



UMEÅ UNIVERSITY

# ON GENERATING THE PROBABILITY MASS FUNCTION USING FIBONACCI POWER SERIES

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## ABSTRACT

This thesis will focus on generating the probability mass function using Fibonacci sequence as the coefficient of the power series. The discrete probability, named Fibonacci distribution, was formed by taking into consideration the recursive property of the Fibonacci sequence, the radius of convergence of the power series, and additive property of mutually exclusive events. This distribution satisfies the requisites of a legitimate probability mass function. Its cumulative distribution function and the moment generating function are then derived and the latter are used to generate moments of the distribution, specifically, the mean and the variance. The characteristics of some convergent sequences generated from the Fibonacci sequence are found useful in showing that the limiting form of the Fibonacci distribution is a geometric distribution. Lastly, the paper showcases applications and simulations of the Fibonacci distribution using MATLAB.

Keywords: Power series distribution, Fibonacci sequence, Fibonacci distribution, probability mass function

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## CHAPTER I

### INTRODUCTION

This chapter introduces the relevant concepts in probability theory, power series and Fibonacci sequence. Related literature and studies, the statement of the problem, significance, scope and limitation of the study are also given in this chapter. Other important details are found in the succeeding chapters.

#### 1.1 Background and Literature Review of the Study

Consider a random experiment of tossing a fair coin such that we achieve a success if we obtained a head. Suppose  $X_1$  is the random variable associated for this experiment that describes the number of tails before obtaining the head and  $p = \frac{1}{2}$  is the Bernoulli probability of having a head in a single toss, then  $X_1$  is commonly known as the geometric random variable.

In 1973, Harold Shane [1] generalized the idea into a Markov process  $X_n$ . The random process is initiated with a marker positioned inside the slot number zero. Then, continuously tossing a fair coin and for every head a marker will move up one slot and for every tail the marker moves back to its original position. Success will only occur if there is  $n$  consecutive heads or the marker moved up  $n$  consecutively.

To summarize the result, let  $X_n$  be a random variable that describes the number of toss (not just tails) needed to obtain  $n$  consecutive heads. Then, the probability distribution of  $X_n$



is given by

$$P(X_n = k) = \frac{F_{n,k}}{2^{n+k}},$$

for  $k = 0, 1, 2, \dots$ , where  $F_{n,k}$  is the  $k^{\text{th}}$  Fibonacci number of order  $n$  defined by

$$F_{n,k+1} = F_{n,k} + F_{n,k-1} + \dots + F_{n,k-n+1}$$

with  $F_{n,0} = 1$  and  $F_{n,-1} = F_{n,-2} = \dots = F_{n,-n+1} = 0$ . For  $n = 2$ ,  $F_{2,k} = F_k$  describes the ordinary Fibonacci number and called the probability distribution as Fibonacci distribution. For  $n > 3$ , the distribution is called the poly-nacci distribution. Shane [1] also generalizes the random variable  $X_n$  with a biased coin, with Bernoulli probability  $0 < p < 1$  of having a head in a single toss. The Fibonacci numbers are not explicitly written on the resulting probability distribution since the probability of success and failure is of different values.

Taillie and Patil (1986) [2] extended the work of Shane by using a different approach on generalizing the Fibonacci distribution. The method is by including the Fibonacci distribution as a member of the power series distribution.

The problem starts by fixing a positive integer  $n$  and a positive real number  $\theta$  satisfying  $\theta + \theta^2 + \dots + \theta^n < 1$ . Consider an  $(n + 1)$ -sided dice whose sides are numbered  $0, 1, 2, \dots, n$ . The dice is biased in such a way that the probability of obtaining 0 is  $1 - \theta - \theta^2 - \dots - \theta^n$ , and the probability of obtaining  $i > 0$  is  $\theta^i$ . The random experiment is to roll the dice until side 0 comes up and  $Y_n$  be the sum of numbers which appears before obtaining 0. The random variable  $Y_n$  has probability distribution

$$P(Y_n = k) = F_{n,k} \theta^k (1 - \theta - \theta^2 - \dots - \theta^n),$$

for  $k = 0, 1, 2, \dots$ , and appropriate values of  $\theta$ . When  $n = 1$ , the distribution reduces to a

geometric distribution. When  $n = 2$  and  $\theta = \frac{1}{2}$ , the distribution  $P(Y_2 = k) = F_k \left(\frac{1}{2}\right)^{k+2}$  reduces to the Fibonacci distribution. The study continues by describing the probability generating function, modes, hazard function and the limiting form of the Fibonacci distribution. Also, discuss the family of power series distribution for the Fibonacci distribution including multiple parameters, higher order of probability space and multivariate distributions.

This paper is an exposition and extensions of the work of Shane[1], Taillie and Patil[2] by focusing on the distribution

$$P(Y_2 = k) = F_k \theta^k (1 - \theta - \theta^2)$$

with different values of the parameter  $\theta$ . The extensions include the derivation of the distribution from the power series, computing the range of the parameter, and derivation of some of its properties. The motivation of the study is to explore on the applications of Fibonacci numbers in the field of probability theory. The use of Fibonacci sequence as the coefficients of the power series is the subject of the study because of some interesting characteristics of the sequence such as recursion, relation to golden ratio, and the convergence of the power series itself. The purpose of the study is to generate a discrete probability distribution using Fibonacci power series, identify its parameter, and discover some of its possible applications. The study will further try to explain some of the application, relation to other distributions, identifying the limiting form of the distribution, and simulations by using MatLab as the main programming language.

## 1.2 Statement and Relevance of the Problem

The main objective of this study is to generate the probability distribution using Fibonacci sequence as the coefficient of the power series. Specifically, this study aims to accomplish the following:

1. Create a random variable that follows the generated probability distribution and identify its parameter.
2. Determine the cumulative distribution function of this random variable.
3. Find the moment generating function and use it to generate the mean and variance of the distribution.
4. Describe scenarios of possible application of the distribution.
5. Simulate the scenarios and compare the results to the distribution.
6. Identify the relation to other distributions by changing the values of the parameter and the random variable.

This problem is relevant because extending the ideas and concept regarding the Fibonacci numbers and dealing with probability distribution at the same time will increase viable options on the statistical models with regards to discrete random variables. This will advance the existing knowledge on the Fibonacci power series and power series distributions. This advancement of knowledge will help the students and practitioners of mathematics to learn more about this distribution. It will also serve as an opportunity for the other mathematicians to improve and to gain more knowledge and possible applications about this distribution.

### 1.3 Scope and Limitation of the Study

The study focused on generating the discrete probability distribution using the Fibonacci power series, and will be named *Fibonacci distribution* all throughout the thesis. This paper is a specification and extension of Taillie and Patil work and will deal with second order Fibonacci sequence and a background on third order and infinite order Fibonacci sequence. The extensions are the computation of the range of parameters, cumulative distribution, moment generating functions, computation of raw and central moments, specifically, the mean and variance, and exploring the applications and simulations of outcomes using MatLab.

Anything unrelated to these is considered outside of the scope of this thesis.

### 1.4 Organization of the Paper

The Introduction (Chapter I) highlights the main problem, background, literature review, scope and limitation of the study and methodology. In Fibonacci Distribution (Chapter II), the preliminaries, initial results of the derivation Fibonacci distribution and its properties are presented. Followed by Application and Simulation (Chapter III), this chapter shows some sample computations, graphs and tables. Also, it includes the replication of the distribution by using Monte Carlo simulation in MatLab. Lastly, the Discussion, Conclusion and Recommendation (Chapter IV) includes the summary and deliberation of the results and the recommendations for further and future studies.

## CHAPTER II

### Fibonacci Distribution

This chapter presents the generalization of Fibonacci distribution as a power series distribution as described by Taillie and Patil [2].

#### 2.1 Fibonacci Sequence

To begin this section, we will present the formal definition of  $n^{\text{th}}$  order Fibonacci number as follows:

**Definition 2.1.** The  $k^{\text{th}}$  Fibonacci number of order  $n$  is defined by the recurrence formula

$$F_{n,k+1} = F_{n,k} + F_{n,k-1} + \dots + F_{n,k-n+1}$$

for  $k = 0, 1, 2, \dots$ , with  $F_{n,0} = 1$  and  $F_{n,-1} = F_{n,-2} = \dots = F_{n,-n+1} = 0$ .

This is an example of a linear homogeneous recurrence relation with constant coefficients. To solve for the characteristic equation of any linear difference equation, we can assume that  $x^m$  be a solution to the recurrence formula. Thus,

$$x^m = x^{m-1} + x^{m-2} + \dots + x^{m-n+1} + x^{m-n}.$$

Since  $x \neq 0$ , we can divide the equation by  $x^{m-n}$ , resulting to

$$x^n = x^{n-1} + x^{n-2} + \dots + x^1 + 1.$$

Transposing each terms to the left-hand side of the equation, we obtained

$$x^n - x^{n-1} - x^{n-2} - \dots - x - 1 = 0$$

and that is the characteristic equation of the  $n^{\text{th}}$  order Fibonacci sequence.

With some manipulation,

$$x^n - x^{n-1} - x^{n-2} - \dots - x - 1 = x^n - \frac{x^n - 1}{x - 1} = \frac{x^{n+1} - 2x^n + 1}{x - 1}.$$

Therefore, the solutions for  $x^{n+1} - 2x^n + 1 = 0$  where  $x \neq 1$  is the same with the solutions for the characteristic equation. Moreover, for  $n > 1$ , at  $x = 1.5$  we have  $x^{n+1} - 2x^n + 1 < 0$ , and at  $x = 2$ , we have  $x^{n+1} - 2x^n + 1 = 1 > 0$  indicating a real root on the interval  $[1.5, 2]$  by extreme value theorem. Observing the change of signs in the characteristic equation, by Descartes' rule of signs, has exactly one real root. We let this root  $\varphi_n$ .

Now, let  $x_k = \frac{F_{n,k+1}}{F_{n,k}}$ , then

$$\begin{aligned} x_k &= \frac{F_{n,k} + F_{n,k-1} + \dots + F_{n,k-n+1}}{F_{n,k}} \\ &= 1 + \frac{1}{x_{k-1}} + \dots + \frac{1}{x_{k-1}x_{k-2} \cdots x_{k-n+1}}. \end{aligned}$$

Suppose  $x = \lim_{k \rightarrow \infty} x_k$ , then

$$x = \lim_{k \rightarrow \infty} 1 + \frac{1}{x_{k-1}} + \dots + \frac{1}{x_{k-1}x_{k-2} \cdots x_{k-n+1}} = 1 + \frac{1}{x} + \dots + \frac{1}{x^{n-1}}.$$

Multiplying  $x^{n-1}$  from the above equation, we get

$$x^n = x^{n-1} + x^{n-2} + \dots + x^1 + 1$$

which is equivalent to the characteristic equation of the  $n^{\text{th}}$  order Fibonacci sequence. Hence, the limit of successive ratio of the  $n^{\text{th}}$  order Fibonacci numbers is  $\varphi_n$ .

