# A new result on boundedness of the Riesz potential in central Morrey-Orlicz spaces 

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#### Abstract

We improve our results on boundedness of the Riesz potential in the central MorreyOrlicz spaces and the corresponding weak-type version. We also present two new properties of the central Morrey-Orlicz spaces: nontriviality and inclusion property.


Keywords Riesz potential • Orlicz functions • Orlicz spaces • Morrey-Orlicz spaces • Central Morrey-Orlicz spaces . Weak central Morrey-Orlicz spaces

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## 1 Central Morrey-Orlicz spaces

A function $\Phi:[0, \infty) \rightarrow[0, \infty]$ is called a Young function, if it is a nondecreasing convex function with $\lim _{u \rightarrow 0^{+}} \Phi(u)=\Phi(0)=0$, and not identically 0 or $\infty$ in $(0, \infty)$. It may have jump up to $\infty$ at some point $u>0$, but then it should be left continuous at $u$.

To each Young function $\Phi$ one can associate another convex function $\Phi^{*}$, i.e., the complementary function to $\Phi$, which is defined by

$$
\Phi^{*}(v)=\sup _{u>0}[u v-\Phi(u)] \text { for } v \geq 0 .
$$

[^0]Then $\Phi^{*}$ is also a Young function and $\Phi^{* *}=\Phi$. Note that $u \leq \Phi^{-1}(u) \Phi^{*^{-1}}(u) \leq$ $2 u$ for all $u>0$, where $\Phi^{-1}$ is the right-continuous inverse of $\Phi$ defined by

$$
\Phi^{-1}(v)=\inf \{u \geq 0: \Phi(u)>v\} \text { with } \inf \emptyset=\infty
$$

We say that Young function $\Phi$ satisfies the $\Delta_{2}$-condition and we write shortly $\Phi \in \Delta_{2}$, if $0<\Phi(u)<\infty$ for $u>0$ and there exists a constant $D_{2}>1$ such that

$$
\Phi(2 u) \leq D_{2} \Phi(u) \text { for all } u>0 .
$$

For any Young function $\Phi$, the number $\lambda \in \mathbb{R}$ and an open ball $B_{r}=\left\{x \in \mathbb{R}^{n}:|x|<\right.$ $r\}, r>0$ we can define central Morrey-Orlicz spaces $M^{\Phi, \lambda}(0)$ as all $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ such that

$$
\|f\|_{M^{\Phi, \lambda}(0)}=\sup _{r>0}\|f\|_{\Phi, \lambda, B_{r}}<\infty,
$$

where

$$
\|f\|_{\Phi, \lambda, B_{r}}=\inf \left\{\varepsilon>0: \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}} \Phi\left(\frac{|f(x)|}{\varepsilon}\right) d x \leq 1\right\} .
$$

Similarly, the weak central Morrey-Orlicz spaces $W M^{\Phi, \lambda}(0)$ are defined as

$$
W M^{\Phi, \lambda}(0)=\left\{f \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right):\|f\|_{W M^{\Phi, \lambda}(0)}=\sup _{r>0}\|f\|_{\Phi, \lambda, B_{r}, \infty}<\infty\right\}
$$

where

$$
\|f\|_{\Phi, \lambda, B_{r}, \infty}=\inf \left\{\varepsilon>0: \sup _{u>0} \Phi\left(\frac{u}{\varepsilon}\right) \frac{1}{\left|B_{r}\right|^{\lambda}} d\left(f \chi_{B_{r}}, u\right) \leq 1\right\},
$$

and $d(f, u)=\left|\left\{x \in \mathbb{R}^{n}:|f(x)|>u\right\}\right|$.
The properties of these spaces can be found in [4]. If $\Phi(u)=u^{p}, 1 \leq p<\infty$ and $\lambda \in \mathbb{R}$, then $M^{\Phi, \lambda}(0)=M^{p, \lambda}(0)$ and $W M^{\Phi, \lambda}(0)=W M^{p, \lambda}(0)$ are classical central and weak central Morrey spaces. Moreover, for $\lambda=0$ the spaces $M^{\Phi, 0}(0)=L^{\Phi}\left(\mathbb{R}^{n}\right)$ and $W M^{\Phi, 0}(0)=W L^{\Phi}\left(\mathbb{R}^{n}\right)$ are classical Orlicz and weak Orlicz spaces.

In the following lemma and later, $B\left(x_{0}, r_{0}\right)$ will denote an open ball with the center at $x_{0} \in \mathbb{R}^{n}$ and radius $r_{0}>0$, that is, $B\left(x_{0}, r_{0}\right)=\left\{x \in \mathbb{R}^{n}:\left|x-x_{0}\right|<r_{0}\right\}$.

Lemma 1 Let $\Phi$ be a Young function, $\Phi^{*}$ its complementary function, $0 \leq \lambda \leq 1$ and $r>0$. Then
(i) $\int_{B_{r}}|f(x) g(x)| d x \leq 2\left|B_{r}\right|^{\lambda}\|f\|_{\Phi, \lambda, B_{r}}\|g\|_{\Phi^{*}, \lambda, B_{r}}$.
(ii) $\left\|\chi_{B\left(x_{0}, r_{0}\right)}\right\|_{\Phi^{*}, \lambda, B_{r}} \leq \frac{\left|B_{r} \cap B\left(x_{0}, r_{0}\right)\right|}{\left|B_{r}\right|^{\lambda}} \Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{r} \cap B\left(x_{0}, r_{0}\right)\right|}\right)$, where $B_{r} \cap B\left(x_{0}, r_{0}\right) \neq \emptyset$ for $x_{0} \in \mathbb{R}^{n}$ and $r_{0}>0$.
In particular, $\left\|\chi_{B_{r}}\right\|_{\Phi^{*}, \lambda, B_{r}} \leq \frac{\Phi^{-1}\left(\left|B_{r}\right|^{\lambda-1}\right)}{\left|B_{r}\right|^{\lambda-1}}$.
(iii) $\left\|\chi_{B_{t}}\right\|_{\Phi, \lambda, B_{r}}=1 / \Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{r} \cap B_{t}\right|}\right)$ and $\left\|\chi_{B_{t}}\right\|_{M^{\Phi, \lambda}(0)}=\frac{1}{\Phi^{-1}\left(\left|B_{t}\right|^{\lambda-1}\right)}$ for any $t>0$.

Proof of this lemma can be found in [4, Lemma 1].

## 2 Riesz potential in the central Morrey-Orlicz spaces

We will work with the central Morrey-Orlicz spaces, defined by the Orlicz functions. A function $\Phi:[0, \infty) \rightarrow[0, \infty)$ is called an Orlicz function, if it is a strictly increasing continuous and convex function with $\Phi(0)=0$.

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a Lebesgue measurable function and $\alpha \in(0, n)$. The Riesz potential is defined as

$$
I_{\alpha} f(x)=\int_{\mathbb{R}^{n}} \frac{f(y)}{|x-y|^{n-\alpha}} d y, \quad \text { for } x \in \mathbb{R}^{n}
$$

The linear operator $I_{\alpha}$ plays an important role in various branches of analysis, including potential theory, harmonic analysis, Sobolev spaces, partial differential equations and can be treated as a special singular integral. That is why it is important to study its boundedness between different spaces. Many authors investigated boundedness of $I_{\alpha}$ in Morrey, Orlicz and Morrey-Orlicz spaces. We present here our main theorem on the boundedness of the Riesz potential in the central Morrey-Orlicz spaces.

In order to prove our result we will use estimate from [13] for the Hardy-Littlewood maximal operator in central Morrey-Orlicz spaces. The Hardy-Littlewood maximal operator $M$ or centred maximal function $M f$ of a function $f$ defined on $\mathbb{R}^{n}$ is defined at each $x \in \mathbb{R}^{n}$ as

$$
M f(x)=\sup _{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)}|f(y)| d y
$$

For any Orlicz function $\Phi$ and $0 \leq \lambda \leq 1$, maximal operator $M$ is bounded on $M^{\Phi, \lambda}(0)$, provided $\Phi^{*} \in \Delta_{2}$, and then there exists a constant $C_{0}>1$ such that

$$
\|M f\|_{M^{\Phi, \lambda}(0)} \leq C_{0}\|f\|_{M^{\Phi, \lambda}(0)}, \quad \text { for all } f \in M^{\Phi, \lambda}(0)
$$

(see [13, Theorem 6(i)]). Moreover, the maximal operator $M$ is bounded from $M^{\Phi, \lambda}(0)$ to $W M^{\Phi, \lambda}(0)$, that is, there exists a constant $c_{0}>1$ such that $\|M f\|_{W M^{\Phi, \lambda}(0)} \leq$ $c_{0}\|f\|_{M^{\Phi, \lambda}(0)}$ for all $f \in M^{\Phi, \lambda}(0)$ (see [13, Theorem 6(ii)]).

Furthermore, in the proof of the main result we will use Hedberg's pointwise estimate from [7, p. 506].

Lemma 2 (Hedberg) If $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a Lebesgue measurable function and $\alpha \in(0, n)$, then for all $x \in \mathbb{R}^{n}$ and $r>0$

$$
\int_{|y-x| \leq r}|f(y)||x-y|^{\alpha-n} d y \leq C_{H} r^{\alpha} M f(x)
$$

with $C_{H}=\frac{2^{n}}{2^{\alpha}-1} v_{n}$, where $v_{n}=|B(0,1)|=\pi^{n / 2} / \Gamma(n / 2+1)$.
Proof For the sake of completeness, we include its proof, taking care about the constant $C_{H}$ in the estimate. For any $x \in \mathbb{R}^{n}$ and $r>0$

$$
\begin{aligned}
\int_{|y-x| \leq r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d y & =\sum_{m=0}^{\infty} \int_{r 2^{-m-1}<|y-x| \leq r 2^{-m}} \frac{|f(y)|}{|x-y|^{n-\alpha}} d y \\
& \leq \sum_{m=0}^{\infty} \int_{B\left(x, r 2^{-m}\right) \backslash B\left(x, r 2^{-m-1}\right)} \frac{|f(y)|}{\left(r 2^{-m-1}\right)^{n-\alpha}} d y \\
& \leq 2^{n-\alpha} r^{\alpha} \sum_{m=0}^{\infty} 2^{-m \alpha}\left(r 2^{-m}\right)^{-n} \int_{B\left(x, r 2^{-m}\right)}|f(y)| d y \\
& =2^{n-\alpha} r^{\alpha} \sum_{m=0}^{\infty} 2^{-m \alpha} \frac{v_{n}}{\left|B\left(x, r 2^{-m}\right)\right|} \int_{B\left(x, r 2^{-m}\right)}|f(y)| d y \\
& \leq 2^{n-\alpha} r^{\alpha} v_{n} \sum_{m=0}^{\infty} 2^{-m \alpha} M f(x)=\frac{2^{n} v_{n}}{2^{\alpha}-1} r^{\alpha} M f(x) .
\end{aligned}
$$

Theorem 1 Let $0<\alpha<n, \Phi, \Psi$ be Orliczfunctions and either $0<\lambda, \mu<1, \lambda \neq \mu$ or $\lambda=0$ and $0 \leq \mu<1$. Assume that there exist constants $C_{1}, C_{2} \geq 1$ such that

$$
\begin{equation*}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \leq C_{1} \Psi^{-1}\left(u^{\mu-1}\right) \text { for all } u>0 \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq C_{2} \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) \text { for all } r>u>0 . \tag{2}
\end{equation*}
$$

(i) If $\Phi^{*} \in \Delta_{2}$, then $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$, that is, there exists a constant $C_{3}=C_{3}\left(n, C_{0}, C_{H}, C_{1}, C_{2}\right) \geq 1$ such that $\left\|I_{\alpha} f\right\|_{M^{\Psi, \mu}(0)} \leq$ $C_{3}\|f\|_{M^{\Phi, \lambda}(0)}$ for all $f \in M^{\Phi, \lambda}(0)$.
(ii) The operator $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $W M^{\Psi, \mu}(0)$, that is, there exists a constant $c_{3}=c_{3}\left(n, c_{0}, C_{H}, C_{1}, C_{2}\right) \geq 1$ such that $\left\|I_{\alpha} f\right\|_{W M^{\Psi, \mu}(0)} \leq c_{3}\|f\|_{M^{\Phi, \lambda}(0)}$ for all $f \in M^{\Phi, \lambda}(0)$.

