicts that \( N \to K \) as \( t \to \infty \).

A delay model compensates for the time it takes for new individuals to reach maturity. It has the form:

\[
\frac{dN(t)}{dt} = f(N(t), N(t-T)),
\]

where \( T \) is a constant called the delay. This model describes a population with a maturation period and then a birth rate independent of the age distribution of the population.

1.1.2 Population models with age structure — time-independent case

In a model with age structure, the death rate \( \mu \) and birth rate \( m \) depends on the age structure of the population. First we present a model where the birth and
death rate are time-independent. Consider \( n(a, t) \), the number of individuals of age \( a \) at time \( t \). If we look at an age group at age \( a \), from time \( t \) to \( t + dt \) the change of individuals in that age group is

\[
dn(a, t) = -\mu(a)n(a, t)dt.
\]

The left hand side can be expressed:

\[
dn(a, t) = n(a + da, t + dt) - n(a, t) = \frac{\partial n(a, t)}{\partial t}dt + \frac{\partial n(a, t)}{\partial a}da.
\]

Our \( a \) represents chronological age, therefore \( da = dt \). If the equation is divided with \( dt \) the result is a partial differential equation:

\[
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a)n(a, t), \quad a, t \geq 0.
\]

The newborn part of the population contributes only to \( n(0, t) \), which gives the boundary condition at \( a = 0 \):

\[
n(0, t) = \int_{0}^{\infty} m(a)n(a, t)da, \quad t \geq 0.
\]

For \( t = 0 \), we have a start population with a certain age distribution \( f(a) \) and the boundary condition at \( t = 0 \) is

\[
n(a, 0) = f(a), \quad a \geq 0.
\]

This model will be studied in the following sections.

### 1.1.3 Population models with age structure — time-dependent case

In a population where the environment changes over time, resulting from for example pollution, the death and birth rates for the population will probably change over time too. The model

\[
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a, t)n(a, t), \quad a, t \geq 0,
\]

describes this. Its boundary conditions are

\[
n(0, t) = \int_{0}^{\infty} m(a, t)n(a, t)da, \quad t \geq 0
\]

\[
n(a, 0) = f(a), \quad a \geq 0.
\]

This model will be examined more closely in the following sections.

### 1.1.4 Other population models with age structure

In a population there can be competition and rivalry among individuals of the same species and \( \mu \) may also depend on \( N = \int_{0}^{\infty} n(a, t)da \), the total number of individuals. The model

\[
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a, t, N)n(a, t)
\]
describes this. Its boundary conditions are:

\[ n(0, t) = \int_0^\infty m(a, t, N)n(a, t) \, dt, \quad t \geq 0 \]
\[ n(a, 0) = f(a), \quad a \geq 0. \]

A population can be influenced by other populations and the birth rate for \( n \), \( m_n \), may also depend on \( P = \int_0^\infty p(a, t) \, da \), the total number of individuals in another population. The model

\[
\begin{cases}
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} &= -\mu_n(a, t, P)n(a, t) \\
\frac{\partial p(a, t)}{\partial t} + \frac{\partial p(a, t)}{\partial a} &= -\mu_p(a, t, N)p(a, t)
\end{cases}
\]

describes this. Its boundary conditions are

\[
\begin{cases}
n(0, t) = \int_0^\infty m_n(a, t)n(a, t) \, dt, & n(a, 0) = f_n(a), \\
p(0, t) = \int_0^\infty m_p(a, t)p(a, t) \, dt, & p(a, 0) = f_p(a).
\end{cases}
\]

1.2 Presentation of the model to study

We will for the most part study the differential equation from section 1.1.3:

\[
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a, t)n(a, t),
\]

with the boundary conditions

\[ n(0, t) = \int_0^\infty m(a, t)n(a, t) \, dt, \quad n(a, 0) = f(a). \]

The function \( n(a, t) \), the number of individuals of age \( a \) at time \( t \), is defined for \( 0 \leq t < \infty \). It is positive and assumed to be continuous. The growth of \( n(a, t) \) is assumed to be at most exponential.

The death rate \( \mu(a, t) \), defined for \( a, t \geq 0 \), is at the lowest no deaths and at most all individuals dead, so \( \mu(a, t) \in [0, 1] \) for all \( a, t \). Since no individual can become infinitely old, the death rate becomes equal to 1 for large \( a \). That is, \( \mu(a, t) = 1, \quad a > A_\mu \) for some constant \( A_\mu \).

The birthrate \( m(a, t) \), defined for \( a, t \geq 0 \), is a positive function. It has compact support since the birthrate for old individuals are 0. That is \( m(a, t) = 0, \quad a > A_m \) for some constant \( A_m \). We also assume that \( m \) is bounded by some constant \( M \).

Since \( f \) describes the start population, it is positive, bounded and has compact support. We will assume that the start population contains some individuals young enough to eventually have children, otherwise we get \( n(0, t) = 0 \) for all \( t \).

From a biological perspective it is interesting to study the increase and decrease of the total population. Therefore we will also study \( N(t) = \int_0^\infty n(a, t) \, da \), the number of individuals at time \( t \).
1.2.1 What to study

In section 2 we will prove that the differential equation (2) has exactly one solution with at most exponential growth. This is proved with help from the Banach fixed point theorem. Therefore we will have to transform the differential equation to an integral equation. We will also consider the stability of the model in section 3, using a corollary to Banach’s theorem.

The asymptotics of the solution can be studied in two ways. In section 4 we will use the Laplace transform. This method requires extra assumptions on the functions and is too difficult for the time-dependent case, so we will use it on the time-independent case. For the time-dependent case, in section 5, we will use the second method. The asymptotics will be estimated using upper and lower bounds. Section 6 compares the solution of the model with age structure with the solution to the model without age structure (1).
2 Existence and uniqueness of solutions

We will prove that the age structure model has exactly one solution, with at most exponential growth. The differential equation (2),(3) described in section 1.2, will be transformed into an integral equation of the form

\[ n(0, t) = (Kn)(t) + F(t) = (Tu)(t), \quad t \geq 0, \]

so that we can apply the Banach fixed point theorem (Appendix A.2) to show that there exists a unique solution to the integral equation.

2.1 The integral equation

In this section we will prove the following theorem.

**Theorem 1** A function \( n(a, t) \), that solves the problem (2),(3), satisfies

\[
\begin{align*}
n(a, t) &= \begin{cases} 
  f(a - t)e^{-\int_{a}^{t} \mu(v, v + t - a) \, dv}, & a \geq t \\
  n(0, t-a)e^{\int_{a}^{t} \mu(v, v + t - a) \, dv}, & a < t 
\end{cases} 
\end{align*}
\]

where \( n(0, t) \) solves the integral equation:

\[
\begin{align*}
n(0, t) &= \int_{0}^{t} m(a, t) n(0, t-a)e^{-\int_{a}^{t} \mu(v, v + t - a) \, dv} \, da \\
&\quad + \int_{t}^{\infty} m(a, t) f(a - t)e^{-\int_{a}^{t} \mu(v, v + t - a) \, dv} \, da.
\end{align*}
\]

If \( m \) and \( f \) are differentiable, then \( n(a, t) \) described by (4),(5) solves problem (2),(3).

**Proof:** To obtain this result, we start by transforming the equation into an ordinary differential equation. Let \( a \) and \( t \) be functions of \( x \) and \( y \):

\[
a = x + y, \quad t = x - y.
\]

Then the partial differential equation can be rewritten as an ordinary differential equation since

\[
\frac{dn(x + y, x - y)}{dx} = \frac{\partial n(a, t)}{\partial t} \frac{dt}{dx} + \frac{\partial n(a, t)}{\partial a} \frac{da}{dx} = \frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a}.
\]

This differential equation has the solution

\[
n(x + y, x - y) = C(2y)e^{-\int_{x}^{t} \mu(s+y, s-y) \, ds},
\]

where \( C \) is to be determined. With \( a \) and \( t \) as variables and \( v = s + \frac{a-t}{2} \), we get

\[
n(a, t) = C(a - t)e^{-\int_{a}^{t} \mu(v, v + t - a) \, dv}.
\]

To determine the function \( C \), let \( t = 0 \) in the expression above:

\[
n(a, 0) = C(a) = f(a).
\]
For $a \geq t$, we get
\[ n(a, t) = f(a - t)e^{-\int_a^t \mu(v, v + t - a) \, dv}. \]

This proves equation (4) for $a \geq t$. For the case $t < a$, we need to find $C(t)$ for negative $t$. Let $a = 0$ in equation (6). This gives us
\[ C(-t) = n(0, t)e^{\int_0^t \mu(v, v + t) \, dv}. \]

If we now replace $t$ with $t - a$, we get
\[ C(a - t) = n(0, t - a)e^{\int_a^t \mu(v, v + t - a) \, dv}. \]

So for $a < t$, we get the following representation of $n(a, t)$:
\[ n(a, t) = n(0, t - a)e^{\int_a^t \mu(v, v + t - a) \, dv} \]
and equation (4) is proved also for $a < t$.

All functions are known in these two expressions for $n(a, t)$ except $n(0, t - a)$.

To find $n(0, \cdot)$, we use the boundary condition at $a = 0$,
\[ n(0, t) = \int_0^\infty m(a)n(a, t) \, da. \tag{7} \]

If we insert the expression for $n(a, t)$ from equation (4) into (7), we obtain the integral equation (5).