### 2.1.1 Fibonacci sequence of order 2

By default, when we say the Fibonacci number, we refer on the 2<sup>nd</sup> order Fibonacci number.

The following is its definition:

**Definition 2.2.** The *Fibonacci sequence*,  $\{F_n\}_{n=0}^{\infty}$ , starts with  $F_0 = 1, F_1 = 1$ , and then each subsequent term is the sum of the two previous ones:

$$F_n = F_{n-1} + F_{n-2}.$$

Hence, the sequence is

$$1, 1, 2, 3, 5, 8, 13, \dots$$

The characteristic equation of the Fibonacci sequence is  $x^2 - x - 1 = 0$ . The positive real root of this equation is the golden ratio  $\varphi_2 = \frac{1 + \sqrt{5}}{2}$ . Moreover,

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}.$$

### 2.1.2 Fibonacci sequence of order 3

The Fibonacci numbers of order 3 is called the Tribonacci numbers and is denoted by  $T_k := F_{3,n}$ .

**Definition 2.3.** The *Tribonacci sequence*,  $\{T_n\}_{n=0}^{\infty}$ , starts with  $T_0 = 1, T_1 = 1, T_2 = 2$ , and then each subsequent term is the sum of the three previous ones:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}.$$

Hence, the sequence is

$$1, 1, 2, 4, 7, 13, 24, 44, 81, \dots$$

The characteristic equation of the Fibonacci sequence is  $x^3 - x^2 - x - 1 = 0$ . The positive real root of this equation is  $\varphi_3 = \frac{1}{3} \left( 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right)$ . Moreover,

$$\lim_{n \rightarrow \infty} \frac{T_{n+1}}{T_n} = \frac{1}{3} \left( 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right).$$

### 2.1.3 Fibonacci sequence of infinite order

The Fibonacci numbers of infinite order is called the infinacci numbers. Denote the infinacci numbers by  $I_n := F_{\infty, n}$ .

**Definition 2.4.** The *infinacci sequence*,  $\{I_n\}_{n=0}^{\infty}$ , starts with  $I_0 = 1$ , and then each subsequent term is the sum of all the previous ones:

$$I_n = \sum_{i=0}^{n-1} I_i.$$

Hence, the sequence is

$$1, 1, 2, 4, 8, 16, 32, \dots$$

We can observe that the infinacci numbers can be defined as

$$I_n = \begin{cases} 1 & \text{for } n = 0, \\ 2^{n-1} & \text{for } n = 1, 2, 3, \dots \end{cases}$$

The limit of successive ratio is

$$\lim_{n \rightarrow \infty} \frac{I_{n+1}}{I_n} = 2.$$

## 2.2 Power Series Distributions

In this section, we will give some preliminaries on power series distribution.

For convention and notation in this paper, we denote  $\{a_n\} := \{a_n\}_{n=0}^{\infty}$ .



Suppose we have a sequence  $\{a_n\}$ , we can define another sequence  $\{s_n\}$  as

$$s_n := \sum_{i=0}^n a_i,$$

and each  $s_n$  is called *partial sum*. The limit of the sequence

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=0}^{\infty} a_i$$

is called an *infinite series*. It is simply the sum of all the terms of sequence  $\{a_n\}$  in order of the index. We say that the infinite series is *convergent* if the partial sum  $\{s_n\}$  is convergent, and the infinite series is *divergent* if the partial sum  $\{s_n\}$  is divergent.

A *power series* is any series written of the form

$$\sum_{i=0}^{\infty} c_n x^n$$

where  $c_n$ 's, known as the coefficient of the series, is an element of the sequence  $\{c_n\}$ .

### 2.2.1 Probability Mass Function

Let  $N$  be a random variable with a power series distribution. The probability mass function of the random variable  $N$  with a parameter  $\theta > 0$  is defined by

$$P(N = n) = \frac{g_n \theta^n}{\sum_{i=0}^{\infty} g_i \theta^i}$$

where  $\{g_n\}$  is the coefficient of the power series and  $g_n \geq 0$ .

You can see that the denominator of the pmf is a power series. And hence the name of the class of distribution.

Note that we want a closed form for the power series so that we can describe well the generated distribution. Such examples are: the geometric power series that generates the

geometric distribution and the Taylor expansion of  $e^x$  that generates the Poisson distribution.

There are many convergent series that exist without closed form. For example, we know that the  $p$ -series converges when  $p > 1$ , however some of the  $p$ -series does not have closed form.

Another examples are functions such as

$$f_1(x) = \sum_{i=1}^{\infty} \frac{1}{3^n - n}, \quad f_2(x) = \sum_{i=1}^{\infty} \frac{n^2 + 2}{n^4 + 5},$$

these functions converge by comparison test but do not have closed forms. With these, we cannot generate or formulate the probability distribution exactly. We can do so by numerical approximation, however it will generate an approximation error. Thus, we cannot describe the generated distribution accurately.

### 2.2.2 Cumulative distribution function

The cumulative distribution function of the random variable  $N$  is defined by

$$P(N \leq n) = \frac{\sum_{i=0}^{\lfloor n \rfloor} g_i \theta^i}{\sum_{i=0}^{\infty} g_i \theta^i}.$$

Observe that the numerator of the cdf is simply the partial sum of the sequence  $\{g_i \theta^i\}$ .

### 2.2.3 Moment generating function and Moments

For a power series distribution with random variable  $N$ , coefficient of the power series  $\{g_n\}$ , and radius of convergence  $R$ , we can generalize the moment generating function. Let  $g(x) = \sum_{i=0}^{\infty} g_i x^i$ , where  $|x| < R$ . Then,

$$P(N = n) = \frac{g_n \theta^n}{\sum_{i=0}^{\infty} g_i \theta^i} = \frac{g_n \theta^n}{g(\theta)}$$

and

$$\begin{aligned}
 M_N(t) = E(e^{tN}) &= \sum_{n=0}^{\infty} e^{tn} P(N = n) \\
 &= \sum_{n=0}^{\infty} e^{tn} \frac{g_n \theta^n}{\sum_{i=0}^{\infty} g_i \theta^i} \\
 &= \frac{1}{g(\theta)} \sum_{n=0}^{\infty} g_n (e^t \theta)^n \\
 &= \frac{g(e^t \theta)}{g(\theta)}.
 \end{aligned}$$

Therefore,

$$E(N^r) = \frac{1}{g(\theta)} \frac{d^r}{dt^r} g(e^t \theta) \Big|_{t=0}.$$

For  $r = 1$ , we have

$$E(N) = \frac{1}{g(\theta)} \frac{d}{dt} g(e^t \theta) \Big|_{t=0} = \frac{e^t \theta g'(e^t \theta)}{g(\theta)} \Big|_{t=0} = \frac{\theta g'(\theta)}{g(\theta)}.$$

This is the mean of the random variable  $N$ .

For  $r = 2$ , we have

$$\begin{aligned}
 E(N^2) &= \frac{1}{g(\theta)} \frac{d^2}{dt^2} g(e^t \theta) \Big|_{t=0} \\
 &= \frac{1}{g(\theta)} \frac{d}{dt} e^t \theta g'(e^t \theta) \Big|_{t=0} \\
 &= \frac{(e^t \theta)^2 g''(e^t \theta) + e^t \theta g'(e^t \theta)}{g(\theta)} \Big|_{t=0} \\
 &= \frac{\theta^2 g''(\theta) + \theta g'(\theta)}{g(\theta)}.
 \end{aligned}$$

We can generalize the variance of any random variable with a power series power series distribution. That is,

$$Var(N) = E(N^2) - [E(N)]^2 = \frac{\theta^2 g''(\theta) + \theta g'(\theta)}{g(\theta)} - \left[ \frac{\theta g'(\theta)}{g(\theta)} \right]^2.$$

## 2.3 Fibonacci Power Series Distribution

The focus of this study revolves on the use of Fibonacci power series and the distribution. In this section, we will combine the concepts presented previously on this chapter.

### 2.3.1 Fibonacci Power Series

**Definition 2.5.** A *Fibonacci power series* is formed using Fibonacci sequence  $\{F_n\}$  as the coefficient of the power series. That is,

$$\sum_{i=0}^{\infty} F_i x^i = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

**Theorem 2.6.** The Fibonacci power series

$$\sum_{i=0}^{\infty} F_i x^i = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

has a radius of convergence  $R = \frac{-1 + \sqrt{5}}{2}$  and converges to  $\frac{1}{1 - x - x^2}$ .

*Proof.* The radius of convergence can easily be solved using ratio test. That is,

$$L = \lim_{n \rightarrow \infty} \left| \frac{F_{n+1} x^{n+1}}{F_n x^n} \right| = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} |x| = \frac{1 + \sqrt{5}}{2} |x|.$$

Choosing  $R = \left( \frac{1 + \sqrt{5}}{2} \right)^{-1} = \frac{-1 + \sqrt{5}}{2}$ , we have

$$L = \frac{1 + \sqrt{5}}{2} |x| < \frac{1 + \sqrt{5}}{2} R = 1$$

Thus, the radius of convergence of the Fibonacci power series is  $\frac{-1 + \sqrt{5}}{2}$ .

Now, suppose  $\sum_{i=0}^{\infty} F_i x^i$  converges to  $f(x)$  for  $|x| < R$ . Then

$$\begin{aligned}
 f(x) &= \sum_{i=0}^{\infty} F_i x^i \\
 &= F_0 x^0 + F_1 x^1 + \sum_{i=2}^{\infty} F_i x^i \\
 &= 1 + x + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) x^i \\
 &= 1 + x + \sum_{i=2}^{\infty} F_{i-1} x^i + \sum_{i=2}^{\infty} F_{i-2} x^i \\
 &= 1 + x + x \sum_{i=2}^{\infty} F_{i-1} x^{i-1} + x^2 \sum_{i=2}^{\infty} F_{i-2} x^{i-2} \\
 &= 1 + x \left( 1 + \sum_{i=1}^{\infty} F_i x^i \right) + x^2 \sum_{i=0}^{\infty} F_i x^i \\
 &= 1 + x f(x) + x^2 f(x) \\
 f(x) - x f(x) - x^2 f(x) &= 1 \\
 f(x)(1 - x - x^2) &= 1 \\
 f(x) &= \sum_{i=0}^{\infty} F_i x^i = \frac{1}{1 - x - x^2}.
 \end{aligned}$$

□

### 2.3.2 Fibonacci Distribution

As stated in introduction, we will explore more on the 2<sup>nd</sup> order Fibonacci power series distribution.

