



On flow of power-law fluids between adjacent surfaces: Why is it possible to derive a Reynolds-type equation for pressure-driven flow, but not for shear-driven flow?

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ABSTRACT

Flows of incompressible Navier–Stokes (Newtonian) fluids between adjacent surfaces are encountered in numerous practical applications, such as seal leakage and bearing lubrication. In seals, the flow is primarily pressure-driven, whereas, in bearings, the dominating driving force is due to shear. The governing Navier–Stokes system of equations can be significantly simplified due to the small distance between the surfaces compared to their size. From the simplified system, it is possible to derive a single lower-dimensional equation, known as the Reynolds equation, which describes the pressure field. Once the pressure field is computed, it can be used to determine the velocity field. This computational algorithm is much simpler to implement than a direct numerical solution of the Navier–Stokes equations and is therefore widely employed by engineers.

The primary objective of this article is to investigate the possibility of deriving a type of Reynolds equation also for non-Newtonian fluids, using the balance of linear momentum. By considering power-law fluids we demonstrate that it is not possible for shear-driven flows, whereas it is feasible for pressure-driven flows. Additionally, we demonstrate that in the full 3D model, a normal stress boundary condition at the inlet/outlet implies a Dirichlet condition for the pressure in the Reynolds equation associated with pressure-driven flow. Furthermore, we establish that a Dirichlet condition for the velocity at the inlet/outlet in the 3D model results in a Neumann condition for the pressure in the Reynolds equation.

1. Introduction

The fundamental governing equations of fluid flow are derived based on the principles of mass, momentum, energy balance and the second law of thermodynamics. When combined with appropriate boundary conditions and constitutive relations, these equations form a comprehensive set that describes the behavior of fluids under various flow conditions. However, this system of equations is highly complex and often poses numerical challenges, even with the aid of modern computers and software, in many realistic applications. Therefore, in engineering applications, it is essential to develop simplified mathematical models that offer computational efficiency, conceptual understanding of the flow, and suitability for optimal design, among other purposes. One area where successful derivation of such simplified models has been accomplished is when the fluid domain is thin, as observed in pipe flow, flow in Hele–Shaw cells, flow through porous media, and flow in channels between adjacent surfaces. In this article

we consider shear-driven and pressure-driven flow of power-law fluids between adjacent curved surfaces. By a “shear-driven” flow, we refer to a flow that is engendered due to a boundary or boundaries of the flow domain being moved which leads to relative motion between the boundaries, while by “pressure-driven” flows we refer to flows in domains where the boundaries are fixed and the flow is caused due to application of a pressure gradient (see the definition of these flows in Section 2.4).

In the late 19th century, the British engineer Henry Selby Hele–Shaw conducted pioneering investigations into fluid flow through narrow gaps between parallel plates (Hele–Shaw, 1898). He conducted experiments using a transparent cell consisting of two closely spaced glass plates. One of Hele–Shaw’s notable findings was the visualization of flow patterns and streamlines around various obstacles, such as cylinders, spheres, and plates. He observed phenomena such as flow separation, vortices, and the interaction between the fluid and the

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Nomenclature

D	Symmetric part of the velocity gradient (s^{-1})
g	Body force per unit mass (ms^{-2})
h	Distance between the surfaces (m)
L	Characteristic length (m)
U	Characteristic speed (ms^{-1})
u, v, w	Velocity components (ms^{-1})
u	Velocity field $\mathbf{u} = ui + vj$ (ms^{-1})
v	Velocity field $\mathbf{v} = ui + vj + wk$ (ms^{-1})
v_l	Velocity of the lower surface, $\mathbf{v}_l = (u_l, v_l, 0)$ (ms^{-1})
v_b	Velocity on Γ_v , $\mathbf{v}_b = (u_b, v_b, w_b)$ (ms^{-1})
u_b	Velocity on γ_v , $\mathbf{u}_b = (u_b, v_b, 0)$ (ms^{-1})
p	Mechanical pressure (Pa)
p_b	Pressure on Γ_T (Pa)
q	Volumetric local flow rate ($m^2 s^{-1}$)
r	Power index (–)
r'	Conjugate power index $1/r + 1/r' = 1$ (–)
T	Cauchy stress tensor (Pa)
η	Apparent viscosity (Pas)
μ	Consistency index in the relation $\eta = \mu \mathbf{D} ^{r-2}$ ($Pa s^{r-2}$)
ρ	Density (constant) ($kg m^{-3}$)
θ	Temperature (K)
Γ_u	Upper surface boundary of Ω where $\mathbf{v} = 0$ (–)
Γ_w	Vertical surface boundaries of Ω where $\mathbf{u} = \mathbf{u}_b$ (–)
Γ_v	Vertical surface boundary where $x = 0$ and $\mathbf{v} = \mathbf{v}_b$ (–)
Γ_T	Vertical surface boundary $\Gamma_w \setminus \Gamma_v$ where $\mathbf{T}\mathbf{n} = -p_b\mathbf{n}$ (–)
γ_v	Boundary part of $\partial\omega$ where $x = 0$ and $\mathbf{u} = \mathbf{u}_b$ (–)
γ_T	Boundary part $\partial\omega \setminus \gamma_v$ where $p = p_b$ (–)
Ω	Full 3D fluid domain (–)
$\partial\Omega$	Boundary of the 3D fluid domain Ω (–)
ω	Lower surface boundary of Ω (–)
$\partial\omega$	Boundary of ω (–)

obstacles. Since Hele-Shaw's groundbreaking investigations, numerous research papers have been dedicated to the mathematical modeling of various types of Hele-Shaw flows. Some examples related to power-law fluids are discussed in the following references: In [Aronsson and Janfalk \(1992\)](#), mathematical results regarding the p-harmonic equation and physical aspects of Hele-Shaw flow of power-law fluids are both discussed. Specifically, several exact solutions of the p-harmonic equation are presented, some of which are associated with Hele-Shaw flow of power-law fluids near a corner. In [Mikelić and Tapiéro \(1995\)](#) and [Fabricius et al. \(2022\)](#), an asymptotic analysis was performed as the gap between the surfaces tends to zero, resulting in the derivation of a type of nonlinear Poiseuille law that relates velocity and pressure gradient, as well as a Reynolds-type equation for the pressure. In [Mikelić and Tapiéro \(1995\)](#), these derivations were based on the balance of linear momentum, whereas in [Fabricius et al. \(2022\)](#), the inertial terms were a priori neglected. A notable difference between [Fabricius et al. \(2022\)](#) and [Mikelić and Tapiéro \(1995\)](#) lies in the boundary conditions applied. In [Fabricius et al. \(2022\)](#), homogeneous Dirichlet conditions

were applied to the velocity at the surfaces, with normal stress conditions on the sides (i.e., the inlet and outlet zones). In contrast, [Mikelić and Tapiéro \(1995\)](#) applied homogeneous Dirichlet conditions for the velocity along the entire boundary (i.e., no inlet or outlet). The stress boundary conditions employed in [Fabricius et al. \(2022\)](#) lead to a Dirichlet condition for the pressure in the corresponding Reynolds equation, which aligns with the physical situation in most applications. Additionally, [Fabricius et al. \(2022\)](#) differs from [Mikelić and Tapiéro \(1995\)](#) in terms of the presence of obstacles within the flow domain.