Now let us consider the last part of Theorem 1. If $m$ and $f$ are differentiable functions, then substituting $t - a = x$ we have
\[
\begin{align*}
n(0, t) &= \int_0^t m(t - x, t)n(0, x)e^{-\int_0^x \mu(v, v + x) \, dv} \, dx \\
&\quad + \int_0^A m(x + t, t)f(x)e^{-\int_x^t \mu(v, v + x) \, dv} \, dx
\end{align*}
\]
is clearly differentiable since the right side is differentiable. From (4) it follows that $n$ is differentiable. \hfill \square

To make the equations more readable, we will sometimes write only $\mu$ when what we really mean is $\mu(v, v + t - a)$. The integral equation (5) can be rewritten as $n(0, t) = (Tn)(t)$, if $T$ is defined as
\[ (Tn)(t) = (Kn)(t) + F(t), \quad t \geq 0, \]
where
\[ (Kn)(t) = \int_0^t m(a, t)n(0, t - a)e^{-\int_0^a \mu(v, v + t - a) \, dv} \, da \]
and
\[ F(t) = \int_t^\infty m(a, t)f(a - t)e^{-\int_a^t \mu(v, v + t - a) \, dv} \, da. \]
2.2 Existence and uniqueness of the solution

In this section, we will prove that the integral equation (5) has exactly one solution at most exponential growth.

Let the space $B_\Lambda$ be the space of all continuous functions $u$ defined on $[0, \infty)$ such that $u(t) = O(e^{\Lambda t})$ as $t \to \infty$. The norm in $B_\Lambda$ is defined by

$$
\|u\|_\Lambda = \sup_{t \geq 0} |u(t)|e^{-\Lambda t}.
$$

**Theorem 2** The equation $n(0,t) = (Tn)(t)$, $t \geq 0$ has exactly one solution in $B_\Lambda$ for a sufficiently large $\Lambda$.

The theorem follows from Banach’s theorem (Theorem 34 in the appendix) if we establish that (i) $B_\Lambda$ is a Banach space and (ii) that $T$ is a contraction on $B_\Lambda$.

**Proposition 3** The space $B_\Lambda$ is a Banach space for all $\Lambda$.

**Proof:** To prove that $B_\Lambda$ is a Banach space we need to prove that $\| \cdot \|_\Lambda$ is in fact a norm and that the space is complete with respect to the metric defined by the norm.

We start by checking that $\| \cdot \|_\Lambda$ is a norm. The norm is obviously non-negative. Only zero has the norm equal to zero, since if $\|u\|_\Lambda = 0$, then $u$ must be identically zero. Conversely $\|0\|_\Lambda = 0$. The norm is homogeneous, since

$$
\|au\|_\Lambda = \sup_{t \geq 0} |au(t)|e^{-\Lambda t} = \sup_{t \geq 0} |a||u(t)|e^{-\Lambda t} = |a|\|u\|_\Lambda.
$$

The triangle inequality holds, since

$$
\|u + v\|_\Lambda = \sup_{t \geq 0} |u(t) + v(t)|e^{-\Lambda t} \leq \sup_{t \geq 0} (|u(t)| + |v(t)|)e^{-\Lambda t}
$$

$$
\leq \sup_{t \geq 0} |u(t)|e^{-\Lambda t} + \sup_{t \geq 0} |v(t)|e^{-\Lambda t} = \|u\|_\Lambda + \|v\|_\Lambda.
$$

So $\| \cdot \|_\Lambda$ is a norm.

To prove that $B_\Lambda$ is a complete space, set $v_i(t) = e^{-\Lambda t}u_i(t)$, where $(u_i) \subset B_\Lambda$ is a Cauchy sequence. Then $(v_i)$ is a Cauchy sequence in $B_0$. Indeed,

$$
\|u_m - u_n\|_\Lambda = \sup_{t \geq 0} |u_m(t) - u_n(t)|e^{-\Lambda t} = \sup_{t \geq 0} |v_m(t) - v_n(t)| = \|v_m - v_n\|.
$$

So $B_\Lambda$ with the norm $\| \cdot \|$ is complete since $B_0 = C[0, \infty)$ with the ordinary supremum norm is complete. □

**Proposition 4** The operator $T$ is a contraction on $B_\Lambda$ for $\Lambda$ sufficiently large.

**Proof:** Since $m$ and $\mu$ are bounded and non-negative, we have

$$
|m(a,t)e^{-\int_a^t \mu(v,v+t-a)dv}| \leq M.
$$

The following inequality comes in handy:

$$
n(0,t)e^{-\Lambda t} \leq \sup_{t \geq 0} |n(0,t)|e^{-\Lambda t} = \|n(0, \cdot)\|
$$

15
For the mapping $T$ we observe:

\[
|(Tn)(t)|e^{-\Lambda t} = \left| \int_0^t m(a,t)e^{-\int_0^a \mu \, dv} n(0, t-a) \, da \right|
+ \left| \int_0^\infty m(a,t)e^{-\int_0^a \mu \, dv} \, da \right| e^{-\Lambda t}
\leq \int_0^t M|n(0, t-a)|e^{-\Lambda(t-a)} e^{-\Lambda a} \, da + \int_t^\infty M|f(a-t)|e^{-\Lambda a} \, da
\leq \int_0^t M\|(n(0, \cdot))\|_\Lambda e^{-\Lambda a} \, da + MN(0).
\]

This gives us

\[
\|Tn\|_\Lambda = \sup_{t \geq 0} |(Tn)(t)|e^{-\Lambda t} = \sup_{t \geq 0} M\|(n(0, \cdot))\|_\Lambda \left( \frac{1}{\Lambda} - \frac{e^{-\Lambda t}}{\Lambda} \right) + MN(0)
\leq \alpha \|(n(0, \cdot))\|_\Lambda + MN(0),
\]

where $\alpha = M/\Lambda$. We have proved that $T : B_\Lambda \rightarrow B_\Lambda$. If we choose $\Lambda > M$ then $\alpha < 1$. We have

\[
\|(Tn_1 - Tn_2)\|_\Lambda = \sup_{t \geq 0} \left| \int_0^t m(a,t)e^{-\int_0^a \mu \, dv} (n_1(0, t-a) - n_2(0, t-a)) \, da \right|
\leq \alpha \|n_1(0, \cdot) - n_2(0, \cdot)\|_\Lambda.
\]

This shows that that $T$ is a contraction. \qed

Finally, since $B_\Lambda$ is a Banach space and $T$ is a contraction for $\Lambda > M$ Theorem 2 is proved.
3 Stability of the model

To examine the stability of the model (2),(3), we use Corollary 35 to the fixed-point theorem in Appendix A.

3.1 Stability, considering \(m\)

If \(m\) is considered as a parameter we have the integral equation

\[
n_m(0, t) = (T_m n_m)(t).
\]

We will show that the model is stable with respect to small disturbances in \(m\).

Let \(P\) be the space of continuous and bounded functions with compact support defined on \([0, \infty) \times [0, \infty)\), with the norm \(\|m\|_P = \sup_{a,t \geq 0} m(a, t)\). We want to prove the following theorem:

**Theorem 5** The integral equation \(n_m(0, t) = (T_m n_m)(t)\) has exactly one solution \(n_m\) and \(n_m\) tends to \(n_{m_0}\) in \(B_\Lambda\) as \(m\) tends to \(m_0\) in \(P\).

**Proof:** We have to check conditions (i)-(iii) described in the corollary to the Banach fixed point theorem.

(i) \(P\) is a metric space.

(ii) The operator \(T_m\) is a contraction on \(B_\Lambda\) with bound dependent on \(M\) and independent of \(m\).

(iii) Let \(\varepsilon > 0\). Then

\[
\|T_m n - T_{m_0} n\|_\Lambda \leq \sup_{t \geq 0} \int_0^t |m(a, t) - m_0(a, t)| |n(0, t - a) e^{-\Lambda (t - a)} e^{-\Lambda a} da
\]

\[
+ \sup_{t \geq 0} \int_{t}^\infty |m(a, t) - m_0(a, t)| |f(a - t) e^{-\Lambda (t - a)} e^{-\Lambda a} da
\]

\[
\leq \sup_{t \geq 0} \int_0^t \|m - m_0\|_P \|n(0, \cdot)\|_\Lambda e^{-\Lambda a} da
\]

\[
+ \sup_{t \geq 0} \int_{t}^\infty \|m - m_0\|_P \|f\|_{-\Lambda} e^{-\Lambda a} da
\]

\[
\leq \frac{\|m - m_0\|_P}{\Lambda} (\|n(0, \cdot)\|_\Lambda + \|f\|_{-\Lambda}) < \varepsilon
\]

if

\[
\|m - m_0\|_P < \delta = \frac{\Lambda \varepsilon}{\|n(0, \cdot)\|_\Lambda + \|f\|_{-\Lambda}}.
\]

This proves the theorem. \(\square\)
4 Asymptotics of \( N \) — the time-independent case

To determine the asymptotic behaviour of \( N \), \( N(t) = \int_{0}^{\infty} n(a, t) \, da \), the number of individuals in a population at time \( t \), in the time-independent case, we will use the Laplace transform. This method will require the extra assumption of bounded derivatives on \( \mu, m \) and \( f \).

