Linking Chains Together

*String Bits and the Bethe Ansatz*

BY

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Abstract

This thesis is divided into two parts. In the first part we focus mainly on certain aspects of the AdS/CFT correspondence. The AdS/CFT correspondence is a proposed duality between Type IIB superstring theory on AdS, x S' and N = 4 supersymmetric Yang-Mills theory. In the BMN limit string states located in the center of AdS, rotate quickly around the equator of the S' and correspond, in the dual theory, to operators constructed as long chains of sub-operators. This structure of the operators can be formulated as a spin chain and by using the Bethe ansatz their properties can be obtained by solving a set of Bethe equations. Having infinitely many sub-operators, there are methods for solving the Bethe equations in certain sectors. In paper III finite size corrections to the anomalous dimensions in the SU(2) sector are calculated to leading order.

Inspired by the chain structure of the corresponding operators, the theory of string bits treats the strings as a discrete sets of points. This theory suffers from the problem of fermion doubling, a general pathology of fermions on a lattice. In paper II we show how to adjust the theory in order to avoid this problem and, in fact, use the fermion doubling to our advantage. The second part of the thesis studies the low energy behaviour of SU(2) Yang-Mills theory in 4 space-time dimensions. In paper I we perform numerical calculations on an effective action for this theory. We propose the existence of a knotted trajectory within the dynamics of this effective action.

Keywords: theoretical physics, string theory, string bit, Bethe ansatz

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To whom it may concern…
List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1 ”Vad är det egentligen du forskar om, Martin?”

Pausen jag alltid gör när någon frågar ”Vad är det egentligen du forskar om, Martin?” brukar oftast räcka för att personen ska tänka om och fråga om annat. Ibland måste man rynka ögonbrynen och låtsas tänka efter innan de ger sig. Antingen ångrar de sig för att jag inte tros kunna svara på ett tillfredsställande vis eller för att personen inte tror sig orka lyssna på ett tillfredsställande vis. Förmodligen är det en kombination av de båda.

Nu har frågan alltså ställts igen. Jag tar min paus...


"Men, en modell formulerad i termer av rabarberkräm, komplimanger och små, små tomtar skulle ju knappast vara vetenskaplig" kanske man vill svara, men det kan den visst vara. 2 Om den gjorde bättre förutsägelser än nuvarande fysikaliska modeller skulle den vara en överlägsen fysikalisk modell. Och en mycket trevligare.

Som det ser ut nu fokuserar många forskare på supersträngteori som uppkom första gången på 70-talet. Det är en modell som formulerats på så vis att dess fundamentala objekt inte är punktformiga (partiklar) utan endimensionella (strängar), som oändligt tunna trådar istället för oändligt små dammkorn.

Strängens rörelse fås genom att ställa krav på den area strängen sveper ut då den rör sig, ungefär som ytan man skulle se om man, med lång slutartid, fotograferade en fallande tråd. Denna area kallas för strängens världsyta och skall i visst hänseende minimeras. Så om man vill veta hur en sträng som ser ut på ett visst sätt vid tiden $t_1$ rör sig för att se ut på ett annat sätt vid tiden $t_2$

\[ \text{Figur 1.1: En sträng rör sig från tiden } t_1 \text{ till } t_2. \]

1 Här invänder säkert många fysiker och säger "Om nu allt beter sig som om världen består av kvarkar så består den väl av kvarkar". Det kan man för all del säga, jag hävdar bara att det inte spelar någon roll för den fysikaliska modellen och dess förutsägelser.

kommer rörelsen svara mot den yta som förbinder de två strängarna och har minst area (se Figur 1.1). Strängen i figuren rör sig i två dimensioner och uppåt/nedåt-dimensionen har ersatts med tiden. Strängens värld vid tiden $t$ är alltså planet på höjden $t$. I figuren kan världsytan tänkas som såphinnan mellan två metallsträngar som automatiskt minimeras av ytspänningen.


För det första kommer vågor som går längs strängarna bara få ha vissa tillåtna frekvenser som vågor på en gitarrsträng med grundton och övertoner. Dessa tillåtna vågor liknar de vågor i kvantmekaniken som beskriver partiklarna vi ser (kvarkar, elektroner, fotoner... etc) och det förefaller som om en sträng på håll kan te sig som olika punktförmiga partiklar beroende på hur den svänger. Olika svängningar på strängarna resulterar i olika typer av partiklar med olika massor. Den kvantmekaniska modellen formulerades i början av förra seklet och beskriver mycket väl hur små partiklar ibland beter sig som vågor och hur vågor (som ljus) ibland beter sig som partiklar.


En sådan modell som förenar kvantmekanikens förutsättelser att små partiklar i själva verket kan ses som små vågpulser och Einsteins allmänna relativiteteori har man aldrig tidigare lyckats finna trots att man letat med ljus och lykta. Det är nämligen just en sådan modell som kan beskriva vad som sker i vår värld eftersom vi hittills har hittat fyra fundamentala typer av växelverkan mellan partiklar: Elektromagnetisk, svag och stark växelverkan som beskrivs med hjälp av kvantmekaniken och gravitationen som beskrivs av Einsteins allmänna relativiteteori.

Nu visar det sig att det finns problem med modellen. För att dessa strängars svängningar ska svara mot kvantmekaniska observationer krävs dessvärre att de rör sig i nio rumsdimensioner (upp/ned, höger/vänster, framåt/bakåt och sex riktningar till) med en tidsdimension (framåt i tiden/bakåt i tiden). Detta kan lätt kritiseras genom att be en strängteoretiker att peka i någon av de sex ytterligare rumsdimensionerna då de ju ska vara vanliga riktningar. På detta svarar strängteoretikern att de sex sista dimensionerna är små och kompaktifierade så det går inte.
Att kompaktifiera en dimension är inte så svårt som det låter, man krullar helt enkelt ihop den. Låt oss säga att en fysiker som lever i två rumsdimensioner vill förklara hur man kompaktifierar den tredje dimensionen i hennes modell för att passa hennes värld. Hon säger då:


På så vis om $R$ skulle vara extremt kort skulle du för det första inte få plats i dimensionen om du inte var platt (tvådimensionell) som jag och avståndet mellan två punkter skulle vara detsamma som avståndet i upp/ned- och höger/vänster-led då avståndet framåt/bakåt är mindre än $\frac{R}{2}$ som är halften av något extreemt kort. Det närmaste objektet kommer ju aldrig vara framför eller bakom dig utan rakt ovanför eller till höger om dig.

Alltså, vad gäller avstånd kommer den kompaktifierade dimensionen inte att spela någon roll och om du är platt och mycket större än längden $R$ kommer du inte ens märka av den. Den spelar bara roll för mycket små partiklar."

Att kompaktifiera sex dimensioner är mycket mer komplicerat, men principen är densamma. En sådan kompaktifierad dimension påverkar vanlig kvantmekanik där partiklar, som ju är oändligt små, ska bete sig som vågor: en våg i den kompaktifierade riktningen kan bara ha våglängder som passar i dimensionens längd så att vågen inte får någon synlig skarv när den gått runt längden $R$. En sträng påverkas på ytterligare ett sätt: En sträng mellan två punkter kan vara *lindad*. För att koppla an till exemplet ovan kan vi låta personen som stod i ett rum med en meters periodicitet hålla ut sina händer och spänna upp en sträng mellan dem. Denna sträng kan spännas upp på oändligt många vis, för om personen tar ena änden av strängen i vänster hand kan han sedan lägga den andra änden i vilken högerhand som helst av den oändliga raden med högerhänder, då alla högerhänder i raden ju är densamma.

Låt säga att strängen går från personens vänsterhand till högerhanden på personen två platser fram i kö. Personen kommer då att hålla en tråd i vänster

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3 Det finns inget lämpligt plural här. Att skriva 'kopior av dig själv' är felaktigt då det ju inte är kopior. Var och en i kö är samma individ.
hand som sträcks framåt och en i höger som sträcks bakåt men det är samma träd, rycker han i den ena rycker det i den andra, och den är lindad två varv i framåt/bakåt-riktningen. Se Figur 1.2 där tre meter ritats ut så att man ser hur personen upplever det, glöm bara inte att var meter faktiskt är samma meter.

Denna lindning är topologiskt invariant, med det menas att om personen håller fast i strängändarna kommer strängen alltid att vara lindad två varv oavsett hur personen och strängen rör sig. För att ändra på det måste man klippa av strängen precis som att man måste klippa av strängen för att knyta upp en knut på strängen (personen fick inte släppa ändarna!). Hur många varv en sträng är lindad kallas för strängens lindningstal. Mer om detta lite senare.

Ett andra problem med strängteori är att bland alla partiklar som kommer fram genom olika svängningar på strängarna finns ett slags partikel, tachyonen (uttalas takiånen), som har imaginär massa. En partikel med imaginär massa ställer till mycket besvär då den kan färdas bakåt i tiden. Detta leder till att tachyonen skulle kunna åka tillbaka i tiden och döda sin mamma (om den haft någon) innan hon födde sin tachyon och på så vis förhindra sin egen existens. Alltså: tachyoner brukar ses som en indikation på att något är galet.

Lösningen på problemet är att man introducerar även fermionska svängningar på strängen. De fermionska svängningarna behövs för att strängteorin ska kunna beskriva fermioner, ett slags partiklar som vi observerat men som den vanliga strängteorin inte skulle kunna beskriva. Faktiskt är de partiklar vi ser till vardags (elektroner och kvarkar) fermioner. Denna utökade strängteori heter supersträngteori och inom den har man hittat fem separata supersträngteorier med var sin speciella familj av svängningar som inte innehåller några svängningar som ger upphov till tachyoner. Om man bara tittar på strängar med svängningar från den första familjen kan inga krockar mellan dessa leda till andra svängningar än de från första familjen. Samma sak gäller de övriga fyra. Detta leder till att vi har fem olika supersträngteorier:

- Typ I
- Typ IIA

Figur 1.2: En person i en periodisk dimension håller en sträng med lindningstal 2.
• Typ IIB
• Heterotisk $E_8 \times E_8$
• Heterotisk $SO(32)$

Ingen av dessa kommer att innehålla tachyoner och om vi studerar en viss supersträngteori kommer inga svängningar från andra supersträngteorier att dyka upp. De olika supersträngteorierna är alltså isolerade från varandra.


Dessvärre påpekas näsvisa fysiker att enligt relativitetsteorin kommer något som rör sig riktigt snabbt också få en större massa, så en liten sak kan vara tung om den rör sig riktigt fort (har riktigt hög energi). Dessa energier som får de minsta partiklarna att bli så massiva att de kröker rummet är förvisso enorma (faktiskt rent kolossalt enorma) men teoretiskt kommer man att behöva kombinera kvantmekanik och Einsteins allmänna relativitetsteori. Även svarta hål är otroligt massiva i jämförelse med deras storlek, så även där behövs en kombination av dem. Som sagt har ju inga andra kända modeller kunnat förena dem två, så inga andra modeller aspirerar på att kunna förutsäga vad som sker vid enorma energier eller i närheten av svarta hål.