The flows discussed above are typically pressure-driven. In lubrication, the flow is driven by surface motion (shear-driven). More precisely, in lubrication one considers flow between two adjacent surfaces that are in relative motion, as shown in [Fig. 1](#). The first lower-dimensional model of lubrication flow was presented by Reynolds in his classical work ([Reynolds, 1886](#)). Using dimensional analysis and scaling, Reynolds motivated the following approximate model of flow in a narrow gap between two solid surfaces that are in relative motion when the fluid is modeled as an incompressible Navier–Stokes fluid:

$$\frac{\partial h}{\partial t} - \operatorname{div} \left(\frac{h^3}{12\mu} \nabla p \right) = -\operatorname{div} \left(\frac{h}{2} V \right) \quad \text{in } \omega \subset \mathbb{R}^2, \quad (1a)$$

$$\frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right) = \frac{\partial p}{\partial x}, \quad \frac{\partial}{\partial z} \left(\mu \frac{\partial v}{\partial z} \right) = \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z} = 0 \quad \text{in } \Omega \subset \mathbb{R}^3, \quad (1b)$$

where p is the pressure, $(u, v, w)^T$ the velocity of the fluid, $h(x, y)$ the distance between the surfaces, V is the sum of the velocities of the upper and lower surface, and μ the constant viscosity. The three-dimensional domain Ω and the two-dimensional domain ω are illustrated in [Fig. 1](#). More rigorous proofs of (1a) and (1b) can be found in [Bayada and Chambat \(1986\)](#) and [Fabricius et al. \(2013\)](#). Reynolds' pioneering work constitutes the backbone of mathematical modeling of lubrication and serves as the basis for thousands of journal articles devoted to lubrication. In particular, these fundamental results have been used to develop models of lubrication that include the effects of elasto-hydrodynamic lubrication (EHL) and hydrodynamic cavitation. For EHL, refer to [Lugt and Morales-Espejel \(2011\)](#), and for hydrodynamic cavitation, refer to [Almqvist et al. \(2014\)](#), [Elrod and Adams \(1975\)](#) and [Giacopini et al. \(2010\)](#). However, most of these models suffer from a common issue: they assume the validity of Reynolds' equation (1a) for the pressure and (1b) even though the assumptions under which they were derived are not satisfied. For example, in EHL, it is simply assumed that the film thickness and viscosity depend on the pressure. Such common misinterpretations and incorrect usage of Reynolds-type equations are discussed in [Almqvist et al. \(2021a,b, 2023\)](#). There is no Reynolds-type equation for power-law fluids. However, attempts to derive an lower-dimensional equation for the pressure have been carried out, see e.g. [Azeez and Bertola \(2021\)](#), [Dien and Elrod \(1983\)](#) and [Yang et al. \(2016\)](#). They are all based on some additional ad hoc assumptions, but the equations are still referred to as Reynolds-type equations.

The main objective of this paper is to address the question whether it is possible to, from the balance of linear momentum, derive a form of the Reynolds equation for fluids that are not Navier–Stokes fluids by just assuming a thin fluid film, as it is for Navier–Stokes fluids.¹ In particular, we will consider power-law fluids. We show that the feasibility of doing this depends on whether the flow is primarily driven by pressure or shear. In order to be able to discuss the difference between these two cases we will analyze them in parallel. We employ scaling and dimensional analysis techniques to derive simplified systems that accurately represents the balance of mass and linear momentum, both for pressure-driven and shear-driven flow scenarios. Furthermore, we establish a lower-dimensional model for the pressure in the case of pressure-driven flow, i.e. a type of nonlinear Reynolds equation. Moreover, we demonstrate the infeasibility of obtaining an explicit Reynolds-type equation for shear-driven flow.

¹ A fluid modeled by the Navier–Stokes constitutive relation is often referred to as a Newtonian fluid but this is a mis-attribution, see for example the discussion in [Dugas \(1988\)](#) and [Truesdell \(1960\)](#).

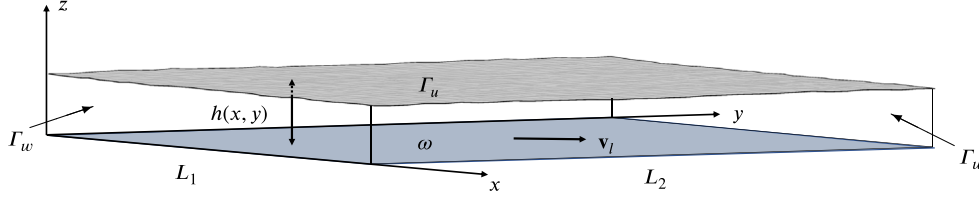


Fig. 1. Illustration of the fluid domain: The lower surface ω is a flat surface in the xy -plane, Γ_u is the upper surface, which may be curved or rough, Γ_w is the wall of the fluid domain, \mathbf{v}_l is the velocity of the lower surface and $h(x, y)$ is the distance between the surfaces.

2. Equations governing incompressible flow of power-law fluids between adjacent surfaces

In this section, we will outline the governing equations for the flow of incompressible power-law fluids. Furthermore, we will provide an illustrative example of a narrow fluid domain positioned between two curved surfaces, which will serve as the fluid domain in our analysis. Moreover, we will introduce two distinct types of boundary conditions: one that gives rise to pressure-driven flow, and another that leads to shear-driven flow.

2.1. The fluid model

Recently Rajagopal introduced a novel theory for modeling the rheological properties of fluids (Rajagopal, 2003, 2006). The main idea is to assume that there is an implicit constitutive relation between the stress \mathbf{T} and the symmetric part of the velocity gradient \mathbf{D} , i.e. that there is a tensor-valued function \mathbf{F} such that $\mathbf{F}(\rho, \theta, \mathbf{T}, \mathbf{D}) = \mathbf{0}$, where ρ is the density of the fluid and θ is the temperature. Requiring that the fluid is isotropic leads to \mathbf{F} being an isotropic function. Standard representation results yields that \mathbf{F} is of the form:

$$\mathbf{F}(\rho, \theta, \mathbf{T}, \mathbf{D}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{D} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{D}^2 + \alpha_5 (\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}) + \alpha_6 (\mathbf{T}^2\mathbf{D} + \mathbf{D}^2\mathbf{T}) + \alpha_7 (\mathbf{T}\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}) + \alpha_8 (\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = \mathbf{0}, \quad (2)$$

where the material moduli α_i , $i = 0, \dots, 8$, may depend on ρ , θ and the invariants

$$\text{tr} \mathbf{T}, \text{tr} \mathbf{D}, \text{tr} \mathbf{T}^2, \text{tr} \mathbf{D}^2, \text{tr} \mathbf{T}^3, \text{tr} \mathbf{D}^3, \text{tr} (\mathbf{T}\mathbf{D}), \text{tr} (\mathbf{T}^2\mathbf{D}), \text{tr} (\mathbf{D}^2\mathbf{T}), \text{tr} (\mathbf{D}^2\mathbf{T}^2).$$

This class of implicit constitutive relations (2) is very rich. In particular, it includes the classical Navier–Stokes fluids (Newtonian fluids), fluids with pressure dependent viscosity, Stokesian fluids, stress power-law fluids, fluids with activation criteria of Bingham or Herschel–Bulkley type, and shear-rate dependent fluids with discontinuities, see Blechta et al. (2020) for a classification of such fluids.

An important subset of implicit constitutive relations of the form (2) is those of the type

$$\alpha_0 \mathbf{I} + \mathbf{T} + \alpha_2 \mathbf{D} = \mathbf{0}. \quad (3)$$

If the fluid can be modeled as incompressible (constant density), then mass balance reduces to $\text{tr} \mathbf{D} = 0$. By considering the trace of both sides in (3) we deduce that $\alpha_0 = -3^{-1} \text{tr} \mathbf{T}$, i.e. α_0 is the mechanical pressure p . By considering α_2 which depends on θ , $\text{tr} \mathbf{T}$ and $\text{tr} \mathbf{D}^2$ we deduce the following constitutive relation

$$\left(-\frac{1}{3} \text{tr} \mathbf{T}\right) \mathbf{I} + \mathbf{T} + \alpha_2(\theta, \text{tr} \mathbf{T}, \text{tr} \mathbf{D}^2) \mathbf{D} = \mathbf{0}.$$

Since $\text{tr} \mathbf{T} = -3p$ and $\text{tr} \mathbf{D}^2 = |\mathbf{D}|^2$ this may be written as

$$\mathbf{T} = -p \mathbf{I} - \alpha_2(\theta, -3p, |\mathbf{D}|^2) \mathbf{D}.$$

In this paper we will consider the case when α_2 is on the form

$$\alpha_2(\theta, -3p, |\mathbf{D}|^2) = -2\mu |\mathbf{D}|^{r-2},$$

i.e. constitutive relations of the form

$$\mathbf{T} = -p \mathbf{I} + 2\mu |\mathbf{D}|^{r-2} \mathbf{D}, \quad (4)$$

where the constants μ and $1 < r < \infty$ are material constants. Corresponding to r we have the conjugate constant r' , where $1/r + 1/r' = 1$. Fluids described by a constitutive relation of the form (4) are called power-law fluids. The constant r is usually called the power index, and μ is called the consistency index. In order to get a shorter notation we define $\eta = \mu |\mathbf{D}|^{r-2}$ (the apparent viscosity). In this notation the constitutive relation becomes

$$\mathbf{T} = -p \mathbf{I} + 2\eta \mathbf{D}. \quad (5)$$

If $r = 2$, then the constitutive relation (4) is reduced to the Navier–Stokes constitutive relation and μ is the viscosity of the fluid.

