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– **MASTER'S THESIS** –

Spinning Correlators at Finite Temperature

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Abstract

This master thesis is framed in the striking correspondence between gravity theories in Anti-de Sitter spacetime (AdS) and Conformal Field Theories (CFT). This is usually known as *AdS/CFT duality* and relates gravity theories in the *bulk* with CFTs that live in their *conformal boundary*. We start by presenting the notion of CFTs and some of the results and techniques that are widely used in this field. This includes conformal correlators for scalar and spin ℓ operators, the state-operator correspondence and the operator product expansion (OPE) of operators. The embedding formalism and the index-free notation to encode tensors in polynomials are also discussed and used throughout this work. The basic notions of AdS are outlined and CFT at finite temperature is then introduced. We include a review of thermal blocks and thermal coefficients for a thermal two-point function between scalar fields in mean field theory, as in [1]. We then analyse the thermal two-point function for conserved currents, which was not known in the literature. Finally, we start a study of its thermal blocks and thermal coefficients for the mean field theory application.

Populärvetenskaplig sammanfattning

Huvudmålet för den teoretiska fysiken under detta århundrade är att hitta en teori som kan förklara allt i universum. Detta kan låta lovande, men utmaningen är enorm! Det handlar om att förena vad som till synes är två helt olika teorier: en teori som förklarar atomvärlden och Einsteins berömda allmänna relativitetsteori som förklarar hur gravitationen fungerar. Forskare har upptäckt ett matematiskt samband mellan specifika teorier om gravitation (i en rumdtid som kallas Anti-de-Sitter) och teorier om icke-gravitation (som kallas konforma fältteorier). Denna dualitet är ett fantastiskt verktyg som ger oss idéer om vilken riktning som är den rätta att utforska i vårt mål för att bättre förstå vad som skulle kunna vara den ultimata teorin om allting. I denna uppsats använder vi denna dualitet och fokuserar på studien av konforma fältteorier med ändlig temperatur. Även om ämnet och de resultat som vi finner är mycket tekniska så är de små steg som gradvis kommer att bidra till ökad förståelse av den teori som så småningom kommer att beskriva hela universum som vi lever i.

Contents

1| Introduction

Conformal Field Theories (CFT) are, in a nutshell, a subset of quantum field theories (QFT) such that there is no preferred length scale. This means, for instance, that we do not expect any massive excitations in these theories. Unlike other areas of physics, such as particle physics, where we are interested in calculating S-matrices and cross sections, here we are going to develop techniques to study correlation functions and the behaviour of the different operators of our theory under the so-called *conformal transformations*. One may think that this is rather abstract and that it would be difficult to apply these techniques to real physical systems. However, notice that, as pointed in [2], quantum field theories become scale-invariant when we explore the limit of low energy regime (i.e. infrared (IR) regime). As a simple example, let us just consider a theory with a particle with mass M. In this case, of course, we do not have a CFT since we have a *mass gap* in the theory. However, if we explore energies E much below than the one corresponding to M, we will actually not *see* the particle and the physics in energies $E < M$ will have no scale at all. Nevertheless, as we will see, scale-invariance is just an example of the larger group of conformal transformations. The interesting point is that, often, scale-invariance also implies invariance under the whole group of *conformal transformations*.[1] The more symmetries we have the better can be our understanding of the theory.

Another motivation to develop the different techniques of CFTs is the following. Suppose we have a CFT in the high energy regime (i.e. ultraviolet (UV)). Once the theory *flows* to the IR regime, we will generally find a CFT. Thus, we can study these CFTs in both the UV and IR regimes to understand better our QFT, i.e. we can study our QFT as the *renormalization group flow* between two different CFTs.

Let us give some examples of nontrivial CFTs. Consider, for instance, a theory from statistical mechanics describing ferromagnetism behaviour, the so-called Ising model. It consists of a cubic lattice in $\mathbb{R}^d$ where we have classical spins taking values $\{s_i = \pm 1\}$ that only interact with the nearest-neighbour. The partition function is given via

$$Z_{\text{Ising}} = \sum_{s_i} \exp\Big(- J \sum_{\langle ij \rangle} s_i s_j \Big) , \tag{1.1}$$

where i,j are indices referring to the points in the lattice. Here the summation is only over pairs of nearest neighbouring points $\langle ij \rangle$. The absolute value of the spin-spin interaction J tells us how strongly neighbouring spins are coupled to each other, while the sign of J determines whether neighbouring spins prefer to align or anti-align (i.e. having *ferromagnetism* or *anti-ferromagnetism*). It can be seen that for a special value of J or, equivalently, for a critical temperature $T = T_c$, the theory become scale (and conformal) invariant and a nontrivial

[1] Under which conditions or requirements this happens is a subject that is currently being studied. Only for $2d$ and $4d$ we know that Lorentz-invariance and unitarity are sufficient conditions [3, 4]. The theory of elasticity in $2d$ is a typical counterexample of a field theory displaying scale but not conformal invariance [5]. Also Maxwell theory in $d \neq 4$ [6].

CFT appears at long distances. Note that to reach the critical point we have to fine-tune the temperature. A similar story holds for the IR CFT in the so-called $g\phi^4$ theory, which in 3d has the Euclidean action

$$S = \int d^3x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \right) . \tag{1.2}$$

Worth noticing is that CFTs are also an extremely interesting tool for explicitly computing critical exponents and correlation functions of statistical mechanics systems at a second order phase transition, or of condensed matter systems at quantum critical points. We find a CFT at the critical point of water and other liquids or in uni-axial magnets at their critical temperatures. Actually, if we study the critical exponents of all the previously examples we mentioned we find exactly the same values, which, in the end, means we have the same CFT for all of them! Thus, even if the realization of all these examples in the UV is clearly very different, all of them flow to the same IR CFT. This universal property in which different UV theories can have the same IR CFT is known as *critical universality*. This is important since we can just study one single realization of this CFT but, at the same time, giving us information for all the other theories which in the UV are very different.

In the present work, one of the main goals is to explore conformal field theories at *finite temperature*. Note that even if the results obtained from QFTs at zero temperature are in good concordance with experimental tests, e.g. QFTs such as QED and QCD allows us to describe and predict quite successfully the experimental data obtained at particle colliders, our world is certainly of non zero temperature, and we could ask what new physics may be found in the presence of a thermal background. The framework of thermal field theories was already introduced in the fifties by Matsubara [7], but allowing the temperature T and chemical potential μ to take non-zero values make calculations a lot more challenging than their zero temperature counterparts. Thus, examining their induced effects on particle physics is still an on-ongoing area of research.

Actually, we may find an original motivation for finite temperature QFTs in cosmological problems. It is thought that the early universe consisted of a quark-gluon plasma (QGP), a strongly interacting deconfined matter [8]. QGP has drawn a lot of attention and is currently being studied at particle accelerators such as the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). In those, they collide heavy nuclei at relativistic energies to produce this hot and dense matter and collect large amount of data. This requires theoretical tools such as thermal field theory, which allows us to study phase transitions such as the confinement-deconfinement phase transition in QCD. It is possible to study how the equation of state of QCD depends on temperature or how fast a hot plasma expands. Moreover, ultra-relativistic collisions between heavy ions are used as a mechanism to study the rate of production of dileptons pairs and photons from the plasma, which can be thought as the analogue of the cosmic microwave background radiation in the primordial universe. Besides, the possibility of the existence of a quark matter core in neutron stars [9] can also be studied using finite temperature field theory. It is thought that cold hadronic matter in the core of neutron stars compressed to sufficiently high densities could eventually undergo a deconfinement transition. Finally, questions such as the asymmetry between matter and antimatter in the observable universe may find an explanation in the violation of the baryon number due to weak interactions [10]. The rate of baryon violation is indeed a challenging mathematical problem, but it could also be addressed using thermal field theory. Therefore, this is a very broad topic that is actively used in many different research areas. Our aim in this work is to introduce this topic from the very basics, which we hope will serve as a bridge between graduate knowledge and current active research.

Let us outline the content of this master thesis. In chapter 2 we start by giving a very pedagogical introduction to CFTs. We derive some general results in great detail for those who are not familiar with the language and the techniques of CFT. In particular, we derive the conformal algebra and study how conformal symmetry constrains two- and three-point functions. In this chapter we consider Lorentzian CFTs in $\mathbb{R}^{1,d-1}$ since we want to treat the topic in full generality. In the remaining chapters we will restrict to Euclidean CFTs in $\mathbb{R}^d$. In chapter 3 we present a very appropriate and useful formalism to deal with spin operators in CFT: the *embedding formalism*. Moreover, in the same chapter, we explain how to get rid of Lorentz indices through encoding tensors in polynomials in a convenient way. This will make expression index-free and look more compact. All the techniques presented here will be used in the following. We end up the first part of the master thesis with more advanced CFT topics in chapter 4, including radial quantization, the state-operator correspondence, a brief discussion on the constraints in a unitary CFT and the introduction of the concept of operator product expansion.

In the second part, we move to Anti-de Sitter spacetime in chapter 5, which is the other pillar of the AdS/CFT correspondence. In this chapter we give the expression for the different scalar propagators. Next, in chapter 6, the notions of CFT at finite temperature are introduced. In particular, the thermal blocks and thermal coefficients of the two-point function for scalar operators are derived in mean field theory [11]. The concepts introduced in these chapters are then used in chapter 7, where we study how a quartic interaction in the gravity bulk leads corrections to the thermal CFT two-point function in the boundary [1].

In the third part of this master thesis, we want to try to reproduce the calculation done in chapter 6 but for spin one operators. For that purpose, we give all the necessary tools for dealing with spinning propagators in AdS in chapter 8, before proceeding to compute the thermal conformal blocks for conserved currents and try to identify the thermal coefficients in mean field theory in chapter 9. We finish with a brief outline of the work in chapter 10.

2 Basics of Conformal Field Theory

2.1 The Conformal Transformations

It is time now to introduce in detail what we understand by a *conformal transformation*. For that, let us consider the metric tensor $g_{\mu\nu}$ in a d dimensional space-time.

Conformal Transformation

A **conformal transformation** is an invertible map from x^μ to $x'^\mu(x)$ that leaves the metric tensor invariant up to a x-dependent scale factor $\Omega(x)$, known as the *conformal factor*:

$$g_{\mu\nu}(x) \;\rightarrow\; g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\, g_{\lambda\rho}(x) = \Omega^2(x)\, g_{\mu\nu}(x)\,. \tag{2.1}$$

In other words, a conformal transformation is a transformation that leaves the metric invariant up to local rescalings. Note that the interpretation of (2.1) depends whether we are considering a fixed or a dynamical background metric $g_{\mu\nu}(x)$ [1], i.e. if our theory has a dynamical graviton or not. For a dynamical metric, the transformation is a diffeomorphism, which is nothing but a change of coordinates $x^\mu \rightarrow x'^\mu(x)$ that induces a change in the metric. As we know, changing coordinates should not affect the physics, and so we regard diffeomorphism invariance as a special case of a gauge symmetry. Recall that a gauge transformation acts trivially on the observables so that all the states related by a gauge transformation are *the same*. On the other hand, CFT do not have a dynamical graviton (i.e. we consider a fixed background metric) so that the transformation should be thought as an honest and *physical* transformation that relates states that may be *equivalent* but not the same. Then we have what is known as a *global symmetry* and, through Noether's theorem, it corresponds to conserved quantities.

It can be easily seen that the set of conformal transformations form a group, which is known as the *conformal group*. Note that, for the special case $\Omega(x) = 1$, we have the subgroup of isometries and, in particular, for $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat metric, we find the Poincaré group (i.e. translations and Lorentz rotations) as a subgroup of the conformal group. If $\Omega(x) =$ constant then we have *scale transformations*, also known as *dilations*. Note that the conformal group preserves angles, i.e. it does not change the angle between two intersecting curves, so that we could say that conformal transformations preserve the *shape* of our system.

Let us try to deepen our understanding of the conformal group by going to an infinitesimal transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$, where $\epsilon = \epsilon^\mu(x)\partial_\mu$ is understood to be an arbitrary infinitesimal vector field. Here we closely follow the analysis in [12]. The corresponding change

[1] In the different metrics that are used throughout this work, we use the *mostly plus* convention.

in the metric tensor $g_{\mu\nu}$ is (to first order in ϵ):

$$g'_{\mu\nu}(x') = (\delta^{\lambda}_{\mu} - \partial_\mu \epsilon^\rho(x))(\delta^{\rho}_{\nu} - \partial_\nu \epsilon^\rho(x))\, g_{\lambda\rho}(x) = g_{\mu\nu}(x) - [\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)]\,, \tag{2.2}$$

which corresponds to a conformal transformation as long as

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = c(x)\, g_{\mu\nu}(x)\,, \tag{2.3}$$

where $c(x)$ is a scalar function. For $c(x) = 0$ this equation is known as the Killing equation, whereas for $c(x) \neq 0$ is called the *conformal* Killing equation. Note that the factor $c(x)$ is determined by contracting both sides with $\delta^{\mu\nu}$, i.e. taking traces on both sides we have:

$$c(x) = \frac{2}{d}\, \partial \cdot \epsilon(x)\,. \tag{2.4}$$

Now, applying an extra derivative ∂_λ on (2.3) and taking a linear combination of the equation with permuted indices as

$$\begin{aligned} &-\Big\{\partial_\rho \partial_\mu \epsilon_\nu(x) + \partial_\rho \partial_\nu \epsilon_\mu(x) = \partial_\rho c(x)\, g_{\mu\nu}\Big\} \\ &+\Big\{\partial_\mu \partial_\nu \epsilon_\rho(x) + \partial_\mu \partial_\rho \epsilon_\nu(x) = \partial_\mu c(x)\, g_{\nu\rho}\Big\} \\ &+\Big\{\partial_\nu \partial_\rho \epsilon_\mu(x) + \partial_\nu \partial_\mu \epsilon_\rho(x) = \partial_\nu c(x)\, g_{\mu\rho}\Big\}\,, \end{aligned} \tag{2.5}$$

we find that

$$2\, \partial_{\mu\nu} \epsilon_\rho(x) = g_{\mu\rho} \partial_\nu c(x) + g_{\nu\rho} \partial_\mu c(x) - g_{\mu\nu} \partial_\rho c(x)\,. \tag{2.6}$$

Contracting with $\eta^{\mu\nu}$ we get

$$2\partial^2 \epsilon_\mu(x) = (2-d)\partial_\mu c(x)\,. \tag{2.7}$$

Applying ∂_ν on the previous expression and, taking into account (2.3) we have that

$$\partial^2 \partial_\mu \epsilon_\nu + \partial^2 \partial_\nu \epsilon_\mu = (2-d)\partial_\mu \partial_\nu c(x) \quad \rightarrow \quad (2-d)\partial_\mu \partial_\nu c(x) = \partial^2 c(x)\, g_{\mu\nu}\,. \tag{2.8}$$

Finally, taking traces on both sides, we end up with

$$(d-1)\, \partial^2 c(x) = 0\,. \tag{2.9}$$

Let us examine the consequences of the previous results. The first thing we can notice is the fact that the relations (2.8) and (2.9) depend on the number of spacetime dimensions d. Thus, depending on the number of dimensions, we will have different solutions. In particular, note that for $d = 1$ there are no constraints on the scalar function $c(x)$, which essentially means that any smooth and invertible map is conformal. Note that this makes perfectly sense with our previous statement that conformal maps preserve angles, since in $1d$ the notion of angle does not exist. The next case is $d = 2$. This is a rather special case in which we find an infinite dimensional group known as the *Virasoro symmetry*. Conformal invariance in $d = 2$ takes a new meaning and would deserve a whole separate chapter. However, for now, we will not discuss this case in detail. Finally, for $d \geq 3$, relations (2.8) and (2.9) imply that $\partial_\mu \partial_\nu \, c(x) = 0$ which means that $c(x)$ has to be, at most, linear in its coordinates, i.e.

$$c(x) = A + B_\mu x^\mu \,, \quad \text{with } A,\, B \text{ constants}\,, \tag{2.10}$$

and, therefore, ϵ_μ is at most quadratic, i.e.

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \,, \quad \text{with } c_{\mu\nu\rho} = c_{\mu\rho\nu} \,. \tag{2.11}$$

Relation (2.11) constitutes the most general solution for the vector field ϵ that generates infinitesimal conformal transformations. Let us study what kind of transformations we find in each case. First of all, the term a_μ is free of constraints. It is easy to see, then, that the vector field $\epsilon^\mu = a^\mu$ generates infinitesimal *translations* $x'^\mu = x^\mu + a^\mu$. Moreover, substituting the linear term of (2.11) in (2.3) we find that

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} \left(\partial \cdot \epsilon\right) g_{\mu\nu} = \frac{2}{d}\, b^\lambda{}_\lambda \, g_{\mu\nu} \,, \tag{2.12}$$

which implies that $b_{\mu\nu}$ can be written as

$$b_{\mu\nu} = \lambda g_{\mu\nu} + \omega_{\mu\nu} \,, \quad \text{with} \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \,. \tag{2.13}$$

Therefore, we have found two more solutions. The first one, with $x'^\mu = x^\mu + \lambda x^\mu$ represents *dilatations*, while the second one with $x'^\mu = x^\mu + \frac{1}{2}(\omega^\mu{}_\nu - \omega_\nu{}^\mu) x^\nu$ represents infinitesimal *rotations* in $\mathbb{R}^d$ or *Lorentz transformations* in $\mathbb{R}^{d-1,1}$. Finally, for the quadratic term of (2.11) we have $c(x) = \frac{4}{d}\, c^\lambda{}_{\lambda\mu} x^\mu$ and, substituting it in (2.6) gives us:

$$c_{\mu\nu\rho} = g_{\mu\rho} b_\nu + g_{\mu\nu} b_\rho - g_{\nu\rho} b_\mu \,, \quad \text{where} \quad b_\mu \equiv \frac{1}{d} c^\lambda{}_{\lambda\mu} \,, \tag{2.14}$$

which corresponds to the infinitesimal conformal transformation with parameter b^μ:

$$x'^\mu = x^\mu + 2(x \cdot b) x^\mu - b^\mu x^2 \,. \tag{2.15}$$

This infinitesimal transformation is known as *special conformal transformation* (SCT).

Finally, if we exponentiate the previous infinitesimal transformation we obtain the set of finite conformal transformations. We may as well count the number of independent parameters of each conformal transformation. For translations, characterized by the vector a^μ, and the SCT, characterized by the parameter b^μ, we have in both cases d parameters. For dilations, it is clear we only need 1 parameter, whereas for Lorentz transformations, which are expressed in terms of the antisymmetric matrix $\Lambda^\mu{}_\nu$, we require $\frac{d(d-1)}{2}$ parameters. A simple algebra exercise gives us then that we need $\frac{(d+1)(d+2)}{2}$ parameters to define the set of finite conformal transformations. Therefore, from a simple counting, we see that the conformal group in $\mathbb{R}^{d-1,1}$ is $\frac{(d+1)(d+2)}{2}$-dimensional. This, together with the explicit expression for the finite conformal transformations, are summarized in Table 2.1.

Conformal transformation	Finite form	nº of parameters
Translations	$x'^{\mu} = x^{\mu} + a^{\mu}$	d
Lorentz transformations	$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ with $\Lambda^{\mu}{}_{\nu} \in SO(1, d-1)$	$\frac{d(d-1)}{2}$
Dilatations	$x'^{\mu} = \alpha x^{\mu}$	1
Special conformal transformations	$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b\cdot x + b^2 x^2}$	d

Table 2.1: Conformal Transformations and its parameters.

It is easy to verify that these are the appropriate finite transformation of the infinitesimal transformations we found previously. In particular, for the SCT, using the geometric series expansion, we immediately recover

$$x'^{\mu} = (1 + 2b\cdot x - b^2 x^2 + \mathcal{O}(b^2))(x^{\mu} - b^{\mu} x^2) = x^{\mu} + 2(x\cdot b)x^{\mu} - b^{\mu} x^2 + \mathcal{O}(b^2)\,. \tag{2.16}$$

Moreover, note that a special conformal transformation is obtained by an inversion, followed by a translation with parameter $-b^{\mu}$ and, finally, another inversion as:

$$x^{\mu} \xrightarrow[\text{inversion}]{} \frac{x^{\mu}}{x^2} \xrightarrow[\text{translation}]{} \frac{x^{\mu}}{x^2} - b^{\mu} \xrightarrow[\text{inversion}]{} \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b\cdot x + b^2 x^2}\,. \tag{2.17}$$

This, together with the fact that conformal transformations form a group, implies that an *inversion* $x^{\mu} \to \frac{x^{\mu}}{x^2}$ is also a conformal transformation. In particular, we have that the Jacobian for inversions is given via

$$\frac{\partial x'^{\mu}}{\partial x^{\lambda}} = \frac{1}{x^2}(\delta^{\mu}{}_{\lambda} - \frac{2x^{\mu} x_{\lambda}}{x^2}) \equiv \frac{1}{x^2} I^{\mu}{}_{\lambda}(x)\,, \tag{2.18}$$

where $I^{\mu}{}_{\lambda}(x)$ is an orthogonal matrix associated to the inversion. We can indeed see that $I^{\mu}{}_{\lambda}(x)$ is an orthogonal matrix by considering, for simplicity, an Euclidean signature and a particular frame where x has only x_1 component. Then, $I^{\mu}{}_{\lambda}(x)$ is just given as

$$I^{\mu}{}_{\lambda}(x) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\,, \tag{2.19}$$

which is an $O(d)$ matrix. Note, however, it is not an $SO(d)$ matrix, meaning that inversion is not continuously connected to the identity, i.e. it is not a connected component of the conformal group.[2] Therefore, we cannot expect inversion to be obtained by exponentiating an element from the conformal Lie algebra. Then, following the definition (2.1) for Euclidean signature, we

[2] As pointed out in [13] this implies that it is possible then to have CFTs that are not invariant under inversion. In particular, CFTs that break parity also break inversion.

have

$$\delta_{\lambda\rho}(x) = \Omega^2(x)\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\rho}\delta_{\mu\nu} = \Omega^2(x)\frac{1}{(x^2)^2}I^\mu{}_\lambda I^\nu{}_\rho\delta_{\mu\nu} = \frac{\Omega^2(x)}{(x^2)^2}(I^T)_{\lambda\nu}I^\nu{}_\rho\,, \tag{2.20}$$

which, since $(I^T)_{\lambda\nu}I^\nu{}_\rho = \delta_{\lambda\rho}(x)$ means that inversions have $\Omega^2(x) = (x^2)^2$ as a conformal factor. Finally, let us give explicitly the conformal factors $\Omega^2(x)$ for each of the conformal transformations. To that end, recall the definition (2.1) and the results in Table 2.1. For *translations*, it is clear that $g_{\mu\nu} = \delta^\lambda{}_\mu\delta^\rho{}_\nu g_{\lambda\rho} = g_{\mu\nu}$ so that $\Omega^2(x) = 1$. For *Lorentz transformations* we have that $x^\lambda = (\Lambda^\lambda{}_\omega)^{-1}\,x'^\omega = \Lambda_\omega{}^\lambda\,x'^\omega$ so that

$$g_{\mu\nu} = \Lambda_\omega{}^\lambda\;\delta_\mu{}^\omega\;\Lambda_\ell{}^\rho\;\delta_\nu{}^\ell\;g_{\lambda\rho} = \Lambda_\mu{}^\lambda\;\Lambda_\nu{}^\rho\;g_{\lambda\rho} = g_{\mu\nu}\,, \tag{2.21}$$

where we have used the definition property of the Lorentz matrices. This again implies that we have $\Omega^2(x) = 1$. For *dilations*, $x^\mu = \frac{1}{\alpha}x'^\mu$ so it is straightforward to see that $\Omega^2(x) = \frac{1}{\alpha^2}$. Finally, for SCTs, the calculation is more cumbersome, but otherwise straightforward. The results are summarized in Table 2.2.

Conformal transformation	Conformal factor $\Omega^2(x)$
Translations	1
Lorentz transformations	1
Dilatations	$\frac{1}{\alpha^2}$
Special conformal transformations	$(1-2b\cdot x+b^2x^2)^2$

Table 2.2: Conformal factors of the conformal group.

2.2 The Conformal Algebra

We have now set all the ingredients to derive the relations that constitute the *conformal algebra*. Since we have the explicit form of the infinitesimal conformal transformations, we can obtain the form of the generators of the conformal algebra acting on functions via [13]

$$\phi(x^\mu + \epsilon^\mu(x)) = \left[1 + ia^\ell P_\ell + \frac{i}{2}\omega^{\ell\kappa}M_{\ell\kappa} + i\lambda D + ib^\ell K_\ell\right]\phi(x^\mu)\,. \tag{2.22}$$

For instance, for translations, we have that

$$\phi(x^\mu + a^\mu) = \phi(x^\mu) + a^\ell\partial_\ell\phi(x^\mu) = [1 + a^\ell\partial_\ell]\phi(x^\mu)\,, \tag{2.23}$$

which implies that $P_\mu = -i\partial_\mu$. Similarly, for Lorentz transformations we find

$$\phi(x^\mu + \omega^{\mu\nu}x_\nu) = \phi(x^\mu) + \omega^{\ell\kappa}x_\kappa\partial_\ell\phi(x^\mu) = \phi(x^\mu) + \frac{1}{2}\omega^{\ell\kappa}[x_\kappa\partial_\ell - x_\ell\partial_\kappa]\phi(x^\mu)\,, \tag{2.24}$$

so that $M_{\mu\nu} = -i(x_\nu\partial_\mu - x_\mu\partial_\nu)$. For dilatations,

$$\phi(x^\mu + \lambda x^\mu) = \phi(x^\mu) + \lambda x^\ell\partial_\ell\phi(x^\mu)\,, \tag{2.25}$$

meaning $D = -ix^\mu\partial_\mu$. Finally, for special conformal transformations, we get

$$\begin{aligned}\phi(x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2) &= \phi(x^\mu) + [2(x\cdot b)x^\kappa\partial_\kappa - x^2 b^\ell\partial_\ell]\phi(x^\mu) = \\ &= \phi(x^\mu) + b^\ell[2(x\cdot\partial)x_\ell - x^2\partial_\ell]\phi(x^\mu)\,,\end{aligned} \tag{2.26}$$

so that $K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$. Thus we have that the generators of the conformal group are

$$\text{(translation)}\quad P_\mu = -i\partial_\mu\,,\qquad \text{(dilatation)}\quad D = -ix^\mu\partial_\mu\,, \tag{2.27}$$

$$\text{(Lorentz)}\quad M_{\mu\nu} = -i(x_\nu\partial_\mu - x_\mu\partial_\nu)\,,\qquad \text{(SCT)}\quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\,. \tag{2.28}$$

These generators obey a set of commutation rules which defines the *conformal algebra*.

Conformal Algebra

The generators of the conformal group satisfy the following non-vanishing commutation relations [12]:

$$[D, P_\mu] = iP_\mu\,, \tag{2.29}$$

$$[D, K_\mu] = -iK_\mu\,, \tag{2.30}$$

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})\,, \tag{2.31}$$

$$[K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)\,, \tag{2.32}$$

$$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\,, \tag{2.33}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})\,. \tag{2.34}$$

All other commutators vanish.

The last commutation relations (2.32)-(2.34) just show that $M_{\mu\nu}$ generates the Lorentz algebra $SO(d,1)$. The ones that are more interesting are (2.29) and (2.30). These say that P_μ and K_μ can be regarded as raising and lowering operators for D, respectively. We will explore this idea in the following sections. Moreover, we can put the previous commutations relations in a simpler form by defining the generators

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu}\,, & J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu)\,, \\ J_{-1,0} &= D\,, & J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu)\,, \end{aligned} \tag{2.35}$$

where $J_{AB} = -J_{BA}$ and $A, B \in \{-1, 0, 1, \cdots, d\}$. It is not difficult to show that these generators satisfy the commutation relations of $SO(d,2)$ or $SO(d+1,1)$, i.e.

$$[J_{AB}, J_{CD}] = i(\eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC})\,, \tag{2.36}$$

where the diagonal metric η_{AB} is $\mathrm{diag}(-1,1,1,\ldots,1)$ if we are in $\mathbb{R}^d$ or $\mathrm{diag}(-1,-1,1,\ldots,1)$ in $\mathbb{R}^{d+1,1}$. The fact that the conformal algebra can be written in such a compact and elegant way is not just a mathematical curiosity but we will develop a nice theoretical framework from this fact. Indeed, the conformal algebra $SO(d+1,1)$ suggests that things may be written in a simpler form if we think in terms of $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$. This is the key idea behind the *embedding space formalism*, which is a powerful tool that we will develop in chapter 3.

2.3 Primaries and Descendants

Now that we know the explicit form of the commutation relations of the conformal algebra, we want to explore the action of the conformal algebra on the operators to later explore the symmetries of our theory. In a general QFT, we have mainly two ways in which we can see how symmetries are realized on operators. The first way is to consider operators acting on a Hilbert space in which states evolve with time. This is the so-called *Schrödinger picture*. Here, we are going to follow a more natural approach, in which Lorentz invariance is manifest, such that operators incorporate a dependency on time, but the state vectors are time-independent. This is known as *Heisenberg picture*, in which a general operator $\mathcal{O}_\alpha(x)$, with Lorentz index α, evolves as:

$$\mathcal{O}^\alpha(x) = e^{-iP\cdot x}\,\mathcal{O}^\alpha(0)\,e^{iP\cdot x}\,. \tag{2.37}$$

Acting with a derivative we find:

$$\partial_\mu \mathcal{O}^\alpha(x) = e^{-iP\cdot x}\Big(-iP_\mu\,\mathcal{O}^\alpha(0) + \mathcal{O}^\alpha(0)\,iP_\mu\Big)e^{iP\cdot x} = -i[P_\mu, \mathcal{O}^\alpha(x)]\,. \tag{2.38}$$

Recall that the generator of the translation in the conformal group was $P_\mu \sim \partial_\mu$. Thus, from the previous result we see that

$$[P_\mu, \mathcal{O}^\alpha(x)] = i\partial_\mu \mathcal{O}^\alpha(x)\,, \tag{2.39}$$

i.e. the action of the momentum generator on a generic field $\mathcal{O}^\alpha(x)$ at the spacetime point x is done through commutators. We would also like to understand what happens with the rest of the operators of the conformal algebra. For that, we need to declare the action of the operators $D, M_{\mu\nu}$ and K_μ on a generic field $\mathcal{O}^\alpha$ at the origin. In a scale-invariant theory, it is natural to diagonalize the dilatation operator D as

$$[D, \mathcal{O}^\alpha(0)] = i\Delta\mathcal{O}^\alpha(0)\,, \tag{2.40}$$

where the eigenvalue Δ is the *scaling dimension* of the operator $\mathcal{O}^\alpha$. Moreover, in a Lorentz (or rotationally) invariant QFT, we expect operators at the origin to transform in irreducible representations of $SO(d-1,1)$ (or $SO(d)$) as

$$[M_{\mu\nu}, \mathcal{O}^\alpha(0)] = i(\mathcal{S}_{\mu\nu})_\beta{}^\alpha\,\mathcal{O}^\beta(0)\,, \tag{2.41}$$

where $\mathcal{S}_{\mu\nu}$ are matrices satisfying the same algebra as $M_{\mu\nu}$ and index contractions are written such that $M_{\mu\nu}$ and $\mathcal{S}_{\mu\nu}$ have the same commutation relations. Before stating the analogue relation for the operator K_μ, let us consider the action of the dilatation operator D and K_μ on a generic field $\mathcal{O}^\alpha$ at the origin:

$$\begin{aligned}[D,[K_\mu, \mathcal{O}^\alpha(0)]] &= [[D,K_\mu], \mathcal{O}^\alpha(0)] + [K_\mu,[D,\mathcal{O}^\alpha(0)]] = \\ &= -i[K_\mu, \mathcal{O}^\alpha(0)] + i\Delta[K_\mu, \mathcal{O}^\alpha(0)] = i(\Delta-1)[K_\mu, \mathcal{O}^\alpha(0)]\,.\end{aligned} \tag{2.42}$$

where we have used the relation from the conformal algebra (2.30) and the defining action of the dilatation operator D on a generic operator $\mathcal{O}^\alpha$ (2.40). This is usually written in literature with the shorthand notation $[Q, \mathcal{O}^\alpha(0)] \to Q\,\mathcal{O}^\alpha(0)$ as

$$DK_\mu\,\mathcal{O}^\alpha(0) = i(\Delta - 1)\,K_\mu \mathcal{O}^\alpha(0)\,. \tag{2.43}$$

From now, we will also omit the index α on the operator $\mathcal{O}^\alpha(0)$ unless it is strictly required. We see, then, that by acting with the operator K_μ we decrease by 1 the scaling dimension of an operator $\mathcal{O}(0)$. If we repeatedly act with K_μ, i.e. $K_{\mu_1}\cdots K_{\mu_n}\mathcal{O}(0)$, we will obtain operators with arbitrarily low dimension. It can be proven that unitarity implies that the scaling dimension Δ is positive. Since in physical (i.e. unitary) theories, then, dimensions are bounded from below, we must get, in the end, an operator such that

$$[K_\mu, \mathcal{O}(0)] = 0\,. \tag{2.44}$$

Such an operator is called a *primary operator*.

