Aspects of Conformal Field theory

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Abstract

Quantum field theories are very good at describing the world around us but use complicated computations that cannot always be solved exactly. Introducing conformal symmetry to quantum field theory can reduce this complexity and allow for quite simple calculation in the best case. This report aims to describe the critical part of the Ising model in 2 dimensions using conformal field theory while assuming only some knowledge of quantum mechanics and complex analysis from the reader. This is done by using the book *Conformal Field Theory* as the source for information about conformal field theory.

Sammanfattning

Kvantfältteorier är mycket bra på att beskriva verkligheten runt om oss men de använder sig av avancerade beräkningar som inte alltid kan lösas exakt. Genom att ge systemet konform symmetri så kan dessa avancerade beräkningar förenklas och bli ganska enkla i de bästa fallen. Målet med denna rapport är att beskriva hur en modell som kallas för "Ising model"kan beskrivas i sitt kritiska tillstånd med hjälp utav konform fältteori. Läsaren antas kunna kvantmekanik samt komplex analys. Informationen om konform fältteori hämtas ifrån boken *Conformal Field Theory*.

Contents

1 Introduction

When it comes to different models of the physical world around us quantum field theories are some of the best theories we have for describing the small fundamental aspects of reality with an example being the standard model used in particle physics. Although these models provide a very accurate description of the world the computations involved are usually quite complicated and sometimes approximations have to be used. By introducing the concept of conformal symmetry, that is a symmetry where angles between curves are locally preserved we can reduce the complexity of the calculations involved. Quantum field theories which are invariant under these conformal transformations are called conformal field theories.

Another interesting theoretical model is the 2D Ising model which describes a lattice of particles with spin up or down. These particles then also have an interaction with their nearest neighbour where the energy of the interaction is based upon if the spins of the two particles agree or disagree. The interesting property of this model is its ability to show a phase transition even while being a very simple statistical model.

The property of conformal field theories to simplify complicated calculations has made them find quite a few applications in areas where one wants to be able to exactly solve a system which is described by quantum field theories. One example of an important theory in theoretical physics which uses conformal field theories is string theory which uses a 2 dimensional conformal field theory in its description of reality. Another example of the uses of conformal field theory is that the critical temperature part of the 2D Ising model can be described using 2D conformal field theory. These uses among others make conformal field theories interesting theoretical models to investigate in order to possibly find new uses for the model or to find solutions to problems which are not solved by the use of more complicated models like quantum field theory.

The goal of this report is to replicate the critical part of the Ising model using conformal field theory along with an explanation of the fundamentals of conformal field theory. This will be achieved by first getting to conformal field theory from a knowledge of quantum mechanics and some complex analysis and then using conformal field theory to construct the critical point of the 2D Ising model. The source for the information about conformal field theory and its relation to the Ising model used in this report is the book *Conformal Field Theory*[1] which is recommended if further reading into the field of conformal field theory is of interest. The notation used in this report will match the notation used in *Conformal Field Theory.*

2 Theory

2.1 Simple quantum fields

One of the simplest systems in quantum field theory is the free scalar field which is described by the action (page 16 equation 2.1[1])

$$S[\varphi] = \int dx dt \mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi) \tag{1}$$

$$\mathcal{L} = \frac{1}{2}\left\{\frac{1}{c^2}\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right\} \tag{2}$$

$\mathcal{L}$ in this equation is the Lagrangian density which is oftentimes just called the Lagrangian, m is the mass of the field and c is the speed of light if we are operating in a relativistic theory otherwise it is some characteristic velocity of the theory. $c = 1, \hbar = 1$ units will be used in this report. We consider a discrete 1 dimensional chain with lattice spacing a, sites at $x = an$ and N total sites which has periodic boundary conditions at these sites. Using the canonical formalism along with discrete Fourier transforms and raising and lowering operators (page 16-18[1] for the full process and definitions) we obtain the following expression for the time dependence of the field:

$$\varphi_n(t) = \sum_{k=0}^{N-1} \sqrt{\frac{2}{Na\omega_k}} \left[e^{i(2\pi kn/N - \omega_k t)} a_k(0) + e^{-i(2\pi kn/N - \omega_k t)} a_k^\dagger(0)\right] \tag{3}$$

Where a_k is the lowering operator and $a_k^\dagger$ is the raising operator. Usually in quantum field theory we operate in the Heisenberg picture where the time dependence of the systems is placed in the operators rather than in the states themselves. They have the commutator

$$[a_k, a_q^\dagger] = \delta_{kq} \tag{4}$$

Where δ_{kq} is the Kronecker delta. The defined ground state is:

$$a_k |0\rangle = 0 \quad \forall k \tag{5}$$

If we want to have a non-discrete system we need to take the continuum limit of the lattice which is done by letting the lattice spacing a go to 0. Using the continuum creation and annihilation operators we then obtain (see page 19[1] for details)

$$\varphi(x) = \int \frac{dp}{2\pi} \left\{a(p)e^{i(px - \omega(p)t)} + a^\dagger(p)e^{-i(px - \omega(p)t)}\right\} \tag{6}$$

Where $a(p)$ is the annihilation operator and $a^\dagger(p)$ is the creation operator. They have the commutator:

$$[a(p), a^\dagger(p')] = 2\pi\delta(p - p') \tag{7}$$

Where δ is the Dirac delta function. Equation (6) permits the splitting of the field into the positive frequency part (the part that only contains annihilation operators) and the negative frequency part (which only contains creation operators). Of the exited states which this field permits the simplest are the elementary excitations: $a^\dagger(p)\left|0\right\rangle$ which have the dispersion relation

$$\omega(p) = \sqrt{m^2 + p^2} \tag{8}$$

This type of dispersion relation is one usually seen for relativistic particles so these different states of the field represents a different amount of free particles. They are free since there is no interaction between the particles because the total energy of the state of the field is just a sum of the individual energies of the particles. The commutator in equation (4) shows us that if we swap the momenta of the two states we obtain the same pair of states that we started with so the particles here are bosons. The ground state is usually called a vacuum in quantum field theory since that state corresponds to having no particles. The Hilbert space created from repeated applications of the creation operator is called a Fock space. A special ordering of operators is often used in quantum field theory called normal ordering and is defined by putting all operators that annihilate the vacuum to right such as with

$$: a(p)a^\dagger(p) : = a^\dagger(p)a(p) \tag{9}$$

The expectation value of a normal ordered operator operating on the vacuum state is zero by definition. This definition however is only true for fields that are free (non-interacting).