2.2. Description of a typical thin domain between adjacent surfaces

This paper considers flow between adjacent surfaces. As a model example we will use the *thin* fluid domain Ω illustrated in Fig. 1. Without loss of generality we assume that one of the surfaces, denoted by ω , is a rectangular flat surface in the xy -plane. More precisely

$$\omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < L_1, 0 < y < L_2\}.$$

The other surface, denoted by Γ_u , is the graph of a smooth function h , i.e. Γ_u is of the form $z = h(x, y)$, where $(x, y) \in \omega$. We are interested in thin domains, i.e. $h(x, y) \ll L$, where $L = \min\{L_1, L_2\}$.

The boundary, $\partial\Omega$, of the fluid domain can be divided into three different parts, namely, the lower surface ω in the xy -plane, the upper surface Γ_u , where $z = h(x, y)$, and the vertical wall of the domain

$$\Gamma_w = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \partial\omega, 0 < z < h(x, y)\}.$$

2.3. Balance of mass and linear momentum for power-law fluids

With the constitutive relation (4) for power-law fluids, the governing equations for balance of mass and linear momentum become:

$$\text{div } \mathbf{v} = 0, \quad (6a)$$

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + 2 \text{div} (\mu |\mathbf{D}(\mathbf{v})|^{r-2} \mathbf{D}(\mathbf{v})), \quad (6b)$$

respectively, where $\mathbf{v} = (u, v, w)$ is the fluid velocity, ρ is the constant density of the fluid and $\mathbf{g} = (g_x, g_y, g_z)$ represents the body force acting per unit mass. The mathematical theory concerning the system (6) can be found in Málek et al. (1996). In the sequel we assume that \mathbf{g} is a constant vector. In component form, the balance of linear momentum for power-law fluids reads:

$$\begin{aligned} \rho \frac{Du}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + 2 \frac{\partial}{\partial x} \left(\eta \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\ &\quad + \frac{\partial}{\partial z} \left(\eta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right), \end{aligned} \quad (7a)$$

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(\eta \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + 2 \frac{\partial}{\partial y} \left(\eta \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(\eta \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right), \quad (7b)$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left(\eta \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right) + \frac{\partial}{\partial y} \left(\eta \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right) + 2 \frac{\partial}{\partial z} \left(\eta \frac{\partial w}{\partial z} \right). \quad (7c)$$

2.4. Boundary conditions

In order to have a well-posed problem we must add boundary conditions. There are many possibilities and it is important to choose the boundary conditions that correctly describe the flow one considers, see e.g. [Boyer and Fabrie \(2013\)](#) and [Temam \(2001\)](#). In this work we will consider two different types of boundary conditions, which render a well-posed problem:

- **Pressure-driven flow:** Both surfaces are stationary. No-slip conditions are imposed at the upper and lower surfaces, i.e. $\mathbf{v} = 0$ on Γ_u and ω . The wall of the fluid domain Γ_w is divided into two distinct parts, denoted by Γ_v and Γ_T . The velocity is given on Γ_v and the normal stress is given on Γ_T . Indeed,

$$\mathbf{v} = \mathbf{v}_b \text{ on } \Gamma_v \quad \text{and} \quad \mathbf{T}\mathbf{n} = -p_b \mathbf{n} \text{ on } \Gamma_T,$$

where $\mathbf{v}_b = \mathbf{v}_b(x, y)$ and $p_b = p_b(x, y)$ are given functions on ω .

- **Shear-driven flow:** One of the bounding surfaces is moving. In our case the lower surface ω is moving in the xy -plane with the velocity $\mathbf{v}_l = (u_l, v_l, 0)$, while the upper surface Γ_u is stationary. No-slip conditions are imposed at the surfaces, i.e. $\mathbf{v} = \mathbf{v}_l$ on ω and $\mathbf{v} = 0$ on Γ_u . The boundary condition on the wall of the fluid domain, Γ_w , is the same as in the pressure-driven case described above.

We assume that Γ_T has nonzero measure, which implies that both the pressure and velocity are unique, see [Fabricius \(2019\)](#).

3. Derivation of lower-dimensional models

In this section we will derive simplified models of thin-film flow by scaling and dimensional analysis. We will explore two specific scenarios: one where the flow is pressure-driven and another where it is shear-driven.

3.1. Independent and dependent dimensionless variables

Let us now discuss how to express the governing system of equations in dimensionless form in an appropriate way. We start by defining new non-dimensional independent and dependent variables:

$$\bar{x} = x/x_*, \quad \bar{y} = y/y_*, \quad \bar{z} = z/z_*, \quad \bar{t} = t/t_*,$$

$$\bar{u} = u/u_*, \quad \bar{v} = v/v_*, \quad \bar{w} = w/w_*, \quad \bar{p} = p/p_*, \quad \bar{\eta} = \eta/\eta_*,$$

where the bar denotes that it is a dimensionless variable and the subscript $*$ denotes that the parameter represents a characteristic scale.

Let us now discuss the characteristic scales associated with the problem at hand:

- The characteristic length scales in the x - and y -direction are naturally $x_* = L_1$ and $y_* = L_2$, see [Fig. 1](#). However, without loss of generality, we introduce a common characteristic scale L , i.e. $x_* = y_* = L$.
- We consider flow between two adjacent surfaces, i.e. the distance between them is significantly smaller than the size of the surfaces. Hence, the characteristic length scale in the z -direction is $z_* = \varepsilon L$, where $\varepsilon \ll 1$. This implies that ε is a small parameter related to the thickness of the fluid domain.

- We assume that the characteristic speed in the xy -plane is U and, therefore, we choose $u_* = v_* = U$.
- We choose $t_* = L/U$, i.e. roughly the time it takes for a typical fluid particle to travel from the inlet to the outlet.

It now remains to determine w_* , p_* , U and η_* .

3.2. Scaling of the balance of mass

In terms of the dimensionless variables and the scaling parameters, the balance of mass (6a) becomes

$$\frac{U}{L} \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{U}{L} \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{w_*}{\varepsilon L} \frac{\partial \bar{w}}{\partial \bar{z}} = 0.$$

To avoid unnecessary restrictions (or unrealistic flow behavior) for the flow we choose $w_* = \varepsilon U$, see the analysis in [Almqvist et al. \(2021b\)](#) and the discussion in [Dowson \(1962\)](#). This yields the following dimensionless equation representing balance of mass

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0.$$

Now it remains to determine p_* , U and η_* .

