On Hodge Theory of Singular Plane Curves

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Abstract. The dimensions of the graded quotients of the cohomology of a plane curve complement $U = \mathbb{P}^2 \setminus C$ with respect to the Hodge filtration are described in terms of simple geometrical invariants. The case of curves with ordinary singularities is discussed in detail. We also give a precise numerical estimate for the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$.

1 Introduction

The Hodge theory of the complement of projective hypersurfaces has received much attention; see, for instance, Griffiths [10] in the smooth case, and Dimca–Saito [5] and Sernesi [13] in the singular case. In this paper we consider the case of plane curves and continue the study initiated by Dimca–Sticlaru [7] in the nodal case and by the author [1] in the case of plane curves with ordinary singularities of multiplicity up to 3.

In the second section we compute the Hodge–Deligne polynomial of a plane curve $C$, the irreducible case in Proposition 2.1 and the reducible case in Proposition 2.2. Using this we determine the Hodge–Deligne polynomial of $U = \mathbb{P}^2 \setminus C$ and then deduce in Theorem 2.7 the dimensions of the graded quotients of $H^2(U)$ with respect to the Hodge filtration.

In Section 3 we consider the case of arrangements of curves having ordinary singularities and intersecting transversely at smooth points. We obtain a formula in Theorem 3.1 generalizing the formulas obtained in [7] and in [1] (for these curves). In fact, the results in [1] show that this formula holds in the more general case of plane curves with ordinary singularities of multiplicity up to 3 (without assuming transverse intersection).

In the fourth section we show that the case of plane curves with ordinary singularities of multiplicity up to 4 (without assuming transverse intersection) is definitely more complicated, and the formula in Theorem 3.1 has to be replaced by the formula in Theorem 4.1 containing a correction term coming from triple points on one component through which another component of $C$ passes.

In the final section we state and prove our main result, Theorem 5.1, which expresses the difference between the Hodge filtration and the pole order filtration on $H^2(U, \mathbb{C})$ in terms of numerical invariants easy to compute in given situations. An example involving a free divisor concludes this note.
2 Hodge Theory of Plane Curve Complements

For the general theory of mixed Hodge structures we refer to [2,15]. Recall the definition of the Hodge–Deligne polynomial of a quasi-projective complex variety $X$:

$$P(X)(u,v) = \sum_{p,q} E^{p,q}(X) u^p v^q,$$

where $E^{p,q}(X) = \sum_i (-1)^i h^{p,q}(H^i_c(X))$, with

$$h^{p,q}(H^i_c(X)) = \dim \mathrm{Gr}^p_k \mathrm{Gr}^W_{p-q} H^i_c(X, \mathbb{C}),$$

the mixed Hodge numbers of $H^i_c(X)$.

This polynomial is additive with respect to constructible partitions, i.e., $P(X) = P(X \setminus Y) + P(Y)$ for a closed subvariety $Y$ of $X$. In this section we determine $P(C)$ for a (reduced) plane curve $C$.

Suppose first that the curve $C$ is irreducible, of degree $N$. Denote by $a_k, k = 1, \ldots, p$ the singular points of $C$, and let $r(C, a_k)$ be the number of irreducible branches of the germ $(C, a_k)$. Let $v: \tilde{C} \to C$ be the normalization mapping. Using the normalization map $v$ and the additivity of the Hodge–Deligne polynomial, it follows that

$$P(C)(u,v) = P(C \setminus (C)_{\mathrm{sing}}) + P((C)_{\mathrm{sing}}) = P(\tilde{C} \setminus (\cup_k v^{-1}(a_k))) + p = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1).$$

Indeed, it is known that for the smooth curve $\tilde{C}$, the genus $g = g(\tilde{C})$ is exactly the Hodge number $h^{1,0}(\tilde{C}) = h^{0,1}(\tilde{C})$. Moreover, it is known that one has the formula

$$(2.1) \quad g = \frac{(N-1)(N-2)}{2} - \sum_k \delta(C, a_k),$$

relating the genus, the degree and the local singularities of $C$, and the $\delta$-invariants can be computed using the formula

$$(2.2) \quad 2\delta(C, a_k) = \mu(C, a_k) + r(C, a_k) - 1,$$

where $\mu(C, a_k)$ is the Milnor number of the singularity $(C, a_k)$. For both formulas above, see [11, p. 85]. This proves the following result.