Inget laboratorium på jorden skulle kunna alstra sådana energier som krävs för att samtidigt se effekter av kvantmekanik och allmän relativitetsteori så vi kan inte mäta hur saker beter sig vid dessa energier. Vi får helt enkelt se till supersträngteorin och dess avknoppningar då det för tillfälle är den mest lojande modellen att förena kvantmekaniken och allmän relativitet. Kanske kan man i framtiden se effekterna av supersträngteorin i universums tidigare skeden eller i svarta hål. Kanske rentav på närmre håll. Vad vi kan göra för tillfälle är att studera modellen och förutsäga så mycket vi kan om den. Och även om den skulle visa sig felaktig leder denna forskning till ny fysik och ny matematik som kan användas inom andra områden.

Så nu vill vi alltså studera en väldigt komplicerad modell. För att kunna räkna på komplicerade modeller krävs approximationer. Sättet man oftast ap-

---

4 Här menar jag de massiva, oändligt täta objekt som kan bli kvar då stora stjärnor kollapsar.
proximerar med kallas störningsteori. Den går ut på att man har någon liten parameter (mycket mindre än 1) i modellen. Om man har en sådan parameter, vi kallar den $\alpha$, kan man ofta få ut ett svar på formen

$$\text{Svaret} = A + B\alpha + C\alpha^2 + D\alpha^3 + E\alpha^4 + \cdots$$

där koefficienterna $(A, B, C, \ldots)$ är lättare att räkna ut än hela Svaret. Om nu $\alpha$ är liten kommer $\alpha^2$ vara ännu mindre och $\alpha^3$ vara otroligt litet... osv. Vad man gör sedan är att man klipper av serien när Svaret är exakt nog och struntar i de följande termerna då de är små. Det är en mycket bra metod om man nu har en sådan liten parameter i modellen. Om $\alpha$ skulle vara stor skulle ju termerna växa och bli mer och mer betydelsefulla och metoden skulle vara oanvändbar. Ibland har man ingen liten parameter men då kan man använda ett knep som uppmärksammandes i mitten av 1990-talet.


Vad som här menas med en dualitet kan jämföras med följande dualitet i verkliga livet. För att göra köttfärslimpa gör följande:
1. Blanda köttfärs och lök i en lagom stor bunke, smaka av med salt.
2. Häll smeten i en form och ställ in i ugnen i en timme. Skölj över med spad då och då tills den är gyllenbrun.
3. Servera med potatis
Kan du göra köttfärslimpa kan du även göra glass. Gör likadant men byt ut följande:

| köttfärs       | → | grädde       |
| lök            | → | hallon       |
| salt           | → | socker       |
| ugn            | → | frys         |
| i en timme     | → | över natten  |
| skölj över med spad | → | rör om       |
| gyllenbrun     | → | genomfrusen  |
| potatis        | → | chokladsås   |
Denna dualitet kan underlätta om man är duktig på att laga köttfärslimpa men dålig på att göra glas. Man behöver inte ens veta att man gör glas, man behöver bara byta ut några ord.


Denna föreslagna dualitet, som kallas AdS/CFT-korrespondensen, var ett samband man anat redan på sjuttiotalet. Då visades att partiklar i en kvantfältteori som den ovan faktiskt tenderade att fara runt och krocka på tänkta ytor. Dessa nätverk av spår av krockande partiklar tyckte man borde kunna ses som en finmaskig världsyta som strängarna hade i supersträngteorier. Där fanns många olikheter, till exempel fanns de båda ytorna i olika antal dimensioner, kvantfältteorier i fyra rum-tids-dimensioner och supersträngteorin i tio, så även om ett samband kunde skönjas vågade man inte hoppas att de båda modellerna skulle kunna vara duala.

AdS/CFT-korrespondensen rönte stor uppmärksamhet då den föreslogs och gör det fortfarande. Nu kan de som jobbat med kvantfältteori jämföra sina resultat med dem som jobbar med supersträngteori. Olika gränsfall utan några

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5 Denna tas upp i avhandlingen i Sektion 4.2 på sidan 24.
6 Detta tas upp i avhandlingen i Sektion 3.2 på sidan 17.
små parametrar för störningsteori kan nu utforskas i den duala kvantfältteorin som har små parametrar.

Där har vi mig. Jag har huvudsakligen studerat en sådan gräns\textsuperscript{7} av kvantfältteorin och motsvarande gräns i den duala strängteorin.

\textsuperscript{7} Denna tas upp i avhandlingen i Sektion 4.3 på sidan 26.
2 Introduction

The goal of this thesis is to explain enough concepts to make clear what is done in [I], [II] and [III]. It is aimed for readers on the level of fresh graduate students in theoretical physics. Unavoidably some concepts are left out and some are explained quite roughly.

The goal of fundamental physics is to find a theory describing the four interactions we observe in nature: electro-magnetic interaction, weak interaction, strong interaction and gravity. As was realized in the beginning of last century a theory describing elementary particles must be quantised in order for the particles of the theory to exhibit observed wave properties. In Quantum Field Theory (QFT) the interactions between elementary particles appears as an exchange of particles effectively changing the momentum of the elementary particles. The exchange particles for the interactions are photons for electro-magnetism, Z-bosons and W-bosons for weak interaction, gluons for strong interaction and gravitons for gravity. In QFT, however, there will be quantum corrections to the masses and the charges of the particles. These corrections turn out to be infinite but there exists well defined prescriptions how to cancel these infinities by introducing counterterms to the Lagrangian describing the theory.

Unfortunately, the procedure of cancelling the infinities only works for the first three interactions. Applying it to gravity one would need for infinitely many counterterms and in order to perform calculations on the theory the contributions from all of the counterterms might have to be considered. Thus, quantised theory would lose its predictability. Basically, gravity is non-renormalisable since its coupling constant has negative mass dimension. So, to sum up, the apparent way of combining gravity to the other interactions does not work.

The suggested theory describing all the four interactions is string theory where the fundamental objects of the theory are not, as for QFT, point like particles but strings. With the new extended objects not only will the theory allow for gravity it will actually demand it. Upholding known symmetries when quantising the theory will lead to the Einstein equations describing gravity, now with small corrections. The strings are very small giving that for the today testable energies a string will appear as a particle since the properties of its inner structure can not be probed. Different oscillations along the string...
will give the apparent particle different properties such as mass and spin.

The grand property now of string theory is that it is formulated with no free parameters, except the string tension $T$ setting the absolute mass scale. The hope among string theorists is that the apparent particles coming out of string theory will be the particles we observe and that the relative masses of the particles will come out naturally. The case for today is that the masses of the particles in the standard model (The QFT that describes the first three interactions for the observed particles) have to be measured and put in by hand. There are more than 20 free parameters in the standard model. Hopefully they will be predicted when string theory is more well understood.

If the apparent particles of string theory really corresponds to observed particles turns out to be hard to see, since quantised string theory dictates the number of space-time dimensions to be $D = 10$ in order to be consistently quantisable. Thus, it will be necessary to compactify 6 dimensions in order to arrive at the 4 observed space-time dimensions. This compactification of 6 space-time dimensions can be done in many ways, each way resulting in different apparent particles.

Among the states of the quantised string one finds tachyons. Tachyons are string oscillations of imaginary mass, and are generally considered a problem. When introducing fermions to string theory an arbitrariness on how to introduce the fermions makes it possible to formulate five different superstring theories$^1$ containing no tachyons.

In the middle of the 1990's it was realised that there existed dualities between the five superstring theories. These dualities were mappings of one superstring theory to another showing that two different superstring theories in certain limits were describing the same physics expressed differently. They proved to be very powerful since the dual to a strongly coupled superstring theory is weakly coupled, making it possible to study strongly coupled theories in their perturbative dual.

Studying the dualities a new type of object in superstring theory was found, the D-brane. D-branes are submanifolds in space-time on which open strings can end. The string ends propagating on the D-branes give rise to QFTs living on the D-branes. This observation led to the proposal of a new duality between a certain superstring theory and the QFT living on D-branes. The duality is called the AdS/CFT-correspondence and claims that Type IIB superstring theory on $\text{AdS}_5 \times S^5$ is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills in 4 space-time dimensions. This proposed duality is quite remarkable since it relates a superstring theory, containing gravity, to a QFT. Since both theories are supposed to describe the same physics, the dynamics of gravity must somehow be contained in $\mathcal{N} = 4$ supersymmetric Yang-Mills. That is, if the proposed

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$^1$ The five superstring theories are called Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$.
Duality turns out to be true. It has been scrutinised the last 7 years and has not yet been proved wrong.

In the AdS/CFT-correspondence string states correspond to operators on the QFT side. The energy of the string state will then correspond to the conformal dimension of the operator. The conformal dimension is the ordinary dimension of the operator with quantum corrections.

Of the three papers in this thesis the last two have been exploring AdS/CFT-correspondence and the first has studied another type of QFT: the $SU(2)$ non-supersymmetric Yang-Mills theory in 4 space-time dimensions.

In [I] A. J. Niemi, K. Torokoff and the author are studying the dynamics of low energy $SU(2)$ Yang-Mills by numerical calculations on a proposed effective potential. The existence of knotted trajectories in this one loop effective action is proposed. Since this paper is not closely related to the other two the introduction to it is in the last chapter of this comprehensive summary.

The topic of [II] by U. H. Danielsson, F. Kristiansson, K. Zarembo and the author is the model of string bits, where Type IIB Green-Schwarz superstrings theory is formulated as a chain of phase points which in the continuum limit will return to the original formulation. The advantage of the string bits formulation is that splitting and joining of strings is easier to view than in the original formulation. It was pointed out that string bits will, as always for a theory containing fermions on a lattice, suffer from fermion doubling. The fermion doubling leads to the amount of fermions in the discretised model will double in the continuum limit. Our solution to this problem is simple and unexpected: we simply start with one half of the fermions using the fermion doubling to our advantage providing for the two types of chiral fermions we want in the string theory.

What inspired the string bits model was the chain structure of the corresponding operators. In [III] K. Zarembo and the author study these corresponding operators mainly in the BMN limit. The operators are then constructed as long closed chains of sub-operators. The conformal dimension of a type of such operators can be transformed into the problem of solving the Bethe equations coming from the Bethe ansatz. These equations can be solved in the continuum limit when the number of sub-operators, $L$, in the chain goes to infinity. In [III] we calculate the leading order finite size corrections ($\sim 1/L^2$) to the conformal dimension for operators in the $SU(2)$ sector.

This comprehensive summary of the thesis is organised as follows: Chapter 3 introduces some crucial concepts of field theory are reviewed. In Chapter 4 a short introduction to string theory is given followed by a short explanation of the AdS/CFT-correspondence, where in particular the BMN limit is studied. With the fundament now explained the theory of string bits can be formulated in Chapter 5. In Chapter 6 we show how to calculate the anomalous dimension of a group of operators containing the BMN operators using
the Bethe ansatz to diagonalise the renormalisation matrix. In this chapter much focus is on the formulation of the Bethe ansatz. In the final Chapter 7 the Poincaré map is introduced and the effective potential in [I] is discussed.
In this thesis many concepts of field theory are required. For a suitable introduction or definitions of concepts read for instance [20]. In this Chapter the most crucial concepts are reviewed starting out with a very widely used kind of field theories: the Yang-Mills theories.