3.3. Scaling of the constitutive relation

Let us now determine the characteristic apparent viscosity η_* . The symmetric part of the velocity gradient is

$$\mathbf{D} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z} \end{pmatrix}.$$

On scaling, we obtain

$$\mathbf{D} = D_* \bar{\mathbf{D}},$$

where

$$D_* = \frac{U}{L\varepsilon}$$

and

$$\bar{\mathbf{D}} = \begin{pmatrix} \varepsilon \frac{\partial \bar{u}}{\partial \bar{x}} & \frac{\varepsilon}{2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) & \frac{1}{2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \varepsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \\ \frac{\varepsilon}{2} \left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right) & \frac{\partial \bar{v}}{\partial \bar{y}} & \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \varepsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \\ \frac{1}{2} \left(\varepsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{z}} \right) & \frac{1}{2} \left(\varepsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{z}} \right) & \varepsilon \frac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix}.$$

We note that the dominating elements in $\bar{\mathbf{D}}$ are of order ε^0 . For small values of ε , i.e. thin fluid domains, we have the following approximation:

$$\bar{\mathbf{D}} \approx \bar{\mathbf{d}} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \frac{\partial \bar{u}}{\partial \bar{z}} \\ 0 & 0 & \frac{1}{2} \frac{\partial \bar{v}}{\partial \bar{z}} \\ \frac{1}{2} \frac{\partial \bar{u}}{\partial \bar{z}} & \frac{1}{2} \frac{\partial \bar{v}}{\partial \bar{z}} & 0 \end{pmatrix}.$$

The constitutive relation (4) involves $|\mathbf{D}|$, which, when expanded, can be written as follows

$$|\mathbf{D}| = \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial w}{\partial y} + \frac{1}{2} \left(\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right)^{1/2}.$$

Hence, scaling leads to

$$|\mathbf{D}| = D_* |\bar{\mathbf{D}}|$$

$$\begin{aligned}
&= \frac{U}{L\epsilon} \left[\epsilon^2 \left(\frac{\partial \bar{u}}{\partial \bar{x}} \right)^2 + \epsilon^2 \left(\frac{\partial \bar{v}}{\partial \bar{y}} \right)^2 + \epsilon^2 \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^2 + \epsilon^2 \frac{\partial \bar{u}}{\partial \bar{y}} \frac{\partial \bar{v}}{\partial \bar{x}} \right. \\
&\quad + \epsilon^2 \frac{\partial \bar{u}}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial \bar{x}} + \epsilon^2 \frac{\partial \bar{v}}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial \bar{y}} + \frac{\epsilon^2}{2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 \\
&\quad \left. + \frac{\epsilon^2}{2} \left(\frac{\partial \bar{v}}{\partial \bar{x}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} \right)^2 + \frac{\epsilon^4}{2} \left(\frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 + \frac{\epsilon^4}{2} \left(\frac{\partial \bar{w}}{\partial \bar{y}} \right)^2 \right]^{1/2},
\end{aligned}$$

and we can, therefore, write

$$\eta = \mu |\mathbf{D}|^{r-2} = \mu D_*^{r-2} |\bar{\mathbf{D}}|^{r-2} = \mu \left(\frac{U}{L\epsilon} \right)^{r-2} |\bar{\mathbf{D}}|^{r-2} = \eta_* \bar{\eta},$$

where

$$\eta_* = \mu \left(\frac{U}{L\epsilon} \right)^{r-2} \quad \text{and} \quad \bar{\eta} = |\bar{\mathbf{D}}|^{r-2}.$$

For small values of ϵ we have

$$\bar{\eta} = |\bar{\mathbf{D}}|^{r-2} \approx \left[\frac{1}{2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} \right)^2 \right]^{(r-2)/2} = |\bar{\mathbf{d}}|^{r-2}.$$

It remains to determine p_* and U .

3.4. Scaling of the balance of linear momentum for power-law fluids

In terms of the dimensionless variables and the scaling parameters, the components (7a)–(7c) of the balance of linear momentum for power-law fluids becomes

$$\begin{aligned}
\rho \frac{U^2}{L} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) &= \rho g_x - \frac{p_*}{L} \frac{\partial \bar{p}}{\partial \bar{x}} \\
&+ \frac{2U^{r-1}\mu}{\epsilon^{r-2}L^r} \frac{\partial}{\partial \bar{x}} \left(\bar{\eta} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \frac{U^{r-1}\mu}{\epsilon^{r-2}L^r} \frac{\partial}{\partial \bar{y}} \left(\bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) \\
&+ \frac{U^{r-1}\mu}{\epsilon^r L^r} \frac{\partial}{\partial \bar{z}} \left(\bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right), \tag{8a}
\end{aligned}$$

$$\begin{aligned}
\rho \frac{U^2}{L} \left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} \right) &= \rho g_y - \frac{p_*}{L} \frac{\partial \bar{p}}{\partial \bar{y}} \\
&+ \frac{U^{r-1}\mu}{\epsilon^{r-2}L^r} \frac{\partial}{\partial \bar{x}} \left(\bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) + \frac{2U^{r-1}\mu}{\epsilon^{r-2}L^r} \frac{\partial}{\partial \bar{y}} \left(\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \\
&+ \frac{U^{r-1}\mu}{\epsilon^r L^r} \frac{\partial}{\partial \bar{z}} \left(\bar{\eta} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right), \tag{8b}
\end{aligned}$$

$$\begin{aligned}
\frac{\epsilon \rho U^2}{L} \left(\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) &= \rho g_z - \frac{p_*}{\epsilon L} \frac{\partial \bar{p}}{\partial \bar{z}} \\
&+ \frac{U^{r-1}\mu}{\epsilon^{r-1}L^r} \frac{\partial}{\partial \bar{x}} \left(\bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right) + \frac{U^{r-1}\mu}{\epsilon^{r-1}L^r} \frac{\partial}{\partial \bar{y}} \left(\bar{\eta} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right) \\
&+ \frac{2U^{r-1}\mu}{\epsilon^{r-1}L^r} \frac{\partial}{\partial \bar{z}} \left(\bar{\eta} \frac{\partial \bar{w}}{\partial \bar{z}} \right). \tag{8c}
\end{aligned}$$

Multiplying (8a) and (8b) by $\epsilon^r L^r / (U^{r-1} \mu)$ and (8c) by $\epsilon^{r-1} L^r / (U^{r-1} \mu)$ and using that $\bar{\eta} = |\bar{\mathbf{D}}|^{r-2}$ yields the following dimensionless components of the balance of linear momentum:

$$\begin{aligned}
\frac{\rho U^{3-r} \epsilon^r L^{r-1}}{\mu} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) &= \frac{\epsilon^r \rho L^r}{\mu U^{r-1}} g_x - \frac{\epsilon^r L^{r-1} p_*}{\mu U^{r-1}} \frac{\partial \bar{p}}{\partial \bar{x}} \\
&+ \epsilon^2 2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \epsilon^2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) \\
&+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right), \tag{9a}
\end{aligned}$$

$$\begin{aligned}
\frac{\rho U^{3-r} \epsilon^r L^{r-1}}{\mu} \left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} \right) &= \frac{\epsilon^r \rho L^r}{\mu U^{r-1}} g_y - \frac{\epsilon^r L^{r-1} p_*}{\mu U^{r-1}} \frac{\partial \bar{p}}{\partial \bar{y}} \\
&+ \epsilon^2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) + \epsilon^2 2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \\
&+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right), \tag{9b}
\end{aligned}$$

$$\frac{\epsilon^r \rho U^{3-r} L^{r-1}}{\mu} \left(\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) = \frac{\rho \epsilon^{r-1} L^r}{\mu U^{r-1}} g_z - \frac{p_* \epsilon^{r-2} L^{r-1}}{\mu U^{r-1}} \frac{\partial \bar{p}}{\partial \bar{z}}$$

$$\begin{aligned}
&+ \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right) + \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right) \\
&+ 2 \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{w}}{\partial \bar{z}} \right). \tag{9c}
\end{aligned}$$

It remains to determine the characteristic pressure p_* and the in-plane characteristic speed U (recall that $U = u_* = v_*$). As we will see, the appropriate scaling depends on whether the flow is pressure-driven or shear-driven.