**Proposition 2.1** With the above notation and assumptions, we have the following for an irreducible plane curve $C \subset \mathbb{P}^2$.

(i) The Hodge–Deligne polynomial of $C$ is given by

$$P(C)(u,v) = uv - gu - gv + 1 - \sum_k (r(C, a_k) - 1),$$

with $g$ given by the formula (2.1).

(ii) $H^0(C) = \mathbb{C}$ is pure of type $(0,0)$.

(iii) $H^2(C) = \mathbb{C}$ is pure of type $(1,1)$.

(iv) The mixed Hodge numbers of the MHS on $H^1(C)$ are given by

$$h^{0,0}(H^1(C)) = \sum_k (r(C, a_k) - 1), \quad h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = g.$$
In particular, one has the following formulas for the first Betti number of \( C \):

\[
b_1(C) = \sum_k \left( r(C, a_k) - 1 \right) + 2g = (N - 1)(N - 2) - \sum_k \mu(C, a_k).
\]

Now we consider the case of a curve \( C \) having several irreducible components. More precisely, let \( C = \bigcup_{j=1}^{r} C_j \) be the decomposition of \( C \) as a union of irreducible components \( C_j \), let \( v_j : \tilde{C}_j \to C_j \) be the normalization mappings, and set \( g_j = g(\tilde{C}_j) \). Suppose that the curve \( C_j \) has degree \( N_j \), denote by \( a^j_k \) for \( k = 1, \ldots, p_j \) the singular points of \( C_j \), and let \( r(C_j, a^j_k) \) be the number of branches of the germ \((C_j, a^j_k)\). Then the formulas (2.1) and (2.2) can be applied to each irreducible curve \( C_j \) as well as Proposition 2.1.

Let \( A \) be the union of the singular sets of the curves \( C_j \). Let \( B \) be the set of points in \( C \) sitting on at least two distinct components \( C_i \) and \( C_j \). For \( b \in B \), let \( n(b) \) be the number of irreducible components \( C_j \) passing through \( b \). By definition, \( n(b) \geq 2 \). Moreover, note that the sets \( A \) and \( B \) are not disjoint in general, and their union is precisely the singular set of \( C \).

Using the additivity of Hodge–Deligne polynomials we get

\[
P(C) = P(C_1 \cup \cdots \cup C_r) = \sum_{j=1}^{r} P(C_j) + \sum_{0 \leq i_1 < \cdots < i_r \leq r} (-1)^{i_1} P(C_{i_1} \cap \cdots \cap C_{i_r}).
\]

The first sum is easy to determine using Proposition 2.1:

\[
\sum_{j=1}^{r} P(C_j)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u + \left( \sum_{j=1}^{r} g_j \right) v + r - \sum_{j,k} \left( r(C_j, a^j_k) - 1 \right).
\]

Consider now the alternating sum, where \( l \geq 2 \). The only points of \( C \) that give a contribution to this sum are the points in \( B \). Now, for a point \( b \in B \), its contribution to the alternating sum is clearly given by

\[
c(b) = \left( \frac{n(b)}{2} \right) + \left( \frac{n(b)}{3} \right) - \cdots + (-1)^{n(b)-1} \left( \frac{n(b)}{n(b)} \right) = -n(b) + 1.
\]

**Proposition 2.2** With the above notation and assumptions, we have the following for a reducible plane curve \( C = \bigcup_{j=1}^{r} C_j \).