Primary operator

A *primary operator* $\mathcal{O}(0)$ is defined through the following properties:

$$\begin{aligned} &[M_{\mu\nu}, \mathcal{O}^\alpha(0)] = i(\mathcal{S}_{\mu\nu})_\beta{}^\alpha\,\mathcal{O}^\beta(0)\,,\\ &[D, \mathcal{O}^\alpha(0)] = i\Delta\mathcal{O}^\alpha(0)\,,\\ &[K_\mu, \mathcal{O}(0)] = 0\,. \end{aligned} \tag{2.45}$$

If, instead of K_μ, we use P_μ, we find that

$$\begin{aligned} DP_\mu\,\mathcal{O}(0) &= [D, P_\mu]\mathcal{O}(0) + P_\mu D\mathcal{O}(0) =\\ &= iP_\mu\mathcal{O}(0) + i\Delta P_\mu\mathcal{O}(0) = i(\Delta+1)P_\mu\mathcal{O}(0)\,, \end{aligned} \tag{2.46}$$

where, in this case, we have used relation (2.29). Then, we see that, by acting with momentum generators $\mathcal{O}(0) \to P_{\mu_1}\cdots P_{\mu_n}\mathcal{O}(0)$, we can construct operators of higher dimension $\Delta \to \Delta + n$, which are known as *descendants*. Moreover, any local operator in a unitary CFT can be written as a linear combination of *only* primaries and descendants (see section 7.4 in [2]). Therefore, to summarize, we see that local operators are divided into *primaries* and *descendants*. The latter ones can be written as linear combinations of derivatives of other local operators, while the *primary operators* can not. Finally, it also interesting to find the explicit action of the operators of the conformal group away from the origin. Consider a generic conformal operator Q. We know that the action of the operator Q on $\mathcal{O}(x)$ is given by

$$\begin{aligned} [Q, \mathcal{O}(x)] &= Q\,\mathcal{O}(x) - \mathcal{O}(x)\,Q = Q\,e^{-iP\cdot x}\,\mathcal{O}(0)\,e^{iP\cdot x} - e^{-iP\cdot x}\,\mathcal{O}(0)\,e^{iP\cdot x}\,Q =\\ &= e^{-iP\cdot x}\Big(e^{iP\cdot x}\,Q\,e^{-iP\cdot x}\,\mathcal{O}(0) - \mathcal{O}(0)\,e^{iP\cdot x}\,Q\,e^{-iP\cdot x}\Big)e^{iP\cdot x} =\\ &= e^{-iP\cdot x}\,[\hat{Q}, \mathcal{O}(0)]\,e^{iP\cdot x}\,, \end{aligned} \tag{2.47}$$

where $\hat{Q} = e^{iP\cdot x}\,Q\,e^{-iP\cdot x}$. Using Hausdorff formula

$$e^A\,B\,e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots\,, \tag{2.48}$$

we get

$$\hat{Q} = Q + ix^\mu [P_\mu, Q] - \frac{1}{2} x^\mu x^\nu [P_\mu, [P_\nu, Q]] + \dots . \tag{2.49}$$

Let us study first the case $Q = M_{\mu\nu}$. Using (2.33) and the fact that $[P_\mu, P_\nu] = 0$, we then have

$$\hat{M}_{\mu\nu} = M_{\mu\nu} + ix^\rho [P_\rho, M_{\mu\nu}] - \frac{1}{2} x^\rho x^\sigma [P_\rho, [P_\sigma, M_{\mu\nu}]] + \dots = M_{\mu\nu} + x_\nu P_\mu - x_\mu P_\nu , \tag{2.50}$$

and

$$[M_{\mu\nu}, \mathcal{O}(x)] = e^{-iP\cdot x} [M_{\mu\nu} + x_\nu P_\mu - x_\mu P_\nu, \mathcal{O}(0)] \, e^{iP\cdot x} = i(\mathcal{S}_{\mu\nu} + x_\nu \partial_\mu - x_\mu \partial_\nu)\mathcal{O}(x) , \tag{2.51}$$

where we have used relations (2.37), (2.39) and (2.41). Analogously, for $Q = D$, using (2.29) we find

$$\hat{D} = D + ix^\mu [P_\mu, D] - \frac{1}{2} x^\mu x^\nu [P_\mu, [P_\nu, D]] + \dots = D + x^\mu P_\mu . \tag{2.52}$$

that, together with relations (2.37), (2.39) and (2.40), gives

$$[D, \mathcal{O}(x)] = e^{-iP\cdot x} [D + x^\mu P_\mu, \mathcal{O}(0)] \, e^{iP\cdot x} = i(\Delta + x^\mu \partial_\mu)\mathcal{O}(x) . \tag{2.53}$$

Finally, for $Q = K_\mu$, using (2.31), together with the previous relations,

$$\begin{aligned}
\hat{K}_\mu &= K_\mu + ix^\nu [P_\nu, K_\mu] - \frac{1}{2} x^\nu x^\rho [P_\nu, [P_\rho, K_\mu]] + \dots = \\
&= K_\mu + 2x^\nu(\eta_{\mu\nu} D - M_{\mu\nu}) + ix^\nu x^\rho [P_\nu, \eta_{\mu\rho} D - M_{\mu\rho}] = \\
&= K_\mu + 2(x_\mu D - x^\nu M_{\mu\nu}) + ix^\nu x^\rho \Big(\eta_{\mu\rho}(-iP_\nu) - i(\eta_{\nu\mu} P_\rho - \eta_{\nu\rho} P_\mu)\Big) = \\
&= K_\mu + 2(x_\mu D - x^\nu M_{\mu\nu}) + x^\nu x^\rho \Big(\eta_{\mu\rho} P_\nu + \eta_{\nu\mu} P_\rho - \eta_{\nu\rho} P_\mu\Big) = \\
&= K_\mu + 2(x_\mu D - x^\nu M_{\mu\nu}) + 2x_\mu (x\cdot P) - x^2 P_\mu ,
\end{aligned} \tag{2.54}$$

which, in the end, gives us

$$[K_\mu, \mathcal{O}(x)] = e^{-iP\cdot x} [\hat{K}_\mu, \mathcal{O}(0)] \, e^{iP\cdot x} = i\Big(2\Delta x_\mu - 2x^\nu \mathcal{S}_{\mu\nu} + 2x_\mu (x\cdot\partial) - x^2 \partial_\mu\Big)\mathcal{O}(x) \tag{2.55}$$

Action of the Conformal Algebra Generators

To summarize, the generators of the conformal algebra act on a field $\mathcal{O}(x)$ as:

$$\begin{aligned}
[P_\mu, \mathcal{O}(x)] &= i\partial_\mu \mathcal{O}(x) , \\
[M_{\mu\nu}, \mathcal{O}(x)] &= i(\mathcal{S}_{\mu\nu} + x_\nu \partial_\mu - x_\mu \partial_\nu)\mathcal{O}(x) , \\
[D, \mathcal{O}(x)] &= i(\Delta + x^\mu \partial_\mu)\mathcal{O}(x) . \\
[K_\mu, \mathcal{O}(x)] &= i\Big(2\Delta x_\mu - 2x^\nu \mathcal{S}_{\mu\nu} + 2x_\mu (x\cdot\partial) - x^2 \partial_\mu\Big)\mathcal{O}(x) .
\end{aligned} \tag{2.56}$$

2.4 The Conformal Symmetry

Once we have seen conformal transformations of spacetime and how conformal generators act on fields, let us explore now the consequences of conformal invariance. We may review first at the level of classical symmetries before proceeding to examine the quantum level. We closely follow the arguments in [12]. Consider, then, a generic action $S = \int d^d x\, \mathcal{L}(\mathcal{O}, \partial_\mu \mathcal{O})$ and generic transformations $x' = x'(x)$, with fields transforming as $\mathcal{O}(x) \to \mathcal{O}'(x') = \mathcal{F}(\mathcal{O}(x))$, where $\mathcal{F}$ is the function relating the new field $\mathcal{O}'$ evaluated at the transformed coordinate x' to the old field $\mathcal{O}$ at x. Then, we have that

$$\begin{aligned} S[\mathcal{O}'] &= \int d^d x\, \mathcal{L}\{\mathcal{O}'(x), \partial_\mu \mathcal{O}'(x)\} = \int d^d x'\, \mathcal{L}\{\mathcal{O}'(x'), \partial'_\mu \mathcal{O}'(x')\} = \\ &= \int d^d x'\, \mathcal{L}\{\mathcal{F}(\mathcal{O}(x)), \partial'_\mu \mathcal{F}(\mathcal{O}(x))\} = \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\{\mathcal{F}(\mathcal{O}(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\mathcal{O}(x))\}\,. \end{aligned} \tag{2.57}$$

The main idea here is to understand under which conditions $S[\mathcal{O}'] = S[\mathcal{O}]$, i.e. if the transformation $x' = x'(x)$ is a *symmetry*, we want to know what consequences or constraints we have in our theory. The easiest example to start with is *translations*, a case we previously explored. They are defined as $x'^\mu = x^\mu + a^\mu$ and $\mathcal{O}'(x+a) = \mathcal{O}(x)$, so that we trivially have

$$\frac{\partial x^\mu}{\partial x'^\mu} = \delta^\mu_\nu; \quad \mathcal{F} = \mathrm{Id}\,, \tag{2.58}$$

i.e. the action is invariant unless it depends explicitly on x. Next, let us try to see what happens for generic infinitesimal transformations

$$x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}; \quad \mathcal{O}'(x') = \mathcal{O}(x) + \epsilon_a \frac{\delta \mathcal{F}}{\delta \epsilon_a(x)}(x)\,, \tag{2.59}$$

where again we are keeping infinitesimal parameters $\{\epsilon_a\}$ to first order. Let us also study in detail the case of dilatations. They are defined through

$$\begin{aligned} x' &= \lambda x \\ \mathcal{O}'(\lambda x) &= \lambda^{-\Delta}\, \mathcal{O}(x)\,, \end{aligned} \tag{2.60}$$

where recall that Δ is the scaling dimension of the field $\mathcal{O}$. The Jacobian is given then via $\left|\frac{\partial x'}{\partial x}\right| = \lambda^d$, and so the transformed action reads as

$$S' = \lambda^d \int d^d x\, \mathcal{L}\{\lambda^{-\Delta}\mathcal{O}, \lambda^{-1-\Delta}\, \partial_\mu \mathcal{O}\}\,. \tag{2.61}$$

In particular, from the action for a free field massless scalar field in flat space

$$S = \int d^d x\, \partial_\mu \mathcal{O}(x) \partial^\mu \mathcal{O}(x) \tag{2.62}$$

we see that, for the action to be a scale invariant dimensionless quantity, the scaling dimension of the field $\mathcal{O}$ must be

$$\Delta = \frac{1}{2} d - 1 \tag{2.63}$$

In classical field theory, the well-known *Noether's theorem* assures us that every continuous symmetry of the Lagrangian gives rise to a current which is classically conserved. Assuming that (2.59) is a symmetry we have that the conserved current is given by [12]

$$j_a^\mu = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{O})}\partial_\nu \mathcal{O} - \delta_\nu^\mu \mathcal{L}\right]\frac{\delta x^\nu}{\delta \epsilon_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathcal{O})}\frac{\delta \mathcal{F}}{\delta \epsilon_a}\,, \tag{2.64}$$

with the variation of the action given by

$$\delta S = -\int d^d x\, j_a^\mu\, \partial_\mu \epsilon_a = \int d^d x\, \partial_\mu j_a^\mu\, \epsilon_a\,. \tag{2.65}$$

If the field configuration satisfy the equations of motion, i.e. $\delta S = 0$, the above relations implies that, for any position-dependent parameters $\epsilon_a(x)$ we have

$$\partial_\mu j_a^\mu = 0\,. \tag{2.66}$$

We can also construct the associated conserved charge as

$$Q_a = \int d^{d-1}x\, j_a^0\,, \tag{2.67}$$

where the integral is over a constant time slice. It is important to note that this conserved current is what is called as *canonical*, in the sense that we have always the freedom to add the divergence of an antisymmetric tensor as

$$j_a^\mu \quad\to\quad j_a^\mu + \partial_\nu B_a^{\nu\mu}, \quad B_a^{\nu\mu} = -B_a^{\mu\nu}\,, \tag{2.68}$$

and still having our current j_a^μ conserved. For infinitesimal translations $x^\mu \to x^\mu + \epsilon^\mu$ we have

$$\frac{\partial x^\mu}{\delta \epsilon^\nu} = \delta_\nu^\mu, \quad \frac{\delta F}{\delta \epsilon^\nu} = 0\,, \tag{2.69}$$

so that the corresponding conserved current is the *stress-energy tensor*

$$T_C^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi\,, \tag{2.70}$$

where the C stands for *canonical*. Therefore, we have seen that translational invariance implies the conservation of the stress-energy tensor, i.e.

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{2.71}$$

Note that, in general, this quantity is not symmetric. However, we have the freedom to modify it by adding the divergence of a tensor $B^{\rho\mu\nu}$ antisymmetric in its first two indices. This *improved* tensor is known as the *Belinfante* stress-energy tensor $T_B^{\mu\nu}$. It can be shown that Poincaré invariance, i.e. translations and Lorentz transformations, requires actually the stress-energy tensor to be symmetric $T^{\mu\nu} = T^{\nu\mu}$. Finally, we can ask what happens if we also include scale and conformal symmetry. Under an arbitrary change of coordinates $x^\mu \to x^\mu + \epsilon^\mu$, and assuming the stress-energy tensor to be symmetric, we have that the variation of the action will be

$$\delta S = -\int d^d x\, T^{\mu\nu} \partial_\mu \epsilon_\nu = -\frac{1}{2}\int d^d x\, T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\,. \tag{2.72}$$

If the infinitesimal transformation is conformal, then using (2.3) we have

$$\delta S = -\frac{1}{2}\int d^d x\, c(x)\, T^{\mu\nu} g_{\mu\nu} = -\frac{1}{2}\int d^d x\, c(x)\, T^{\mu}{}_{\mu}\,, \tag{2.73}$$

which implies that $\delta S = 0$ if $T^{\mu}{}_{\mu} = 0$, i.e. a conformal field theory must have a conserved stress-energy tensor, symmetric and *traceless*. This is, in fact, a key feature of conformal field theory in any dimension. Even if the tracelessness of the stress-energy tensor is present at the classical level, e.g. Maxwell and Yang-Mills theory in four dimensions, the story is much more complicated at the quantum level, where the need to introduce a scale for regulating the theories make them to fail to be conformal [14].

As we have reviewed, at a classical level, continuous symmetries of the action implies the existence of a conserved current. We can now turn to the quantum level, where, in this case, the objects of interest turn out to be N-point correlation functions of *local operators*. Invariance under continuous symmetries will now lead to constraints relating different correlation functions. Consider a general N-point correlation function

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int [d\Phi]\; \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\; e^{-S[\Phi]}\,, \tag{2.74}$$

where $Z = \int [d\Phi]\; e^{-S[\Phi]}$ is the partition function of the vacuum and we have collectively denoted by Φ the set of all fields in the theory. Under the general transformation

$$\begin{aligned} x &\to x'\,, \\ \mathcal{O}(x) &\to \mathcal{O}'(x') = \mathcal{F}(\mathcal{O}(x))\,, \end{aligned} \tag{2.75}$$

we have

$$\begin{aligned} \langle \mathcal{O}_1(x_1')\cdots\mathcal{O}_n(x_n')\rangle &= \frac{1}{Z}\int [d\Phi]\; \mathcal{O}_1(x_1')\cdots\mathcal{O}_n(x_n')\; e^{-S[\Phi]} = \\ &= \frac{1}{Z}\int [d\Phi']\; \mathcal{O}_1'(x_1')\cdots\mathcal{O}_n'(x_n')\; e^{-S[\Phi']} = \\ &= \frac{1}{Z}\int [d\Phi]\; \mathcal{F}(\mathcal{O}_1(x_1))\cdots\mathcal{F}(\mathcal{O}_n(x_n))\; e^{-S[\Phi]} = \\ &= \langle \mathcal{F}(\mathcal{O}_1(x_1))\cdots\mathcal{F}(\mathcal{O}_n(x_n))\rangle\,, \end{aligned} \tag{2.76}$$

where in the second line we have just renamed the integration variable $\Phi \to \Phi'$. Note that from the second to the third line there are two main hypothesis we have assumed. The first one is that the action is invariant under the change of $\Phi(x) \to \mathcal{F}(\Phi(x))$, which we assumed by definition since we are interested in studying the consequences of such symmetries. The second one, however, is much more subtle: we need the functional integration measure $[d\Phi]$ to be also invariant, i.e. the Jacobian of this change of variable be trivial and do not depend on the fields Φ. This is not always true and the failure of this is often the reason why conformal invariance fails at the quantum level.

In summary, the consequences of the symmetry of the action and the functional integral measure gives us the identity

$$\langle \mathcal{O}_1(x_1') \cdots \mathcal{O}_n(x_n') \rangle = \langle \mathcal{F}(\mathcal{O}_1(x_1)) \cdots \mathcal{F}(\mathcal{O}_n(x_n)) \rangle \,. \tag{2.77}$$

Let us see some examples for primary scalar operators. For translation transformations $x \to x+a$, as we have already seen, $\mathcal{F}$ is trivial, so we have that translation invariance implies

$$\langle \mathcal{O}_1(x_1 + a) \cdots \mathcal{O}_n(x_n + a) \rangle = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,. \tag{2.78}$$

i.e. correlation functions can only depend on the relative positions $(x_i - x_j)$. Since scalar fields have no Lorentz index, we have that Lorentz invariance gives us

$$\langle \mathcal{O}_1(\Lambda^\mu{}_\nu\, x_1^\nu) \cdots \mathcal{O}_n(\Lambda^\mu{}_\nu\, x_n^\nu) \rangle = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,. \tag{2.79}$$

And, as a last example, scale invariance implies the following relation

$$\langle \mathcal{O}_1(\lambda x_1) \cdots \mathcal{O}_n(\lambda x_n) \rangle = \lambda^{-\Delta_1} \cdots \lambda^{-\Delta_n} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,. \tag{2.80}$$

All these relations can be written in a compact form if we note that, from relation (2.1), for the conformal factor $\Omega^2(x)$, the Jacobian of the conformal transformation $x \to x'(x)$ is given by

$$\left| \frac{\partial x'}{\partial x} \right| = (\Omega^2(x))^{-\frac{d}{2}} \,, \tag{2.81}$$

(check Table 2.2 for the values of each conformal transformation). Assuming conformal invariance of the action and the functional integration measure, correlation functions of scalar primary operators obey

$$\begin{aligned} \langle \mathcal{O}_1(x_1') \cdots \mathcal{O}_n(x_n') \rangle &= \left| \frac{\partial x'}{\partial x} \right|_{x_1}^{-\frac{\Delta_1}{d}} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x_n}^{-\frac{\Delta_n}{d}} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \\ &= \Omega(x_1)^{\Delta_1} \cdots \Omega(x_n)^{\Delta_n} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \,. \end{aligned} \tag{2.82}$$

The above identity, which is a consequence of conformal symmetries, heavily constrains the allowed form of the correlation functions, which is going to be the main subject we are going to review in the next section.

However, before we go any further, let us briefly explain how things change when considering spinning fields, since, so far, we have omitted some technicalities due to the presence of an intrinsic spin regarding *finite* conformal transformations (for the *infinitesimal* ones see, for instance, (2.41)). Recall that the defining property of conformal transformations was $g'_{\mu\nu}(x') = \Omega^2 g_{\mu\nu}(x)$, which means that the Jacobian of the coordinate transformation can be written as

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{\Omega(x)} R^\mu{}_\nu(x), \quad R^\mu{}_\nu \in SO(d) \,. \tag{2.83}$$

i.e. conformal transformations can be regarded locally as a rotation and scale transformation.

Consider primary operators with an intrinsic spin. It is natural, and it can be explicitly derived from the conformal algebra (see [2]), that their *finite* conformal transformation rule should depend on the rotation matrix $R^\mu{}_\nu(x)$ in (2.83).

An operator in an irreducible representation D of $SO(d)$ transforms as

$$\mathcal{O}^{\alpha_1\alpha_2\cdots}(x) \quad \to \quad \mathcal{O}'^{\alpha_1\alpha_2\cdots}(x') = \Omega(x)^{\Delta}\, D[R^{\mu}{}_{\nu}(x)]_{\beta_1\beta_2\ldots}{}^{\alpha_1\alpha_2\cdots}\, \mathcal{O}^{\beta_1\beta_2\cdots}(x)\,, \tag{2.84}$$

where $D[R^{\mu}{}_{\nu}(x)]$ is a matrix implementing the action of D in the $SO(d)$ representation of $\mathcal{O}$. For instance,

$$\begin{aligned} D(R) &= 1 && \text{(scalar representation)} \\ D(R)_{\mu}{}^{\nu} &= R_{\mu}{}^{\nu} && \text{(vector representation)} \\ \ldots & && \ldots \end{aligned} \tag{2.85}$$

and so on. A spin-one field $J_\mu(x)$, for instance, transforms as

$$J'_\mu(x') = \Omega(x)^{\Delta}\, R^{\nu}{}_{\mu}(x) J_\nu(x)\,. \tag{2.86}$$

2.5 Conformal Correlators

2.5.1 Scalar operators

Let us start by examining conformal correlators for scalar fields. The simplest case is the 1-point correlation function. If we apply the transformation rule (2.82) to inversions, we have that

$$\langle \mathcal{O}(\frac{x}{x^2})\rangle = -(x^2)^{\Delta}\langle \mathcal{O}(x)\rangle\,, \tag{2.87}$$

where, from (2.18), we have used that the Jacobian for inversions is given by

$$\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right|_{\text{inv}} = \frac{1}{\Omega^{d}_{\text{inv}}(x)} = -\frac{1}{x^{2d}}\,. \tag{2.88}$$

If correlators are to be conformally invariant, this implies that vacuum one-point function $\langle \mathcal{O}(x)\rangle$ in $\mathbb{R}^d$ vanish except for those operators with $\Delta = 0$. As we will see in section (4.3), if we assume unitarity, only the identity operator has scaling dimension 0.[3]

Next, consider two-point functions. Poincaré invariance implies that

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f(|x_1 - x_2|)\,, \tag{2.89}$$

while, for dilatations, we have that (2.82) becomes

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\rangle\,. \tag{2.90}$$

Thus, we have that the symmetries of conformal field theory fixes two-point functions to be of the form

$$f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)\,, \tag{2.91}$$

[3] Note that when we will study CFTs in more generic manifolds $\mathcal{M}^d$ this will no longer be true.

i.e.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{\mathcal{O}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,, \tag{2.92}$$

where $c_{\mathcal{O}}$ is some normalization constant in the two point function. Moreover, we have that, for a conformal transformation $x \to x'$

$$(x_1 - x_2)^2 = \Omega(x_1)\,\Omega(x_2)\,(x'_1 - x'_2)^2\,. \tag{2.93}$$

This is trivial to see for translations, rotation and scale transformations if we check the values of the conformal factors in Table 2.2. It suffices to check this relation for inversions $x \to \frac{x}{x^2}$, since special conformal transformations are just a composition of translation and inversions. Indeed, for inversions we have

$$\begin{aligned}(x'_1 - x'_2)^2 &= (\frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2})^2 = (\frac{1}{x_1^2} + \frac{1}{x_2^2} - \frac{2x_1\cdot x_2}{x_1^2 x_2^2}) = \frac{x_1^2 + x_2^2 - 2x_1\cdot x_2}{x_1^2\,x_2^2} \\ &= \frac{(x_1 - x_2)^2}{x_1^2\,x_2^2} = \frac{(x_1-x_2)^2}{\Omega_{\text{inv}}(x_1)\,\Omega_{\text{inv}}(x_2)}\,,\end{aligned} \tag{2.94}$$

as we expected. Then, using (2.93) we find

$$\frac{c_{\mathcal{O}}}{|x'_1 - x'_2|^{\Delta_1+\Delta_2}} = \Omega(x_1)^{\frac{\Delta_1+\Delta_2}{2}}\,\Omega(x_2)^{\frac{\Delta_1+\Delta_2}{2}}\,\frac{c_{\mathcal{O}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,, \tag{2.95}$$

so that, together with relation (2.82), we have

$$\frac{c_{\mathcal{O}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{\Omega(x_1)^{\frac{\Delta_1+\Delta_2}{2}}\,\Omega(x_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\Omega(x_1)^{\Delta_1}\,\Omega(x_2)^{\Delta_2}}\,\frac{c_{\mathcal{O}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{2.96}$$

This identity is only satisfied if $\Delta_1 = \Delta_2$ or $c_{\mathcal{O}} = 0$. In other words,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{\mathcal{O}}}{x_{12}^{2\Delta_1}}\delta_{\Delta_1,\Delta_2}\,, \tag{2.97}$$

where $x_{ij} = x_i - x_j$ is a notation we will use from now on. Note that the two-point function constant $c_{\mathcal{O}}$ can be set to 1 with a simple field redefinition.

We can apply a similar reasoning for three-point functions. Poincaré and scaling invariance constrain the three-point function up to an overall coefficient f_{123} as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{f_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c}\,, \tag{2.98}$$

with $a+b+c = \Delta_1 + \Delta_2 + \Delta_3$. Relations (2.82) and (2.93) will lead in this case to the identity

$$\frac{f_{123}}{|x_{12}|^a\,|x_{23}|^b\,|x_{13}|^c} = \frac{(\Omega(x_1)\Omega(x_2))^{a/2}\,(\Omega(x_2)\Omega(x_3))^{b/2}\,(\Omega(x_1)\Omega(x_3))^{c/2}}{\Omega(x_1)^{\Delta_1}\,\Omega(x_2)^{\Delta_2}\,\Omega(x_3)^{\Delta_3}}\,\frac{f_{123}}{|x_{12}|^a\,|x_{23}|^b\,|x_{13}|^c}\,, \tag{2.99}$$

which impose the following constraints

$$a + c = 2\Delta_1, \qquad a + b = 2\Delta_2, \qquad b + c = 2\Delta_3\,. \tag{2.100}$$

The solution to these constraints is unique and given via

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \,, \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \,, \\ c &= \Delta_1 + \Delta_3 - \Delta_2 \,. \end{aligned} \tag{2.101}$$

Thus, we see that conformal invariance is again powerful enough to completely fix the three-point function of primary scalars, up to an overall constant f_{123}, as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{f_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} \,. \tag{2.102}$$

Note that now we cannot get rid of the factor f_{123} as we did for the two-point function coefficient $c_{\mathcal{O}}$ since we already used our field redefinition. Thus f_{123} is not a normalization factor, but it is a physical quantity. Actually, this object, as we will see, is very important and encodes a lot of information of our conformal field theory.

So far, we have seen that the spacetime structure of the two- and three- point functions is completely fixed up to two numbers, which are the conformal dimension of our primary operators Δ_i and the three-point coefficient f_{123}. One may feel encouraged to think that a similar story holds for higher-point functions, but, unfortunately, that is not true. With already four points, there are non-trivial conformally invariant combinations called *conformal cross-ratios*,

$$u = \frac{(x_{12})^2\,(x_{34})^2}{(x_{13})^2\,(x_{24})^2}\,, \qquad v = \frac{(x_{23})^2\,(x_{14})^2}{(x_{13})^2\,(x_{24})^2}\,, \tag{2.103}$$

which can be used to build an arbitrary dependence (i.e. not fixed by conformal symmetry) of n-point functions.

2.5.2 Spinning operators

However, in the following sections, we are not going to deal with higher-point functions, but try to understand how the story changes when we start introducing operators with spin. Similar computations as we did for scalar operators can be done to derive the expressions for spinning conformal correlators. For a spin-ℓ traceless symmetric tensor we have [2]

$$\langle J^{\mu_1\ldots\mu_\ell}(x)J_{\nu_1\ldots\nu_\ell}(0)\rangle = c_J\Big(\frac{I^{(\mu_1}{}_{\nu_1}(x)\cdots I^{\mu_\ell)}{}_{\nu_\ell}(x)}{x^{2\Delta}} - \text{traces}\Big)\,, \tag{2.104}$$

where $I_{\mu\nu}$ is the inversion matrix we found in (2.18) and the symmetrization can be done, indistinctly, in the μ's, the ν's or both. Note that with *traces* we mean adding terms proportional to $\delta^{\mu_i\mu_j}$ and $\delta_{\nu_i\nu_j}$ so that the expression is traceless in the μ and ν indices separately, not necessarily under $\mu-\nu$ contractions. For instance, the two-point function for a symmetric traceless field is given by

$$\langle J_{\mu\nu}(x)J_{\lambda\rho}(0)\rangle = \frac{1}{x^{2\Delta}}[I_{\mu\lambda}(x)I_{\nu\rho}(x) + (\mu\leftrightarrow\nu) - \frac{2}{d}\delta_{\mu\nu}\delta_{\lambda\rho}]\,, \tag{2.105}$$

where the numerical factor in the $\delta_{\mu\nu}\delta_{\lambda\rho}$ term is fixed by the tracelessness condition. Finally, let us mention that, in most of the cases, we set the constant c_J to 1. Note that it could also be fixed by some Ward identities, which, for some notable cases such as the normalization constant

c_T of the stress tensor $T^{\mu\nu}$, is the preferred option.

Similarly, an interesting example of a three-point function is the one that involves scalars ϕ_1, ϕ_2 and a spin-ℓ operator $J_{\mu_1 \ldots \mu_\ell}$. It is given via

$$\langle \phi_1(x_1)\phi_2(x_2)J_{\mu_1\ldots\mu_\ell}(x_3)\rangle = \frac{f_{\phi_1\phi_2 J}\,(Z_{\mu_1}\cdots Z_{\mu_\ell} - \text{traces})}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3+\ell)}\,(x_{13}^2)^{\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2-\ell)}\,(x_{23}^2)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1-\ell)}}\,, \quad (2.106)$$

with $Z^\mu \equiv \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$. Here we have not gone through the explicit derivations since there is a transparent way to derive the above results. For that, we will have to shift to a new formalism in the next chapter, called *embedding formalism*, which will turn out to be very practical for our purpose.

3| The Embedding Formalism

As have seen in the previous chapter, conformal symmetry imposes strong constraints on the correlation function for scalar primary fields. Even if those constraints were easy to work out for primary scalars, it is less transparent how to deal with non-zero spin primary fields. In this chapter, we are going to develop a useful formalism, called the *embedding formalism*, which will make the case for primary fields with spin easier. The main idea was first noticed by Dirac [15]: as we saw, conformal algebra is, in fact, isomorphic to $SO(d+1,1)$, the algebra of Lorentz transformations in $\mathbb{R}^{d+1,1}$ space, which will be referred as the *embedding space*. This means that, while the d-dimensional conformal group acts in a somewhat non-trivial way on $\mathbb{R}^d$, it actually acts much more naturally on the $\mathbb{R}^{d+1,1}$ space. Consider the embedding space coordinates

$$X^0, X^1, \dots X^d, X^{d+1}\,, \tag{3.1}$$

where X^0 is the timelike direction. Explicitly, the action of the conformal group in the embedding space $\mathbb{R}^{d+1,1}$ in the vector representation is simply

$$X^A \to \Lambda^A{}_B X^B\,, \tag{3.2}$$

with $\Lambda^A{}_B$ an $SO(d+1,1)$ matrix. The main point here is whether we can get the action of the conformal group on $\mathbb{R}^d$ out of this simple action, so that conformal symmetry constraints in $\mathbb{R}^d$ arise from simple Lorentz constraints in $\mathbb{R}^{d+1,1}$. For that, we need to get rid of two coordinates and we are going to do it via a sort of stereographic projection. The material in this chapter is closely based on references [16] and [17]. As a final remark, it is worth noticing that, even in the previous chapter we have been considering both conformal transformations in *Euclidean* $\mathbb{R}^d$ and *Lorentzian* $\mathbb{R}^{1,d-1}$ space, from now on, we are going to restrict our attention to *Euclidean Conformal Field Theories* in $\mathbb{R}^d$.

3.1 Null cone formalism for scalar fields

Let us consider lightcone coordinates $X^A = (X^+, X^-, X^i)$ defined as

$$X^+ = X^0 + X^{d+1}, \qquad X^- = X^0 - X^{d+1}\,, \tag{3.3}$$

in which the mostly plus metric in $\mathbb{R}^{d+1,1}$ reads as

$$ds^2 = \sum_{i=1}^{d} (dX^i)^2 - dX^+ dX^-\,. \tag{3.4}$$

We shall consider the null cone in $\mathbb{R}^{d+1,1}$, i.e. the space of light rays through the origin defined via

$$X^2 = -(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = 0\,. \tag{3.5}$$

This gets rids of one coordinate. Next, we shall take a generic section of the light-cone, so that we remove another coordinate. The section is parametrized by X^μ, i.e.

$$X^+ = f(X^\mu)\,, \tag{3.6}$$

so that we identify coordinates X^μ with the $\mathbb{R}^d$ coordinates x^μ. The metric ds^2 on the section induced from the Minkowksi metric in $\mathbb{R}^{d+1,1}$ is then

$$ds^2\mid_{\text{section}} = dx^2 - dX^+dX^-\mid_{X^+=f(x),\, X^-=\frac{x^2}{X^+}}\,. \tag{3.7}$$

Our goal is to put a point $x \in \mathbb{R}^d$ in correspondence with the coordinates X^A in $\mathbb{R}^{d+1,1}$. For that, let us analyse how the group $SO(d+1,1)$ acts on a generic section. First, a point x^μ on our section defines a light-ray, consisting of vectors X^A. Applying an $SO(d+1,1)$ transformation $X^A \to \Lambda^A{}_B X^B$, the light-ray is mapped to a new one. From there, we have then to *rescale* by a factor $\lambda(X)$, which in general depends on X, to get back into a point x'^μ in the section. In all these steps, the only thing that changes the metric is the rescaling to get back into the section. It changes as

$$ds'^2 = d(\lambda(X)\,X)^2 = (\lambda dX + X(\nabla\lambda\cdot dX))^2 = \lambda^2 dX^2 = \lambda^2 ds^2\,, \tag{3.8}$$

where we have used that $X^2 = 0$ and $X \cdot dX = 0$. Therefore we have that, the action of $SO(d+1,1)$ on the section implies a change in the metric as

$$ds'^2 = \lambda(X)^2 ds^2\,. \tag{3.9}$$

Now, when is this a conformal transformation? Recall that now are we considering *Euclidean* CFT so that relation (3.9) is a conformal transformation as long as ds^2 is the Euclidean metric. From expression (3.7), it is clear that the particular choice of section that achieves this is the one with $f(X) = \text{constant}$, so that $dX^+ = 0$ and the metric is just $ds^2 = \sum_{i=1}^d (dX^i)^2$. Without loss of generality we can take the constant to be 1. Therefore, the Euclidean section is parametrized as

$$X^A = (X^+, X^-, X^\mu) = (1, x^2, x^\mu), \qquad x \in \mathbb{R}^d\,, \tag{3.10}$$

known as the *Poincaré section*. This is for the coordinates. Which conditions should we demand for the fields? Consider scalar fields $\phi(X)$ defined on the cone. The $SO(d+1,1)$ action on such fields is

$$X \to X', \qquad \phi(X) \to \phi'(X') = \phi(X)\,. \tag{3.11}$$

We will assume that the field on the Euclidean section is the d-dimensional field:

$$\phi(X)\mid_{\text{section}} = \phi(x), \tag{3.12}$$

and also that ϕ is homogeneous of degree $-\Delta$ on X:

$$\phi(\lambda X) = \lambda^{-\Delta}\phi(X)\,. \tag{3.13}$$

It can be seen that these two conditions are enough to imply the correct transformation rules for the fields in $\mathbb{R}^d$. Indeed, from relations (2.1) and (2.93) we have that for conformal transformations

$$ds'^2 = g'_{\mu\nu}(x')dx'^{\mu}dx'^{\nu} = \Omega(x)^2 g_{\mu\nu}(x)\frac{dx^{\mu}}{\Omega(x)^2}\frac{dx^{\nu}}{\Omega(x)^2} = \frac{1}{\Omega(x)^2}ds^2\,, \tag{3.14}$$

so that we can identify, from (3.9), that $\lambda(X) \equiv \frac{1}{\Omega(x)}$. Then,

$$\phi(\lambda X)\,|_{\text{section}} = \lambda^{-\Delta}\phi(X)\,|_{\text{section}} \to \phi(x') = \Omega(x)^{\Delta}\phi(x)\,, \tag{3.15}$$

which agrees with the expected behaviour of the fields in $\mathbb{R}^d$ (c.f. relation (2.82)). In summary, with this *projective light cone* formalism, any conformally invariant quantity in $\mathbb{R}^d$ can be *lifted* to an $SO(d+1,1)$-invariant quantity in $\mathbb{R}^{d+1,1}$. Finding conformally invariant quantities in $\mathbb{R}^d$, which can be hard, is then reduced to write down Lorentz-invariant expressions in $\mathbb{R}^{d+1,1}$, which, as we will see next, is a much easier process.