3.5. A simplified model of pressure-driven flow between two adjacent surfaces

Assume that the flow is driven by an external pressure, which is given on the lateral part of the fluid domain. Let p_A denote the difference between the maximal and minimal external pressure. In this case, we choose the scaling $p_* = p_A$. Balancing the pressure and viscous forces so that we obtain 1 in front of $\partial_x p$ and $\partial_y p$ yields the characteristic in-plane speed

$$U = L \left(\frac{p_A}{\mu} \right)^{1/(r-1)} \epsilon^{r/(r-1)} = L \left(\frac{p_A}{\mu} \right)^{1/(r-1)} \epsilon^{r'},$$

and the components (9a)–(9c) of the balance of linear momentum become

$$\begin{aligned}
\frac{\rho L^2 p_A^{(3-r)/(r-1)}}{\mu^{2/(r-1)}} \epsilon^{2r/(r-1)} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) &= \frac{\rho L}{p_A} g_x - \frac{\partial \bar{p}}{\partial \bar{x}} \\
&+ \epsilon^2 2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \epsilon^2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) \\
&+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right), \tag{10a}
\end{aligned}$$

$$\begin{aligned}
\frac{\rho L^2 p_A^{(3-r)/(r-1)}}{\mu^{2/(r-1)}} \epsilon^{2r/(r-1)} \left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} \right) &= \frac{\rho L}{p_A} g_y - \frac{\partial \bar{p}}{\partial \bar{y}} \\
&+ \epsilon^2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) + \epsilon^2 2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \\
&+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right), \tag{10b}
\end{aligned}$$

$$\begin{aligned}
\frac{\rho L^2 p_A^{(3-r)/(r-1)}}{\mu^{2/(r-1)}} \epsilon^{2r/(r-1)} \left(\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) &= \frac{1}{\epsilon} \frac{\rho L}{p_A} g_z - \frac{1}{\epsilon^2} \frac{\partial \bar{p}}{\partial \bar{z}} \\
&+ \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right) + \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right) \\
&+ 2 \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{w}}{\partial \bar{z}} \right). \tag{10c}
\end{aligned}$$

If we only keep the dominating terms in each equation, we obtain the following simplified system of balance of linear momentum:

$$0 = \frac{\rho L}{p_A} g_x - \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{d}}|^{r-2} \frac{\partial \bar{u}}{\partial \bar{z}} \right), \tag{11a}$$

$$0 = \frac{\rho L}{p_A} g_y - \frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{d}}|^{r-2} \frac{\partial \bar{v}}{\partial \bar{z}} \right), \tag{11b}$$

$$0 = \frac{\partial \bar{p}}{\partial \bar{z}}. \tag{11c}$$

In dimensional form this becomes

$$0 = \rho g_x - \frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial z} \left(|\mathbf{d}|^{r-2} \frac{\partial u}{\partial z} \right), \tag{12a}$$

$$0 = \rho g_y - \frac{\partial p}{\partial y} + \mu \frac{\partial}{\partial z} \left(|\mathbf{d}|^{r-2} \frac{\partial v}{\partial z} \right), \tag{12b}$$

$$0 = \frac{\partial p}{\partial z}. \tag{12c}$$

where

$$|\mathbf{d}|^{r-2} = \left[\frac{1}{2} \left(\frac{\partial u}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial z} \right)^2 \right]^{(r-2)/2}. \tag{13}$$

In particular, (12c) implies that the pressure does not depend on z , i.e., $p = p(x, y)$.

Remark. p_* is of order ϵ^0 and U is of order $\epsilon^{r'}$.

3.6. A simplified model of shear-driven between two adjacent surfaces

Assume that the flow is driven by a relative motion between the surfaces. For example, if the lower surface is moving with velocity $\mathbf{v}_l = (u_l, v_l, 0)$ and the upper surface is stationary, then a natural choice of U is $|\mathbf{v}_l|$. To obtain 1 in front of $\partial_x p$ and $\partial_y p$ in (9a) and (9b) the characteristic pressure should be chosen as

$$p_* = \frac{\mu U^{r-1}}{L^{r-1}} \epsilon^{-r},$$

which leads to

$$\begin{aligned} \frac{\rho U^{3-r} \epsilon^r L^{r-1}}{\mu} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{u}}{\partial \bar{z}} \right) &= \frac{\epsilon^r \rho L^r}{\mu U^{r-1}} g_x - \frac{\partial \bar{p}}{\partial \bar{x}} \\ &+ \epsilon^2 2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \epsilon^2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}} \right) \right) \\ &+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{x}} \right) \right), \end{aligned} \quad (14a)$$

$$\begin{aligned} \frac{\rho U^{3-r} \epsilon^r L^{r-1}}{\mu} \left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{v}}{\partial \bar{z}} \right) &= \frac{\epsilon^r \rho L^r}{\mu U^{r-1}} g_y - \frac{\partial \bar{p}}{\partial \bar{y}} \\ &+ \epsilon^2 \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} \right) \right) + \epsilon^2 2 \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{v}}{\partial \bar{y}} \right) \\ &+ \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \epsilon^2 \frac{\partial \bar{w}}{\partial \bar{y}} \right) \right), \end{aligned} \quad (14b)$$

$$\begin{aligned} \frac{\epsilon^r \rho U^{3-r} L^{r-1}}{\mu} \left(\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) &= \frac{\rho \epsilon^{r-1} L^r}{\mu U^{r-1}} g_z - \frac{1}{\epsilon^2} \frac{\partial \bar{p}}{\partial \bar{z}} \\ &+ \frac{\partial}{\partial \bar{x}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{w}}{\partial \bar{x}} + \epsilon^2 \frac{\partial \bar{u}}{\partial \bar{z}} \right) \right) + \frac{\partial}{\partial \bar{y}} \left(|\bar{\mathbf{D}}|^{r-2} \left(\frac{\partial \bar{w}}{\partial \bar{y}} + \epsilon^2 \frac{\partial \bar{v}}{\partial \bar{z}} \right) \right) \\ &+ 2 \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{D}}|^{r-2} \frac{\partial \bar{w}}{\partial \bar{z}} \right). \end{aligned} \quad (14c)$$

Keeping only the dominating terms renders the following approximation of the balance of linear momentum:

$$0 = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{d}}|^{r-2} \frac{\partial \bar{u}}{\partial \bar{z}} \right), \quad (15a)$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{z}} \left(|\bar{\mathbf{d}}|^{r-2} \frac{\partial \bar{v}}{\partial \bar{z}} \right), \quad (15b)$$

$$0 = \frac{\partial \bar{p}}{\partial \bar{z}}, \quad (15c)$$

which in dimensional form becomes

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial z} \left(|\mathbf{d}|^{r-2} \frac{\partial u}{\partial z} \right), \quad (16a)$$

$$0 = -\frac{\partial p}{\partial y} + \mu \frac{\partial}{\partial z} \left(|\mathbf{d}|^{r-2} \frac{\partial v}{\partial z} \right), \quad (16b)$$

$$0 = \frac{\partial p}{\partial z}, \quad (16c)$$

where \mathbf{d} is the same as in (13) and, as a consequence of (16c), $p = p(x, y)$.

Remark. U is of order ϵ^0 and p_* is of order ϵ^{-r} .

4. A lower-dimensional equation for the pressure

In this section we will investigate if it is possible to derive a form of the Reynolds equation, for flow of power-law fluids between two adjacent surfaces, i.e. if it is possible to deduce a lower-dimensional equation for the pressure. We will consider the cases of pressure-driven flow and shear-driven flow separately.