If \( m \) and \( \mu \) are independent of time we have the differential equation:

\[
\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a)n(a, t)
\]

with the solution

\[
n(a, t) = \begin{cases} 
  f(a - t)e^{-\int_{a}^{\infty} \mu(v) \, dv}, & a \geq t \\
  n(0, t - a)e^{-\int_{0}^{a} \mu(v) \, dv}, & a < t
\end{cases}
\]

where \( n(0, t) \) satisfies the integral equation

\[
n(0, t) = \int_{0}^{t} m(a)n(0, t - a)e^{-\int_{a}^{\infty} \mu(v) \, dv} \, da + \int_{t}^{\infty} m(a)f(a - t)e^{-\int_{a}^{\infty} \mu(v) \, dv} \, da.
\]

We write \( A(t) \sim B(t) \) if \( \lim_{t \to \infty} A(t)/B(t) = 1 \).

**Theorem 6** The function \( N \) has the following asymptotic behaviour:

\[
N(t) \sim Ce^{\sigma_{0}t} \quad \text{if} \quad \sigma_{0} > -1 \\
N(t) \sim Cte^{-t} \quad \text{if} \quad \sigma_{0} = -1 \\
N(t) \sim Ce^{-t} \quad \text{if} \quad \sigma_{0} < -1
\]

where \( \sigma_{0} \) satisfies

\[
\int_{0}^{\infty} m(a)e^{-\sigma_{0}a - \int_{a}^{\infty} \mu(v) \, dv} \, da = 1.
\]

We first prove a special case of Riemann-Lebesgue lemma that will be useful later.

**Lemma 7** Let \( R(a) \) be a continuous function on \( \mathbb{R} \) with compact support and \( \sigma \in \mathbb{R} \). Then

\[
\int_{0}^{\infty} R(a)e^{-\sigma a}e^{-i\omega a} \, da \to 0, \quad \text{when} \quad |\omega| \to \infty
\]

and

\[
\int_{-\infty}^{0} R(a)e^{-\sigma a}e^{-i\omega a} \, da \to 0, \quad \text{when} \quad |\omega| \to \infty.
\]

**Proof:** The function \( a \mapsto R(a)e^{-\sigma a} \) is uniformly continuous. For \( \varepsilon > 0 \), let \( \alpha = \frac{4\varepsilon}{\pi + \pi} \). Choose \( \omega_{0} \) so that for \( \omega > \omega_{0} \) \( |R(a_{1})e^{-\sigma a_{1}} - R(a_{2})e^{-\sigma a_{2}}| < \alpha \) when \( |a_{1} - a_{2}| \leq \frac{2\pi}{\omega_{0}} \). Choose \( M \in \mathbb{N} \) so that \( \frac{4\omega}{\pi} < M \leq \frac{4\omega_{0}}{\pi} + 1 \).

We divide the integral into \( M \) integrals. Suppose that \( \omega \geq \omega_{0} \). Then

\[
\int_{0}^{\infty} R(a)e^{-\sigma a - i\omega a} \, da = \sum_{k=0}^{M-1} \int_{\frac{2\pi k}{\pi}}^{\frac{2\pi (k+1)}{\pi}} R(a)e^{-\sigma a}(\cos(\omega a) - \sin(\omega a)) \, da.
\]
For an arbitrary integral in the sum, we have
\[
\left| \int_{2\pi k}^{2\pi (k+1)} R(a)e^{-\sigma a} \cos(\omega a) \, da \right| \leq \left| \int_{2\pi k}^{2\pi (k+1)} R\left(\frac{2\pi k}{\omega}\right)e^{-\sigma \frac{2\pi k}{\omega}} \cos(\omega a) \, da \right|
\]
\[
+ \left| \int_{2\pi k}^{2\pi (k+1)} (R(a)e^{-\sigma a} - R\left(\frac{2\pi k}{\omega}\right)e^{-\sigma \frac{2\pi k}{\omega}}) \cos(\omega a) \, da \right| \leq 2\pi \frac{\alpha}{\omega}.
\]
The integral is bounded by \(\varepsilon\):
\[
\left| \int_{0}^{\infty} R(a)e^{-\sigma a} \cos(\omega a) \, da \right| \leq \sum_{0}^{M-1} 2\pi \frac{\alpha}{\omega} < \frac{(A\omega + 2\pi \alpha)}{\omega} < (A + 2\pi)\alpha = \varepsilon.
\]

The same calculations can be used for the term with \(\sin(a\omega)\). With the change of variables \(a = -x\) the same calculations also hold for the second integral. This proves Lemma 7.

4.1 Representation of \(n(0, \cdot)\)

In this section, we will find an asymptotic representation for \(n(0, \cdot)\). Set
\[
g(t) = m(t)e^{-\int_{0}^{t} \mu(v) \, dv}, \quad t \geq 0,
\]
and
\[
F(t) = \int_{t}^{\infty} m(a)f(a-t)e^{\int_{a-t}^{\infty} \mu(v) \, dv} \, da, \quad t \geq 0.
\]

**Theorem 8** The function \(n(0, \cdot)\) has the asymptotic representation:
\[
n(0, t) = Ke^{\sigma_0 t} + O(e^{(\sigma_0 - \delta)t}) \quad \text{as } t \to \infty
\]
where
\[
K = \left. \frac{\mathcal{L}F(\sigma)}{\sigma - (1 - \mathcal{L}g(\sigma))} \right|_{\sigma = \sigma_0}.
\]
where \(\mathcal{L}\) denotes the one-sided Laplace transform.

4.1.1 Domain of convergence

We start by examining where the Laplace transform of \(n(a, \cdot)\) converges. We know that \(n\) is a function with at most exponential growth:
\[
\|n(a, \cdot)\|_\Lambda = \sup_{t \geq 0} |n(a, t)|e^{-\Lambda t} = B < \infty,
\]
which gives us an upper bound for \(n\):
\[
n(a, t) \leq Be^{\Lambda t}.
\]

The Laplace transform of \(n\), \(\mathcal{L}n(a, s) = \int_{0}^{\infty} n(a, t)e^{-st} \, dt\), converges if \(\text{Re}(s) \geq \Lambda\).
4.1.2 The Laplace transform of \( n(0, \cdot) \)

We now find the Laplace transform of \( n(0, \cdot) \) by transforming the integral equation for \( n(0, \cdot) \):

\[
n(0, t) = \int_0^t m(a)n(0, t-a)e^{-f_a^t \mu(v) dv} da + \int_t^\infty m(a)f(a-t)e^{-f_a^t \mu(v) dv} da.
\]

To transform \( n(0, t) \), use (8) and (9) to rewrite the equation as

\[
n(0, t) = g(t) * n(0, t) + F(t).
\]

Observe that \( F \) is not identically zero, since some individuals are young enough to eventually have children. Therefore \( LF \) is not identically zero. Using the Laplace transform, we get

\[
\mathcal{L}n(0, s) = \frac{LF(s)}{1 - Lg(s)}
\]

where

\[
Lg(s) = \int_0^\infty m(t)e^{-st-f_0^t \mu(v) dv} dt
\]

and

\[
LF(s) = \int_0^\infty \int_t^\infty m(a)f(a-t)e^{-st-f_a^t \mu(v) dv} da dt.
\]

The term \( Ln(0, s) \) is defined where \( \text{Re}(s) > \Lambda \) but the right hand side of (10) has an extension to the whole complex plane. In [5], page 423, the following sufficient condition for inversion of the Laplace transformation is found. If there exist positive constants \( h, R_0 \) and \( k \) such that \( |Ln(0, s)| \leq h/|s|^k \) when \( |s| > R_0 \), we can find \( n(0, \cdot) \) by using the inversion formula for the Laplace transformation:

\[
n(0, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Ln(0, s)e^{st} ds,
\]

where \( \sigma > \Lambda \). In Lemma 12 we will see that the condition is satisfied. The integral can be calculated by integration along a rectangle and calculations of the residues of the poles. Therefore we need to determine the poles of the right hand side of (10) inside the area of integration.

**Lemma 9** The functions \( LF \) and \( Lg \) are entire.

**Proof:** Since \( m \) has compact support, we have

\[
LF(s) = \int_0^{A_m} \int_t^{A_m} m(a)f(a-t)e^{-f_a^t \mu(v) dv} e^{-st} da dt.
\]

By differentiating under the integral signs using the fact that \( e^{-st} \) is entire, it follows that \( LF \) is entire. The same argument applies to \( Lg \):

\[
Lg(s) = \int_0^{A_m} m(t)e^{-f_0^t \mu(v) dv} e^{-st} dt.
\]

Now we examine where \( Lg(s)=1 \).
Lemma 10 The function $1 - Lg$ has exactly one real zero, $\sigma_0$, and the zero is of order one.

Proof: Let $s = \sigma + i\omega$. Recall that $m$ is not allowed to be 0 everywhere. For $s = \sigma$, $Lg$ is a real-valued, continuous and decreasing function of $\sigma$:

$$Lg(\sigma_1) - Lg(\sigma_2) = \int_{0}^{\infty} m(t)e^{-\int_{0}^{t} \mu(v) \, dv}(e^{-\sigma t} - e^{-\sigma_2 t}) \, dt > 0 \quad \text{if } \sigma_1 < \sigma_2.$$

If $\sigma > M = \sup_{a \geq 0} m(a)$, then $Lg(s) < 1$:

$$Lg(\sigma) \leq \int_{0}^{\infty} Me^{-\sigma t} \, dt = \frac{M}{\sigma} < 1. \quad (11)$$

There exists $\sigma$ so that $Lg(\sigma) = \int_{0}^{\infty} m(a)e^{-\sigma t - \int_{0}^{t} \mu(v) \, dv} \, dt > 1$, since $m$ is not identically zero and the last factor is greater than $e^{-\sigma t}$.