(i) The Hodge–Deligne polynomial of \( C \) is given by

\[
P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u + \left( \sum_{j=1}^{r} g_j \right) v + r - \sum_{j,k} \left( r(C_j, a^j_k) - 1 \right) - \sum_{b \in B} \left( n(b) - 1 \right).
\]

with \( g_j \) given by the formula (2.1).

(ii) \( H^0(C) = \mathbb{C} \) is pure of type \((0, 0)\).

(iii) \( H^2(C) = \mathbb{C}^r \) is pure of type \((1,1)\).

(iv) The mixed Hodge numbers of the MHS on \( H^1(C) \) are given by

\[
h^{0,0}(H^1(C)) = \sum_{j,k} \left( r(C_j, a^j_k) - 1 \right) + \sum_{b \in B} \left( n(b) - 1 \right) - r + 1,
\]

\[
h^{1,0}(H^1(C)) = \sum_{j=1}^{r} g_j.
\]
In particular, one has the following formula for the first Betti number of $C$:

$$b_1(C) = \sum_{j,k} \left((r(C_j, a_k^j) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1 + 2 \sum_{j=1}^{r} g_j \right).$$

Note that a point in the intersection $A \cap B$ will give a contribution to the last two sums in the above formula for $P(C)$.

**Example 2.3** Suppose $C$ is a nodal curve. Then for each singularity $a_k^j \in A$ one has $a_k^j \notin B$ (otherwise we get worse singularities than nodes) and $r(a_k^j) = 2$. Moreover, each point $b \in B$ satisfies $n(b) = 2$. It follows that in this case we get

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2,$n_2$$

with $n_2$ the number of nodes of $C$. More precisely, in this case we have $n_2 = n_2' + n_2''$, where $n_2'$ (resp. $n_2''$) is the number of nodes of $C$ in $A_0$ (resp. in $B$), and one clearly has

$$n_2' = S_1 := \sum_{j,k} \left((r(C_j, a_k^j) - 1), \quad n_2'' = S_2 := \sum_{b \in B} (n(b) - 1).$$

**Example 2.4** Suppose $C$ has only nodes and ordinary triple points as singularities. Then let $n_3$ be the number of triple points and note that we can write, as above, $n_3 = n_3' + n_3''$, where $n_3'$ (resp. $n_3''$) is the number of triple points of $C$ in $A_0 = A \setminus B$ (resp. in $B$). For a point $a \in A_0$, the contribution to the sum $S_1$ is 2, while the contribution to the sum $S_2$ is 0.

A point $b \in B$ can be of two types. The first type, corresponding to the partition $3 = 1 + 1 + 1$, is when $b$ is the intersection of three components $C_j$, all smooth at $b$. The contribution of such a point $b$ is 0 to the sum $S_1$ and 2 to the sum $S_2$.

The second type, corresponding to the partition $3 = 2 + 1$, is when $b$ is the intersection of two components, say $C_i$ and $C_j$, such that $C_i$ has a node at $b$, and $C_j$ is smooth at $b$. The contribution of such a point $b$ is 1 to the sum $S_1$ and 1 to the sum $S_2$.

It follows that the contribution of any triple point to the sum $S_1 + S_2$ is equal to 2. Since the double points in $C$ can be treated exactly as in Example 2.3, this yields the following:

$$P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2 - 2n_3.$$

When there are only triple points in $B$ of the first type, we obviously have the following additional relations

$$S_1 = n_2' + 2n_3', \quad S_2 = n_2'' + 2n_3''.$$

**Example 2.5** Suppose $C$ has only ordinary points of multiplicity 2, 3, and 4 as singularities. Then let $n_4$ be the number of points of multiplicity 4 and note that we can write, as above, $n_4 = n_4' + n_4''$, where $n_4'$ (resp. $n_4''$) is the number of points of multiplicity 4 of $C$ in $A_0 = A \setminus B$ (resp. in $B$). For a point $a \in A_0$ of multiplicity 4, the contribution to the sum $S_1$ is 3, while the contribution to the sum $S_2$ is 0.
A point \( b \in B \) can be of 4 types. The first type, corresponding to the partition \( 4 = 1 + 1 + 1 + 1 \), is when \( b \) is the intersection of 4 components \( C_j \), all smooth at \( b \). The contribution of such a point \( b \) is 0 to the sum \( S_1 \) and 3 to the sum \( S_2 \).