4.1. A variant of the Reynolds equation for pressure-driven flow of power-law fluids

We will now try to derive a lower-dimensional model for the pressure from the simplified system (12), when we have pressure-driven flow. To this end, we introduce the following notation: $\mathbf{u} = (u, v)$, $\mathbf{g} = (g_x, g_y)$ and $\mathbf{x} = (x, y)$, in which (12) become

$$\frac{\partial}{\partial z} \left(\left| \frac{\partial \mathbf{u}}{\partial z} \right|^{r-2} \frac{\partial \mathbf{u}}{\partial z} \right) = \mathbf{f}, \quad (17a)$$

$$\frac{\partial p}{\partial z} = 0, \quad (17b)$$

where

$$\mathbf{f} = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} = \frac{2^{(r-2)/2}}{\mu} (\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}). \quad (18)$$

Integrating (17a) with respect to z yields

$$\left| \frac{\partial \mathbf{u}}{\partial z} \right|^{r-2} \frac{\partial \mathbf{u}}{\partial z} = \mathbf{f} z + \mathbf{a}(\mathbf{x}), \quad (19)$$

where $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}))$. By taking absolute value of both sides of (19), we obtain

$$\left| \frac{\partial \mathbf{u}}{\partial z} \right|^{r-1} = |\mathbf{f} z + \mathbf{a}(\mathbf{x})|,$$

and consequently,

$$\left| \frac{\partial \mathbf{u}}{\partial z} \right| = |\mathbf{f} z + \mathbf{a}(\mathbf{x})|^{\frac{1}{r-1}}.$$

When inserted into (19) together with that $(2-r)/(r-1) = r'-2$ leads to

$$\frac{\partial \mathbf{u}}{\partial z} = \left| \frac{\partial \mathbf{u}}{\partial z} \right|^{2-r} (\mathbf{f} z + \mathbf{a}(\mathbf{x})) = |\mathbf{f} z + \mathbf{a}(\mathbf{x})|^{r'-2} (\mathbf{f} z + \mathbf{a}(\mathbf{x})). \quad (20)$$

Let us now compute \mathbf{a} . Integrating across the film yields

$$\int_0^h \frac{\partial \mathbf{u}}{\partial z} dz = \int_0^h |\mathbf{f} z + \mathbf{a}|^{r'-2} (\mathbf{f} z + \mathbf{a}) dz,$$

and the no-slip boundary conditions, see Section 2.4, on the surfaces implies that

$$0 = \int_0^h |\mathbf{f} z + \mathbf{a}|^{r'-2} (\mathbf{f} z + \mathbf{a}) dz. \quad (21)$$

The problem is now to find \mathbf{a} . It can be observed that (21) is the Euler equation associated with the variational problem

$$\min_{\mathbf{a} \in \mathbb{R}^2} \left\{ \int_0^h \frac{1}{r'} |\mathbf{f} z + \mathbf{a}|^{r'} dz \right\}. \quad (22)$$

From the variational problem (22) we see that \mathbf{a} must point in the negative direction of the vector \mathbf{f} , i.e. \mathbf{a} is of the form

$$\mathbf{a} = -t\mathbf{f}, \quad 0 \leq t \leq h,$$

which reduces the Euler equation (21) to finding t such that

$$\int_0^h |z - t|^{r'-2} (z - t) dz = 0. \quad (23)$$

Indeed,

$$0 = \int_0^h |z - t|^{r'-2} (z - t) dz = \frac{1}{r'} \int_0^h \frac{d}{dz} |z - t|^{r'} dz = \frac{1}{r'} (|h - t|^{r'} - |t|^{r'}),$$

which yields $t = h/2$. Hence,

$$\mathbf{a} = -\frac{h}{2} \mathbf{f}.$$

When inserted into (20) yields

$$\frac{\partial \mathbf{u}}{\partial z} = \left| \mathbf{f} z - \frac{h}{2} \mathbf{f} \right|^{r'-2} \left(\mathbf{f} z - \frac{h}{2} \mathbf{f} \right) = |\mathbf{f}|^{r'-2} \mathbf{f} \left| z - \frac{h}{2} \right|^{r'-2} \left(z - \frac{h}{2} \right).$$

To obtain an expression for \mathbf{u} we integrate with respect to z . Indeed,

$$\begin{aligned}\mathbf{u}(\mathbf{x}, z) &= |\mathbf{f}|^{r'-2} \mathbf{f} \int_0^z \left| s - \frac{h}{2} \right|^{r'-2} \left(s - \frac{h}{2} \right) ds \\ &= |\mathbf{f}|^{r'-2} \mathbf{f} \frac{1}{r'} \int_0^z \frac{d}{ds} \left| s - \frac{h}{2} \right|^{r'} ds \\ &= |\mathbf{f}|^{r'-2} \mathbf{f} \frac{1}{r'} \left(\left| z - \frac{h}{2} \right|^{r'} - \left(\frac{h}{2} \right)^{r'} \right).\end{aligned}\quad (24)$$

The volumetric effective local flow rate \mathbf{q} is

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \int_0^h \mathbf{u} dz.$$

Let us compute q_1 :

$$\begin{aligned}q_1 &= \int_0^h u dz = |\mathbf{f}|^{r'-2} f_1 \frac{1}{r'} \int_0^h \left(\left| z - \frac{h}{2} \right|^{r'} - \left(\frac{h}{2} \right)^{r'} \right) dz \\ &= |\mathbf{f}|^{r'-2} f_1 \frac{1}{r'} \left(\int_0^h \left| z - \frac{h}{2} \right|^{r'} dz - \frac{h^{r'+1}}{2^{r'}} \right).\end{aligned}\quad (25)$$

The integral on the right-hand side is determined as follows:

$$\begin{aligned}\int_0^h \left| z - \frac{h}{2} \right|^{r'} dz &= \int_0^{h/2} \left(\frac{h}{2} - z \right)^{r'} dz + \int_{h/2}^h \left(z - \frac{h}{2} \right)^{r'} dz \\ &= \left[-\frac{1}{r'+1} \left(\frac{h}{2} - z \right)^{r'+1} \right]_0^{h/2} + \left[\frac{1}{r'+1} \left(z - \frac{h}{2} \right)^{r'+1} \right]_{h/2}^h \\ &= \frac{h^{r'+1}}{2^{r'}(r'+1)}.\end{aligned}\quad (26)$$

Inserting (26) into (25) yields

$$q_1 = -\frac{h^{r'+1}}{2^{r'}(r'+1)} |\mathbf{f}|^{r'-2} f_1.$$

In the same way we obtain

$$q_2 = -\frac{h^{r'+1}}{2^{r'}(r'+1)} |\mathbf{f}|^{r'-2} f_2.$$

Thus we have derived the following local type of Poiseuille law, i.e., a relation between the volumetric effective local flow rate \mathbf{q} and $\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}$:

$$\mathbf{q} = -\frac{h^{r'+1}}{2^{r'}(r'+1)} |\mathbf{f}|^{r'-2} \mathbf{f} = -\psi |\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}|^{r'-2} (\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}), \quad (27)$$

where the flow factor ψ is

$$\psi = \frac{2^{1-r'/2} h^{r'+1}}{(r'+1) \mu^{r'-1}}.$$

Let us now derive a Reynolds-type equation for the pressure. Integrating (6a), i.e. the balance of mass, across the film gives

$$\int_0^{h(x,y)} \frac{\partial w}{\partial z} dz = - \int_0^{h(x,y)} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} dz,$$

and by using the Leibniz rule, stating that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = f(x, b(x)) \frac{db}{dx} - f(x, a(x)) \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x, t) dt$$

we obtain

$$w(x, y, h) - w(x, y, 0) = -\frac{\partial}{\partial x} \int_0^h u dz - \frac{\partial}{\partial y} \int_0^h v dz. \quad (28)$$

Due to the homogeneous Dirichlet boundary conditions for the velocity at the surfaces, the left-hand side is zero. The integrals on the right-hand side are by definition q_1 and q_2 . Hence, the equation (28) may be written as

$$\operatorname{div}_{\mathbf{x}} \mathbf{q} = 0, \quad (29)$$

where $\operatorname{div}_{\mathbf{x}}$ is the divergence operator with respect to x and y . By inserting (27) into (29) we obtain a lower-dimensional equation for the

pressure, namely

$$\operatorname{div}_{\mathbf{x}} \left(\psi |\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}|^{r'-2} (\nabla_{\mathbf{x}} p(\mathbf{x}) - \rho \mathbf{g}) \right) = 0 \quad \text{in } \omega. \quad (30)$$

The equation (30) reduces to the classical Reynolds equation (1a) for stationary surfaces when $r = 2$ (Navier–Stokes fluids) and $\mathbf{g} = 0$.