In order to obtain a fermionic field we will first need to understand the fundamentals of Grassmann variables (for a more complete explanation of Grassmann variables see chapter 2.B[1]). A Grassmann algebra is a vector space where the generators θ_i (often called Grassmann variables) have a defined anticommutative (oftentimes called antisymmetric instead) product:

$$\theta_i\theta_j = -\theta_j\theta_i \tag{10}$$

For Grassmann variables used in the description of fermionic fields ψ_i is usually used. We also need to use the anticommutator

$$\{a, b\} = ab + ba \tag{11}$$

instead of the regular commutator. This will give us the following anticommutators for the mode operators(the creation and annihilation operators) of a free fermionic field(for a full description including examples see chapter 2.1.2[1]):

$$\{a(p), a^\dagger(q)\} = (2\pi)2\omega_p\delta(p - q) \tag{12}$$

$$\{a(p), a(q)\} = \{a^\dagger(p), a^\dagger(q)\} = 0 \tag{13}$$

With a generic Lagrangian (repeated indices are summed over)

$$L = \frac{i}{2}\psi_i T_{ij}\dot{\psi}_j - V(\psi) \tag{14}$$

For complex Grassmann variables we have the Lagrangian

$$L = i\bar{\psi}_i T_{ij}\dot{\psi}_j - V(\psi) \tag{15}$$

having the defined vacuum state

$$\psi_i \left|0\right\rangle = 0 \quad \forall i \tag{16}$$

and the anticommutation relations

$$\{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0 \tag{17}$$

$$\{\psi_i, \psi_j^\dagger\} = (T^{-1})_{ij} \tag{18}$$

2.2 Path integrals

A method widely used in quantum field theory is the method of path integrals which can provide a easy way to bridge the gap between fundamental fields and statistical physics. The probability amplitude of a field going from the state $\varphi_i(x, t_i)$ to $\varphi_f(x, t_f)$ in path integrals is

$$\langle\varphi_f(x, t_f)|\varphi_i(x, t_i)\rangle = \int [d\varphi(x, t)]e^{iS[\varphi]} \tag{19}$$

The action $S[\varphi]$ is the action of one specific path from $\varphi_i(x, t_i)$ to $\varphi_f(x, t_f)$ and the differential $[d\varphi(x, t)]$ implies that the integral is taken over all possible paths from $\varphi_i(x, t_i)$ to $\varphi_f(x, t_f)$. This computation can be quite difficult, if not impossible to solve exactly and is one of the reasons a simplification is very welcome. The path integral method has the advantage of easily being able to be compatible with special relativity since we do not make time a special dimension so we can easily transfer over to 4 dimensional spacetime. Also if the field used is Lorentz invariant in a non-quantum theory then that invariance is preserved by the path integral method. For fermions we need to translate this expression into one that uses Grassmann variables (see chapter 2.2.2[1] for the full derivation)

$$\langle\psi_f(x, t_f)|\psi_i(x, t_i)\rangle = \int [d\bar{\psi}d\psi]e^{iS[\bar{\psi},\psi]} \tag{20}$$

With the same interpretation as the bosonic path integral. More common than these types of path integrals is the scattering amplitude between a number of free particles (also called asymptotic states). These scattering amplitudes are obtained in practise from correlation functions. The n-point correlation function for a point particle is

$$\langle x(t_1)x(t_2)...x(t_n)\rangle = \langle 0|\mathcal{T}(\hat{x}(t_1)\hat{x}(t_2)...\hat{x}(t_n))|0\rangle \tag{21}$$

Where we used $\mathcal{T}$ as the time ordering operator which places the operators in increasing order of time from right to left. For the remainder of this report this time ordering operator will not be explicitly written out but is always implicit inside correlation functions. Path integrals can be used to calculate the correlation function in the following way (see chapter 2.3.1 and 2.3.2[1] for derivation)

$$\langle x(\tau_1)x(\tau_2)...x(\tau_n)\rangle = \frac{\int[dx]x(\tau_1)x(\tau_2)...x(\tau_n)exp(-S[x(\tau)])}{\int[dx]exp(-S[x(\tau)])} \tag{22}$$

We used the change of variable known as the Wick rotation:

$$t \rightarrow -i\tau \tag{23}$$

Where τ is real so we must integrate over time along the imaginary axis. We then obtain the real part of the correlation functions through analytic continuation. One advantage of this change of variable is that our metric changes from a Minkowski metric to a Euclidean metric and this is why this procedure is also called the Euclidean formalism. For the rest of the report we will be working in the Euclidean formalism but we will be replacing τ with t so that the dimensions of the systems will be clear.