The second type, corresponding to the partition \( 4 = 2 + 1 + 1 \), is when \( b \) is the intersection of 3 components, say \( C_i, C_j, \) and \( C_k \), such that \( C_i \) has a node at \( b \), and \( C_j \) and \( C_k \) are smooth at \( b \). The contribution of such a point \( b \) is 1 to the sum \( S_1 \) and 2 to the sum \( S_2 \).

The third type, corresponding to the partition \( 4 = 2 + 2 \), is when \( b \) is the intersection of 2 components, say \( C_i \) and \( C_k \), such that \( C_i \) and \( C_k \) have a node at \( b \). The contribution of such a point \( b \) is 2 to the sum \( S_1 \) and 1 to the sum \( S_2 \).

The fourth type, corresponding to the partition \( 4 = 3 + 1 \), is when \( b \) is the intersection of 2 components, say \( C_i \) and \( C_k \), such that \( C_i \) has a triple point at \( b \), and \( C_k \) is smooth at \( b \). The contribution of such a point \( b \) is 2 to the sum \( S_1 \) and 1 to the sum \( S_2 \).

It follows that the contribution of any point of multiplicity 4 to the sum \( S_1 + S_2 \) is equal to 3. Since the double and triple points in \( C \) can be treated exactly as in Example 2.4, this yields the following:

\[
P(C)(u, v) = ruv - \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v + r - n_2 - 2n_3 - 3n_4.
\]

When there are only points of multiplicity 4 in \( B \) of the first type, then we obviously have the following additional relations

\[
S_1 = n'_2 + 2n'_3 + 3n'_4, \quad S_2 = n''_2 + 2n''_3 + 3n''_4.
\]

Let us now look at the cohomology of the smooth surface \( U = \mathbb{P}^2 \setminus C \). By the additivity we get \( P(U) = P(\mathbb{P}^2) - P(C) \), where \( P(\mathbb{P}^2) = u^2v^2 + uv + 1 \). This yields the following consequence.

**Corollary 2.6**

\[
P(U)(u, v) = u^2v^2 - (r - 1)uv + \left( \sum_{j=1}^{r} g_j \right) u + \left( \sum_{j=1}^{r} g_j \right) v - (r - 1) + \sum_{j,k} \left( (r(C_j, a_k) - 1) + \sum_{b \in B} (n(b) - 1) \right).
\]

The contribution of \( H^2(U, \mathbb{C}) \) to \( P(U) \) is the term \( u^2v^2 \), and that of \( H^2(U, \mathbb{C}) \) is the term \( -(r - 1)uv \). Moreover, the dimension \( \dim \text{Gr}_1^k H^3(U, \mathbb{C}) \) is the number of independent classes of type \((1, 2)\), which correspond to classes of type \((1, 0)\) in \( H^2(U) \), and hence to the terms in \( u \) in \( P(U) \). For both statements see the proof of [1, Theorem 2.1]. This proves the following result.

**Theorem 2.7**

\[
\dim \text{Gr}_1^k H^3(U, \mathbb{C}) = \sum_{j=1}^{r} g_j
\]
and
\[
\dim \text{Gr}^2_r H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j + \sum_{j,k}(r(C_j, a^j_k) - 1) + \sum_{b \in B} (n(b) - 1) - r + 1.
\]

In particular, all the components \( C_j \) of the curve \( C \) are rational if and only if \( H^2(U) \) is pure of type \((2, 2)\).