#### Probability mass function

**Definition 2.7.** Let  $N$  be a random variable that follows the 2<sup>nd</sup> order Fibonacci distribution with parameter  $0 < \theta < \frac{-1 + \sqrt{5}}{2}$ . The probability of  $N = n$  is given by the probability mass function:

$$P(N = n) := F_n \theta^n (1 - \theta - \theta^2)$$

for  $n = 0, 1, 2, \dots$ . We call  $N$  a Fibonacci random variable with parameter  $\theta$ .

The distribution has 3 factors and each can be understood as follows:  $1 - \theta - \theta^2$  describes the independent probability of obtaining success,  $F_n$  is the number of composition of  $n$  by using 1 and 2 and  $\theta^n$  is the total probability of each compositions of  $n$  with weights  $\theta$  and  $\theta^2$ , respectively.

For example, let  $n = 3$ . The compositions of 3 by using 1 and 2 are

$$1 + 1 + 1, \quad 1 + 2, \quad 2 + 1,$$

so  $F_3 = 3$  and the total probability of each composition is

$$\theta \cdot \theta \cdot \theta = \theta \cdot \theta^2 = \theta^2 \cdot \theta = \theta^3$$

. Thus,  $P(N = 3) = 3\theta^3(1 - \theta - \theta^2)$ .

### Cumulative distribution function

**Theorem 2.8.** If  $N$  is a Fibonacci random variable with parameter  $\theta$ , then the cumulative distribution function of  $N$  is

$$P(N \leq k) = \begin{cases} 0 & \text{for } k < 0 \\ 1 - F_{n+1}\theta^{n+1} - F_n\theta^{n+2} & \text{for } n \leq k < n + 1 \end{cases}$$

*Proof.* The proof is by direct computation. Let  $n = 0, 1, 2, \dots$ , we have

$$\begin{aligned}
P(N \leq n) &= \sum_{i=0}^n P(N = i) \\
&= \sum_{i=0}^n F_i \theta^i (1 - \theta - \theta^2) \\
&= \sum_{i=0}^n F_i \theta^i - \sum_{i=0}^n F_i \theta^{i+1} - \sum_{i=0}^n F_i \theta^{i+2} \\
&= F_0 + F_1 \theta + F_2 \theta^2 + F_3 \theta^3 + \dots + F_{n-2} \theta^{n-2} + F_{n-1} \theta^{n-1} + F_n \theta^n \\
&\quad - F_0 \theta - F_1 \theta^2 - F_2 \theta^3 - F_3 \theta^4 - \dots - F_{n-2} \theta^{n-1} - F_{n-1} \theta^n - F_n \theta^{n+1} \\
&\quad - F_0 \theta^2 - F_1 \theta^3 - F_2 \theta^4 - F_3 \theta^5 - \dots - F_{n-2} \theta^n - F_{n-1} \theta^{n+1} - F_n \theta^{n+2} \\
&= F_0 + (F_1 - F_0) \theta + (F_2 - F_1 - F_0) \theta^2 + (F_3 - F_2 - F_1) \theta^3 + \dots \\
&\quad + (F_{n-1} - F_{n-2} - F_{n-3}) \theta^{n-1} + (F_n - F_{n-1} - F_{n-2}) \theta^n \\
&\quad - (F_n + F_{n-1}) \theta^{n+1} - F_n \theta^{n+2}.
\end{aligned}$$

Note that  $F_{n+2} = F_{n+1} + F_n$ , so  $F_{n+2} - F_{n+1} - F_n = 0$  for all  $n$ , thus

$$P(N \leq n) = F_0 + (F_1 - F_0) \theta - (F_n + F_{n-1}) \theta^{n+1} - F_n \theta^{n+2}.$$

Simplifying further by substituting  $F_0 = F_1 = 1$  and  $F_n + F_{n-1} = F_{n+1}$ , we get

$$P(N \leq n) = 1 - F_{n+1} \theta^{n+1} - F_n \theta^{n+2}.$$

Since  $N$  is discrete random variable, for any real  $k$  with  $n \leq k < n + 1$ ,  $P(N \leq k) = P(N \leq n)$  which is the desired result.  $\square$

### Moment generating function, Mean and Variance

**Theorem 2.9.** If  $N$  is a Fibonacci random variable with parameter  $\theta$ , then

1. mgf of  $N$ ,  $M_N(t) = \frac{1 - \theta - \theta^2}{1 - e^t \theta - (e^t \theta)^2}$  for  $t < \ln \left( \frac{-1 + \sqrt{5}}{2\theta} \right)$ .
2. mean of  $N$ ,  $E(N) = \frac{\theta + 2\theta^2}{1 - \theta - \theta^2}$ .

3. variance of  $N$ ,  $Var(N) = \frac{\theta + 4\theta^2 - \theta^3}{(1 - \theta - \theta^2)^2}$ .

*Proof.* The proof is by direct computation. Note that  $\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2}$  for  $|x| < \frac{-1 + \sqrt{5}}{2}$ . We have

$$\begin{aligned} M_N(t) &= E(e^{tN}) \\ &= \sum_{n=0}^{\infty} e^{tn} P(N = n) \\ &= \sum_{n=0}^{\infty} e^{tn} F_n \theta^n (1 - \theta - \theta^2) \\ &= (1 - \theta - \theta^2) \sum_{n=0}^{\infty} F_n (e^t \theta)^n. \end{aligned}$$

If  $|e^t \theta| < \frac{-1 + \sqrt{5}}{2}$ , then

$$M_N(t) = \frac{1 - \theta - \theta^2}{1 - e^t \theta - (e^t \theta)^2} = \frac{1 - \theta - \theta^2}{1 - e^t \theta - e^{2t} \theta^2}.$$

Note that  $e^t > 0$  and  $\theta > 0$ , thus

$$\begin{aligned} e^t \theta &< \frac{-1 + \sqrt{5}}{2} \\ e^t &< \frac{-1 + \sqrt{5}}{2\theta} \\ \ln(e^t) &< \ln\left(\frac{-1 + \sqrt{5}}{2\theta}\right) \\ t &< \ln\left(\frac{-1 + \sqrt{5}}{2\theta}\right). \end{aligned}$$

Since  $\theta < \frac{-1 + \sqrt{5}}{2}$ , then

$$\begin{aligned} 1 &< \left(\frac{-1 + \sqrt{5}}{2\theta}\right) \\ 0 &< \ln\left(\frac{-1 + \sqrt{5}}{2\theta}\right). \end{aligned}$$

Therefore, it is possible to choose  $t = 0$ .

Consequently,

$$M_N(t) = \frac{1 - \theta - \theta^2}{1 - e^t \theta - e^{2t} \theta^2},$$



when  $0 < \theta < \frac{-1 + \sqrt{5}}{2}$  and  $t < \ln \left( \frac{-1 + \sqrt{5}}{2\theta} \right)$ .

From here, we can compute for the mean as  $E(N) = \left. \frac{d}{dt} M_N(t) \right|_{t=0}$ .

$$\begin{aligned} \frac{d}{dt} M_N(t) &= (1 - \theta - \theta^2)(-1) \frac{-e^t \theta - 2e^{2t} \theta^2}{(1 - e^t \theta - e^{2t} \theta^2)^2} \\ &= (1 - \theta - \theta^2) \frac{e^t \theta + 2e^{2t} \theta^2}{(1 - e^t \theta - e^{2t} \theta^2)^2} \\ \left. \frac{d}{dt} M_N(t) \right|_{t=0} &= (1 - \theta - \theta^2) \frac{\theta + 2\theta^2}{(1 - \theta - \theta^2)^2} \\ E(N) &= \frac{\theta + 2\theta^2}{1 - \theta - \theta^2}. \end{aligned}$$

Now, to solve for the variance, we need the second raw moment,  $E(N^2)$ . We can solve

this using the mgf as  $E(N^2) = \left. \frac{d^2}{dt^2} M_N(t) \right|_{t=0}$ .

Starting from  $\frac{d}{dt} M_N(t) = (1 - \theta - \theta^2) \frac{e^t \theta + 2e^{2t} \theta^2}{(1 - e^t \theta - e^{2t} \theta^2)^2}$ , we have

$$\begin{aligned} \frac{d^2}{dt^2} M_N(t) &= (1 - \theta - \theta^2) \left[ \frac{(1 - e^t \theta - e^{2t} \theta^2)^2 (e^t \theta + 4e^{2t} \theta^2)}{(1 - e^t \theta - e^{2t} \theta^2)^4} \right. \\ &\quad \left. - \frac{(e^t \theta + 2e^{2t} \theta^2)(2)(1 - e^t \theta - e^{2t} \theta^2)(-e^t \theta - 2e^{2t} \theta^2)}{(1 - e^t \theta - e^{2t} \theta^2)^4} \right] \\ \left. \frac{d^2}{dt^2} M_N(t) \right|_{t=0} &= (1 - \theta - \theta^2) \left[ \frac{(1 - \theta - \theta^2)^2 (\theta + 4\theta^2)}{(1 - \theta - \theta^2)^4} \right. \\ &\quad \left. - \frac{(\theta + 2\theta^2)(2)(1 - \theta - \theta^2)(-\theta - 2\theta^2)}{(1 - \theta - \theta^2)^4} \right] \\ &= \frac{(1 - \theta - \theta^2)(\theta + 4\theta^2) + 2(\theta + 2\theta^2)^2}{(1 - \theta - \theta^2)^2} \\ &= \frac{\theta + 5\theta^2 + 3\theta^3 + 4\theta^4}{(1 - \theta - \theta^2)^2}. \end{aligned}$$

Finally,

$$\begin{aligned} Var(N) &= E(N^2) - [E(N)]^2 \\ &= \frac{\theta + 5\theta^2 + 3\theta^3 + 4\theta^4}{(1 - \theta - \theta^2)^2} - \left[ \frac{\theta + 2\theta^2}{1 - \theta - \theta^2} \right]^2 \\ &= \frac{\theta + 5\theta^2 + 3\theta^3 + 4\theta^4 - (\theta^2 + 4\theta^3 + 4\theta^4)}{(1 - \theta - \theta^2)^2} \\ &= \frac{\theta + 4\theta^2 - \theta^3}{(1 - \theta - \theta^2)^2}. \end{aligned}$$

In summary, we obtained the mgf  $M_N(t) = \frac{1 - \theta - \theta^2}{1 - e^t \theta - (e^t \theta)^2}$  for  $t < \ln \left( \frac{-1 + \sqrt{5}}{2\theta} \right)$ ,

the mean  $E(N) = \frac{\theta + 2\theta^2}{1 - \theta - \theta^2}$  and the variance  $Var(N) = \frac{\theta + 4\theta^2 - \theta^3}{(1 - \theta - \theta^2)^2}$ .  $\square$

### Distribution using different parameters

We will show some sample computations of the Fibonacci distribution, its pmf, cdf, mean, variance and plots with different values of  $\theta$ .