### 3.1 Yang-Mills Theories

To describe gauge invariance we want a gauge invariant language much like in the covariant notation in relativity. Looking at a field transforming as

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$$  \hspace{1cm} (3.1)

we see that its derivative will transform in a not too elegant way

$$\partial_\mu \psi(x) \rightarrow \partial_\mu e^{i\alpha(x)}\psi(x) = e^{i\alpha(x)}(\partial_\mu + i\partial_\mu \alpha)\psi(x).$$  \hspace{1cm} (3.2)

Introducing now a new gauge field $A_\mu$ which parallel to the transformation in 3.1 is transformed as

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}(\partial_\mu \alpha(x))$$  \hspace{1cm} (3.3)

and a covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, where $e$ is the charge, the transformation is more tractable

$$D_\mu \psi(x) \rightarrow [\partial_\mu + ie(A_\mu - \frac{1}{e}(\partial_\mu \alpha))]e^{i\alpha}\psi(x) = e^{i\alpha}(\partial_\mu + ieA_\mu)\psi(x) = e^{i\alpha}D_\mu \psi(x).$$  \hspace{1cm} (3.4)

Now the derivative of $\psi$ transform as $\psi$ itself. The electromagnetic field tensor can be expressed as the commutator of the covariant derivatives

$$[D_\mu, D_\nu]\psi(x) = \left( [\partial_\mu, \partial_\nu] + ie([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] - e^2[A_\mu, A_\nu]) \right)\psi(x)$$

$$= ie((\partial_\mu A_\nu - (\partial_\nu A_\mu))\psi(x) = ieF_{\mu\nu}\psi(x).$$  \hspace{1cm} (3.5)

This concept of gauge invariance was generalised by Yang and Mills to work not only for the $U(1)$ case of the phase as above but also for more complex
groups. Now our field is a vector of fields in the representation space of the symmetry and transform as

$$\psi(x) \rightarrow \exp(i\alpha^a(x) T^a) \psi(x) \quad (3.6)$$

where $T^a$ are the generators of the group and $a = 1 \ldots n$, with $n$ the dimension of the group, is being summed over.

Now we would like to do the same things as above, but deriving the factor $\exp(i\alpha^a(x) T^a)$ is very complicated since the $T^a$s do not necessarily commute. For small $\alpha^a$s we can do it up to linear terms:

$$\psi(x) \rightarrow (1 + i\alpha^a(x) T^a + \cdots) \psi(x). \quad (3.7)$$

For the linear case the gauge field now will be $A_\mu^a(x) = A_\mu^a(x) T^a$ and transform as

$$A_\mu^a = A_\mu^b T^a \rightarrow A_\mu^b T^a + \frac{1}{g}(\partial_\mu \alpha^a)T^a + i\alpha^a A_\mu^b [T^a, T^b] + \cdots. \quad (3.8)$$

It is here conventional to denote the charge $g$. This last term occurs now since the $A_\mu^a$s are matrices which might not commute. Gauge theories for non-abelian groups are quite naturally called *non-abelian gauge theories*. The transformation in 3.8 can be written on component form

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{g}(\partial_\mu \alpha^a) - f^{abc} \alpha^b A_\mu^c + \cdots \quad (3.9)$$

where $f^{abc}$ are the *structure constants* of the group satisfying

$$[T^a, T^b] = i f^{abc} T^c. \quad (3.10)$$

Now we can define the covariant derivative $D_\mu = \partial_\mu + igA_\mu^a T^a$ and get

$$D_\mu \psi(x) \rightarrow \left[\partial_\mu + ig(A_\mu^a + \frac{1}{g}(\partial_\mu \alpha^a) - f^{abc} \alpha^b A_\mu^c)T^a\right](1 + i\alpha^a T^a)\psi(x)$$

$$= (1 + i\alpha^a T^a)\left[\partial_\mu + igA_\mu^a T^a\right]\psi(x)$$

$$= (1 + i\alpha^a T^a)D_\mu \psi(x) \quad (3.11)$$

keeping only up to linear terms in $\alpha$.

The field strength will now contain one more term. The fourth term in the top row of 3.5 will not vanish since $[A_\mu, A_\nu] \neq 0$. We end up with

$$F_{\mu\nu} = \frac{1}{ig} [D_\mu, D_\nu] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) - ig[A_\mu, A_\nu] \quad (3.12)$$
which in component form looks like
\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \tag{3.13}
\]

Now, the action describing the motion of fermions coupled to the gauge fields can be copied from the Lagrangian describing electro-magnetism using this new formalism:
\[
\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 + i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \tag{3.14}
\]

Throughout this thesis such theories are referred to as gauge theories or gauge field theories.

Later on, when discussing the AdS/CFT correspondence we will refer to \( \mathcal{N} = 4 \) supersymmetric Yang-Mills. This is a Yang-Mills theory with supersymmetry, that is a theory that is invariant under a certain transformation mixing the fermionic and the bosonic contributions. Furthermore it has a \( U(N) \) symmetry giving the fields matrix properties. In order for the theory to be supersymmetric we will have four types of fields in the theory: the \( A_\mu \)'s, six scalar fields \( \Phi^i \) (\( i = 1 \ldots 6 \) is the \( SO(6) \) index) and fermionic fields \( \psi_{a\alpha} \) and \( \tilde{\psi}_{a\dot{\alpha}} \) (\( a = 1 \ldots 4 \) is the \( SU(4) \) index and \( \alpha, \dot{\alpha} = 1 \ldots 2 \) are \( SU(2) \) index). Throughout the thesis we focus mainly on the scalar fields, \( \Phi^i \). The terms relevant to us will be
\[
\mathcal{L}_{\mathcal{N}=4} = \frac{1}{2} \text{Tr} D^\mu \Phi^i D_\mu \Phi^j - \frac{g^2}{4} \text{Tr} [\Phi^i, \Phi^j] [\Phi^i, \Phi^j] \tag{3.15}
\]
\[+ \text{other contributions containing field strengths and fermions.}
\]

Here the traces are in the \( U(N) \) index. The first term shows that the scalar field in order to render the action \( S = \int d^4x \mathcal{L} \) dimensionless must have bare dimension 1. Furthermore \( \mathcal{N} = 4 \) supersymmetric Yang-Mills is conformal, not only classically but also for the quantised theory since its \( \beta \) functions are zero.

### 3.2 Large \( \mathcal{N} \) Limit

To study what happens for large \( N \) in \( U(N) \) Yang-Mills theory it is convenient to look at the adjoint representation where the \( U(N) \) index \( a \) is replaced by the two indexes \( i \) and \( j \) of the matrix as
\[
a \text{ in } A_\mu^a \rightarrow i, j \text{ in } (A_\mu)_j^i = \left( \sum_a A_\mu^a T^a \right)_j^i. \tag{3.16}
\]
Where $T^a$ are the generators of $U(N)$. This is perhaps a more natural choice or parametrisation since the representations really transforms in those indices.

To express this in the Feynman diagrams we introduce the double line notation where the line of a propagator $(\overleftarrow{a'})$ which is proportional to $\delta_{aa'}$ is replaced by a double line $(\overleftrightarrow{a'j})$ proportional to $\delta_{aa'}\delta_{jj'}$. For such a Feynman diagram consisting of $V$ vertices, $L$ propagators and $F$ closed loops we will end up a contribution proportional to

$$N^{L+V-L}f(g^2N).$$

(3.17)

Now the exponent of the naked $N$ is quite special. It is called the Euler characteristic, $\chi = F + V - L$. Triangulating any two dimensional surface with $V$ vertices, $L$ lines and $F$ faces, the Euler characteristic of the triangulation will always be the same. Playing around with triangulations it is easy to convince oneself that the Euler characteristic simply corresponds to the number of holes in the three dimensional body enclosed by the triangulated surface. That is for a triangulation of a closed surface $\chi = 2 - 2G$ where $G$ is the number of holes, or handles, of the enclosed body. $G$ is called the genus of the body.

Since $N$ is large we can now make a new kind of expansion of the correlation functions

$$Z = \sum_G N^{2-2G}Z_G(\lambda) \quad \text{where} \quad \lambda = g^2N.$$  

(3.18)

The ’t Hooft limit is when $N \to \infty$ and $g \to 0$ in such a way that $\lambda = g^2N$ is kept finite. For this limit we see that the main contribution comes from planar diagrams with lowest genus. That is, the most important diagrams are the ones triangulating $S^2$.

The fact that the Feynman diagrams of a Yang-Mills theory with $U(N)$ symmetry tend to be planar hinted researchers that these sheets of dense Feynman diagrams could in fact reflect the world sheets of strings that will be presented in Section 4.1
The AdS/CFT-correspondence is a proposed duality between the two different models, Type IIB superstring theory in $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ conformal supersymmetric Yang-Mills. This correspondence was first conjectured by J. Maldacena [17]. More details were then worked out by S. S. Gubser, I. R. Klebanov, A. M Polyakov [15] and E. Witten [28].

This correspondence is quite astonishing since it relates string theory, which contains gravity, to quantum field theory. Further it relates models of different dimensionality. To provide some information on this correspondence it is necessary to first explain the basics of string theory.

4.1 Basic String Theory

For a more complete description of the basics of string theory there are many sources of information. To mention a few: [29, 26, 16, 21, 14]. What follows will be a quick review on the topics being most important to the rest of this thesis.

The motion of a relativistic particle can be achieved by extremising the action

$$S_{\text{particle}} = -m \int d\ell = -m \int d\tau \sqrt{-(\partial_\tau X)^\mu (\partial_\tau X)_\mu}$$  \hspace{1cm} (4.1)$$

where $\tau$ is any parameter for the particles trajectory in space-time and $X = X(\tau)$ is the space-time coordinate at parameter value $\tau$. The minus sign in the square root is so that for time like motion we get a positive argument since we are using the space-time signature $(-+++\cdots)$. Note that we have not specified any dimension for the space-time. The space-time where the particle lives is called the target space.

The action in 4.1 is proportional to the length of the trajectory (the particles world line) telling us that the physical path a particle will take is really the one with extremised length in target space. This geometric way of viewing the mechanics is easy to generalise to higher dimensional objects. We simply extremise the volume the object sweeps out in target space, which is a length for a zero dimensional object (particle), an area for a one dimensional object (string), a three dimensional volume for a two dimensional object (membrane)… and so on.
As suggested by the name of this section we will look at the one-dimensional object and extremise the area of its world sheet. This action is called the **Nambu-Goto action** and is defined as

\[ S_{\text{NG}} = -T \int dA \]

where again \( X \) is the space-time coordinate which is now parametrised by two coordinates so that \( X = X(\tau, \sigma) \). This is very much like 4.1 but now the \( m \) is called \( T \) which is the tension of the string.\(^1\) Now the world sheet is parametrised by two parameters \( \tau \) and \( \sigma \) and by geometry the area spanned in target space by \( d\tau \) and \( d\sigma \) will be the square root of the determinant of the induced metric

\[ h = \det h_{ab}, \quad h_{ab} = (\partial_a X)^\mu (\partial_b X)_\mu \]

where \( a \) and \( b \) goes through \( \tau \) and \( \sigma \).

The action in 4.2 has the unfortunate property of containing a square root which is a bad thing since we in the end want to quantise it. Ideally we would like it to be quadratic in \( X \) (linear in \( h_{ab} \)). A similar action, the **Polyakov action**, with a non-determined metric \( \gamma_{ab} \) was introduced by Brink, Di Vecchia, Deser and Zumino:

\[ S_p = -\frac{T}{2} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} h_{ab} \quad \text{where} \quad \gamma = \det \gamma_{ab}. \]

Inserting now in this the solution for \( \gamma_{ab} \) which extremise the action we will actually end up with the Nambu-Goto action. The action in 4.4 looks more complicated and contains indeed a square root but it is linear in \( h_{ab} \). This, Polyakov pointed out, proves very useful for quantisation.