We want to show that the zero at $s = \sigma_0$ is of order 1. We have

$$\frac{d}{da}(1 - Lg(\sigma)) = \int_{0}^{\infty} tm(t)e^{-\sigma t - \int_{0}^{t} \mu(v) \, dv} \, dt$$

and the last integral is not 0 since the integrand is non-negative and not identically zero. \(\square\)

Notice that (11) gives us the information $\sigma_0 \leq M$. We need to prove that there are no complex zeroes in the area of integration.

Lemma 11 There exists a $\delta$ such that the function $1 - Lg$ has no complex zero in the half plane $\text{Re}(s) > \sigma_0 - \delta$.

Proof: We consider five cases.

For $s = \sigma + i\omega$ and $\sigma > \sigma_0$, we have

$$\text{Re}(Lg(s)) = \int_{0}^{\infty} m(a)e^{-\int_{0}^{t} \mu(v) \, dv}e^{-\sigma a} \cos(a\omega) \, da$$

$$< \int_{0}^{\infty} m(a)e^{-\int_{0}^{t} \mu(v) \, dv}e^{-\sigma a} \, da = 1$$

For $s = \sigma_0 + i\omega$ and $\omega \neq 0$, we have, since $\cos(a\omega) < 1$ almost everywhere,

$$\text{Re}(Lg(s)) = \int_{0}^{\infty} m(a)e^{-\int_{0}^{t} \mu(v) \, dv}e^{-\sigma a} \cos(a\omega) \, da$$

$$< \int_{0}^{\infty} m(a)e^{-\int_{0}^{t} \mu(v) \, dv}e^{-\sigma a} \, da = 1.$$

For $s = \sigma_0 - 1 + i\omega$ there exist a $\omega_0$ such that $\text{Re}(Lg(s)) < 1$ for $|\omega| > |\omega_0|$. This follows from Lemma 7 and the fact that $m$ has compact support.

For $s = \sigma + i\omega$, where $\sigma_0 - 1 \leq \sigma \leq \sigma_0$ we can use Lemma 7 and the fact that $Lg(s)$ is a integral over $e^{-\sigma t}$ to obtain that $\text{Re}(Lg(s)) < 1$ for $|\omega| > |\omega_0|$.

The function $1 - Lg$ has no zeros on the segment $\text{Re}(s) = \sigma_0$, $|\omega| \leq |\omega_0|$. Therefore there exists open disks with radius < 1/2, where $1/(1 - Lg)$ is analytic, around each point on the line segment. It follows from Heine-Borel theorem that the segment can be covered by a finite number of these disks. Therefore there exists a constant $\delta$ such that the function is analytic on $\sigma_0 - \delta < \text{Re}(s) \leq \sigma_0$, $|\omega| \leq |\omega_0|$. \(\square\)

Choose $\delta$ such that $\delta \neq \sigma_0 + 1$. 21
4.1.3 Asymptotic behaviour of $\mathcal{L}n(0, \cdot)$

We will use integration by parts to find a representation of $\mathcal{L}n(0, \cdot)$ that will make it possible to calculate the integral in the inversion formula. To be able to integrate by parts we must make the extra assumptions on $f$, $m$ and $\mu$.

**Lemma 12** Let $m$ and $\mu$ be differentiable with bounded derivatives and $f$ two times differentiable with bounded derivatives. Then

$$\mathcal{L}n(0, s) = \frac{n(0, 0)}{s} + O \left( \frac{1}{s^2} \right) \quad \text{as} \quad |\text{Re}(s)| \to \infty.$$  

**Proof:** We start with $\mathcal{L}g(s)$:

$$\mathcal{L}g(s) = \int_0^\infty m(t)e^{-st} - \int_0^t \mu(v)dv \, dt = \left[ -\frac{m(t)e^{-\int_0^t \mu(v)dv}}{s}e^{-st} \right]_0^A_m$$

$$+ \frac{1}{s} \int_0^A_m (m'(t) - m(t)\mu(t))e^{-\int_0^t \mu(v)dv}e^{-st} \, dt = O \left( \frac{1}{s} \right).$$

This gives us

$$\frac{1}{1 - \mathcal{L}g(s)} = 1 + O \left( \frac{1}{s} \right).$$

Now look at $\mathcal{L}F(s)$:

$$\mathcal{L}F(s) = \int_0^A_m \int_t^A_m m(a)f(a-t)e^{-st} - \int_0^t \mu(v)dv \, da \, dt = \int_0^A_m F(t)e^{-st} \, dt,$$

where $F(t)$ has bounded derivatives. So we have

$$\mathcal{L}F(s) = \left[ -\frac{F(t)}{s}e^{-st} + \frac{F'(t)}{s^2}e^{-st} \right]_0^A_m$$

$$+ \frac{1}{s^2} \int_0^A_m F''(t)e^{-st} \, dt$$

$$= \frac{1}{s} \int_0^A_m m(a)f(a) \, da + O \left( \frac{1}{s^2} \right) = \frac{n(0, 0)}{s} + O \left( \frac{1}{s^2} \right).$$

The result for $\mathcal{L}n(0, s)$ is

$$\mathcal{L}n(0, s) = \left( \frac{n(0, 0)}{s} + O \left( \frac{1}{s^2} \right) \right) \left( 1 + O \left( \frac{1}{s^2} \right) \right) = \frac{n(0, 0)}{s} + O \left( \frac{1}{s^2} \right). \quad \square$$

4.1.4 The representation of $n(0, \cdot)$

**Proof of Theorem 8:** To find $n(0, t)$ we will use Laplace inversion formula and integrate along a line in the half plane $\sigma > \Lambda$:

$$n(0, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{L}n(0, s)e^{st} \, ds.$$  

To calculate this we will integrate along a rectangle and use the fact that $s = \sigma_0$ is the only pole inside the area of integration.
We have

\[ n(0, t) = -\lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{L_2} \mathcal{L}n(0, s)e^{st}ds - \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{L_3} \mathcal{L}n(0, s)e^{st}ds - \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{L_4} \mathcal{L}n(0, s)e^{st}ds + \text{Res} (\mathcal{L}n(0, \sigma_0)e^{\sigma_0 t}). \]

According to Lemma 12, the vertical integral along \( L_3 \) can be written

\[ n(0, 0) \int_{\sigma_0 - \delta - i\infty}^{\sigma_0 - \delta + i\infty} \frac{e^{-st}}{s} ds + \frac{1}{2\pi i} \int_{\sigma_0 - \delta - i\infty}^{\sigma_0 - \delta + i\infty} \frac{e^{-st}}{s^2} O(1) ds. \]

If we change the integration variable to \( \omega \) and use the fact that the integral of an odd function is zero, we see that these integrals equal

\[ \frac{n(0, 0)}{\pi} e^{(\sigma_0 - \delta)t} \left( \int_{0}^{\infty} \frac{(\sigma_0 - \delta) \cos(\omega t)}{\omega^2 + (\sigma_0 - \delta)^2} d\omega + \int_{0}^{\infty} \frac{\omega \sin(\omega t)}{\omega^2 + (\sigma_0 - \delta)^2} d\omega \right) + ie^{(\sigma_0 - \delta)t} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{(\sigma_0 - \delta + i\omega)^2} O(1) d\omega. \]

Observe that the last integral is bounded since

\[ \int_{-\infty}^{\infty} \frac{1}{(\sigma_0 - \delta)^2 + \omega^2} d\omega < \infty. \]

So with help from [4], page 181, we get

\[ \frac{1}{2\pi i} \int_{\sigma_0 - \delta - i\infty}^{\sigma_0 - \delta + i\infty} \mathcal{L}n(0, s)e^{st}ds = 2n(0, 0)H(\sigma_0 - \delta) + O(e^{(\sigma_0 - \delta)t}) = O(e^{(\sigma_0 - \delta)t}), \]

where \( H \) denotes the heaviside function. The residue of the pole at \( \sigma_0 \) is

\[ \text{Res} (\mathcal{L}n(0, \sigma_0)e^{\sigma_0 t}) = \frac{\mathcal{LF}(\sigma_0)e^{\sigma_0 t}}{\sigma - \sigma_0} = Ke^{\sigma_0 t}. \]

For the two horizontal integrals along \( L_2 \) and \( L_4 \), Lemma 1 shows that \( \mathcal{L}g(s) \to 0 \) and \( \mathcal{LF}(s) \to 0 \) when \( \omega \to \infty \), since

\[ m(t)e^{-\pi t - \int_0^t \mu(\nu) d\nu} \]
and
\[ \int_1^\infty m(a)f(a-t)e^{-\sigma t-f_a^\mu(v) dv} da \]
are continuous and have compact support. Therefore \( L n(0, s) \to 0 \) when \( \omega \to \infty \) and the two horizontal integrals tend to 0. The result is
\[ n(0, t) = O(e^{\sigma_0 - \delta t}) + Ke^{\sigma_0 t} \]
and Theorem 8 is proved. \( \square \)

4.2 Asymptotics of \( N \)

We are now ready to look at the asymptotics of \( N \), which is defined by:
\[ N(t) = \int_0^\infty n(a, t) da = \int_0^t n(0, t-a)e^{\sigma_0(t-a)-f_a^\mu(v) dv} dt + \int_t^\infty f(a-t)e^{-f_a^\mu(v) dv} da. \]