4.2. A variant of the Reynolds equation for shear-driven flow of power-law fluids

Let us next consider the case when the flow is shear-driven. We start from the simplified flow model (16), which in vector notation reads:

$$\frac{\partial}{\partial z} \left(\left| \frac{\partial \mathbf{u}}{\partial z} \right|^{r-2} \frac{\partial \mathbf{u}}{\partial z} \right) = \mathbf{f}, \quad (31a)$$

$$\frac{\partial p}{\partial z} = 0, \quad (31b)$$

where

$$\mathbf{f} = \mathbf{f}(\mathbf{x}) = \frac{2^{(r-2)/2}}{\mu} \nabla_{\mathbf{x}} p(\mathbf{x}).$$

In the same way as for the pressure-driven case, see Section 4.1, we obtain

$$\frac{\partial \mathbf{u}}{\partial z} = |\mathbf{f} z + \mathbf{a}(\mathbf{x})|^{r-2} (\mathbf{f} z + \mathbf{a}(\mathbf{x})). \quad (32)$$

Let us try to determine \mathbf{a} by the same procedure that we used in the pressure-driven case. Indeed, by integrating across the fluid and using the no-slip boundary conditions at the surfaces yields

$$-\mathbf{u}_l = \int_0^h |\mathbf{f} z + \mathbf{a}|^{r-2} (\mathbf{f} z + \mathbf{a}) dz, \quad (33)$$

where $\mathbf{u}_l = (u_l, v_l)$. Equation (33) is the Euler equation associated with the variational problem

$$\min_{\mathbf{a} \in \mathbb{R}^2} \left\{ \int_0^h \frac{1}{r'} |\mathbf{f} z + \mathbf{a}|^{r'} + \frac{1}{h} \mathbf{u}_l \cdot \mathbf{a} dz \right\}. \quad (34)$$

Here we note a fundamental difference compared to the pressure-driven case. Namely, we cannot conclude that \mathbf{a} must point in the negative direction of the vector \mathbf{f} , i.e. \mathbf{a} is not always of the form $\mathbf{a} = -t\mathbf{f}$, $0 \leq t \leq h$. This means that we cannot use the homogeneity of the function $f(\cdot) = |\cdot|^{r'}$ to reduce the Euler equation as it was done in (23), which means that we are not able to determine \mathbf{a} explicitly, except when $r' = 2$. In the case $r' = 2$ we obtain

$$\mathbf{a} = -\frac{1}{h} \mathbf{u}_l - \frac{h}{2} \mathbf{f}.$$

Since \mathbf{a} cannot, in general, be determined explicitly, we conclude that it is not possible to derive a Reynolds-type equation for the pressure in the shear-driven case.

We have just found out that it is not, in general, feasible to derive a form of Reynolds' equation for shear-driven flow of non-Newtonian fluids. Nevertheless, there are numerous papers presenting modified forms of Reynolds' equations, obtained by introducing additional assumptions. Often, these assumptions are evidently incorrect; however, the resulting outcome can provide a pragmatic approach for computing the pressure, which may prove useful for engineering purposes. Let us briefly discuss a few of these attempts. In Yang et al. (2016), the authors initiate their analysis by introducing a simplified governing equation for the flow, where certain elements in the vector \mathbf{d} are omitted. Subsequently, they model the flow by combining the Poiseuille and Couette flows through a superposition approach, despite the nonlinearity of the governing equation. Another version of Reynolds equation for power-law fluids is derived in Azeez and Bertola (2021) and Dien and Elrod (1983). The starting point of the analysis is a simplified model of the flow. To analyze this model a small parameter is introduced in order to model the pressure gradient; subsequently a standard perturbation procedure is used to obtain the velocity field, which in turn is used to obtain a type of Reynolds equation for power-law fluids. It is

claimed that the equation applies better to Couette dominated flow. A weakness is that there are implicitly two small parameters involved in the problem, one related to the film thickness and the other which is introduced by the authors. Therefore it is necessary to start the analysis from the balance of linear momentum for power-law fluids. For example as in the case where one small parameter is related to the film thickness and one is related to the wavelength of the surface roughness, see e.g. Fabricius et al. (2017). In the above-mentioned works (Azeez and Bertola, 2021; Dien and Elrod, 1983; Yang et al., 2016) numerical results are presented. However, they do not provide any comparisons between their approximate solutions and the corresponding solutions to the full balance of linear momentum, which is a crucial step in justifying the use of the simplified models.

4.3. Boundary conditions for the Reynolds-type equation associated with pressure driven flow

As we saw in Section 4.1 it is possible to derive a lower-dimensional equation for the pressure, i.e., equation (30) when the flow is pressure driven. We will now determine the appropriate boundary conditions that should be imposed on the boundary $\partial\omega$.

On Γ_T we have normal stress boundary condition $T\mathbf{n} = -p_b\mathbf{n}$. By using the constitutive relation (4) this can be written as

$$(-p\mathbf{I} + 2\mu|\mathbf{D}|^{r-2}\mathbf{D})\mathbf{n} = -p_b\mathbf{n}$$

The dimensionless form reads

$$\left(-\bar{p}\mathbf{I} + \frac{2\mu}{p_\Delta}\left(\frac{U}{L\varepsilon}\right)^{r-1}|\bar{\mathbf{D}}|^{r-2}\bar{\mathbf{D}}\right)\mathbf{n} = -\frac{p_b}{p_\Delta}\mathbf{n},$$

where

$$U = L\left(\frac{p_\Delta}{\mu}\right)^{1/(r-1)}\varepsilon^{r/(r-1)},$$

that is

$$\left(-\bar{p}\mathbf{I} + 2\varepsilon|\bar{\mathbf{D}}|^{r-2}\bar{\mathbf{D}}\right)\mathbf{n} = -\frac{p_b}{p_\Delta}\mathbf{n}.$$

Recall that the elements in $\bar{\mathbf{D}}$ are of different orders of ε , but the dominating elements are of order ε^0 . For small values of ε we obtain the approximate dimensionless boundary condition $\bar{p} = p_b/p_\Delta$, which in dimensional form is $p = p_b$. In particular, $p = p_b$ on γ_T , where γ_T is the projection of Γ_T onto the xy -plane.

On Γ_v the velocity is given. More precisely $(u, v, w) = (u_b, v_b, w_b)$. We will now determine what this means for the pressure boundary condition on γ_v , i.e., on the projection of Γ_v onto the xy -plane. Indeed, integrating the first two components in the boundary condition across the fluid film yields

$$\int_0^h u \, dz = u_b h \quad \text{and} \quad \int_0^h v \, dz = v_b h$$

According to the analysis in Section 4.1, see (27), we obtain that for small values of ε this is approximately

$$-\psi|\nabla_x p(\mathbf{x}) - \rho\mathbf{g}|^{r-2}(\nabla_x p(\mathbf{x}) - \rho\mathbf{g}) = h\mathbf{u}_b,$$

where $\mathbf{u}_b = (u_b, v_b)$. From this we deduce the following boundary condition of Neumann-type for the pressure:

$$|\nabla_x p(\mathbf{x}) - \rho\mathbf{g}|^{r-2}(\nabla_x p(\mathbf{x}) - \rho\mathbf{g}) \cdot \mathbf{v} = -\frac{h}{\psi}\mathbf{u}_b \cdot \mathbf{v} \quad \text{on } \gamma_v,$$

where \mathbf{v} is the outward unit normal vector to the boundary of ω .

5. Conclusions

The primary objective of this study was to explore the possibility of deriving a Reynolds-type equation for non-Newtonian fluids, where the apparent viscosity depends on the symmetric part of the velocity gradient. To accomplish this, we focused on investigating power-law fluids.

We have successfully demonstrated that the balance of linear momentum for power-law fluids can be simplified considerably, for both pressure-driven and shear-driven flows, by assuming that the distance between the surfaces is significantly smaller than the lateral size of the surfaces. Our analysis further shows that the potential to derive a form of the Reynolds equation from this simplified governing system of equations depends on the dominant driving force behind the flow, specifically whether it is primarily driven by shear or pressure. The main conclusion is that it is not possible to derive a Reynolds-type equation from the simplified system in the case of shear-driven flow of power-law fluids, as it is in the case with Navier–Stokes fluids. However, for pressure-driven flow, we have successfully derived a type of Reynolds equation applicable to the flow of power-law fluids between closely situated surfaces, which may not be inherently smooth.