In our earlier paper [4, Theorem 3] it was proved result under conditions (1) and (3), and the latter means that

$$
\begin{equation*}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq C_{4} \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) \quad \text { for all } u>0, \quad r>0 \tag{3}
\end{equation*}
$$

The condition (3) is stronger than the assumption (2) because

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} & \geq \int_{u}^{2 u} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \geq u^{\frac{\alpha}{n}} \int_{u}^{2 u} \Phi^{-1}\left(\frac{r^{\lambda}}{2 u}\right) \frac{d t}{t} \\
& \geq \frac{u^{\frac{\alpha}{n}}}{2} \int_{u}^{2 u} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right) \frac{d t}{t}=\frac{\ln 2}{2} u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)
\end{aligned}
$$

and clearly the integral in (2) is smaller than the integral in (3). This improvement provides us with larger classes of Orlicz functions $\Phi$ and $\Psi$, defining central MorreyOrlicz spaces where the operator $I_{\alpha}$ is bounded.

In the simplest case, when $\Phi(u)=u^{p}, \Psi(u)=u^{q}$ where $1<p<q<\infty$, then the convergence of the integral in (1) means $p<\frac{n(1-\lambda)}{\alpha}$ and the assumption itself gives equality $\frac{\alpha}{n}+\frac{\lambda-1}{p}=\frac{\mu-1}{q}$. Assumptions (2) and (3) are both equivalent and give the following equations: $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $\frac{\lambda}{p}=\frac{\mu}{q}$. Of course, with the above assumptions, the operator $I_{\alpha}$ is bounded from $M^{p, \lambda}(0)$ to $M^{q, \mu}(0)$.

Only later, on the Examples 2 and 3, we will see that the conditions (1) and (2) hold but estimate (3) fails, which shows that our Theorem 1 improves Theorem 3 in [4].

Let us comment on what we can get when the numbers $\lambda$ and $\mu$ come from "boundaries".

Remark 1 If $\lambda=\mu=0$ we come to the same conclusion as in [4, Remark 4], that is, condition (1) is sufficient for the boundedness of $I_{\alpha}$ from Orlicz space $L^{\Phi}\left(\mathbb{R}^{n}\right)$ to weak Orlicz space $W L^{\Psi}\left(\mathbb{R}^{n}\right)$. If, in addition $\Phi^{*} \in \Delta_{2}$, then $I_{\alpha}$ is bounded from $L^{\Phi}\left(\mathbb{R}^{n}\right)$ to $L^{\Psi}\left(\mathbb{R}^{n}\right)$. Note that in this case condition (2) follows from (1).

Remark 2 If $\lambda=0$ and $0<\mu<1$, then the condition (3) is not satisfied, as we already mentioned in [4, Remark 3] and therefore the result proved in [4] does not include boundedness of the Riesz potential in this case. On the other hand, in this case, assumption (1) is stronger than (2). Indeed,

$$
\begin{aligned}
u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{1}{u}\right) & \leq 4 \int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{1}{t}\right) \frac{d t}{t} \\
& \leq 4 C_{1} \Psi^{-1}\left(u^{\mu-1}\right) \leq 4 C_{1} \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) \text { for all } r>u>0
\end{aligned}
$$

Therefore, if (1) holds, then $I_{\alpha}$ is bounded from $L^{\Phi}\left(\mathbb{R}^{n}\right)$ to $W M^{\Psi, \mu}(0)$. If, in addition $\Phi^{*} \in \Delta_{2}$, then $I_{\alpha}$ is bounded from $L^{\Phi}\left(\mathbb{R}^{n}\right)$ to $M^{\Psi, \mu}(0)$. In particular, when $\Phi(u)=$
$u^{p}, \Psi(u)=u^{q}, 0<\frac{\alpha}{n}<\frac{1}{p}$ and $\frac{\alpha}{n}-\frac{1}{p}=\frac{\mu-1}{q}$, then (1) holds. In fact, for all $u>0$

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{1}{t}\right) \frac{d t}{t} & =\int_{u}^{\infty} t^{\frac{\alpha}{n}-\frac{1}{p}-1} d t=\frac{1}{\frac{1}{p}-\frac{\alpha}{n}} u^{\frac{\alpha}{n}-\frac{1}{p}} \\
& =\frac{q}{1-\mu} u^{\frac{\mu-1}{q}}=C_{1} \Psi^{-1}\left(u^{\mu-1}\right)
\end{aligned}
$$

and we obtain (1) with $C_{1}=\frac{q}{1-\mu}$. Thus, from Theorem 1 we get that $I_{\alpha}$ is bounded from $L^{p}\left(\mathbb{R}^{n}\right)$ to $M^{q, \mu}(0)$. This result, in particular, was proved in [3, Theorem 2].

Remark 3 If $0<\lambda<1$ and $\mu=0$, the conditions (2) and (3) are not satisfied. Additionally, $I_{\alpha}$ is not bounded from $M^{\Phi, \lambda}(0)$ to $L^{\Psi}\left(\mathbb{R}^{n}\right)$ by applying the necessary condition for boundedness of $I_{\alpha}$ given in [4, Theorem 2(ii)]. In fact, let $R \geq 1$, $x_{R}=(R, 0, \ldots, 0) \in \mathbb{R}^{n}$ and $f_{R}(x)=\chi_{B\left(x_{R}, 1\right)}(x)$. Following the same arguments as in [9, Proposition 1] and [4, Theorem 2(ii)] we obtain that

$$
\left\|f_{R}\right\|_{M^{\Phi, \lambda}(0)} \leq \frac{1}{\Phi^{-1}\left(\frac{v_{n}^{\lambda}}{2^{n} v_{n-1}} R^{\lambda n}\right)} \text { and }\left\|I_{\alpha} f_{R}\right\|_{L^{\Psi}\left(\mathbb{R}^{n}\right)} \geq \frac{2^{\alpha-n} v_{n}}{\Psi^{-1}\left(\frac{1}{v_{n}}\right)}
$$

Thus,

$$
\begin{aligned}
\liminf _{R \rightarrow \infty} \frac{\left\|I_{\alpha} f_{R}\right\|_{L^{\Psi}\left(\mathbb{R}^{n}\right)}}{\left\|f_{R}\right\|_{M^{\Phi, \lambda}(0)}} & \geq \frac{2^{\alpha-n} v_{n}}{\Psi^{-1}\left(\frac{1}{v_{n}}\right)} \liminf _{R \rightarrow \infty} \Phi^{-1}\left(\frac{v_{n}^{\lambda}}{2^{n} v_{n-1}} R^{\lambda n}\right) \\
& \geq \frac{2^{\alpha-n} v_{n}}{\Psi^{-1}\left(\frac{1}{v_{n}}\right)} \min \left(1, \frac{v_{n}^{\lambda}}{2^{n} v_{n-1}}\right) \liminf _{R \rightarrow \infty} \Phi^{-1}\left(R^{\lambda n}\right)=\infty
\end{aligned}
$$

and therefore $I_{\alpha}$ is not bounded from $M^{\Phi, \lambda}(0)$ to $L^{\Psi}\left(\mathbb{R}^{n}\right)$.
Remark 4 If $0<\lambda=\mu<1$, then the assumption (2) does not hold. Indeed, let $r>u=r^{\lambda}$ with $r>1$. Then

$$
u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right) \leq C_{2} \Psi^{-1}\left(\frac{r^{\lambda}}{u}\right), \quad \text { for all } r>u>0
$$

means

$$
r^{\frac{\alpha \lambda}{n}} \Phi^{-1}(1) \leq C_{2} \Psi^{-1}(1), \text { for all } r>1,
$$

which is not true when $r \rightarrow \infty$. Moreover, if either $a=\liminf _{t \rightarrow 0^{+}} \frac{\Phi^{-1}(t)}{\Psi^{-1}(t)}>0$ or $b=\liminf _{t \rightarrow \infty} \frac{\Phi^{-1}(t)}{\Psi^{-1}(t)}=\infty$, then by Theorem 2 in [4] the Riesz potential $I_{\alpha}$ is not bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \lambda}(0)$. In particular, $I_{\alpha}$ is not bounded from $M^{p, \lambda}(0)$
to $M^{q, \lambda}(0)$ for any $1 \leq p, q<\infty$ (see also [9]). There remains an unresolved case when $a=0$ and $b<\infty$.

Proof of Theorem 1 (i) For any $x \in B_{r}$ and $f \in M^{\Phi, \lambda}(0)$ we consider two disjoint subsets

$$
B_{r}^{1}=\left\{x \in B_{r}: \Phi\left(\frac{M f(x)}{C_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \leq\left|B_{r}\right|^{\lambda-1}\right\}
$$

and

$$
B_{r}^{2}=\left\{x \in B_{r}: \Phi\left(\frac{M f(x)}{C_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right)>\left|B_{r}\right|^{\lambda-1}\right\} .
$$

We estimate the Riesz potential $I_{\alpha} f(x)$ by a sum of two integrals

$$
\begin{aligned}
\left|I_{\alpha} f(x)\right| & \leq \int_{|y| \leq 2 r}|f(y)||x-y|^{\alpha-n} d y+\int_{|y|>2 r}|f(y)||x-y|^{\alpha-n} d y \\
& =: I_{1} f(x)+I_{2} f(x) .
\end{aligned}
$$

For $x \in B_{r}^{1}$ and $|y| \leq 2 r$ we have $|y-x| \leq|y|+|x| \leq 3 r$, and so

$$
I_{1} f(x)=\int_{|y| \leq 2 r}|f(y)||x-y|^{\alpha-n} d y \leq \int_{|y-x| \leq 3 r}|f(y)||x-y|^{\alpha-n} d y
$$

By Hedberg's pointwise estimate, given in Lemma 2, we obtain

$$
I_{1} f(x) \leq C_{5}\left|B_{r}\right|^{\frac{\alpha}{n}} M f(x), \text { where } C_{5}=C_{H} 3^{\alpha} v_{n}^{-\alpha / n}
$$

This implies, for $x \in B_{r}^{1}$, that

$$
I_{1} f(x) \leq C_{0} C_{5}\|f\|_{M^{\Phi, \lambda}(0)}\left|B_{r}\right|^{\frac{\alpha}{n}} \Phi^{-1}\left(\left|B_{r}\right|^{\lambda-1}\right) .
$$