**Example 2.8** Suppose \( C \) has only ordinary points of multiplicity 2, 3, and 4 as singularities. Let \( n_k \) be the number of points of multiplicity \( k \), for \( k = 2, 3, 4 \); then using Example 2.5, we get the formula
\[
\dim \text{Gr}^2_r H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j - r + 1 + n_2 + 2n_3 + 3n_4.
\]

### 3 Arrangements of Transversely Intersecting Curves

Recall that \( C = \bigcup_{j=1}^r C_j \) is the decomposition of \( C \) as a union of irreducible components \( C_j \), and the curve \( C_j \) has degree \( N_j \). In this section we assume that any curve \( C_j \) has only ordinary multiple points as singularities and let \( n_k(C_j) \) be the number of ordinary points on \( C_j \) of multiplicity \( k \). We also assume that the intersection of any two distinct components \( C_i \) and \( C_j \) is transverse, i.e., the points in \( C_i \cap C_j \) are nodes of the curve \( C_i \cup C_j \). This implies in particular that \( A \cap B = \emptyset \). The formulas (2.1) and (2.2) yield the equality:
\[
g_j = \frac{(N_j - 1)(N_j - 2)}{2} - \frac{1}{2} \sum_{k}(\mu(C_j, a^j_k) + r(C, a^j_k) - 1).
\]

Using this, Theorem 2.7 gives the formula
\[
\dim \text{Gr}^2_r H^2(U, \mathbb{C}) = \sum_{j=1}^r \frac{(N_j - 1)(N_j - 2)}{2} - \frac{1}{2} \sum_{j,k}(\mu(C_j, a^j_k) - r(C, a^j_k) + 1)
+ \sum_{b \in B}(n(b) - 1) - r + 1.
\]

If \( a^j_k \) is an ordinary \( m \)-multiple point on the curve \( C_j \), one has \( \mu(C_j, a^j_k) = (m - 1)^2 \), and hence
\[
\mu(C_j, a^j_k) - r(C, a^j_k) + 1 = (m - 1)(m - 2).
\]

If we denote by \( n'_m \) (resp. \( n''_m \)) the number of \( m \)-multiple points of \( C \) coming from just one component \( C_j \) (resp. from the intersection of several components \( C_j \)), we see that we have
\[
\sum_{j,k}(\mu(C_j, a^j_k) - r(C, a^j_k) + 1) = \sum_m (m - 1)(m - 2)n'_m.
\]

This equality explains the contribution of the points in \( A \). Now let \( b \in B \) such that \( n(b) = m \). The number of such points is precisely \( n''_m \). It follows that
\[
\sum_{b \in B}(n(b) - 1) = \sum_m (m - 1)n''_m.
\]
Let \(1 \leq i < j \leq r\) and consider the intersection \(C_i \cap C_j\). It contains exactly \(N_i N_j\) points, since \(C_i\) and \(C_j\) intersects transversely. The sum \(S = \sum_{1 \leq i < j \leq r} N_i N_j\) represents the number of all such intersection points. Note that a point \(b \in B\) is counted in this sum exactly \(\binom{m(b)}{2}\) times. This yields the formula

\[
2S = \sum_m m(m-1)n''_m.
\]

These formulas give the following result.

**Theorem 3.1** With the above assumptions and notation, one has

\[
\dim \text{Gr}_p^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,
\]

with \(n_m = n'_m + n''_m\) the number of ordinary \(m\)-tuple points of \(C\).

The following consequence of Theorems 2.7 and 3.1 applies in particular to any projective line arrangement.

**Corollary 3.2** Assume that \(C = \bigcup_{j=1}^r C_j\) is the decomposition of \(C\) as a union of irreducible components \(C_j\), with any curve \(C_j\) having only ordinary multiple points as singularities and being rational, i.e., \(g_j = 0\). If the intersection of any two distinct components \(C_i\) and \(C_j\) is transverse, i.e., the points in \(C_i \cap C_j\) are nodes of the curve \(C_i \cup C_j\), then one has

\[
\dim H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,
\]

with \(n_m\) the number of ordinary \(m\)-tuple points of \(C\).