Table 2.1. The mean and variance of Fibonacci random variable  $N$

$\theta$	0.2	0.4	0.6
Mean, $E(N)$	$\frac{7}{19}$	$\frac{18}{11}$	33
Variance, $Var(N)$	$\frac{220}{361}$	$\frac{610}{121}$	1140

Table 2.2. The pmf and cdf of Fibonacci random variable,  $N$ , with parameter  $\theta = 0.2$

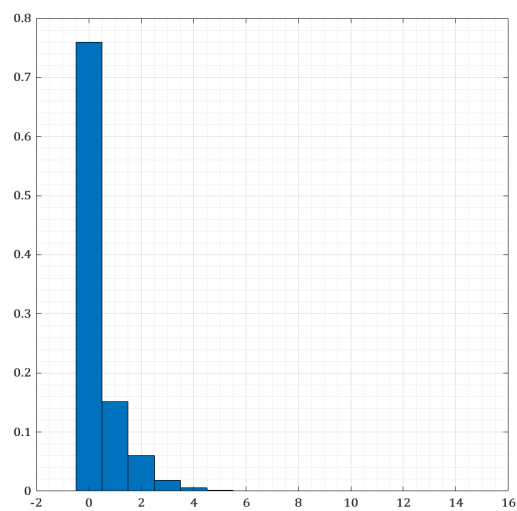
$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.76000000	0.76000000	8	0.00006615	0.99996836
1	0.15200000	0.91200000	9	0.00002140	0.99998976
2	0.06080000	0.97280000	10	0.00000693	0.99999669
3	0.01824000	0.99104000	11	0.00000224	0.99999892
4	0.00608000	0.99712000	12	0.00000073	0.99999965
5	0.00194560	0.99906560	13	0.00000023	0.99999989
6	0.00063232	0.99969792	14	0.00000008	0.99999996
7	0.00020429	0.99990221	15	0.00000002	0.99999999

Table 2.3. The pmf and cdf of Fibonacci random variable,  $N$ , with parameter  $\theta = 0.4$ 

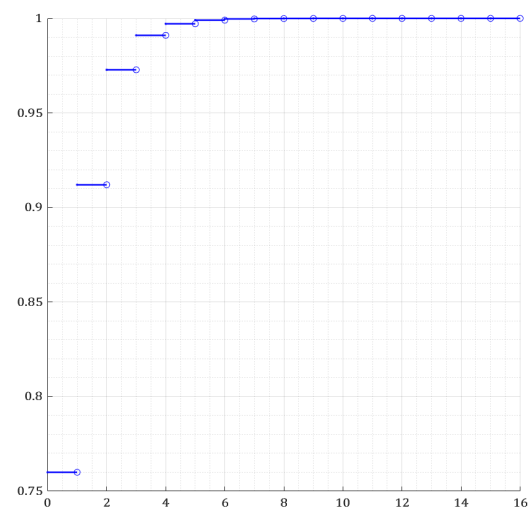
$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.44000000	0.44000000	8	0.00980419	0.98201692
1	0.17600000	0.61600000	9	0.00634388	0.98836081
2	0.14080000	0.75680000	10	0.00410622	0.99246703
3	0.08448000	0.84128000	11	0.00265751	0.99512454
4	0.05632000	0.89760000	12	0.00172000	0.99684454
5	0.03604480	0.93364480	13	0.00111320	0.99795774
6	0.02342912	0.95707392	14	0.00072048	0.99867822
7	0.01513882	0.97221273	15	0.00046630	0.99914453

Table 2.4. The pmf and cdf of Fibonacci random variable,  $N$ , with parameter  $\theta = 0.6$ 

$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.04000000	0.04000000	8	0.02284278	0.24014172
1	0.02400000	0.06400000	9	0.02217093	0.26231265
2	0.02880000	0.09280000	10	0.02152596	0.28383861
3	0.02592000	0.11872000	11	0.02089711	0.30473572
4	0.02592000	0.14454000	12	0.02028761	0.32502333
5	0.02488320	0.16952320	13	0.01969553	0.34471886
6	0.02426112	0.19378432	14	0.01912086	0.36383972
7	0.02351462	0.21729894	15	0.01856290	0.38240262

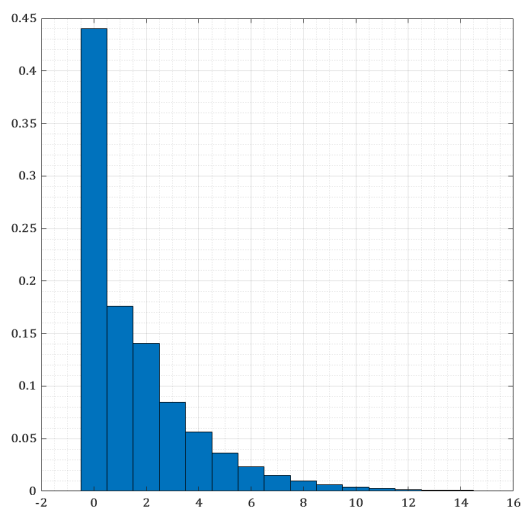


(a)

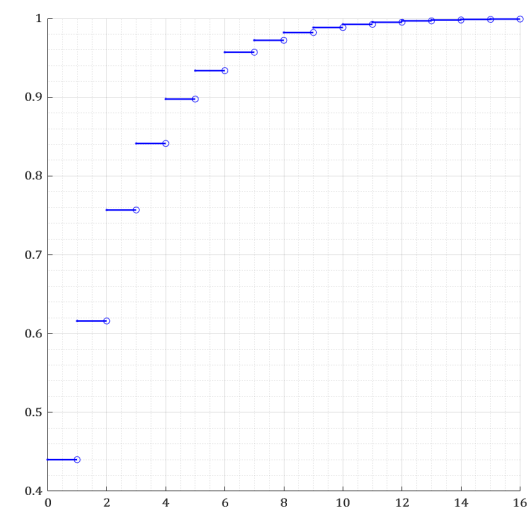


(b)

Figure 2.1. Plot of (a) pmf, and (b) cdf for Fibonacci random variable  $N$  with  $\theta = 0.2$ , for  $n = 0, 1, 2, \dots, 15$ .



(a)



(b)

Figure 2.2. Plot of (a) pmf, and (b) cdf for Fibonacci random variable  $N$  with  $\theta = 0.4$ , for  $n = 0, 1, 2, \dots, 15$ .

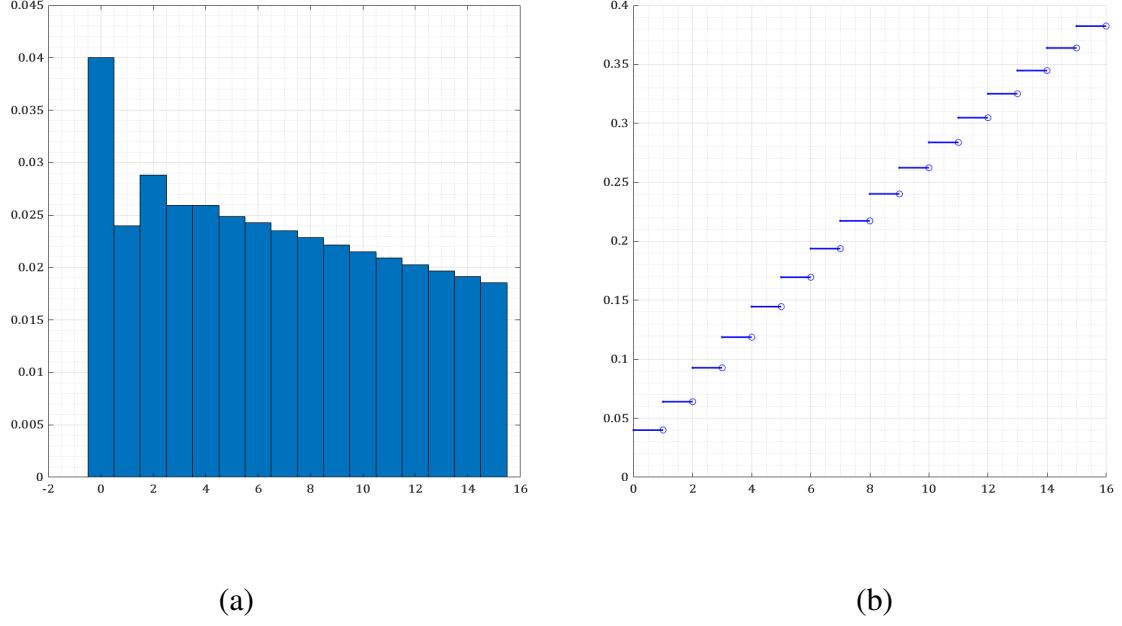


Figure 2.3. Plot of (a) pmf, and (b) cdf for Fibonacci random variable  $N$  with  $\theta = 0.6$ , for  $n = 0, 1, 2, \dots, 15$ .

By observation of the results, we can see that as the parameter  $\theta$  close to zero, the pmf approaches zero and cdf approaches one faster than with  $\theta$  close to  $\frac{-1 + \sqrt{5}}{2}$ . That is because the ratio of consecutive probability is

$$\frac{P(N = n + 1)}{P(N = n)} = \frac{F_{n+1}\theta^{n+1}(1 - \theta - \theta^2)}{F_n\theta^n(1 - \theta - \theta^2)} = \frac{F_{n+1}\theta}{F_n}.$$

Thus, for a fixed  $\theta$  close to zero, and  $1 \leq \frac{F_{n+1}}{F_n} \leq 2$ , the next probability decreases by a maximum factor of  $2\theta$ . Therefore, as  $\theta \rightarrow 0$ , the faster the  $P(N = n)$  approaches zero.

Now, for the cdf,

$$\begin{aligned} P(N \leq n) &= 1 - F_{n+1}\theta^{n+1} - F_n\theta^{n+2} \\ &= 1 - \frac{F_{n+1}\theta^{n+1}(1 - \theta - \theta^2)}{(1 - \theta - \theta^2)} - \theta^2 \frac{F_n\theta^n(1 - \theta - \theta^2)}{(1 - \theta - \theta^2)} \\ &= 1 - \frac{P(N = n + 1)}{1 - \theta - \theta^2} - \theta^2 \frac{P(N = n)}{1 - \theta - \theta^2}. \end{aligned}$$

Hence, if  $\theta$  is small and close to zero, as  $n$  increases, the  $P(N = n)$  approaches 0, and  $P(N \leq n)$  approaches 1 faster than with larger  $\theta$ .