The parameters \( \tau \) and \( \sigma \) are viewed as the time and the space coordinate for the string. Hence, varying \( \sigma \) keeping \( \tau \) fixed can be seen as going along the string at a fixed time. This is of course not necessarily true for observers in other Lorentz frames.

Another novel feature of the Polyakov action is that it possesses more symmetries than the Nambu-Goto action. It has Poincaré invariance since it is expressed covariantly and invariance under \( \tau \) and \( \sigma \) reparametrisation (Diff invariance). These invariances are inherited from the Nambu-Goto action and the mass of a particle can also be viewed as a tension on the particles world line. The heavier it is the more effort it takes to change its motion which corresponds to bending the world line. So the mass is describing how hard it is to bend the world line.

\(^1\) Actually the mass of a particle can also be viewed as a tension on the particles world line. The heavier it is the more effort it takes to change its motion which corresponds to bending the world line.
are expected. A third invariance is exclusive for the Polyakov action: It is invariant under scaling of $\gamma_{ab} \rightarrow f(\tau, \sigma) \gamma_{ab}$ for any $f(\tau, \sigma)$. This Weyl invariance is easily seen from 4.4 considering that $\gamma \rightarrow f^2 \gamma$ and $\gamma^{ab}$ being the inverse of $\gamma_{ab}$ transforms as $\gamma^{ab} \rightarrow f^{-1} \gamma^{ab}$.

The last two symmetries allows us to select the convenient conformal gauge where the metric is on the form

\[
\gamma_{ab} = e^{\phi(\tau, \sigma)} \eta_{ab} = e^{\phi(\tau, \sigma)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(4.5)

For this particular choice of gauge the Polyakov action 4.4 will look like

\[
S_P = - \frac{T}{2} \int d\tau d\sigma \eta^{ab} h_{ab} = \frac{T}{2} \int d\tau d\sigma \left( \dot{X}_\mu \dot{X}^\mu - (X'')_\mu (X')^\mu \right)
\]

(4.6)

where the dot is a derivative in $\tau$ and a prime is a derivative in $\sigma$. As Polyakov pointed out we now see that it is indeed suitable for quantisation rendering the path integrals Gaussian.

Now we consider a string of finite length evolving through time. Its world-sheet can be parametrised as $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \pi$ where $\sigma = 0, \pi$ are the end points of the string. Setting the variation of the action 4.6 to zero we get

\[
0 = \delta S_P = \frac{T}{2} \int d\tau d\sigma \left( \dot{X}_\mu \delta \dot{X}^\mu - (X'')_\mu (\delta X')^\mu \right)
\]

(4.7)

\[
= -T \int d\tau d\sigma \left( \dot{X}_\mu - X''_\mu \right) \delta X^\mu - T \int d\tau \left[ X'_\mu \delta X^\mu \right]_{\sigma=0}^{\sigma=\pi}
\]

giving the equation of motion

\[
0 = \ddot{X}_\mu - X''_\mu.
\]

(4.8)

But also the end points must be treated so that the second term of the variation will also vanish.

We have so far not considered open and closed strings. An open string will trace out a world sheet like a ribbon where a closed string traces out a cylinder. The simplest case is the closed string. For a parametrisation where $\sigma$ is viewed as the parameter along the string in space the two ends of the string are joined as

\[
X_\mu(0, \tau) = X'_\mu(\pi, \tau)
\]

\[
X''_\mu(0, \tau) = X''_\mu(\pi, \tau)
\]

(4.9)

so that the ends are connected with no kink. As we see this works well and the second term in 4.7 vanishes.

For open strings we are left with two alternatives. For the second term in
4.7 to vanish we can have either
\[ X'_\mu(0, \tau) = 0 = X'_\mu(\pi, \tau) \quad \text{or} \quad \delta X_\mu(0, \tau) = 0 = \delta X_\mu(\pi, \tau). \quad (4.10) \]

The first condition is the Neumann condition and corresponds to a completely free end. The second case is the Dirichlet condition is a fixed end since there is no variation at the end points.

In fact, nothing tells us that we have to have the same condition on both ends of the string. Not even that we have the same conditions in all directions. The condition is really \( D \) conditions and we can choose for a Neumann or a Dirichlet condition for every direction.

The most natural choice is the Neumann condition. To start with it conserves the momentum of the oscillations on the string where the Dirichlet condition does not. Furthermore to assume that the ends are fixed we would need something to fix them on and what would that be? If we would then have Dirichlet conditions in some directions and Neumann conditions in the others the strings end points should be fixed so that they slide around on lower dimensional submanifolds in space-time. Unnatural as it might be, it turned out to be the case.

4.1.1 D-Branes

One might hope for simplicity that we could settle for only Neumann conditions but we can not. To uphold that \( \gamma_{ab} \) in the Polyakov action 4.4 is extremised after we quantise the string we get restrictions on the dimension on space-time. For bosonic strings that we have looked at \( D = 26 \) and after introducing also fermions and supersymmetry we arrive at \( D = 10 \). Both critical dimensions are larger than one might expect, so, to arrive at the world we see we will need to compactify directions.

The simplest case of compactification is to compactify one dimension on a circle. If we identify \( x_{25} \sim x_{25} + 2\pi R \) for some compactification radius \( R \) the modes on a closed string in that direction will be characterised by two integers \( n \) and \( w \). \( n \) is the wave number counting the harmonic waves being single valued on the string. \( w \) is the winding number counting how many times the string is wound up in the compactified direction so that for the world sheet \( X_{25}(\tau, 0) = X_{25}(\tau, \pi) + 2\pi Rw \). The winding number is shown in Figure 1.2 back on page 5 where the string the man is holding would be a closed string with \( w = 2 \) if only he put his hands together.

Studying the contribution to the string's characteristics from the two numbers we see that the mass contribution from them will be on the form
\[ m^2 = \frac{n^2}{R^2} + \frac{R^2w^2}{(\alpha')^2} + \text{contribution from other directions} \quad (4.11) \]
where $\alpha' = (2\pi T)^{-1}$ is the Regge slope. Now we see that changing $R$, which only affects the 25th dimension, as $R \to \alpha' / R$ the effect of $n$ and $w$ will interchange. This is called T-duality and tells us that a closed string in space-time compactified on a very small circle will have the same properties as if space-time was compactified on a very large circle but with $n$ and $w$ interchanged.

A problem, now, is for open strings. If the open string would have a Neumann condition in the compactified direction it is free to unwind and $w$ will not make sense. So, the infinitely massive $n$-modes at $R \to 0$ will have no correspondence at $R \to \infty$ for the T-duality.

The solution is that the T-duality will actually change the Neumann condition to a Dirichlet condition so we will need for the submanifolds mentioned in the end of last subsection for the string ends to be fixed to.

These submanifolds are called D-branes or D$p$-branes where the $p$ is the spatial dimensionality of the submanifold. The case above where a string end in 25 spatial dimensions and 1 time dimension is restricted with a Dirichlet condition in only the compactified direction is free to move in the other 24 dimensions and is hence fixed to a D24-brane.

Now we look at a D$p$-brane on which open strings end. The string ends have $p$ directions of Neumann conditions ($x_\mu$ where $\mu = 0 \ldots p$) and $25 - p$ directions of Dirichlet conditions ($x_\mu$ where $\mu = p + 1 \ldots d$). Hence, the end points are free to move in the tangential directions of the brane ($0 \ldots p$) but restricted in the normal directions ($p + 1 \ldots d$). The zero mass vector mode of such a string will give rise to a $U(1)$ gauge vector field as well as $(d - p)$ scalar fields. All the fields are living on the D$p$-brane itself.

Having then $N$ different D$p$-branes will then lead to a vector field of $U(1)^N$ symmetry which, if the $N$ D$p$-branes coincide will become a $U(N)$ symmetry. This is a first glance of the main topic of this section, that open strings ending on D-branes will lead to field theories on the branes.

Studying the D-branes further it turns out that they will have a dynamic on their own. This motion is described by the Born-Infeld action.

4.1.2 Genus Expansion

The action describing our string can be generalised further. We can add a geometric term which possesses the three symmetries of the Polyakov action, 4.4. Such a term is the Einstein-Hilbert action

$$\chi = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{-\gamma} R$$

(4.12)

where $R$ is the Ricci scalar describing the local curvature of the world sheet.

Adding the Einstein-Hilbert action to the Polyakov action will not lead to any new dynamics of the worldsheet since it for two dimensional surfaces is
purely topological. What it will amount to is a natural expansion of expressions. The new term gives the genus of the world surface and the natural expansion will be a genus expansion of the world surface. So, we will arrive at a genus expansion for our correlation functions

\[ Z = \sum_{G} g_{s}^{2G-2} Z_{G} \]  

(4.13)

which we recognise the from 3.18 but now with the genus of the world sheet instead of the genus of the planar Feynman diagram as for the field theory case.

Here \( g_{s} \) might appear to be a free parameter, but it is not. As it turns out it will be predicted by the theory.

### 4.2 The Correspondence

When quantising superstring theory one must for consistency put the space-time dimensions to 10 which is definitely not the number we observe. How could ever such a model give rise to a world which effectively works as a 4 dimensional quantum field theory? We would like for some method to relate models of different dimensionality. In fact such a method exists and is referred to as the *holographic principle*. The holographic principle claims that all information in certain spaces is also obtainable on its boundary so that a model describing what happening on the boundary can be used also to tell what is happening inside (in the bulk of) the space.

It might seem far fetched that this could be used to save us but as it turned out, it did. In [17] J. Maldacena proposed what is now referred to as the AdS/CFT-correspondence where he claims such a duality between Type IIB superstrings living in \( \text{AdS}_{5} \times S^{5} \) and conformal supersymmetric \( \mathcal{N} = 4 \) Yang-Mills theory. Usually the superstring side of the correspondence is referred to as the gravity side, since it contains gravity, and the Yang-Mills side is referred to as the gauge side, since it describes a gauge theory.

To see why this could be we first observe that \( \text{AdS}_{5} \) which is a 5 dimensional space-time of constant negative cosmological constant actually has a boundary with the Minkowski metric. In global coordinates \( \text{AdS}_{5} \) can be described by the metric

\[ ds^{2} = R^{2} \left[ - \cosh^{2} \rho \, d\tau^{2} + d\rho^{2} + \sinh^{2} \rho \, d\Omega_{3}^{2} \right] \]  

(4.14)

where \( \tau \) is time, \( \rho \) is the radial coordinate and \( d\Omega_{3}^{2} \) is the angular contribution. The boundary of this space is where \( \rho \to \infty \) and fixed where

\[ ds^{2} = \left( \frac{Re^{\rho}}{2} \right)^{2} \left[ - d\tau^{2} + d\Omega_{3}^{2} \right] . \]  

(4.15)
Now we see that the boundary is $\mathbb{R} \times S^3$ which is the compactified Minkowski metric.

Another way of testing if such a duality is even possible is to compare the global symmetries of the two models. They would have to agree exactly. For the bosonic part of string theory the symmetry comes from the isometries of the space. The isometry group of $\text{AdS}_5 \times S^5$ is $SO(4, 2)$ from $\text{AdS}_5$ and $SO(6)$ from $S^5$. Introducing also fermions the full symmetry of Type IIB superstring theory is $SU(2, 2|4)$.