Scalar fields in embedding formalism

To establish a correspondence between *scalar* fields on $\mathbb{R}^d$ and $\mathbb{R}^{d+1,1}$ we need the following ingredients. A field $\phi(X)$ with the properties:

1. Defined on the null cone: $X^2 = 0\,.$
2. Homogeneous of degree $-\Delta$: $\phi(\lambda X) = \lambda^{-\Delta}\phi(X), \quad \lambda > 0\,.$

We then define $\phi(x)$ to be the field $\phi(X)$ in the Poincaré section:

$$\phi(X)\,|_{\text{section}} = \phi(x)$$

Applications in scalar fields

Let's study a simple example of the embedding formalism, outlined above, in order to understand it better. We can start by deriving the expression of the two-point function. On the light-cone, the most general Lorentz invariant expression containing two operators $\phi(X)$ and $\phi(Y)$ with scaling dimension Δ is

$$\langle\phi(X)\phi(Y)\rangle = \frac{c}{(-2X\cdot Y)^{\Delta}}\,, \tag{3.16}$$

where c is just a constant and the need of the numerical factor 2 will become clear in a few lines. Note that terms with $X^2 = Y^2 = 0$ cannot appear. Also, note that we can construct nothing like (3.16) if the fields have different scaling dimension $\Delta_1 \neq \Delta_2$, thus we expect the two-point function vanishes in that case (as we know it does). Now, let us project this expression into the physical space in $\mathbb{R}^d$. For that, we know that

$$X = (X^+, X^-, X^{\mu}) = (1, x^2, x^{\mu}), \qquad Y = (Y^+, Y^-, Y^{\mu}) = (1, y^2, y^{\mu})\,, \tag{3.17}$$

so that[1]

$$X \cdot Y = X^\mu Y_\mu - \frac{1}{2}(X^+Y^- + X^-Y^+) = x^\mu y_\mu - \frac{1}{2}(x^2 + y^2) = -\frac{1}{2}(x-y)^2 \,. \tag{3.18}$$

The projected two-point function then reads as

$$\langle \phi(x)\phi(y)\rangle = \frac{c}{(x-y)^{2\Delta}}\delta_{\Delta_1,\Delta_2}, \tag{3.19}$$

as expected (c.f. with relation (2.97)).

We can proceed similarly for the three-point function of primary scalar fields with scaling dimensions $\Delta_1, \Delta_2, \Delta_3$. The most general Lorentz-invariant expression is given by

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\rangle = \frac{f_{123}}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}}\,, \tag{3.20}$$

where again the numerical factors 2 have no important consequences since we can always absorb them within the proportionality constant. Consistency with scaling leads to the following constraints

$$\begin{aligned} \alpha_{123} + \alpha_{132} &= \Delta_1, \\ \alpha_{123} + \alpha_{231} &= \Delta_2, \\ \alpha_{132} + \alpha_{231} &= \Delta_3\,, \end{aligned} \tag{3.21}$$

which admit the unique solution:

$$\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2}\,. \tag{3.22}$$

Projecting to the Euclidean section as before, we know that

$$-2X_i \cdot X_j = (x_i - x_j)^2 \equiv x_{ij}^2\,, \tag{3.23}$$

so that we recover the well-known result (2.102):

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\rangle = \frac{f_{123}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\,. \tag{3.24}$$

Clearly, the embedding space derivation is much more efficient and economical than the physical space one.

3.2 Null cone formalism for fields with spin

Even if we have easily recovered the previous results for correlations functions with primary scalar fields, the power of the embedding formalism arises when considering primaries *with spin*. Here, we will only consider *symmetric traceless* primary fields in $\mathbb{R}^d$. Of course, primaries in other representations of $SO(d)$ can be considered, such as antisymmetric tensors or fermions, but it will not be the subject of this work. As before, we have to study which conditions we have to impose on the fields $F_{MNL\ldots}(X)$ that live in the light-cone of the embedding space

[1] Bear in mind that we are working in light-cone coordinates and the metric we have to use for computing the scalar product is the one given by expression (3.4).

$\mathbb{R}^{d+1,1}$ to put them in correspondence with the symmetric and traceless fields[2] $f_{\mu\nu\lambda\dots}(x)$ in $\mathbb{R}^d$. We consider the fields $F_{MNL\dots}(X)$ to be also symmetric and traceless, but it is clear that they have two extra components per index than the d dimensional ones. To remove one of them, we impose *transversality* of the null cone fields

$$(X\cdot F(X))_{NL\dots} = X^M F_{MNL\dots}(X) = 0\,. \tag{3.25}$$

Then, we define the physical fields $f_{\mu\nu\lambda\dots}(x)$ to be the projection on the Euclidean section of the fields $F_{MNL\dots}(X)$ via

$$f_{\mu\nu\lambda\dots}(x) = \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\cdots F_{MNL\dots}(X)|_{X=X(x)}\,. \tag{3.26}$$

Note that this definition implies a *redundancy*. Indeed, anything proportional to X^M gives zero since

$$X^2=0 \quad\rightarrow\quad X_M\frac{\partial X^M}{\partial x^\mu}=0\,, \tag{3.27}$$

so that $F_{MNL\dots}(X)\to F_{MNL\dots}(X)+X_M\Lambda_{NL\dots}(X)$ projects to the same physical operator $f_{\mu\nu\lambda\dots}(x)$ for any tensor $\Lambda_{NL\dots}(X)$[3]. This $SO(d+1,1)$ tensors are sometimes referred as *pure gauge* in the literature. It is this *gauge redundancy* that reduces another degree of freedom per index, matching, then, the degrees of freedom of the fields $f_{\mu\nu\lambda\dots}(x)$.

We should also make sure that with rule (3.26) the tracelessness condition is preserved. Indeed, the projected tensor $f_{\mu\nu\lambda\dots}(x)$ is traceless as long as $F_{MNL\dots}(X)$ is traceless and transverse. From $X^M=(X^+,X^-,X^\mu)=(1,x^2,x^\mu)$,

$$\frac{\partial X^M}{\partial x^\nu} = (0,2x_\nu,\delta^\mu_\nu)\,, \tag{3.28}$$

so that for each pair of indices we have the identity:

$$\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \eta^{MN}+X^MK^N+X^NK^M\,, \tag{3.29}$$

with the auxiliary vector $K_M=(0,2,0)$ and the metric explicitly given by

$$\eta^{MN} = \begin{pmatrix} 0 & -2 & 0 & 0 & \cdots \\ -2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 0 & 1 & \\ & & & & \ddots \end{pmatrix}\,. \tag{3.30}$$

We see, then, that (3.29) contracted with $F_{MN\dots}(X)$ will vanish by tracelessness and transversality, so that $f_{\mu\nu\dots}(x)$ will be also traceless as we claimed.

We are only left to study how the $SO(d+1,1)$ group acts on the null cone and see under what conditions this reproduces the correct transformation law for the projected fields $f_{\mu\nu\lambda\dots}(x)$.

[2]Note that we will use capital letters in indices to denote quantities in the embedding space $\mathbb{R}^{d+1,1}$ and lower case letters for the physical quantities in $\mathbb{R}^d$.

[3]For spin 1, consistently with what is explained below, we should demand the scalar function $\Lambda(X)$ to be homogeneous of degree $-\Delta-1$, i.e. $\Lambda(\lambda X)=\lambda^{-\Delta-1}\Lambda(X)$.

Under an $SO(d+1,1)$ transformation,

$$F'_{MNL\dots}(X') = \Lambda_M{}^{M'}\Lambda_N{}^{N'}\dots F_{M'N'L'\dots}(X)\,. \tag{3.31}$$

It turns out that, as in the case of scalar fields, the homogeneity condition in X does the job:

$$F_{MNL\dots}(\lambda X) = \lambda^{-\Delta}F_{MNL\dots}(X)\,. \tag{3.32}$$

Indeed, let us see that this condition implies transformation rule (2.84). For simplicity, let us consider the spin 1 case, although the proof is completely analogous for general spin. Relation (2.84) for spin 1 is reduced to

$$f'_\mu(x') = \Omega(x)^\Delta R^{\mu'}{}_\mu(x) f_{\mu'}(x)\,. \tag{3.33}$$

However, we know, from (2.83), that the line element transforms as

$$dx' = \frac{\partial x'}{\partial x}\cdot dx = \frac{1}{\Omega(x)}R(x)\cdot dx\,, \tag{3.34}$$

so that it is enough to prove that

$$\begin{aligned} f'(x')\cdot dx' &= f'_\mu(x')\frac{1}{\Omega(x)}R_\lambda{}^\mu dx^\lambda = \Omega(x)^\Delta R^{\mu'}{}_\mu f_{\mu'}(x)\frac{1}{\Omega(x)}R_\lambda{}^\mu dx^\lambda \\ &= \Omega(x)^{\Delta-1}(R\cdot R^T)_\lambda{}^{\mu'} f_{\mu'}(x)dx^\lambda = \Omega(x)^{\Delta-1}f(x)\cdot dx\,. \end{aligned} \tag{3.35}$$

Indeed, the projection rule implies that

$$f(x)\cdot dx = f_\mu(x)\,dx^\mu = F_M(X)\frac{\partial X^M}{\partial x^\mu}dx^\mu = F_M(X)\,dX^M = F(X)\cdot dX\,. \tag{3.36}$$

We already known how is $SO(d+1,1)$ acts on the Poincaré section. When we transform $X\to\lambda\cdot X$ note that the scalar product $F(X)\cdot dX$ is preserved (since it is a "Lorentz" invariant quantity),

$$F(Y)\cdot dY = F(X)\cdot dX, \quad Y=\Lambda\cdot X\,, \tag{3.37}$$

but to get from Y back to the section we have to rescale $X' = \lambda Y \equiv \frac{1}{\Omega}Y$ (as we saw previously). Then, taking into account homogeneity and transversality we have

$$\begin{aligned} f(x)\cdot dx &= F(X)\cdot dX = F(Y)\cdot dY = F(\Omega X')\cdot d(\Omega X') = \\ &= \Omega^{-\Delta}F(X')\cdot[d\Omega\, X' + \Omega\, dX'] = \Omega^{-\Delta+1}F(X')\cdot dX' = \Omega^{-\Delta+1}f(x')\cdot dx'\,, \end{aligned} \tag{3.38}$$

ending with result (3.35).

Tensor fields in embedding formalism

To establish a correspondence between *tensor* fields on $\mathbb{R}^d$ and $\mathbb{R}^{d+1,1}$ we need the following ingredients. A field $F_{A_1 \dots A_\ell}$, a tensor of $SO(d+1,1)$, with the properties:

1. Defined on the null cone: $X^2 = 0\,.$
2. Symmetric and Traceless.
3. Transverse: $X^A\, F_{AA_2 \dots A_\ell}(X) = 0\,.$
4. Homogeneous of degree $-\Delta$: $F_{A_1 \dots A_\ell}(\lambda X) = \lambda^{-\Delta} F_{A_1 \dots A_\ell}(X), \quad \lambda > 0\,.$

We then define $f_{a_1 \dots a_\ell}(x)$ to be related to $F_{A_1 \dots A_\ell}(X)$ by projecting to the Poincaré section as:

$$f_{a_1 \dots a_\ell}(x) = \frac{\partial X^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial X^{A_\ell}}{\partial x^{a_\ell}} F_{A_1 \dots A_\ell}(X)\,. \tag{3.39}$$

Applications in tensor fields

Let us explore the power of the embedding formalism and derive the two point function of a vector field $J_M(X)$. The most general Lorentz invariant tensor, consistent with scaling, we can write down is

$$\langle J_M(X) J_N(Y) \rangle = \frac{1}{(-2X \cdot Y)^\Delta} \left[c_1 W_{MN} + c_2 \frac{X_M Y_N}{X \cdot Y} \right], \tag{3.40}$$

with

$$W_{MN} = \eta_{MN} + \alpha \frac{Y_M X_N}{X \cdot Y}\,. \tag{3.41}$$

Note that the second term will project to zero due to (3.27). Transversality also allows us to fix the value of the constant α as

$$X^M \langle J_M(X) J_N(Y) \rangle = Y^N \langle J_M(X) J_N(Y) \rangle = 0 \quad \to \quad \alpha = -1\,, \tag{3.42}$$

so, on the cone, we finally have

$$\langle J_M(X) J_N(Y) \rangle = c_J \, \frac{\eta_{MN} - \frac{Y_M X_N}{X \cdot Y}}{(-2X \cdot Y)^\Delta}\,, \tag{3.43}$$

where c_J is a constant. Let us now use (3.26) to project the two-point function to the physical space, i.e.

$$\langle J_\mu(x) J_\nu(y) \rangle = \frac{\partial X^M}{\partial x^\mu} \frac{\partial Y^N}{\partial y^\nu} \langle J_M(X) J_N(Y) \rangle\,. \tag{3.44}$$

Let us work out explicitly one of the terms. From $X^M = (1, x^2, x^\lambda)$ we have that $\partial_\mu X^M = (0, 2x_\mu, \delta^\lambda_\mu)$ and

$$Y_M = \eta_{MN} Y^N = (-\frac{1}{2} y^2, -\frac{1}{2}, y_\lambda)\,, \tag{3.45}$$

where recall we are considering metric (3.4). Therefore, the factor Y_M in the two-point function will project to

$$\frac{\partial X^M}{\partial x^\mu} Y_M = -x_\mu + y_\mu \,. \tag{3.46}$$

Similarly, we will have that

$$\begin{aligned} \eta_{MN} &\to \delta_{\mu\nu}, \\ Y_M &\to -x_\mu + y_\mu, \\ X_N &\to x_\nu - y_\nu \\ X\cdot Y &\to \frac{1}{2}(x-y)^2 \end{aligned} \tag{3.47}$$

Therefore, the two-point function in physical space will be given via

$$\langle J_\mu(x) J_\nu(y)\rangle = c_J \frac{\delta_{\mu\nu} - \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x-y)^2}}{(x-y)^{2\Delta}} = c_J \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}} \,, \tag{3.48}$$

where

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \,, \tag{3.49}$$

is the orthogonal matrix associated with an inversion (see (2.18)). The two-point function for higher spin primaries can be computed analogously. Interestingly, two-point function for higher spin operators can be constructed from the above, i.e. apart from $I_{\mu\nu}$ no new conformally covariant tensors appear. For spin-ℓ traceless symmetric tensor we have the result in (2.104).

Another interesting correlator to study is the three-point function of two scalars and one spin ℓ operator. This is a correlator that will appear in the next chapters so it is worth it to understand it properly. Again we will do the derivation for spin 1 explicitly and generalize to general spin afterwards. On the null cone we will have

$$\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle = \frac{f_{\phi_1\phi_2 J}}{(-2X_1\cdot X_2)^{\alpha_{123}}(-2X_1\cdot X_3)^{\alpha_{132}}(-2X_2\cdot X_3)^{\alpha_{231}}} \times W_M \,, \tag{3.50}$$

where the powers α_{ijk} of the scalar factor are determined by the correct scaling given by (3.21) and the tensor structure W_M equals to

$$W_M = \frac{(-2X_2\cdot X_3)X_{1M} - (-2X_1\cdot X_3)X_{2M}}{(-2X_1\cdot X_2)^{\frac{1}{2}}(-2X_1\cdot X_3)^{\frac{1}{2}}(-2X_2\cdot X_3)^{\frac{1}{2}}} \,. \tag{3.51}$$

Let us comment a few things on the tensor structure. The relative sign is, as before, fixed by transversality. Note also that no term proportional to X_{3M} has been included since would project to zero anyway. The scaling is completely determined in the scalar part so the tensor structure have scaling 0 in all variables. Finally, it is immediate to check that the tensor structure is transverse, i.e. $(X_3)^M W_M = 0$. Projecting to physical space as

$$\langle \phi_1(x_1)\phi_2(x_2)J_\mu(x_3)\rangle = \frac{\partial X_3^M}{\partial x_3^\mu}\langle \phi_1(X_1)\phi_2(X_2)J_M(X_3)\rangle \,, \tag{3.52}$$

we find, as explicitly computed before,

$$\begin{aligned} \frac{\partial X_3^M}{\partial x_3^\mu} X_{iM} &= (x_i - x_3)_\mu \,, \qquad && i = 1, 2 \,, \\ -2X_i \cdot X_j &= (x_i - x_j)^2 \,, \qquad && i = 1, 2, 3 \; (i < j) \,, \end{aligned} \tag{3.53}$$

so that we end up with the tensor structure

$$W_\mu = \frac{|x_2 - x_3|^2 (x_1 - x_3)_\mu - |x_1 - x_3|^2 (x_2 - x_3)_\mu}{|x_1 - x_2||x_1 - x_3||x_2 - x_3|} = \frac{|x_{23}||x_{13}|}{|x_{12}|} Z_\mu \,, \tag{3.54}$$

where we have defined

$$Z^\mu \equiv \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} \,. \tag{3.55}$$

Therefore, the three-point function of two scalars and one spin 1 operators in physical space is given by

$$\begin{aligned} \langle \phi_1(x_1) \phi_2(x_2) J_\mu(x_3) \rangle &= \frac{f_{\phi_1 \phi_2 J}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}} \times \frac{|x_{23}||x_{13}|}{|x_{12}|} Z_\mu \\ &= \frac{f_{\phi_1 \phi_2 J} \; Z_\mu}{(x_{12}^2)^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3 + 1)} (x_{13}^2)^{\frac{1}{2}(\Delta_1 + \Delta_3 - \Delta_2 - 1)} (x_{23}^2)^{\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1 - 1)}} \,. \end{aligned} \tag{3.56}$$

The three-point function of higher-spin operators $J_{\mu_1 \ldots \mu_\ell}$ is constructed from the above, analogously as what we did for the two-point functions, since it turns out that Z_μ is the only indexed object for three points that is conformal invariant. The general result was already given in (2.106).

3.3 Encoding tensors in polynomials

Working with indices may lead to cumbersome calculations when considering three- and four-point correlation functions. To keep computations more transparent, some compact *index-free* formalism has also been developed, which consists on encoding the information of tensors into polynomials. This is sketched in the following section, where we summarize the results presented in [17]. This formalism will be useful from chapter 5 onwards. We will extend our previous convention that upper case letters refer to quantities in the embedding space, while lower case letters refer to quantities in the physical space.

3.3.1 Tensors in physical space

The main idea is that any symmetric tensor in $\mathbb{R}^d$ can be put in one-to-one correspondence with a d-dimensional polynomial by contracting the tensor with a reference vector h^a:

$$f_{a_1 \ldots a_\ell}(x) \text{ symmetric } \rightarrow \; f(x, h) \equiv f_{a_1 \ldots a_\ell}(x) \, h^{a_1} \cdots h^{a_\ell} \,. \tag{3.57}$$

Now, as we have already seen, in CFT we are particularly interested in symmetric and *traceless* tensors. For them, we can restrict the respective polynomial $f(h)$ to the submanifold $h^2 = 0$:

$$f_{a_1 \ldots a_\ell}(x) \text{ symmetric and } \textit{traceless} \;\rightarrow\; f(x,h) \equiv f_{a_1 \ldots a_\ell}(x)\, h^{a_1} \cdots h^{a_\ell}|_{h^2=0} \,. \tag{3.58}$$

To see why, let us consider a symmetric *traceless* tensor $f_{a_1 \ldots a_\ell}$ and a symmetric (not *traceless*) tensor $\tilde{f}_{a_1 \ldots a_\ell}$ such that their associated polynomials $f(h)$ and $\tilde{f}(h)$ differ by

$$f(x,h) = \tilde{f}(x,h) + \{\text{terms vanishing at } h^2 = 0\} \,. \tag{3.59}$$

Then, $f_{a_1 \ldots a_\ell}(x)$ can be recovered from $\tilde{f}_{a_1 \ldots a_\ell}(x)$.[4] This means that, for symmetric traceless tensors, we can drop any polynomial term proportional to h^2 without any ambiguity.

Now, the important question is how we recover $f_{a_1 \ldots a_\ell}(x)$ from a given $f(x,h)$. For that we need what is called the *Todorov* operator defined as

$$D_a \equiv \left(\frac{d}{2} - 1 + h \cdot \frac{\partial}{\partial h} \right) \frac{\partial}{\partial h^a} - \frac{1}{2}\, h_a \frac{\partial^2}{\partial h \cdot \partial h} \,, \tag{3.60}$$

which is constructed as an *interior* operator on the cone, i.e. it maps $O(h^2)$ functions to themselves $D_a O(h^2) = O(h^2)$. We then have that

$$f_{a_1 \ldots a_\ell}(x) = \frac{1}{\ell! (\frac{d}{2} - 1)_\ell} D_{a_1} \cdots D_{a_\ell} f(x,h) \,, \tag{3.61}$$

where $(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)}$ is the Pochhammer symbol.

3.3.2 Tensors in embedding space

The same strategy can be applied for the embedding space quantities. We can encode a tensor $F_{A_1 \ldots A_\ell}$ in the embedding space $\mathbb{R}^{d+1,1}$ in a $(d+2)$-dimensional polynomial,

$$F_{A_1 \cdots A_\ell}(X) \text{ symmetric} \rightarrow F(X,H) = F_{A_1 \cdots A_\ell}(X)\, H^{A_1} \cdots H^{A_\ell} \,, \tag{3.62}$$

i.e. contracting the tensor with a $(d+2)$-dimensional reference vector H^A. Recall that we were interested in a particular subclass of embedding space tensors, i.e. those that were symmetric, traceless and transverse (STT). For that kind of tensors, we can restrict the polynomial to a submanifolds satisfying $H^2 = 0$ and $H \cdot X = 0$:

$$F_{A_1 \cdots A_\ell}(X) \text{ STT} \rightarrow F(X,H) = F_{A_1 \cdots A_\ell}(X)\, H^{A_1} \cdots H^{A_\ell}|_{H^2=0, H\cdot X=0} \,. \tag{3.63}$$

As before, we can reformulate the statement above by considering a STT tensor $F_{A_1 \cdots A_\ell}(X)$ and *any* tensor $\tilde{F}_{A_1 \cdots A_\ell}(X)$ such that their associated polynomials $F(X,H)$ and $\tilde{F}(X,H)$ only differ by terms proportional to $H^2 = 0$ and $H \cdot X$, i.e.

$$F(X,H) = \tilde{F}(X,H) + \{\text{terms vanishing at } H^2 = 0 \text{ and } H \cdot X = 0\} \,. \tag{3.64}$$

Then, it can be proved that $F_{A_1 \cdots A_\ell}(X)$ can be recovered from $\tilde{F}_{A_1 \cdots A_\ell}(X)$.

Analogously as in the physical space, we recover $F_{A_1 \ldots A_\ell}(X)$ from a given $F(X,H)$ via the

[4]See [17] for an intuition behind this fact and hints for a formal proof.

same differential operator as (3.63) made to act in the $(d+2)$-dimensional space:

$$F_{A_1\dots A_\ell}(x) = \frac{1}{\ell!(\frac{d}{2}-1)_\ell}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_\ell}F(X,H)\,, \tag{3.65}$$

with

$$\mathcal{D}_A \equiv \left(\frac{d}{2}-1+H\cdot\frac{\partial}{\partial H}\right)\frac{\partial}{\partial H^A}-\frac{1}{2}\,H_A\frac{\partial^2}{\partial H\cdot\partial H}\,. \tag{3.66}$$

Note that we have the same d as (3.60).

Finally, for completeness, let us give the explicit connection between an expression written in index-free notation embedding coordinates $F(X,H)$ and the same expression in index-free physical space coordinates $f(x,h)$. From relations (3.26) and (3.28) we can see that we can move from an expression $F(X,H)$ to $f(x,h)$ by using

$$\begin{aligned} X^M &= (X^+,X^-,X^\mu) = (1,x^2,x^\mu) \\ H^M &= (H^+,H^-,H^\mu) = (0,2x\cdot h,h^\mu)\,, \end{aligned} \tag{3.67}$$

where recall that here we are considering light-cone coordinates. The above expressions will become useful, for instance, when considering mean field theory for currents at finite temperate in chapter 9.

4| More on CFT topics

Until now we have been dealing mostly with how conformal symmetry constrains the spacetime structure of the correlation functions of local operators. We are going to understand an alternative view of correlation functions, not in a statistical mechanics sense as we have seen so far, but based on Hilbert spaces and quantum mechanical evolution. We will develop all the necessary tools to understand the concepts related to what is called the *Operator Product Expansion (OPE)*, which is the ultimate goal of this chapter.

4.1 Radial Quantization

In QFT, to specify a quantization in a theory, we usually have to foliate our d-dimensional spacetime into $(d-1)$-dimensional surfaces. Each $(d-1)$-dimensional leaf is endowed with its own Hilbert space. In a rotationally-invariant Euclidean theory in $\mathbb{R}^d$, for instance, any direction can be chosen as *time*. Then, we can think of the orthogonal slices to our time direction (i.e. surfaces of equal time, the *spatial* slices) as slices endowed with their own Hilbert space in which the states live. Each choice is then a quantization of the theory. In practice, it is common to choose foliations that respect the symmetries of the theory so that the slices are related by symmetry transformations, meaning the Hilbert space is the same on each surface.

We can create *in* states $|\Psi_{in}\rangle$ by inserting operators in a far away past of a given surface and *out* states $\langle\Psi_{out}|$ when inserting operators in the future of the same surface. The operator insertions create then a Hilbert space of states, whose overlap $\langle\Psi_{out}|\Psi_{in}\rangle$ is then equal to the S-matrix elements used in scattering processes. These are related to the correlation function of operators which create these *in* and *out* states through the so-called *LSZ formula*.

More generally, for a given foliation, if we have operators in different slices, an Euclidean correlation function $\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle$ is interpreted as a time-ordered expectation values

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \langle 0|T\{\hat{\mathcal{O}}_1(t_1,\mathbf{x}_1)\cdots\hat{\mathcal{O}}_n(t_n,\mathbf{x}_n)\}|0\rangle\,, \tag{4.1}$$

where $|0\rangle$ is the vacuum in the Hilbert space $\mathcal{H}$ living on a spatial slice and $\hat{\mathcal{O}}_i(x):\mathcal{H}\to\mathcal{H}$ are quantum operators corresponding to the path integral insertions $\mathcal{O}_i(x)$. Different quantizations would imply, of course, different Hilbert space or quantum operators, but as long as we arrange the operators as on the right-hand side of (4.1) we will get the correlator on the left-hand side (see Appendix A in [2] for explicit realizations of these ideas). Using

$$\mathcal{O}(x) = e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}\,, \tag{4.2}$$

(c.f. (2.37), here we are working in *Euclidean* signature) the time-ordered correlator (4.1) with

$t_i - t_j > 0$ when $i > j$ becomes

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= \langle 0|e^{t_nP^0}\mathcal{O}_n(0,\mathbf{x}_n)e^{-t_nP^0}\cdots e^{t_1P^0}\mathcal{O}_1(0,\mathbf{x}_1)e^{-t_1P^0}|0\rangle = \\ &= \langle 0|\mathcal{O}_n(0,\mathbf{x}_n)e^{-(t_n-t_{n-1})P^0}\cdots e^{-(t_2-t_1)P^0}\mathcal{O}_1(0,\mathbf{x}_1)|0\rangle\,.\end{aligned} \tag{4.3}$$

As we can see, the path integral between spatial slices encodes the time evolution operator $U(t) = e^{-tP^0}$, i.e. if the *in* and *out* states live on different surfaces there will be a unitary evolution operator U connecting the two Hilbert states $\langle\Psi_{out}|U|\Psi_{in}\rangle$. From (4.3) we can see that in Euclidean QFT only time-order correlators make sense. Indeed, had we chosen $t_1 > t_2 > \cdots > t_n$ in (4.1), we would have obtained unbounded exponential operators in (4.3).

In CFT, there is a more natural choice of foliation: one that respects scale invariance. To achieve that, we can foliate spacetime by S^{d-1} spheres around the origin. Of course, quantizing around any other point should give the same correlators. The unitary operator that will move us from one sphere to another will not use the Hamiltonian P^0 but the dilatation operator D. This is what is known as *radial quantization*, where now correlation functions are interpreted as radially ordered product,

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= \langle 0|\mathcal{R}\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}|0\rangle = \\ &= \theta(|x_n|-|x_{n-1}|)\cdots\theta(|x_2|-|x_1|)\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \\ &+ \text{permutations}\,.\end{aligned} \tag{4.4}$$

Note that operators living on the same sphere, i.e. same radius but different angle, commute just as spacelike-separated operators commute in usual quantization.

There is an alternative view of radial quantization that is worth to mention. Let us introduce radial coordinates $r > 0$ on $\mathbb{R}^d$ and

$$\tau = \log(r) \leftrightarrow r = e^{\tau}\,. \tag{4.5}$$

It is easy to see that the metric of the cylinder $\mathbb{R}\times S^{d-1}$ is equivalent to the flat space metric $\mathbb{R}^d$ by a Weyl transformation [2]

$$\begin{aligned}ds^2_{\mathbb{R}^d} &= dr^2 + r^2 ds^2_{S^{d-1}} = r^2\Big(\frac{dr^2}{r^2} + ds^2_{S^{d-1}}\Big) = \\ &= e^{2\tau}(d\tau^2 + ds^2_{S^{d-1}}) = e^{2\tau}ds^2_{\mathbb{R}\times S^{d-1}}\,.\end{aligned} \tag{4.6}$$

Note that dilations $r \to \lambda r$ become shifts of radial time $\tau \to \tau + \log\lambda$. In this sense, radial quantization in flat space is equivalent to usual quantization on the cylinder, where the time evolution operator is given by $U = e^{-D\tau}$. Since we have performed a nontrivial Weyl rescaling the theory needs to be conformal invariant (not only scale invariant). It is then possible to define operators on the cylinder and relate correlation functions between both geometries in a surprisingly easy way [2]. But for now we should move to study the relation between operators and states inside spheres in radial quantization.

4.2 State-Operator correspondence

In radial quantization, as we have explained, we deal with spheres as spacetime foliations. How do we generate states living on the sphere in this formalism? The simplest way is to perform a path integral over the interior of the sphere B. If there are no operator insertion inside B then we generate the vacuum state $|0\rangle$ on the boundary ∂B. The dilatation eigenvalue, which plays the role of the energy in usual quantization, is zero for this state.

To create a state $|\Delta\rangle$ we insert an operator $\mathcal{O}_\Delta$ at the origin $x = 0$, so that we have

$$\mathcal{O}(0)_\Delta \quad \to \quad |\Delta\rangle \equiv \mathcal{O}_\Delta(0)\,|0\rangle \ . \tag{4.7}$$

Note that the generated state will have energy equal to the scaling dimension Δ:

$$D\,|\Delta\rangle = D\mathcal{O}_\Delta(0)\,|0\rangle = [D, \mathcal{O}_\Delta(0)]\,|0\rangle + \mathcal{O}_\Delta(0)D\,|0\rangle = \Delta\mathcal{O}_\Delta\,|0\rangle = \Delta\,|\Delta\rangle \ , \tag{4.8}$$

where we have used the *Euclidean* version of (2.40) and that $|0\rangle$ is killed by D. Recall our previous discussion in section 2.3 where we showed that acting with momentum generators $\mathcal{O}(0) \to P_{\mu_1}\cdots P_{\mu_n}\mathcal{O}(0)$ allowed us to construct operators of higher dimension $\Delta \to \Delta + n$. Since we have now established a relation *operator* $\to$ *state* we have that, in radial quantization, a *conformal multiplet* is obtained by acting with momentum generators on primary states

$$\{|\mathcal{O}\rangle\}_\Delta \to \{P_\mu\,|\mathcal{O}\rangle\}_{\Delta+1} \to \{P_\mu P_\nu\,|\mathcal{O}\rangle\}_{\Delta+2} \to \cdots \ . \tag{4.9}$$

This is, actually, equivalent to acting with derivatives of $\mathcal{O}(x)$ at the origin:

$$P_\mu\,|\mathcal{O}\rangle = [P_\mu, \mathcal{O}(0)]\,|0\rangle = \partial_\mu\mathcal{O}(x)|_{x=0}\,|0\rangle \ , \tag{4.10}$$

where we have used the *Euclidean* version of (2.39). If we act, instead, with operator K_μ we lower the dimension of states by 1 until we eventually hit zero, and this will give us a primary.

If we insert an operator $\mathcal{O}_\Delta(x)$ with $x \neq 0$, the resulting state $|\Psi\rangle = \mathcal{O}_\Delta(x)\,|0\rangle$ is no longer an eigenstate of the dilation operator D, but a superposition of descendant states with different "energies"

$$|\Psi\rangle = \mathcal{O}_\Delta(x)\,|0\rangle = e^{x\cdot P}\mathcal{O}(0)e^{-x\cdot P}\,|0\rangle = e^{x\cdot P}\,|\mathcal{O}\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(x\cdot P)^n\,|\mathcal{O}\rangle \ . \tag{4.11}$$

We have seen that we can generate states by inserting operators with definite scaling dimension Δ at the origin. If the operator is a primary it will then be killed by K_μ. The construction turns out to work backwards too: given a state of "energy" Δ, annihilated by K_μ, we can construct a local primary operator of dimension Δ by defining its correlation functions with other operators:

$$\langle\phi(x_1)\phi(x_2)\ldots\mathcal{O}_\Delta(0)\rangle = \langle 0|\phi(x_1)\phi(x_2)\ldots|\Delta\rangle \, . \tag{4.12}$$

It can be seen that this definition satisfy all usual transformation properties expected from conformal invariance. One may be concerned whether the operator $\mathcal{O}_\Delta(0)$ will be local or not. But remember that, as we briefly discussed in the previous section, a CFT on the cylinder $\mathbb{R}\,\times S^{d-1}$ quantized on equal time slices can be described equivalently in terms of a CFT on $\mathbb{R}^d$ quantized on equal radius slices. Thus, the operator $\mathcal{O}_\Delta$ inserted in a distant past surface in the cylinder quantization would become an operator *localized* at the origin in the radial quantization.