We can support this claim with the mean and the variance as well. Notice that for  $\theta = 0.2$  the mean and variance are small and for  $\theta = 0.6$  the mean, especially the variance are large. Empirically, most of the probability is around the mean  $E(N)$  within some standard deviations away. Thus, for small  $\theta$ , with small variance, most of the probability are within some small interval about the mean or dense around the mean. While, for large  $\theta$ , with large variance, most of the probability are of bigger interval about the mean or sparse values. One can observe further by analysing Figure 2.1, Figure 2.2, and Figure 2.3.

Another observation is that for  $\theta < 0.5$ , the distribution is strictly decreasing while for  $\theta \geq 0.5$  there where finite number of oscillations from the start then gradually become a decreasing function.

Note that  $\frac{F_n}{F_{n+1}} \geq 0.5$  for all  $n = 0, 1, 2, \dots$ . Thus, if  $\theta < 0.5$ , then  $\frac{F_n}{F_{n+1}} > \theta$ . It follows that  $F_n > \theta F_{n+1}$  or  $F_n \theta^n (1 - \theta - \theta^2) > \theta^{n+1} F_{n+1} (1 - \theta - \theta^2)$  for all  $n = 0, 1, 2, \dots$ . Therefore,  $P(N = n) > P(N = n + 1)$  for all  $n = 0, 1, 2, \dots$

Meanwhile, for  $\theta = 0.5$ , we have  $2\theta = 1$ . Then  $F_2 \theta^2 = F_1 \theta$  implying that  $P(N = 2) = P(N = 1)$ . Moreover, for  $n = 2, 3, 4, \dots$ , we have  $F_n \theta^n = 2F_n \theta^{n+1} > F_{n+1} \theta^{n+1}$  implying that  $P(N = n) > P(N = n + 1)$ . Now, for  $\theta > 0.5$ , there exists an integer  $k$  such that  $\theta > \frac{F_k}{F_{k+1}} \geq 0.5$  implying that  $P(N = k) < P(N = k + 1)$ . Note that  $k = 1$  is on the list of those integers. We are interested with  $m = \max \left\{ k : \theta > \frac{F_k}{F_{k+1}} \right\}$ . With this, for  $n \leq m + 1$ , the  $P(N = n)$  oscillates because of the monotonicity of  $\left\{ \frac{F_{2n}}{F_{2n+1}} \right\}$  and  $\left\{ \frac{F_{2n+1}}{F_{2n+2}} \right\}$ . Lastly, for  $n > m + 1$ ,  $P(N = n) > P(N = n + 1)$  since  $\theta < \frac{F_{m+1}}{F_{m+2}}$ .

## 2.4 Tribonacci Power Series Distribution

In this section, we will use the  $3^{rd}$  order Fibonacci number into the Fibonacci distribution.

### 2.4.1 Tribonacci Power Series

**Definition 2.10.** A *Tribonacci power series* is formed using Tribonacci sequence  $\{T_n\}$  as the coefficient of the power series. That is,

$$\sum_{i=0}^{\infty} T_i x^i = 1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + \dots$$

**Theorem 2.11.** The Tribonacci power series

$$\sum_{i=0}^{\infty} T_i x^i = 1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + \dots$$

has a radius of convergence  $R = \frac{1}{\varphi_3}$  and converges to  $\frac{1}{1 - x - x^2 - x^3}$ .

*Proof.* Let  $\varphi_3 = \frac{1}{3} \left( 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right)$ , which is the only real solution to the characteristic equation of  $3^{rd}$  order Fibonacci sequence. The radius of convergence can easily be solved using ratio test. That is,

$$L = \lim_{n \rightarrow \infty} \left| \frac{T_{n+1} x^{n+1}}{T_n x^n} \right| = \lim_{n \rightarrow \infty} \frac{T_{n+1}}{T_n} |x| = \varphi_3 |x|.$$

Choosing  $R = \frac{1}{\varphi_3}$ , we have

$$L = \varphi_3 |x| < \varphi_3 R = 1$$

Thus, the radius of convergence of the Tribonacci power series is  $\frac{1}{\varphi_3}$ .

Now, suppose  $\sum_{i=0}^{\infty} T_i x^i$  converges to  $g(x)$  for  $|x| < R$ . Then

$$\begin{aligned}
 g(x) &= \sum_{i=0}^{\infty} T_i x^i \\
 &= T_0 x^0 + T_1 x^1 + T_2 x^2 + \sum_{i=3}^{\infty} T_i x^i \\
 &= 1 + x + 2x^2 + \sum_{i=3}^{\infty} (T_{i-1} + T_{i-2} + T_{i-3}) x^i \\
 &= 1 + x + 2x^2 + x \sum_{i=3}^{\infty} T_{i-1} x^{i-1} + x^2 \sum_{i=3}^{\infty} T_{i-2} + x^3 \sum_{i=3}^{\infty} T_{i-3} x^i \\
 &= 1 + x \left( 1 + x + \sum_{i=2}^{\infty} T_i x^i \right) + x^2 \left( 1 + \sum_{i=1}^{\infty} T_i x^i \right) + x^3 \sum_{i=3}^{\infty} T_{i-3} x^i \\
 &= 1 + xg(x) + x^2g(x) + x^3g(x) \\
 g(x) &= \frac{1}{1 - x - x^2 - x^3}.
 \end{aligned}$$

□

### 2.4.2 Tribonacci Distribution

The proofs and computation for the Tribonacci distribution is similar to the Fibonacci distribution.

#### Probability mass function

**Definition 2.12.** Let  $N$  be a random variable that follows the 3<sup>rd</sup> order Fibonacci distribution with parameter  $0 < \theta < \frac{1}{\varphi_3}$ . The probability of  $N = n$  is given by the probability mass function:

$$P(N = n) := T_n \theta^n (1 - \theta - \theta^2 - \theta^3)$$

for  $n = 0, 1, 2, \dots$ . We call  $N$  a Tribonacci random variable with parameter  $\theta$ .



### Cumulative distribution function

The cumulative distribution is given by the following theorem

**Theorem 2.13.** If  $N$  is a Tribonacci random variable with parameter  $\theta$ , then the cumulative distribution function of  $N$  is

$$P(N \leq k) = \begin{cases} 0 & \text{for } k < 0 \\ 1 - T_{n+1}\theta^{n+1} - (T_n + T_{n-1})\theta^{n+2} - T_n\theta^{n+3} & \text{for } n \leq k < n+1 \end{cases}$$

### Moment generating function, Mean and Variance

**Theorem 2.14.** If  $N$  is a Tribonacci random variable with parameter  $\theta$ , then

1. mgf of  $N$ ,  $M_N(t) = \frac{1 - \theta - \theta^2 - \theta^3}{1 - e^t\theta - (e^t\theta)^2 - (e^t\theta)^3}$  for  $t < \ln\left(\frac{1}{\varphi_3\theta}\right)$ .
2. mean of  $N$ ,  $E(N) = \frac{\theta + 2\theta^2 + 3\theta^3}{1 - \theta - \theta^2 - \theta^3}$ .
3. variance of  $N$ ,  $Var(N) = \frac{\theta + 4\theta^2 + 8\theta^3 - 4\theta^4 - \theta^5}{(1 - \theta - \theta^2 - \theta^3)^2}$ .

### Distribution using different parameters

We will show some sample computations of the Tribonacci distribution, its pmf, cdf, mean, variance and plots with different values of  $\theta$ .

Table 2.5. The mean and variance of Fibonacci random variable  $N$

$\theta$	0.18	0.36	0.54
Mean, $E(N)$	$\frac{666}{1985}$	$\frac{3076}{1879}$	$\frac{10505}{72}$
Variance, $Var(N)$	$\frac{840}{1459}$	$\frac{2274}{415}$	$\frac{194159}{9}$

Table 2.6. The pmf and cdf of Tribonacci random variable,  $N$ , with parameter  $\theta = 0.18$ 

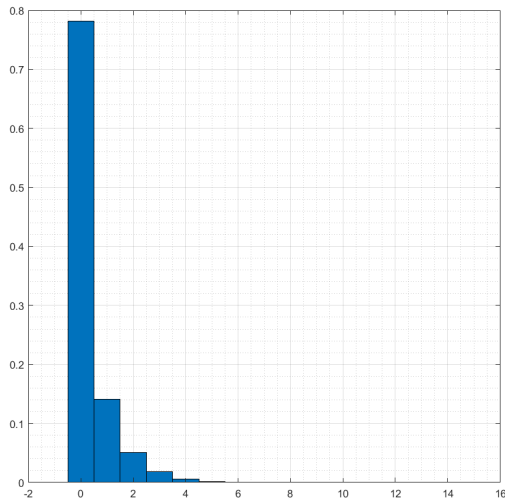
$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.78176800	0.78176800	8	0.00006978	0.99996546
1	0.14071824	0.92248624	9	0.00002310	0.99998856
2	0.05065856	0.97314480	10	0.00000764	0.99999621
3	0.01823708	0.99138189	11	0.00000253	0.99999874
4	0.00574468	0.99712657	12	0.00000083	0.99999958
5	0.00192036	0.99904693	13	0.00000027	0.99999986
6	0.00063815	0.99968508	14	0.00000009	0.99999995
7	0.00021059	0.99989567	15	0.00000003	0.99999998

Table 2.7. The pmf and cdf of Tribonacci random variable,  $N$ , with parameter  $\theta = 0.36$ 

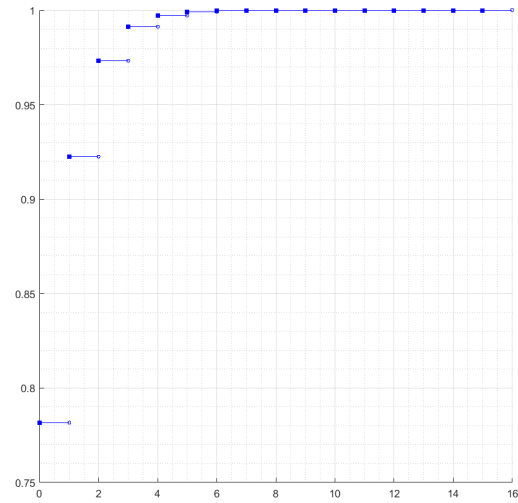
$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.46374400	0.46374400	8	0.01059700	0.97923123
1	0.16694784	0.63069184	9	0.00701757	0.98624880
2	0.12020244	0.75089428	10	0.00464572	0.99089453
3	0.08654576	0.83744004	11	0.00307635	0.99397089
4	0.05452382	0.89196387	12	0.00203698	0.99600787
5	0.03645307	0.92841694	13	0.00134876	0.99735663
6	0.02422727	0.95264422	14	0.00089307	0.99824971
7	0.01599000	0.96863422	15	0.00059134	0.99884106