For the bosonic part of conformal supersymmetric Yang-Mills we have conformal invariance giving us the symmetry group $SO(4, 2)$. Yet another symmetry of conformal SYM is the $R$-symmetry which has in fact the symmetry group $SO(6)$. Introducing again supersymmetry the global group is in fact $SU(2, 2|4)$ indicating that both models has the same global symmetry. This does not prove J. Maldacena’s proposal but it indicates that it might be possible.

Now, having a stack of $N$ coincident D 3-branes they will actually generate the curved $\text{AdS}_5 \times S^5$. And as was discussed in Section 4.1.1 such a stack of branes will have a field theory of symmetry $U(N)$ living on the branes. The D3-branes have space-time dimension $3 + 1$.

The elements compared in the duality are string states on the gravity side and operators on the gauge side. It was found that the energy of the string state is equivalent to the conformal dimension of the corresponding operator on the gauge side. The conformal dimension, $\Delta$, is the dimension of the operator. This dimension will have quantum corrections to it so that the conformal dimension is the sum of the classical dimension (bare dimension) and the quantum contributions (anomalous dimension). The parameters of the two models are related as

$$\frac{R^4}{\ell_{s}^4} = \lambda = g^2_{\text{YM}}N \quad \text{and} \quad g_s = \frac{g^2_{\text{YM}}}{4\pi}.$$ (4.16)

On the gravity side $R$ is the radius of AdS$_5$ and S$^5$, $\ell_{s} = (2\pi T)^{-1/2}$ is the characteristic string length and $g_s$ is the parameter for string interaction introduced in Section 4.1.2. On the gauge side $\lambda$, $g_{\text{YM}}$ and $N$ are the same entities explained in Section 3.2.\footnote{Now the Yang-Mills coupling $g_{\text{YM}}$ is equipped with an index to avoid confusion with the string coupling $g_s$.}

A problem for testing the AdS/CFT-correspondence, which is indeed quite an asset when using the correspondence, is that in the regime where one side is possible to handle with perturbation theory the other is not. For the gauge side we have the $\lambda = g^2_{\text{YM}}N$ as our expansion parameter. On the gravity side the parameter for expansion is the string scale is the Regge slope $\alpha' = \ell_{s}^2 = \ldots$
$(2\pi T)^{-1}$. Now the AdS/CFT-correspondence dictates
\[ \lambda = \frac{R^4}{\ell^4} = \frac{R^4}{(\alpha')^2} \quad (4.17) \]
showing us that when the gauge side is perturbative, $\lambda \to 0$, the gravity side is not. The opposite case $\alpha' \to 0$ with a near zero string length is described by supergravity but is not perturbative on the gauge side.

4.3 The BMN limit
What one would need to simultaneously study both sides perturbatively is some small parameter to expand in so that both sides’ expansions could be compared term by term. In 2002 D. Berenstein, J. Maldacena and H. Nastase [7] proposed that one could in fact use the $R$ charge, $J$, on the gauge side which corresponds to the angular momentum around $S^5$ on the gravity side. What we are then interested in are operators of large $J$ on the gauge side and, thus, strings spinning quickly around the sphere on the gravity side.

Having now a large $J$ and a large $\lambda$ but with $\lambda' = \lambda/J^2$ being small we can use this for our expansion. What we now hope for is that we will be able to compare term by term in this expansion for the gauge side and for the gravity side. On the first side we will have a single trace of a long product of operators and on the other a small string rotating quickly around $S^5$.

We start by exploring the gravity side.

4.3.1 The Penrose limit
The Penrose limit of a space-time is the space-time describing the neighbourhood of a point moving very fast. Since we are interested in the space-time a string will experience when spinning very quickly around $S^5$ we have to work out a suitable Penrose limit of our space-time. Starting out with the metric for $AdS_5 \times S^5$ of radius $R$ for both spaces
\[ ds^2 = R^2 \left[ -cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2 + \cos^2 \theta \, d\phi^2 + d\theta^2 + \sin^2 \theta \, d\tilde{\Omega}_3^2 \right] \quad (4.18) \]
where for the $AdS_5$ contributions (first row) $t$ is the time, $\rho$ is the radial coordinate and $d\Omega_3^2$ is the angular contribution. For the $S^5$ contributions (second row) $\theta$ is the azimuthal coordinate, $\phi$ the equatorial coordinate and $d\tilde{\Omega}_3^2$ the other angular contribution. We are now interested in the geometry of a system in the middle of $AdS_5$ rotating rapidly around the equator of the $S^5$ part of space.
Moving fast motion along the equatorial coordinate $\phi$ we go to light cone coordinates, $x^+ = \frac{1}{2}(t + \phi)$ and $x^- = \frac{1}{2}(t - \phi)$. Furthermore we rescale $x^+, x^-, \rho$ and $\theta$ as follows

$$
x^+ = \frac{1}{\mu} \frac{t + \phi}{2}, \quad x^- = \mu R^2 \frac{t - \phi}{2}, \quad r = R \rho, \quad y = R \theta
$$

where now $\mu$ is a new variable and $R \to \infty$. This leaves us with the metric

$$
ds^2 = -R^2(d\tau^2 - d\phi^2 - (r^2 + y^2)d\rho^2 + r^2d\Omega^2_3 + dy^2 + y^2d\Omega^2_3)
$$

where we used that the unscaled coordinates $\rho, \theta$, in order to render the new coordinates finite, are small, and also that $t - \phi$ is small so that $dt \approx d\phi \approx \mu dx^+$. Now, $\vec{x}$ is a vector in $\mathbb{R}^8$ describing $\vec{r}$ and $\vec{y}$ which are both vectors in $\mathbb{R}^4$.

The novel feature of this metric is that it is the so called pp-wave background metric. This is a very interesting background for string theory since in it the Polyakov action 4.4 simplifies greatly in the light cone gauge where the world sheet time is fixed to the light cone time, $\tau = x^+$. For this gauge choice we end up with an action that is quadratic. If we look at the Type IIB Green-Schwarz string the action will be

$$
S = \frac{T}{2} \int d\tau d\sigma \left[ \dot{x}^2 - (\dot{x}')^2 - \mu^2 \dot{\rho}^2 - i \theta (\dot{\theta} + \dot{\theta}') - i \tilde{\theta} (\dot{\tilde{\theta}} - \dot{\tilde{\theta}}') - 2i \mu \theta \Pi \tilde{\theta} \right]
$$

where we now excluded the vector bar on the $\vec{x}$ and $\Pi = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. In this gauge we will also have that the length of the string is $p^+$. The bosonic part of this action has the mode expansion

$$
x = \cos(\mu \tau)x_0 + \frac{\sin(\mu \tau)}{2\pi \mu T}p_0
$$

$$
+ \frac{i}{2\sqrt{\pi T}} \sum_{p \neq 0} \left[ \exp \left( -i\sqrt{\mu^2 + k^2} \tau - ikp \sigma \right) \frac{\alpha_p}{p} 
+ \exp \left( -i\sqrt{\mu^2 + k^2} \tau + ikp \sigma \right) \frac{\tilde{\alpha}_p}{p} \right]
$$

where $k = \frac{2\pi T}{p}$ and $x_0, p_0, \alpha_n$ and $\tilde{\alpha}_n$ are the strings center of mass, its momentum and the polarisation vectors for left movers and right movers respectively. From the expansion we see that the bosonic energy contribution from
the modes will be

$$H = p^- = \frac{1}{2} \sum_{p=-\infty}^{\infty} N_p \sqrt{\mu^2 + k^2 p^2} = \frac{1}{2} \sum_{p} N_p \sqrt{\mu^2 + \left(\frac{2\pi T p}{p^+}\right)^2}. \quad (4.23)$$

Actually the fermionic contribution will have the similar form so that also fermions can be included in the occupation numbers $N_p$.

For the string bits in the next section we will use Hamiltonian formalism. The corresponding light cone Hamiltonian to action 4.21 is

$$H = \frac{T}{2} \int d\sigma \left[ p^2 + (\chi')^2 + \mu^2 \chi^2 + i \theta \theta' - i \bar{\theta} \bar{\theta}' + 2 i \mu \theta \Pi \bar{\theta} \right]. \quad (4.24)$$

4.3.2 BMN operators

To study the corresponding operators we see that global coordinates in $\text{AdS}_5 \times \text{S}^5$ energy and angular momentum around the equator of $\text{S}^5$ is defined as

$$E = i \partial_t \quad \text{and} \quad J = -i \partial_{\phi}.$$ \quad (4.25)

The energy of a string state on the gravity side corresponds to the conformal dimension $\Delta$ of the operator on the gauge side. The angular momentum of the string corresponds to the $R$ charge of the operator. Using the metric elements $g^{+\pm} = g^{-+} = -\frac{1}{2}$ from 4.20 and the reparametrisation 4.19 we find

$$p^+ = -\frac{1}{2} p^- = \frac{i}{2} \partial^+ = \frac{i}{2} \left( \partial_t - \partial_{\phi} \right) = \frac{\Delta + J}{2\mu R^2},$$ \quad (4.26)

$$p^- = -\frac{1}{2} p^+ = \frac{i}{2} \partial^- = \frac{i}{2} \mu \left( \partial_t + \partial_{\phi} \right) = \frac{\Delta - J}{2}. \quad (4.27)$$

The BMN operators are single trace operators with $J \gg 1$ and $\Delta - J$ is kept finite. Now $\lambda = g_{\text{sym}} N \gg 1$ but $\lambda' = \lambda / J^2$ is small. For the case $\Delta - J = 0$ there exists only one such operator:

$$\text{Tr} Z^J \quad \text{where} \quad Z = \Phi^5 + i \Phi^6 \quad (4.28)$$

where the trace is in colour indices and $\Phi^i$ for $i = 1 \ldots 6$ are the six scalars of the $\mathcal{N} = 4$ SYM presented in 3.15. This operator is a chiral primary operator assuring it no corrections to its dimension due to supersymmetry. On the gravity side this operator corresponds to a string state of $p^+ = \frac{J}{\mu R^2}$ and $p^- = 0$ which is the vacuum state of the string.

$$|0, p^+ \rangle_{lc} \quad \longleftrightarrow \quad \text{Tr} Z^J \quad (4.29)$$

Excited states of the string will now be impurities in this long line of $J$ Zs. For example the dual picture to a bosonic mode in the 8th direction on the
vacuum state is the insertion of the 4th scalar $\Phi^4$, summing over all positions multiplied with a phase term as

$$\langle a_n^\dagger \rangle^4 |0,p^+\rangle_{lc} \longleftrightarrow \sum_{j=1}^J \text{Tr}(Z^j \Phi^4 Z^{J-j}) e^{2\pi inj/J}. \quad (4.30)$$

Now, for only one impurity the cyclicity of the trace will make the operator identically zero unless $n = 0$. This reflects the requirement that the total momentum along the string must vanish so that there is no such state on the gravity side.

Having two string modes with a vanishing total momentum along the string the cyclicity of the trace can be used to write down the operator with only one summation

$$\langle a_n^\dagger \rangle^6 (a_{-n}^\dagger)^7 |0,p^+\rangle_{lc} \longleftrightarrow \sum_{j=1}^J \text{Tr}((\Phi^3 Z^j \Phi^4 Z^{J-j}) e^{2\pi inj/J}. \quad (4.31)$$

For this string state one ends up with a non-zero operator.