In order to be able to switch between the normal ordering and the time ordering of operators we need some relation between the two for the case of free fields. This relation comes in the form of Wick's theorem which makes use of contractions which are defined in the following way

$$:\phi_1\overbrace{\phi_2\phi_3\phi_4}:=:\phi_1\phi_3:\langle\phi_2\phi_4\rangle \tag{24}$$

So the contraction of two operators just means that we take them out of the normal ordering and multiply with their 2-point function. For bosons Wick's theorem then states the following: The time ordered product is equal to the normal ordered product, plus all the possible ways one could contract the fields within the normal ordered product. For example

$$\begin{aligned}\mathcal{T}(\phi_1\phi_2\phi_3) =&: \phi_1\phi_2\phi_3 : + : \overbrace{\phi_1\phi_2}\phi_3 : + \\ &: \overbrace{\phi_1\phi_2\phi_3} : + : \phi_1\overbrace{\phi_2\phi_3} :\end{aligned} \tag{25}$$

In the case of fermionic fields we also need to put a sign before each term equal to the sign obtain via anticommutation for the amount of times we need to swap places of neighbouring operators in order to get the contracted fields next to each other. Equation (25) then becomes

$$\begin{aligned}\mathcal{T}(\psi_1\psi_2\psi_3) =&: \psi_1\psi_2\psi_3 : + : \overbrace{\psi_1\psi_2}\psi_3 : - \\ &: \overbrace{\psi_1\psi_2\psi_3} : + : \psi_1\overbrace{\psi_2\psi_3} :\end{aligned} \tag{26}$$

2.3 General symmetric invariance

Symmetries play a huge role in physics as a whole therefore looking at the effects of different symmetries on fields and their properties is in order. A general action of a field depends only on the field $\boldsymbol{\phi}$ and the first derivative of the field $\partial_\mu \boldsymbol{\phi}$

$$S = \int d^d x \mathcal{L}(\boldsymbol{\phi}, \partial_\mu \boldsymbol{\phi}) \tag{27}$$

Where the d in the exponent is referring to the number of dimensions of the system. A general transformation of the position and the field will look like

$$\begin{aligned} \boldsymbol{x} &\to \boldsymbol{x}' \\ \boldsymbol{\phi}(\boldsymbol{x}) &\to \boldsymbol{\phi}'(\boldsymbol{x}') \end{aligned} \tag{28}$$

To note here is that the new field $\boldsymbol{\phi}'$ at $\boldsymbol{x}'$ can be expressed as a function of the initial field $\boldsymbol{\phi}$ at $\boldsymbol{x}$

$$\boldsymbol{\phi}'(\boldsymbol{x}') = \mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x})) \tag{29}$$

The new action can then be calculated to be

$$S' = \int d^d x \left| \frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}} \right| \mathcal{L}(\mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x})), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x}))) \tag{30}$$

For infinitesimal transformations of the general kind

$$\begin{aligned} x'^\mu &= x^\mu + \omega_a \frac{\partial x^\mu}{\partial \omega_a} \\ \boldsymbol{\phi}'(\boldsymbol{x}') &= \boldsymbol{\phi}(\boldsymbol{x}) + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}(\boldsymbol{x}) \end{aligned} \tag{31}$$

Where $\{\omega_a\}$ is a set of infinitesimally small parameters for the transformation which are all first order only. One can define a Generator of this transformation in the following way (equation 2.128[1])

$$iG_a \boldsymbol{\phi} = \frac{\partial x^\mu}{\partial \omega_a} \partial_\mu \boldsymbol{\phi} - \frac{\partial \mathcal{F}}{\partial \omega_a} \tag{32}$$

For the variation in the action $\delta S = S' - S$ obtained from the infinitesimal transformation in equation (31) we can derive the formula

$$\delta S = \int d^d x \partial_\mu j_a^\mu \omega_a \tag{33}$$

Where j_a^μ is known as the current associated with this infinitesimal transformation and is in this case equal to

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \boldsymbol{\phi})} \partial_\nu \boldsymbol{\phi} - \delta_\nu^\mu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \boldsymbol{\phi})} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{34}$$

According to Noether's theorem every symmetry of the system should correspond to some classically conserved quantity of the system with a symmetry of the system implying an invariance of the action for any variance of the fields. An invariant action simply means

$$\delta S = 0 \tag{35}$$

Noether's theorem then implies that for position dependant values of ω_a the action is invariant. This can be shown to lead to the relation

$$\partial_\mu j^\mu_a = 0 \tag{36}$$

and the conserved quantity associated with the current j^μ_a (called the conserved charge):

$$Q_a = \int d^{d-1}x j^0_a \tag{37}$$

Where $d - 1$ in the exponent refers to only taking into account the spatial dimensions of the system.

As discussed earlier correlation functions are important objects in quantum field theory so seeing how they transform under transformations like equation (28) would be good. If we consider the general correlation function (equation 2.147[1]):

$$\langle \boldsymbol{\phi}(\boldsymbol{x}_1)\boldsymbol{\phi}(\boldsymbol{x}_2)...\boldsymbol{\phi}(\boldsymbol{x}_n)\rangle = \frac{1}{Z}\int [d\boldsymbol{\phi}]\boldsymbol{\phi}(\boldsymbol{x}_1)\boldsymbol{\phi}(\boldsymbol{x}_2)...\boldsymbol{\phi}(\boldsymbol{x}_n)exp(-S[\boldsymbol{\phi}]) \tag{38}$$

Where Z is the partition function (also called the vacuum functional). One can then show that the invariance of the action and of the measure under transformations like equation (28) leads to the following result:

$$\langle \boldsymbol{\phi}(\boldsymbol{x}'_1)\boldsymbol{\phi}(\boldsymbol{x}'_2)...\boldsymbol{\phi}(\boldsymbol{x}'_n)\rangle = \langle \mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x}_1))\mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x}_2))...\mathcal{F}(\boldsymbol{\phi}(\boldsymbol{x}_n))\rangle \tag{39}$$

Another way to express the effects of this symmetry is through the so-called Ward identities which will be useful later for conformal symmetry. Any infinitesimal transformation can be expressed using its generators in the following way

$$\boldsymbol{\phi}'(\boldsymbol{x}) = \boldsymbol{\phi}(\boldsymbol{x}) - i\omega_a G_a \boldsymbol{\phi}(\boldsymbol{x}) \tag{40}$$