Table 2.8. The pmf and cdf of Tribonacci random variable,  $N$ , with parameter  $\theta = 0.54$ 

$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.01093600	0.01093600	8	0.00640462	0.06249962
1	0.00590544	0.01684144	9	0.00636192	0.06886155
2	0.00637787	0.02321931	10	0.00631752	0.07517907
3	0.00688810	0.03010742	11	0.00627509	0.08145416
4	0.00650925	0.03661667	12	0.00623251	0.08768668
5	0.00652785	0.04314453	13	0.00619015	0.09387684
6	0.00650777	0.04965230	14	0.00614818	0.10002503
7	0.00644269	0.05609500	15	0.00610646	0.10613150

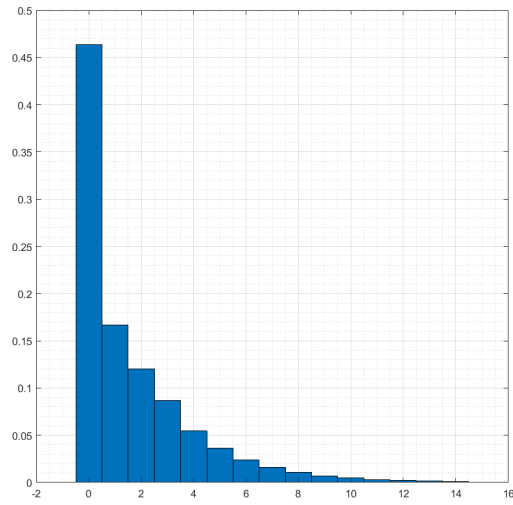


(a)

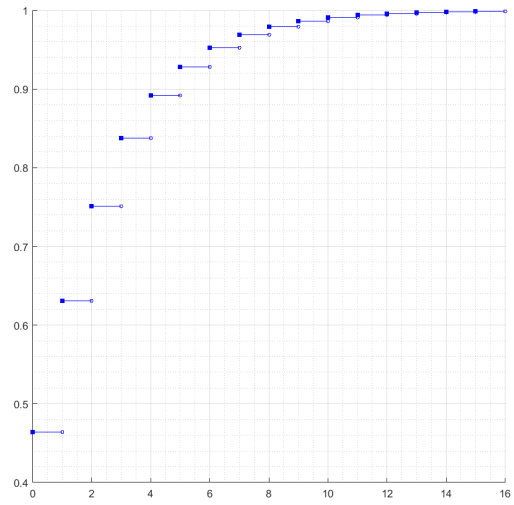


(b)

Figure 2.4. Plot of (a) pmf, and (b) cdf for Tribonacci random variable  $N$  with  $\theta = 0.18$ , for  $n = 0, 1, 2, \dots, 15$ .

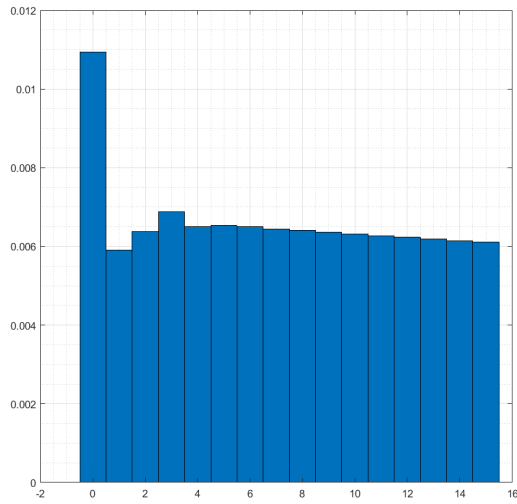


(a)

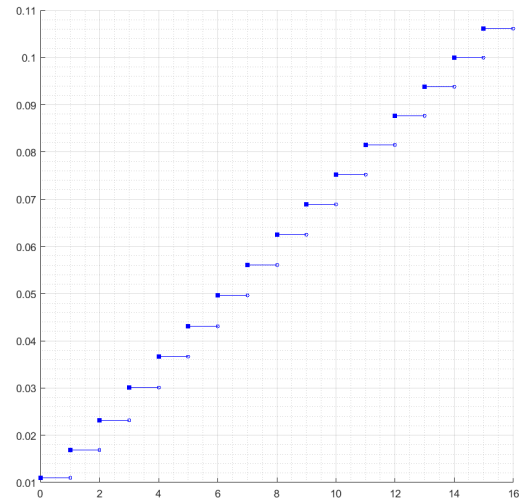


(b)

Figure 2.5. Plot of (a) pmf, and (b) cdf for Tribonacci random variable  $N$  with  $\theta = 0.36$ , for  $n = 0, 1, 2, \dots, 15$ .



(a)



(b)

Figure 2.6. Plot of (a) pmf, and (b) cdf for Tribonacci random variable  $N$  with  $\theta = 0.54$ , for  $n = 0, 1, 2, \dots, 15$ .

## 2.5 Infinacci Power Series Distribution

In this section, we will observe the infinite order Fibonacci number into the Fibonacci distribution.

### 2.5.1 Infinacci Power Series

**Definition 2.15.** A *infinacci power series* is formed using infinacci sequence  $\{I_n\}$  as the coefficient of the power series. That is,

$$\sum_{i=0}^{\infty} I_i x^i = 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \dots$$

**Theorem 2.16.** The infinacci power series

$$\sum_{i=0}^{\infty} I_i x^i = 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \dots$$

has a radius of convergence  $R = \frac{1}{2}$  and converges to  $\frac{1-x}{1-2x}$ .

*Proof.* The radius of convergence can easily be solved using ratio test. That is,

$$L = \lim_{n \rightarrow \infty} \left| \frac{I_{n+1} x^{n+1}}{I_n x^n} \right| = \lim_{n \rightarrow \infty} \frac{I_{n+1}}{I_n} |x| = 2|x|.$$

Choosing  $R = \frac{1}{2}$ , we have

$$L = 2|x| < 2R = 1$$

Thus, the radius of convergence of the infinacci power series is  $\frac{1}{2}$ .

Now, suppose  $\sum_{i=0}^{\infty} I_i x^i$  converges to  $h(x)$  for  $|x| < R$ . Then

$$\begin{aligned} h(x) &= 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + \dots \\ &= 1 + \frac{x}{1-2x} \\ h(x) &= \frac{1-x}{1-2x}. \end{aligned}$$

□

### 2.5.2 Infinacci Distribution

The proofs and computation for the infinacci distribution is similar to the Fibonacci distribution.

#### Probability mass function

**Definition 2.17.** Let  $N$  be a random variable that follows the infinite order Fibonacci distribution with parameter  $0 < \theta < \frac{1}{2}$ . The probability of  $N = n$  is given by the probability mass function:

$$P(N = n) := \begin{cases} \frac{1 - 2\theta}{1 - \theta} & \text{for } n = 0, \\ (2\theta)^n \frac{1 - 2\theta}{2(1 - \theta)} & \text{for } n = 1, 2, 3, \dots \end{cases}$$

We call  $N$  an infinacci random variable with parameter  $\theta$ .

#### Cumulative distribution function

The cumulative distribution is given by the following theorem

**Theorem 2.18.** If  $N$  is an infinacci random variable with parameter  $\theta$ , then the cumulative distribution function of  $N$  is

$$P(N \leq k) = \begin{cases} 0 & \text{for } k < 0 \\ \frac{1 - 2\theta}{1 - \theta} & \text{for } 0 \leq k < 1 \\ 1 - \frac{\theta}{1 - \theta}(2\theta)^n & \text{for } n \leq k < n + 1 \end{cases}$$

#### Moment generating function, Mean and Variance

**Theorem 2.19.** If  $N$  is an infinacci random variable with parameter  $\theta$ , then

1. mgf of  $N$ ,  $M_N(t) = \frac{(1 - e^t\theta)(1 - 2\theta)}{(1 - 2e^t\theta)(1 - \theta)}$  for  $t < \ln\left(\frac{1}{2\theta}\right)$ .
2. mean of  $N$ ,  $E(N) = \frac{\theta}{(1 - \theta)(1 - 2\theta)}$ .

3. variance of  $N$ ,  $Var(N) = \frac{\theta - 2\theta^3}{(1 - \theta)^2(1 - 2\theta)^2}$ .

### Distribution using different parameters

We will show some sample computations of the infinacci distribution, its pmf, cdf, mean, variance and plots with different values of  $\theta$ .

Table 2.9. The mean and variance of infinacci random variable  $N$

$\theta$	0.12	0.32	0.48
Mean, $E(N)$	$\frac{100}{357}$	$\frac{200}{153}$	$\frac{300}{13}$
Variance, $Var(N)$	$\frac{516}{1109}$	$\frac{845}{199}$	$\frac{23929}{40}$

Table 2.10. The pmf and cdf of infinacci random variable,  $N$ , with parameter  $\theta = 0.12$

$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.80952380	0.80952380	8	0.00004450	0.99997905
1	0.12952380	0.93904761	9	0.00001424	0.99999329
2	0.04144761	0.98049523	10	0.00000455	0.99999785
3	0.01326323	0.99375847	11	0.00000145	0.99999931
4	0.00424423	0.99800271	12	0.00000046	0.99999978
5	0.00135815	0.99936086	13	0.00000014	0.99999992
6	0.00043460	0.99979547	14	0.00000004	0.99999997
7	0.00013907	0.99993455	15	0.00000001	0.99999999

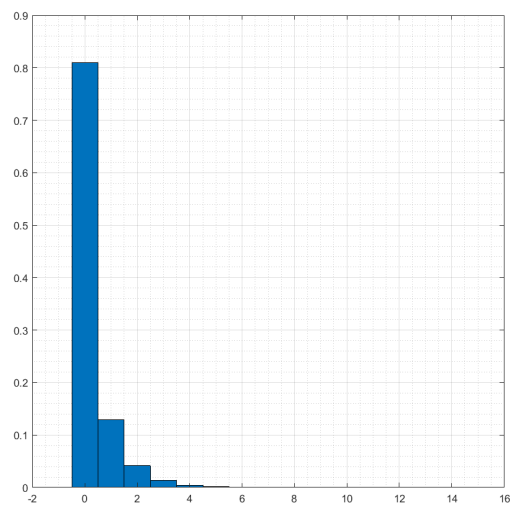
Table 2.11. The pmf and cdf of infinacci random variable,  $N$ , with parameter  $\theta = 0.32$ 