On the other hand,

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & \geq \int_{u}^{2 u} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \geq \ln 2 u^{\frac{\alpha}{n}} \Phi^{-1}\left((2 u)^{\lambda-1}\right) \\
& \geq 2^{\lambda-1} \ln 2 u^{\frac{\alpha}{n}} \Phi^{-1}\left(u^{\lambda-1}\right) \geq \frac{1}{4} u^{\frac{\alpha}{n}} \Phi^{-1}\left(u^{\lambda-1}\right)
\end{aligned}
$$

for any $u>0$. Thus, applying assumption (1) we obtain

$$
\begin{aligned}
I_{1} f(x) & \leq 4 C_{0} C_{1} C_{5}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\left|B_{r}\right|^{\mu-1}\right) \\
& \leq \frac{4}{2^{n(\mu-1)}} C_{0} C_{1} C_{5}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\left|B_{2 r}\right|^{\mu-1}\right) \\
& \leq 4 \cdot 2^{n} \cdot C_{0} C_{1} C_{5}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\left|B_{2 r}\right|^{\mu-1}\right)
\end{aligned}
$$

To estimate the second integral $I_{2} f(x)$, first note that when $x \in B_{r}^{1}$ and $|y|>2 r$ we have $|x|<r<|y| / 2$ and $|y-x| \geq|y|-|x|>|y| / 2$, and so $|x-y|^{\alpha-n}<2^{n-\alpha}|y|^{\alpha-n}$. Thus, following Hedberg's method, as in [4, pp. 18-20], we obtain

$$
\begin{aligned}
I_{2} f(x) & \leq 2^{n-\alpha} \int_{|y|>2 r}|f(y)||y|^{\alpha-n} d y=2^{n-\alpha} \sum_{k=0}^{\infty} \int_{r 2^{k+1}<|y| \leq r 2^{k+2}}|f(y)||y|^{\alpha-n} d y \\
& \leq 2^{n-\alpha} \sum_{k=0}^{\infty}\left(2^{k+1} r\right)^{\alpha-n} \int_{|y| \leq 2^{k+2} r}|f(y)| d y .
\end{aligned}
$$

Then, from Lemma 1, it follows that

$$
\begin{aligned}
& I_{2} f(x) \leq 2^{n-\alpha+1} \sum_{k=0}^{\infty}\left(2^{k+1} r\right)^{\alpha-n}\left|B_{2^{k+2}}\right|^{\lambda}\|f\|_{\Phi, \lambda, B_{2^{k+2}}}\left\|\chi_{B_{2^{k+2}}}\right\|_{\Phi^{*}, \lambda, B_{2} k+2_{r}} \\
& \leq 2^{n-\alpha+1}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty}\left(2^{k+1} r\right)^{\alpha-n}\left|B_{2^{k+2} r}\right|^{\lambda} \frac{\Phi^{-1}\left(\left|B_{2^{k+2} r}\right|^{\lambda-1}\right)}{\left|B_{2^{k+2} r}\right|^{\lambda-1}} \\
& =2^{2 n-\alpha+1} v_{n}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty}\left(2^{k+1} r\right)^{\alpha} \Phi^{-1}\left(\left|B_{2^{k+2} r}\right|^{\lambda-1}\right) \\
& =\frac{2^{2 n-\alpha+1} v_{n}^{1-\frac{\alpha}{n}}}{n \ln 2}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty}\left|B_{2^{k+1} r}\right|^{\frac{\alpha}{n}} \Phi^{-1}\left(\left|B_{2^{k+2_{r}}}\right|^{\lambda-1}\right) \int_{\left|B_{2^{k+1}}\right|}^{\left|B_{2^{k+2}}\right|} \frac{d t}{t} \\
& \leq 2^{2 n-\alpha+2} v_{n}^{1-\frac{\alpha}{n}}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty} \int_{\left|B_{2^{k+1}}\right|}^{\left|B_{2^{k+2},}\right|} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \\
& \leq C_{6}\|f\|_{M^{\Phi, \lambda}(0)} \int_{\left|B_{2 r}\right|}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \text {, where } C_{6}=2^{2 n-\alpha+2} v_{n}^{1-\frac{\alpha}{n}} \text {. }
\end{aligned}
$$

Applying assumption (1) we get

$$
I_{2} f(x) \leq C_{1} C_{6}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\left|B_{2 r}\right|^{\mu-1}\right)
$$

Thus, for $x \in B_{r}^{1}$, we obtain

$$
\left|I_{\alpha} f(x)\right| \leq I_{1} f(x)+I_{2} f(x) \leq 2 C_{7}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\left|B_{2 r}\right|^{\mu-1}\right)
$$

where $C_{7}=C_{1} \cdot \max \left\{4 \cdot 2^{n} \cdot C_{0} C_{5}, C_{6}\right\}$. Since $2^{n(\mu-1)}<1$ it follows that

$$
\int_{B_{r}^{1}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{2 C_{7}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \leq\left|B_{r}^{1}\right|\left|B_{2 r}\right|^{\mu-1} \leq 2^{n(\mu-1)}\left|B_{r}\right|^{\mu}<\left|B_{r}\right|^{\mu} .
$$

Let now $x \in B_{r}^{2}$. We can write $I_{\alpha} f(x)$ as follows

$$
\begin{aligned}
\left|I_{\alpha} f(x)\right| & \leq \int_{|x-y| \leq \delta}|f(y)||x-y|^{\alpha-n} d y+\int_{|x-y|>\delta}|f(y)||x-y|^{\alpha-n} d y \\
& =: I_{3} f(x)+I_{4} f(x)
\end{aligned}
$$

where $\delta$ is defined in the following way

$$
\begin{equation*}
\Phi\left(\frac{M f(x)}{C_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right)=\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{\delta}\right|} . \tag{4}
\end{equation*}
$$

Since $x \in B_{r}^{2}$ it follows that $\left|B_{\delta}\right|<\left|B_{r}\right|$. Hedberg's pointwise estimate from Lemma 2 to $I_{3} f(x)$ gives

$$
I_{3} f(x) \leq C_{H} \delta^{\alpha} M f(x)=C_{H}\left(\delta^{n} v_{n}\right)^{\alpha / n} v_{n}^{-\alpha / n} M f(x)=v_{n}^{-\alpha / n} C_{H}\left|B_{\delta}\right|^{\frac{\alpha}{n}} M f(x),
$$

and from the assumption (2) we get

$$
I_{3} f(x) \leq v_{n}^{-\alpha / n} C_{2} C_{H} \frac{\Psi^{-1}\left(\frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|}\right)}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{\delta}\right|}\right)} M f(x)
$$

Next, since equality (4) holds it follows that

$$
I_{3} f(x) \leq v_{n}^{-\alpha / n} C_{0} C_{2} C_{H}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|}\right)
$$

Applying again Hedberg's method for $I_{4} f(x)$ we obtain

$$
\begin{aligned}
I_{4} f(x) & =\int_{|x-y|>\delta}|f(y)||x-y|^{\alpha-n} d y=\sum_{k=0}^{\infty} \int_{2^{k} \delta<|x-y| \leq 2^{k+1} \delta}|f(y)||x-y|^{\alpha-n} d y \\
& \leq \sum_{k=0}^{\infty}\left(2^{k} \delta\right)^{\alpha-n} \int_{|x-y| \leq 2^{k+1} \delta}|f(y)| d y \\
& \leq \sum_{k=0}^{\infty}\left(2^{k} \delta\right)^{\alpha-n} \int_{B_{|x|+2^{k+1} \delta}}|f(y)| \chi_{B\left(x, 2^{k+1} \delta\right)}(y) d y
\end{aligned}
$$

where $B_{|x|+2^{k+1} \delta}$ is the smallest ball with the centre at origin containing $B\left(x, 2^{k+1} \delta\right)$. From Lemma 1, using the fact that $B_{|x|+2^{k+1} \delta} \cap B\left(x, 2^{k+1} \delta\right)=B\left(x, 2^{k+1} \delta\right)$, we get

$$
\begin{aligned}
I_{4} f(x) & \leq 2 \sum_{k=0}^{\infty}\left(2^{k} \delta\right)^{\alpha-n}\left|B_{|x|+2^{k+1} \delta}\right|^{\lambda}\|f\|_{\Phi, \lambda, B_{|x|+2^{k+1}}}\left\|\chi_{B\left(x, 2^{k+1} \delta\right)}\right\|_{\Phi^{*}, \lambda, B_{|x|+2^{k+1}}} \\
& \leq 2\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty}\left(2^{k} \delta\right)^{\alpha-n}\left|B\left(x, 2^{k+1} \delta\right)\right| \Phi^{-1}\left(\frac{\left|B_{|x|+2^{k+1} \delta}\right|^{\lambda}}{\left|B\left(x, 2^{k+1} \delta\right)\right|}\right) \\
& =\frac{2^{n+1} v_{n}}{n \ln 2}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty}\left(2^{k} \delta\right)^{\alpha} \Phi^{-1}\left(\frac{\left|B_{|x|+2^{k+1} \delta}\right|^{\lambda}}{\left|B\left(x, 2^{k+1} \delta\right)\right|}\right)_{\left|B\left(x, 2^{k} \delta\right)\right|}^{\left|B\left(x, 2^{k+1} \delta\right)\right|} \int_{t} \frac{d t}{t} .
\end{aligned}
$$

Since $|x| \leq r$ and $2^{k} \delta \leq\left(\frac{t}{v_{n}}\right)^{\frac{1}{n}} \leq 2^{k+1} \delta$ it follows that

$$
\begin{aligned}
\left|B_{|x|+2^{k+1} \delta}\right| & \leq v_{n}\left(r+2^{k+1} \delta\right)^{n} \leq v_{n}\left(r+2 \frac{t^{\frac{1}{n}}}{v_{n}^{\frac{1}{n}}}\right)^{n}=\left(v_{n}^{\frac{1}{n}} r+2 t^{\frac{1}{n}}\right)^{n} \\
& =\left(\left|B_{r}\right|^{\frac{1}{n}}+2 t^{\frac{1}{n}}\right)^{n} \leq 2^{n}\left(\left|B_{r}\right|^{\frac{1}{n}}+t^{\frac{1}{n}}\right)^{n} \leq 4^{n} \max \left\{\left|B_{r}\right|, t\right\} .
\end{aligned}
$$