**Proof Theorem 6:** Let us start by inserting our new expression for \( n(0, t) \), found in the previous section, into the equation for \( N(t) \):
\[ N(t) = K \int_0^t e^{\sigma_0(t-a)-f_a^\mu(v) dv} dt + \int_0^t e^{(\sigma_0-\delta)(t-a)-f_a^\mu(v) dv} O(1) dt + \int_t^\infty f(a-t)e^{-f_a^\mu(v) dv} da = I_1 + I_2 + I_3. \]

To estimate these integrals, we will use the fact that \( \mu(a) = 1 \) for \( a \geq A_\mu \), no individual can be infinitely old. Consider \( N(t) \) for \( t > A_\mu \). We will use \( c_i \) to denote constants. We start with \( I_1 \):
\[ I_1 = K \int_0^t e^{\sigma_0(t-a)-f_a^\mu(v) dv} da = e^{\sigma_0 t} K \int_0^{A_\mu} e^{-\sigma_0 a-f_a^\mu(v) dv} da + e^{\sigma_0 t} K \int_{A_\mu}^t e^{-\sigma_0 a-(a-A_\mu)-f_a^\mu(v) dv} da. \]
So for \( \sigma_0 \neq -1 \) we get
\[ I_1 = c_1(\sigma_0)e^{\sigma_0 t} - c_2(\sigma_0)e^{-t} \]
and for \( \sigma_0 = -1 \)
\[ I_1 = c_3e^{-t}t + c_4(\sigma_0)e^{-t}. \]

For \( I_2 \) we can use the same calculations as for \( I_1 \). We have chosen \( \delta \) so that \( \sigma_0 - \delta \neq -1 \). We have
\[ I_2 = \int_0^t e^{(\sigma_0-\delta)(t-a)-f_a^\mu(v) dv} O(1) dt = O(1)e^{(\sigma_0-\delta)t} \int_0^t e^{-(\sigma_0-\delta)a-f_a^\mu(v) dv} da = O(1)e^{(\sigma_0-\delta)t} \left( \int_{A_\mu}^t e^{-(\sigma_0-\delta+1)a-f_a^\mu(v) dv} da + \int_0^{A_\mu} e^{-(\sigma_0-\delta)a-f_a^\mu(v) dv} \right) = O(1)e^{(\sigma_0-\delta)t} - O(1)e^{-t}, \]
where \( O(1) \) denotes a bounded function.
Finally, we calculate the integral $I_3$:

$$I_3 = \int_t^\infty f(a-t) e^{-\int_{a-t}^\infty \mu(v) \, dv} \, da = \int_0^\infty f(x) e^{-\int_x^\infty \mu(v) \, dv - \int_x^0 \mu(v) \, dv} \, dx$$

$$= e^{-t} \int_0^\infty f(x) e^{-x - \int_x^\infty \mu(v) \, dv + \mu_x} \, dx = c_5 e^{-t}.$$

We have the following expressions for $N(t)$. For $\sigma_0 \neq -1$:

$$N(t) = c_1(\sigma_0) e^{\sigma_0 t} - c_2(\sigma_0) e^{-t} + O(1) e^{(\sigma_0 - \delta) t} - O(1) e^{-t} + c_5 e^{-t}$$

and for $\sigma_0 = -1$:

$$N(t) = c_3 e^{-t} - c_4(\sigma_0) e^{-t} + O(1) e^{-1 - \delta t} - O(1) e^{-t} + c_5 e^{-t}. \quad \Box$$

Notice that the population is increasing if $\sigma_0 > 0$. 

25
5 Asymptotics of $N$—the time-dependent case

In the previous section we found the asymptotic behaviour of $N$, in the time-independent case, by using the Laplace transform. This method does not work when $m$ or $\mu$ depend on time, and it requires that $m$, $f$ and $\mu$ are differentiable functions. In this section we will instead try to find upper and lower bounds for $N$ to get information about the asymptotic behaviour. We need first to consider bounds $n_+(t)$ and $n_-(t)$ to $n(0, t)$.

Recall that $n(0, t)$ satisfies the integral equation (5):

$$n(0, t) = \int_0^t m(a, t)n(0, t - a)e^{-\int_a^t \mu dv} da + \int_0^\infty m(a, t)f(a - t)e^{-\int_a^t \mu dv} da$$

$$= (Kn)(t) + F(t).$$

For a function $n$ we use the notation $E(n)(t) = n(0, t) - (Kn)(t) - F(t)$. We will first prove two lemma that describe sufficient conditions for such bounds $n_+$ and $n_-$.

Lemma 13 If $n_+$ satisfies $E(n_+)(t) \geq 0$ for all $t \geq 0$, and $n$, $n_+$ is of at most exponential growth, then $n_+ \geq n(0, \cdot)$, where $n(0, \cdot)$ is the solution to the integral equation (5).

Proof: Let $n_1, n_2, \ldots$ denote the sequence of approximations to $n$ obtained from the Picard iteration scheme with $n_+$ as the initial approximation. Then $n_1 = (Kn_+)+F \leq n_+$. Suppose that $n_{k-1} \leq n_+$ for some $k \geq 2$. Then $n_k = (Kn_{k-1}) + F \leq (Kn_+) + F \leq n_+$. By induction, $n_k \leq n_+$ for all $k$, and thus by letting $k \to \infty$, $n \leq n_+$. □

In an analog way, we can prove the following lemma.

Lemma 14 If $n_-$ satisfies $E(n_-)(t) \leq 0$ for all $t \geq 0$, and $n_-$ is of at most exponential growth, then $n_- \leq n(0, \cdot)$ where $n(0, \cdot)$ is the solution to the integral equation (5).

Let $\sigma_0(t_1) = -\infty$ if $m(a, t_1) \equiv 0$ and otherwise, for a given $t$, let $\sigma_0(t)$ be the function that satisfies the following equation:

$$\int_0^\infty m(a, t)e^{-\sigma_0(t)a - \int_0^a \mu(v, v + t - a) dv} da = 1.$$ 

Because of Lemma 10, there exists a unique $\sigma_0(t)$ for every $t$. We will seek bounds under certain conditions on $\sigma_0$.

5.1 Upper bounds for $n(0, \cdot)$

We will seek upper bounds for $n(0, \cdot)$. Let

$$F_{f_1}(t) = \int_t^\infty m(a, t)f_1(a - t)e^{-\int_a^t \mu dv} da.$$

We will need a lemma.
\textbf{Lemma 15} Suppose that $n_{f_1}(0, \cdot)$ solve the integral equation (5), with $f_1$ as start population, where $f_1(a) > 0$ for $a \in [0, A_m]$. Then for the solution $n_f(0, \cdot)$ to (5) with another start population $f$, there exists a constant $C$ such that

$$n_f(0, t) \leq C n_{f_1}(0, t) \quad \text{for all } t \geq 0.$$  

\textbf{Proof:} We have

$$n_f(0, t) = \int_0^t m(a, t)n_{f_1}(0, t-a)e^{-\int_0^t \mu(v, v+t-a)dv} da + F_{f_1}(t).$$

Since $f_1(a) > 0$ for $x \in [0, A_m]$ and $f(a)$ is bounded, there exists a constant $C$ so that, for another $f$,

$$CF_{f_1}(t) - F_f(t) = \int_0^\infty m(a, t)(CF_{f_1}(a-t) - f(a-t))e^{-\int_0^{x-a} \mu(v, v+t-a)dv} da \geq 0.$$  

Therefore, for the other start population $f$, we get

$$E(Cn_{f_1})(t) = Cn_{f_1}(0, \cdot) - K(Cn_{f_1}) - F_f = C(n_{f_1}(0, \cdot) - K(n_{f_1}) - F_{f_1}) + CF_{f_1} - F_f.$$  

Since

$$n_{f_1}(0, \cdot) - K(n_{f_1}) - F_{f_1} = 0,$$

we have $E(Cn_{f_1}) \geq 0$. Thus $Cn_{f_1}(0, \cdot)$ is an upper bound for $n_f(0, \cdot)$. This is true for an arbitrary $f$ and therefore $Cn_{f_1}(0, \cdot) \geq n(0, \cdot).$ \Box

Now we can seek upper bounds for $n$ under some conditions on $\sigma_0(t)$. We start with $\sigma_0(t)$ that eventually has a constant upper boundary.

\textbf{Lemma 16} If $\sigma_0(t) \leq c$ for all $t > T$, then there exists constants, $D_1$, $D_2$, $c$ and $\sigma$, such that

$$n(0, t) \leq n_+(t) = D_1 e^{ct} H(T-t) + D_2 e^{ct} H(t-T)$$

where $n$ is the solution to (5) and $H$ denotes the Heaviside function.