With both constructions, states and local operators are in one-to-one correspondence, which is known as *state-operator correspondence*. Note, however, that the key requirement of the state-operator correspondence is, indeed, that the operator is local, which ultimately relies on the conformal equivalence between the cylinder and the radial plane. Therefore, this state-operator correspondence is a non-trivial result which is not satisfied for QFTs in general.

State-operator correspondence

Since $|0\rangle$ is killed by K, D, and M, we have that the conformal group act on states as

$$\begin{aligned} [M_{\mu\nu}, \mathcal{O}(0)] &= \mathcal{S}_{\mu\nu}\, \mathcal{O}(0) && \leftrightarrow && M_{\mu\nu}\, |\mathcal{O}\rangle = \mathcal{S}_{\mu\nu}\, |\mathcal{O}\rangle\ , \\ [D, \mathcal{O}(0)] &= \Delta\, \mathcal{O}(0) && \leftrightarrow && D\, |\mathcal{O}\rangle = \Delta\, |\mathcal{O}\rangle\ , \\ [K_\mu, \mathcal{O}(0)] &= 0 && \leftrightarrow && K_\mu\, |\mathcal{O}\rangle = 0\, , \end{aligned} \tag{4.13}$$

4.3 Unitary bounds

Unitarity in quantum mechanics implies that the norm of states have to be non-negative. We could ask what kind of constraints unitarity imposes in a conformal theory. It turns out that unitarity imposes lower bounds for the conformal dimensions Δ of operators. As we saw, different states in a conformal multiplet can be connected through the action of the generators P and K, that act as raising and lowering operator, respectively. We can actually argue that the relation

$$P_\mu^\dagger = K_\mu\, , \qquad \Longleftrightarrow \qquad (P_\mu\, |\psi\rangle)^\dagger = \langle\psi|\, K_\mu\, , \tag{4.14}$$

must hold so that raising and lowering work properly (see [2] for a proper general proof). With this relation in mind, we can start examining the following matrix element for a primary scalar operator $\mathcal{O}$:

$$\langle\mathcal{O}|\, K_\mu P_\nu\, |\mathcal{O}\rangle = \langle\mathcal{O}|\, [K_\mu, P_\nu]\, |\mathcal{O}\rangle = 2\, \langle\mathcal{O}|\, (\delta_{\mu\nu} D - M_{\mu\nu})\, |\mathcal{O}\rangle = 2\delta_{\mu\nu}\Delta\, \langle\mathcal{O}|\mathcal{O}\rangle\ , \tag{4.15}$$

where in the first step we have used the fact that $K\, |\mathcal{O}\rangle = 0$ since $\mathcal{O}$ is a primary field, and we have used the *Euclidean* version of (2.31). Then, demanding the first descendant of the conformal multiplet to also have non-negative norm implies that

$$|P\, |\mathcal{O}\rangle\, |^2 \geq 0 \quad \to \quad \Delta \geq 0\, . \tag{4.16}$$

Here we learn that for the particular case of the identity operator $\mathbb{1}$, which does not have any descendant since $[P_\mu, \mathbb{1}] = \partial_\mu \mathbb{1} = 0$, we have that $\Delta_{\mathbb{1}} = 0$, as we anticipated in chapter 2.

If we restrict to the case $\Delta \neq 0$ we can find stronger restrictions. Indeed,

$$\begin{aligned} \left\| P_\mu P^\mu\, |\mathcal{O}\rangle \right\|^2 &= \langle\mathcal{O}|\, K_\alpha K^\alpha P_\mu P^\mu\, |\mathcal{O}\rangle = \langle\mathcal{O}|\, K_\alpha P_\mu K^\alpha P^\mu + K^\alpha [K_\alpha, P_\mu] P^\mu\, |\mathcal{O}\rangle = \\ &= \langle\mathcal{O}|\, K_\alpha P_\mu [K^\alpha, P^\mu] + 2K^\alpha (\delta_{\mu\alpha} D + M_{\mu\alpha}) P^\mu\, |\mathcal{O}\rangle = \\ &= \langle\mathcal{O}|\, 2K_\mu P^\mu D + (2K_\mu P^\mu D + 2K_\mu [D, P^\mu]) + 2K^\alpha M_{\mu\alpha} P^\mu\, |\mathcal{O}\rangle = \\ &= \langle\mathcal{O}|\, 4K\cdot P\Delta + 4K\cdot P - 2dK\cdot P\, |\mathcal{O}\rangle = \\ &= 2(2\Delta + 2 - d)\, \langle\mathcal{O}|\, K\cdot P\, |\mathcal{O}\rangle \geq 0\, , \end{aligned} \tag{4.17}$$

where we have used the usual commutation relation of the conformal algebra. Thus, we have found that

$$\Delta \geq \frac{d-2}{2}\,. \tag{4.18}$$

For spin ℓ *traceless* symmetric operators we can take inner products between first-level descendants $P_\mu\,|\mathcal{O}^{\mu\mu_2\ldots\mu_\ell}\rangle$ and use the conformal algebra to find the unitary bound [2]

$$\Delta \geq d-2+\ell\,. \tag{4.19}$$

Of course we could ask whether exploring further level descendants would bring us new and stronger contraints. The answer is that for *traceless* symmetric tensors this is not the case and no more interesting bounds can't be derived. However, for theories with more symmetries, such as supersymmetric theories, new unitary bounds can be found. In summary, we have that

$$\begin{aligned} &\Delta_{\mathcal{O}} = 0 \quad \text{for} \quad \mathcal{O} = \mathbb{1}\,, \\ &\Delta_{\mathcal{O}} \geq \begin{cases} \frac{d-2}{2} & \ell = 0\,, \\ \ell + d - 2 & \ell > 0\,. \end{cases} \end{aligned} \tag{4.20}$$

What happens when the unitary bound is saturated? Once Δ saturates the bound, we have a null state in the conformal multiplet. Indeed, in the case of scalars, we can see from (4.17) that the null state is $P^2\,|\mathcal{O}\rangle$, which translates to $\partial^2\mathcal{O}(x) = 0$ in operator language. This is nothing but the Klein-Gordon equation describing a free scalar field. On the other hand, for spin ℓ operators, even if we did not go through the explicit derivation, the null state in this case is given by $P_\mu\,|\mathcal{O}^{\mu\mu_2\ldots\mu_\ell}\rangle = 0$ which, in operator language, becomes $\partial_\mu\mathcal{O}^{\mu\mu_2\ldots\mu_\ell}(x) = 0$, i.e. when the bound is saturated we have a conserved current. Moreover, it is not difficult to see that the reverse is also true. Indeed, let us assume that $J_{\mu_1\ldots\mu_\ell}$ is a conserved primary current. We have that [18]

$$\begin{aligned} 0 &= [\,P_\beta,[K_\alpha,\,\mathcal{O}_{\mu_1\ldots\mu_\ell}]\,] = [\,K_\alpha,[P_\beta,\,\mathcal{O}_{\mu_1\ldots\mu_\ell}]\,] + [\,\mathcal{O}_{\mu_1\ldots\mu_\ell},[K_\alpha,\,P_\beta]\,] = \\ &= [\,K_\alpha,[P_\beta,\,\mathcal{O}_{\mu_1\ldots\mu_\ell}]\,] - 2\,\eta_{\alpha\beta}\,\Delta_{\mathcal{O}}\,\mathcal{O}_{\mu_1\ldots\mu_\ell} - 2\,[\mathcal{S}_{\alpha\beta},\mathcal{O}_{\mu_1\ldots\mu_\ell}] \\ &= [\,K_\alpha,[P_\beta,\,\mathcal{O}_{\mu_1\ldots\mu_\ell}]\,] - 2\,\eta_{\alpha\beta}\,\Delta_{\mathcal{O}}\,\mathcal{O}_{\mu_1\ldots\mu_\ell} - 2\sum_i(\eta_{\alpha\mu_i}\mathcal{O}_{\mu_1\ldots\beta\ldots\mu_\ell} - \eta_{\beta\mu_i}\mathcal{O}_{\mu_1\ldots\alpha\ldots\mu_\ell})\,. \end{aligned} \tag{4.21}$$

Tracing with $\eta^{\beta\mu_1}$ the above expression makes the first commutator vanish since we are considering a conserved current. It is not difficult then to show that we finally get

$$0 = \Delta_{\mathcal{O}}\,\mathcal{O}_{\alpha\mu_2\ldots\mu_\ell} - (d+\ell-2)\,\mathcal{O}_{\alpha\mu_2\ldots\mu_\ell}\,. \tag{4.22}$$

Therefore, we have shown that for a conserved current, the conformal dimension $\Delta_{\mathcal{O}}$ saturates the unitary bound. As we have seen, since the implication works both ways, we have that $\Delta_{\mathcal{O}} = \ell + d - 2$ *if and only if* $\mathcal{O}^{\mu_1\cdots\mu_\ell}$ is a conserved current. An important example is when we have a global symmetry current with $\ell = 1, \Delta = d-1$, case we are going to study from chapter 8 onwards, or when we have a stress-energy tensor with $\ell = 2$ and $\Delta = d$.

4.4 Operator Product Expansion (OPE)

It is time we study what happens when local operators approach each other. Let us suppose we have two operators $\mathcal{O}_i(x)\mathcal{O}_j(0)$ inserted inside a sphere. If we perform the path integral over the interior we get some state $|\Psi\rangle = \mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle$ on the boundary. This state will have an expansion into a basis of "energy" (in fact, dilatation) eigenstates $|E_n\rangle$, i.e.

$$|\Psi\rangle = \mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle = \sum_n c_n(x)\,|E_n\rangle \ , \tag{4.23}$$

By the state-operator correspondence, the states $|E_n\rangle$ are in one-to-one correspondence with operators that are either primaries or descendants (derivatives of primaries) so that we can write

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12},\partial_2)\mathcal{O}_k(x_2)\,. \tag{4.24}$$

where the sum k runs over primary operators and $C_{ijk}(x_{12},\partial_2)$ is an operator understood as a power series of ∂_2, which depends only on the separation between the two operators on grounds of translation invariance. In other words, this tells us that two local operators inserted at nearby points can be closely approximated by an infinite sum of operators at one of these points.[1] This is called *Operator Product Expansion* (OPE). The relation (4.24) is valid inside any correlation function

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\cdots\rangle = \sum_k C_{ijk}(x_{12},\partial_2)\langle\mathcal{O}_k(x_2)\cdots\rangle\,. \tag{4.25}$$

as long as the other operators insertions $\mathcal{O}_n(x_n)$ have $|x_n - x_2| \geq |x_2 - x_1|$.

Let us now see how consistency with conformal invariance, as we could expect, constrains the form of the OPE. Let us suppose, that $\mathcal{O}_i, \mathcal{O}_j$ are scalar operators while $\mathcal{O}_k$ have non-zero spin ℓ. Let us focus, for instance, on the following term:

$$\begin{aligned}\mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle &= \sum_k c_{ijk}(x,\partial)x_{\mu_1}\ldots x_{\mu_\ell}\,\mathcal{O}_k^{\mu_1\ldots\mu_\ell}(0) = \\ &= \frac{\text{const.}}{|x|^n}\Big(x_{\mu_1}\ldots x_{\mu_\ell}\,\mathcal{O}_k^{\mu_1\ldots\mu_\ell}(0)+\cdots\Big)\,|0\rangle + \{\text{other primaries}\}\,,\end{aligned} \tag{4.26}$$

where by $\cdots$ we mean the contribution of derivatives (descendants) of the primary $\mathcal{O}_k^{\mu_1\ldots\mu_\ell}$. Let us see how scaling fixes the power n of the denominator by applying the operator D on both sides on the above expression. From the LHS,

$$\begin{aligned}D\mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle &= [D,\mathcal{O}_i(x)]\,\mathcal{O}_j(0)\,|0\rangle + \mathcal{O}_i(x)[D,\mathcal{O}_j(0)]\,|0\rangle = \\ &= \Big((\Delta_i + x^\lambda\partial_\lambda)+\Delta_j\Big)\,\mathcal{O}_i(x)\mathcal{O}_j(0)\,|0\rangle = \\ &= (\Delta_i+\Delta_j-n+\ell)\frac{\text{const.}}{|x|^n}\Big(x_{\mu_1}\ldots x_{\mu_\ell}\,\mathcal{O}_k^{\mu_1\ldots\mu_\ell}+\cdots\Big)\,|0\rangle + \cdots\,,\end{aligned} \tag{4.27}$$

where we have used the *Euclidean* version of (2.40)-(2.53), and (4.26) in the last step.

[1] Actually, we could have expanded around a different point x_3 as $\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C'_{ijk}(x_{13},x_{23},\partial_3)\mathcal{O}_k(x_3)$, although the form (4.24) is more practical for computations.

On the other hand, if we act with D on the RHS of (4.26)

$$D\left(\frac{\text{const.}}{|x|^n}\left(x_{\mu_1}\dots x_{\mu_\ell}\,\mathcal{O}_k^{\mu_1\dots\mu_\ell}(0)+\cdots\right)|0\rangle = \frac{\text{const.}}{|x|^n}\Delta_k\, x_{\mu_1}\dots x_{\mu_\ell}\,\mathcal{O}_k^{\mu_1\dots\mu_\ell}(0)\,|0\rangle+\cdots\,. \quad (4.28)$$

By comparing (4.27) and (4.28) we find that

$$n=\Delta_i+\Delta_j-\Delta_k+\ell\,. \quad (4.29)$$

We could also act with K_μ. It can be shown, actually, that K_μ completely fixes C_{ijk} up to an overall coefficient. However, there is a more interesting way to see how the OPE is fixed. The method is based on the fact that we can use the OPE to reduce an n- point function $\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle$ to $(n-1)$-point function recursively:

$$\begin{aligned}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle &= \sum_{k_1} C_{12k_1}\langle\mathcal{O}_{k_1}\mathcal{O}_3\cdots\mathcal{O}_n\rangle = \\ &= \sum_{k_1}\cdots\sum_{k_{n-1}} C_{12k_1}C_{k_13k_2}\cdots C_{k_{n-2}nk_{n-1}}\langle\mathcal{O}_{k_{n-1}}\rangle\,,\end{aligned} \quad (4.30)$$

where we have suppressed the position dependence of the $\mathcal{O}_i$ for simplicity. This works recursively until we get a one-point function, which, as we say, were given by

$$\langle\mathcal{O}\rangle_{\mathbb{R}^d} = \begin{cases}1 & \text{if } \mathcal{O}=\mathbb{1}\,,\\ 0 & \text{otherwise}\,.\end{cases} \quad (4.31)$$

Note that each time we apply an OPE we have to find a pair of operators $\mathcal{O}_i\mathcal{O}_j$ and a sphere surrounding them, such that all the other operators lie outside of it. This is always possible in $\mathbb{R}^d$.[2]

Let us apply the above idea with the OPE

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{k'}[C_{ijk'}(x_{12},\partial_2)]_{\mu_1\dots\mu_\ell}\,\mathcal{O}_{k'}^{\mu_1\dots\mu_\ell}(x_2) \quad (4.32)$$

to the three-point function

$$\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_{k\nu_1\dots\nu_\ell}(x_3)\rangle = \sum_{k'}[C_{ijk'}(x_{12},\partial_2)]_{\mu_1\dots\mu_\ell}\,\langle\mathcal{O}_{k'}^{\mu_1\dots\mu_\ell}(x_2)\,\mathcal{O}_{k\nu_1\dots\nu_\ell}(x_3)\rangle\,. \quad (4.33)$$

as long as $|x_{23}|>|x_{12}|$ so that the OPE is valid. As we saw, two-point functions are non-zero as long as the operators have the same spin and dimension Δ. Three-point functions are also fixed by conformal invariance:

$$\frac{f_{ijk}\,(Z_{\nu_1}\cdots Z_{\nu_\ell}-\text{traces})}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}} = \frac{[C_{ijk}(x_{12},\partial_2)]_{\mu_1\dots\mu_\ell}}{x_{23}^{2\Delta_k}}c_{\mathcal{O}}I^{(\mu_1}{}_{\nu_1}(x_{23})\cdots I^{\mu_\ell)}{}_{\nu_\ell}(x_{23})\,, \quad (4.34)$$

where $c_{\mathcal{O}}$ is the coefficient in the two-point function and we have used expressions (2.104) and (2.106). This enables us to see that C_{ijk} has to be proportional to $\frac{f_{ijk}}{c_{\mathcal{O}}}$ times a differential operator. This operator can, in fact, be obtained by matching in the $x_1\to x_2$ limit expansion of

[2] Depending on the configuration of operator insertions, this procedure would not work for other manifolds that are not $\mathbb{R}^d$, which is an additional complication that arises when studying CFT at nonzero temperature (see chapter 6).

both sides of (4.34). For instance, for the simplest case in which we have only scalar operators and $\Delta_i = \Delta_j = \Delta_\phi$, $\Delta_k = \Delta$ we find that[2]

$$C_{ijk}(x,\partial) = \frac{f_{ijk}}{c_{\mathcal{O}}}\, x^{\Delta-2\Delta_\phi}\Big(1+\frac{1}{2}x\cdot\partial+\alpha x^\mu x^\nu \partial_\mu\partial_\nu+\beta x^2\partial^2+\dots\Big)\,, \tag{4.35}$$

with

$$\alpha = \frac{\Delta+2}{8(\Delta+1)}\,, \qquad \beta = -\frac{\Delta}{16(\Delta-\frac{(d-2)}{2})(\Delta+1)}\,. \tag{4.36}$$

Of course, when considering spin the expressions become more complicated. For the construction of the operator $C_{ijk}(x,\partial)$ when the exchanged operator has $\ell = 1, 2$, see, for instance, [19, Appendix A].

In summary, we have seen that the conformal invariance restricts the OPE in (4.26) to have the following form:

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \frac{f_{ijk}}{c_{\mathcal{O}}}|x|^{\Delta_k-\Delta_i-\Delta_j-\ell}\left(x_{\mu_1}\cdots x_{\mu_\ell}\mathcal{O}_k^{\mu_1\dots\mu_\ell}(0)+\cdots\right). \tag{4.37}$$

Note that here we have considered only one kind of term. In general, there are contributions of a scalar operator $\mathcal{O}_k$ and primary operators of all spins.

5| Basics of AdS

In the first part of this work we have explored the basics about Conformal Field Theories. One of the reasons why it is interesting to study CFT's is, among others, that there is a duality between this type of QFT's and gravity in *Anti-de Sitter space* (AdS), known as *AdS/CFT correspondence*, *holographic duality* or *gauge/gravity correspondence*. More precisely, this duality relates dynamics of gravity in $(d+1)$-dimensions, usually refereed as the *bulk spacetime* to strongly correlated quantum matter in d-dimensions, at the *boundary* of the spacetime. For a pedagogical motivation of this correspondence from the point of view of the Kadanoff-Wilson renormalization group approach in lattice systems see [20]. In this chapter, we are going to briefly review the basic features of Anti-de Sitter geometry to quickly move to consider QFT on the AdS background. Our goal here is to describe the different kind of propagators in AdS between scalar operators so that we can use them in future chapters. Most of the material in this chapter is based on [13] and [20].

5.1 Anti-de Sitter Spacetime

Relating a general QFT to a given geometry can be a very complex problem. For the case of CFT's this turns out to be easier. Indeed, let us consider a CFT in d-dimensions. The most general metric in $(d+1)$-dimensions[1] we can write down that is rotationally-invariant in d-dimensions is

$$ds^2 = \kappa^2(z)\Big(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu\Big)\,, \tag{5.1}$$

where $x^\mu = (x^0, \cdots x^{d-1})$, z is the coordinate of the extra dimension and $\kappa(z)$ is a function to be determined. If z is to be a scale length, then, since we expect the theory to be conformal invariant, the metric should be invariant under the transformation

$$x^\mu \to \lambda x^\mu, \qquad z \to \lambda z\,. \tag{5.2}$$

Invariance of the metric (5.1) under this transformation imposes the function $\kappa(z)$ to transform as

$$\kappa(z) \to \frac{1}{\lambda}\,\kappa(z)\,, \tag{5.3}$$

[1] The use of $(d+1)$ dimensions is just a convention in the context of AdS/CFT correspondence, so that the dual CFT is taken to have d spacetime dimensions.

so that it restricts the function to have the following form

$$\kappa(z) = \frac{R}{z}\,, \tag{5.4}$$

where R is a constant. We then have that the line element (5.1) reads as

$$ds^2 = \frac{R^2}{z^2}\,(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu)\,. \tag{5.5}$$

which is the metric for the Euclidean AdS space in $(d+1)$-dimensions that we will denote as $EAdS_{d+1}$. The global constant R is usually known as the *anti-de Sitter radius*. Note that AdS is conformally equivalent to $\mathbb{R}^+ \times \mathbb{R}^d$ where the time-like boundary, located at $z = 0$, is just $\mathbb{R}^d$. Notice also that the metric is singular at $z = 0$, so that we will have to introduce some kind of regularization procedure to define quantities in the AdS boundary, i.e. the fields that live in the CFT.

But, what is AdS_{d+1} spacetime? Anti-de Sitter spacetime is a maximally symmetric spacetime with negative curvature, solution of the equation of motion of a gravity action such as

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\,(-2\Lambda + \mathcal{R})\;, \tag{5.6}$$

where G_N is the Newton gravity constant, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ the Ricci scalar. We have that the cosmological constant and the scalar curvature equal to (see [20] for details):

$$\Lambda = -\frac{d(d-1)}{2R^2}, \qquad \mathcal{R} = -\frac{d(d+1)}{R^2}\,, \tag{5.7}$$

so that both the cosmological constant and the scalar curvature are negative. Moreover, by maximally symmetric, we mean that AdS_{d+1} has the maximal number of spacetime symmetries, i.e. $\frac{1}{2}(d+1)(d+2)$. One convenient way to see the symmetries of AdS spacetime is to use the embedding space formalism introduced in chapter 3. Let us, then, embed AdS spacetime as the solution of the hyperboloid in $\mathbb{R}^{d+1,1}$:

$$P_A\,P^A \equiv -P_0^2 + P_1^2 + \cdots + P_d^2 + P_{d+1}^2 = -R^2\,, \qquad P_0 > 0\,, \tag{5.8}$$

where we can define the *Poincaré coordinates* by

$$\begin{aligned} P_0 &= \frac{R^2 + x^2 + z^2}{2z}\,, \\ P_\mu &= R\,\frac{x^\mu}{z}\,, \\ P_{d+1} &= \frac{R^2 - x^2 - z^2}{2z}\,, \end{aligned} \tag{5.9}$$

where $x^\mu \in \mathbb{R}^d$ and $z > 0$[2], so that the metric reads as (5.5) in these coordinates. From the definition of (5.8), it is clear that $EAdS$ is invariant under $SO(d+1,1)$, whose dimension is $\frac{1}{2}(d+1)(d+2)$. Moreover, the generators are given by

$$J_{AB} = -i\Big(P_A\frac{\partial}{\partial P^B} - P_B\frac{\partial}{\partial P^A}\Big)\,. \tag{5.10}$$

[2]Note that z runs from 0 to ∞ so that P_0 has a fixed sign, which it is required by expression (5.8).

5.2 QFT in AdS for a free scalar field

The AdS/CFT duality relates fields in AdS to QFT operator sources on the boundary. Let us study the simplest case: we consider a free massive scalar field Φ in the AdS_{d+1} space with action

$$S_E = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left[g^{\alpha\beta} \partial_\alpha \Phi \, \partial_\beta \Phi + m^2 \Phi^2 \right] . \tag{5.11}$$

where we will use the greek letters $\{\alpha, \beta, \gamma\}$ to denote AdS_{d+1} indices. We can derive the equations of motion from the action (5.11) obtaining

$$\frac{1}{\sqrt{g}} \partial_\alpha \left(\sqrt{g} g^{\alpha\beta} \partial_\beta \Phi \right) - m^2 \Phi = 0 , \tag{5.12}$$

or, more explicitly, in Poincaré coordinates,

$$z^{d+1} \, \partial_z (\frac{1}{z^{d-1}} \, \partial_z \Phi) + z^2 \partial^2 \Phi - m^2 R^2 \Phi = 0 , \tag{5.13}$$

where R is the AdS-radius. We can solve the above equation by performing a Fourier decomposition as

$$\Phi(z, x^\mu) = \int \frac{d^d k}{(2\pi)^d} \, e^{ik \cdot x} \, f_k(z) . \tag{5.14}$$

Equation (5.13) becomes then

$$z^{d+1} \, \partial_z (\frac{1}{z^{d-1}} \partial_z f_k) - k^2 z^2 f_k - m^2 R^2 f_k = 0 . \tag{5.15}$$

To solve the above equation near the boundary $z = 0$ we make an ansatz such that $f_k \sim z^\Delta$. The leading terms then gives us the constraint

$$\Delta(\Delta - d) = m^2 R^2 , \tag{5.16}$$

which has the following solutions

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 \, R^2} , \tag{5.17}$$

where $\Delta_\pm \in \mathbb{R}$. Thus, we find that the modes $f_k(z)$ near $z \sim 0$ behave as

$$f_k(z) \approx A(k) z^{\Delta_-} + B(k) z^{\Delta_+} . \tag{5.18}$$

Note that the condition $\Delta_\pm \in \mathbb{R}$ restricts the allowed values for the mass m to be

$$m^2 \geq - \left(\frac{d}{2R} \right)^2 , \tag{5.19}$$

which is known as the *Breitenlohner-Freedman (BF) bound.* Notice that negative values of the mass in the bulk field are allowed since they do not imply that theory is unstable. Unlike flat space, the negative curvature of AdS compensates the unstability as long as the BF bound is satisfied.

Moreover, it can be shown that the action (5.11) integrated from $z = 0$ to a cut-off $z = \epsilon$ yields a finite result for $\Delta \geq d/2$, i.e. $\Delta = \Delta_+$ is a solution for all masses that satisfy the BF bound. However, the boundary term that results from integrating by parts (5.11) is only non-zero for $\Delta \leq d/2$, meaning that for $\Delta = \Delta_-$ we have a different action than we have for $\Delta = \Delta_+$. This makes the case $\Delta = \Delta_-$ more subtle. When one integrates this inequivalent action from $z = 0$ to a cut-off, it is found that it is finite for $\Delta \geq d/2 - 1$, i.e. for the range of masses [21]

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1\,. \tag{5.20}$$

The above upper limit corresponds to $\Delta = d/2 - 1$, which is the lower bound in Δ imposed by unitarity in CFTs. Therefore, from masses below the unitary bound, but above the BF bound, we can choose either Δ_+ or Δ_-, which means that we have two different bulk theories for the same CFT!

This fact where any holographic CFT admits two bulk duals is sometimes omitted from the literature, where it is usally assumed that only $\Delta = \Delta_+$ is admissible. For this case, note that the solution $z^{\Delta_+} \xrightarrow[z\to 0]{} 0$ represents the *normalizable* solution, where the bulk excitations of the field decay at the boundary. However, since $d - \Delta \leq 0$, this does not happen with the other solution, which is called a *non-normalizable* solution. By inverse Fourier transforming, we can get then the behaviour of the scalar field ϕ near the boundary in position space:

$$\Phi(z, x) = [A(x) + O(z^2)]z^{d-\Delta} + [B(x) + O(z^2)]z^{\Delta}\,. \tag{5.21}$$

As we noticed, $d - \Delta \leq \Delta$, so that the dominant term at the boundary $z = \epsilon$ is given by

$$\Phi(z = \epsilon, x) \approx \epsilon^{d-\Delta} A(x)\,. \tag{5.22}$$

Since $d - \Delta$ is negative if $m^2 > 0$, we see that the leading term is typically divergent as we approach the boundary $z = \epsilon \to 0$. Here is when the correspondence between the gravity dynamics and the QFT comes: we can identify the QFT source $\varphi(x)$ (defined in (5.23)) as the value of the bulk field Φ at the boundary once we remove its divergences, i.e. the QFT source $\varphi(x)$ can be identified with $A(x)$ so that

$$\varphi(x) = \lim_{z\to 0} z^{\Delta-d}\,\Phi(z, x)\,. \tag{5.23}$$

or, equivalently, at leading order, $\Phi(z, x) = z^{d-\Delta}\varphi(x)$. We see, then, that the source φ of an operator in a QFT is the boundary value of the bulk field Φ at leading order of the non-normalizable solution.

To be more explicit about the AdS/CFT correspondence, let us consider $\mathcal{S}(\Phi(P))$ as the action of the bulk field, i.e. the field living in AdS, with an associate generating functional $Z_{\text{AdS}} = \int [d\Phi]\, e^{-\mathcal{S}(\Phi(P))}$. The duality, in a nutshell, then, states that

$$Z_{\text{AdS}}\big|_{\{\Phi = z^{d-\Delta}\varphi\}} = \left\langle \exp\Big(\int dx^d \varphi(x)\phi(x) \Big) \right\rangle_{\text{field theory}}, \tag{5.24}$$

so that correlation functions are obtained by computing repeated derivatives with respect to the source

$$\frac{\partial}{\partial \varphi(x_1)} \cdots \frac{\partial}{\partial \varphi(x_n)} Z_{\text{AdS}}|_{\{\Phi = z^{d-\Delta}\varphi\}} = \langle \phi(x_1) \cdots \phi(x_n) \rangle_{\text{field theory}} \,. \tag{5.25}$$

Now, if ϕ is the operator dual to Φ, the action in the boundary is given by

$$S_{\text{boundary}} \sim \int d^d x \, \sqrt{\gamma_\epsilon}\, \Phi(\epsilon, x)\, \phi(\epsilon, x)\,, \tag{5.26}$$

where $\gamma_\epsilon = \left(\frac{R}{\epsilon}\right)^{2d}$ is the determinant of the induced metric at the boundary $z = \epsilon$. Having identified the relation between the source φ and the bulk field Φ, in (5.23), we can then write

$$S_{\text{boundary}} \sim R^d \int d^d x \;\varphi(x) \epsilon^{-\Delta}\, \phi(\epsilon, x)\,. \tag{5.27}$$

Note that in order for the action in the boundary to be finite and independent of ϵ as $\epsilon \to 0$ we should require that

$$\phi(z = \epsilon, x) = \epsilon^{\Delta} \phi(z = 0, x)\,. \tag{5.28}$$

In terms of the QFT this is nothing but a scale transformation, thus Δ is interpreted as the mass scaling dimension of the dual operator ϕ. Also, from relation $\Phi(\epsilon, x) = \epsilon^{d-\Delta}\varphi(x)$ we see tat $d - \Delta$ is the mass scaling dimension of the source φ.

Finally, with regard to the normalizable modes, we have seen that they correspond to the bulk excitations at the boundary. It can be argued that they determine the vacuum expectation value of the operators with scaling dimension Δ in the CFT (see [21] for some clarifications). Explicitly, using embedding coordinates introduced in chapter 3, we can write[3]:

$$\phi(X) = C_\Delta^{-1/2} \lim_{z \to 0} z^{-\Delta}\, \Phi(P)\,, \tag{5.29}$$

where we have made explicit the distinction between the coordinates P in the bulk space given by (5.9) and the coordinates X at the boundary $z = 0$, which can be related through the limit

$$X^A = \lim_{z \to 0} z\, P^A\,. \tag{5.30}$$

Finally, let us mention that for the range of masses (5.20), choosing the alternative boundary condition $\Delta = \Delta_-$ interchanges the role of the coefficients in the expansion (5.21).

5.3 AdS propagators for scalar fields

From the generators in (5.10) we have that the action of the quadratic Casimir on a scalar field is given by

$$\frac{1}{2} J_{AB} J^{BA} \Phi = [-P^2 \partial_P^2 + P \cdot \partial_P \,(d + P \cdot \partial_P)]\Phi\,. \tag{5.31}$$

[3]The purpose of the extra factor $C_\Delta^{-1/2}$ is to recover the usual form for the two-point functions in the CFT, as we shall soon see.

Using that the Laplacian in the embedding space can be written as [13]

$$\partial_P^2 = -\frac{1}{R^{d+1}}\frac{\partial}{\partial R}R^{d+1}\frac{\partial}{\partial R} + \nabla^2_{AdS}\,. \tag{5.32}$$

Note that from the definition of the hypersurface $P^2 = -R^2$ we have that

$$\frac{\partial R}{\partial P} = -\frac{P}{R} \quad \rightarrow P\cdot\partial_P = P\cdot\frac{\partial R}{\partial P}\partial_R = -\frac{P^2}{R}\partial_R = R\,\partial_R\,. \tag{5.33}$$

Substituting (5.32) in (5.31),

$$\begin{aligned}
\frac{1}{2}J_{AB}J^{BA}\Phi &= \Big\{R^2\Big(-\frac{1}{R^{d+1}}\frac{\partial}{\partial R}R^{d+1}\frac{\partial}{\partial R} + \nabla^2_{AdS}\Big) + R\frac{\partial}{\partial R}(d + R\frac{\partial}{\partial R})\Big\}\Phi = \\
&= R^2\,\nabla^2_{AdS}\Phi - \frac{R^2}{R^{d+1}}\frac{\partial}{\partial R}\Big(R^{d+1}\frac{\partial\Phi}{\partial R}\Big) + d\,R\frac{\partial\Phi}{\partial R} + R\frac{\partial\Phi}{\partial R} + R^2\frac{\partial^2\Phi}{\partial R^2} = \\
&= R^2\,\nabla^2_{AdS}\Phi - (d+1)R\frac{\partial\Phi}{\partial R} - R^2\frac{\partial^2\Phi}{\partial R^2} + (d+1)\,R\frac{\partial\Phi}{\partial R} + R^2\frac{\partial^2\Phi}{\partial R^2} = \\
&= R^2\,\nabla^2_{AdS}\Phi\,,
\end{aligned} \tag{5.34}$$

thus we have

$$\frac{1}{2}J_{AB}J^{BA}\Phi = R^2\,\nabla^2_{AdS}\Phi\,. \tag{5.35}$$

With this in mind, let us again consider a free scalar field with action

$$S = \frac{1}{2}\int_{AdS} dP\Big[(\nabla\Phi)^2 + m^2\Phi^2\Big]\,. \tag{5.36}$$

We know that the two-point function $\langle\Phi(P)\Phi(Q)\rangle$ obeys

$$[\nabla_P^2 - m^2]\,\langle\Phi(P)\Phi(Q)\rangle = -\delta(P,Q)\,, \tag{5.37}$$

and that, by the symmetry of the problem, can only depend on the product $P\cdot Q$ or, equivalently, on what is known as the *chordal distance* $u = \frac{(P-Q)^2}{R^2}$. To not carry the quantity R, we are going to assume, from now on, that $R = 1$ so that all lengths will be given in units of the AdS radius.