So using the concavity of $\Phi^{-1}$, we get

$$
\begin{aligned}
I_{4} f(x) & \leq \frac{2^{n+1} v_{n}^{1-\frac{\alpha}{n}}}{n \ln 2}\|f\|_{M^{\Phi, \lambda}(0)} \sum_{k=0}^{\infty} \int_{\left|B\left(x, 2^{k} \delta\right)\right|}^{\left|B\left(x, 2^{k+1} \delta\right)\right|} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{4^{\lambda n}\left(\max \left\{\left|B_{r}\right|, t\right\}\right)^{\lambda}}{t}\right) \frac{d t}{t} \\
& \leq \frac{4^{\lambda n} 2^{n+1} v_{n}^{1-\frac{\alpha}{n}}}{n \ln 2}\|f\|_{M^{\Phi, \lambda}(0)} \int_{\left|B_{\delta}\right|}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{\left(\max \left\{\left|B_{r}\right|, t\right\}\right)^{\lambda}}{t}\right) \frac{d t}{t} \\
& \leq C_{8}\|f\|_{M^{\Phi, \lambda}(0)}\left[\int_{\left|B_{\delta}\right|}^{\left|B_{r}\right|} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{t}\right) \frac{d t}{t}+\int_{\left|B_{r}\right|}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t}\right],
\end{aligned}
$$

where $C_{8}=\frac{4^{\lambda n} 2^{n+1} v_{n}^{1-\frac{\alpha}{n}}}{n \ln 2} \leq \frac{4^{n} \cdot 2^{n+2} \cdot v_{n}^{1-\frac{\alpha}{n}}}{n}$. Based on the assumptions of (1), (2) and the fact that $\left|B_{\delta}\right|<\left|B_{r}\right|$ we get

$$
\begin{aligned}
I_{4} f(x) & \leq C_{8}\|f\|_{M^{\Phi, \lambda}(0)}\left[C_{2} \Psi^{-1}\left(\frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|}\right)+C_{1} \Psi^{-1}\left(\left|B_{r}\right|^{\mu-1}\right)\right] \\
& \leq C_{8}\left(C_{2}+C_{1}\right)\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|}\right)
\end{aligned}
$$

Thus, for $x \in B_{r}^{2}$ we obtain

$$
\left|I_{\alpha} f(x)\right| \leq I_{3} f(x)+I_{4} f(x) \leq C_{9}\|f\|_{M^{\Phi, \lambda}(0)} \Psi^{-1}\left(\frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|}\right)
$$

with $C_{9}=v_{n}^{-\alpha / n} C_{0} C_{2} C_{H}+C_{8}\left(C_{1}+C_{2}\right)$. Then

$$
\begin{aligned}
\int_{B_{r}^{2}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{C_{9}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x & \leq \int_{B_{r}} \frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|} d x \\
& =\left|B_{r}\right|^{\mu-\lambda} \int_{B_{r}} \Phi\left(\frac{M f(x)}{C_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \\
& \leq\left|B_{r}\right|^{\mu-\lambda} \int_{B_{r}} \Phi\left(\frac{M f(x)}{\|M f\|_{M^{\Phi, \lambda}(0)}}\right) d x \leq\left|B_{r}\right|^{\mu} .
\end{aligned}
$$

Finally, since $B_{r}=B_{r}^{1} \cup B_{r}^{2}$ and the last two sets are disjoint, and by the convexity of $\Psi$ it follows that

$$
\begin{aligned}
\int_{B_{r}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{C_{3}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x= & \int_{B_{r}^{1}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{C_{3}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \\
& +\int_{B_{r}^{2}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{C_{3}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \\
\leq & \frac{1}{2} \int_{B_{r}^{1}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{2 C_{7}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \\
& +\frac{1}{2} \int_{B_{r}^{2}} \Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{C_{9}\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \\
\leq & \left|B_{r}\right|^{\mu}
\end{aligned}
$$

where $C_{3}=2 \max \left\{2 C_{7}, C_{9}\right\}$. Hence, $\left\|I_{\alpha} f\right\|_{M^{\Psi, \mu}(0)} \leq C_{3}\|f\|_{M^{\Phi, \lambda}(0)}$.
(ii) Similarly to the previous case, we will present $B_{r}$ as a union of two disjoint subsets $B_{r}=B_{r}^{1} \cup B_{r}^{2}$, where $B_{r}^{1}$ and $B_{r}^{2}$ are defined in the same way as in the first part of the proof with respect to the constant $c_{0}$, that is,

$$
B_{r}^{1}=\left\{x \in B_{r}: \Phi\left(\frac{M f(x)}{c_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \leq\left|B_{r}\right|^{\lambda-1}\right\}
$$

and

$$
B_{r}^{2}=\left\{x \in B_{r}: \Phi\left(\frac{M f(x)}{c_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right)>\left|B_{r}\right|^{\lambda-1}\right\} .
$$

For $x \in B_{r}$ we get

$$
\begin{aligned}
\Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{c_{3}\|f\|_{M^{\Phi, \lambda}(0)}}\right) & \leq \frac{1}{2} \Psi\left(\frac{\left|I_{\alpha} f(x)\right| \chi_{B_{r}^{1}}(x)}{4 c_{7}\|f\|_{M^{\Phi, \lambda}(0)}}\right)+\frac{1}{2} \Psi\left(\frac{\left|I_{\alpha} f(x)\right| \chi_{B_{r}^{2}}(x)}{2 c_{9}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \\
& =: \frac{1}{2}\left(I_{5}+I_{6}\right),
\end{aligned}
$$

where $c_{3}=2 \max \left\{4 c_{7}, 2 c_{9}\right\}, c_{7}=C_{1} \max \left\{4 \cdot 2^{n} \cdot c_{0} C_{5}, C_{6}\right\}, c_{9}=v_{n}^{-\alpha / n} c_{0} C_{2} C_{H}+$ $C_{8}\left(C_{1}+C_{2}\right)$. We follow the same calculations as in the proof of Theorem 3(ii) in [4] and we get

$$
d\left(\Psi\left(\frac{\left|I_{\alpha} f(x)\right|}{c_{3}\|f\|_{M^{\Phi, \lambda}(0)}}\right), u\right) \leq d\left(I_{5}, u\right)+d\left(I_{6}, u\right)
$$

and

$$
\sup _{u>0} \frac{\Psi(u)}{\left|B_{r}\right|^{\mu}} d\left(\frac{\left|I_{\alpha} f(x)\right|}{c_{3}\|f\|_{M^{\Phi, \lambda}(0)}}, u\right) \leq \sup _{u>0} \frac{u}{\left|B_{r}\right|^{\mu}} d\left(I_{5}, u\right)+\sup _{u>0} \frac{u}{\left|B_{r}\right|^{\mu}} d\left(I_{6}, u\right),
$$

where we used the property $\Psi(u) d(g, u)=v d\left(g, \Psi^{-1}(v)\right)=v d(\Psi(g), v)$ for any $u>0$ with $v=\Psi(u)$.

From the first part of the proof of this theorem for any $r>0$ we have

$$
I_{5}=\Psi\left(\frac{\left|I_{\alpha} f(x)\right| \chi_{B_{r}^{1}}(x)}{2 \cdot 2 c_{7}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \leq \frac{1}{2}\left|B_{2 r}\right|^{\mu-1} \chi_{B_{r}^{1}}(x)<\frac{1}{2}\left|B_{r}\right|^{\mu-1} \chi_{B_{r}^{1}}(x)
$$

and

$$
\begin{aligned}
\sup _{u>0} \frac{u}{\left|B_{r}\right|^{\mu}} d\left(I_{5}, u\right) & \leq \frac{1}{2} \sup _{u>0} \frac{u}{\left|B_{r}\right|^{\mu}} d\left(\left|B_{r}\right|^{\mu-1} \chi_{B_{r}^{1}}(x), u\right) \\
& =\frac{1}{2} \sup _{u>0} u d\left(\frac{1}{\left|B_{r}\right|} \chi_{B_{r}^{1}}(x), u\right) \leq \frac{1}{2}
\end{aligned}
$$

For $I_{6}$ from the first part of the proof of this theorem we obtain

$$
I_{6}=\Psi\left(\frac{\left|I_{\alpha} f(x)\right| \chi_{B_{r}^{2}}(x)}{2 c_{9}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \leq \frac{1}{2} \frac{\left|B_{r}\right|^{\mu}}{\left|B_{\delta}\right|} \chi_{B_{r}^{2}}(x),
$$

where $\delta$ is defined as in (4) with respect to $c_{0}$, that is,

$$
\Phi\left(\frac{M f(x)}{c_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right)=\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{\delta}\right|} .
$$

Thus,

$$
I_{6} \leq \frac{1}{2}\left|B_{r}\right|^{\mu-\lambda} \Phi\left(\frac{M f(x)}{c_{0}\|f\|_{M^{\Phi, \lambda}(0)}}\right) \chi_{B_{r}^{2}}(x)
$$

and doing the same calculations as in the proof of Theorem 3(ii) in [4] we get

$$
\sup _{u>0} \frac{u}{\left|B_{r}\right|^{\mu}} d\left(I_{6}, u\right) \leq \frac{1}{2}
$$

Hence,

$$
\sup _{u>0} \frac{\Psi(u)}{\left|B_{r}\right|^{\mu}} d\left(\frac{\left|I_{\alpha} f(x)\right|}{c_{3}\|f\|_{M^{\Phi, \lambda}(0)}}, u\right) \leq 1
$$

and $\left\|I_{\alpha} f\right\|_{W M^{\Psi, \mu}(0)} \leq c_{3}\|f\|_{M^{\Phi, \lambda}(0)}$.
Below we present examples for our Theorem 1. In our earlier paper [4] we have shown that Example 1 holds under conditions (1) and (3), which clearly means that it also holds under conditions (1) and (2) of Theorem 1.

Example 1 Let $0<\alpha<n, 0 \leq \lambda<1,1<p<\frac{n(1-\lambda)}{\alpha}, 0 \leq a \leq \sqrt{1-\frac{1}{p}}-\left(1-\frac{1}{p}\right)$ and

$$
\Phi^{-1}(u)= \begin{cases}u^{\frac{1}{p}} & \text { for } 0 \leq u \leq 1, \quad \Psi^{-1}(u)=u^{\frac{1}{q}} \text { with } 1<p<q<\infty . \\ u^{\frac{1}{p}}(1+\ln u)^{-a} & \text { for } u \geq 1,\end{cases}
$$

If $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}, \frac{\lambda}{p}=\frac{\mu}{q}$, then conditions (1) and (2) of Theorem 1 are satisfied, and the Riesz potential $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$. We note that condition $0 \leq a \leq \frac{1}{p}$ ensures that function $\Phi^{-1}(u)$ is increasing on $(0, \infty)$ and $\Phi^{-1}(u) / u$ is decreasing on $(0, \infty)$. Then the function $\Phi^{-1}(u)$ is equivalent to a concave function on $(0, \infty)$ (cf. [1, pp. 117-118] or [11, p. 49]). On the other hand, if we have stronger requirement $0 \leq a \leq \sqrt{1-\frac{1}{p}}-\left(1-\frac{1}{p}\right)$, then it is possible to prove that the function $\Phi^{-1}(u)$ is concave on $(0, \infty)$. In particular, if $a=0$ we get the Spanne-Peetre type result [15] proved in [5, Proposition 1.1], that is, the Riesz potential $I_{\alpha}$ is bounded from $M^{p, \lambda}(0)$ to $M^{q, \mu}(0)$ under the conditions $1<p<\frac{n(1-\lambda)}{\alpha}, 0 \leq \lambda<1, \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $\frac{\lambda}{p}=\frac{\mu}{q}$.

The next two examples satisfy conditions (1) and (2), but the requirement (3) does not hold for them.