From this we may derive the Ward identity for the current j^μ_a associated with the transformation

$$\frac{\partial}{\partial x^\mu}\langle j^\mu_a(x)\boldsymbol{\phi}(\boldsymbol{x}_1)\boldsymbol{\phi}(\boldsymbol{x}_2)...\boldsymbol{\phi}(\boldsymbol{x}_n)\rangle = \tag{41}$$
$$-i\sum_{i=1}^{n}\delta(x - x_i)\langle \boldsymbol{\phi}(\boldsymbol{x}_1)\boldsymbol{\phi}(\boldsymbol{x}_2)...G_a\boldsymbol{\phi}(\boldsymbol{x}_i)...\boldsymbol{\phi}(\boldsymbol{x}_n)\rangle$$

For translations the conserved current is the important object known as the canonical energy-momentum tensor which is one of the central objects which will be discussed in later sections of this report along with different fields. It can be shown to be (equation 2.165)

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \boldsymbol{\phi})}\partial^\nu \boldsymbol{\phi} \tag{42}$$

Where $\eta^{\mu\nu}$ is the diagonal metric tensor of d dimensional flat spacetime. It has the corresponding conserved charge

$$P^\nu = \int d^{d-1}x T_c^{0\nu} \tag{43}$$

Which is just the regular 4-momentum obtain in relativity.

2.4 Two dimensional conformal invariance

In the rest of the report we will mostly be working in 2 dimensions since conformal invariance is much more interesting in 2 dimensions due to that fact that any analytic function is a locally conformal transformation in 2D. A transformation which is conformal, that is one that preserves the angles between curves must have the following effect on the metric tensor

$$g^{\mu\nu\prime}(\boldsymbol{x}') = \Lambda(\boldsymbol{x})g^{\mu\nu}(\boldsymbol{x}) \tag{44}$$

So it's invariant up to a scale factor.For a set of coordinates (z^0, z^1) any change of coordinates $z^\mu \to w^\mu(x)$ implies a change in the metric tensor

$$g^{\mu\nu} \to \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta} \tag{45}$$

Which combined implies the following two possible solutions:

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad and \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{46}$$

or

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad and \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{47}$$

Where equation (46) is simply the Cauchy-Riemann equations which are true for holomorphic (analytic) functions. The other equations describe what we call antiholomorphic functions. This suggest using a complex coordinate system z and $\bar{z}$ with the translation rules

$$\begin{aligned} z &= z^0 + iz^1 \\ \bar{z} &= z^0 - iz^1 \\ \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1) \end{aligned} \tag{48}$$

An alternative notation $\partial_z = \partial$ and $\partial_{\bar{z}} = \bar{\partial}$ will be used instead if the differentiation variable is clearly implied elsewhere. The metric tensor described earlier is in these coordinates

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{49}$$

Where the order of variables for μ is first $z$and then $\bar{z}$ and this also applies to μ in the energy-momentum tensor. Infinitesimal conformal transformations are all of the form(equation 5.15[1])

$$z' = z + \epsilon(z) \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1} \tag{50}$$

and with $\bar{z}'$ and $\bar{\epsilon}(\bar{z})$ being defined similarly. With the change in a dimensionless (scalar) field with no spin being

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_n \{c_n \ell_n \phi(z, \bar{z}) + \bar{c}_n \bar{\ell}_n \phi(z, \bar{z})\} \tag{51}$$

With the generators

$$\ell_n = -z^{n+1}\partial \qquad \bar{\ell_n} = -\bar{z}^{n+1}\bar{\partial} \tag{52}$$

Which have the commutation relations (equation 5.19[1])

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \\ [\ell_n, \bar{\ell}_m] &= 0 \end{aligned} \tag{53}$$

The algebra made up of these generators is called the Witt algebra.

A field that transforms in the following way when a globally conformal transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ is applied to it is called quasi-primary (equation 5.23[1]):

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{54}$$

A quasi-primary field is one whose components transform in the same way as a covariant tensor which has rank $h + \bar{h}$ where h is known as the holomorphic conformal dimension of the field while $\bar{h}$ is the antiholomorphic conformal dimension and the number of different indices for h is z and similarly for $\bar{h}$. There are also fields which show this behaviour even under locally conformal transformations and these fields are called primary.

2.5 Conformal symmetry of correlation functions

We now once again return to correlation functions to study the effect of conformal symmetry on them. Using equation (38) with the same invariance of the action and the measure and using the new holomorphic and antiholomorphic coordinates we obtain: (equation 5.24[1])

$$\langle \phi_1(w_1, \bar{w}_1)\phi_2(w_2, \bar{w}_2)...\phi_n(w_n, \bar{w}_n)\rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\phi_n(z_n, \bar{z}_n)\rangle \tag{55}$$

The conformal symmetry allows us to give a exact unique solution to the 2-point correlator:

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z_1} - \bar{z_2})^{2\bar{h}}} \tag{56}$$

Where C_{12} is some constant. However if the conformal dimensions of the fields are not the same this correlator vanishes which implies that

$$h_1 = h_2 = h \qquad \bar{h_1} = \bar{h_2} = \bar{h} \tag{57}$$

The 3-point correlator can also be calculated uniquely and exactly and is

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{58}$$

Where C_{123} once again is some constant and z_{ij} is

$$z_{ij} = z_i - z_j \tag{59}$$

The 4-point correlator however cannot be uniquely determined by conformal symmetry alone and the same goes for any higher order correlator as well.