## 4 Curves with Ordinary Singularities of Multiplicity \(\leq 4\)

Let \(C \in \mathbb{P}^2\) be a curve of degree \(N\) having only ordinary singular points of multiplicity at most 4. Set \(U = \mathbb{P}^2 \setminus C\), and let \(C = \bigcup_{j=1}^r C_j\) be the decomposition of \(C\) in irreducible components. Then

\[
P(C) = \sum_{j=1}^r P(C_j) - \sum_{0 \leq i < j \leq r} P(C_i \cap C_j) + \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) - \sum_{0 \leq i < j < k < \ell \leq r} P(C_i \cap C_j \cap C_k \cap C_\ell).
\]

Let \(a^m_j\) denote the number of singular points of multiplicity \(m\) that belong to the component \(C_j\) (note that a point can be singular on two components, being a node on each of them).

Denote by \(b^k_j\) (resp. \(b^k\)) the number of triple points (resp. points of multiplicity 4) of \(C\) that are intersection of exactly \(k\) components, for \(k = 2, 3\) (respectively \(k = 3, 4\)). Let \(b^4_j\) (resp. \(b^4\)) be the number of singular points \(p\) of multiplicity 4 in \(C\) representing the intersection of exactly 2 components, such that one of which has a triple point at
$p$ (resp. each one has a node at $p$). Then one has
\[ \sum_{0 \leq i < k < l \leq r} P(C_i \cap C_j) = \sum_{0 \leq i < j \leq r} N_i N_j - b_3^2 - 3b_4^2 - 2b_4^3. \]

Indeed, a point of type $b_3^2$ (resp. $b_4^2$, resp. $b_4^3$) occurs only in one intersection $C_i \cap C_j$ and has the multiplicity 2 (resp. 3, resp. 4) in this intersection. A point of type $b_4^3$ occurs in 3 intersections $C_i \cap C_j$ with multiplicities 1, 2, 2, and this accounts for the correction term $-2b_4^3$. Then one has
\[ \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) = b_3^2 + b_4^3 + \binom{4}{3} b_4^4 \]
and
\[ \sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l) = b_4^4. \]

Hence, by Proposition 2.1, we get the following:
\[
P(C)(u, v) = \frac{r^2}{r - 1} \left( \sum_{j=1}^{r} g_j \right) u - \left( \sum_{j=1}^{r} g_j \right) v - \left( \sum_{j=1}^{r} (a_j^3 + 3a_j^4) \right) v + \left( \sum_{j=1}^{r} (a_j^3 + 3a_j^4 + 6a_j^4) \right)
\]
\[ - \sum_{j=1}^{r} (a_j^3 + 3a_j^4) + \sum_{j=1}^{r} N_i N_j - b_3^2 - 3b_4^2 - 2b_4^3 - 3b_4^3 - b_4^4. \]

Finally, we get
\[
\dim Gr_{r, 2} H^2(U) = \frac{r}{2} \left( \sum_{j=1}^{r} (g_j + a_j^3 + 3a_j^4 + 6a_j^4 - 1) \right) + \sum_{j=1}^{r} N_i N_j + 1 - \left( \sum_{j=1}^{r} (a_j^3 + b_j^3) \right)
\]
\[ - 3 \left( \sum_{j=1}^{r} (a_j^3 + b_j^3 + b_j^4 + b_j^4) \right) + b_4^2 = \frac{(N - 1)(N - 2)}{2} - n_3 - 3n_4 + b_4^2, \]
with $n_m$ the number of ordinary $m$-tuple points of $C$.