Example 2 Let $0<\alpha<n, 0<\lambda, \mu<1,1<p_{1}<p_{2}<\frac{n(1-\lambda)}{\alpha}, 1<q_{1}<q_{2}<$ $\infty$ and

$$
\Phi(u)=\max \left(u^{p_{1}}, u^{p_{2}}\right), \quad \Psi(u)=\max \left(u^{q_{1}}, u^{q_{2}}\right) .
$$

If $\frac{1}{p_{1}}-\frac{\alpha}{n}=\frac{1}{q_{1}}, \frac{\lambda}{p_{1}}<\frac{\mu}{q_{1}}$ and $\frac{1}{p_{2}}-\frac{\alpha}{n}=\frac{1}{q_{2}}, \frac{\lambda}{p_{2}}=\frac{\mu}{q_{2}}$, then conditions (1) and (2) of Theorem 1 are satisfied and the Riesz potential $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$.

Example 3 Let $0<\alpha<n, 0<\lambda, \mu<1,1<p_{1}<p_{2}<\infty, 1<q_{1}<q_{2}<$ $\infty, a, b>0$ and

$$
\begin{aligned}
& \Phi^{-1}(u)= \begin{cases}u^{\frac{1}{p_{1}}}(1-\ln u)^{a} & \text { for } 0<u \leq 1, \\
u^{\frac{1}{p_{2}}}(1+\ln u)^{-b} & \text { for } u \geq 1,\end{cases} \\
& \Psi^{-1}(u)= \begin{cases}u^{\frac{1}{q_{1}}}\left(1-\frac{1-\lambda}{1-\mu} \ln u\right)^{a} & \text { for } 0<u \leq 1, \\
u^{\frac{1}{q_{2}}} & \text { for } u \geq 1 .\end{cases}
\end{aligned}
$$

If $\frac{1}{p_{1}}-\frac{\alpha}{n}=\frac{1}{q_{1}}, \frac{1}{p_{2}}-\frac{\alpha}{n}=\frac{1}{q_{2}}, \frac{\lambda}{p_{2}}=\frac{\mu}{q_{2}}, \frac{\lambda}{p_{1}}<\frac{\mu}{q_{1}}$ and $0<a \leq \frac{1-\mu}{1-\lambda}\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right), 0<$ $b \leq \frac{1}{p_{2}}$, then conditions (1) and (2) of Theorem 1 are satisfied and the Riesz potential $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$.

The technical details related to the proofs in Examples 2 and 3 are shifted to the "Appendix" in Sect. 4.

## 3 Two properties of central Morrey-Orlicz spaces

Properties of Morrey and central Morrey spaces were considered by several authors (for example V. I. Burenkov, V. S. Guliyev, E. Nakai, Y. Sawano and others). Here we will present some properties of central Morrey-Orlicz spaces. It is known that $M^{p, \lambda}(0) \neq\{0\}$ if and only if $\lambda \geq 0$ (see [2]). In the next proposition we describe when the central Morrey-Orlicz space $M^{\Phi, \lambda}(0)$ is nontrivial.

Proposition 1 Let $\Phi$ be an Orlicz function and $\lambda \in \mathbb{R}$. The space $M^{\Phi, \lambda}(0) \neq\{0\}$ if and only if $\lambda \geq 0$.

Proof Let first $\lambda<0$ and $f \in M^{\Phi, \lambda}(0)$, such that $f \not \equiv 0$. Then

$$
\sup _{r>0}\left\|\frac{f}{\|f\|_{M^{\Phi, \lambda}(0)}}\right\|_{\Phi, \lambda, B_{r}}=1
$$

and therefore

$$
\left\|\frac{f}{\|f\|_{M^{\Phi, \lambda}(0)}}\right\|_{\Phi, \lambda, B_{r}} \leq 1 \quad \text { for all } r>0 .
$$

Thus,

$$
\frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}} \Phi\left(\frac{|f(x)|}{\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \leq 1 \text { for all } r>0
$$

On the other hand, there exists $t_{0}>0$, such that $\int_{B_{t_{0}}} \Phi\left(\frac{|f(x)|}{\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x>0$ and for any $r>t_{0}$ and $\lambda<0$ we have

$$
\int_{B_{t_{0}}} \Phi\left(\frac{|f(x)|}{\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \leq \int_{B_{r}} \Phi\left(\frac{|f(x)|}{\|f\|_{M^{\Phi, \lambda}(0)}}\right) d x \leq\left|B_{r}\right|^{\lambda} \rightarrow 0 \text { as } r \rightarrow \infty
$$

which means that $f(x)=0$ on $B_{t_{0}}$ and we are done.
Let now $\lambda \geq 0$. Then we will show that there exists $f \in M^{\Phi, \lambda}(0)$, such that $f \not \equiv 0$. We follow ideas from [9, Proposition 1] and consider function $f_{R}(x)=\chi_{B\left(x_{R}, 1\right)}(x)$, where $R>1$ and $x_{R}=(R, 0, \ldots, 0)$. We will show that $f \in M^{\Phi, \lambda}(0)$ for any $\lambda \geq 0$. In our previous paper we have shown that

$$
\left\|f_{R}\right\|_{M^{\Phi, \lambda}(0)}=\sup _{r>R-1} \frac{1}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\mid}}{\left|B_{r} \cap B\left(x_{R}, 1\right)\right|}\right)},
$$

for details we refer to the proof of Theorem 2 in [4]. Since $\left|B_{r} \cap B\left(x_{R}, 1\right)\right| \leq$ $\left|B\left(x_{R}, 1\right)\right|=v_{n}$ it follows that $\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{r} \cap B\left(x_{R}, 1\right)\right|} \geq \frac{\left|B_{r}\right|^{\lambda}}{v_{n}}$ and $\frac{1}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{\left|B_{r} \cap B\left(x_{R}, 1\right)\right|}\right)} \leq \frac{1}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{v_{n}}\right)}$. Thus,

$$
\left\|f_{R}\right\|_{M^{\Phi, \lambda}(0)}=\sup _{r>R-1} \frac{1}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\mid}}{\left|B_{r} \cap B\left(x_{R}, 1\right)\right|}\right)} \leq \sup _{r>R-1} \frac{1}{\Phi^{-1}\left(\frac{\left|B_{r}\right|^{\lambda}}{v_{n}}\right)}=\frac{1}{\Phi^{-1}\left(\frac{\left|B_{R-1}\right|^{\lambda}}{v_{n}}\right)}
$$

where the last equality is true since $\lambda \geq 0$. Therefore, $f_{R} \in M^{\Phi, \lambda}(0)$.
Next we consider inclusion properties of central Morrey-Orlicz spaces. In the case of classical Morrey and classical central Morrey spaces it is known that if $1 \leq p<$ $q<\infty, 0 \leq \mu<\lambda<1$ and $\frac{1-\lambda}{p}=\frac{1-\mu}{q}$, then

$$
\begin{equation*}
M^{q, \mu}\left(\mathbb{R}^{n}\right) \stackrel{1}{\hookrightarrow} M^{p, \lambda}\left(\mathbb{R}^{n}\right) \text { and } M^{q, \mu}(0) \stackrel{1}{\hookrightarrow} M^{p, \lambda}(0) . \tag{5}
\end{equation*}
$$

Both inclusions are proper (see, for example, [6]). We also note that the second embedding in (5) is also true for $1<\lambda<\mu$. We have shown in [4] that the embeddings (5) follow by the Hölder-Rogers inequality with $\frac{q}{p}>1$. In the next theorem we present inclusion properties of central Morrey-Orlicz spaces.

Proposition 2 Let $\Phi$ and $\Psi$ be Orlicz functions, $0 \leq \lambda, \mu<1$. Then $M^{\Psi, \mu}(0) \hookrightarrow$ $M^{\Phi, \lambda}(0)$ if and only if there are constants $A_{1}, A_{2}>0$, such that
(i) $\Phi\left(\frac{u}{A_{1}}\right) \leq \Psi(u)^{\frac{\lambda-1}{\mu-1}}$ for all $u>0$ and
(ii) $\Phi\left(\frac{u}{A_{2}}\right) \leq \Psi(u) r^{\lambda-\mu}$ for all $u, r>0$, satisfying $\Psi^{-1}\left(r^{\mu-1}\right)<u$.

Proof Let first $f \in M^{\Psi, \mu}(0), f \not \equiv 0, B_{r}$ be any open ball in $\mathbb{R}^{n}$ and functions $\Phi$ and $\Psi$ satisfy conditions (i) and (ii). Then

$$
\frac{1}{\left|B_{r}\right|^{\mu}} \int_{B_{r}} \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right) d x \leq 1
$$

and so $\Psi\left(\frac{|f(x)|}{\|f\|_{M}^{\Psi, \mu_{(0)}}}\right)<\infty$ a.e. in $B_{r}$. We divide $B_{r}$ into two disjoint subsets

$$
B_{r}^{3}:=\left\{x \in B_{r}:\left|B_{r}\right|^{\mu-1}<\Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right)<\infty \text { a.e. }\right\}
$$

and

$$
B_{r}^{4}:=\left\{x \in B_{r}:\left|B_{r}\right|^{\mu-1} \geq \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right) \text { a.e. }\right\} .
$$

Let us denote $t=\left|B_{r}\right|$ and $u=\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}$. Then, for $x \in B_{r}^{3}$ we have $0<t^{\mu-1}<$ $\Psi(u)$ and from (ii) it follows that

$$
\begin{aligned}
& \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}^{3}} \Phi\left(\frac{|f(x)|}{A_{2}\|f\|_{M^{\Psi, \mu}(0)}}\right) d x \leq \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}^{3}} \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right)\left|B_{r}\right|^{\lambda-\mu} d x \\
& \quad=\frac{1}{\left|B_{r}\right|^{\mu}} \int_{B_{r}^{3}} \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right) d x \leq \frac{1}{\left|B_{r}\right|^{\mu}} \int_{B_{r}} \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right) d x \leq 1 .
\end{aligned}
$$

For $x \in B_{r}^{4}$ we get $0<\Psi(u) \leq t^{\mu-1}$ and from (i) it follows that

$$
\begin{aligned}
\frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}^{4}} \Phi\left(\frac{|f(x)|}{A_{1}\|f\|_{M^{\Psi, \mu}(0)}}\right) d x & \leq \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}^{4}} \Psi\left(\frac{|f(x)|}{\|f\|_{M^{\Psi, \mu}(0)}}\right)^{\frac{\lambda-1}{\mu-1}} d x \\
& \leq \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}^{4}}\left(\left|B_{r}\right|^{\mu-1}\right)^{\frac{\lambda-1}{\mu-1}} d x=\frac{\left|B_{r}^{4}\right|}{\left|B_{r}\right|} \leq 1 .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \frac{1}{\left|B_{r}\right|^{\lambda}} \int_{B_{r}} \Phi\left(\frac{|f(x)|}{2 \max \left(A_{1}, A_{2}\right)\|f\|_{M^{\Psi, \mu}(0)}}\right) d x \\
& \leq \frac{1}{2\left|B_{r}\right|^{\lambda}}\left[\int_{B_{r}^{3}} \Phi\left(\frac{|f(x)|}{A_{1}\|f\|_{M^{\Psi, \mu}(0)}}\right) d x+\int_{B_{r}^{4}} \Phi\left(\frac{|f(x)|}{A_{2}\|f\|_{M^{\Psi, \mu}(0)}}\right) d x\right] \leq 1
\end{aligned}
$$

and so $\|f\|_{M^{\Phi, \lambda}(0)} \leq 2 \max \left(A_{1}, A_{2}\right)\|f\|_{M^{\Psi, \mu}(0)}$.
Let now $\|f\|_{M^{\Phi, \lambda}(0)} \leq C\|f\|_{M^{\Psi, \mu}(0)}$ for any $f \in M^{\Psi, \mu}(0)$ and some constant $C>0$. First for any $t>0$ we consider $f_{t}(x)=\chi_{B_{t}}(x)$. Then, using Lemma 1(iii) we obtain

$$
\frac{1}{\Phi^{-1}\left(\left|B_{t}\right|^{\lambda-1}\right)} \leq \frac{C}{\Psi^{-1}\left(\left|B_{t}\right|^{\mu-1}\right)},
$$

which also means that $\Psi^{-1}\left(s^{\mu-1}\right) \leq C \Phi^{-1}\left(s^{\lambda-1}\right)$ for all $s>0$ or $s^{\mu-1} \leq$ $\Psi\left(C \Phi^{-1}\left(s^{\lambda-1}\right)\right)$. By change of variables $\Phi^{-1}\left(s^{\lambda-1}\right)=u$ we get

$$
\Phi(u)^{\frac{\mu-1}{\lambda-1}} \leq \Psi(C u) \text { for all } u>0,
$$

so we have condition (i) with $A_{1}=C$.
For the proof of the second part, i.e. to prove the necessity of the condition (ii), we refer to the proof of Lemma 4.12 and Theorem 4.1 in [8] and to Theorem 4.9 and Lemma 4.10 in [14].