$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.52941176	0.52941176	8	0.00745080	0.98675411
1	0.16941176	0.69882352	9	0.00476851	0.99152263
2	0.10842352	0.80724705	10	0.00305185	0.99457448
3	0.06939105	0.87663811	11	0.00195318	0.99652767
4	0.04441027	0.92104839	12	0.00125003	0.99777770
5	0.02842257	0.94947097	13	0.00080002	0.99857773
6	0.01819044	0.96766142	14	0.00051201	0.99908974
7	0.01164188	0.97930331	15	0.00032769	0.99941743

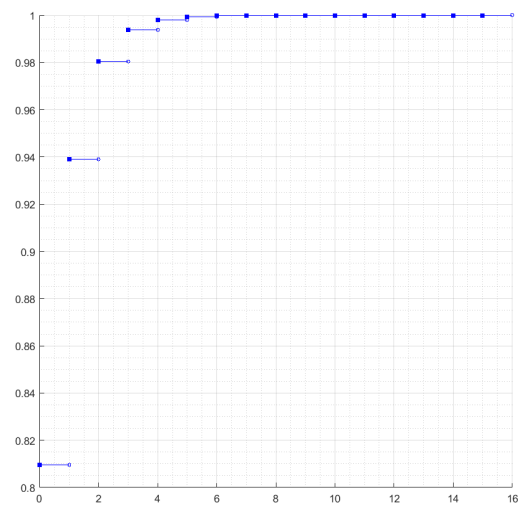
Table 2.12. The pmf and cdf of infinacci random variable,  $N$ , with parameter  $\theta = 0.48$ 

$n$	$P(N = n)$	$P(N \leq n)$	$n$	$P(N = n)$	$P(N \leq n)$
0	0.07692307	0.07692307	8	0.02774575	0.33410192
1	0.03692307	0.11384615	9	0.02663592	0.36073785
2	0.03544615	0.14929230	10	0.02557048	0.38630833
3	0.03402830	0.18332061	11	0.02454766	0.41085600
4	0.03266717	0.21598779	12	0.02356575	0.43442176
5	0.03136048	0.24734827	13	0.02262312	0.45704489
6	0.03010606	0.27745434	14	0.02171820	0.47876309
7	0.02890182	0.30635617	15	0.02084947	0.49961257



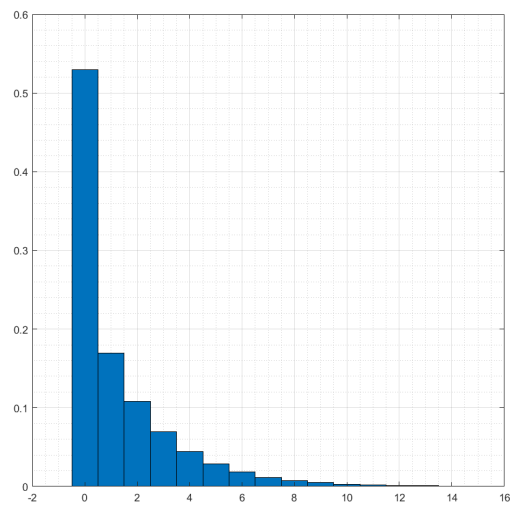


(a)

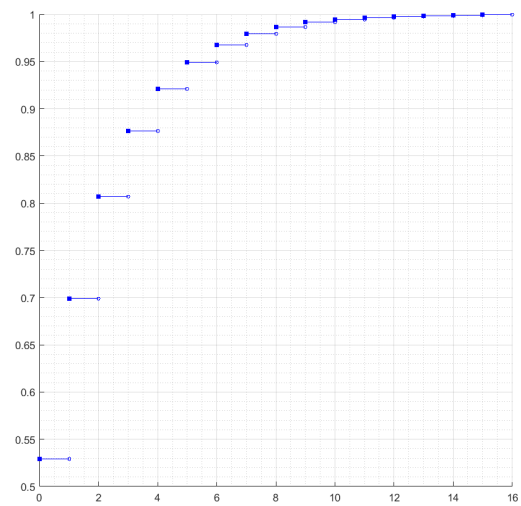


(b)

Figure 2.7. Plot of (a) pmf, and (b) cdf for infinacci random variable  $N$  with  $\theta = 0.12$ , for  $n = 0, 1, 2, \dots, 15$ .

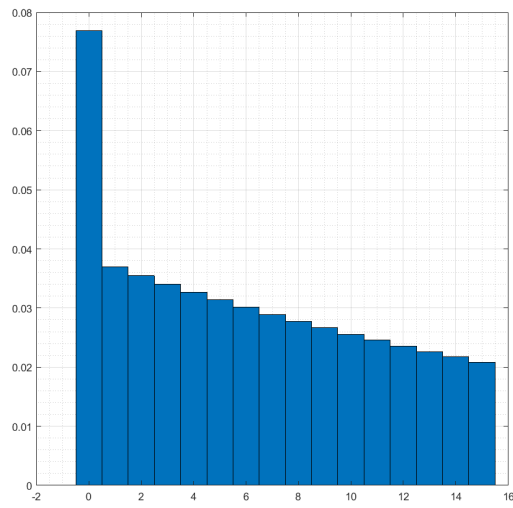


(a)

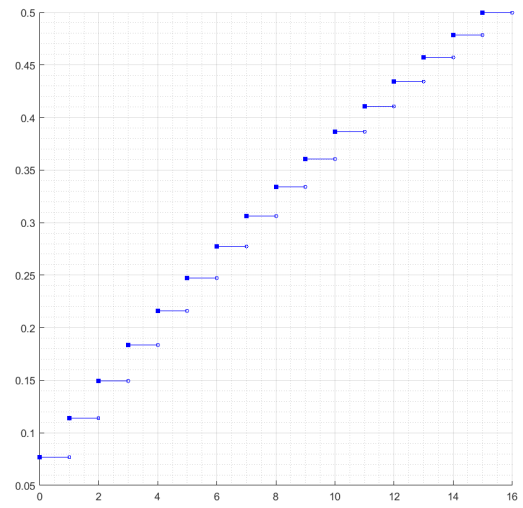


(b)

Figure 2.8. Plot of (a) pmf, and (b) cdf for infinacci random variable  $N$  with  $\theta = 0.36$ , for  $n = 0, 1, 2, \dots, 15$ .



(a)



(b)

Figure 2.9. Plot of (a) pmf, and (b) cdf for infinacci random variable  $N$  with  $\theta = 0.54$ , for  $n = 0, 1, 2, \dots, 15$ .

## CHAPTER III

### APPLICATION AND SIMULATION

In this chapter, we will compare the Fibonacci distribution to other distribution. Moreover, we will show some simulation for its application.

#### 3.1 Fibonacci Distribution and Geometric Distribution

Consider the Fibonacci sequence  $\{F_n\}$ . Based on limit of successive ratio of Fibonacci number, we have

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}.$$

By Hadamard's formula, we can conclude

$$\lim_{n \rightarrow \infty} |F_n|^{\frac{1}{n}} = \frac{1 + \sqrt{5}}{2}.$$

Now,

$$\lim_{n \rightarrow \infty} |F_n|^{\frac{1}{n}} = 1 + \frac{-1 + \sqrt{5}}{2} = \lim_{\theta \rightarrow \frac{-1 + \sqrt{5}}{2}} (1 + \theta).$$

That is, when  $n$  is sufficiently large and  $\theta$  is about its maximum value,  $F_n^{\frac{1}{n}}$  and  $1 + \theta$  approaches the same value  $\frac{1 + \sqrt{5}}{2}$ .

Consequently, when  $N$  is a Fibonacci random variable with  $n$  is sufficiently large and  $\theta$

is about its maximum value,

$$\begin{aligned}
 P(N = n) &= F_n \theta^n (1 - \theta - \theta^2) \\
 &= \left(F_n^{\frac{1}{n}}\right)^n \theta^n (1 - \theta - \theta^2) \\
 &\approx (1 + \theta)^n \theta^n (1 - \theta - \theta^2) \\
 &= [(1 + \theta)\theta]^n (1 - \theta - \theta^2) \\
 &= (\theta + \theta^2)^n (1 - \theta - \theta^2).
 \end{aligned}$$

The symbol ‘ $\approx$ ’ means approximately equal to. Let  $p = 1 - \theta - \theta^2$  and  $q = 1 - p$ . Also,

$$\begin{aligned}
 0 < \theta &< \frac{-1 + \sqrt{5}}{2} \\
 0 < \theta^2 &< \left[\frac{-1 + \sqrt{5}}{2}\right]^2 = \frac{3 - \sqrt{5}}{2} \\
 0 < \theta + \theta^2 &< 1 \\
 0 < q &< 1 \\
 0 < p &< 1.
 \end{aligned}$$

Implying that  $P(N = n) \approx pq^n$ , which is a geometric distribution. Hence, the Fibonacci probability distribution is nearly a geometric distribution whenever  $n$  is large and its parameter  $\theta$  is close to its maximum possible value.

Now, for any order of Fibonacci distribution, if you observe the distribution for large values of  $n$ , we can see that the pmf looks like a geometric distribution. We can support this claim by the successive ratio of the Fibonacci probability distribution. That is,

$$\frac{P(N = n + 1)}{P(N = n)} = \frac{F_{k,n+1} \theta^{n+1} (1 - \theta - \theta^2 - \dots - \theta^k)}{F_{k,n} \theta^n (1 - \theta - \theta^2 - \dots - \theta^k)} = \frac{F_{k,n+1}}{F_k} \theta,$$

where  $k$  is the order of Fibonacci number. For large enough  $n$ , we can see that  $\frac{F_{k,n+1}}{F_k}$  is closer and closer to a constant value  $\varphi_k$ . Therefore, the successive ratio of the Fibonacci probability distribution is almost a constant at the tail of the distribution which is a property of geometric

distribution. For infinite order Fibonacci distribution, it is clear that for  $n > 0$ , the distribution looks like a geometric distribution. And hence, explaining the claim.