Now we return back to the Ward identities described by equation (41) since it turns out that for conformal symmetry we can write one identity for all of it which is valid for any conformal transformation $\epsilon, \bar{\epsilon}$. This identity is appropriately named the conformal ward identity (equation 5.46[1]):

$$\delta_{\epsilon,\bar{\epsilon}} \langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle \tag{60}$$

Where we used the renormalised energy-momentum tensors

$$T(z) = -2\pi T_{zz} \qquad \bar{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}} \tag{61}$$

X here is an arbitrary amount of primary fields and $\delta_{\epsilon,\bar{\epsilon}} \langle X\rangle$ is the variation of X when the transformation $\epsilon, \bar{\epsilon}$ is applied. The contour C is taken counterclockwise and its only condition is that it must include all of the positions of the fields included in X.

2.6 Operator product expansion

A property that one encounters when dealing with multiple fields inside correlation functions is that if their positions overlap the correlation function tends to have a singularity at that point. By looking specifically at this singular behaviour we can say something about the correlation functions of fields whose positions overlap. The way this is done is through the use of the so called operator product expansion which is a representation of a product of operators by a sum of different regular (when $z \to w$) operators which are multiplied by a function of $z - w$ which may diverge. The OPE of any primary field with the energy-momentum tensor can be shown to be (equation 5.71[1])

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}) \end{aligned} \tag{62}$$

It is implied with all operator product expansions that they only make sense within a correlator so we have dropped the brackets $\langle ... \rangle$. The $\sim$ symbol implies that they are equal modulo regular terms which do not have a singularity at $z = w$. Specifically for a free bosonic field with the action

$$S = \frac{1}{2}g\int d^2x\partial_\mu\varphi\partial^\mu\varphi \tag{63}$$

we can show it has the 2-point correlator (equation 5.75[1])

$$\langle\varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\{\ln(z-w) + \ln(\bar{z}-\bar{w})\} \tag{64}$$

Where g is some renormalisation constant and the formula is true up to some additive constant. By taking the derivatives $\partial_z\varphi$ and $\partial_{\bar{z}}\varphi$ and then taking only the holomorphic part we obtain the operator product expansion

$$\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2} \tag{65}$$

The original field can be shown to instead have the operator product expansion

$$\varphi(z)\varphi(w) \sim -\ln(z-w) \tag{66}$$

This field has the energy-momentum tensor (equation 5.79[1])

$$T(z) = -2\pi g : \partial\varphi\partial\varphi : \tag{67}$$

Using this along with wick theorem, described by equation (25) we can calculate the operator product expansion of the energy-momentum tensor with $\partial\varphi$

$$\begin{aligned} T(z)\partial\varphi(w) = -2\pi g : \partial\varphi(z)\partial\varphi(z) : \partial\varphi(w) = \\ -2\pi g(: \partial\varphi(z)\partial\varphi(z)\partial\varphi(w) : + : \partial\varphi(z)\partial\varphi(z)\partial\varphi(w) : + \\ : \partial\varphi(z)\partial\varphi(z)\partial\varphi(w) : \sim \frac{\partial\varphi(z)}{(z-w)^2} \end{aligned} \tag{68}$$

Finally we expand $\partial\varphi(z)$ around w and obtain the full operator product expansion

$$T(z)\partial\varphi(w) \sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{z-w} \tag{69}$$

Comparing this to equation (62) we see that $\partial\varphi$ has a conformal dimension of 1 and is a primary field. Similarly we can solve for the operator product expansion of the energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{70}$$

For a free fermion we instead have the action (equation 5.84[1])

$$S = \frac{1}{2}g \int d^2x \mathbf{\Psi}^\dagger \gamma^0 \gamma^\mu \partial_\mu \mathbf{\Psi} \tag{71}$$

With the Dirac matrices γ^μ

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{72}$$

and using the definition for the spinor $\mathbf{\Psi} = (\psi, \bar{\psi})$ we obtain

$$S = g \int d^2x (\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi) \tag{73}$$

The holomorphic operator product expansion of the fermionic field with itself can then be calculated to be

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g}\frac{1}{z-w} \tag{74}$$

and by using the energy-momentum tensor

$$T(z) = -\pi g : \psi(z)\partial\psi(z) : \tag{75}$$

we can obtain the operator product expansion of the energy-momentum tensor with the field ψ by using Wick's theorem:

$$T(z)\psi(w) \sim \frac{\psi(w)}{2(z-w)^2} + \frac{\partial\psi(w)}{z-w} \tag{76}$$

We once again also derive the operator product expansion of the energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{1}{4(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{77}$$

Which shows that the energy-momentum tensor is not primary field in this case either. It can in fact be shown that the energy-momentum tensor is a quasi-primary field in general under conformal transformations.

By looking at equations (70) and (77) we can see the following general pattern for the operator product expansion of the energy-momentum tensor with itself:

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{78}$$

Where the constant c is know as the central charge of the theory and is one of the main parameters of theories with conformal invariance. We will later see its values for different descriptions of the critical Ising model.

2.7 The operator formalism

In order to be able to use some of the simplifications offered by conformal symmetry we will need to switch to a view that makes the fields into operators acting upon different states. The first step to doing this is to use the so called radial quantisation which is a coordinate transformation from the regular complex coordinates to ones similar to polar coordinates except that our radial coordinate represents time and our angular coordinate represents space (for more information see chapter 6.1.1[1]). See figure 1 below for an illustration.