This is the recipe given by D. Berenstein, J. Maldacena and H. Nastase, that the string modes are translated into inserted operators with the contribution $\Delta - J = 1$ in the long row of $Z$s as above with the corresponding operators being

$$\langle a^\dagger \rangle^i \longleftrightarrow \begin{cases} D_i Z & \text{for } i = 1\ldots 4 \\ \Phi_{j-4} & \text{for } i = 5\ldots 8 \end{cases} \quad (4.32)$$

Similarly the fermionic string modes are matched to the fermionic operators in $\mathcal{N} = 4$ SYM.

In their paper D. Berenstein, J. Maldacena and H. Nastase argue that these building blocks are sufficient and that inserting operators with the contribution $\Delta - J > 1$ will give rise to very large masses as $N \to \infty$.

Using now the correspondence we see from 4.23 that our contributions from the states are

$$\langle \Delta - J \rangle_p = \frac{2H_p}{\mu} = \sqrt{1 + \left(\frac{2\pi \mathcal{T} p}{\mu p^+}\right)^2} = \sqrt{1 + \frac{\lambda}{T^2} p^2} = \sqrt{1 + \lambda' p^2} \quad (4.33)$$

where now the effective coupling constant comes out in a natural way.
5 String Bits

In their original paper D. Berenstein, J. Maldacena and H. Nastase refers to BMN states as “strings of Zs” implying that the objects on both sides of the duality have a similar structure. In 2002 H. Verlinde [27] pushes this even further introducing string bits to describe the string side so that not only would both sides be string like, they could both be viewed as chains constructed by linked elements. On the string side the string is described as a linked set of phase points which in the continuum limit can be seen as a parametrisation of the string.

To express the discretised Hamiltonian for the BMN limit, 4.24, one needs not only a set of phase points but also an ordering of these points telling us how the points are linked together. Having J phase points (string bits) we write them \{p_n, x_n, θ_n, ˜θ_n\} where p_n and x_n are 10 dimensional space-time vectors, θ and ˜θ SO(8) spinors and n = 1…J the index of the phase point. Each string bit traces out a strip of the world sheet when evolving in time. Having now also a one-to-one function γ: \{1…J\} ↦→ \{1…J\} showing the order of the phase points we can now, using a as the string length between the phase points express the discretised derivatives along the string:

\[ \partial_σ X_n \rightarrow \frac{1}{a} \left( X_γ(n) - X_n \right) \] (5.1)

Now, the discretised Hamiltonian can be expressed as a sum instead of an integral. If we rewrite 4.24 with \( T = 1 \) and \( μ = 1 \) we get

\[ H_{SB} = \frac{a}{2} \sum_{n=1}^{J} \left[ p_n^2 + x_n^2 + \frac{1}{a^2} (x_γ(n) - x_n)^2 \right. \\
\left. + \frac{i}{a} θ_n θ_γ(n) - \frac{i}{a} ˜θ_n ˜θ_γ(n) + 2iθ_n Π ˜θ_n \right] \] (5.2)

and the canonical commutation relations will be

\[ [p_i^n, x_j^m] = \frac{1}{ia} δ^{ij} δ_{nm}, \quad \{ θ_a^n, θ_b^m \} = \frac{1}{a} δ^{ab} δ_{mn}, \quad \{ ˜θ_a^n, ˜θ_b^m \} = \frac{1}{a} δ^{ab} δ_{mn} \] (5.3)

with i and j being the vector indices and a and b the spinor indices.

The upshot of all this is that this method automatically allows for several separate strings. If γ has a group of string bits which are all linked together
it represents a closed string (As in Figure 5.1 where the arrows represents $\gamma$). There is no need for any 2-D parametrisation of the world sheet and no genus expansion, it is all taken care of.

Still, we have the desired reparametrisation invariance from string theory but now in the shape of a permutation invariance where the string bits are re-ordered. The permutation group $S_J$ will then be used to divide our Hilbert space in the different conjugacy classes of the permutation. Every conjugacy class is then the group of $\gamma$s which arrange the string bits in the same configuration of strings. This means that the Hilbert space can be written as

$$\mathcal{H} = \sum_{i \in \{\text{conj. classes}\}} \mathcal{H}_i$$

where each $\mathcal{H}_i$ is a twisted sector.

These strings can now join and split up using the same type of operators, one simply exchange two string bits. If the two exchanged bits are on the same string, the string divides in two and if they are on separate strings, the two strings merge into one. The process of connecting two strands like that with the permutation operator $P_{ab}$ is shown in Figure 5.2. In his original paper H. Verlinde shows how to introduce interaction terms to the discretised Hamiltonian fulfilling the required properties.

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5.1 Fermion Doubling

There is however a problem with the model. To see this problem we study one single string gauge fixed so that \( \gamma(n) = n + 1 \) for \( n < J \) and \( \gamma(J) = 1 \). For this case we can express the Hamiltonian expressed with the Fourier transformed coordinates defined by

\[
p_n = \frac{1}{\sqrt{J}} \sum_{p=J/2}^{J/2-1} p_p e^{2\pi i p n / J}, \quad x_n = \frac{1}{\sqrt{J}} \sum_{p=J/2}^{J/2-1} x_p e^{2\pi i p n / J},
\]

\[
\theta_n = \frac{1}{\sqrt{J}} \sum_{p=J/2}^{J/2-1} \theta_p e^{2\pi i p n / J}, \quad \tilde{\theta}_n = \frac{1}{\sqrt{J}} \sum_{p=J/2}^{J/2-1} \tilde{\theta}_p e^{2\pi i p n / J}.
\]

This Hamiltonian will be

\[
H_{SB} = a \frac{J}{2} \sum_{p=-J/2}^{J/2-1} \left[ p_p p_{-p} + x_p x_{-p} + \frac{4}{a^2} x_p x_{-p} \sin^2 \frac{p \pi}{J} \right. \\
+ \left. \frac{1}{a} \left( \theta_p \theta_{-p} + \tilde{\theta}_p \tilde{\theta}_{-p} \right) \sin \frac{2p \pi}{J} + 2i \theta_p \Pi \theta_{-p} \right].
\]

Already we can catch a glimpse of the problem. If we look at the fermionic part of the Hamiltonian, which is the part of the brackets in 5.7 on the second line, we see that the \( \sin \frac{2p \pi}{J} \) factor will be zero twice in the allowed span of \( p \) (the Brillouin zone). This reveals that if we look at low energy there will be two regions of small mass, \( p \sim 0 \) and \( p \sim J/2 \). This results in twice as many light states as desired — we will at the continuum limit get out the double amount of fermions. This fermion doubling problem for string bits was pointed out by S. Bellucci and C. Sochichiu [6]. The problem does not arise for the bosons since the factor \( \left( \sin^2 \frac{2p \pi}{J} \right) \) has only one zero in the Brillouin zone.

To view the problem more clearly, we express the Hamiltonian in terms of creation and annihilation operators. For this we will need the commutation relations for the Fourier transformed coordinates which are

\[
[p'_p, x'_{-q}] = \frac{1}{ia} \delta^{ij} \delta_{pq}, \quad \{ \theta'_p, \theta'_{-q} \} = \frac{1}{a} \delta^{ab} \delta_{pq}, \quad \{ \tilde{\theta}'_p, \tilde{\theta}'_{-q} \} = \frac{1}{a} \delta^{ab} \delta_{pq}.
\]

Starting with the simpler bosonic part, using the commutator relations in 5.8 we can express

\[
H_{\text{bosonic}} = a \frac{J}{2} \sum_{p=-J/2}^{J/2-1} \left[ (p_p + i \omega_p x_p)(p_{-p} - i \omega_p x_{-p}) + \frac{\omega_p D}{a} \right]
\]

with \( \omega_p = \sqrt{1 + \frac{4}{a^2} \sin^2 \frac{2p \pi}{J}} \). The creation operator \( (p_p + i \omega_p x_p) \) then creates
states of energy $\omega_p$. For the light states of small $p \ll J/2$ we will arrive at the bosonic energy

$$
\omega_p = \sqrt{1 + \frac{4}{a^2} \sin^2 \frac{p \pi}{J}} \approx \sqrt{1 + \frac{4\pi^2 p^2}{J^2 a^2}} = \sqrt{1 + \left(\frac{2\pi p}{p^+}\right)^2} \quad (5.10)
$$

remembering that the total length of the string is $p^+ = Ja$. This is in perfect agreement with what was expected from the string side in 4.33 with $T = \mu = 1$.

For the fermionic part we rewrite the Hamiltonian on another familiar form:

$$
H_{\text{fermionic}} = \frac{a}{2} \sum_{p=-J/2}^{J/2-1} \left[ \frac{1}{a} (\theta_p \theta_{-p} + \bar{\theta}_p \bar{\theta}_{-p}) \sin \frac{2p \pi}{J} + 2i \bar{\theta}_p \Pi \theta_{-p} \right]
$$

$$
= \frac{a}{2} \sum_{p=-J/2}^{J/2-1} \left[ \frac{1}{a} (\theta_p \theta_{-p} + \bar{\theta}_p \bar{\theta}_{-p}) \sin \frac{2p \pi}{J} + i \bar{\theta}_p \Pi \theta_{-p} - i \theta_p \Pi \bar{\theta}_{-p} \right]
$$

$$
= \frac{a}{2} \sum_{p=-J/2}^{J/2-1} \left( \theta_p \bar{\theta}_p \left( \frac{1}{a} \sin \frac{2p \pi}{J} \right) \left( \frac{1}{a} \sin \frac{2p \pi}{J} \right) \right) \left( \theta_{-p} \bar{\theta}_{-p} \right). \quad (5.11)
$$

Now, the energy of the eigenstates, $\tilde{\omega}_p$, are simply the eigenvalues of the matrix which can be found since $\Pi^2 = 1$

$$
\tilde{\omega}_p = \sqrt{1 + \frac{1}{a^2} \sin^2 \frac{2p \pi}{J}}. \quad (5.12)
$$

For small $p$ this coincides with $\omega_p$. A setback, though is that within the Brillouin zone the lightest state is degenerate for $p = 0$ and for $p = J/2$. As was told before this is highly unsatisfactory and an indication that something is wrong.

As it turns out the vacuum part of the bosonic Hamiltonian, $\frac{D}{2} \sum \omega_p$, will for the light states be cancelled by a similar term, $-\frac{D}{2} \sum \tilde{\omega}_p$, in the fermionic Hamiltonian.

It should be mentioned that the problem with fermion doubling is nothing new nor is it an exclusive problem of string bits. It is a general pathology for any model having fermions on a lattice.

### 5.2 The Solution

There is, however, for the case of string bits a simpler solution to the problem of fermion doubling than one might suspect. We have a model that comes up with twice as many fermions as we put into it but on the other hand we want two types of fermions at the end since that is the situation for Type IIB Green-Schwarz superstrings with the two chiralities (the $\theta$s and the $\bar{\theta}$s). Maybe one
could start off with only a half of the fermions (say the $\theta$s) and hope that the fermion doubling will provide for the rest (the $\bar{\theta}$s). What U. H. Danielsson, F. Kristiansson, K. Zarembo and the author proposed in [II] is exactly this.