\textbf{Proof:} Because of Lemma 15, it is enough to prove that $n_{f_1} \leq n_+$, for

$$f_1(a) = D_1 e^{-\sigma a - \int_0^a \mu(v, v-a)dv}$$

for $a \in [0, A_m]$. Choose the constants so that $D_2 e^{ct} \geq D_1 e^{ct}$ for $t \in [0, T]$. Then

$$E(n_+) = D_1 e^{ct} H(T-t) - D_1 e^{ct} \int_0^t m(a, t)e^{-\sigma a - \int_0^a \mu dv} H(T-t+a) da$$

$$+ D_2 e^{ct} H(t-T) - D_2 e^{ct} \int_0^t m(a, t)e^{-\sigma a - \int_0^a \mu dv} H(t-T-a) da$$

$$+ \int_0^\infty m(a, t)f_1(a-t)e^{-\int_0^{x-t-a} \mu dv} da.$$  

For $t > T$, we get

$$E(n_+) = D_2 e^{ct} - \int_0^\infty m(a, t)D_1 e^{\sigma(t-a)} e^{-\int_0^a \mu dv} da$$

$$- D_2 e^{ct} \int_0^{t-T} m(a, t)e^{-\sigma a} e^{-\int_0^a \mu dv} da.$$  

27
Using the fact that $D_1 e^{\sigma(t-a)} \leq D_2 e^{\sigma(t-a)}$, we get

$$E(n_+) \geq D_2 e^{\sigma t} - D_2 e^{\sigma t} \int_0^t m(a, t)e^{-\sigma a} e^{-\int_0^a \mu dv} da \geq 0.$$ 

Since, by assumption, the last integral is not greater than 1.

For $t \leq T$, we get

$$E(n_+) = D_1 e^{\sigma t} - D_1 e^{\sigma t} \int_0^t m(a, t)e^{-\sigma a} e^{-\int_0^a \mu dv} da - F_{f_1}(t).$$

There exists a $\sigma$ such that $E(n_+) \geq 0$, $\sigma \geq M$ will always do. Since

$$E(n_+) \geq D_1 e^{Mt} - D_1 e^{\sigma t} \int_0^\infty Me^{-Ma} e^{-\int_0^a \mu(v+v-t-a) dv} da \geq 0. \quad \square$$

We can prove the following theorem.

**Theorem 17** If $\sigma_0(t) \leq c$ for all $t > T$, then there exists constants, $D_2$ and $c$, such that $n(0, t) \leq n_+(t) = D_2 e^{\sigma t}$

where $n$ is the solution to (5) and $H$ denotes the Heaviside function.

**Proof:** Since we chose $D_2 e^{\sigma t} \geq D_1 e^{\sigma t}$ for $t \in [0, T]$, we have $D_1 e^{\sigma t} H(T - t) + D_2 e^{\sigma t} H(t - T) \leq D_2 e^{\sigma t}$ and $n_+(t) = D_2 e^{\sigma t}$ is also a upper bound for $n(0, \cdot)$.

We will continue to look for a bound for $n$ under some other conditions on $\sigma_0$. For every $t$ let $m(a, t) \neq 0$ for some interval on $a$. This will mean that $\sigma_0(t) \neq -\infty$. Under this condition we can find an upper bound for $n(0, \cdot)$ for a $\sigma_0$ that eventually becomes a decreasing function.

**Lemma 18** If $\sigma_0$ is a decreasing function on $[T, \infty]$, and if, for every fixed $t$, $m(\cdot, t)$ is positive on some interval, then there exists constants $D_1$, $D_2$ and $\sigma$ such that

$$n(0, t) \leq n_+(t) = D_1 e^{\sigma t} H((T + A_m) - t) + D_2 e^{\int_0^T \sigma_0(\tau)d\tau} H(t - (T + A_m)).$$

**Proof:** Because of lemma 15, it is also here enough to prove that $n_{f_1} \leq n_+$, for

$$f_1(a) = D_1 e^{-\sigma a} - \int_0^a \mu(v, v-a) dv \quad \text{for} \quad a \in [0, A_m].$$

Choose $D_1$, $D_2$ and $\sigma$ such that $D_2 e^{\int_0^T \sigma_0(\tau)d\tau} \geq D_1 e^{\sigma t}$ for $t \in [0, T + A_m]$. For $t \leq T + A_m$ we get $E(n_{f_1}) \geq 0$ using the same calculations as in the previous case for $t \leq T$. Consider $E(n_+)$ for $t \geq T + A_m$:

$$E(n_+) = D_2 e^{\int_0^T \sigma_0(\tau)d\tau} - \int_{t-(T+A_m)}^\infty m(a, t)D_1 e^{\sigma(t-a)} e^{-\int_0^a \mu dv} da$$

$$- D_2 \int_0^{t-(T+A_m)} m(a, t)e^{\int_0^a \sigma_0(\tau)d\tau} e^{-\int_0^a \mu dv} da.$$
If we now use the fact that $D_2 e^{\int_0^t \sigma_0(\tau) d\tau} \geq D_1 e^{\sigma t}$ for $t \in [0, T + A_m]$, we get

$$E(n_+) \geq D_2 e^{\int_0^t \sigma_0(\tau) d\tau} - D_2 \int_0^\infty m(a, t) e^{\int_0^t \sigma_0(\tau) d\tau} e^{-\int_0^a \mu dv} da$$

$$= D_2 e^{\int_0^t \sigma_0(\tau) d\tau} \left(1 - \int_0^\infty m(a, t) e^{-\int_0^a \sigma_0(\tau) d\tau} e^{-\int_0^a \mu dv} da\right).$$

Since $\sigma_0$ is a decreasing function, we know that $e^{-\int_0^a \sigma_0(\tau) d\tau} \leq e^{-\sigma_0(t)}$. Using this, we obtain

$$E(n_+) \geq D_2 e^{\int_0^t \sigma_0(\tau) d\tau} \left(1 - \int_0^\infty m(a, t) e^{-\sigma_0(t)} e^{-\int_0^a \mu dv} da\right) = 0. \quad \square$$

**Theorem 19** If $\sigma_0$ is a decreasing function on $[T, \infty]$, and if, for every fixed $t$, $m(\cdot, t)$ is positive on some interval, then there exists a constant $D_2$ such that $n(0, t) \leq n_+(t) = D_2 e^{\int_0^t \sigma_0(\tau) d\tau}$.

**Proof:** We chose $D_1$, $D_2$ and $\sigma$ such that $D_2 e^{\int_0^t \sigma_0(\tau) d\tau} \geq D_1 e^{\sigma t}$ for $t \in [0, T + A_m]$.

\[\square\]

### 5.2 Lower bounds for $n(0, \cdot)$

We will now seek lower bounds for $n(0, \cdot)$. We seek bounds under the condition that $m(a, t) \geq 0$ for an interval of length $\varepsilon$ for all $t$ and a small constant $\varepsilon$.

**Lemma 20** Suppose that $f(x) > 0$ for $x$ in an interval of length $\delta$ and $g(u, x) > 0$ for every $x$ for $u$ in an interval of length $\varepsilon$, where $\varepsilon$ and $\delta$ are positive. Also suppose that both $g(u, x)$ and $f(x) \geq 0$ for all $x$. Then

$$\int_0^\varepsilon g(u, x)f(x - u)du > 0$$

for $x$ in an interval of length $\delta + \varepsilon$.

**Proof:** Suppose that $f(x) > 0$ for $x \in [x_1, x_1 + \delta]$ and $g(u, x) > 0$ for $u \in [u_2, u_2 + \varepsilon]$. Let $x \in (x_1 + u_2, x_1 + u_2 + \delta + \varepsilon)$. For $x \geq u_2 + \varepsilon$ we know that $g(u, x) > 0$ when we integrate over the interval $u \in [u_2, u_2 + \varepsilon]$. When we integrate over the interval $u \in [u_2, u_2 + \varepsilon]$, $x - u$ will vary in the interval $[x - u_2 - \varepsilon, x - u_2]$, which contains an interval where $f(u - x) > 0$. For $x = x_1 + u_2 + \rho < u_2 + \varepsilon$ we know that $g(u, x) > 0$ when we integrate over the interval $u \in [u_2, u_2 + x_1 + \rho]$. When we integrate over the interval $u \in [u_2, u_2 + x_1 + \rho]$, $x - u$ will vary in the interval $[x - u_2 - x_1 - \rho, x - u_2]$, which contains an interval where $f(u - x) > 0$. \[\square\]

Consider $(K^p F)(t) = K(K^{p-1} F)(t)$, where $p$ is a positive integer. Notice that $F(t) > 0$ for an interval of length $\delta$ otherwise we get the solution $n(0, t) = 0$. We also know that $g(u, t) = m(a, t)e^{-\int_0^a \mu dv} > 0$ for an interval of length $\varepsilon$. Therefore, using the lemma above, we can choose $p$ such that $(K^p F)(t) > 0$ for an interval of length $A_m$.

We will start to look for a lower bound for $n$, for $\sigma_0(t)$ that eventually is bounded from below.
Lemma 21 Suppose that \( \sigma_0(t) > c \) for all \( t > T \). Then there exists a positive integer \( p \) and constants \( C \) and \( T_F \) such that
\[
n(0, t) \geq n(t) = H(t - T_F)Ce^{ct} + F(t) + (KF)(t) + \ldots + (K^{p-1}F)(t)
\]

Proof: Choose \( T_F > T \) such that \( (K^pF)(t) > 0 \) for \( t \in [T_F, T_F + A_m] \) this is possible according to Lemma 20. Consider first \( E(n)(t) \) for \( t < T_F \):
\[
E(n)(t) = H(t - T_F)Ce^{ct} - (KH(t - T_F)Ce^{ct})(t) - (KF)(t) \leq 0.
\]
For \( t \geq T_F \),
\[
E(n)(t) = Ce^{ct} - Ce^{ct} \int_0^t m(a)e^{-ca}e^{-\int_0^a \mu(v) dv} H(t - T_F - a) da - (K^pF)(t),
\]
which can be rewritten:
\[
E(n)(t) = Ce^{ct} \left( 1 - \int_0^\infty m(a, t)e^{-ca} - \int_0^a \mu(v) dv da \right) + Ce^{ct} \int_0^\infty m(a, t)e^{-ca} - \int_0^a \mu(v) dv da - (K^pF)(t). \tag{12}
\]
Notice that the term in (12) is negative since \( \sigma_0(t) > c \) for \( t \geq T \). The first term in (13) equals 0 for \( t \geq T_F + A_m \) since \( m(a, t) = 0 \) for \( a \geq A_m \). Notice that \( (K^pF)(t) > 0 \) for \( t \in [T_F, T_F + A_m] \) and the first term in (13) is bounded. Therefore we can choose \( C \) such that \( E(n)(t) \leq 0 \).