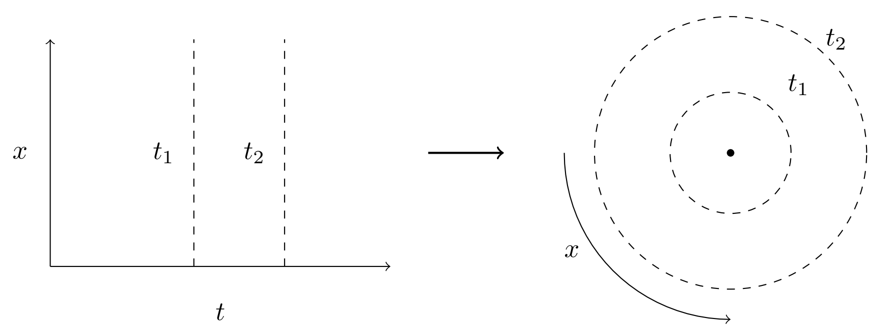

Figure 1: An illustration of radial quantisation. Source for TikZ code[2]

When it comes to the vacuum state with these new field operators we assume that as $t \to \pm\infty$ any interactions between the fields goes to 0 so that at infinity the fields can be treated as being free. For the interacting field $\phi(z, \bar{z})$ we thus get the operator and state (equation 6.3[1])

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})\, |0\rangle \tag{79}$$

We can then define hermitian conjugation of a quasi-primary field as the following (equation 6.4[1]):

$$[\phi(z, \bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z) \tag{80}$$

We can then also do something called mode expansion of a field in the following way (equation 6.7[1]):

$$\phi(z,\bar{z}) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\bar{z}^{-n-\bar{h}}\phi_{m,n} \tag{81}$$

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+\bar{h}-1}\phi(z,\bar{z})$$

With the vacuum state criteria

$$\phi_{m,n}\left|0\right\rangle = 0 \quad (m > -h, n > -\bar{h}) \tag{82}$$

In order to make equations clearer we will be ignoring the antiholomorphic part of future equations and simply writing out the holomorphic part since the antiholomorphic part usually has exactly the same structure. The field described in equation (81) then becomes:

$$\phi(z) = \sum_{m\in\mathbb{Z}} z^{-m-h}\phi_m \tag{83}$$

$$\phi_m = \frac{1}{2\pi i}\oint dz z^{m+h-1}\phi(z)$$

What was previously time ordered fields within correlators will now instead be radially ordered operators since time is now represented by our radial coordinate, with radial ordering being defined as putting the operators in order of increasing radial coordinate from right to left. Just as time ordering was implied before we now instead imply a radial ordering of operators within correlators.

From this we can obtain some relations for the commutator for the holomorphic fields $a(z)$ and $b(w)$

$$\oint_w dz a(z)b(w) = [A, b(w)] \tag{84}$$

$$[A,B] = \oint_0 dw \oint_w dz a(z)b(w) \tag{85}$$

where

$$A = \oint a(z)dz \qquad B = \oint b(z)dz \tag{86}$$

Where the subscript for the counterclockwise contour integrals determines around which point the integral is taken at some fixed time and if no subscript is specified it is taken at a fixed time around the origin.

2.8 The Virasoro algebra

By using the mode expansion described by equation (81) on the energy-momentum tensor we obtain

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \\ \bar{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \\ L_n &= \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \\ \bar{L}_n &= \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}) \end{aligned} \tag{87}$$

Where L_n and $\bar{L}_n$ are the generators of the local conformal transformations in the operator formalism on the Hilbert space, similar to how equation (52) describes the local conformal generators in the path integral formalism on the space of functions. These generators on the Hilbert space make up the famous Virasoro algebra which can be shown to have the commutation rules (see equation 6.25[1] for proof):

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0 \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \end{aligned} \tag{88}$$

Where we once again encounter the central charge c of the theory. The vacuum state is obtained from the following relation (equation 6.26[1])

$$\begin{aligned} L_n \left|0\right\rangle &= 0 \quad (n \geq -1) \\ \bar{L}_n \left|0\right\rangle &= 0 \end{aligned} \tag{89}$$

We can also see that the free state obtained from the field operator is an eigenstate of the $n = 0$ generators:

$$\begin{aligned} \left|h, \bar{h}\right\rangle &= \phi(0,0) \left|0\right\rangle \\ L_0 \left|h, \bar{h}\right\rangle &= h \left|h, \bar{h}\right\rangle \\ \bar{L}_0 \left|h, \bar{h}\right\rangle &= \bar{h} \left|h, \bar{h}\right\rangle \end{aligned} \tag{90}$$

For these eigenstates of L_0 we have two different raising operators: ϕ_m $(m < 0)$ and L_n $(n < 0)$. So any exited state can be constructed by repeated application of these operators who on each application increase the conformal dimensions of the state by $-m$ or $-n$ respectively. States obtained this way are called descendant states of the asymptotic state.

When we defined normal ordering back in equation (9) we said that that definition only holds for non-interacting fields. The generalised normal ordering for two operators $A(z)$ and $B(z)$ is denoted by $(AB)(z)$ and can be show to be (equation 6.130[1])

$$(AB)(w) = \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} A(z)B(w) \tag{91}$$

and also equal to (equation 6.144[1])

$$(AB)(z) = \sum_n z^{-n-h_A-h_B} (AB)_n \tag{92}$$

for:

$$(AB)_m = \sum_{n \leq -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n \tag{93}$$

We can now define a descendant fields for some descendant state as the normal ordering of operators which operates on the vacuum to produce the corresponding descendant state:

$$L_{-n} |h\rangle = (L_{-n}\phi)(0) |0\rangle \tag{94}$$

The reason we have been focusing so much on primary fields specifically is made clear in chapter 6.6.1[1] where two important things are noted: under a conformal transformation a primary field and its descendants only transform to that primary field or one of its descendants, A correlator consisting of descendant fields can always be reduced to a correlator containing only of the respective primary fields. A set consisting of a primary field ϕ and all of its descendants is denoted $[\phi]$ and is called the conformal family of that field.