What we really need is a Hamiltonian describing the dynamics of the string bits which in the continuum limit describes Type IIB Green-Schwarz superstrings. The Hamiltonian we proposed have been stripped of the $\bar{\theta}$s and the last term has been replaced by a staggered term similar to the original one:

$$H = \frac{a}{2} \sum_{n=1}^{J} \left[ p_{n}^{2} + x_{n}^{2} + \frac{1}{a^{2}}(x_{n+1} - x_{n})^{2} + \frac{i}{a} \theta_{n} \theta_{n+1} + i(-1)^{n} \theta_{n} \Pi \theta_{n+1} \right]$$

(5.13)

$$= \frac{a}{2} \sum_{p=-J/2}^{J/2-1} \left[ p_{p} p_{-p} + x_{p} x_{-p} + \frac{4}{a^{2}} x_{p} x_{-p} \sin^{2} \frac{p \pi}{J} \right. + \frac{1}{a} \theta_{p} \theta_{-p} \sin \frac{2p \pi}{J} + i \theta_{p+j/2} \Pi \theta_{-p} \cos \frac{2p \pi}{J} \right]$$

(5.14)

Now, we are again looking at one single string of $J$ string bits which is gauge fixed so that $\gamma(n) = n + 1$ and the Fourier transforms are still the ones in 5.5 and 5.6.

Making now the identification

$$\bar{\theta}_{p} = \theta_{p+j/2}$$

(5.15)

this new Hamiltonian is in the continuum limit identical to the original one in 5.2 and 5.7 which makes it a suitable Hamiltonian. Furthermore a slight change of coordinates is required so that the original coordinate representations will change as follows:

$$\theta_{n} \rightarrow \frac{1}{2}(\theta_{n} + \theta_{n+1})$$

(5.16)

$$\bar{\theta}_{n} \rightarrow \frac{1}{2}(-1)^{n}(\theta_{n} - \theta_{n+1})$$

(5.17)

Here a remark is on its place, all the staggered terms need for $J$ to be even for them to be globally defined. This might lead to problems when interaction is turned on but might be remedied by pairing up stringbits which will not give any drastic changes in the continuum limit.

It is also shown in [II] that the supersymmetry algebra is intact and the supercharges still commute with the Hamiltonian in the continuum limit.
6 Integrability in Yang-Mills Theory

Yet another method to study the AdS/CFT correspondence in the BMN limit was proposed by J. Minahan and K. Zarembo [18]. They showed that the matrix of anomalous dimensions for the $SO(6)$ sector could actually be diagonalised using the Bethe ansatz, an old method to find exact solutions to a spin chain. This makes it possible to identify the eigenstates of the matrix of anomalous dimensions. These eigenstates will correspond to the energy eigenstates of the strings. This task would else have been virtually impossible since BMN operators are single trace operators containing very long rows of field operators and are heavily mixed when renormalised.

In the $SO(6)$ sector of conformal supersymmetric Yang-Mills theory BMN operators can be written

$$\Theta[\psi] = \psi_{i_{1}...i_{L}} \text{Tr} \Phi^{i_{1}}...\Phi^{i_{L}}$$

where $\Phi^{i}$ are the scalar fields. This group of operators does not only contain BMN operators but also the chiral primary operators. Chiral primary operators are traceless and symmetric in all index pairs and they have no anomalous dimension due to supersymmetry. The corresponding string states to a chiral primary operators are string states that survive in the supergravity limit.

When renormalising operators in our model they will only mix among operators of the same bare dimensionality, and for one-loop renormalisation the scalars $\Phi^{i}$ mix only among themselves. So, for operators of the type in 6.1 the renormalised operators will be linear combinations of the bare, unnormalised operators

$$\Theta^{i}_{\text{ren}} = Z^{i}_{j} \Theta^{j}_{\text{bare}} .$$

From this $Z$ we can now deduce the matrix of anomalous dimension

$$\Gamma = \frac{dZ}{d(\log \Lambda)} Z^{-1}$$

where $\Lambda$ is the renormalisation scale. The eigenvalues, $\gamma$, of the matrix, $\Gamma$, are then the anomalous dimension of the corresponding eigenvectors, $\Theta$, so that the eigenvectors we have

$$\langle \Theta(x) \Theta(y) \rangle = \frac{\text{constant}}{|x-y|^{2(L+\gamma)}} .$$
Figure 6.1: The contributing one loop planar diagrams.

From this we read off the conformal dimension $\Delta = L + \gamma$.

The planar, one loop contributions to $Z$ are then the one loop diagrams for neighbouring $\Phi_s$ in the operator 6.1. The three contributing diagrams are depicted in Figure 6.1, where we have a crossing, a gluon exchange and a self energy. In the diagrams the operator $\Theta$ is represented by a horizontal bar and the $l$th and the $(l+1)$th elements are the only interacting part of the chain. The contribution from the diagrams are taken in Feynman gauge and due to the colour blindness of the gluon only the first diagram will contain any mixing of the $l$th and the $(l+1)$th operator.

Calculating now the Feynman diagrams we find the contribution from the first diagram to be

$$Z_{\ldots, h, l+1, \ldots} = 1 - \frac{\lambda}{16\pi^2} \left( 2\delta_{lj}^{h, l+1} \delta_{l+1}^{h, l} - \delta_{lj}^{h, l} \delta_{l+1}^{h, l+1} - \delta_{l+1}^{h, l} \delta_{lj}^{h, l+1} \right) \log \Lambda.$$  (6.5)

For the second contribution in Figure 6.1 there is no mixing of colour index and we end up with

$$Z_{\ldots, \delta h, l+1, \ldots} = 1 - \frac{\lambda}{16\pi^2} \left( \delta_{lj}^{h} \delta_{l+1}^{h} \right) \log \Lambda$$  (6.6)

The third part, the one loop self energy, was calculated in [10]:

$$Z_{\ldots, \delta h, l+1, \ldots} = 1 + \frac{\lambda}{16\pi^2} \left( 2\delta_{lj}^{h} \delta_{l+1}^{h} \right) \log \Lambda$$  (6.7)

Summing up the contributions from the diagrams we get

$$Z_{\ldots, \delta h, l+1, \ldots} = 1 + \frac{\lambda}{16\pi^2} \left( \delta_{lj}^{h} \delta_{l+1}^{h} + 2\delta_{lj}^{h} \delta_{l+1}^{h+1} - 2\delta_{lj}^{h+1} \delta_{l+1}^{h} \right) \log \Lambda$$

$$= 1 + \frac{\lambda}{16\pi^2} \left( K_{lj}^{h, l+1} + 2I_{lj}^{h, l+1} - 2P_{lj}^{h, l+1} \right) \log \Lambda$$  (6.8)

where we now defined the matrix elements of the trace operator $K$, the identity
1 \text{ and the exchange operator } P \text{ being } SO(6) \otimes SO(6) \text{ matrices operating on two operators in } \Theta. \text{ More compactly we can write } Z_{l,l+1} = Z_{l,l+1}^{ij} P_{ij} \text{ operating on the } l\text{th and the } (l+1)\text{th operator in } \Theta. \text{ In this language we have} 

Z_{l,l+1} = 1_{l,l+1} + \frac{\lambda}{16\pi^2} (K_{l,l+1} + 21_{l,l+1} - P_{l,l+1}) \log \Lambda \quad (6.9)

The full Z, being the product of all neighbouring Zs, can now be expressed up to terms linear in \lambda

Z = \prod_l Z_{l,l+1} = 1 + \frac{\lambda}{16\pi^2} \sum_l (K_{l,l+1} + 21_{l,l+1} - P_{l,l+1}) \log \Lambda. \quad (6.10)

from this we can easily derive up to first order in \lambda

\frac{dZ}{d(\log \Lambda)} = \sum_l \frac{\lambda}{16\pi^2} (K_{l,l+1} + 21_{l,l+1} - P_{l,l+1}), \quad (6.11)

Z^{-1} = 1 - \frac{\lambda}{16\pi^2} \sum_l (K_{l,l+1} + 21_{l,l+1} - P_{l,l+1}) \log \Lambda. \quad (6.12)

Using this in the definition of the renormalisation matrix 6.3 we find up to one loop corrections

\Gamma = \frac{\lambda}{16\pi^2} \sum_l (K_{l,l+1} + 21_{l,l+1} - P_{l,l+1}). \quad (6.13)

This is the matrix we want to diagonalise. If we find the eigenvectors and the eigenvalues of it they are the multiplicatively normalisable operators and their anomalous dimension. The conformal dimension of the operators corresponds to the energy of the corresponding string state. As it turns out there is a method from 1931 to diagonalise such a matrix: the Bethe ansatz. To explain the idea behind it we explain it in detail for the \text{SU}(2) sector in next sections to then sketch how to do it for SO(6) in the Section 6.3.

The story of diagonalising our matrix starts with the Heisenberg spin chain.

6.1 The Heisenberg Spin Chain

The Heisenberg spin chain is defined by a Hamiltonian operator, \( H \), operating on a Hilbert space

\[ \mathcal{H} = \prod_{l=1}^{L} h_l \quad (6.14) \]
being a product space of $L$ local spaces. The Hamiltonian is made up of a sum of neighbour Hamiltonians each acting on two adjacent local spaces, linking them together to a chain. For a closed chain we use the periodicity condition $h_{l+L} = h_l$ to link together all the local spaces

$$H = \sum_{l=1}^{L} H_{l,l+1}. \quad (6.15)$$

To explain the Bethe ansatz we consider the simplest case, the XXX $1/2$ spin chain. Hence, in this section and in 6.2 we consider the XXX $1/2$ spin chain and then use that results for a the more general case.

For the XXX $1/2$ case each local space is $\mathbb{C}^2$ and the neighbour Hamiltonian is constructed as

$$H_{ij} = \frac{J_0}{4} (\vec{\sigma}_i \cdot \vec{\sigma}_j - 1_{ij}) \quad (6.16)$$

with $\vec{\sigma}_l$ defined to be the vector of Pauli matrices

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z, \sigma_z) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right) \quad (6.17)$$

acting on the $l$th element of $\mathcal{H}$ only such that

$$\vec{\sigma}_l \mathcal{H} = h_1 \otimes \cdots \otimes (\vec{\sigma}_l h_l) \otimes \cdots \otimes h_L. \quad (6.18)$$

Here the XXX part of the name indicates the independence on direction in the vector product of $\vec{\sigma}$. The system could be generalised as

$$H_{ij} = \frac{J_0}{4} \left( \sum_k A_k (\vec{\sigma}_i)_k (\vec{\sigma}_j)_k - 1_{ij} \right), \quad (6.19)$$

Here $k$ is summed over $x, y$ and $z$. Such a generalised spin chain would then be a XXY or a XYZ spin chain depending on the three $A_i$'s. The $1/2$ part of the name XXX $1/2$ is simply to tell that each link in the chain has spin $1/2$ properties.