5.3.1 Bulk-to-bulk propagator

Let us try to solve equation (5.37). Using relations (5.31) and (5.35), we have that

$$\nabla_P^2\,\Phi(P) = [-P^2\partial_P^2 + P\cdot\partial_P\,(d + P\cdot\partial_P)]\Phi(P)\,. \tag{5.38}$$

Moreover, using that $P\cdot\partial_P = u\,\partial_u$ and $\partial_P^2 = 2(d+1)\partial_u + 4u\,\partial_u^2$, and substituting (5.16), we have that (5.37) can be rewritten as

$$\Big((u^2+4u)\partial_u^2 + [(d+1)(u+2)]\partial_u - \Delta(\Delta-d)\Big)\,\langle\Phi(P)\Phi(Q)\rangle = -\delta(u)\,, \tag{5.39}$$

which has the form of the hypergeometric differential equation

$$(u^2+4u)\frac{d^2\Pi}{du^2}+((a+b+1)u+4c)\frac{d\Pi}{du}+ab\,\Pi=0\,, \tag{5.40}$$

with $a=\Delta, b=d-\Delta$ and $c=\frac{d+1}{2}$. This differential equation has two well-known solution given by the hypergeometric functions $u^{-a}{}_2F_1(a,1+a-c,1+a-b;\frac{-4}{u})$ and $u^{-b}{}_2F_1(b,1+b-c,1+b-a;\frac{-4}{u})$, being the latter non-physical since it blows up at $u\to\infty$. Therefore, the bulk-to-bulk propagator is given via

$$\langle\Phi(P)\,\Phi(Q)\rangle=C_\Delta\,u^{-\Delta}\,{}_2F_1\left(\Delta,\Delta+\frac{1-d}{2},2\Delta-d+1;\frac{-4}{u}\right)\,, \tag{5.41}$$

where u is the chordal distance $u=(P-Q)^2=-2-2P\cdot Q$, and the normalization factor is given by

$$C_\Delta=\frac{\Gamma(\Delta)}{2\pi^{d/2}\,\Gamma(\Delta-\frac{d}{2}+1)}\,. \tag{5.42}$$

5.3.2 Bulk-to-boundary propagator

The bulk-to-boundary propagator is obtained by taking one of the fields to the boundary following the definition (5.29):

$$\begin{aligned}\langle\phi(X)\Phi(Q)\rangle&=C_\Delta^{-1/2}\lim_{\substack{z\to0\\u\to\infty}}z^{-\Delta}C_\Delta\,u^{-\Delta}\,{}_2F_1\left(\Delta,\Delta+\frac{1-d}{2},2\Delta-d+1;\frac{-4}{u}\right)\\&=\lim_{z\to0}\frac{C_\Delta^{1/2}}{(-2z-2zP\cdot Q)^\Delta}=\frac{C_\Delta^{1/2}}{(-2X\cdot Q)^\Delta}\,.\end{aligned} \tag{5.43}$$

5.3.3 Boundary-to-boundary propagator

By taking both points to the boundary we get the boundary-to-boundary propagator

$$\langle\phi(X)\phi(Y)\rangle\;=C_\Delta^{-1/2}\lim_{z\to0}z^{-\Delta}\frac{C_\Delta^{1/2}}{(-2X\cdot Q)^\Delta}=\lim_{z\to0}\frac{1}{(-2X\cdot zQ)^\Delta}=\frac{1}{(-2X\cdot Y)^\Delta}\,, \tag{5.44}$$

which is nothing else but the two-point function in embedding space we introduced in equation (3.16). Here we see that the extra factor $C_\Delta^{-1/2}$ in our definition in (5.29) was indeed convenient to recover the standard convention of the two-point functions in CFTs.

6 | CFT at finite temperature

In this chapter, we are going to go one step further in our study of CFTs and we are going to consider CFT at nonzero temperature. As we mentioned in the Introduction, CFT at finite temperature turned out to be a necessary and interesting tool to study multiple kind of systems. This, as we shall see, will imply some additional complexities, but it will eventually allow us to study finite temperature dynamics of interacting systems, which are of great interest both theoretically and experimentally. Most of the material here follows the notation and line of argumentation of [11].

6.1 Compactifying the Euclidean time

Before starting developing the formalism of QFT at finite temperature, let us first discuss a subtle point about what notion of temperature we will be dealing with. Consider first a d-dimensional (only *space* dimensions) classical statistical system. For instance, the 3d Ising model away from criticality describing the ferromagnets in real world, which is not a CFT. Here we have not a real-time, dynamical description but a statistical description in thermodynamic equilibrium at a temperature T. As we know, we can tune the temperature to a critical value, the *critical temperature* T_c in which second-order phase transitions will happen. The correlation functions of this model are then described by a CFT, such as the 3d Ising CFT model. The issue here is that as soon as you leave the critical point at $T = T_c$, the CFT is no longer valid to describe the system. Here, the notion of temperature is understood as a relevant operator which is tuned to get the CFT.

However, there is a second way in which CFTs appear. *Euclidean* d-dimensional CFTs can arise as Wick rotations of Lorentzian ($(d-1)+1$)-dimensional CFT (emerging as the low-energy limit of a generic QFT). This Lorentzian QFTs describe dynamics of *spatial* $(d-1)$-dimensional systems, such as the dynamics of a $(3+1)$-dimensional ferromagnets. Now, we can actually put this QFT at finite temperature $T = \frac{1}{\beta}$, meaning going from a dynamical description to a statistical description of thermal equilibrium. As we shall explain soon, this can be achieved by Wick rotating and *compactifying* the Euclidean time direction in a circle of length β. If the QFT was, in particular, a CFT, we get an Euclidean CFT with one compactified dimension. Note that, in contrast with the classical statistical systems, even if we perturb the temperature, the theory will still be described by a CFT. In this case then, the notion of temperature becomes a compactification scale.

Let us stress that the difference between the two cases is not about the CFT itself but about the system they describe, meaning that the same Euclidean CFT can describe the statistical field theory of the 3d Ising model or some Wick rotation. Finally, note that finite temperature CFTs are technically no longer CFTs since there is already a scale. However, we still refer to them as CFTs due to the fact that temperature is the only scale (i.e. they were conformal before).

Let us then start. In order to formalize QFT at nonzero temperature we can use, for simplicity, the *canonical ensemble* by assuming that the chemical potentials are zero. Consider a dynamical system characterized by a Hamiltonian $\mathcal{H}$. The statistical properties of the QFT at finite T can be extracted from the partition function

$$\mathcal{Z} = \mathrm{tr}\left(e^{-\mathcal{H}/T}\right) = \mathrm{tr}\left(e^{-\beta\mathcal{H}}\right) = \sum_n \langle n|e^{-\beta\mathcal{H}}|n\rangle \,, \tag{6.1}$$

where the trace is over the Hilbert space living on $\mathbb{R}^{d-1}$ if the CFT is in d-dimensions. Regarding the thermal correlators, they are calculated as the ensemble average running over the Boltzmann weight factor as

$$\langle\mathcal{O}\rangle_\beta = \frac{1}{\mathcal{Z}}\sum_n \langle n|\mathcal{O}|n\rangle e^{-\beta\mathcal{H}} = \frac{1}{\mathcal{Z}}\,\mathrm{tr}\left(e^{-\beta\mathcal{H}}\mathcal{O}\right) . \tag{6.2}$$

Ultimately, we want to construct a path integral formulation of correlation functions. The path integral picture works, as previously mentioned, by going to Euclidean signature and making our time periodic, i.e. computing the path integral on $S^1 \times \mathbb{R}^{d-1}$, where β is the circumference of circle S^1. This is equivalent to compute the traces in (6.2). For a derivation and proof in detail of this result see Section 4.2 in [22]. It is worth to mention to not confuse the manifold $\mathcal{M}_\beta = S^1 \times \mathbb{R}^{d-1}$, in which we compute our path integrals in thermal CFT, with the manifold $\mathbb{R} \times S^{d-1}$ we introduced when talking about radial quantization. Notice that the latter is conformally equivalent to flat space and, therefore, conformal invariance is not broken, unlike for $\mathcal{M}_\beta$.

The path integral in thermal CFT is the same as the one in Euclidean CFT in flat space, except that now we have a different manifold. In particular, we will have the same set of local operators but some things will change. First, note that we have introduced a dimensionful parameter in the theory, thus scale invariance is no longer present, although we can still relate results on circles of different radius (different temperatures). Without scale invariance it means that some operators $\mathcal{O}$ with non-zero scaling dimension $\Delta \neq 0$ can get vacuum expectation values, i.e.

$$\langle\mathcal{O}\rangle_\beta \neq 0 \qquad \text{for } \Delta \neq 0\,, \tag{6.3}$$

where the notation $\langle\cdots\rangle_\beta$ denotes a correlator in $\mathcal{M}_\beta$. Moreover, rotation invariance is also broken, having now a preferred direction, i.e. the periodic time direction. This implies that, besides scalars, also operators with spin will get vacuum expectation values.

Finally, the OPE procedure of two operators $\mathcal{O}_i\mathcal{O}_j$ described in section 4.4 may not always be possible in the geometry of $\mathcal{M}_\beta$. For that, we would need to find a sphere with flat interior containing only $\mathcal{O}_i$ and $\mathcal{O}_j$, which it may be impossible.

6.2 Low-point functions at nonzero temperature

For studying CFT on the manifold $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$, we will use coordinates $x = (\tau, \mathbf{x})$, where τ is periodic $\tau \in [0, \beta)$ and $\mathbf{x} \in \mathbb{R}^{d-1}$. As in nonzero temperature, let us see how the symmetries we have in $\mathcal{M}_\beta$ constrain the low-point functions. Regarding one-point functions, note that translation-invariance implies that

$$\langle P^\mu \mathcal{O}(x)\rangle_\beta = \partial^\mu \langle \mathcal{O}(x)\rangle_\beta = \partial^\mu \langle \mathcal{O}(0)\rangle_\beta = 0\,, \tag{6.4}$$

i.e descendant operators have vanishing one-point functions. However, an interesting difference in comparison to flat-space is that one-point functions on the manifold $\mathcal{M}_\beta$ can acquire non-zero values for some kind of operators, which are restricted by the symmetries of the theory. Note that, even if we have lost rotational invariance in $SO(d)$, we still have invariance under $SO(d-1)$ in $\mathcal{M}_\beta$. If we have a parity-invariant theory, this residual symmetry, together with the discrete symmetry $\tau \to -\tau$, would combine into the $\mathbb{Z}_2 \times O(d-1)$ symmetry group. In general, however, we can expect our CFT not to be parity-invariant. In that case, we have a symmetry when the discrete symmetry $\tau \to -\tau$ is performed together with a reflection in one of the directions of $\mathbb{R}^{d-1}$. The symmetry group of the theory is then $O(d-1) \subset SO(d)$, where a reflection on $O(d-1)$ also affects the sign of τ. To have non-vanishing one-point functions, the restriction of representation of $SO(d)$ to a representation of $O(d-1) \subset SO(d)$ must contain the identity, which only happens for symmetric traceless tensors (STT) $\mathcal{O}^{\mu_1\cdots\mu_\ell}(x)$ with even spin ℓ [11]. Of course, the one-point functions can only depend on our preferred direction $e^\mu = (1,0,\dots)$, the unit vector in the τ-direction, and the scale introduced in our theory $T = \frac{1}{\beta}$. Thus, by symmetry and dimensional analysis, we have that

$$\langle \mathcal{O}^{\mu_1\cdots\mu_\ell}(x)\rangle_\beta = \frac{b_\mathcal{O}}{\beta^\Delta}(e^{\mu_1}\cdots e^{\mu_\ell} - \text{traces})\,, \tag{6.5}$$

where, from now on, the normalization of the correlators by the partition function $\mathcal{Z}(\beta)$ is implicitly understood. Here Δ is the dimension of $\mathcal{O}^{\mu_1\cdots\mu_\ell}(x)$ and $b_\mathcal{O}$ is a dynamical constant that depends on the theory we are considering.

Let us now study two-point functions. Another difference with respect $\mathbb{R}^d$ is that, in $\mathcal{M}_\beta$, two-point functions of non-identical operators may be nonvanishing. However, for simplicity, let us restrict anyway to two-point functions of identical operators:

$$g(\tau,\mathbf{x}) \equiv \langle \phi(x)\,\phi(0)\rangle_\beta\,. \tag{6.6}$$

As we mentioned already, if we want to perform the OPE we need to find a sphere whose interior is flat containing both operators. In our geometry, the largest possible sphere has diameter β, wrapping entirely around S^1 and being tangent to itself. Therefore, assuming

$$|x| = \sqrt{\tau^2 + \mathbf{x}^2} < \beta\,, \tag{6.7}$$

we can apply the OPE to find

$$g(\tau,\mathbf{x}) = \sum_{\mathcal{O}\in\phi\times\phi} \frac{f_{\phi\phi\mathcal{O}}}{c_\mathcal{O}} |x|^{\Delta-2\Delta_\phi-\ell}\, x_{\mu_1}\cdots x_{\mu_\ell}\langle \mathcal{O}^{\mu_1\cdots\mu_\ell}(0)\rangle_\beta\,, \tag{6.8}$$

where $\mathcal{O}$ runs over primary operators appearing in the $\phi\times\phi$ OPE, $c_\mathcal{O}$ is the two-point function coefficient appearing in (2.104) and $f_{\phi\phi\mathcal{O}}$ is the three-point function coefficient as in (2.106). One may find this expression familiar since we already derived the structure of such kind of terms in (4.37). Note, however, that here descendants have vanishing one-point functions, so we only need the leading (non-derivative) terms in the OPE.

Substituting (6.5) into (6.8) we find that

$$g(\tau,\mathbf{x}) = \sum_{\mathcal{O}\in\phi\times\phi} \frac{f_{\phi\phi\mathcal{O}}\, b_{\mathcal{O}}}{c_{\mathcal{O}}\,\beta^{\Delta}} |x|^{\Delta-2\Delta_\phi}\, \frac{x_{\mu_1}\cdots x_{\mu_\ell}}{|x|^\ell} (e^{\mu_1}\cdots e^{\mu_\ell} - \text{traces})\,. \tag{6.9}$$

We are left, then, with an interesting exercise to compute the following contraction

$$\frac{x_{\mu_1}\cdots x_{\mu_\ell}}{|x|^\ell}\,(e^{\mu_1}\cdots e^{\mu_\ell} - \text{traces})\,. \tag{6.10}$$

To derive the result more generally, let us consider $x, y \in \mathbb{R}^d$ to be unit vectors, i.e. $|x| = |y| = 1$, so that we want to compute the contraction

$$(x_{i_1}\cdots x_{i_\ell})\,(y^{i_1}\cdots y^{i_\ell} - \text{traces})\,. \tag{6.11}$$

For that purpose, we may use some of the ideas developed in 3.3.1, where we introduced a formalism to encode symmetric and traceless tensor in polynomials so that we are spared dealing with indices. First, note that, from definition (3.58),

$$y(h) \equiv (y_{i_1}\cdots y_{i_\ell} - \text{traces})\, h^{i_1}\cdots h^{i_\ell} = (h\cdot y)^\ell\,, \tag{6.12}$$

where we have used h as a reference null vector, $h^2 = 0$. Now, define the quantity $C(\ell)$ to be

$$C(\ell) \equiv (x\cdot\mathcal{D})^\ell (h\cdot y)^\ell = x^{i_1}\cdots x^{i_\ell}\,\mathcal{D}_{i_1}\cdots\mathcal{D}_{i_\ell}\, y(h)\,, \tag{6.13}$$

where $\mathcal{D}_i$ is the Todorov operator defined in (3.60). On one hand, using (3.61) and defining $\nu \equiv \frac{d}{2} - 1$, we see that

$$C(\ell) = \ell!\,(\nu)_\ell\,(x^{i_1}\cdots x^{i_\ell})\,(y_{i_1}\cdots y_{i_\ell} - \text{traces})\,, \tag{6.14}$$

where recall that $(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)}$ is the Pochhammer symbol. On the other hand, as it is shown in detail in A.1.1, it can be proved that

$$C(\ell) = \frac{(\ell!)^2}{2^\ell}\,\mathcal{C}_\ell^{(\nu)}(x\cdot y)\,, \tag{6.15}$$

where $\mathcal{C}_\ell^{(\nu)}(z)$ are orthogonal polynomials on the interval $z\in[-1,1]$ with respect the weight function $(1-z^2)^{(\nu-\frac{1}{2})}$ that are known as *Gegenbauer Polynomials*. Using both results (6.14) and (6.15) we then find that

$$x^{i_1}\cdots x^{i_\ell}\,(y_{i_1}\cdots y_{i_\ell} - \text{traces}) = \frac{\ell!}{2^\ell\,(\nu)_\ell}\,\mathcal{C}_\ell^{(\nu)}(x\cdot y)\,. \tag{6.16}$$

If we apply result (6.16) to (6.10) we have that

$$\left(\frac{x_{\mu_1}}{|x|}\cdots\frac{x_{\mu_\ell}}{|x|}\right)(e^{\mu_1}\cdots e^{\mu_\ell} - \text{traces}) = \frac{\ell!}{2^\ell\,(\nu)_\ell}\,\mathcal{C}_\ell^{(\nu)}(\eta)\,, \qquad \text{where} \qquad \eta = \frac{x\cdot e}{|x|} = \frac{\tau}{|x|}\,. \tag{6.17}$$

With result (6.17) we can finally write the expression for the thermal two-point function between two identical scalar operators as

$$g(\tau,\mathbf{x}) = \sum_{\mathcal{O}\in\phi\times\phi} \frac{a_{\mathcal{O}}}{\beta^{\Delta}} |x|^{\Delta-2\Delta_\phi}\, \mathcal{C}_\ell^{(\nu)}(\eta)\,, \qquad \text{with}$$
$$a_{\mathcal{O}} \equiv \frac{f_{\phi\phi\mathcal{O}}\, b_{\mathcal{O}}}{c_{\mathcal{O}}} \frac{\ell!}{2^\ell\,(\nu)_\ell}\,. \tag{6.18}$$

We call each kinematical factor $|x|^{\Delta-2\Delta_\phi}\, \mathcal{C}_\ell^{(\nu)}(\eta)$ in (6.18) a *thermal block*, and we will refer to the dynamical data $a_{\mathcal{O}}$ as *thermal coefficients*.

6.3 The KMS condition

Let us now take a small detour to review a fundamental relation in finite temperature theory. Consider a thermal two point function in Euclidean time $\langle\phi(\tau,\mathbf{x})\,\phi(0,\mathbf{y})\rangle_\beta$ with $\tau>0$. Then,

$$\begin{aligned}\langle\phi(\tau,\mathbf{x})\,\phi(0,\mathbf{y})\rangle_\beta &= \mathrm{tr}\Big(e^{-\beta\mathcal{H}}\, e^{\tau\mathcal{H}}\phi(0,\mathbf{x})e^{-\tau\mathcal{H}}\,\phi(0,\mathbf{y})\Big)\\ &= \mathrm{tr}\Big(e^{-(\beta-\tau)\mathcal{H}}\phi(0,\mathbf{x})e^{-\tau\mathcal{H}}\,\phi(0,\mathbf{y})\Big)\,,\end{aligned} \tag{6.19}$$

where $\mathcal{H}$ is the Hamiltonian. Notice that the convergence of the exponential factors implies not only $\tau>0$ but also $\tau<\beta$, so that the thermal two-point function is defined for $\tau\in(0,\beta)$. We can massage the above relationship a little more, such as

$$\begin{aligned}\langle\phi(\tau,\mathbf{x})\,\phi(0,\mathbf{y})\rangle_\beta &= \mathrm{tr}\Big(e^{-(\beta-\tau)\mathcal{H}}\phi(0,\mathbf{x})e^{-\beta\mathcal{H}}e^{(\beta-\tau)\mathcal{H}}\,\phi(0,\mathbf{y})\Big)\\ &= \mathrm{tr}\Big(e^{-\beta\mathcal{H}}e^{(\beta-\tau)\mathcal{H}}\,\phi(0,\mathbf{y})e^{-(\beta-\tau)\mathcal{H}}\phi(0,\mathbf{x})\Big)\\ &= \mathrm{tr}\Big(e^{-\beta\mathcal{H}}\,\phi(\beta-\tau,\mathbf{y})\,\phi(0,\mathbf{x})\Big)\\ &= \langle\phi(\beta-\tau,\mathbf{y})\phi(0,\mathbf{x})\,\rangle_\beta\,,\end{aligned} \tag{6.20}$$

where we have applied cyclicity property of the trace in the second step. This is known as the *KMS condition*, named after Kubo, Martin and Schwinger. Setting $\mathbf{y}=0$, we have that

$$\langle\phi(\tau,\mathbf{x})\,\phi(0,0)\rangle_\beta = \langle\phi(\beta-\tau,0)\phi(0,\mathbf{x})\,\rangle_\beta = \langle\phi(\beta-\tau,-\mathbf{x})\phi(0,0)\,\rangle_\beta\,, \tag{6.21}$$

by translation invariance. Thus, with notation from (6.6), we conclude that

$$g(\tau,\mathbf{x}) = g(\beta-\tau,-\mathbf{x})\,. \tag{6.22}$$

Moreover, recall we have a residual $SO(d-1)$ symmetry, so that the correlator will only depend on $|\mathbf{x}|$, so that it will be invariant under $\mathbf{x}\to-\mathbf{x}$. Thus, we can further see that the Euclidean KMS condition for the two-point function of identical scalar operators is given via

$$g(\tau,\mathbf{x}) = g(\beta-\tau,\mathbf{x})\,. \tag{6.23}$$

Actually, note that the fact that the correlator only depends on $|\mathbf{x}|$ can be seen in our construction of the thermal block decomposition in (6.18). However, relation (6.18) does not manifestly satisfy (6.23) since the thermal blocks are not invariant under thermal translations. If we impose the KMS condition, this gives us restrictions usually known as *thermal crossing equations* that constrain the thermal coefficients $a_{\mathcal{O}}$, relating the $b_{\mathcal{O}}$'s terms in terms of the OPE coefficients $f_{\phi\phi\mathcal{O}}$ and the dimensions $\Delta_{\mathcal{O}}$ (see [11] for more details).

6.4 Mean Field Theory

For simplicity, let us set $\beta = 1$ for what follows. Let us introduce a set of coordinates that will be useful in what follows. Taking advantage of the residual symmetry in $\mathbb{R}^{d-1}$, we can set $x^\mu = (\tau, |\mathbf{x}|, 0, \dots, 0)$ and define

$$z = \tau + i|\mathbf{x}|, \qquad \bar{z} = \tau - i|\mathbf{x}|\,. \tag{6.24}$$

The thermal two-point function becomes then a function of z and $\bar{z}$,

$$g(z,\bar{z}) = \langle \phi(z,\bar{z})\phi(0)\rangle_\beta\,. \tag{6.25}$$

There is an important and simple solution for the above correlator given in mean field theory (MFT). The $\phi \times \phi$ OPE of the MFT two-point function of scalar primaries ϕ is particularly simple, involving only the unit operator $\mathbb{1}$ and double-trace operators $[\phi\phi]_{n,\ell}$ of the form [23]

$$[\phi\phi]_{n,\ell} = \partial_{\alpha_1}\dots\partial_{\alpha_k}\partial_{\mu_1}\dots\partial_{\mu_m}(\partial^2)^{u_1}\phi\,\partial_{\alpha_{k+1}}\dots\partial_{\alpha_\ell}\partial^{\mu_1}\dots\partial^{\mu_m}(\partial^2)^{u_2}\phi - (\text{traces})\,, \tag{6.26}$$

with $m + u_1 + u_2 = n$ so that the operators have dimension $\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell$ and even spin ℓ. We can compute the thermal two-point function using the method of images

$$g(z,\bar{z})^{\text{MFT}} = \langle \phi(z,\bar{z})\phi(0)\rangle_\beta^{\text{MFT}} = \sum_{m=-\infty}^{\infty} \frac{1}{[(z+m)(\bar{z}+m)]^{\Delta_\phi}}\,. \tag{6.27}$$

To explicitly expand the thermal two-point function we can go back to $(\tau, \mathbf{x})$ coordinates so that we have, for $m \neq 0$,

$$\begin{aligned} \frac{1}{\Big[(z+m)(\bar{z}+m)\Big]^{\Delta_\phi}} &= \frac{1}{\Big[(m+\tau)^2 + |\mathbf{x}|^2\Big]^{\Delta_\phi}} = \frac{1}{|m|^{2\Delta_\phi}\Big[1 + \frac{2\tau m}{|m|^2} + \frac{|\mathbf{x}|^2}{|m|^2}\Big]^{\Delta_\phi}} = \\ &= \frac{1}{|m|^{2\Delta_\phi}\Big[1 - 2\frac{\tau}{|\mathbf{x}|}\Big(-\frac{|\mathbf{x}|}{|m|}\,\text{sgn}(m)\Big) + \frac{|\mathbf{x}|^2}{|m|^2}\Big]^{\Delta_\phi}} \\ &\equiv \frac{1}{|m|^{2\Delta_\phi}\Big[1 - 2\eta\omega + \omega^2\Big]^{\Delta_\phi}}\,, \end{aligned} \tag{6.28}$$

where $\text{sgn}(m) = \frac{m}{|m|}$, $\eta = \frac{\tau}{|x|}$ and we have identified $\omega \equiv -\frac{|\mathbf{x}|}{|m|}\,\text{sgn}(m)$. In this explicit form, we can directly apply identity (A.4) to (6.28) to find

$$\frac{1}{\Big[(z+m)(\bar{z}+m)\Big]^{\Delta_\phi}} = \sum_{j=0}^{\infty}(-1)^j \mathcal{C}_j^{(\Delta_\phi)}(\eta)\,\text{sgn}(m)^j \frac{|x|^j}{|m|^{2\Delta_\phi+j}}\,. \tag{6.29}$$

Therefore, the two-point functions is given via

$$g(\tau,\mathbf{x})=\frac{1}{|x|^{2\Delta_\phi}}+\sum_{j=0}^{\infty}(-1)^j\Big(\sum_{m\neq 0}\frac{\mathrm{sgn}(m)^j}{|m|^{2\Delta_\phi+j}}\Big)\mathcal{C}_j^{(\Delta_\phi)}(\eta)\,|x|^j\,. \tag{6.30}$$

Notice that for $j=$ odd the contributions for $m>0$ and $m<0$ in $\Big(\sum_{m\neq 0}\frac{\mathrm{sgn}(m)^j}{|m|^{2\Delta_\phi+j}}\Big)$ cancel each other, ending up with

$$g(\tau,\mathbf{x})=\frac{1}{|x|^{2\Delta_\phi}}+\sum_{j=0,2,\ldots}2\zeta(2\Delta_\phi+j)\,\mathcal{C}_j^{(\Delta_\phi)}(\eta)\,|x|^j\,, \tag{6.31}$$

where we have introduced $\zeta(s)$ as the *Riemann ζ-function*

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}\,. \tag{6.32}$$

Fortunately, the Gegenbauer polynomials $\mathcal{C}_j^{(\Delta_\phi)}(\eta)$ can be written in terms of the Gegenbauer polynomials $\mathcal{C}_j^{(\nu)}(\eta)$, which are the ones appearing in (6.18). They are related through the following expansion [11],

$$\mathcal{C}_j^{(\Delta_\phi)}(\eta)=\sum_{\ell=j,j-2,\ldots,j\,\mathrm{mod}\,2}\frac{(\ell+\nu)(\Delta_\phi)_{\frac{j+\ell}{2}}(\Delta_\phi-\nu)_{\frac{j-\ell}{2}}}{(\frac{j-\ell}{2})!(\nu)_{\frac{j+\ell+2}{2}}}\,\mathcal{C}_\ell^{(\nu)}(\eta)\,. \tag{6.33}$$

Inserting the above expansion into (6.31) and changing variables as $j=2n+\ell$ we get

$$g(\tau,\mathbf{x})=\frac{1}{|x|^{2\Delta_\phi}}+\sum_{n=0}^{\infty}\sum_{\ell=0,2\ldots}\frac{2\zeta(2\Delta_\phi+2n+\ell)(\ell+\nu)(\Delta_\phi)_{\ell+n}(\Delta_\phi-\nu)_n}{n!(\nu)_{\ell+n+1}}\,\mathcal{C}_\ell^{(\nu)}(\eta)|x|^{2n+\ell}\,. \tag{6.34}$$

Note that the above expansion has the structure of the thermal block decomposition in (6.18) including the unit operator (with $\Delta_{\mathbb{1}}=0$), and the double-trace operators (with $\Delta_{[\phi\phi]_{n,\ell}}=2\Delta_\phi+2n+\ell$) we mentioned before. Thus, we can identify the thermal coefficients

$$\begin{aligned} a_{\mathbb{1}}&=1\,,\\ a_{[\phi\phi]_{n,\ell}}&=\frac{2\zeta(2\Delta_\phi+2n+\ell)(\ell+\nu)(\Delta_\phi)_{\ell+n}(\Delta_\phi-\nu)_n}{n!(\nu)_{\ell+n+1}}\,. \end{aligned} \tag{6.35}$$

An analogous and *novel* computation for identifying the thermal coefficients in the case of a propagator for spin-one conserved currents is done in section 9.3.

6.5 Thermal AdS propagators for scalar fields

Let us finish this chapter giving the explicit expression for the propagators in thermal AdS between two identical scalar operators. In this case, the AdS/CFT correspondence implies that free fields propagating in thermal AdS are dual to MFT in the boundary. Indeed, consider a real scalar field $\phi(X)$ of dimension Δ and its dual bulk field $\Phi(P)$, where recall that coordinates in the bulk P and in the boundary X are related via (5.30).

6.5.1 Thermal bulk-to-bulk propagator

From the bulk-to-bulk propagator expression in AdS given in (5.41) and (5.42) we can construct the bulk-to-bulk in thermal AdS via the method of images (see [1] for a suggestion of a more general way to construct the propagator) as

$$\begin{aligned}\langle\Phi(P)\Phi(Q)\rangle_\beta &\equiv \sum_{m=-\infty}^{\infty}\langle\Phi(P_m)\Phi(Q)\rangle = \\ &= \sum_{m=-\infty}^{\infty}\frac{C_\Delta}{(P_m-Q)^\Delta}\,{}_2F_1\left(\Delta,\Delta+\frac{1-d}{2},2\Delta-d+1;\frac{-4}{u}\right)\end{aligned} \tag{6.36}$$

where P_m stands for the image of a point P after m thermal translations $\tau\to\tau+m$, where recall we have set $\beta=1$.

6.5.2 Thermal bulk-to-boundary propagator

Similarly, we can use expression (5.43) to directly see that the bulk-to-boundary propagator in thermal Ads is given via

$$\langle\phi(X)\Phi(Q)\rangle_\beta = \sum_{m=-\infty}^{\infty}\langle\phi(X_m)\Phi(Q)\rangle = \sum_{m=-\infty}^{\infty}\frac{C_\Delta^{1/2}}{(-2X_m\cdot Q)^\Delta} \tag{6.37}$$

6.5.3 Thermal boundary-to-boundary propagator

Finally, from (5.44), we have that the boundary-to-boundary propagator in thermal AdS reads as

$$\langle\phi(X)\phi(Y)\rangle_\beta = \sum_{m=-\infty}^{\infty}\frac{1}{(-2X_m\cdot Y)^\Delta} \tag{6.38}$$

7| Holographic solution from local quartic interaction in the bulk

In this chapter, we review the work published in [1]. We fill in some intermediate calculations and give more details of the computations involved.