2.9 Simple minimal models

For theories describing physical systems we are mostly interested in finite representations of the Virasoro algebra which are usually called Verma modules. In order to construct these finite representation we need to define a inner product between two descendant states:

$$L_{-k_1} L_{-k_2} ... L_{-k_m} |h\rangle \qquad L_{-l_1} L_{-l_2} ... L_{-l_n} |h\rangle \tag{95}$$

Which have the defined inner product (equation 7.9[1])

$$\langle h| L_{k_m} ... L_{k_2} L_{k_1} L_{-l_1} L_{-l_2} ... L_{-l_n} |h\rangle \tag{96}$$

Using the Hermitian conjugate $L_m^\dagger = L_{-m}$ and with the condition:

$$\langle h| L_i = 0 \quad (i < 0) \tag{97}$$

These finite descriptions can sometimes be simplified since there could be a subspace inside the Verma module which contains all of the information of the larger module and as such can be used as a representation for it. These Verma modules are called reducible. Sometimes these Verma modules can permit states which have a negative norm (these states are usually called ghosts when discussed in the context of string theory). The Verma modules which lack negative norm states are called unitary and its these that we are interested in. It can be shown that a *non-reducible unitary finite* Verma module can be achieved with the following constraints of the central charge and the conformal dimension (equation 7.65[1]):

$$c = 1 - 6\frac{(p-p')^2}{pp'} \tag{98}$$

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \tag{99}$$

Where p and p' are two coprime integers and where r, s is bounded by:

$$1 \leq r < p' \qquad 1 \leq s < p \tag{100}$$

With $h_{r,s}$ also having the symmetry:

$$h_{r,s} = h_{p'-r,p-s} \tag{101}$$

Verma modules which have all these properties are called minimal models and are usually described by the central charge along with the two numbers p and p' as $\mathcal{M}(p, p')$ with the usual convention $p > p'$. A way of describing how the limited number of conformal families of primary fields behaves under operator product expansions is through so called fusion rules. For example if we have four fields $\phi_1, \phi_2, \phi_3, \phi_4$ and the following fusion rule:

$$\phi_1 \times \phi_2 = \phi_3 + \phi_4 \tag{102}$$

This fusion rule above means that when we take the operator product expansion of ϕ_1 with ϕ_2 (the left hand side is just an operator product expansion) the field obtained will belong to either the conformal family of ϕ_3 or to the conformal family of ϕ_4. These fusion rules describe the short range behaviour of operator product expansions in Verma modules by specifying what conformal families they can belong to.

3 Results

3.1 The 2D statistical Ising model

In the normal two dimensional Ising model we have spin variables σ_i which can be either -1 or 1 sitting at the vertexes of a two dimensional crystal lattice of size $N \times M$ and interact with their nearest neighbour with the energy per interaction $\langle ij \rangle$ being (equation 12.1[1]):

$$E_{\langle ij \rangle} = -J\sigma_i\sigma_j \tag{103}$$

One of the interesting things about this model is the fact that it undergoes a phase transition when the coupling strength $K = \beta J$ ($\beta = 1/(k_B T)$) reaches some critical value K_c which is mainly achieved by changing the temperature of the system. This critical point is what we will describe with a minimal model later. First we expand the model into two different phases one for high-temperature (where K is small) and one for low temperature (where K is large) and look at the criteria for when the two phases overlap. For the high temperature expansion we can obtain the following partition function (equation 12.4[1]):

$$Z_{high} = [2\cosh(K)]^{NM} \sum_{loops} [\tanh(K)]^{\text{length of loop}} \tag{104}$$

Where we sum over all closed loops of vertexes with the same spin. In the high temperature phase the magnetisation at zero field called the spontaneous magnetisation is zero. The spins in this phase will be in a disordered state with the spins being randomly distributed over the lattice. The low temperature expansion is instead (equation 12.5[1]):

$$Z_{low} = 2e^{NMK} \sum_{loops} e^{-2K(\text{length of loop})} \tag{105}$$

Where the loops this time are the boundaries of the different spin 1 or -1 regions that appear for the low temperature expansion. As said the spins in this low temperature phase will form droplets of a certain size which are local regions of the lattice where only one spin appears. In this phase the spontaneous magnetisation is non-zero with it having a maximum at $T = 0$ and going to zero as $T \to T_c$. The two directions of the spontaneous magnetisation have the same energy and which one appears will depend on the way that the external field was brought to zero. This ordered phase is also called the ferromagnetic phase. The overlap of the two different expansions thus occurs when:

$$e^{-2K'} = \tanh K \tag{106}$$

At this point we will start finding droplets of all sizes so we can have droplets inside other droplets. This can be viewed as having both the disordered nature of the randomly distributed spins in the high temperature phase and the ordered nature of the droplets in the low temperature phase.

3.2 Ising model as a minimal model

When the Ising model reaches the critical point and we take the continuum limit the spin operator obtains a non-local element and the operators start anticommuting (page 442[1]). This then motivates describing the theory with a massless free fermion which has the action as described in equation (73)($g = 1/(2\pi)$ has been used):

$$S = \frac{1}{2\pi} \int d^2z (\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi) \tag{107}$$

The free fermion has a conformal dimension of $h = 1/2$, as seen in equation (76) and a central charge of $c = 1/2$ as seen in equation (78) and (77). This implies that the minimal model describing the critical Ising model is $\mathcal{M}(4, 3)$ which gives us the following fusion rules:

$$\begin{aligned} \sigma \times \sigma &= \mathbb{I} + \epsilon \\ \sigma \times \epsilon &= \sigma \\ \epsilon \times \epsilon &= \mathbb{I} \end{aligned} \tag{108}$$

With the energy operator $\epsilon(z, \bar{z})$ being a continuum version of the interaction energy $E_{\langle ij \rangle}$ and $\mathbb{I}$ being the identity field (primary field in the conformal family of the energy-momentum tensor). We also have the following correlators as seen in equation (74):

$$\begin{aligned} \langle \psi(z)\psi(w) \rangle &= \frac{1}{z - w} \\ \langle \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) \rangle &= \frac{1}{\bar{z} - \bar{w}} \end{aligned} \tag{109}$$