**Theorem 4.1** Let $C \subset \mathbb{P}^2$ be a curve of degree $N$ having only ordinary singular points of multiplicity at most 4. If $U = \mathbb{P}^2 \setminus C$, then one has
\[
\dim Gr_{r, 2} H^2(U, C) = \frac{(N - 1)(N - 2)}{2} - \sum_{m=3}^{4} \binom{m-1}{2} n_m + b_4^2, \]
with $n_m$ the number of ordinary $m$-tuple points of $C$ and $b_4^2$ the number of singular points $p$ of $C$ that are smooth on one component $C_i$ of $C$ and have multiplicity 3 on the other component $C_j$ of $C$ passing through $p$. 
5 Pole Order Filtration Versus Hodge Filtration for Plane Curve Complements

For any hypersurface $V$ in a projective space $\mathbb{P}^n$, the cohomology groups $H^*(U, \mathbb{C})$ of the complement $U = \mathbb{P}^n \setminus V$ have a pole order filtration $P^k$; see, for instance, [8]. By the work of Deligne, Dimca [3] and M. Saito [12], one has

$$P^k H^m(U, \mathbb{C}) \subset P^{k+1} H^m(U, \mathbb{C})$$

for any $k$ and any $m$. For $m = 0$ and $m = 1$, the above inclusions are in fact equalities (the case $m = 0$ is obvious and the case $m = 1$ follows from the equality $H^1(U, \mathbb{C}) = H^1(U, \mathbb{C})$). For $m = 2$, we have again that $P^k H^2(U, \mathbb{C}) = P^{k+1} H^2(U, \mathbb{C})$ for $k = 0, 1$ for obvious reasons, but one can get strict inclusions

$$P^2 H^2(U, \mathbb{C}) \subsetneq P^2 H^2(U, \mathbb{C})$$

already in the case when $V = C$ is a plane curve; see [5], Remark 2.5, or [4]. However, to give such examples of plane curves was until now rather complicated. We give below a numerical condition that tells us exactly when the above strict inclusion holds.

We first need to recall some basic definitions. Let $S = \mathbb{C}[x, y, z]$ be the graded ring of polynomials with complex coefficients, where $S_r$ is the vector space of homogeneous polynomials of $S$ of degree $r$. For a homogeneous polynomial $f$ of degree $N$, define the Jacobian ideal of $f$ to be the ideal $I_f$ generated in $S$ by the partial derivatives $f_x, f_y, f_z$ of $f$ with respect to $x$, $y$, and $z$. The graded Milnor algebra of $f$ is given by

$$M(f) = \bigoplus_r M(f)_r = S/I_f.$$

Note that the dimensions $\dim M(f)_r$ can be easily computed in a given situation using some computer software e.g., Singular.

Let $C \subset \mathbb{P}^2$ be the curve defined by $f = 0$, and suppose that $P$ is a singular point of $C$ with local equation $g = 0$. Define the Tjurina number $\tau(C, P)$ of $C$ at the point $P$ by

$$\tau(C, P) = \dim_{\mathbb{C}} \frac{O_P}{(g, I_g)},$$

where $O_P$ is the local ring of germs of regular functions at $P$ and $(g, I_g)$ is the ideal generated by $g$ and its Jacobian $I_g$. The Tjurina number $\tau(C)$ of a curve $C$ is given by the sum of the Tjurina numbers of all the singularities of $C$. Now we can state the main result of this section.

**Theorem 5.1** Let $C : f = 0$ be a reduced curve of degree $N$ in $\mathbb{P}^2$ having only weighted homogeneous singularities and let $C_i$ for $i = 1, \ldots, r$ be the irreducible components of $C$. If $U = \mathbb{P}^2 \setminus C$, then

$$\dim P^2 H^2(U, \mathbb{C}) - \dim P^2 H^2(U, \mathbb{C}) = \tau(C) + \sum_{i=1}^r g_i - \dim M(f)_{2N-3},$$

where $\tau(C)$ is the global Tjurina number of $C$ and $g_i$ is the genus of the normalization of $C_i$ for $i = 1, \ldots, r$. 


In particular we get the following result, which yields a new proof for [7, Theorem 1.3].