We have provided justification for which boundary conditions ought to be imposed in the Reynolds equation for pressure-driven flow. Of course, these are related to the boundary conditions that are used in the full 3D model, i.e., the full balance of linear momentum. We have shown that a normal stress boundary condition in the full 3D model implies a Dirichlet condition for the pressure in the simplified system of equations. Furthermore, we proved that a Dirichlet condition for the velocity in the full 3D model results in a Neumann condition for the pressure in the simplified system of equations. In particular, this determines which boundary conditions should be imposed in the Reynolds-type equation for pressure-driven flow of power-law fluids.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Andreas Almqvist reports financial support was provided by Swedish Research Council.

Data availability

No data was used for the research described in the article.

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References

- Almqvist, A., Burtseva, E., Rajagopal, K., Wall, P., 2021a. On lower-dimensional models in lubrication, Part A: Common misinterpretations and incorrect usage of the Reynolds equation. *Proc. IMechE Part J: J. Eng. Tribol.* 235 (8), 1692–1702. <http://dx.doi.org/10.1177/1350650120973792>.
- Almqvist, A., Burtseva, E., Rajagopal, K., Wall, P., 2021b. On lower-dimensional models in lubrication, Part B: Derivation of a Reynolds type of equation for incompressible piezo-viscous fluids. *Proc. IMechE Part J: J. Eng. Tribol.* 235 (8), 1703–1718. <http://dx.doi.org/10.1177/1350650120973800>.
- Almqvist, A., Burtseva, E., Rajagopal, K., Wall, P., 2023. On lower-dimensional models of thin film flow, Part C: Derivation of a Reynolds type of equation for fluids with temperature and pressure dependent viscosity. *Proc. IMechE Part J: J. Eng. Tribol.* 237 (3), 514–526. <http://dx.doi.org/10.1177/13506501221135269>.
- Almqvist, A., Fabricius, J., Larsson, R., Wall, P., 2014. A new approach for studying cavitation in lubrication. *J. Tribol.* 136 (1), 011706. <http://dx.doi.org/10.1115/1.4025875>.
- Aronsson, G., Janfalk, U., 1992. On hele-shaw flow of power-law fluids. *European J. Appl. Math.* 3 (4), 343–366. <http://dx.doi.org/10.1017/S095679250000905>.
- Azeez, S., Bertola, V., 2021. Lubrication of journal bearings by shear thinning lubricants using different constitutive models. *Proc. IMechE Part J: J. Eng. Tribol.* 235 (6), 1203–1210. <http://dx.doi.org/10.1177/1350650120950521>.
- Bayada, G., Chambat, M., 1986. The transition between the Stokes equations and the Reynolds equation: A mathematical proof. *Appl. Math. Optim.* 14 (1), 73–93. <http://dx.doi.org/10.1007/BF01442229>.
- Blechta, J., Málek, J., Rajagopal, K., 2020. On the classification of incompressible fluids and a mathematical analysis of the equations that govern their motion. *SIAM J. Math. Anal.* 52 (2), 1232–1289. <http://dx.doi.org/10.1137/19M1244895>.

- Boyer, F., Fabrie, P., 2013. Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models. In: Applied Mathematical Sciences, (183), Springer, New York.
- Dien, I.K., Elrod, H.G., 1983. A generalized steady-state Reynolds equation for non-Newtonian fluids, with applications to journal bearings. *J. Lubr. Technol.* 105, 385–390. <http://dx.doi.org/10.1115/1.3254619>.
- Dowson, D., 1962. A generalized Reynolds equation for fluid-film lubrication. *Int. J. Mech. Sci.* 4 (2), 159–170. [http://dx.doi.org/10.1016/S0020-7403\(62\)80038-1](http://dx.doi.org/10.1016/S0020-7403(62)80038-1).
- Dugas, R., 1988. *A History of Mechanics*. Dover Publications, New York.
- Elrod, H.G., Adams, M.L., 1975. A computer program for cavitation and starvation problems. In: Dowson, D., Godet, M., Taylor, C. (Eds.), *Cavitation Relat. Phenom. Lubr. - Proc. - Mech. Eng. Publ.* 37–43.
- Fabricsius, J., 2019. Stokes flow with kinematic and dynamic boundary conditions. *Quart. Appl. Math.* 77 (3), 525–544. <http://dx.doi.org/10.1090/qam/1534>.
- Fabricsius, J., Koroleva, Y., Wall, P., 2013. A rigorous derivation of the time-dependent Reynolds equation. *Asymptot. Anal.* 84 (1–2), 103–121. <http://dx.doi.org/10.1115/1.4035868>.
- Fabricsius, J., Manjate, S., Wall, P., 2022. On pressure-driven hele-shaw flow of power-law fluids. *Appl. Anal.* 101 (14), 5107–5137. <http://dx.doi.org/10.1080/00036811.2021.1880570>.
- Fabricsius, J., Tsandzana, A., Perez-Rafols, F., Wall, P., 2017. A Comparison of the Roughness Regimes in Hydrodynamic Lubrication. *J. Tribol.* 139 (5), 1–10. <http://dx.doi.org/10.1115/1.4035868>.
- Giacopini, M., Fowell, M.T., Dini, D., Strozzi, A., 2010. A mass-conserving complementary formulation to study lubricant films in the presence of cavitation. *J. Tribol.* 132 (4), 041702. <http://dx.doi.org/10.1115/1.4002215>.
- Hele-Shaw, H.S., 1898. The flow of water. *Nature* 58, 34–36. <http://dx.doi.org/10.1038/058034a0>.
- Lugt, P.M., Morales-Espejel, G.E., 2011. A review of elasto-hydrodynamic lubrication theory. *Tribol. Trans.* 54 (3), 470–496. <http://dx.doi.org/10.1080/10402004.2010.551804>.
- Málek, J., Nečas, J., Rokyta, M., Růžička, M., 1996. Weak and Measure-Valued Solutions to Evolutionary PDEs. In: *Applied Mathematics and Mathematical Computation*, (13), Taylor & Francis Group.
- Mikelić, A., Tapiéro, R., 1995. Mathematical derivation of the power law describing polymer flow through a thin slab. *RAIRO Modél. Math. Anal. Numér.* 29 (1), 3–21, URL http://www.numdam.org/item/M2AN_1995_29_1_3_0/.
- Rajagopal, K.R., 2003. On implicit constitutive theories. *Appl. Math.* 48 (4), 279–319. <http://dx.doi.org/10.1023/A:1026062615145>.
- Rajagopal, K.R., 2006. On implicit constitutive theories for fluids. *J. Fluid Mech.* 550, 243–249. <http://dx.doi.org/10.1017/S0022112005008025>.
- Reynolds, O., 1886. On the theory of lubrication and its application to mr. Beauchamps tower's experiments, including an experimental determination of the viscosity of olive oil. *Proc. R. Soc. Lond.* 40 (242–245), 191–203. <http://dx.doi.org/10.1098/rstl.1886.0005>.
- Temam, R., 2001. *Navier-Stokes Equations*. AMS Chelsea Publishing, Providence, RI, p. xiv+408. <http://dx.doi.org/10.1090/chel/343>, Theory and numerical analysis, Reprint of the 1984 edition.
- Truesdell, C., 1960. A program toward rediscovering the rational mechanics of the age of reason. *Arch. Hist. Exact Sci.* 1 (1), 1–36. <http://dx.doi.org/10.1007/BF00357393>.
- Yang, Q., Huang, P., Fang, Y., 2016. A novel Reynolds equation of non-Newtonian fluid for lubrication simulation. *Trib. Int.* 94, 458–463. <http://dx.doi.org/10.1016/j.triboint.2015.10.011>.