Here we consider a thermal QFT in AdS_{d+1} where a massive real scalar field Φ has a nontrivial quartic interaction with the following Euclidean action

$$S = \int d^{d+1}x\sqrt{g}\Big(\frac{1}{2}\Phi(-\nabla^2+m^2)\Phi + \frac{\lambda}{4!}\Phi^4\Big)\,. \tag{7.1}$$

We will work in lightcone embedding coordinates in Euclidean AdS

$$P^A = (P_+, P_-, P_i)\,, \tag{7.2}$$

where $P_\pm = P_0 \pm P_{d+1}$ and $i = 1, \dots, d$. We will make use of the Poincaré patch coordinates given in (5.9) so that

$$P^A = (\frac{1}{z}, \frac{z^2+x^2}{z}, \frac{x^\mu}{z})\,, \tag{7.3}$$

with $x^\mu = (\tau, \mathbf{x})$ and where we have set $R = 1$. Consequently, the boundary coordinates X^A given via the limit (5.30) will read as

$$X^A = \lim_{z\to 0} zP^A = (1, \tau^2+\mathbf{x}^2, \tau, \mathbf{x})\,. \tag{7.4}$$

One of the goals of the paper is to compute the leading correction to the boundary thermal two-point function (6.38) due to the presence of the $\lambda\Phi^4$ bulk interaction. The correction to the thermal two-point function is given by the thermal AdS Witten diagram

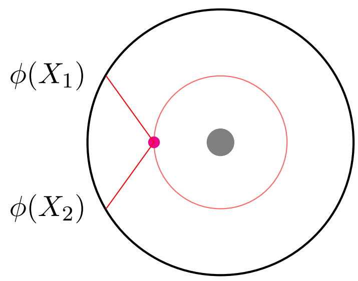

Figure 7.1: Thermal AdS Witten diagram for the leading correction to the boundary thermal two-point function $\langle\phi(X_1)\phi(X_2)\rangle_\beta|_\lambda$ with a $\lambda\Phi^4$ bulk interaction. The gray circle in the middle is just a reminder that the propagators (red lines) can wind around that direction since there is a periodic direction in the bulk.

which is evaluated by the integral

$$\langle\phi(X_1)\phi(X_2)\rangle_\beta|_\lambda = -\frac{1}{2}\int_{\mathrm{AdS}_\beta} d^{d+1}P\sqrt{g}\langle\phi(X_1)\Phi(P)\rangle_\beta\langle\Phi(P)\Phi(P)\rangle_\beta\langle\Phi(P)\phi(X_2)\rangle_\beta\,, \tag{7.5}$$

where the factor $\frac{1}{2}$ takes into account the symmetry of the thermal AdS Witten diagram. Writing out the propagators using (6.36) and (6.37) we have

$$\begin{aligned}\langle\phi(X_1)\phi(X_2)\rangle_\beta|_\lambda = -\frac{1}{2}\int_{\mathrm{AdS}_\beta} d^{d+1}P\sqrt{g}\sum_{m,n}\sum_{p\neq 0}\frac{C_\Delta^2}{(-2X_{1,m}\cdot P)^\Delta(-2X_{2,n}\cdot P)^\Delta}\\ \times u_p^{-\Delta}\,{}_2F_1\left(\Delta,\Delta+\frac{1-d}{2},2\Delta-d+1;\frac{-4}{u_p}\right)\,,\end{aligned} \tag{7.6}$$

where

$$u_p = (P_p - P)^2\,. \tag{7.7}$$

As a regularization scheme, note that the divergent term coming from the bulk-to-bulk propagator when the image number is $p=0$ in (7.6) is set to zero. To integrate the hypergeometric function ${}_2F_1(a,b,c;-x)$ it is useful to use the Barnes representation

$${}_2F_1(a,b,c;-x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_{s_0-i\infty}^{s_0-i\infty}\frac{ds}{2\pi i}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-2)}{\Gamma(c-s)}x^{-s}\,, \tag{7.8}$$

where $0 < s_0 < \min(\mathrm{Re}\,a,\,\mathrm{Re}\,b)$. In our case this means that the contour integral is at $0 < s_0 < \Delta - \frac{d-1}{2}$. We then can write

$$\begin{aligned}&\langle\phi(X_1)\phi(X_2)\rangle_\beta|_\lambda =\\ &= -\int_{s_0-i\infty}^{s_0-i\infty}\frac{ds}{2\pi i}\frac{\Gamma(2\Delta-d+1)\,C_\Delta^2}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+\frac{1}{2})}\frac{2^{-2s-1}\Gamma(s)\Gamma(\Delta-s)\Gamma(\Delta-\frac{d}{2}+\frac{1}{2}-s)}{\Gamma(2\Delta-d+1-s)}f_{s-\Delta,\Delta}(X_1,X_2)\end{aligned} \tag{7.9}$$

where we have defined

$$f_{a,\Delta}(X_1,X_2) \equiv \int_{AdS_\beta} d^{d+1}P\sqrt{g}\sum_{m,n}\sum_{p\neq 0}\frac{1}{(-2X_{1,m}\cdot P)^\Delta(-2X_{2,n}\cdot P)^\Delta}u_p^a\,. \tag{7.10}$$

To compute the integral (7.10) we use Poincaré patch coordinates (7.3) and boundary coordinates

$$X_i^A = (1,\tau_i^2+\mathbf{x}_i^2,\tau_i,\mathbf{x}_i)\,, \tag{7.11}$$

where $X_{i,m}$ denotes the boundary point after an i-thermal translation $\tau_i \to \tau_i + m$. With these coordinates, we have that

$$\begin{aligned}(P_p-P)^A &= (0,\frac{2\tau p+p^2}{z},\frac{p}{z},0)\,,\\ (P_p-P)_A &= \left(-\frac{1}{2}\left(\frac{2\tau p+p^2}{z}\right),0,\frac{p}{z},0\right)\,.\end{aligned} \tag{7.12}$$

where, since we are using light-cone coordinates, we have used the metric

$$ds^2 = \sum_{i=1}^{d} (dP^i)^2 - dP^+ dP^- \,, \tag{7.13}$$

to lower vector Lorentz indices. From (7.12) we then conclude that

$$u_p = (P_p - P)^2 = \frac{p^2}{z^2} \,. \tag{7.14}$$

Similarly, we have

$$\begin{aligned} P_A &= \left(-\frac{1}{2}\left(\frac{z^2+\tau^2+\mathbf{x}^2}{z}\right), -\frac{1}{2z}, \frac{\tau}{z}, \frac{\mathbf{x}}{z} \right) \,, \\ X_i^A &= (1, \tau_i^2 + \mathbf{x}_i^2, \tau_i, \mathbf{x}_i) \,, \end{aligned} \tag{7.15}$$

so that

$$-2X_i \cdot P = \frac{(\tau-\tau_i)^2 + (\mathbf{x}-\mathbf{x}_i)^2 + z^2}{z} \,. \tag{7.16}$$

Note that from the metric in Poincaré coordinates (5.5) we have that $\sqrt{g} = z^{-d-1}$. Moreover, using the identity

$$\frac{1}{A^\Delta} = \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds}{s}\, s^\Delta\, e^{-sA} \,, \tag{7.17}$$

we can write

$$\frac{1}{(-2X_i \cdot P)} = z^\Delta \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds_i}{s_i}\, s_i^\Delta\, e^{-s_i((\tau-\tau_i)^2 + (\mathbf{x}-\mathbf{x}_i)^2 + z^2)} \,. \tag{7.18}$$

Using results (7.14), (7.16) and (7.18), we can write relation (7.10) as

$$\begin{aligned} f_{a,\Delta} = \frac{2\zeta(-2a)}{\Gamma^2(\Delta)} \int_0^\infty ds_1\, s_1^{\Delta-1} \int_0^\infty ds_2\, s_2^{\Delta-1} \int_0^1 d\tau \sum_{m,n=-\infty}^{\infty} e^{-s_1(\tau-\tau_1-m)^2 - s_2(\tau-\tau_2-n)^2} \\ \times \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, e^{-s_1(\mathbf{x}-\mathbf{x}_1)^2 - s_2(\mathbf{x}-\mathbf{x}_2)^2} \int_0^\infty dz\, z^{2\Delta-2a-d-1}\, e^{-(s_1+s_2)z^2} \,. \end{aligned} \tag{7.19}$$

where we have performed the sum in p as

$$\sum_{p\neq 0} p^{2a} = \sum_{p=1}^{\infty} \frac{2}{p^{-2a}} = 2\zeta(-2a) \,, \tag{7.20}$$

with the Riemann ζ-function defined in (6.32).

On one hand, the $\mathbf{x}$ and z integrals are evaluated as follows:

$$\begin{aligned}\int_0^\infty dz\, z^{2\Delta-2a-d-1}\, e^{-(s_1+s_2)z^2} &= \frac{1}{2}\Gamma(\Delta-\frac{d}{2}-a)(s_1+s_2)^{a-\Delta+\frac{d}{2}}\,,\\ \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x}\, e^{-s_1(\mathbf{x}-\mathbf{x}_1)^2-s_2(\mathbf{x}-\mathbf{x}_2)^2} &= \frac{\pi^{\frac{d}{2}}}{(s_1+s_2)^{\frac{d}{2}}}\, e^{-\frac{s_1s_2}{s_1+s_2}(\mathbf{x}_1-\mathbf{x}_2)^2}\,.\end{aligned} \tag{7.21}$$

On the other hand, regarding the τ integrals, if we sum over the images first we find

$$\sum_{m=-\infty}^{\infty} e^{-s_1(\tau-\tau_1-m)^2} = e^{-s_1(\tau-\tau_1)^2}\sum_{m=-\infty}^{\infty} e^{-s_1m^2+2s_1(\tau-\tau_1)m} = e^{\frac{i\pi z^2}{\bar{\tau}}}\,\vartheta_3(z;\bar{\tau})\,, \tag{7.22}$$

where we have defined $\bar{\tau}=-\frac{s_1}{i\pi}$ and $z=-\frac{s_1}{i\pi}(\tau_1-\tau)$ and where the Jacobi theta function $\vartheta_3(z;\tau)$ is given by

$$\vartheta_3(z;\tau)=\sum_{m=-\infty}^{\infty} e^{2\pi i z m}\, e^{i\pi\tau m^2}\,. \tag{7.23}$$

A completely analogous result is found for the variable s_2. Using the modular s-transformation of the theta function ϑ_3

$$\vartheta_3(\frac{z}{\tau};-\frac{1}{\tau})=\sqrt{-i\tau}e^{\frac{i\pi z^2}{\tau}}\,\vartheta_3(z,\tau)\,, \tag{7.24}$$

we can express the sum over m,n in (7.19) as

$$\sum_{m,n=-\infty}^{\infty} e^{-s_1(\tau-\tau_1-m)^2-s_2(\tau-\tau_2-n)^2}=\vartheta_3\left(\tau_1-\tau;\frac{i\pi}{s_1}\right)\vartheta_3\left(\tau_2-\tau;\frac{i\pi}{s_2}\right)\frac{\pi}{\sqrt{s_1s_2}}\,. \tag{7.25}$$

The integral over τ is then evaluated as

$$\int_0^1 d\tau\,\vartheta_3\left(\tau_1-\tau;\frac{i\pi}{s_1}\right)\theta_3\left(\tau_2-\tau;\frac{i\pi}{s_2}\right)=\vartheta_3\left(\tau_1-\tau_2;i\pi\frac{(s_1+s_2)}{s_1s_2}\right)\,. \tag{7.26}$$

Applying a modular s-transformation again, and using the previous results, we have

$$\begin{aligned}f_{a,\Delta}(X_1,X_2)&=\frac{\zeta(-2a)\pi^{\frac{d}{2}}}{\Gamma^2(\Delta)\Gamma(\Delta-\frac{d}{2}-a)}\int_0^\infty ds_1\, s_1^{\Delta-1}\int_0^\infty ds_2\, s_2^{\Delta-1}\\ &\quad\times\sum_{m=-\infty}^{\infty} e^{-\frac{s_1s_2}{s_1+s_2}((\tau_1-\tau_2+m)^2+(\mathbf{x}_1-\mathbf{x}_2)^2)}(s_1+s_2)^{a-\Delta}\,.\end{aligned} \tag{7.27}$$

Finally, redefining $s_2\to s_1s_2$ we obtain

$$f_{a,\Delta}(X_1,X_2)=\frac{\pi^{\frac{d}{2}}\zeta(-2a)\Gamma(-a)^2\Gamma(-a+\Delta-\frac{d}{2})}{\Gamma(-2a)\Gamma(\Delta)^2}\,g(z,\bar{z})^{\text{MFT}}_{\Delta_\phi=a+\Delta}\,. \tag{7.28}$$

where $g(z,\bar{z})^{\text{MFT}}_{\Delta_\phi=a+\Delta}$ is the thermal two-point function defined in (6.27). Plugging this result for $f_{s-\Delta,\Delta}$ in (7.9) and applying the Legendre duplication formula

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(z+\frac{1}{2})\,. \tag{7.29}$$

to the following factors

$$\begin{aligned} \Gamma\big[2\big(\Delta - \frac{d}{2} + \frac{1}{2}\big)\big] &= \pi^{-\frac{1}{2}}\, 2^{2\Delta-1}\, \Gamma(\Delta - \frac{d}{2} + \frac{1}{2})\, \Gamma(\Delta - \frac{d}{2} + 1)\,, \\ \Gamma[2(\Delta - s)] &= \pi^{-\frac{1}{2}}\, 2^{2\Delta-2s-1} \Gamma(\Delta - s)\, \Gamma(\Delta - s + \frac{1}{2})\,, \end{aligned} \tag{7.30}$$

and simplifying the following expression as

$$\frac{2^{-2s-1}\, \Gamma(2\Delta - d + 1)\, \Gamma(\Delta - s)}{4\Gamma(\Delta - \frac{d}{2} + 1)^2\, \Gamma(\Delta - \frac{d}{2} + \frac{1}{2})\, \Gamma(2\Delta - 2s)} = \frac{2^{-d-2}}{\Gamma(\Delta - \frac{(d-2)}{2})\, \Gamma(\Delta + \frac{1}{2} - s)}\,, \tag{7.31}$$

we get the final answer

$$\langle \phi(X_1)\phi(X_2)\rangle_\beta|_\lambda = \int_{s_0-i\infty}^{s_0-i\infty} \frac{ds}{2\pi i} \Gamma(s)^2 \Gamma(\Delta-2)^2\, M_\beta(s)\, g(z,\bar{z})^{\mathrm{MFT}}_{\Delta_\phi = s}\,, \tag{7.32}$$

with

$$M_\beta(s) = -\frac{2^{-d-2}\, \pi^{-\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta - \frac{(d-2)}{2})} \frac{\zeta(2\Delta - 2s)\, \Gamma(\Delta - \frac{d-1}{2} - s)\, \Gamma(2\Delta - \frac{d}{2} - s)}{\Gamma(\Delta + \frac{1}{2} - s)\, \Gamma(2\Delta - (d-1) - s)}\,. \tag{7.33}$$

8| Spinning AdS propagators

In the previous chapters we have dealt with bulk scalar fields in AdS and thermal AdS. So far no spin was involved. Now, we are going to derive analogous results but for operators with spin. This, of course, will bring more difficulties. We are going to start in this chapter by explaining how to deal with tensors in AdS in the embedding formalism and encode them in polynomials. This will give us the right tools to work with operators with spin in AdS without worrying about indices. Finally, we will give the expression for the thermal AdS propagators for spin 1 fields. Most of the results presented here and the derivation in detail for the embedding formalism for AdS can be found in [24].

8.1 Embedding formalism for AdS

Let us consider tensor in Euclidean $(d+1)$-dimensional Anti de Sitter space AdS_{d+1}. As we defined in (5.8), AdS_{d+1} can be embedded as a solution of the hyperboloid $P^2 = -1$, $P^0 > 0$ (recall that we set $R = 1$) in $\mathbb{R}^{d+1,1}$. We want then to establish a relation between fields in AdS_{d+1} and fields in $\mathbb{R}^{d+1,1}$, as we did between tensors in $\mathbb{R}^d$ and $\mathbb{R}^{d+1,1}$ in section 3.2. Regarding notation, we will again use capital letters to denote embedding space indices in $\mathbb{R}^{d+1,1}$ while greek letters $\{\alpha, \beta, \gamma\}$ to denote AdS_{d+1} indices and greek letters $\{\mu, \nu, \rho\}$ to denote indices in the boundary $\mathbb{R}^d$.

Let us consider symmetric traceless tensors in $\mathbb{R}^{d+1,1}$ with components $G_{A_1 \dots A_\ell}(P)$ and transverse to the surface $P^2 = -1$, i.e.

$$P^{A_1} G_{A_1 \dots A_\ell}(P) = 0 \,. \tag{8.1}$$

The components of the tensor in AdS_{d+1} are then obtained by projecting as

$$g_{\alpha_1 \dots \alpha_\ell}(p) = \frac{\partial P^{A_1}}{\partial p^{\alpha_1}} \cdots \frac{\partial P^{A_\ell}}{\partial p^{\alpha_\ell}} G_{A_1 \dots A_\ell}(P) \,, \tag{8.2}$$

(c.f. (3.39)). Note that the embedding tensor $G(P)$ is defined away from the AdS_{d+1} surface $P^2 = -1$. All the components that are transverse to the hyperboloid are, of course, unphysical.

Now, let us proceed explaining how to encode these AdS_{d+1} tensor in polynomials, so that we have a more economical way to deal with them. The symmetric, traceless and transverse (STT) tensors $G_{A_1 \dots A_\ell}(P)$ defined on $P^2 = -1$ can be encoded by $(d+1)$-dimensional polynomials as

$$G_{A_1 \dots A_\ell}(P) \text{ STT} \to G(P, W) = G_{A_1 \dots A_\ell}(P)\, W^{A_1} \cdots W^{A_\ell}|_{W^2=0, W\cdot P=0} \,, \tag{8.3}$$

where the traceless and transverse condition allows us to restrict the polynomials to a submanifold satisfying $W^2 = 0$ and $P \cdot W = 0$, respectively (c.f.(3.66)).

To recover a tensor from a given polynomial we need an operator such that its action on a function only depends on the values of the function on our submanifold $P^2+1 = W^2 = W\cdot P = 0$. This operator is given by

$$
\begin{aligned}
K_A = &\frac{d-1}{2}\Big(\frac{\partial}{\partial W^A} + P_A\Big(P\cdot\frac{\partial}{\partial W}\Big)\Big) + \Big(W\cdot\frac{\partial}{\partial W}\Big)\frac{\partial}{\partial W^A} + \\
&+ P_A\Big(W\cdot\frac{\partial}{\partial W}\Big)\Big(P\cdot\frac{\partial}{\partial W}\Big) - \frac{1}{2}W_A\Big(\frac{\partial^2}{\partial W\cdot\partial W} + \Big(P\cdot\frac{\partial}{\partial W}\Big)\Big(P\cdot\frac{\partial}{\partial W}\Big)\Big)\,.
\end{aligned}
\tag{8.4}
$$

Note that the operator K_A is symmetric, traceless and transverse, i.e.

$$
K_A\,K_B = K_B\,K_A \qquad K_A\,K^A = 0, \qquad P^A\,K_A = 0\,, \tag{8.5}
$$

so that its action on any polynomial of W will give us a STT AdS tensor. Explicitly, we have that[24]

$$
G_{A_1\cdots A_\ell}(P) = \frac{1}{\ell!(\frac{d-1}{2})_\ell} K_{A_1}\cdots K_{A_\ell}\,G(P,W)\,, \tag{8.6}
$$

where $(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)}$ is the Pochhammer symbol.

8.2 AdS propagators for massless spin 1 fields

We are now interested in deriving some expressions for the AdS propagators of *massless* spin 1 fields. It is worthy to derive their expressions since we will use some of the results in the following chapter. It will also be an excellent opportunity to put into practice the encoding notation we have developed in the previous section. However, let us first mention some features of the correspondence between AdS and CFT in spin 1 fields. As we will see, massless particles in AdS correspond to conserved currents in CFT. Recall that conserved currents satisfy

$$
\partial_\mu J^{\mu\mu_2\ldots\mu_\ell} = 0 \quad \leftrightarrow \quad [P_\mu, J^{\mu\mu_2\ldots,\mu_\ell}] = 0\,. \tag{8.7}
$$

This has some consequences at the level of CFT states. The action of $J_{\mu_1\mu_2\ldots\mu_\ell}\,|0\rangle$ corresponds to a particle of spin ℓ in AdS in its ground state. If the current is conserved, i.e $\partial^\nu J_{\nu\mu_2\ldots\mu_\ell}\,|0\rangle = 0$, some of the descendants will vanish, meaning that a large number of states are removed from the Hilbert space of the theory. Recall also from the analysis in section 4.3 that a conserved current J_μ ($\ell = 1$) has a conformal dimension $\Delta_J = d-1$.

Let us start by considering a massive vector field $\mathcal{A}_\alpha$ in the bulk with action

$$
\int_{AdS} dP\sqrt{g}\,\Big[\frac{1}{2}(D_\alpha\mathcal{A}_\beta)^2 - \frac{1}{2}(D_\alpha\mathcal{A}^\alpha)^2 + \frac{1}{2}m^2\mathcal{A}_\alpha\mathcal{A}^\alpha\Big]\,, \tag{8.8}
$$

In the AdS/CFT correspondence for scalar fields, we found that the mass m of the bulk field Φ is given via $m^2 = \Delta(\Delta - d)$, where Δ is the scaling dimension of the dual scalar field ϕ in CFT. The relation between the mass and the conformal dimension between general bulk spin fields and their dual fields in four dimensions can be found, for instance, in [25]. In general, for a vector field $\mathcal{A}_\alpha$ in AdS_{d+1}, we find the following relation [26]

$$
m^2 = (\Delta-1)(\Delta-d+1)\,. \tag{8.9}
$$

We want to relate our bulk field $\mathcal{A}_\alpha$ with its dual field J_μ that lives in the CFT. For that, note that since the generating functional in AdS_{d+1} is given by $Z_{AdS} = \int [d\mathcal{A}_\alpha] e^{-\mathcal{S}(\mathcal{A}_\alpha(P))}$, the AdS/CFT correspondence for vector fields reads as

$$Z_{AdS}|_{\substack{\mathcal{A}_\mu = A_\mu^{\text{bg}} \\ \mathcal{A}_z = 0}} = \Big\langle \exp\Big(\int dx^d A_\mu^{\text{bg}}(x) J^\mu(x) \Big) \Big\rangle_{\text{field theory}} , \tag{8.10}$$

where we have gauge fixed the z-component of the bulk vector field $\mathcal{A}_\alpha$ to zero and treat the other components as a background gauge field $A_\mu^{\text{bg}}(x)$ in the CFT (note that this background field plays the role of the source $\varphi(x)$ in the case of scalar fields, c.f. (5.24)). However, even if we take the gauge $\mathcal{A}_z = 0$, we still have a residual gauge freedom: namely those transformations $A_\mu^{\text{bg}} \to A_\mu^{\text{bg}} + \partial_\mu \Lambda$ for scalar functions Λ that do not depend on the coordinate z. Then, since, in general, we expect that[1]

$$Z_{CFT}[A_\mu^{\text{bg}}(x)] = Z_{CFT}[A_\mu^{\text{bg}}(x) + \partial\Lambda(x)] , \tag{8.11}$$

one could integrate by parts the integral in the exponential (8.10), to see straightforwardly that, if the vector field $\mathcal{A}$ is massless, the residual gauge freedom of A_μ^{bg} implies that the dual operator J_μ is a *conserved current*, i.e. $\partial_\mu J^\mu = 0$. Finally, it can be shown ([18]) that the precise relation between the bulk and the boundary field is given via $J_\mu(x) = \lim_{z\to 0} z^{2-d} \mathcal{A}_\mu(z, x)$.

8.2.1 Bulk-to-bulk propagator

Once we have reviewed the basic features of the AdS/CFT correspondence for spin 1 fields $\mathcal{A}_\alpha$ it is time to derive the expressions for the different propagators between the bulk fields $\mathcal{A}_\alpha$ and the boundary fields J_μ. The task to study the propagators in Euclidean AdS_{d+1} for spin-1 and spin-2 fields was first studied by [27]. However, here we follow the notation of the later approach by [24]. To construct the bulk-to-bulk propagator of a field with spin ℓ between points P and Q, with polarization vectors W_1 and W_2 respectively, we need to consider polynomials of degree ℓ in both W_1 and W_2, constructed from the scalar products $P \cdot W_2$, $Q \cdot W_1$ and $W_1 \cdot W_2$. For spin $\ell = 1$ we can write

$$\langle \mathcal{A}(P, W_1) \mathcal{A}(Q, W_2) \rangle = W_{12}\, g_0(u) + (W_1 \cdot Q)(W_2 \cdot P)\, g_1(u) , \tag{8.12}$$

where $W_{12} = W_1 \cdot W_2$ and recall that u is the chordal distance $u = -2 - 2P \cdot Q$. To study the massless limit $\Delta \to d - 1$ it is convenient to note that the above expression can be rewritten as

$$\langle \mathcal{A}(P, W_1) \mathcal{A}(Q, W_2) \rangle = W_{12}\, G(u) + W_1 \cdot \nabla_1((W_2 \cdot P) L_1(u)) . \tag{8.13}$$

Comparing with expression (8.12) we have

$$g_0(u) = G(u) + L_1(u); \quad g_1(u) = -2L_1'(u) . \tag{8.14}$$

Note that, if the current is *conserved*, the function $L_1(u)$ in (8.13) does not contribute to physical processes since their contribution vanishes after integrating by parts. The function

[1] In general, this relation may not be satisfied for the background gauge field, making the current J^μ to cease to be conserved. This is known as a *'t Hooft anomaly*. In this case, the 't Hooft anomaly is present due to the coupling of a global symmetry to a background gauge field. More problematic would be if we were to couple the symmetry to a dynamical gauge field, since the 't Hooft anomaly would become a *gauge anomaly*, making the theory meaningless.

$G(u)$, which turns out to be the only physical degree of freedom, can be proved to be given via

$$G(u)=\frac{\Gamma(\Delta)}{2\pi^{d/2}\,\Gamma(\Delta-\frac{d}{2}+1)}(u)^{-\Delta}\;{}_2F_1\left(\Delta,\Delta+\frac{1-d}{2},2\Delta-d+1;\frac{-4}{u}\right)\,, \tag{8.15}$$

so that the bulk-to-bulk propagator for massless spin 1 fields $\mathcal{A}$ reads

$$\langle\mathcal{A}(P,W_1)\mathcal{A}(Q,W_2)\rangle=G(u)\;W_1\cdot W_2\,. \tag{8.16}$$

To recover the components of the propagator we just act with the projector (8.4)

$$\langle\mathcal{A}_A(P)\mathcal{A}_B(Q)\rangle=\frac{1}{(\frac{d-1}{2})^2}K_BK_A\;\langle\mathcal{A}(P,W_1)\mathcal{A}(Q,W_2)\rangle\,, \tag{8.17}$$

so that

$$\begin{aligned}
&\langle\mathcal{A}_A(P)\mathcal{A}_B(Q)\rangle=\\
&=\left(\frac{\partial}{\partial W_2^B}+Q_B\left(Q\cdot\frac{\partial}{\partial W_2}\right)\right)\left(\frac{\partial}{\partial W_1^A}+P_A\left(P\cdot\frac{\partial}{\partial W_1}\right)\right)G(u)W_1\cdot W_2=\\
&=\left(\frac{\partial}{\partial W_2^B}+Q_B\left(Q\cdot\frac{\partial}{\partial W_2}\right)\right)\;\left(W_2^D\eta_{AD}+P_A\,P^C\,W_2^D\;\eta_{CD}\right)G(u)=\\
&=\left(\eta_{AB}+P_AP_B+Q_AQ_B-(P\cdot Q)P_AQ_B\right)G(u)\,.
\end{aligned} \tag{8.18}$$

Therefore,

$$\langle\mathcal{A}_A(P)\mathcal{A}_B(Q)\rangle=\left(\eta_{AB}+P_AP_B+Q_AQ_B-(P\cdot Q)P_A\,Q_B\right)G(u)\,, \tag{8.19}$$

where $G(u)$ in the massless limit $\Delta\to d-1$ reads as

$$G(u)=\frac{\Gamma(d-1)}{2\pi^{d/2}\,\Gamma(\frac{d}{2})}(u)^{-d+1}\;{}_2F_1\left(d-1,\frac{d-1}{2},d-1;\frac{-4}{u}\right)\,. \tag{8.20}$$

8.2.2 Bulk-to-boundary propagator

The bulk-to-boundary propagator of a spin 1 and dimension Δ field is given by [24]

$$\langle J(X,H)\mathcal{A}(P,W)\rangle=\mathcal{C}_{\Delta,1}\;\frac{(-2X\cdot P)(W\cdot H)+2(W\cdot X)(H\cdot P)}{(-2X\cdot P)^{\Delta+1}}\,, \tag{8.21}$$

with

$$\mathcal{C}_{\Delta,1}=\frac{\Delta\;\Gamma(\Delta-1)}{2\pi^{d/2}\;\Gamma(\Delta+1-\frac{d}{2})}\,. \tag{8.22}$$

To recover the components of the propagator we act with both the operators (3.66) and (8.4)

$$\langle J_A(X)\mathcal{A}_B(P)\rangle=\left(\frac{\partial}{\partial H^A}\right)\left(\frac{\partial}{\partial W^B}+P_B\,(P\cdot\frac{\partial}{\partial W})\right)\langle J(X,H)\mathcal{A}(P,W)\rangle\,, \tag{8.23}$$

so that

$$\langle J_A(X)\mathcal{A}_B(P)\rangle = \frac{\mathcal{C}_{\Delta,1}}{(-2X\cdot P)^{\Delta+1}}\left(\frac{\partial}{\partial H^A}\right)[(-2X\cdot P)H_B + 2X_B(H\cdot P)+$$

$$(-2X\cdot P)P_B(P\cdot H) + 2P_B(P\cdot X)(H\cdot P)] = \tag{8.24}$$

$$= \frac{\mathcal{C}_{\Delta,1}}{(-2X\cdot P)^{\Delta+1}}[(-2X\cdot P)\eta_{AB} + 2P_A X_B]\,.$$

Therefore, the bulk-to boundary propagator is given by

$$\langle J_A(X)\mathcal{A}_B(P)\rangle = \mathcal{C}_{\Delta,1}\frac{(-2X\cdot P)\eta_{AB} + 2P_A X_B}{(-2X\cdot P)^{\Delta+1}}\,. \tag{8.25}$$

In the massless limit $\Delta \to d-1$ we have

$$\langle J_A(X)\mathcal{A}_B(P)\rangle = \frac{(d-1)\,\Gamma(d-2)}{2\pi^{d/2}\,\Gamma(\frac{d}{2})}\frac{(-2X\cdot P)\,\eta_{AB} + 2P_A X_B}{(-2X\cdot P)^d}\,. \tag{8.26}$$

8.2.3 Boundary-to-boundary propagator

Finally, the boundary-to-boundary propagator of a spin 1 and dimension Δ field reads as

$$\langle J(X,H)J(Y,H')\rangle = \mathcal{C}_{\Delta,1}\,\frac{(-2X\cdot Y)(H\cdot H') + 2(H\cdot Y)(X\cdot H')}{(-2X\cdot Y)^{\Delta+1}}\,, \tag{8.27}$$

or, written in components,

$$\langle J_A(X)J_B(Y)\rangle = \mathcal{C}_{\Delta,1}\frac{(-2X\cdot Y)\,\eta_{AB} + 2X_B Y_A}{(-2X\cdot Y)^{\Delta+1}}\,. \tag{8.28}$$

In the massless limit we have

$$\langle J_A(X)J_B(Y)\rangle = \frac{(d-1)\,\Gamma(d-2)}{2\pi^{d/2}\,\Gamma(\frac{d}{2})}\frac{(-2X\cdot Y)\,\eta_{AB} + 2Y_A X_B}{(-2X\cdot Y)^d}\,. \tag{8.29}$$

Note that the spacetime structure in (8.29) is the same as in (3.43) with $\Delta = d-1$, as we expected.

8.3 Thermal AdS propagators for massless spin 1 fields

Let us finnish this chapter by constructing the *thermal* AdS propagators for a massless spin 1 field $\mathcal{A}_\alpha$. We can easily contract the expressions for the propagators in *thermal* AdS from the previous AdS propagators by just applying the method of images, analogously as we did for the scalar propagators. We will refer to the normalization constant C_Δ in the massless limit where $\Delta \to d-1$ as

$$\mathcal{C}_{\Delta\to d-1} = \frac{(d-1)\,\Gamma(d-2)}{2\pi^{d/2}\,\Gamma(\frac{d}{2})}\,. \tag{8.30}$$

8.3.1 Bulk-to-bulk propagator

The bulk-to-bulk propagator in thermal AdS can be built from expression (8.19). Using the method of images we and defining P_m as the image of a point P after a shift of m (actually βm if we don't consider $\beta = 1$) in the Euclidean time coordinate τ, i.e $\tau \to \tau + m$, we find

$$\begin{aligned} \langle \mathcal{A}_A(P)\mathcal{A}_B(Q)\rangle_\beta &= \sum_{m=-\infty}^{\infty} \langle \mathcal{A}_A(P_m)\mathcal{A}_B(Q)\rangle = \\ &= \sum_{m=-\infty}^{\infty} \big(\eta_{AB} + (P_m)_A(P_m)_B + Q_A Q_B - (P_m\cdot Q)(P_m)_A\, Q_B\big)\, G(u_p)\,, \end{aligned} \tag{8.31}$$

with $G(u)$ given in (8.20).

8.3.2 Bulk-to-boundary propagator

Likewise, applying the method of images to (8.26), we get that the bulk-to boundary propagator in thermal AdS is given via

$$\begin{aligned} \langle J_A(X)\mathcal{A}_B(P)\rangle_\beta &= \sum_{m=-\infty}^{\infty} \langle J_A(X_m)\mathcal{A}_B(P)\rangle = \\ &= \sum_{m=-\infty}^{\infty} \mathcal{C}_{\Delta\to d-1} \frac{(-2X_m\cdot P)\,\eta_{AB} + 2P_A(X_m)_B}{(-2X_m\cdot P)^d}\,. \end{aligned} \tag{8.32}$$

However, it is actually more accurate to write this propagator in index-free notation. Indeed, from (8.21) we have

$$\begin{aligned} \langle J(X,H)\mathcal{A}(P,W)\rangle_\beta &= \sum_{m=-\infty}^{\infty} \langle J(X_m,H_m)\mathcal{A}(P,W)\rangle = \\ &= \sum_{m=-\infty}^{\infty} \mathcal{C}_{\Delta\to d-1} \frac{(-2X_m\cdot P)(W\cdot H_m) + 2(W\cdot X_m)(H_m\cdot P)}{(-2X_m\cdot P)^d}\,. \end{aligned} \tag{8.33}$$

Note that the polarization vector H, whose explicit connexion with physical coordinates was given in (3.67), also depends on x, and thus on τ. This means that the vector H gets also affected by the thermal shift, feature that was no transparent in expression (8.32).

8.3.3 Boundary-to-boundary propagator

Finally, for the boundary-to-boundary propagator in thermal AdS, from (8.27) we have

$$\begin{aligned} \langle J(X,H)J(Y,H')\rangle_\beta &= \sum_{m=-\infty}^{\infty} \langle J(X_m,H_m)J(Y,H')\rangle = \\ &= \sum_{m=-\infty}^{\infty} \mathcal{C}_{\Delta\to d-1} \frac{(-2X_m\cdot Y)(H_m\cdot H') + 2(H_m\cdot Y)(X_m\cdot H')}{(-2X_m\cdot Y)^d}\,. \end{aligned} \tag{8.34}$$

9| Thermal conformal blocks for spinning operators

We are now ready to derive the analogous expansion in thermal blocks for spin 1 operators as we did for scalar fields in (6.18). First of all, recall that, as we saw in section 4.4 for the case of scalar fields, we could obtain the differential operator C_{ijk} in the OPE expansion of two operators $\mathcal{O}_i\mathcal{O}_j$ by matching, in the $x_1 \to x_2$ limit, the three-point function $\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k^{(\ell)}\rangle$ with the same three-point function having the OPE expansion inserted in it (see (4.33) and (4.34)). We will apply the same idea for spin 1 operators J_μ and we will to obtain the OPE expansion of two operators J_μ from the three-point function $\langle JJ\mathcal{O}\rangle$, where $\mathcal{O}$ will be a spin ℓ operator . Next, we will expand in thermal conformal blocks as we did in (6.18). Finally, we will focus on mean field theory and will identify the thermal coefficients, analogous as in (6.35).