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## Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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## 4 Appendix

We will present here all the technical proofs related to Examples 2 and 3.
Proof of Example 2 The function

$$
\Phi(u)=\max \left(u^{p_{1}}, u^{p_{2}}\right)= \begin{cases}u^{p_{1}} & \text { for } 0 \leq u \leq 1, \\ u^{p_{2}} & \text { for } u \geq 1,\end{cases}
$$

is an Orlicz function, $\Phi^{*} \in \Delta_{2}$ and

$$
\Phi^{-1}(u)=\min \left(u^{\frac{1}{p_{1}}}, u^{\frac{1}{p_{2}}}\right)= \begin{cases}u^{\frac{1}{p_{1}}} & \text { for } 0 \leq u \leq 1, \\ u^{\frac{1}{p_{2}}} & \text { for } u \geq 1\end{cases}
$$

If $u \geq 1$, then estimate (1) holds since

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & =\int_{u}^{\infty} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{1}}} \frac{d t}{t}=\int_{u}^{\infty} t^{-\frac{1}{q_{1}}+\frac{\lambda}{p_{1}}-1} d t \\
& =\frac{1}{\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}+\frac{\lambda}{p_{1}}}<\frac{1}{\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}} u^{\frac{\mu-1}{q_{1}}}=\frac{1}{\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}} \Psi^{-1}\left(u^{\mu-1}\right)
\end{aligned}
$$

where the last integral is convergent because $p_{1}<\frac{n(1-\lambda)}{\alpha}$.
If $0<u \leq 1$, then estimate (1) holds since

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & =\int_{u}^{1} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t}+\int_{1}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \\
& =\int_{u}^{1} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{2}}} \frac{d t}{t}+\int_{1}^{\infty} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{1}}} \frac{d t}{t} \\
& =\int_{u}^{1} t^{\frac{\mu-1}{q_{2}}-1} d t+\int_{1}^{\infty} t^{-\frac{1}{q_{1}}+\frac{\lambda}{p_{1}}-1} d t \\
& =\frac{q_{2}}{1-\mu}\left(u^{\frac{\mu-1}{q_{2}}}-1\right)+\frac{1}{\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}} \\
& =\frac{q_{2}}{1-\mu} u^{\frac{\mu-1}{q_{2}}}+\left(\frac{1}{\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}}-\frac{1}{\frac{1-\mu}{q_{2}}}\right) \\
& \leq \frac{q_{2}}{1-\mu} u^{\frac{\mu-1}{q_{2}}}=\frac{q_{2}}{1-\mu} \Psi^{-1}\left(u^{\mu-1}\right)
\end{aligned}
$$

where the last inequality is true since $p_{1}<p_{2}$ and therefore $\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}>\frac{1-\mu}{q_{2}}$.
Estimate (2) also holds for any $r>u>0$. Indeed, let first $r>1$, then $1<r^{\lambda}<$ $r^{\mu}<r$. We consider three cases on $u$.
$1^{\circ}$. If $u<r^{\lambda}$, then we get

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \\
& \quad=r^{\frac{\mu}{q_{2}}} u^{-\frac{1}{q_{2}}}+r^{\frac{\lambda}{p_{2}}} \int_{u}^{r^{\lambda}} t^{\frac{\alpha}{n}-\frac{1}{p_{2}}-1} d t+r^{\frac{\lambda}{p_{1}}} \int_{r^{\lambda}}^{r} t^{\frac{\alpha}{n}-\frac{1}{p_{1}}-1} d t \\
& \quad=\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+q_{2} r^{\frac{\lambda}{p_{2}}}\left(u^{-\frac{1}{q_{2}}}-r^{\lambda\left(\frac{\alpha}{n}-\frac{1}{p_{2}}\right)}\right)+q_{1} r^{\frac{\lambda}{p_{1}}}\left(r^{\lambda\left(\frac{\alpha}{n}-\frac{1}{p_{1}}\right)}-r^{\frac{\alpha}{n}-\frac{1}{p_{1}}}\right) \\
& \quad=\left(1+q_{2}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+\left(q_{1}-q_{2}\right) r^{\lambda \frac{\alpha}{n}}-q_{1} r^{\frac{\lambda-1}{p_{1}}+\frac{\alpha}{n}} \leq\left(1+q_{2}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right)
\end{aligned}
$$

where the last inequality is true since $q_{1}<q_{2}$.
$2^{\circ}$. If $1<r^{\lambda}<r^{\mu}<u<r$, then we obtain

$$
\begin{aligned}
u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right) & +\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}=r^{\frac{\lambda}{p_{1}}} u^{\frac{\alpha}{n}-\frac{1}{p_{1}}}+r^{\frac{\lambda}{p_{1}}} \int_{u}^{r} t^{\frac{\alpha}{n}-\frac{1}{p_{1}}-1} d t \\
& <r^{\frac{\mu}{q_{1}}} u^{-\frac{1}{q_{1}}}+q_{1} r^{\frac{\mu}{q_{1}}}\left(u^{-\frac{1}{q_{1}}}-r^{-\frac{1}{q_{1}}}\right)<\left(1+q_{1}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{1}}} \\
& =\left(1+q_{1}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) .
\end{aligned}
$$

$3^{\circ}$. If $1<r^{\lambda}<u<r^{\mu}<r$, then taking into account that $\frac{1}{p_{1}}>\frac{1}{p_{2}}$ and $\frac{r^{\lambda}}{u}<1$ we obtain

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}=r^{\frac{\lambda}{p_{1}}} u^{\frac{\alpha}{n}-\frac{1}{p_{1}}}+r^{\frac{\lambda}{p_{1}}} \int_{u}^{r} t^{\frac{\alpha}{n}-\frac{1}{p_{1}}-1} d t \\
& \quad=u^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{u}\right)^{\frac{1}{p_{1}}}+q_{1}\left(\left(\frac{r^{\lambda}}{u}\right)^{\frac{1}{p_{1}}} u^{\frac{\alpha}{n}}-r^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}}\right) \\
& \quad<\left(1+q_{1}\right) u^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{u}\right)^{\frac{1}{p_{2}}}=\left(1+q_{1}\right) r^{\frac{\mu}{q_{2}}} u^{-\frac{1}{q_{2}}}=\left(1+q_{1}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right)
\end{aligned}
$$

Let now $0<r \leq 1$. Then $0<u<r \leq r^{\mu} \leq r^{\lambda} \leq 1$ and (2) holds since

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}=r^{\frac{\lambda}{p_{2}}} u^{-\frac{1}{q_{2}}}+r^{\frac{\lambda}{p_{2}}} \int_{u}^{r} t^{\frac{\alpha}{n}-\frac{1}{p_{2}}-1} d t \\
& =\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+q_{2} r^{\frac{\mu}{q_{2}}}\left(u^{-\frac{1}{q_{2}}}-r^{-\frac{1}{q_{2}}}\right) \leq\left(1+q_{2}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right)
\end{aligned}
$$

Thus, all conditions of Theorem 1 are satisfied and $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$.

Let us note that this example does not satisfy the condition (3), that is, the condition

$$
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq C_{4} \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) \text { for all } u>0, r>0
$$

does not hold in this case. Indeed, let $0<r<1$ and $u>r$. We choose any number $v$, such that $0<\lambda<v<\mu<1$ and let $u=r^{\nu}$. Then $0<r<r^{\mu}<u=r^{\nu}<r^{\lambda}<1$ and we get

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} & =r^{\frac{\lambda}{p_{2}}} \int_{r^{\nu}}^{r^{\lambda}} t^{-\frac{1}{q_{2}}-1} d t+r^{\frac{\lambda}{p_{1}}} \int_{r^{\lambda}}^{\infty} t^{-\frac{1}{q_{1}}-1} d t \\
& =q_{2} r^{\frac{\lambda}{p_{2}}-\frac{v}{q_{2}}}-q_{2} r^{\frac{\lambda}{p_{2}}-\frac{\lambda}{q_{2}}}+q_{1} r^{\frac{\lambda}{p_{1}}-\frac{\lambda}{q_{1}}} \\
& =q_{2} r^{\frac{\mu}{q_{2}}-\frac{v}{q_{2}}}+\left(q_{1}-q_{2}\right) r^{\frac{\lambda \alpha}{n}}
\end{aligned}
$$

Since $\frac{\lambda \alpha}{n}=\frac{\lambda}{p_{2}}-\frac{\lambda}{q_{2}}=\frac{\mu-\lambda}{q_{2}}>\frac{\mu-v}{q_{2}}$ and $\frac{1}{q_{1}}>\frac{1}{q_{2}}$ it follows that

$$
\begin{aligned}
\lim _{r \rightarrow 0+} \frac{\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}}{\Psi^{-1}\left(\frac{r^{\mu}}{u}\right)} & =\lim _{r \rightarrow 0+} \frac{q_{2} r^{\frac{\mu-v}{q_{2}}}+\left(q_{1}-q_{2}\right) r^{\frac{\lambda \alpha}{n}}}{r^{\frac{\mu-v}{q_{1}}}} \\
& =\lim _{r \rightarrow 0+} r^{(\mu-v)\left(\frac{1}{q_{2}}-\frac{1}{q_{1}}\right)}\left(q_{2}+\left(q_{1}-q_{2}\right) r^{\frac{\lambda \alpha}{n}-\frac{\mu-v}{q_{2}}}\right)=\infty .
\end{aligned}
$$

Thus, this example does not satisfy condition (3), which shows that Theorem 1 improves our result proved in [4, Theorem 3].