Theorem 22 Suppose that \( \sigma_0(t) \geq c \) for all \( t > T \). Then there exists constants \( C \) and \( T_F \) such that
\[
n(0, t) \geq n(t) = H(t - T_F)Ce^{ct}.
\]

Proof: We know that
\[
H(t - T_F)Ce^{ct} \leq H(t - T_F)Ce^{ct} + F(t) + (KF)(t) + \ldots + (K^{p-1}F)(t).
\]
Therefore \( H(t - T_F)Ce^{ct} \) is a lower bound as well. \( \Box \)

There is a lower bound for an eventually increasing \( \sigma_0 \).

Lemma 23 Suppose that \( \sigma_0 \) is increasing on \([T, \infty]\). Then there exists a positive integer \( p \) and constants \( T_F \) and \( C \) such that
\[
n(0, t) \geq H(t - T_F)Ce^{-\int_0^\infty \sigma_0(\tau) d\tau} + F(t) + (KF)(t) + \ldots + (K^pF)(t).
\]

Proof: Choose \( T_F > T + A_m \) such that \( (K^pF)(t) > 0 \) for \( t \in [T_F, T_F + A_m] \) this is possible according to Lemma 20. As in the previous lemma, \( E(n)(t) \leq 0 \) for \( t \leq T_F \). For \( t > T_F \), we get
\[
E(n)_+ = Ce^{-\int_0^t \sigma_0(\tau) d\tau} \left( 1 - \int_0^\infty m(a, t)e^{-\int_0^a \sigma_0(\tau) d\tau} - \int_0^a \mu(v) dv da \right) + C \int_0^\infty m(a, t)e^{-\int_0^a \sigma_0(\tau) d\tau} - \int_0^a \mu(v) dv da - (K^{p+1}F)(t).
\]
Since $\sigma_0(t)$ is increasing for $t \geq T$ we know that $e^{-\int_{t-a}^{t} \sigma_0(\tau) d\tau} \geq e^{-\sigma_0(t)}$. For $t \geq T_F$, this implies that

$$E(n_-) \leq C e^{-\int_{0}^{t} \sigma_0(\tau) d\tau} \left(1 - \int_{0}^{\infty} m(a, t) e^{-\sigma_0(t)a - \int_{0}^{a} \mu dv} da\right) + C \int_{t-T_F}^{\infty} m(a, t) e^{-\int_{t-a}^{t} \sigma_0(\tau) d\tau - \int_{0}^{a} \mu dv} da - (K^{p+1}) F(t)$$

$$= C \int_{t-T_F}^{\infty} m(a, t) e^{-\int_{t-a}^{t} \sigma_0(\tau) d\tau - \int_{0}^{a} \mu dv} da - (K^{p+1}) F(t).$$

As in the proof for Lemma 21 we can choose $C$ and $p$ so that $E(n_-) \leq 0$. □

Clearly the following theorem is true.

**Theorem 24** Suppose that $\sigma_0$ is increasing on $[T, \infty]$. Then there exists constants $C$ an $T_F$ such that

$$n(0, t) \geq H(t - T_F) C e^{-\int_{0}^{t} \sigma_0(\tau) d\tau}.$$

### 5.3 Upper and lower bounds for $N$

Now we are ready to look at $N$, which has the following equation:

$$N(t) = \int_{0}^{\infty} n(a, t) da = \int_{0}^{t} n(0, t-a) e^{-\int_{0}^{a} \mu dv} da + \int_{t}^{\infty} f(a-t) e^{-\int_{0}^{a} \mu dv} da.$$

We will find bounds for some combinations of conditions on $\sigma_0(t)$.

**Theorem 25** Suppose that $\sigma_0(t) \leq c$ for all $t \geq T$. Then there exists constants $C_1$ and $C_2$ such that, for $t > T + A_\mu$, we have

$$N(t) \leq C_1 e^{ct} + C_2 e^{-t}, \quad \text{if } c \neq -1,$$

and

$$N(t) \leq C_1 e^{-t} + C_2 t e^{-t}, \quad \text{if } c = -1.$$

**Proof:** Insert our first upperbound $n_+ (t)$, from Theorem 17, which was calculated under the given condition on $\sigma_0$, into the equation for $N$. Then

$$N(t) \leq \int_{0}^{t} D_2 e^{c(t-a)} e^{-\int_{0}^{a} \mu \{v_v + t-a\} dv} da + \int_{t}^{\infty} f(a-t) e^{-\int_{0}^{a} \mu \{v_v + t-a\} dv} da.$$

For $t > T + A_\mu$, using the fact that there exists a constant $F$ such that $f < F$, we get

$$N(t) \leq \int_{0}^{A_\mu} D_2 e^{c(t-a)} e^{-\int_{0}^{a} \mu \{v_v + t-a\} dv} da + \int_{A_\mu}^{t} D_2 e^{c(t-a)} e^{-\sigma_0 + A_\mu - \int_{0}^{a} \mu \{v_v + t-a\} dv} da$$

$$+ \int_{0}^{\infty} f(x) e^{-\int_{x}^{A_\mu} \mu \{v_v + x\} dv} da + F \int_{A_\mu}^{\infty} e^{-(x+t-A_\mu)} dx.$$

This proves the theorem. □
\textbf{Theorem 26} Suppose that \( \sigma_0(t) \geq c \) for all \( t \geq T \). Then there exists constants \( C_1, C_2 \) and \( T_F \) such that, for \( t > T_F + A_\mu \), we have

\[ N(t) \geq C_1e^{\sigma t} + C_2e^{-t}, \quad \text{if } c \neq -1 \]

and

\[ N(t) \geq C_1e^{-t} + C_3e^{-t}, \quad \text{if } c = -1. \]

\textbf{Proof:} Insert \( n_-(0, t) \) from Theorem 22 into the equation for \( N \).

\[ N(t) \geq \int_0^t H(t-a-T_F)C_2e^{\sigma(t-a)}e^{-\int_0^a \mu dv} da + \int_t^\infty f(a-t)e^{-\int_{a-t}^a \mu dv} da. \]

Since \( e^{-\int_0^a \mu dv} \geq e^{-a} \), we have

\[ N(t) \geq \int_0^{t-T_F} C_2e^{\sigma(t-a)}e^{-a} da + \int_t^\infty f(a-t)e^{-t} da. \]

\[ \square \]

\textbf{Theorem 27} Suppose that \( \sigma_0(t) \geq c \) and that \( \sigma_0 \) is decreasing on \( [T - A_\mu, \infty] \). Then there exists constants \( C_1, C_2 \) and \( C_3 \) such that, for \( t > T + A_\mu \), we have

\[ N(t) \leq C_1e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} + C_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau}(t - A_\mu) + C_3e^{-t}, \quad \text{if } c \neq -1, \]

and

\[ N(t) \leq C_1e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} + C_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau}(t - A_\mu) + C_3e^{-t}, \quad \text{if } c = -1. \]

\textbf{Proof:} We use the fact that \( e^{-\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \leq e^{-\sigma_0(t)A} \leq e^{-cA} \). We obtain

\[ N(t) = \int_0^{A_\mu} D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} e^{-\int_0^a \mu(v, v+\tau-a) dv} da + \int_{A_\mu}^\infty f(a-t)e^{-\int_{a-t}^a \mu dv} da \]

\[ \quad \quad \quad + \int_{A_\mu}^{t-A_\mu} D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} e^{-\int_{t-A_\mu}^{t-A_\mu} A_\mu - A_\mu - \int_{t-A_\mu}^{t-A_\mu} \mu(v, v+\tau-a) dv} da \]

\[ \leq D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \left( \int_0^{A_\mu} e^{-\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} da + \int_{A_\mu}^t e^{-\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} e^{-a} da \right) \]

\[ + C_3e^{-t}. \]

For \( t > T + A_\mu \) this bounded from above by

\[ D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \int_0^{A_\mu} e^{-\sigma_0(t)A} da + D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \int_{A_\mu}^t e^{-\sigma_0(t)(t+1)A} da + C_4e^{-t} \]

\[ \leq D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \int_0^{A_\mu} e^{-\sigma_0(t)A} da + D_2e^{\int_{t-A_\mu}^{t-A_\mu} \sigma_0(\tau)d\tau} \int_{A_\mu}^t e^{-\sigma_0(t)(t+1)A} da + C_4e^{-t}. \]

This proves the theorem. \( \square \)

\section{5.4 Comparison with the time-independent case}

In the time-independent case we have \( \sigma_0(t) = \sigma_0 \) for all \( t \geq 0 \). This means that we have the following estimate:

\[ C_1e^{\sigma_0 t} \leq n(0, t) \leq C_2e^{\sigma_0 t}. \]

This gives the same asymptotic behaviour for \( N \) as we found in Section 4.
6 Comparison with the model without age structure

Here we will compare the age structure population model, time-independent case, with the model without age structure (1). First we examine what happens if we have \(m\) and \(\mu\) constant. Let \(m(a) = b\) and \(\mu(a) = d\). We know that \(\sigma_0\) satisfies:

\[
1 = \int_0^\infty be^{-\sigma_0 a - \int_0^a d\,dv} \, da = \int_0^\infty be^{-(\sigma_0 + d)\,a} \, da = \frac{b}{\sigma_0 + d}.
\]

Nicely enough \(\sigma_0 = b - d\). So for \(m\) and \(\mu\) constant and \(\sigma_0 > -1\) the solution to the age structure population model, for the time-independent case, has the same asymptotics, \(N(t) \sim e^{\sigma_0 t}\), as the most simple model.