9.1 OPE from three-point functions in general dimensions

Before engaging into explicit calculations, let us briefly outline the main idea of how to extract an OPE from a three-point function. Let us consider an operator J_μ of spin 1, i.e. an operator in the symmetric traceless representation of rank 1 of $SO(d)$. The goal in this section is to compute the OPE $J \times J$ taking only into account symmetric traceless primaries.[1] In order to extract the primary contribution to the OPE we should keep the leading terms of the three-point function $\langle JJ\mathcal{O}^{(\ell)}\rangle$ in the limit $x_1 \to x_2$, where $\mathcal{O}^{(\ell)}$ is an operator in the spin ℓ representation.

On one hand, the three-point function in the $x_1 \to x_2$ limit is given as

$$\langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}^{(\ell)}(x_3,h_3)\rangle = \frac{1}{(x_{12}^2)^{\Delta_J-\frac{\Delta}{2}}(x_{23}^2)^{\Delta}}\sum_{i=1}^{n_{\rm str}}\lambda_i\,\mathcal{T}_i(x;h)\,, \tag{9.1}$$

where Δ_J and Δ are the conformal dimensions of the operators J and $\mathcal{O}$, respectively, λ_i are constants and $\mathcal{T}_i$ represents $n_{\rm str}$ tensor structures. Here we have adopted the free-index notation $\mathcal{O}(x,h) = h^{\mu_1}\cdots h^{\mu_\ell}\,\mathcal{O}_{\mu_1\cdots\mu_\ell}(x)$, where h is a null vector, that we introduced in (3.57). On the other hand, the primary contribution to the OPE would have the generic form

$$J(x_1,h_1)\times J(x_2,h_2) \;\ni\; \mathcal{C}_\ell(x_{12},h_1,h_2)\,\mathcal{O}^{(\ell)}(x_2,h)\,. \tag{9.2}$$

Then, the differential operator $\mathcal{C}_\ell$ can be determined by inserting this OPE inside the three-point function and comparing

$$\langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}^{(\ell)}(x_3,h_3)\rangle \;= \mathcal{C}_\ell(x_{12},h_1,h_2)\langle\mathcal{O}^{(\ell)}(x_2,h)\mathcal{O}^{(\ell)}(x_3,h_3)\rangle\,, \tag{9.3}$$

[1] This is because those are the only ones with non-vanishing thermal expectation value (see section 6.2).

i.e. finding the differential operator $\mathcal{C}_\ell$ that, acting on the two point function in (9.3), gives the three-point (9.1) function in the $x_1 \to x_2$ limit.

This procedure can be done explicitly in four dimensions using the formalism developed in [28], where it is shown that the three-point function of a vector-vector-tensor can be expressed in terms of building blocks I_{ij} and J^i_{jk}. For general d, since the aforementioned building blocks are only defined in four dimensions, we need to use the embedding formalism of [17][2]. In this case, we will have the following tensor structures

$$H_{ij} = x_{ij}^2 \left(h_i \cdot h_j - 2\frac{(h_i \cdot x_{ij})(h_j \cdot x_{ij})}{x_{ij}^2} \right) , \tag{9.4a}$$

$$V_{k,ij} = \frac{x_{ki}^2 x_{kj}^2}{x_{ij}^2} \left(\frac{h_k \cdot x_{ki}}{x_{ki}^2} - \frac{h_k \cdot x_{kj}}{x_{kj}^2} \right) . \tag{9.4b}$$

Note that $H_{ji} = H_{ij}$ and $V_{k,ji} = -V_{k,ij}$.

9.1.1 Three-point function of 2 conserved currents and spin ℓ operator

Let us then start doing the explicit calculation of the three-point function of two conserved currents J (then $\Delta_J = d-1$) with a spin ℓ operator $\mathcal{O}$ in this section. For simplicity, let us just assume that the theory is parity invariant so that we have a dimension-independent number of tensor structures. The tensor structures for vector-vector-tensor are[3]

$$\begin{aligned} \langle J(x_1,h_1)\, J(x_2,h_2)\, \mathcal{O}(x_3,h_3)\rangle = \frac{(V_{3,12})^{\ell-2}}{(x_{12}^2)^{\frac{1}{2}(2d-\Delta-\ell)}(x_{13}^2)^{\frac{1}{2}(\Delta+\ell)}(x_{23}^2)^{\frac{1}{2}(\Delta+\ell)}} \Big[\\ \lambda_1\, V_{1,23}V_{2,31}(V_{3,12})^2 + \\ \lambda_2\, H_{23}V_{1,23}V_{3,12} + \lambda_3\, H_{13}V_{2,31}V_{3,12} + \\ \lambda_4\, H_{13}H_{23} + \lambda_5\, H_{12}(V_{3,12})^2 \Big] . \end{aligned} \tag{9.5}$$

where recall the definition $\mathcal{O}(x,h) = h^{\mu_1}\cdots h^{\mu_\ell}\, \mathcal{O}_{\mu_1\cdots\mu_\ell}(x)$ and similarly for J. To work with these structures, let us define the following quantities

$$\begin{gathered} \langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}(x_3,h_3)\rangle = \mathcal{K}_\ell\, (\lambda_1\, \Lambda_1 + \lambda_2\, \Lambda_2 + \lambda_3\, \Lambda_3 + \lambda_4\, \Lambda_4 + \lambda_5\, \Lambda_5)\,; \\ \mathcal{K}_\ell = \frac{(V_{3,12})^{(\ell-2)}}{(x_{12}^2)^{\frac{1}{2}(2d-\Delta-\ell)}(x_{13}^2)^{\frac{1}{2}(\Delta+\ell)}(x_{23}^2)^{\frac{1}{2}(\Delta+\ell)}}\,, \quad \Lambda_1 = V_{1,23}V_{2,31}(V_{3,12})^2\,, \\ \Lambda_2 = H_{23}V_{1,23}V_{3,12}\,, \quad \Lambda_3 = H_{13}V_{2,31}V_{3,12}\,, \quad \Lambda_4 = H_{13}H_{23}\,, \quad \Lambda_5 = H_{12}(V_{3,12})^2\,. \end{gathered} \tag{9.6}$$

Conservation of the currents and permutation symmetry will impose some linear relations between the various coefficients λ_i. Indeed, first note that if we impose Bose symmetry between the conserved currents, i.e.

$$\langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}(x_3,h_3)\rangle \overset{!}{=} \langle J(x_2,h_2)J(x_1,h_1)\mathcal{O}(x_3,h_3)\rangle\,. \tag{9.7}$$

[2] The map between these two formalisms is summarized in [29, Appendix A].

[3] See, for example, (2.3) of [30]. This is only in four dimensions, but the parity even structures are universal.

This implies

$$
\begin{aligned}
&(V_{3,12})^{\ell-2} \to (V_{3,21})^{\ell-2} = (-1)^{\ell-2}(V_{3,12})^{\ell-2} \ \to \ \ell \text{ even}\,, \\
&\Lambda_2 = H_{23}V_{1,23}V_{3,12} \to H_{13}V_{2,13}V_{3,21} = H_{13}V_{2,31}V_{3,12} = \Lambda_3\,, \\
&\Lambda_3 = H_{13}V_{2,31}V_{3,12} \to H_{23}V_{1,32}V_{3,21} = H_{23}V_{1,23}V_{3,12} = \Lambda_2\,,
\end{aligned}
\tag{9.8}
$$

while the other tensor structures remain invariant. Therefore, we have the following constraints:

$$
\ell \text{ even} \quad \text{and} \quad \lambda_2 = \lambda_3\,. \tag{9.9}
$$

Now let us impose that J_μ is conserved up to contact terms, i.e.

$$
\frac{\partial}{\partial h_{i,\mu}} \frac{\partial}{\partial x_i^\mu} \langle J(x_1,h_1)\, J(x_2,h_2)\, \mathcal{O}(x_3,h_3)\rangle = 0\,, \qquad i = 1,2\,. \tag{9.10}
$$

Notice that some of the possible tensor structures we expect to get after acting with $\frac{\partial}{\partial h_{1,\mu}} \frac{\partial}{\partial x_1^\mu}$ on the three-point function are:

$$
\mathcal{K}_\ell\, V_{2,31}V_{3,12}^2\,, \quad \mathcal{K}_\ell\, H_{2,3}V_{3,12}\,, \quad \mathcal{K}_\ell\, V_{2,31}V_{3,12}\,, \quad \mathcal{K}_\ell\, H_{23}\,, \quad \mathcal{K}_\ell\, V_{3,12}^2\,. \tag{9.11}
$$

The calculation is quite cumbersome and time-consuming, so the best strategy turns out to be to write code in Mathematica. After some work we get the following result:

$$
\begin{aligned}
&\frac{\partial}{\partial h_{1,\mu}} \frac{\partial}{\partial x_1^\mu} \langle J(x_1,h_1) J(x_2,h_2) \mathcal{O}(x_3,h_3)\rangle = \\
&\mathcal{K}_\ell \Big\{ \lambda_1 \left[\, (\Delta - d + 1)\ V_{2,31}V_{3,12}^2\, \right] + \lambda_2 \left[\, (\Delta - d + 1)\ H_{23}V_{3,12}\, \right] + \\
&\lambda_3 \left[\, (-2 + \ell + 2d - \Delta)\ V_{2,31}V_{3,12}^2 + H_{23}V_{3,12}\, \right] + \\
&\lambda_4 \left[\, (-4 + \ell + 2d - \Delta)\ H_{23}V_{3,12}\, \right] + \lambda_5 \left[\, -(\ell + \Delta)\ V_{2,31}V_{3,12}^2 - \ell\ H_{23}V_{3,12}\, \right] \Big\} \stackrel{!}{=} 0\,,
\end{aligned}
\tag{9.12}
$$

leading to the following constraints:

$$
\begin{aligned}
&\lambda_1\ (\Delta - d + 1) + \lambda_3\ (-2 + \ell + 2d - \Delta) - \lambda_5\ (\ell + \Delta) = 0 \\
&\lambda_2\ (\Delta - d + 1) + \lambda_3 + (-4 + \ell + 2d - \Delta)\lambda_4 + \ell\lambda_5 = 0\,.
\end{aligned}
\tag{9.13}
$$

Together with $\lambda_2 = \lambda_3$, the above constraints have the following solution:

$$
\begin{aligned}
\lambda_1 &= \frac{(2d - \Delta + \ell - 2)}{d - \Delta - 1}\ \lambda_2 - \frac{(\Delta + \ell)}{d - \Delta - 1}\ \lambda_5 \\
\lambda_4 &= \frac{d - \Delta - 2}{2d - \Delta + \ell - 4}\ \lambda_2 + \frac{\ell}{2d - \Delta + \ell - 4}\ \lambda_5\,.
\end{aligned}
\tag{9.14}
$$

Therefore, defining,

$$\begin{aligned}\mathcal{A} &\equiv \frac{\ell}{2d-\Delta+\ell-4}\,, \quad \mathcal{B} \equiv \frac{(\Delta+\ell)}{d-\Delta-1}\,,\\ \mathcal{C} &\equiv \frac{d-\Delta-2}{2d-\Delta+\ell-4}\,, \quad \mathcal{D} \equiv \frac{(2d-\Delta+\ell-2)}{d-\Delta-1}\,,\end{aligned} \tag{9.15}$$

the previous constraints lead the three-point function in (9.6) to have the following form:

$$\begin{aligned}&\langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}(x_3,h_3)\rangle = \mathcal{K}_\ell\,(\hat{\lambda}_1\,\mathcal{T}_1\;+\hat{\lambda}_2\,\mathcal{T}_2)\,, \qquad \text{with}\\ &\mathcal{T}_1 = \mathcal{A}\;H_{13}H_{23} + (V_{3,12})^2\left(H_{12} - \mathcal{B}\,V_{1,23}V_{2,31}\right)\,,\\ &\mathcal{T}_2 = \mathcal{C}\;H_{13}H_{23} + \mathcal{D}\;V_{1,23}V_{2,31}(V_{3,12})^2 + H_{23}V_{1,23}V_{3,12} + H_{13}V_{2,31}V_{3,12}\,.\end{aligned} \tag{9.16}$$

Therefore, the end result, is that ℓ is forced to be even and the three-point function depends on only two independent structures, which we have called $\mathcal{T}_1$ and $\mathcal{T}_2$.

9.1.2 OPE from a 3pt function vector-vector-tensor

As we previously explained, in order to obtain the OPE we have to "reverse" equation (9.16). In other words we want to find the differential operator $\mathcal{C}_\ell(x_{12},h_1,h_2,\mathcal{D}_h)$ that gives the above three-point function when acting on the two-point function of $\mathcal{O}_\ell$. The operator $\mathcal{D}_h$ is the Todorov operator defined in (3.60), whose job is to take into account the $h^2=0$ constraint. The differential operator that we seek must satisfy, in the OPE limit $x_1\to x_2$,

$$\lim_{x_1\to x_2}\langle J(x_1,h_1)J(x_2,h_2)\mathcal{O}(x_3,h_3)\rangle = \mathcal{C}_\ell(x_{12},h_1,h_2,\mathcal{D}_h)\frac{[H_{23}(h,h_3)]^\ell}{(x_{23}^2)^{\Delta+\ell}}\,, \tag{9.17}$$

where $[H_{23}(h,h_3)]^\ell/(x_{23}^2)^{\Delta+\ell}$ is the two-point function $\langle\mathcal{O}^{(\ell)}(x_2)\mathcal{O}^{(\ell)}(x_3)\rangle$. Let us first explore the OPE limit $x_1\to x_2$ for the three-point function. Note that since all descendants will have vanishing thermal expectation value, it is enough to consider the leading singularity.

In the limit $x_1\to x_2$, the building blocks H_{ij}, $V_{k,ij}$ get the following expressions

$$\begin{aligned}H_{12} &= (h_1\cdot h_2)x_{12}^2 - 2(h_1\cdot x_{12})(h_2\cdot x_{12}) &&\to\quad H_{12}\,,\\ H_{i3} &= (h_i\cdot h_3)x_{i3}^2 - 2(h_i\cdot x_{i3})(h_3\cdot x_{i3}) &&\to\quad (h_i\cdot h_3)x_{23}^2 - 2(h_i\cdot x_{23})(h_3\cdot x_{23}) \quad i=1,2\,,\\ V_{1,23} &= \frac{x_{13}^2}{x_{23}^2}(h_1\cdot x_{12}) - \frac{x_{12}^2}{x_{23}^2}(h_1\cdot x_{13}) &&\to\quad (h_1\cdot x_{12})\,,\\ V_{2,31} &= \frac{x_{12}^2}{x_{13}^2}(h_2\cdot x_{23}) + \frac{x_{23}^2}{x_{13}^2}(h_2\cdot x_{12}) &&\to\quad (h_2\cdot x_{12})\,,\\ V_{3,12} &= \frac{x_{13}^2}{x_{12}^2}(h_3\cdot x_{23}) - \frac{x_{23}^2}{x_{12}^2}(h_3\cdot x_{13}) &&\to\quad \frac{2(h_3\cdot x_{23})(x_{12}\cdot x_{23}) - (h_3\cdot x_{12})x_{23}^2}{x_{12}^2}\,.\end{aligned} \tag{9.18}$$

Inspired by the above expressions, we can make the formal replacements in the three-point function (9.17)

$$
\begin{aligned}
H_{12} \quad &\rightarrow \quad \widetilde{H}_{12} = (h_1 \cdot h_2) x_{12}^2 - 2(h_1 \cdot x_{12})(h_2 \cdot x_{12})\,, \\
H_{i3} \quad &\rightarrow \quad \widetilde{H}_{i3} = h_i \cdot \mathcal{D}_h \quad i = 1,2\,, \\
V_{1,23} \quad &\rightarrow \quad \widetilde{V}_{1,23} = (h_1 \cdot x_{12})\,, \\
V_{2,31} \quad &\rightarrow \quad \widetilde{V}_{2,31} = (h_2 \cdot x_{12})\,, \\
V_{3,12} \quad &\rightarrow \quad \widetilde{V}_{3,12} = -\frac{x_{12} \cdot \mathcal{D}_h}{x_{12}^2}\,,
\end{aligned}
\tag{9.19}
$$

where we have defined the building blocks $\widetilde{H}_{ij}$, $\widetilde{V}_{k,ij}$ to be the blocks in the differential operator $\mathcal{C}_\ell(x_{12}, h_1, h_2, \mathcal{D}_h)$ that act on the two-point function in (9.17). After this formal replacement we can then write our differential operator as

$$
\begin{aligned}
\mathcal{C}_\ell(x_{12}, h_1, h_2, \mathcal{D}_h) &= \frac{\alpha_\ell}{(x_{12}^2)^{\frac{1}{2}(2d-\Delta-\ell)}} \left(-\frac{x_{12} \cdot \mathcal{D}_h}{x_{12}^2} \right)^{\ell-2} \left(\hat{\lambda}_1\, \tilde{\mathcal{T}}_1 + \hat{\lambda}_2\, \tilde{\mathcal{T}}_2 \right) \\
&= \frac{\alpha_\ell}{(x_{12}^2)^{d-1-\frac{\Delta}{2}}} \left(-\frac{x_{12} \cdot \mathcal{D}_h}{|x_{12}|} \right)^{\ell-2} \left(\hat{\lambda}_1\, \tilde{\mathcal{T}}_1 + \hat{\lambda}_2\, \tilde{\mathcal{T}}_2 \right)\,,
\end{aligned}
\tag{9.20}
$$

where $\tilde{\mathcal{T}}_i$ is just $\mathcal{T}_i$ after the replacement (9.19). There is only a constant α_ℓ to be determined so that both sides in (9.17) match. Is it not difficult to show that by applying the operator $\mathcal{C}_\ell$ for different ℓ we can find a general closed expression for α_ℓ, which is given by

$$
\alpha_\ell = \prod_{\substack{k=2 \\ k\text{even}}}^{\ell} \frac{4}{k(k-1)(2k+d-6)(2k+d-4)} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\ell!\,\Gamma\left(\frac{d-2}{2}+\ell\right)}\,. \tag{9.21}
$$

9.2 Thermal conformal blocks

Now that we have an expression for the differential operator $\mathcal{C}_\ell$ we can make it act on an exchange operator $\mathcal{O}_\ell$ to obtain the primary contributions of the OPE of two conserved currents $J \times J$ (see (9.2)). Note that since our interest is actually to compute the thermal expectation value

$$
g(x_{12}, h_1, h_2) = \langle J(x_1, h_1) J(x_2, h_2) \rangle_\beta\,, \tag{9.22}
$$

we should apply our differential operator $\mathcal{C}_\ell$ to the thermal one-point function

$$
\langle \mathcal{O}(x, h) \rangle_\beta = \frac{d_{\mathcal{O}}}{\beta^\Delta} (h \cdot e)^\ell\,, \tag{9.23}
$$

analogously as we did in (6.8) with (6.5). Here $d_{\mathcal{O}}$ is just a dynamical constant that depends on the theory and recall that $e^\mu = (1, 0, \ldots, 0)$ is the unit vector in the τ (i.e. circle) direction. In contrast with what we had in (6.8), appliying the differential operator $\mathcal{C}_\ell$ to the one-point function (9.23) will lead to an expansion with several tensor structures. Each of these tensor structures will be split into tensor basis structures $\mathbb{T}_i$ times other functions $g^{(i)}_{\Delta,\ell}(\xi)$ that depend on the *cross-ratio* η

$$\eta = \frac{x_{12} \cdot e}{|x_{12}|} = \frac{\tau}{|x_{12}|}\,, \tag{9.24}$$

in the same fashion as we defined in (6.17). Explicitly, we will have that (9.22) will read as

$$g_{\Delta,\ell}(x_{12}, h_1, h_2) = \frac{1}{\beta^\Delta}(x_{12}^2)^{\frac{\Delta}{2}-d+1} \sum_{i=1}^{5} a^{(i)}_{\Delta,\ell}\, g^{(i)}_{\Delta,\ell}(\eta)\, \mathbb{T}_i(h_1, h_2, x_{12})\,. \tag{9.25}$$

The functions $g^{(i)}_{\Delta,\ell}(\eta)$ are called the *thermal conformal blocks* and computing them is precisely the goal of this section.

First, let us figure out what possible tensor structures $\mathbb{T}_i$ could appear in the thermal two-point function of spinning operators. For *traceless* symmetric tensors of spin s, the most general tensor structure that is $\mathbb{R}^{d-1}$ rotational and parity invariant is given by

$$\frac{(h_1 \cdot h_2)^{j_1}(h_1 \cdot x_{12})^{j_2}(h_2 \cdot x_{12})^{j_3}(h_1 \cdot e)^{j_4}(h_2 \cdot e)^{j_5}}{(x_{12}^2)^{\frac{(j_2+j_3)}{2}}}\,, \tag{9.26}$$

subject to the following constraints

$$j_1 + j_2 + j_4 = j_1 + j_3 + j_5 = s\,. \tag{9.27}$$

Note that, having fixed, for instance, the first triple, i.e. (j_1, j_2, j_4) we still have some degrees of freedom in the values of j_3 and j_5. The only restriction is that they have to add up to $j_3 + j_5 = s - j_1$. The possible combinations of j_3 and j_5 that sum to a certain value $s - j_1$ is $s - j_1 + 1$. Of course, the argument is the same if we first fix the triplet (j_1, j_3, j_5). Thus, for every triplet we fix that sums to s we have $n = s - j_1 + 1$ tensor structures. In our case, for $s = 1$ we have

$$n_{(1,0,0)} + n_{(0,1,0)} + n_{(0,0,1)} = 1 + 2 + 2 = 5\,. \tag{9.28}$$

We then choose our tensor basis structures to be

$$\begin{aligned} &\mathbb{T}_1 = h_1 \cdot h_2\,, \qquad \mathbb{T}_2 = \frac{(h_1 \cdot x_{12})(h_2 \cdot x_{12})}{x_{12}^2}\,, \qquad \mathbb{T}_3 = \frac{(h_1 \cdot x_{12})(h_2 \cdot e)}{|x_{12}|}\,,\\ &\mathbb{T}_4 = \frac{(h_1 \cdot e)(h_2 \cdot x_{12})}{|x_{12}|}\,, \qquad \mathbb{T}_5 = (h_1 \cdot e)(h_2 \cdot e)\,. \end{aligned} \tag{9.29}$$

Having defined our tensor basis structures, we can now proceed to make the differential operator $\mathcal{C}_\ell$ in (9.20) act to the one-point function (9.23). Let us do this in several steps. First, let us partially apply $\mathcal{C}_\ell$. The differential operators that we have in the tensor structures $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ in (9.20) are just of the type

$$(h_1 \cdot \mathcal{D}_h)(h_2 \cdot \mathcal{D}_h),\ \left(-\frac{x_{12} \cdot \mathcal{D}_h}{x_{12}^2}\right)^2,\ (h_i \cdot \mathcal{D}_h)\left(-\frac{x_{12} \cdot \mathcal{D}_h}{x_{12}^2}\right) \quad \text{for} \quad i = 1, 2\,. \tag{9.30}$$

Letting the above tensor structures to act on the one-point function (9.23) will give us a complicated expression. However, we will have only 6 different types of terms. For instance, we will have terms proportional to:

$$\begin{aligned}
&\left(-\frac{x_{12}\cdot\mathcal{D}_h}{|x_{12}|}\right)^L (h\cdot e)^L\,,\\
&\left(-\frac{x_{12}\cdot\mathcal{D}_h}{|x_{12}|}\right)^L (h\cdot e)^{L-1}\,\frac{h\cdot x_{12}}{|x_{12}|}\,,\\
&\left(-\frac{x_{12}\cdot\mathcal{D}_h}{|x_{12}|}\right)^L (h\cdot e)^{L-2}\,\frac{(h\cdot x_{12})^2}{x_{12}^2}\,.
\end{aligned} \tag{9.31}$$

Note that so far we have only considered the action of the tensor structures (9.30) on the one-point function (9.23). We still need to let the operator $\left(-\frac{x_{12}\cdot\mathcal{D}_h}{|x_{12}|}\right)^L$ act. Its application will result in either a scalar function of η or a combination of tensor structures times a function of η. For all the examples mentioned in (9.31) we will end up with scalar functions $F_i^{(L)}(\eta)$,

$$F_1^{(L)}(\eta), \qquad F_2^{(L)}(\eta), \qquad F_3^{(L)}(\eta)\,, \tag{9.32}$$

respectively. Notice also that we have written the expressions above for a general L, which simply corresponds to $\ell-2$ in the notation we have been using. The only problem left is then determine these functions $F_i^{(L)}(\eta)$. For that, let us define the functions $\widehat{C}_\ell^{(\nu)}(\eta)$ as

$$\widehat{C}_\ell^{(\nu)}(\eta) = \frac{(\ell!)^2}{2^\ell}\,\mathcal{C}_\ell^{(\nu)}(\eta)\,, \tag{9.33}$$

where $\mathcal{C}_\ell^{(\nu)}(\eta)$ are the Gegenbauer polynomials. It can be proven that the functions $F_i^{(L)}(\eta)$ can be expressed in terms of the functions (9.33). The exact proof, which is based on induction, is shown in Appendix A.1. Actually, we had already shown in A.1.1 that

$$F_1^{(L)}(\eta) = \widehat{C}_L^{(\frac{d}{2}-1)}(\eta)\,. \tag{9.34}$$

For $F_2^{(L)}(\eta)$ we have the following result:

$$F_2^{(L)}(\eta) = \frac{1}{2}(d+L-3)L\,\widehat{C}_{L-1}^{(\frac{d}{2}-1)}(\eta)\,. \tag{9.35}$$

which is proved in A.1.2. Finally, it is proved in A.1.3 that $F_3^{(L)}(\eta)$ is given via:

$$F_3^{(L)}(\eta) = \frac{1}{4}L(L-1)(d+L-4)(d+L-3)\,\widehat{C}_{L-2}^{(\frac{d}{2}-1)}(\eta)\,. \tag{9.36}$$

This concludes our task of finding the thermal conformal blocks decomposition of (9.22). One subtlety that arise in the calculation is that Gegenbauer polynomials $\mathcal{C}_n^{(\alpha)}$ with different weights α are mixed in the same expression. In order to shift all Gegenbauer polynomials to have the same weight, we can use the identity relating Gegenbauer polynomials of different weights that we already used in (6.33). In terms of the functions (9.33), the identity reads as

$$\widehat{C}_j^{(\lambda)}(\eta) = \sum_{L=j,j-2,\ldots,j \bmod 2} \frac{2^{L-j}(j!)^2}{(L!)^2}\,\frac{(L+\alpha)(\lambda)_{\frac{j+L}{2}}(\lambda-\alpha)_{\frac{j-L}{2}}}{(\frac{j-L}{2})!\,(\alpha)_{\frac{j+L+2}{2}}}\,\widehat{C}_L^{(\alpha)}(\eta)\,. \tag{9.37}$$

The final result is, as expected, of the form of expression (9.25), with the explicit expression for the thermal blocks given in Appendix B. Note that, in stark contrast with expression (6.18) the thermal blocks now involve more than one kind of Gegenbauer polynomials, which will make the task of identifying the thermal coefficients more challenging.

9.3 Mean field theory for conserved currents

Let us now explore one particular application of the thermal conformal blocks we found in the previous section. We will consider the simplest example of mean field theory, in analogy as we did section 6.4. The propagator of a spin-one conserved current at finite temperature reads was already derived in (8.34). We have that

$$\begin{aligned}\langle J(X_1,H_1)J(X_2,H_2)\rangle_\beta^{\text{MFT}} &= \\ &= \sum_{m\in\mathbb{Z}} \mathcal{C}_{\Delta\to d-1}\,\frac{(-2X_{1,m}\cdot X_2)(H_{1,m}\cdot H_2)+2(H_{1,m}\cdot X_2)(X_{1,m}\cdot H_2)}{(-2X_{1,m}\cdot X_2)^d}\,.\end{aligned} \tag{9.38}$$

Let us project the above expression to physical coordinates $x=(\tau,\mathbf{x})$ where, as usual, $\mathbf{x}$ denotes the spatial vector and x the Euclidean spacetime vector. Using (3.67) we can compute the following scalar products

$$\begin{aligned}(-2X_{1,m}\cdot X_2) &= x_{12}^2+2m(\tau_1-\tau_2)+m^2\,,\\ (H_{1,m}\cdot H_2) &= (h_1\cdot h_2)\,,\\ (H_{1,m}\cdot X_2) &= -(h_1\cdot x_{12})-m(h_1\cdot e)\,,\\ (X_{1,m}\cdot H_2) &= (h_2\cdot x_{12})+m(h_2\cdot e)\,,\end{aligned} \tag{9.39}$$

where e is the unit vector in the circle direction. Then, using the tensor structures $\mathbb{T}_i$ defined in (9.29), we can rewrite the propagator (9.38) as

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\text{MFT}} &= \\ &= \sum_{m\in\mathbb{Z}} \mathcal{C}_{\Delta\to d-1}\left(\frac{\mathbb{T}_1}{(x_{12}^2+2\,m\,(x_{12}\cdot e)+m^2)^{d-1}}-2\,\frac{x_{12}^2\,\mathbb{T}_2+m\,|x_{12}|\,(\mathbb{T}_3+\mathbb{T}_4)+m^2\,\mathbb{T}_5}{(x_{12}^2+2\,m\,(x_{12}\cdot e)+m^2)^d}\right)\,.\end{aligned} \tag{9.40}$$

Note that, using the definition in (9.24), all terms in (9.40) can be written in the following general form (except for numerical factors)

$$\Pi(\lambda,\kappa,\tau)=\frac{m^\tau(x_{12}^2)^\kappa}{(x_{12}^2+2m|x_{12}|\eta+m^2)^\lambda}\,, \tag{9.41}$$

for some λ,κ and τ. In particular, the first term has $\lambda=d-1$ and $\kappa=\tau=0$. The rest of terms have all $\lambda=d$. For the second one we have $\kappa=1,\tau=0$, for the third and fourth terms $\kappa=\frac{1}{2},\tau=1$, while for the fifth term we have $\kappa=0,\tau=2$. Similarly as we did in (6.28), we can rewrite the above propagator structure as

$$\Pi(\lambda,\kappa,\tau)=\frac{m^\tau(x_{12}^2)^\kappa}{|m|^{2\lambda}\left(1-2\eta\left(-\frac{x_{12}}{|m|}\text{sgn}(m)\right)+\frac{x_{12}^2}{|m|^2}\right)^\lambda}\,, \tag{9.42}$$

where $\mathrm{sgn}(m) = \frac{m}{|m|}$. In this explicit form, we can apply the definition of the Gegenbauer polynomials as the generating function (A.4) to (9.42) to get

$$\Pi(\lambda,\kappa,\tau) = m^{\tau}(x_{12}^2)^{\kappa} \sum_{j=0}^{\infty} \frac{(-2)^j}{(j!)^2} \frac{x_{12}^j}{|m|^{2\lambda}} \left(\frac{\mathrm{sgn}(m)}{|m|}\right)^j \widehat{C}_j^{(\lambda)}(\eta)\,. \tag{9.43}$$

Note that, when substituting (9.43) back to (9.40), we have a sum over j and a sum over m. Permuting both sums we can actually remove the m dependence by using the Riemann function (6.32). Recall that, as we argued in (6.31), only contributions for $j=$ even do not cancel each other. Thus, in general, we will have

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{m^{\tau}}{|m|^{2\lambda}} \left(\frac{\mathrm{sgn}(m)}{|m|}\right)^j = 2\zeta(j+2\lambda-\tau)\,, \quad \text{for } j = \text{even}\,. \tag{9.44}$$

The goal now is to match this mean field theory expansion with the thermal conformal decomposition (9.25), i.e. we want to rewrite our MFT expansion in the form (here $\beta=1$)

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\mathrm{MFT}} &= \mathbb{T}_1 \sum_{\Delta,\ell\,\in\,\Sigma_1} a_{\Delta,\ell}^{(1)}\,(x_{12}^2)^{\frac{\Delta}{2}-d+1} g_{\Delta,\ell}^{(1)}(\eta) + \\ &+ \mathbb{T}_2 \sum_{\Delta,\ell\,\in\,\Sigma_2} a_{\Delta,\ell}^{(2)}\,(x_{12}^2)^{\frac{\Delta}{2}-d+1} g_{\Delta,\ell}^{(2)}(\eta) + (\mathbb{T}_3+\mathbb{T}_4) \sum_{\Delta,\ell\,\in\,\Sigma_{3,4}} a_{\Delta,\ell}^{(3,4)}\,(x_{12}^2)^{\frac{\Delta}{2}-d+1} g_{\Delta,\ell}^{(3,4)}(\eta) \\ &+ \mathbb{T}_5 \sum_{\Delta,\ell\,\in\,\Sigma_5} a_{\Delta,\ell}^{(5)}\,(x_{12}^2)^{\frac{\Delta}{2}-d+1} g_{\Delta,\ell}^{(3)}(\eta)\,,\end{aligned} \tag{9.45}$$

and identify both the coefficients $a_{\Delta,\ell}^{(i)}$ and the spectrums Σ_i.

9.3.1 Thermal block $\mathbb{T}_1$

If we restrict our attention to the structure $\mathbb{T}_1$ in the mean field expansion of the propagator (9.40) and we apply the previous results we have that

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\mathrm{MFT}}\,|_{\mathbb{T}_1} = \mathcal{C}_{\Delta\to d-1} \Big(&\frac{1}{(x_{12}^2)^{d-1}} + \\ &+ \sum_{j=0,2,\ldots} \frac{2^{j+1}}{(j!)^2}\,\zeta(j+2d-2)\,x_{12}^j\,\widehat{C}_j^{(d-1)}(\eta)\Big)\end{aligned} \tag{9.46}$$

Applying identity (9.37) with $\alpha = \left(\frac{d}{2}+1\right)$[4] and defining $\Delta_J \equiv d-1$ we get

$$\begin{aligned}&\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\mathrm{MFT}}\,|_{\mathbb{T}_1} = \mathcal{C}_{\Delta\to d-1} \Big(\frac{1}{(x_{12}^2)^{d-1}} + \\ &+ \sum_{j=0,2,\ldots} \sum_{\ell=j,j-2,\ldots,j \bmod 2} \frac{2^{\ell+1}}{(\ell!)^2} \frac{(\ell+\alpha)(\Delta_J)_{\frac{j+\ell}{2}}(\Delta_J-\alpha)_{\frac{j-\ell}{2}}}{(\frac{j-\ell}{2})!\,(\alpha)_{\frac{j+\ell+2}{2}}}\,\zeta(j+2\Delta_J)\,x_{12}^j\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,.\end{aligned} \tag{9.47}$$

[4] In our previous decomposition in thermal conformal blocks we managed to bring all Gegenbauer polynomials to have weight $\left(\frac{d}{2}+1\right)$. This was indeed necessary for the task of identifying the thermal coefficients.