Proof of Example 3 Since $\frac{\lambda}{p_{1}}<\frac{\mu}{q_{1}}$ and $1<q_{1}<q_{2}<\infty$ it follows that

$$
\frac{\mu-\lambda}{q_{2}}=\frac{\lambda}{p_{2}}-\frac{\lambda}{q_{2}}=\frac{\lambda}{p_{1}}-\frac{\lambda}{q_{1}}<\frac{\mu}{q_{1}}-\frac{\lambda}{q_{1}}=\frac{\mu-\lambda}{q_{1}},
$$

and so $\lambda<\mu$. Moreover, note that conditions $a \leq \frac{1}{p_{1}}, b \leq \frac{1}{p_{2}}$ and $a \leq \frac{1}{q_{1}} \frac{1-\mu}{1-\lambda}$ ensure that functions $\Phi^{-1}(u)$ and $\Psi^{-1}(u)$ are increasing on $(0, \infty)$, and functions $\Phi^{-1}(u) / u$
and $\Psi^{-1}(u) / u$ are decreasing on $(0, \infty)$. Then $\Phi^{-1}(u)$ and $\Psi^{-1}(u)$ are equivalent to concave functions on $(0, \infty)$ (cf. [1, pp. 117-118] or [11, p. 49]). There exist concrete parameters $a$ and $b$, for which functions $\Phi^{-1}(u)$ and $\Psi^{-1}(u)$ are concave on $(0, \infty)$, but it requires long calculations to prove this, so we omit such details. In addition, for further estimations we require that $0<a \leq \frac{1-\mu}{1-\lambda}\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right)$.

First we will show that (1) holds for any $u>0$. Let $\varepsilon>0$ be sufficiently small such that $0<\varepsilon<\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}-a(1-\lambda)$. Observe that

$$
\begin{aligned}
\frac{1}{q_{1}}-\frac{\lambda}{p_{1}}-a(1-\lambda) & \geq \frac{1}{q_{1}}-\frac{\lambda}{p_{1}}-(1-\mu)\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right) \\
& >\frac{1}{q_{1}}-\frac{\mu}{q_{1}}-(1-\mu)\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right)=(1-\mu) \frac{1}{q_{2}}>0 .
\end{aligned}
$$

Then the function

$$
\begin{equation*}
f_{1}(t)=t^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln t^{1-\lambda}\right)^{a} \text { is decreasing on }[1, \infty) . \tag{6}
\end{equation*}
$$

We consider separately two cases on $u$. For $u \geq 1$ we have

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & =\int_{u}^{\infty} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{1}}}\left(1+\ln t^{1-\lambda}\right)^{a} \frac{d t}{t}=\int_{u}^{\infty} t^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln t^{1-\lambda}\right)^{a} \frac{d t}{t^{1+\varepsilon}} \\
& \leq u^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln u^{1-\lambda}\right)^{a} \int_{u}^{\infty} \frac{d t}{t^{1+\varepsilon}}=\frac{1}{\varepsilon} u^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}}\left(1+\ln u^{1-\lambda}\right)^{a} .
\end{aligned}
$$

Since $\frac{\lambda}{p_{1}}<\frac{\mu}{q_{1}}$ it follows that

$$
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \leq \frac{1}{\varepsilon} u^{\frac{\mu}{q_{1}}-\frac{1}{q_{1}}}\left(1+\frac{1-\lambda}{1-\mu} \ln u^{1-\mu}\right)^{a}=\frac{1}{\varepsilon} \Psi^{-1}\left(u^{\mu-1}\right)
$$

and therefore estimate (1) holds for $u \geq 1$.

Let $0<u<1$. Then

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & =\int_{u}^{1} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{2}}}\left(1+\ln t^{\lambda-1}\right)^{-b} \frac{d t}{t}+\int_{1}^{\infty} t^{\frac{\alpha}{n}+\frac{\lambda-1}{p_{1}}}\left(1-\ln t^{\lambda-1}\right)^{a} \frac{d t}{t} \\
& =\int_{u}^{1} t^{\frac{\mu}{q_{2}}-\frac{1}{q_{2}}}\left(1+\ln t^{\lambda-1}\right)^{-b} \frac{d t}{t}+\int_{1}^{\infty} t^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}}\left(1+\ln t^{1-\lambda}\right)^{a} \frac{d t}{t}
\end{aligned}
$$

The function $\left(1+\ln t^{\lambda-1}\right)^{-b}$ is increasing for any $0<t<1$ and by (6) we obtain

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} & \leq \int_{u}^{1} t^{\frac{\mu-1}{q_{2}}-1} d t+\int_{1}^{\infty} t^{\frac{\lambda}{p_{1}}-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln t^{1-\lambda}\right)^{a} \frac{d t}{t^{1+\varepsilon}} \\
& \leq \frac{q_{2}}{1-\mu}\left(u^{\frac{\mu-1}{q_{2}}}-1\right)+\int_{1}^{\infty} t^{-1-\varepsilon} d t \\
& =\frac{q_{2}}{1-\mu} u^{\frac{\mu-1}{q_{2}}}+\left(\frac{1}{\varepsilon}-\frac{q_{2}}{1-\mu}\right) .
\end{aligned}
$$

If $\frac{1}{\varepsilon}-\frac{q_{2}}{1-\mu} \leq 0$, then we are ready with (1) for $0<u<1$. If $\frac{1}{\varepsilon}-\frac{q_{2}}{1-\mu}>0$, then since $1<u^{\frac{\mu-1}{q_{2}}}$ we get

$$
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(t^{\lambda-1}\right) \frac{d t}{t} \leq \frac{q_{2}}{1-\mu} u^{\frac{\mu-1}{q_{2}}}+\left(\frac{1}{\varepsilon}-\frac{q_{2}}{1-\mu}\right) u^{\frac{\mu-1}{q_{2}}}=\frac{1}{\varepsilon} \Psi^{-1}\left(u^{\mu-1}\right)
$$

and therefore estimate (1) holds for $0<u<1$.
Next, we will show that (2) holds. Let first $0<r \leq 1$. Then $0<r \leq r^{\mu} \leq r^{\lambda} \leq 1$ and for any $0<u<r$ we obtain

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \\
& \quad=r^{\frac{\lambda}{p_{2}}} u^{-\frac{1}{q_{2}}}\left(1+\ln \frac{r^{\lambda}}{u}\right)^{-b}+\int_{u}^{r} t^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{t}\right)^{\frac{1}{p_{2}}}\left(1+\ln \frac{r^{\lambda}}{t}\right)^{-b} \frac{d t}{t}
\end{aligned}
$$

Since $\left(1+\ln \frac{r^{\lambda}}{u}\right)^{-b} \leq 1$ and the function $\left(1+\ln \frac{r^{\lambda}}{t}\right)^{-b}$ is increasing for any $t \in\left(0, r^{\lambda}\right)$ it follows that

$$
\begin{aligned}
u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} & \leq r^{\frac{\mu}{q_{2}}} u^{-\frac{1}{q_{2}}}+r^{\frac{\mu}{q_{2}}}\left(1+\ln r^{\lambda-1}\right)^{-b} \int_{u}^{r} t^{-\frac{1}{q_{2}}-1} d t \\
& \leq\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+q_{2} r^{\frac{\mu}{q_{2}}}\left(u^{-\frac{1}{q_{2}}}-r^{-\frac{1}{q_{2}}}\right) \\
& \leq\left(1+q_{2}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}} \\
& =\left(1+q_{2}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) .
\end{aligned}
$$

Thus, we get that (2) holds for any $0<u<r \leq 1$.
Let now $r>1$. Then $1<r^{\lambda}<r^{\mu}<r$. We consider three cases on $u$.
$1^{\circ}$. Let first $0<u<r^{\lambda}<r^{\mu}<r$. Then

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}=r^{\frac{\lambda}{p_{2}}} u^{-\frac{1}{q_{2}}}\left(1+\ln \frac{r^{\lambda}}{u}\right)^{-b} \\
& +\int_{u}^{r^{\lambda}} t^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{t}\right)^{\frac{1}{p_{2}}}\left(1+\ln \frac{r^{\lambda}}{t}\right)^{-b} \frac{d t}{t}+\int_{r^{\lambda}}^{r} t^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{t}\right)^{\frac{1}{p_{1}}}\left(1-\ln \frac{r^{\lambda}}{t}\right)^{a} \frac{d t}{t} \\
& <\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+r^{\frac{\lambda}{p_{2}}} \int_{u}^{r^{\lambda}} t^{-\frac{1}{q_{2}}}\left(1+\ln \frac{r^{\lambda}}{t}\right)^{-b} \frac{d t}{t} \\
& +r^{\frac{\lambda}{p_{1}}} \int_{r^{\lambda}}^{r} t^{-\frac{1}{q_{1}}}+\varepsilon \\
&
\end{aligned}
$$

Let $0<\varepsilon<\frac{1}{q_{1}}-a$. Since the function $\left(1+\ln \frac{r^{\lambda}}{t}\right)^{-b}$ is increasing for any $t \in\left(0, r^{\lambda}\right)$ and

$$
\begin{equation*}
f_{2}(t)=t^{-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln \frac{t}{r^{\lambda}}\right)^{a} \text { is decreasing on }\left(r^{\lambda}, \infty\right) \tag{7}
\end{equation*}
$$

it follows that

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \\
& \leq\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+r^{\frac{\mu}{q_{2}}} \int_{u}^{r^{\lambda}} t^{-\frac{1}{q_{2}}-1} d t+r^{\frac{\lambda}{p_{1}}-\frac{\lambda}{q_{1}}+\lambda \varepsilon} \int_{r^{\lambda}}^{r} t^{-1-\varepsilon} d t \\
&=\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}+q_{2} r^{\frac{\mu}{q_{2}}}\left(u^{-\frac{1}{q_{2}}}-r^{-\frac{\lambda}{q_{2}}}\right)+\frac{1}{\varepsilon} r^{\lambda\left(\frac{1}{p_{1}}-\frac{1}{q_{1}}\right)}-\frac{1}{\varepsilon} r^{\lambda\left(\frac{1}{p_{1}}-\frac{1}{q_{1}}\right)+\varepsilon(\lambda-1)} \\
& \leq\left(1+q_{2}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}-r^{\frac{\alpha \lambda}{n}}\left(q_{2}-\frac{1}{\varepsilon}+\frac{1}{\varepsilon} r^{\varepsilon(\lambda-1)}\right),
\end{aligned}
$$

where the last expression follows from two equalities $\frac{\mu-\lambda}{q_{2}}=\frac{\lambda}{p_{2}}-\frac{\lambda}{q_{2}}=\frac{\alpha \lambda}{n}$ and $\lambda\left(\frac{1}{p_{1}}-\frac{1}{q_{1}}\right)=\frac{\alpha \lambda}{n}$. Observe that

$$
\frac{1}{q_{1}}-a \geq \frac{1}{q_{1}}-\frac{1-\mu}{1-\lambda}\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right)=\frac{1}{q_{1}} \frac{\mu-\lambda}{1-\lambda}
$$

$$
+\frac{1}{q_{2}} \frac{1-\mu}{1-\lambda}=\frac{\mu-\lambda}{1-\lambda}\left(\frac{1}{q_{1}}-\frac{1}{q_{2}}\right)+\frac{1}{q_{2}}>\frac{1}{q_{2}}
$$

and we choose $\varepsilon>0$ such that $\frac{1}{q_{2}}<\varepsilon<\frac{1}{q_{1}}-a$. Then $q_{2}-\frac{1}{\varepsilon}+\frac{1}{\varepsilon} r^{\varepsilon(\lambda-1)}>0$ for any $r>1$. Thus, for $r^{\lambda}>u$ we obtain
$u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq\left(1+q_{2}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}=\left(1+q_{2}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right)$,
which shows that (2) holds for $u<r^{\lambda}<r^{\mu}<r$.
$2^{\circ}$. Let now $1<r^{\lambda}<u<r^{\mu}<r$. By (7) we have