The spin operator will have the conformal dimension of 1/16 (equation 7.83[1]) and from this we obtain its propagator:

$$\langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2) \rangle = \frac{1}{|z_1 - z_2|^{\frac{1}{4}}} \tag{110}$$

By using the fusion rules in equation (108) we can obtain the following operator product expansions

$$\begin{aligned} \epsilon(z, \bar{z})\epsilon(w, \bar{w}) &\sim \frac{1}{|z - w|^2} \\ \psi(z)\sigma(w, \bar{w}) &\sim \frac{1}{(z - w)^{\frac{1}{2}}}\mu(w, \bar{w}) \\ \bar{\psi}(\bar{z})\mu(w, \bar{w}) &\sim \frac{1}{(\bar{z} - \bar{w})^{\frac{1}{2}}}\sigma(w, \bar{w}) \end{aligned} \tag{111}$$

Where we have μ as the disorder operator which is the operator dual to the spin operator. Since the previously used fermionic field was free we could take two copies of the same system and since they don't interact this theory obtains a central charge of $c = 1$. This then leads to a single bosonic field description for the critical Ising model. This combined bosonic field φ is described as:

$$\varphi(z, \bar{z}) = \phi(z) - \bar{\phi}(\bar{z}) \tag{112}$$

Where

$$e^{i\phi(z)} = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad e^{i\bar{\phi}(\bar{z})} = \frac{1}{\sqrt{2}}(\bar{\psi}_1 + i\bar{\psi}_2)$$

By doing this we can see that the correlators of the combined spin operator $\sigma = \sigma_1 \times \sigma_2$ with itself is (equation 12.59[1]):

$$\langle \sigma(z,\bar{z})\sigma(w,\bar{w})\rangle^2 = N \langle \cos \frac{\varphi}{2}(z,\bar{z}) \cos \frac{\varphi}{2}(w,\bar{w})\rangle \tag{113}$$

The fact that the correlator is squared is because of the two underlying fermionic theories. The 4-spin correlator when $z_1 \to z_2$ and $z_3 \to z_4$ can then be shown to be

$$\begin{aligned} &\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle^2 = \\ &\frac{1}{|z_{12}z_{34}|^{\frac{1}{2}}}[C^2_{\sigma\sigma\mathbb{I}} + 2|z_{12}z_{34}|C^2_{\sigma\sigma\epsilon} \langle \epsilon(z_2,\bar{z}_2)\epsilon(z_4,\bar{z}_4)\rangle] \end{aligned} \tag{114}$$

From this we can calculate the following two constants for correlation functions:

$$C_{\sigma\sigma\mathbb{I}} = 1 \qquad C_{\sigma\sigma\epsilon} = \frac{1}{2} \tag{115}$$

Using this and the fact that in the high-low temperature duality which occurs at the critical point, the energy operator changes sign and the spin and disorder operators are swapped. We can then calculate the following correlator

$$\begin{aligned} &\langle \sigma(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\mu(z_4,\bar{z}_4)\rangle^2 = \\ &\frac{|z_{13}z_{24}|^{\frac{1}{2}}}{2|z_{14}z_{23}z_{12}z_{34}|^{\frac{1}{2}}}\left[-1 + \frac{|z_{12}z_{34}|}{|z_{13}z_{24}|} + \frac{|z_{14}z_{23}|}{|z_{13}z_{24}|}\right] \end{aligned} \tag{116}$$

By taking the limits of $z_1 \to z_2$ and $z_3 \to z_4$ we can calculate the exact operator product expansions of equation (111):

$$\begin{aligned} \psi(z)\sigma(w,\bar{w}) &= \frac{e^{i\pi/4}}{\sqrt{2}(z-w)^{\frac{1}{2}}}\mu(w,\bar{w}) \\ \psi(z)\mu(w,\bar{w}) &= \frac{e^{-i\pi/4}}{\sqrt{2}(z-w)^{\frac{1}{2}}}\sigma(w,\bar{w}) \\ \bar{\psi}(\bar{z})\sigma(w,\bar{w}) &= \frac{e^{-i\pi/4}}{\sqrt{2}(\bar{z}-\bar{w})^{\frac{1}{2}}}\mu(w,\bar{w}) \\ \bar{\psi}(\bar{z})\mu(w,\bar{w}) &= \frac{e^{i\pi/4}}{\sqrt{2}(\bar{z}-\bar{w})^{\frac{1}{2}}}\sigma(w,\bar{w}) \end{aligned} \tag{117}$$

4 Discussion

The goal of the report was to replicate the critical part of the Ising model using conformal field theory which has been done. The amount of material within conformal field theory which fell out of the scope of the report was more that initially thought though since quite a few of the derivations would take too much time to go trough in any meaningful way. Other applications of conformal field theory were also not discussed however they were almost certainly outside of the scope of this report. If a larger more thorough review of the topic is of interest the source used in this report *Conformal Field Theory* is highly recommended as a very in-depth book about the matter which has a much bigger scope than this article.

Of the other applications of conformal field theory that one could look deeper into a good starting point are some of the other more advanced minimal models like the WZW models have applications in for example explaining the integer quantum hall effect[3]. String theory is one of the more prominent theories which makes use of conformal field theory in the modelling of string worldsheets[1]. There is also the concept of combining conformal field theory with supersymmetry, usually called superconformal field theory.

5 Conclusion

While quantum field theory is a very powerful models in describing the world around us its flaw of being computationally complicated can be alleviated by applying conformal symmetry. The theory then obtained has more niche uses but makes up for it by having much less complicated calculations. Of the applications of conformal field theory the Ising model is perhaps not the most interesting but it is still a illustrative example of how a highly advanced theory based upon a specific symmetry still can find quite a lot of useful applications in physics.

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