For non constant \(m\) and \(\mu\) \(\sigma_0\) is the asymptotic birthrate minus death rate for the total population:

\[
b \equiv d = \lim_{t \to \infty} \left( \int_0^t m(a)n(a, t) \, da - \int_0^t \mu(a)n(a, t) \, da \right)
\]

\[
= \lim_{t \to \infty} \left( \int_0^t m(a)Ke^{\sigma_0(t-a)}e^{-\int_0^a \mu(v) \, dv} \, da - \int_0^t \mu(a)Ke^{\sigma_0(t-a)}e^{-\int_0^a \mu(v) \, dv} \, da + \int_0^t O(1)e^{(\sigma_0-\delta)(t-a)} \, da + \int_t^\infty n(a, t) \, da \right)
\]

\[
= \lim_{t \to \infty} \left( 1 - \int_0^t e^{-\sigma_0 a} \left( e^{\int_0^a \mu(v) \, dv} \right) \, da \right) = \lim_{t \to \infty} \frac{e^{\sigma_0 t - \int_0^a \mu(v) \, dv}}{e^{-\sigma_0 a - \int_0^a \mu(v) \, dv} + \sigma_0} = \sigma_0 = \sigma_0
\]

It is interesting to find out when the simplest model has the same asymptotic solution as the solution to the age structure population model. That is, what conditions on \(m\) and \(\mu\) correspond to \(\sigma_0 > -1\)?

If \(\sigma_0 > -1\) we have:

\[
1 = \int_0^\infty m(a)e^{-\sigma_0 a - \int_0^a \mu(v) \, dv} \, da < \int_0^\infty m(a)e^{\sigma_0 a - \int_0^a \mu(v) \, dv} \, da,
\]

and if \(\int_0^\infty m(a)e^{\sigma_0 a - \int_0^a \mu(v) \, dv} \, da > 1\), we have for \(\sigma \leq -1\):

\[
Lg(\sigma) = \int_0^\infty m(a)e^{-(\sigma_0 + \sigma + 1)(t-a) + \int_0^a \mu(v) \, dv} \, da \geq \int_0^\infty m(a)e^{\sigma_0 a - \int_0^a \mu(v) \, dv} \, da > 1.
\]

This gives us \(\sigma_0 > -1\). Therefore we can state:

**Theorem 28** \(\sigma_0 > -1 \iff \int_0^\infty m(a)e^{\sigma_0 a - \int_0^a \mu(v) \, dv} \, da > 1\)
6.1 Biological analysis of the requirement $\sigma_0 > -1$

We will here discuss the biological properties of a population that has $\sigma_0 > -1$. Hopefully it is reasonable to assume that most populations have $\sigma_0 > -1$.

Consider an individual that lives to be older than $a = A_m$ under its lifetime it gets $\int_0^\infty m(a) \, da$ children. If $\int_0^\infty m(a) \, da$ is less than 1 we surely have a population that will die out and biologically its not a possible scenario. If instead $\int_0^\infty m(a) \, da$ is greater than 1 we have:

$$1 < \int_0^\infty m(a) \, da \leq \int_0^\infty m(a)e^{a - \int_0^a \mu(v) \, dv} \, da$$

and $\sigma_0 > -1$ for this population. So we can assume that $\sigma_0 > -1$ for biological populations.
7 Conclusions

We have proved the existence of a unique solution to the model with age structure. We have found the asymptotics for the time-independent case, and bounds for some cases of the time-dependent case.

It would be interesting to find more bounds and eventually the complete asymptotics for the time-dependent case. We would like to find out when the age structure model gives the same asymptotics as the simple model without age structure. The next step could be to take in consideration birth and death rates that depend on the total number of individuals.

The growth of a population probably depend on its spatial structure. Therefore analysis of how migration influences the asymptotics is interesting. The goal is to determine under which conditions the approximation with the simple model without age structure is valid.
References


A Banach theory

To examine if integral equation has solutions, the Banach fixed point theorem can be used. In order to prepare for this theorem, a couple of definitions are needed.

A.1 Definitions

The following definitions are presented in [2].

Definition 29 (Metric space) A metric space is a pair \((X,d)\), where \(X\) is a set and \(d\) is a metric on \(X\). The metric \(d\) is a function from \(X \times X\) to \(\mathbb{R}\) with the following properties, for \(x, y, z \in X\):

\[
M(1) \quad d \text{ is real-valued, finite and nonnegative} \\
M(2) \quad d(x, y) = 0 \iff x = y \\
M(3) \quad d(x, y) = d(y, x) \\
M(4) \quad d(x, y) \leq d(x, z) + d(z, y).
\]

Definition 30 (Completeness) A sequence in a metric space is called Cauchy if for every \(\varepsilon > 0\) there is a \(N \in \mathbb{N}\), \(N = N(\varepsilon)\) such that

\[d(x_m, x_n) < \varepsilon \text{ for } m, n > N\]

The space \(X\) is said to be complete if every Cauchy sequence has a limit which belongs to \(X\).

Definition 31 (Norm) A norm on a vector space is a real-valued function \(\| \cdot \|\) on \(X\), with the properties:

\[
N(1) \quad \|x\| \geq 0 \\
N(2) \quad \|x\| = 0 \iff x = 0 \\
N(3) \quad \|\alpha x\| = |\alpha|\|x\| \\
N(4) \quad \|x + y\| \leq \|x\| + \|y\|.
\]

A norm on a vector space \(X\) induce a metric \(d(x, y) = \|x - y\|\), on \(X\).

Definition 32 (Banach space) A Banach space is a normed vector space which is complete with respect to the metric defined by the norm.

Definition 33 (Contraction) Let \(X = (X,d)\) be a metric space. A mapping \(T : X \to X\) is a contraction on \(X\) if there is a \(\alpha, \alpha \in [0, 1)\) such that, for \(x, y \in X\),

\[d(Tx, Ty) \leq \alpha d(x, y).
\]

A.2 The Banach fixed point theorem

Theorem 34 (Banach fixed point theorem) Consider a complete metric space \(X = (X,d)\). Let \(T : X \to X\) be a contraction on \(X\). Then \(T\) has precisely one fixed point, i.e., a point \(x \in X\) such that \(T(x) = x\).
Proof: Take a $x_0 \in X$. Construct $x_1 = T(x_0)$, $x_2 = T(x_1)$, ..., $x_n = T(x_{n-1})$. Since $T$ is a contraction,

$$d(x_n, x_{n+1}) = d(T^n x_0, T^n x_1) \leq \alpha^n d(x_0, x_1).$$

So for $m \geq n$ we have:

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + \ldots + d(x_{n-1}, x_n)$$

$$\leq (\alpha^m + \alpha^{m+1} + \ldots + \alpha^n) d(x_0, x_1)$$

$$\leq \frac{\alpha^m}{1-\alpha} d(x_0, x_1) \to 0 \quad \text{when } m, n \to \infty.$$

The sequence $(x_n)$ is therefore a Cauchy sequence. By assumption $X$ is complete and therefore the limit $x$ to $(x_n)$ exists. Since $x_n = T(x_{n-1})$ and $T$ is continuous, it follows that $x = T(x)$. We have now found a fixed point, $x$, and the only thing left is to prove that it is unique.

Assume that there is another fixed point $y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).$$

Since $\alpha < 1$, this gives $d(x, y) = 0$ and hence $x = y$. □

If the contraction $T_p$ depends on an additional parameter, $p \in P$, we have $T_p : X \to X$ and the equation $T_p x_p = x_p$, $x_p \in X$. In [3] the following corollary is presented.

Corollary 35 Suppose that

(i) $P$ is a metric space, called the parameter space,

(ii) for each $p$ the operator $T_p$ is a contraction on $X$ with $\alpha$ independent of $p$,

(iii) for $p_0 \in P$ and $x \in X$, $\lim_{p \to p_0} T_p x = T_{p_0} x$.

Then, for each $p \in P$, the equation $T_p x_p = x_p$ has exactly one solution $x_p \in X$, and $\lim_{p \to p_0} x_p = x_{p_0}$.

Proof: Let $x_p$ be the solution of the integral equation, $T_p x_p = x_p$. We get

$$d(x_p, x_{p_0}) = d(T_p x_p, T_{p_0} x_{p_0}) \leq d(T_p x_p, T_p x_{p_0}) + d(T_p x_{p_0}, T_{p_0} x_{p_0})$$

$$\leq \alpha d(x_p, x_{p_0}) + d(T_p x_{p_0}, T_{p_0} x_{p_0})$$

and therefore

$$d(x_p, x_{p_0}) \leq \frac{1}{1-\alpha} d(T_p x_{p_0}, T_{p_0} x_{p_0}) \to 0 \quad \text{when } p \to p_0. \quad \square$$
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