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \\
&=r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a}+\int_{u}^{r} t^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{t}\right)^{\frac{1}{p_{1}}}\left(1+\ln \frac{t}{r^{\lambda}}\right)^{a} \frac{d t}{t} \\
&=r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a}+r^{\frac{\lambda}{p_{1}}} \int_{u}^{r} t^{-\frac{1}{q_{1}}}+\varepsilon \\
& u \\
&\left.\leq r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{t}{r^{\lambda}}\right)^{a} \frac{d t}{t^{1+\varepsilon}} \frac{u}{r^{\lambda}}\right)^{a}+r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a} \int_{u}^{r} t^{-1-\varepsilon} d t \\
& \leq\left(1+\frac{1}{\varepsilon}\right) r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a} .
\end{aligned}
$$

Since $\ln \frac{u}{r^{\lambda}} \leq \frac{u}{r^{\lambda}}-1$ for any $\frac{u}{r^{\lambda}}>1$ and $a<\frac{1}{p_{1}}-\frac{1}{p_{2}}$ it follows that

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq\left(1+\frac{1}{\varepsilon}\right) r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(\frac{u}{r^{\lambda}}\right)^{a} \\
& \leq\left(1+\frac{1}{\varepsilon}\right) r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(\frac{u}{r^{\lambda}}\right)^{\frac{1}{p_{1}}-\frac{1}{p_{2}}}=\left(1+\frac{1}{\varepsilon}\right) r^{\frac{\lambda}{p_{2}}} u^{-\frac{1}{q_{1}}+\frac{1}{q_{1}}-\frac{1}{q_{2}}} \\
&=\left(1+\frac{1}{\varepsilon}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{2}}}=\left(1+\frac{1}{\varepsilon}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right),
\end{aligned}
$$

where we used equality $\frac{1}{p_{1}}-\frac{1}{p_{2}}=\frac{1}{q_{1}}-\frac{1}{q_{2}}$. Thus, condition (2) is true also for $1<r^{\lambda}<u<r^{\mu}<r$.
$3^{\circ}$. Finally, we will show that (2) holds for $1<r^{\lambda}<r^{\mu}<u<r$. Again by (7) we get

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \\
& \quad=r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1-\ln \frac{r^{\lambda}}{u}\right)^{a}+\int_{u}^{r} t^{\frac{\alpha}{n}}\left(\frac{r^{\lambda}}{t}\right)^{\frac{1}{p_{1}}}\left(1+\ln \frac{t}{r^{\lambda}}\right)^{a} \frac{d t}{t} \\
& \quad=r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a}+r^{\frac{\lambda}{p_{1}}} \int_{u}^{r} t^{-\frac{1}{q_{1}}+\varepsilon}\left(1+\ln \frac{t}{r^{\lambda}}\right)^{a} \frac{d t}{t^{1+\varepsilon}} \\
& \quad \leq\left(1+\frac{1}{\varepsilon}\right) r^{\frac{\lambda}{p_{1}}} u^{-\frac{1}{q_{1}}}\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a} .
\end{aligned}
$$

Since the function $g(u)=1+\ln \frac{u}{r^{\lambda}}-\left(1+\ln \frac{u}{r^{\mu}}\right)\left(1+\ln r^{\mu-\lambda}\right)$ is decreasing for any $r^{\mu}<u<r$ it follows that $g(u) \leq g\left(r^{\mu}\right)=0$. Thus, taking into account that $\ln r^{\mu-\lambda} \leq r^{\mu-\lambda}-1$ for any $r>1$ and $\mu>\lambda$ we obtain

$$
\left(1+\ln \frac{u}{r^{\lambda}}\right)^{a} \leq\left(1+\ln \frac{u}{r^{\mu}}\right)^{a}\left(1+\ln r^{\mu-\lambda}\right)^{a} \leq r^{a(\mu-\lambda)}\left(1+\ln \frac{u}{r^{\mu}}\right)^{a} .
$$

Applying estimates $a<\frac{1}{p_{1}}-\frac{1}{p_{2}}$ and $\frac{1-\lambda}{1-\mu}>1$ we get

$$
\begin{aligned}
& u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{u}\right)+\int_{u}^{r} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq\left(1+\frac{1}{\varepsilon}\right) u^{-\frac{1}{q_{1}}} r^{a(\mu-\lambda)+\frac{\lambda}{p_{1}}}\left(1+\ln \frac{u}{r^{\mu}}\right)^{a} \\
& \leq\left(1+\frac{1}{\varepsilon}\right) u^{-\frac{1}{q_{1}}} r^{\frac{\mu}{p_{1}}-\frac{\mu}{p_{2}}+\frac{\lambda}{p_{2}}}\left(1+\ln \frac{u}{r^{\mu}}\right)^{a} \\
& \quad \leq\left(1+\frac{1}{\varepsilon}\right) u^{-\frac{1}{q_{1}}} r^{\frac{\mu}{q_{1}}}-\frac{\mu}{q_{2}}+\frac{\mu}{q_{2}}\left(1+\frac{1-\lambda}{1-\mu} \ln \frac{u}{r^{\mu}}\right)^{a} \\
& \quad=\left(1+\frac{1}{\varepsilon}\right)\left(\frac{r^{\mu}}{u}\right)^{\frac{1}{q_{1}}}\left(1-\frac{1-\lambda}{1-\mu} \ln \frac{r^{\mu}}{u}\right)^{a}=\left(1+\frac{1}{\varepsilon}\right) \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) .
\end{aligned}
$$

Thus, all conditions of Theorem 1 are satisfied and $I_{\alpha}$ is bounded from $M^{\Phi, \lambda}(0)$ to $M^{\Psi, \mu}(0)$.

It is important to mention that this example does not satisfy the condition (3), that is, the condition

$$
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} \leq C_{4} \Psi^{-1}\left(\frac{r^{\mu}}{u}\right) \text { for all } u>0 \text { and for all } r>0
$$

is not true for functions $\Phi^{-1}$ and $\Psi^{-1}$ defined in Example 3. Indeed, let $0<r<1$ and $u=1$. Then taking into account that $\left(1-\ln \frac{r^{\lambda}}{t}\right)^{a}$ is increasing function on $[1, \infty)$
we get

$$
\begin{aligned}
\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t} & =r^{\frac{\lambda}{p_{1}}} \int_{1}^{\infty} t^{-\frac{1}{q_{1}}}\left(1-\ln \frac{r^{\lambda}}{t}\right)^{a} \frac{d t}{t} \\
& \geq r^{\frac{\lambda}{p_{1}}}\left(1+\ln \frac{1}{r^{\lambda}}\right)^{a} \int_{1}^{\infty} t^{-\frac{1}{q_{1}}-1} d t=q_{1} r^{\frac{\lambda}{p_{1}}}\left(1+\ln \frac{1}{r^{\lambda}}\right)^{a}
\end{aligned}
$$

Since $\frac{\lambda}{p_{1}}<\frac{\mu}{q_{1}}$ it follows that

$$
\begin{aligned}
\lim _{r \rightarrow 0+} \frac{\int_{u}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^{\lambda}}{t}\right) \frac{d t}{t}}{\Psi^{-1}\left(\frac{r^{\mu}}{u}\right)} & \geq \lim _{r \rightarrow 0+} \frac{q_{1} r^{\frac{\lambda}{p_{1}}}\left(1+\ln \frac{1}{r^{\lambda}}\right)^{a}}{r^{\frac{\mu}{q_{1}}}\left(1+\frac{1-\lambda}{1-\mu} \ln \frac{1}{r^{\mu}}\right)^{a}}=q_{1} \lim _{s \rightarrow \infty} \frac{s^{\frac{\mu}{q_{1}}-\frac{\lambda}{p_{1}}}(1+\lambda \ln s)^{a}}{\left(1+\frac{(1-\lambda) \mu}{1-\mu} \ln s\right)^{a}} \\
& =q_{1} \lim _{s \rightarrow \infty} \frac{s^{\frac{\mu}{q_{1}}-\frac{\lambda}{p_{1}}}(\lambda \ln s)^{a}}{\left(\frac{(1-\lambda) \mu}{1-\mu} \ln s\right)^{a}}=\infty .
\end{aligned}
$$

Thus, the condition (3) is not satisfied for the functions $\Phi$ and $\Psi$, defined in Example 3. However, as it was shown above, all conditions of Theorem 1 are satisfied, which shows that Theorem 1 improves our result proved in [4, Theorem 3].

## References

1. Bergh, J., Löfström, J.: Interpolation Spaces: An Introduction. Springer, Berlin (1976)
2. Burenkov, V.I., Guliyev, H.V.: Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces. Studia Math. 163(2), 157-176 (2004)
3. Burenkov, V.I., Guliev, V.S., Guliev, G.V.: Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces. Dokl. Akad. Nauk 412(5), 585-589 (2007) (in Russian); English translation in Dokl. Math. 75(1), 103-107 (2007)
4. Burtseva, E., Maligranda, L., Matsuoka, K.: Boundedness of the Riesz potential in central MorreyOrlicz spaces. Positivity 26(1), Paper No. 22 (2022)
5. Fu, Z.W., Lin, Y., Lu, S.: $\lambda$-central BMO estimates for commutators of singular integral operators with rough kernels. Acta Math. Sin. (Engl. Ser.) 24(3), 373-386 (2008)
6. Gunawan, H., Hakim, D.I., Idris, M.: Proper inclusions of Morrey spaces. Glas. Mat. Ser. III 53(73), 143-151 (2018)
7. Hedberg, L.I.: On certain convolution inequalities. Proc. Am. Math. Soc. 36, 505-510 (1972)
8. Kita, H.: Some inclusion relations of Orlicz-Morrey spaces and the Hardy-Littlewood maximal function. In: Banach and Function Spaces IV (ISBFS 2012), pp. 81-116. Yokohama Publishers, Yokohama (2014)
9. Komori-Furuya, Y., Sato, E.: Fractional integral operators on central Morrey spaces. Math. Inequal. Appl. 20(3), 801-813 (2017)
10. Krasnoselskii, M.A., Rutickii, Ja.B.: Convex Functions and Orlicz Spaces. Noordhoff, Groningen (1961)
11. Krein, S.G., Petunin, Yu.I., Semenov, E.M.: Interpolation of Linear Operators. American Mathematical Society, Providence (1982)
12. Maligranda, L.: Orlicz Spaces and Interpolation, Seminars in Mathematics, vol. 5. Universidade Estadual de Campinas, Campinas (1989)
13. Maligranda, L., Matsuoka, K.: Maximal function in Beurling-Orlicz and central Morrey-Orlicz spaces. Colloq. Math. 138(2), 165-181 (2015)
14. Nakai, E.: Orlicz-Morrey spaces and the Hardy-Littlewood maximal function. Stud. Math. 188(3), 193-221 (2008)
15. Peetre, J.: On the theory of $\mathcal{L}_{p, \lambda}$ spaces. J. Funct. Anal. 4, 71-87 (1969)

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