### 3.1.1 Comparison of Fibonacci distribution and geometric distribution

The underlying condition for the Fibonacci distribution to have geometric distribution as a limiting form is to have  $\theta \rightarrow \frac{-1 + \sqrt{5}}{2}$  and sufficiently large  $n$ . See the following table with  $\theta = 0.612$  and  $p = 1 - \theta - \theta^2 = 0.013456$

Table 3.1. The comparison of the pmf of Fibonacci random variable  $N$  with  $\theta = 0.612$  and Geometric random variable  $N'$  with  $p = 0.013456$

$n$	$P(N = n)$	$P(N' = n)$	Percentage Error(%)
80	0.00444164	0.00455232	2.4312
81	0.00439828	0.00449106	2.0660
82	0.00435534	0.00443063	1.6994
83	0.00431281	0.00437101	1.3315
84	0.00427071	0.00431220	0.9621
85	0.00422901	0.00425417	0.5914
86	0.00418772	0.00419693	0.2193
87	0.00414684	0.00414045	0.1542
88	0.00410635	0.00408474	0.5291
89	0.00406626	0.00402978	0.9054
90	0.00402656	0.00397555	1.2831

The sufficiency of  $n$  lies on the approximation that  $(F_n)^{\frac{1}{n}} \approx 1 + \theta$ . In this simulation, since we have  $\theta = 0.612$ , the value of  $n$  such that  $(F_n)^{\frac{1}{n}} - 1$  is closest to  $\theta$  when  $n = 87$ . Therefore, around  $n = 87$  the probability of Fibonacci random variable  $N$  is close to the

probability of Geometric random variable  $N'$ .

The following table will give a guide on some values of  $\theta$  and  $n$  and the given values are not limited on whats on the table.

Table 3.2. The values of  $n$  to have accurate approximation of the Fibonacci distribution with specific  $\theta$  to a Geometric distribution

$n$	29	35	43	58	87	172
$\theta$	0.600	0.603	0.606	0.609	0.612	0.615

### 3.2 Convergence of Fibonacci random variable

Let  $\{N_n\}$  be a sequence of Fibonacci random variables such that  $N_n$  has a parameter  $\theta_n$  and

$\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$  and  $N$  be a Fibonacci random variable with parameter  $\theta$ . Then,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} P(N_n = k) &= \lim_{n \rightarrow \infty} F_k \theta_n^k (1 - \theta_n - \theta_n^2) \\
 &= F_k \lim_{n \rightarrow \infty} \theta_n^k \lim_{n \rightarrow \infty} (1 - \theta_n - \theta_n^2) \\
 &= F_k \theta^k (1 - \theta - \theta^2) \\
 &= P(N = k).
 \end{aligned}$$

Note that this is valid since  $k$  is finite and  $\lim_{n \rightarrow \infty} \theta_n$  assumes to exist.

If  $\theta_n \rightarrow \frac{-1 + \sqrt{5}}{2}$ , then

$$\lim_{n \rightarrow \infty} P(N_n = k) = P(N = k) = F_k \left( \frac{-1 + \sqrt{5}}{2} \right)^k \left( 1 - \frac{-1 + \sqrt{5}}{2} - \left( \frac{-1 + \sqrt{5}}{2} \right)^2 \right) = 0.$$

Therefore, when  $\theta_n \rightarrow \frac{-1 + \sqrt{5}}{2}$ , the Fibonacci random variable  $N_n \xrightarrow{d} 0$ , and so  $N_n \xrightarrow{p} 0$ .

Consider  $\theta_n \rightarrow 0$  and fixed  $r \geq 1$ . Let  $g(x) = \frac{1}{1 - x - x^2}$ , then

$$\begin{aligned}
 \lim_{n \rightarrow \infty} E(N_n^r) &= \lim_{n \rightarrow \infty} \frac{1}{g(\theta_n)} \frac{d^r}{dt^r} g(e^t \theta_n) \Big|_{t=0} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{g(\theta_n)} \lim_{n \rightarrow \infty} \frac{d^r}{dt^r} g(e^t \theta_n) \Big|_{t=0}.
 \end{aligned}$$

Since  $r$  is finite, the order of limit and derivative can be changed. Also,  $\lim_{n \rightarrow \infty} g(\theta_n) = g(0) = 1$ .

Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} E(N_n^r) &= \frac{1}{g(0)} \frac{d^r}{dt^r} \lim_{n \rightarrow \infty} g(e^t \theta_n) \Big|_{t=0} \\ &= \frac{d^r}{dt^r} g(0) \Big|_{t=0} \\ &= \frac{d^r}{dt^r} (1) \Big|_{t=0} \\ &= 0 \end{aligned}$$

Hence, when  $\theta_n \rightarrow 0$ , the sequence of Fibonacci random variable  $\{N_n\}$  converges in  $r^{\text{th}}$  mean to the zero random variable, and therefore converges in probability and in distribution to the zero random variable.

### 3.3 Application of the distribution

The original application of this distribution came from the paper of Shane [1]. The random experiment came from a fair coin tossing. Suppose you will toss a coin repeatedly and stops when you obtained an outcome of consecutive heads, then the sample space looks like

$$\{HH, THH, HTHH, TTHH, HTTHH, THTHH, TTTHH, \dots\}.$$

Since we have a fair coin, the probability of having a head on a single toss is  $\frac{1}{2}$  and the probability of having a tail on a single toss is also  $\frac{1}{2}$ . Thus,  $P(\{HH\}) = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$ ,  $P(\{THH\}) = \frac{1}{2} \frac{1}{2} \frac{1}{2} = \frac{1}{8}$ , and so on.

This random experiment can represent a random variable  $N$  that describes the number of trials before reaching two consecutive heads. That is,  $n = 0$  implies that there are no coin toss before getting two consecutive heads,  $n = 1$  implies that there is 1 coin toss before getting two consecutive heads,  $n = 2$  implies that there are 2 coin tosses before getting two consecutive heads, and so on.

This random variable is a Fibonacci random variable with parameter  $\theta = \frac{1}{2}$  or  $1 - \theta - \theta^2 = \frac{1}{4}$ . Since the probability is mapped as

$$P(N = 0) = P(\{HH\}) = \frac{1}{4} = F_0 \left(\frac{1}{2}\right)^0 \frac{1}{4},$$

$$P(N = 1) = P(\{THH\}) = \frac{1}{8} = F_1 \left(\frac{1}{2}\right)^1 \frac{1}{4},$$

$$\begin{aligned} P(N = 2) &= P(\{HTHH, TTHH\}) = P(\{HTHH\}) + P(\{TTHH\}) \\ &= \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = F_2 \left(\frac{1}{2}\right)^2 \frac{1}{4}, \end{aligned}$$

and so  $P(N = n) = F_n \left(\frac{1}{2}\right)^n \frac{1}{4} = \frac{F_n}{2^{n+2}}$ .

We can create a simulation of similar experiment using MatLab. Suppose we want to observe the distribution for  $n = 0, 1, 2, \dots, k$  and  $m$  experiments. The algorithm is as follows

Step 1. Randomly generate  $m$  binary strings of length  $k + 2$  and embed it into a matrix  $A$

Step 2. Create a frequency table  $n$  vs  $f(n)$

Step 3. For  $i = 1 : k + 1$

1. Find all rows with 1 on both the  $i^{\text{th}}$  and  $(i + 1)^{\text{st}}$  columns
2. Count the number of rows, this is the frequency of  $i-1$ , stored in  $f(n = i - 1)$
3. Delete all these rows

Step 4. For  $i = k + 2$ , we have  $f(n = k + 1)$  which pertains to the frequency of a binary strings with no consecutive 1's which is the remaining number of rows of  $A$ .

Step 5. The probability,  $F(N = n) = f(n)/m$ .



Note that MatLab index starts at 1, that is why we start  $i$  with 1. Moreover, random number generator gives different values depending on the seed number or the device. This is the result for  $k = 15$ ,  $m = 1000$  and  $m = 10000$  with seed number 0,  $-rng(0)-$ .

Table 3.3. Comparison of Fibonacci distribution with  $\theta = 0.5$  and simulation of binary string with success of two consecutive 1's in 1000 AND 10000 experiments

	Fibonacci distribution	Simulation: Frequency Distribution	
$n$	$P(N = n)$	1000 experiments	10000 experiments
0	0.25000000	0.249	0.2505
1	0.12500000	0.138	0.1267
2	0.12500000	0.133	0.1273
3	0.09375000	0.091	0.0940
4	0.07812500	0.075	0.0768
5	0.06250000	0.058	0.0625
6	0.05078125	0.046	0.0500
7	0.04101563	0.032	0.0415
8	0.03320313	0.037	0.0328
9	0.02685547	0.026	0.0262
10	0.02172852	0.020	0.0211
11	0.01757813	0.018	0.0166
12	0.01422119	0.018	0.0128
13	0.01150512	0.014	0.0109
14	0.00930786	0.006	0.0091
15	0.00753021	0.004	0.0095
$n \geq 16$	0.03189850	0.035	0.0317

On Table 3.3, we can see clearly, as we increase the number of experiments in a single simulation, the closer the simulated values to the theoretical values. These simulations supports the existence of Fibonacci random variable with parameter  $\theta = \frac{1}{2}$ .

## CHAPTER IV

### DISCUSSION, CONCLUSION AND RECOMMENDATION

The recursive characteristics of a Fibonacci sequence and the convergence of the Fibonacci power series provide a helpful tool in generating the probability mass function of Fibonacci random variable and deriving the moment generating function, the mean and the variance of this distribution. The recursion property shows an important role in developing its application on some values of  $\theta$ .

The function  $P(N = n) = F_{k,n}\theta^n(1 - \theta - \theta^2 - \dots - \theta^k)$ , for  $n = 0, 1, 2, \dots$  is a power series distribution derived using the  $k^{\text{th}}$  order Fibonacci sequence  $\{F_{k,n}\}$  provided that  $0 < \theta < \frac{1}{\varphi_k}$  and has an additive property. Moreover, the moment generating function of a random variable,  $N$ , having this distribution is  $M_N(t) = \frac{1 - \theta - \theta^2 - \dots - \theta^k}{1 - e^t\theta - e^{2t}\theta^2 - \dots - e^{kt}\theta^k}$  and will exist only when  $t < \ln\left(\frac{1}{\varphi_k\theta}\right)$ . This moment generating function is useful in finding the raw moments of a random variable. The closed form pmf, cdf, mgf of the infinite order Fibonacci distribution are also present in the paper and provides a clear comparison to the Fibonacci distribution.

The tail of the Fibonacci distribution follows a nearly geometric distribution. This indicates that the geometric distribution can also be approximated by a Fibonacci distribution when  $n$  is large. While limited, the application with  $\theta = \frac{1}{2}$  supports the theory and the simulation provided supports the existence of the distribution. With these, the Fibonacci distribution might be able to model actual real life probability and statistical problems.

To improve this study, it is recommended to study further the behavior of the discrete probability distribution especially its raw and central moments extending to the distribution's skewness and kurtosis. It is also recommended to extend the cumulative distribution to study the survival function and hazard rate function and deeper on survival and risk analysis. It is also suggested to study further its possible applications in statistical modelling and data fitting. Another interesting recommendation is to generate a power series distribution using Lucas numbers or other sequences with converging power series.

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