**Corollary 5.2** If a reduced plane curve has only nodes as singularities, then one has

$$\dim M(f)_{2N-3} = \tau(C) + \sum_{i=1}^{r} g_i.$$ 

**Proof** Indeed, it is known that for a nodal curve one has the equality $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$; see [2] or [12].

Note that we have the following obvious consequence of Theorem 2.7.

**Corollary 5.3** For a reduced plane curve $C$, one has

$$\dim P^2H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) \leq \sum_{i=1}^{r} g_i.$$ 

**Proof** Indeed, Theorem 2.7 can be restated as

$$\dim H^2(U, \mathbb{C}) - \dim F^2H^2(U, \mathbb{C}) = \sum_{i=1}^{r} g_i$$

in view of the equality $F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$; see proof of [4, Cor. 1.32, p. 185].

**Remark 5.4** If a reduced plane curve $C$ has only rational irreducible components, i.e., $g_i = 0$ for all $i$, then the above inequality implies $F^2H^2(U, \mathbb{C}) = P^2H^2(U, \mathbb{C})$. This result can be regarded as an improvement of a part of [5, Remark 2.5], where the result is claimed only for curves with nodes and cusps as singularities.

The above discussion also implies the following result, which can be regarded as a generalization of [1, Theorem 4.1 (A)].

**Corollary 5.5** If a reduced plane curve $C : f = 0$ has only weighted homogeneous singularities, then one has

$$0 \leq \dim M(f)_{2N-3} - \tau(C) \leq \sum_{i=1}^{r} g_i.$$ 

In particular, if in addition the curve $C$ has only rational irreducible components, then one has $\dim M(f)_{2N-3} = \tau(C)$.

Now we give the proof of Theorem 5.1. Corollary 1.3 in [8] implies that

$$\dim P^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) + \tau(C) - \dim M(f)_{2N-3}.$$ 

On the other hand, Theorem 2.7 and the fact $\dim F^1H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})$ yield

$$\dim F^2H^2(U, \mathbb{C}) = \dim H^2(U, \mathbb{C}) + \sum_{i=1}^{r} g_i,$$

which clearly completes the proof of Theorem 5.1.
**Example 5.6** In this example we present a free divisor $C : f = 0$, whose irreducible components consist of 12 lines and one elliptic curve and where

$$F^2H^2(U, C) \neq P^2H^2(U, C).$$

Let $f = xyz(x^3 + y^3 + z^3) \cdot [(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3]$. If we consider the pencil of cubic curves $(x^3 + y^3 + z^3, xyz)$, then the curve $C$ contains all the singular fibers of this pencil, and this accounts for the 12 lines given by

$$xyz\left[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3\right] = 0$$

and the elliptic curve (hence of genus 1) given by $x^3 + y^3 + z^3 = 0$. Then $C$ is a free divisor (see [14]) or by a direct computation using Singular, which shows that $I = J_f$, where $I$ is the saturation of the Jacobian ideal $J_f$; see [6, Remark 4.7]. The direct computation by Singular also yields $r(C) = 156$ and $\dim M(f)_{2N-3} = \dim M(f)_{27} = 156$. Moreover, applying [9, Corollary 1.5], we see via a Singular computation that all singularities of the curve $C$ are weighted homogeneous. Alternatively, there are 12 nodes, 3 in each of the 4 singular fibers of the pencils (which are triangles), and the 9 base points of the pencil, each an ordinary point of multiplicity 5. Each of the 12 lines contains exactly 3 of these base points, and they are exactly the intersection of the elliptic curve with the line. This description implies that there are no other singularities in accord with $12 + 9 \times 16 = 156 = r(C)$. It follows from Theorem 5.1 that $\dim P^2H^2(U, C) - \dim F^2H^2(U, C) = 1$. Hence, the presence of a single irrational component of $C$ leads to $F^2H^2(U, C) \neq P^2H^2(U, C)$.

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