Now, we we make the change of variables $j = 2n + \ell$, so that

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\text{MFT}}|_{\mathbb{T}_1} &= \mathcal{C}_{\Delta\to d-1}\Big(\frac{1}{(x_{12}^2)^{d-1}}+ \\ &+\sum_{n=0}^{\infty}\sum_{\ell=0,2,\dots}\frac{2^{\ell+1}}{(\ell!)^2}\frac{(\ell+\alpha)(\Delta_J+1)_{\ell+n}(\Delta_J-\alpha+1)_n}{n!\,(\alpha)_{\ell+n+1}}\,\zeta(2\Delta_J+2n+\ell)\,(x_{12}^2)^{n+\frac{\ell}{2}}\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,.\end{aligned} \tag{9.48}$$

Note that, comparing with (9.25), we can easily see which operators appears in the mean field expansion for the structure $\mathbb{T}_1$. Indeed, we can identify the unit operator (with $\Delta_{\mathbb{1}} = 0$) in the first term and the double trace operators with $\Delta_{[JJ]_{n,\ell}} = 2\Delta_J + 2n + \ell$, as expected. Comparing to the sum in (B.1) we see that the spectrum of the double trace operators must have ℓ even so that it matches with the Gegenbauers in (9.48). Roughly, we have then that the spectrum Σ_1 in the MFT expansion (9.45) is

$$\Sigma_1 = \begin{cases}\ell = 0; & \Delta_{\mathbb{1}} = 0\,,\\ \ell = \text{even}; & \Delta_{[JJ]_{n,\ell}}\,.\end{cases} \tag{9.49}$$

As defined in (9.45), the thermal coefficients $a^{(1)}_{\Delta,\ell}$ have the following expressions: $a^{(1)}_{\mathbb{1}} = \hat{\lambda}_1$ while we leave the coefficients $a^{(1)}_{[JJ]_{n,\ell}}$ to be determined in future work.

9.3.2 Thermal block $\mathbb{T}_2$

We can follow an analogous procedure as we followed for $\mathbb{T}_1$. From the expression

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\text{MFT}}|_{\mathbb{T}_2} &= \mathcal{C}_{\Delta\to d-1}\Big(\frac{-2}{(x_{12}^2)^{d-1}} \\ &-\sum_{j=0,2,\dots}\;\sum_{\ell=j,j-2,\dots,j \bmod 2}\frac{2^{\ell+2}}{(\ell!)^2}\frac{(\ell+\alpha)(\Delta_J+1)_{\frac{j+\ell}{2}}(\Delta_J-\alpha+1)_{\frac{j-\ell}{2}}}{(\frac{j-\ell}{2})!\,(\alpha)_{\frac{j+\ell+2}{2}}}\,\zeta(j+2\Delta_J+2)\,x_{12}^{j+2}\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,.\end{aligned} \tag{9.50}$$

but changing variables now as $j = 2n + \ell - 2$, we find for the thermal block in $\mathbb{T}_2$

$$\begin{aligned}\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\text{MFT}}|_{\mathbb{T}_2} &= \mathcal{C}_{\Delta\to d-1}\Big(\frac{-2}{(x_{12}^2)^{d-1}} \\ &-\sum_{n=1}^{\infty}\sum_{\ell=0,2,\dots}\frac{2^{\ell+2}}{(\ell!)^2}\frac{(\ell+\alpha)(\Delta_J+1)_{\ell+n-1}(\Delta_J-\alpha+1)_{n-1}}{(n-1)!\,(\alpha)_{\ell+n}}\,\zeta(2\Delta_J+2n+\ell)\,(x_{12}^2)^{n+\frac{\ell}{2}}\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,.\end{aligned} \tag{9.51}$$

Here, comparing with (9.25), we can again identify the identity operator (with $\Delta_{\mathbb{1}} = 0$) and the double trace operators with $\Delta_{[JJ]_{n,\ell}} = 2\Delta_J + 2n + \ell$. An exact argument as we had for $\mathbb{T}_1$ holds here to roughly determine the spectrum Σ_2 in the MFT expansion (9.45). In this case reads

$$\Sigma_2 = \begin{cases}\ell = 0; & \Delta_{\mathbb{1}} = 0\,,\\ \ell = \text{even}; & \Delta_{[JJ]_{n,\ell}}\,.\end{cases} \tag{9.52}$$

As defined in (9.45), the thermal coefficients $a^{(2)}_{\Delta,\ell}$ have again the following expressions: $a^{(2)}_{\mathbb{1}} = \hat{\lambda}_1$ while we leave the coefficients $a^{(2)}_{[JJ]_{n,\ell}}$ to be determined in future work.

9.3.3 Thermal block $\mathbb{T}_3$ and $\mathbb{T}_4$

Let us now study the case of the thermal block structures $\mathbb{T}_3$ and $\mathbb{T}_4$. We now have the following starting point

$$\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\rm MFT}|_{\mathbb{T}_{(3,4)}} = \mathcal{C}_{\Delta\to d-1}\times$$
$$\Big(-\sum_{j=0,2,\ldots}\ \sum_{\ell=j,j-2,\ldots,j \bmod 2}\frac{2^{\ell+2}}{(\ell!)^2}\frac{(\ell+\alpha)(\Delta_J+1)_{\frac{j+\ell}{2}}(\Delta_J-\alpha+1)_{\frac{j-\ell}{2}}}{(\frac{j-\ell}{2})!\,(\alpha)_{\frac{j+\ell+2}{2}}}\,\zeta(j+2\Delta_J+1)\,x_{12}^j|x_{12}|\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,. \tag{9.53}$$

Here we first shift $\ell\to\ell-1$ (to match the Gegenbauer polynomial we find in (B.3)) and then we change $j=2n+\ell-1$. We get

$$\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\rm MFT}|_{\mathbb{T}_{(3,4)}} = \mathcal{C}_{\Delta\to d-1}\times$$
$$-\Big(\sum_{n=0}^{\infty}\sum_{\ell=1,3,\ldots}\frac{2^{\ell+1}}{((\ell-1)!)^2}\frac{(\ell+\alpha-1)(\Delta_J+1)_{\ell+n-1}(\Delta_J-\alpha+1)_n}{n!\,(\alpha)_{\ell+n}}\,\zeta(2\Delta_J+2n+\ell)\,(x_{12}^2)^{n+\frac{\ell}{2}}\,\widehat{C}_{\ell-1}^{(\alpha)}(\eta)\Big)\,. \tag{9.54}$$

In this case, comparing with (9.25), we can identify only the double trace operators with $\Delta_{[JJ]_{n,\ell}}=2\Delta_J+2n+\ell$. Note the identity here is missing. Roughly, the spectrum $\Sigma_{(3,4)}$ in the MFT expansion (9.45) is given via

$$\Sigma_{(3,4)}=\Big\{\ell=\text{odd};\quad \Delta_{[JJ]_{n,\ell}}=2\Delta_J+2n+\ell\,. \tag{9.55}$$

The thermal coefficients $a_{\Delta,\ell}^{(3,4)}$ can be then found by matching both sums, which we leave as future work.

9.3.4 Thermal block $\mathbb{T}_5$

Finally, for the thermal block in the $\mathbb{T}_5$ tensor structure we have

$$\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\rm MFT}|_{\mathbb{T}_5} = \mathcal{C}_{\Delta\to d-1}\times$$
$$-\Big(\sum_{j=0,2,\ldots}\ \sum_{\ell=j,j-2,\ldots,j \bmod 2}\frac{2^{\ell+2}}{(\ell!)^2}\frac{(\ell+\alpha)(d)_{\frac{j+\ell}{2}}(d-\alpha)_{\frac{j-\ell}{2}}}{(\frac{j-\ell}{2})!\,(\alpha)_{\frac{j+\ell+2}{2}}}\,\zeta(j+2d-2)\,x_{12}^j\,\widehat{C}_\ell^{(\alpha)}(\eta)\Big)\,. \tag{9.56}$$

Here first we shift $\ell\to\ell-2$ (to match the same kind of Gegenbauer polynomial we find in (B.4)) and then we change $j=2n+\ell$. Finally, we get

$$\langle J(x_1,h_1)J(x_2,h_2)\rangle_\beta^{\rm MFT}|_{\mathbb{T}_5} = \mathcal{C}_{\Delta\to d-1}\times$$
$$-\Big(\sum_{n=-1}^{\infty}\sum_{\ell=2,4,\ldots}\frac{2^{\ell}}{((\ell-2)!)^2}\frac{(\ell+\alpha-2)(d)_{\ell+n-1}(d-\alpha)_{n+1}}{(n+1)!\,(\alpha)_{\ell+n}}\,\zeta(2\Delta_J+2n+\ell)\,(x_{12}^2)^{n+\frac{\ell}{2}}\,\widehat{C}_{\ell-2}^{(\alpha)}(\eta)\Big)\,. \tag{9.57}$$

In this case, comparing with (9.25), we can only identify the double trace operators with $\Delta_{[JJ]_{n,\ell}}=2\Delta_J+2n+\ell$, so that the spectrum Σ_5 in the MFT expansion (9.45) is given via

$$\Sigma_5=\Big\{\ell=2,4,6,\ldots;\quad \Delta_{[JJ]_{n,\ell}}=2\Delta_J+2n+\ell,\quad n\in[-1,\infty)\,. \tag{9.58}$$

Only for this tensor structure, identifying a close form for the thermal coefficients $a^{(5)}_{\Delta,\ell}$ can be immediately found by matching both sums. As defined in (9.45), these have the following expression

$$a^{(5)}_{\Delta_{[JJ]_{n,\ell}}} = \frac{8(n+1)}{\ell(\ell-1)d(d-2)}(\frac{1}{\ell}\hat{\lambda}_1 - \frac{1}{\ell+d+2n}\hat{\lambda}_2) \times \frac{(d-1)\,\Gamma(d-2)}{2\pi^{d/2}\,\Gamma(\frac{d}{2})}\times \\ \times \frac{2^{\ell}\,(\ell+\alpha-2)(d)_{\ell+n-1}(d-\alpha)_{n+1}\,\zeta(2\Delta_J+2n+\ell)}{((\ell-2)!)^2\,(n+1)!\,(\alpha)_{\ell+n}}\,, \tag{9.59}$$

which can be recast as,

$$a^{(5)}_{\Delta_{[JJ]_{n,\ell}}} = \frac{2^{\ell+2}\,\zeta(2\Delta_J+2n+\ell)\,(\ell+\alpha-2)}{\ell(\ell-1)(\Delta_J+1)(\Delta_J-1)\,\Gamma(\ell-1)^2}\left(\frac{1}{\ell}\hat{\lambda}_1 - \frac{1}{\Delta_J+2n+\ell+1}\hat{\lambda}_2\right)\times \\ \times \frac{\Gamma(\Delta_J-1)\,(\Delta_J)_{\ell+n}(\Delta_J-\alpha+1)_{n+1}}{\pi^{\alpha-1}\,\Gamma(\alpha-1)\,n!\,(\alpha)_{\ell+n}}\,. \tag{9.60}$$

10| Discussion

In this work we have taken a long journey from the basics of Conformal Field Theories to the computation of thermal conformal blocks in CFTs at finite temperature. We have presented the different results and techniques that are widely used in the field of CFTs, including the explicit expressions for two- and three- point functions, or the so-called embedding formalism, which as we have seen, is natural way of embedding the action of the conformal group that makes the expressions to look simpler and more elegant. We have also gone through some of the most important results in CFT such the state-operator correspondence, which is a one-to-one correspondence that is not present in general QFTs between all the operators at any point and all the states in the Hilbert space. This, together with the operator product expansion, a way of rewriting the *product* of two operators $\mathcal{O}(x_1)\mathcal{O}(x_2)$ as a convergent sum over all the operators in the theory at any point between x_1 and x_2, are the cornerstones of CFTs. As we already explained, the motivations for studying CFT are diverse, but, without doubt, AdS/CFT is one of the most interesting ones. In that sense, we introduced some of the basic facts of the Anti-de Sitter space and stated how the mathematical correspondence between a theory of bulk gravity and a CFT at its boundary is achieved.

Note that a chapter discussing the importance of the presence of a stress-energy tensor $T_{\mu\nu}$ in a CFT or the role of the so-called *Ward identities* could have been included in this work. However, we chose to just outline the results that had direct impact on our final goal, which consisted on computing the thermal blocks of a propagator of conserved currents. Indeed, about halfway through the work, we became involved in discussing CFTs at finite temperature. The introduction of a scale in the theory brings new features that were not present in *pure* CFTs. We introduce the concept of thermal blocks and thermal coefficients and we go through the work of [11] in deriving the thermal coefficients in the mean field expansion for the two-point function of scalar fields. We also review the work by [1] in which the expression of the holographic thermal two-point function of scalar operators in the boundary theory of a weakly-coupled QFT in AdS is computed. One direction for future work would be to perform an analogous calculation for higher spin operators. In this thesis we have explained and presented all the ingredients necessary to carry out this task for spin 1 operators. This, in particular, includes the explicit expression for the bulk-to-bulk and bulk-to-boundary propagators for vector fields.

Finally, we derive the explicit expressions for thermal conformal blocks for conserved currents. Although we go through most of the steps on how to achieve that, due to the challenging nature of the computation, most of the explicit calculations are not shown since they are performed with Mathematica. After identifying the thermal blocks we try to reproduce the analogue computation of [11] for conserved currents in identifying thermal coefficients in mean field theory. Here we find a major complication with respect to the case of scalar fields: the thermal blocks depend on more than one type of Gegenbauer polynomials. This makes the task of identifying most of the thermal coefficients a hard problem that we leave as a future direction of work.

A| The Gegenbauer Polynomials

A.1 Generalities of the Gegenbauer Polynomials

The *Gegenbauer polynomials* $\mathcal{C}_\ell^{(\nu)}(z)$ are solutions to the Gegenbauer differential equation

$$(1-z^2)f'' - (2\nu+1)zf' + \ell(\ell+2\nu)f = 0. \tag{A.1}$$

They actually generalize the Legendre polynomials, since for $\nu = 1/2$, (A.1) reduce to the Legendre equation. They also reduce to the to the Chebyshev polynomials of the second kind when $\nu = 1$.

For a fixed ν, the polynomials are orthogonal on $[-1,1]$ with respect the weight function $\omega(z) = (1-z^2)^{\nu-\frac{1}{2}}$ so that, for $\nu \neq \lambda$,

$$\int_{-1}^{1} \mathcal{C}_\ell^{(\nu)}(z)\, \mathcal{C}_\ell^{(\lambda)}(z)\, (1-z^2)^{\nu-\frac{1}{2}}\, dz = 0\,. \tag{A.2}$$

Besides, they are normalized by

$$\int_{-1}^{1} \left[\mathcal{C}_\ell^{(\nu)}(z)\right]^2 (1-z^2)^{\nu-\frac{1}{2}}\, dz = \frac{\pi\, 2^{1-2\nu}\, \Gamma(\ell+2\nu)}{\ell!(\ell+\nu)[\Gamma(\nu)]^2}\,. \tag{A.3}$$

Another characterization of the Gegenbauer polynomials is given through the following generation function:

$$\frac{1}{(1-2xy+y^2)^\alpha} = \sum_{j=0}^{\infty} \mathcal{C}_j^{(\alpha)}(x)\, y^j\,, \tag{A.4}$$

for $0 \leq |x| \leq 1, |y| \leq 1, \alpha > 0$.

However, for the purpose of this section we just need to know that the Gegenbauer polynomials satisfy the following recurrence relation:

$$\begin{aligned} \mathcal{C}_0^{(\nu)}(z) &= 1\,,\\ \mathcal{C}_1^{(\nu)}(z) &= 2\nu z\,,\\ \mathcal{C}_\ell^{(\nu)}(z) &= \frac{1}{\ell}\Big[2z(\ell+\nu-1)\, \mathcal{C}_{\ell-1}^{(\nu)}(z) - (\ell+2\nu-2)\, \mathcal{C}_{\ell-2}^{(\nu)}(z)\Big]\,. \end{aligned} \tag{A.5}$$

A.1.1 Induction proof I

Let us then derive a useful result that introduce us to the Gegenbauer polynomials. This first proof is taken from [31]. Given the unit vectors $x, y \in \mathbb{R}^d$, i.e. $|x| = |y| = 1$, a null vector h, i.e. $h^2 = 0$, and the Todorov operator $\mathcal{D}$ defined in (3.60) (here we will leave the h subscript in $\mathcal{D}_h$ implicit), let us define

$$C(\ell) = (x \cdot \mathcal{D})^\ell (h \cdot y)^\ell \,. \tag{A.6}$$

Defining $\nu = \frac{d}{2} - 1$ and $\xi = x \cdot y$, the aim is to prove that $C(\ell)$ is given via

$$C(\ell) = \frac{(\ell!)^2}{2^\ell}\, \mathcal{C}_\ell^{(\nu)}(\xi)\,. \tag{A.7}$$

where $\mathcal{C}_\ell^{(\nu)}(\xi)$ are the Gegenbauer polynomials. We are going to prove the above result by deriving a recurrence relation for $C(\ell)$ and comparing it to the one for the Gegenbauers in (A.5). First note that

$$C(0) = 1, \qquad C(1) = (x \cdot \mathcal{D})(h \cdot y) = \frac{d-2}{2}\, x \cdot y \tag{A.8}$$

Then, we have that

$$\begin{aligned} C(\ell+1) &= (x \cdot \mathcal{D})^\ell (x \cdot \mathcal{D})(h \cdot y)^{\ell+1} = \\ &= (x \cdot \mathcal{D})^\ell \Big\{ x^i \Big[\Big(\nu + h \cdot \frac{\partial}{\partial h}\Big) \frac{\partial}{\partial h^i} - \frac{1}{2}\, h_i \frac{\partial^2}{\partial h \cdot \partial h}\Big] (h \cdot y)^{\ell+1} \Big\} \\ &= (x \cdot \mathcal{D})^\ell \Big\{ x^i \nu (\ell+1)(h \cdot y)^\ell y_i + x^i \ell(\ell+1) h^j (h \cdot y)^{\ell-1} y_j y_i - \frac{1}{2} x^i h_i (\ell+1)\ell (h \cdot y)^{\ell-1} y^j y_j \Big\} \\ &= (x \cdot \mathcal{D})^\ell \Big\{ (\ell+1)(\ell+\nu)(x \cdot y)(h \cdot y)^\ell - \frac{1}{2}\ell(\ell+1)(h \cdot x)(h \cdot y)^{\ell-1} \Big\} \\ &= (\ell+1)(\ell+\nu)\xi (x \cdot \mathcal{D})^\ell (h \cdot y)^\ell - \frac{1}{2}\ell(\ell+1)\left[(x \cdot \mathcal{D})^\ell, (h \cdot x)\right](h \cdot y)^{\ell-1}\,, \end{aligned} \tag{A.9}$$

where we have used the fact that $(x \cdot \mathcal{D})^\ell (h \cdot y)^{\ell-1} = 0$ since we have (at least) ℓ derivatives acting on an $(\ell-1)$-polynomial. Moreover, we have that

$$\begin{aligned} [x \cdot \mathcal{D}, h \cdot x] &= (x \cdot \mathcal{D})(h \cdot x) + (h \cdot x)(x \cdot \mathcal{D}) - (h \cdot x)(x \cdot \mathcal{D}) = \\ &= x^i \Big(\nu + h \frac{\partial}{\partial h}\Big) \frac{\partial}{\partial h^i}(h \cdot x) = x^i \Big(\nu + h \frac{\partial}{\partial h}\Big) x_i = \nu + h \frac{\partial}{\partial h}\,, \end{aligned} \tag{A.10}$$

which is just an operator that counts the homogeneity weight of h on whatever expression there is on its right. Now, using the properties of commutators and the above result, it follows from (A.9) that

$$\begin{aligned} C(\ell+1) &= \\ &= (\ell+1)(\ell+\nu)\xi\, C(\ell) - \frac{1}{2}\ell(\ell+1) \sum_{k=0}^{\ell-1} (x \cdot \mathcal{D})^k [x \cdot \mathcal{D}, h \cdot x](x \cdot \mathcal{D})^{\ell-1-k}\, (h \cdot y)^{\ell-1} \\ &= (\ell+1)(\ell+\nu)\xi\, C(\ell) - \frac{1}{2}\ell(\ell+1) \sum_{k=0}^{\ell-1} (\nu + k)(x \cdot \mathcal{D})^{\ell-1}\, (h \cdot y)^{\ell-1} \\ &= (\ell+1)(\ell+\nu)\xi\, C(\ell) - \frac{1}{4}\ell^2(\ell+1)(\ell+2\nu-1)(x \cdot \mathcal{D})^{\ell-1}\, (h \cdot y)^{\ell-1}\,, \end{aligned} \tag{A.11}$$

where we have performed the sum

$$\sum_{k=0}^{\ell-1}(\nu+k) = \ell\nu + \frac{(\ell-1)\ell}{2}\,. \tag{A.12}$$

Therefore, we have obtained the recursion relation

$$\begin{aligned} C(0) &= 1\,, \\ C(1) &= \nu\,\xi, \\ C(\ell+1) &= (\ell+1)(\ell+\nu)\,\xi\,C(\ell) - \frac{1}{4}\ell^2(\ell+1)(\ell+2\nu-1)\,C(\ell-1)\,. \end{aligned} \tag{A.13}$$

By plugging (A.7) into (A.13) we can easily check that this is the same recursion relation that the one of the Gegenbauer polynomials. Uniqueness of its solution ends the proof.

A.1.2 Induction proof II

Let us now prove a slightly different problem. Keeping the definitions for $\nu = \frac{d}{2} - 1$ and $\xi = x \cdot y$, in this case let us consider our function $D(\ell)$ to be given by

$$D(\ell) = (x \cdot \mathcal{D})^\ell (h \cdot y)^{\ell-1}(h \cdot x)\,. \tag{A.14}$$

where recall that $\mathcal{D}$ is the Todorov operator $\mathcal{D}$ defined in (3.60). Then, we can show that

$$D(\ell) = \frac{1}{2}(d+\ell-3)\ell\frac{((\ell-1)!)^2}{2^{\ell-1}}\mathcal{C}^{(\nu)}_{\ell-1}(\xi)\,. \tag{A.15}$$

The proof is completely analogous as the previous one. The aim is to derive a recurrence relation for $D(\ell)$ that is equivalent to the one that holds for the Gegenbauer polynomials. First note that for $\ell = 1$ we have

$$D(\ell=1) = (x \cdot \mathcal{D})^\ell (h \cdot x) = \nu\,, \tag{A.16}$$

and for $\ell = 2$,

$$\begin{aligned} D(\ell=2) &= (x \cdot \mathcal{D})^2 (h \cdot y)(h \cdot x) = \\ &= (x \cdot \mathcal{D})[\nu(h \cdot x)(x \cdot y) + \frac{d}{2}(h \cdot y)] = \\ &= \left[\nu^2 + \frac{d}{2}\left(\frac{d}{2} - 1\right)\right](x \cdot y) = (d-1)\left(\frac{d}{2} - 1\right)(x \cdot y)\,, \end{aligned} \tag{A.17}$$

which agrees with (A.15) for $\ell = 1, 2$.

Next, we can move to the inductive step with

$$\begin{aligned}
&D(\ell+1) = (x\cdot\mathcal{D})^\ell (x\cdot\mathcal{D})(h\cdot y)^\ell (h\cdot x) = \\
&= (x\cdot\mathcal{D})^\ell \Big\{ \ell(\nu+\ell)(h\cdot y)^{\ell-1}(x\cdot y)(h\cdot x) + (\ell+\nu)(h\cdot y)^\ell \\
&- \frac{1}{2}\ell(\ell-1)(h\cdot y)^{\ell-2}(h\cdot x)^2 - \ell(h\cdot y)^{\ell-1}(x\cdot y)(h\cdot x)\Big\} = \\
&= \ell(\nu+\ell)\,\xi D(\ell) + (\ell+\nu)C(\ell) - \frac{1}{2}\ell(\ell-1)[(x\cdot\mathcal{D})^\ell,(h\cdot x)](h\cdot y)^{\ell-2}(h\cdot x) - \ell\xi\, D(\ell)\,,
\end{aligned} \tag{A.18}$$

where again we have used the fact that $(x\cdot\mathcal{D})^\ell(h\cdot y)^{\ell-2}(h\cdot x)=0$ since we have (at least) ℓ derivatives with respect to h acting on an $(\ell-1)$-polynomial in h. Note that in deriving the recurrence relation for $D(\ell)$ we have found the function $C(\ell)$ from the previous proof. Now we can proceed exactly in the same way as we did in (A.11). Using (A.10) we find that

$$\begin{aligned}
&D(\ell+1) = \ell(\nu+\ell-1)\,\xi D(\ell) + (\ell+\nu)C(\ell) \\
&- \frac{1}{2}\ell(\ell-1)\sum_{k=0}^{\ell-1}(x\cdot\mathcal{D})^k[x\cdot\mathcal{D},h\cdot x](x\cdot\mathcal{D})^{\ell-1-k}\,(h\cdot y)^{\ell-2}(h\cdot x) = \\
&= \ell(\nu+\ell-1)\,\xi D(\ell) + (\ell+\nu)C(\ell) - \frac{1}{2}\ell(\ell-1)\sum_{k=0}^{\ell-1}(\nu+k)(x\cdot\mathcal{D})^{\ell-1}\,(h\cdot y)^{\ell-2}(h\cdot x)\,.
\end{aligned} \tag{A.19}$$

Performing the (A.12) sum we end up with

$$D(\ell+1) = \ell(\nu+\ell-1)\,\xi\, D(\ell) + (\ell+\nu)C(\ell) - \frac{1}{4}\ell^2(\ell-1)(\ell+2\nu+1)D(\ell-1)\,. \tag{A.20}$$

Substituting the expressions (A.7) and (A.15) for $C(\ell)$ and $D(\ell)$, respectively, into (A.20) we get the Gegenbauer recursion relation (A.5).

A.1.3 Induction proof III

Let us give one more example on how to prove this kind of relations. We now consider the case in which the function $E(\ell)$ reads as

$$E(\ell) = (x\cdot\mathcal{D})^\ell (h\cdot y)^{\ell-2}(h\cdot x)^2\,. \tag{A.21}$$

Then, we can prove that $E(\ell)$ is related to the Gegenbauer polynomials through the relation

$$E(\ell) = \frac{1}{4}\ell(\ell-1)(d+\ell-4)(d+\ell-3)\frac{((\ell-2)!)^2}{2^{\ell-2}}\mathcal{C}^{(\nu)}_{\ell-2}(\xi)\,. \tag{A.22}$$

Again the proof is completely analogous as the previous ones. We first check that for $\ell=2$:

$$E(\ell=2) = (x\cdot\mathcal{D})^2(h\cdot x)^2 = (x\cdot\mathcal{D})(2\nu+1)(h\cdot x) = \left(\frac{d}{2}-1\right)(d-1)\,, \tag{A.23}$$

which coincides with expression (A.22). For $\ell+1$ we have:

$$
\begin{aligned}
&E(\ell+1) = (x\cdot\mathcal{D})^{\ell}(x\cdot\mathcal{D})(h\cdot y)^{\ell-1}(h\cdot x)^2 = (x\cdot\mathcal{D})^{\ell}\Big\{(\ell-1)(\ell+\nu)\,\xi(h\cdot y)^{\ell-2}(h\cdot x)^2 \\
&+2(\ell+\nu)(h\cdot y)^{\ell-1}(h\cdot x) - \frac{1}{2}(\ell-1)(\ell-2)(h\cdot x)(h\cdot y)^{\ell-3}(h\cdot x)^2 \\
&-(\ell-1)\,2\,\xi(h\cdot y)^{\ell-2}(h\cdot x)^2 - (h\cdot y)^{\ell-1}(h\cdot x)\Big\} = \\
&= (\ell-1)(\ell+\nu-2)\,\xi E(\ell) + 2(\ell+\nu)D(\ell) \\
&-\frac{1}{2}(\ell-1)(\ell-2)[(x\cdot\mathcal{D})^{\ell},(h\cdot x)](h\cdot y)^{\ell-3}(h\cdot x)^2 - (\ell-1)2\xi\,E(\ell) - D(\ell)\,,
\end{aligned}
\tag{A.24}
$$

where we have used once more the fact that $(x\cdot\mathcal{D})^{\ell}(h\cdot y)^{\ell-3}(h\cdot x)^2 = 0$. Note that in deriving the recurrence relation for $E(\ell)$ we have found the function $D(\ell)$ from the previous proof. Proceeding exactly in the same way as we did in the previous proofs we see that

$$
\begin{aligned}
&E(\ell+1) = (\ell-1)(\ell+\nu-2)\,\xi E(\ell) + (2(\ell+\nu)-1)D(\ell) \\
&-\frac{1}{2}(\ell-1)(\ell-2)\sum_{k=0}^{\ell-1}(x\cdot\mathcal{D})^{k}[x\cdot\mathcal{D},h\cdot x](x\cdot\mathcal{D})^{\ell-1-k}\,(h\cdot y)^{\ell-3}(h\cdot x)^2 = \\
&= (\ell-1)(\ell+\nu-2)\,\xi E(\ell) + (2(\ell+\nu)-1)D(\ell) - \frac{1}{2}(\ell-1)(\ell-2)\sum_{k=0}^{\ell-1}(\nu+k)E(\ell-1)
\end{aligned}
\tag{A.25}
$$

After doing the sum in the last term we get

$$
E(\ell+1) = (\ell-1)(\ell+\nu-2)\,\xi\,E(\ell) + (2(\ell+\nu)-1)D(\ell) - \tfrac{1}{4}\ell(\ell-1)(\ell-2)(\ell+2\nu-1)E(\ell-1) \tag{A.26}
$$

Substituting the expressions (A.15) and (A.22) for $D(\ell)$ and $E(\ell)$, respectively, into (A.26) we get the Gegenbauer recursion relation (A.5).

B| Thermal Blocks for spin 1 operators

Here we give the explicit form of the thermal blocks corresponding to the decomposition (9.25). For the thermal block corresponding to the $\mathbb{T}_1$ structure we find:

$$
\begin{aligned}
g^{(1)}_{\Delta,\ell} = & \left(\frac{d(d-2)\hat{\lambda}_1}{(2\ell+d-2)(2\ell+d)}\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell} \\
& - \left(\frac{\ell(\ell-1)d(d-2)}{4(2\ell+d)(2\ell+d-4)(\ell+2d-\Delta-4)}\times\right. \\
& \left.\times\left[\ell(d(4\ell-3)+2(\ell^2-\ell(\Delta+4)+\Delta+4))\hat{\lambda}_1+(2\ell+d)(d-\Delta-2)\hat{\lambda}_2\right]\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell-2}+ \\
& + \left(\frac{\ell(\ell-2)^2(\ell-3)^2(\ell-1)d(d-2)}{16(2\ell+d-4)(2\ell+d-2)(\ell+2d-\Delta-4)}\times\right. \\
& \left.\times\left[\ell(d(2\ell-1)+(\ell-1)(\ell-\Delta-2))\hat{\lambda}_1+(2\ell+d-2)(d-\Delta-2)\hat{\lambda}_2\right]\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell-4}\,.
\end{aligned}
\tag{B.1}
$$

A similar structure holds for the second thermal block in $\mathbb{T}_2$:

$$
\begin{aligned}
g^{(2)}_{\Delta,\ell} = & \left(\frac{d(d-2)(\ell+2d-\Delta-2)}{(2\ell+d)(2\ell+d-2)(\Delta-d+1)}(\hat{\lambda}_1-\hat{\lambda}_2)\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell}+ \\
& + \left(\frac{\ell(\ell-1)^2d(d-2)}{2(2\ell+d)(2\ell+d-4)(d-\Delta-1)}\times\right. \\
& \left.\times\left[(\ell(\ell+2d-\Delta-2))\hat{\lambda}_1-(\ell^2-d^2+d+(d+\ell)\Delta)\hat{\lambda}_2\right]\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell-2}+ \\
& + \left(\frac{\ell(\ell-2)^2(\ell-3)^2(\ell-1)d(d-2)(-\ell+d-\Delta)}{16(2\ell+d-4)(2\ell+d-2)(d-\Delta-1)(\ell+2d-\Delta-4)}\times\right. \\
& \left[(\ell^3+\ell(d(d-3)-\Delta+2)+\ell^2(d+\Delta-3))\hat{\lambda}_1+\right. \\
& \left.\left.(-\ell^3+(d-2)^2(d-\Delta-1)-\ell^2(d+\Delta-3)+\ell(d(d-2\Delta-3)+5\Delta+2))\hat{\lambda}_2\right]\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell-4}\,.
\end{aligned}
\tag{B.2}
$$

We find the same thermal block for both tensor structures $\mathbb{T}_3$ and $\mathbb{T}_4$:

$$\begin{aligned} g^{(3,4)}_{\Delta,\ell} &= -\frac{\ell\, d(d-2)}{8(2\ell+d-2)(\ell+2d-\Delta-4)}\Big\{ \Big(4(\ell+2d-\Delta-4)\hat{\lambda}_2\Big)\, \widehat{C}^{(\frac{d}{2}+1)}_{\ell-1} + \\ &+ \Big((\ell-2)^2(\ell-1)\Big[\ell(2\ell+d-2)\hat{\lambda}_1 + (d^2-d(\Delta+2)-(\ell-1)(\ell+\Delta))\hat{\lambda}_2\Big]\Big)\, \widehat{C}^{(\frac{d}{2}+1)}_{\ell-3}\Big\}\,. \end{aligned} \tag{B.3}$$

And, finally, the last thermal block$\mathbb{T}_5$ has the simplest structure, given by:

$$g^{(5)}_{\Delta,\ell} = \left(\frac{\ell(\ell-1)d(d-2)}{4(\ell+2d-\Delta-4)}\Big[\ell\hat{\lambda}_1 + (d-\Delta-2)\hat{\lambda}_2\Big]\right)\widehat{C}^{(\frac{d}{2}+1)}_{\ell-2}\,. \tag{B.4}$$

Note that all thermal blocks have Gegenbauer polynomials with the same weight $\alpha = \frac{d}{2}+1$.

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