### 6.1.1 The Neighbour Hamiltonian

To visualise the meaning of the neighbour Hamiltonian we look at the explicit matrix formulation for our two neighbouring spaces $\mathcal{H} = h_1 \otimes h_2$. For this we have only two $\vec{\sigma}_l$'s:

$$\vec{\sigma}_1 = \vec{\sigma} \otimes 1 \quad \text{and} \quad \vec{\sigma}_2 = 1 \otimes \vec{\sigma}. \quad (6.20)$$
Using the definitions of the tensor product for matrices \( A \) and \( B \)

\[
A \otimes B = \begin{pmatrix}
A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\
A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\
A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\
A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22}
\end{pmatrix}
\]  

(6.21)

and for vectors \( \vec{u}, \vec{v} \)

\[
\vec{u} \otimes \vec{v} = \begin{pmatrix}
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\
\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} u_2 \\ v_1 \end{pmatrix} \\
\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}
\end{pmatrix}
\]  

(6.22)

we can express the neighbour Hamiltonian in 6.16 as

\[
H_{12} = \frac{J_0}{4} \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 - 1_{12} \right) = \frac{J_0}{4} \left( (\vec{\sigma} \otimes 1) \cdot (1 \otimes \vec{\sigma}) - 1 \otimes 1 \right)
\]

\[
= \frac{J_0}{4} \left( \sum_i \sigma_i \otimes \sigma_i - 1 \otimes 1 \right) = \frac{J_0}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]  

(6.23)

The eigenvalues of this Hamiltonian (i.e. the energies of the system) are 0 and \( -J_0 \) with multiplicity 3 and 1 respectively. This represents the triplet and the singlet state of two added spin 1/2 systems and can be seen rewriting the Hamiltonian in terms of total spin \( S = (\vec{\sigma}_1 + \vec{\sigma}_2)/2 \):

\[
H_{12} = \frac{J_0}{4} \left( \vec{\sigma}_1 \cdot \vec{\sigma}_2 - 1_{12} \right) = \frac{J_0}{4} \left( \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2)^2 - \frac{1}{2} \vec{\sigma}_1^2 - \frac{1}{2} \vec{\sigma}_2^2 - 1_{12} \right)
\]

\[
= \frac{J_0}{2} (S^2 - 21_{12}) = \frac{J_0}{2} (S(S+1) - 2) 1_{12}
\]  

(6.24)

where we used \( \vec{\sigma} \cdot \vec{\sigma} = 31 \). Again we see that the triplet \( (S = 1) \) has the energy 0 and the singlet \( (S = 0) \) has energy \(-J_0\) with \( S \) being the total spin eigenvalue of the two local spaces.

Furthermore observing that the exchange operator \( P_{12} \) defined as

\[
P_{12} = \frac{1}{2} \left( \sum_i \sigma_i \otimes \sigma_i + 1 \otimes 1 \right)
\]  

(6.25)

can be constructed as

\[
P_{12} = \frac{1}{2} \left( \sum_i \sigma_i \otimes \sigma_i + 1 \otimes 1 \right)
\]  

(6.26)
the neighbour Hamiltonian can be expressed in terms of $I$ and $P$

$$H_{12} = \frac{J_0}{2} (P_{12} - I_{12})$$  \hspace{1cm} (6.27)

So, the neighbour Hamiltonian can be expressed in terms of (up to a sign) swapping the two local spaces and leaving them intact. As it stands it is the projection onto antisymmetric state products: $H_{12} = -J_0 \text{Proj}_{as}$.

6.1.2 The Lax Operator and the $R$ Operator

Our hope now is to diagonalise the Hamiltonian of the whole chain (6.15), as we did for the two neighbour case above. The bad thing when it comes to diagonalising the whole Hamiltonian is that it is a $2^L \times 2^L$ matrix. For large $L$ (being the case we are interested in) it will quickly be out of reach for diagonalisation, using even the fastest computer.

To study global properties of the spin chain it will turn out fruitful to study long products of operators acting on neighbour pairs, going through all neighbouring pairs in the whole chain.

Often when studying integrability one tries to construct a Lax operator. Our Lax operator ($L$) operates on the tensor product of two local spaces (as was the neighbour Hamiltonian). Let us again look at $h_1 \otimes h_2$ where the Lax operator looks like

$$L_{12}(\lambda) = \lambda I_{12} + \frac{i}{2} \sum \sigma_i \otimes \tilde{\sigma}_i = (\lambda - \frac{i}{2}) I_{12} + i P_{12}$$  \hspace{1cm} (6.28)

where we again used 6.26. The parameter $\lambda$ is called the spectral parameter.

We will now construct long words built up of such Lax operators to study global properties of the system, so we need to study how they behave when their order is changed. First we observe that two Lax operators operating on the same two local spaces commute and do not cause any problem (remember $L_{ij}(\lambda) = L_{ji}(\lambda)$). If they operate on four different local spaces they will also commute. Problem will arise, though, when they have one vector space in common. To visualise this we need to operate on a product space of three local spaces, $h_1 \otimes h_2 \otimes h_3$. On this product space, how do $L_{12}(\lambda)$ and $L_{13}(\mu)$ behave when changing their order?

Let us look at the simplest case when $\lambda = \mu = i/2$ so that $L_{12}(i/2) = i P_{12}$ and $L_{13}(i/2) = i P_{13}$. Using the properties of the exchange operator $P_{12}P_{13} = P_{13}P_{23}$ and its permutation as well as $P_{ij}^2 = I_{ij}$ we try to exchange $L_{12}(i/2)$
and $L_{13}(i/2)$ as well as possible:

\[
L_{12}(i/2)L_{13}(i/2) = iL_{12}iL_{13} = iL_{13}iL_{12} = -iL_{23}iL_{13}iL_{23} = (iL_{23}(i/2))^{-1}L_{13}(i/2)L_{12}(i/2)L_{23}(i/2) \tag{6.29}
\]

so, in this simple case we have

\[
L_{23}(i/2)L_{12}(i/2)L_{13}(i/2) = L_{13}(i/2)L_{12}(i/2)L_{23}(i/2). \tag{6.30}
\]

The fact that we will need a $L_{23}$ term to change the order of $L_{12}$ and $L_{13}$ is true also for the case with arbitrary $\lambda$ and $\mu$. It is simple, although tedious, to show that for the general case we have

\[
L_{23}(\lambda - \mu + i/2)L_{12}(\lambda)L_{13}(\mu) = L_{13}(\mu)L_{12}(\lambda)L_{23}(\lambda - \mu + i/2) \tag{6.31}
\]

which coincides to the case $\lambda = \mu = i/2$. This operator that allows us to change the order is called the $R$ operator

\[
R_{ij}(\lambda) = L_{ij}(\lambda + i/2) = \lambda L_{ij} + iP_{ij}. \tag{6.32}
\]

Using this we can express 6.31 as

\[
R_{23}(\lambda - \mu)L_{12}(\lambda)L_{13}(\mu) = L_{13}(\mu)L_{12}(\lambda)R_{23}(\lambda - \mu) \tag{6.33}
\]

which is called the Yang-Baxter equation. In fact, we can leave the Lax operator altogether now and express the Yang-Baxter equation in $R$ operators only

\[
R_{23}(\lambda - \mu)R_{12}(\lambda)R_{13}(\mu) = R_{13}(\mu)R_{12}(\lambda)R_{23}(\lambda - \mu) \tag{6.34}
\]

simply by letting $\lambda \rightarrow \lambda + i/2$ and $\mu \rightarrow \mu + i/2$.

\section*{6.1.3 The Transfer Matrix}

To be able to shuffle and manipulate our local spaces it is convenient to take use of auxiliary spaces being container spaces that we will get rid of at the end. So instead of looking at our Hilbert space defined in 6.14 we expand it with one auxiliary space, $a$ and look at $a \otimes \mathcal{H}$. Using this expanded Hilbert space the transfer matrix, $T_a(\lambda)$, is defined as

\[
T_a(\lambda) = \prod_{l=1}^{L} L_{wl}(\lambda). \tag{6.35}
\]

Also the transfer matrix is obeying a relation similar to the Yang-Baxter relation which is easily shown for $L = 2$. Now we look at two transfer matrices
(\(T_a\) and \(T_b\)) each having its own auxiliary space \((a\) and \(b)\). We write it out

\[
R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) = R_{ab}(\lambda - \mu) L_{a1}(\lambda) L_{a2}(\lambda) L_{b1}(\mu) L_{b2}(\mu)
\]

(6.36)

noting that \(R\) has argument \(\lambda - \mu\), all \(Ls\) with index \(a\) (\(b\)) has argument \(\lambda\) (\(\mu\)) we can skip all the arguments. Also remembering that \(L_{ij} = L_{ji}\) we can with the use of 6.33 express 6.36 as

\[
R_{ab} L_{a1}(\lambda) L_{a2}(\lambda) L_{b1}(\mu) L_{b2}(\mu) = R_{ab} L_{b1}(\mu) L_{b2}(\mu) L_{a1}(\lambda) L_{a2}(\lambda)
\]

(No equal indices)

(6.37)

writing it out with all the arguments we end up with

\[
R_{ab}(\lambda - \mu) T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda) R_{ab}(\lambda - \mu)
\]

(6.38)

which is referred to as the Fundamental Commutator Relation (FCR). The process of showing the same result for \(L > 2\) is more or less identical and it can be shown by induction that it is true for all \(L\).

Now to construct operators made out of \(T_a(\lambda)\)s acting on our Hilbert space, 6.14, and not also on the auxiliary spaces we can take the trace over the auxiliary spaces where the trace of an operator \(V_1 \otimes \cdots \otimes V_L\) is defined as

\[
\text{Tr}_l V_1 \otimes \cdots \otimes V_L = (\text{Tr} V_l) V_1 \otimes \cdots \otimes V_{l-1} \otimes V_{l+1} \otimes \cdots \otimes V_L.
\]

(6.39)

Taking the auxiliary traces of 6.38 acted on by \(R_{ab}^{-1}(\lambda - \mu)\) from the left we have

\[
\text{Tr}_a \text{Tr}_b T_a(\lambda) T_b(\mu) = \text{Tr}_a \text{Tr}_b R_{ab}^{-1}(\lambda - \mu) T_b(\mu) T_a(\lambda) R_{ab}(\lambda - \mu)
\]

(6.40)

where only the \(T_a(\lambda)\) and the \(T_b(\mu)\) act on our Hilbert space. Using this the cyclicity of the trace allows us to put the \(R_{ab}^{-1}\) in the end and \(\text{Tr}_a\) can pass all operators containing only \(b\) index and vice versa. We end up with

\[
(\text{Tr}_a T_a(\lambda)) (\text{Tr}_b T_b(\mu)) = (\text{Tr}_b T_b(\mu)) (\text{Tr}_a T_a(\lambda)).
\]

(6.41)

Remembering now that \(a\) and \(b\) are only dummy indices whose spaces we have traced away we see that the operator acting on our Hilbert space (and no auxiliary spaces) \(t(\lambda) = \text{Tr}_a T_a(\lambda)\) obeys the relation

\[
[t(\lambda), t(\mu)] = 0.
\]

(6.42)

This might not look too impressive at first sight, but what we have found
is a one parameter family of operators working on the whole chain which all mutually commute. So, all the $t(\lambda)$s have the same eigenvectors in our Hilbert space, and also all operators we can construct of the $t(\lambda)$s. As we will see, our Hamiltonian can also be expressed in terms of the $t(\lambda)$s giving that if we manage to diagonalise $t(\lambda)$ we have diagonalised also the Hamiltonian.

To see what operators we can find in the $t(\lambda)$ we remember that it is a polynomial in $(\lambda - i/2)$ of degree $L$ (see 6.28 and 6.35). So, let us study it term by term:

$$t(\lambda) = \sum_{i=0}^{L} (\lambda - i/2)^i t_i$$  \hspace{1cm} (6.43)

with the first term being

$$t_0 = t(i/2) = \text{Tr}_a T_a(i/2) = \text{Tr}_a \prod_{i=1}^{L} L_{ai}(i/2) = \lambda^L \text{Tr}_a \prod_{i=1}^{L} P_{ai}. \hspace{1cm} (6.44)$$

The product of permutation will go through $a \otimes \mathcal{H}$ as

$$\prod_{i=1}^{L} P_{ai}(a \otimes \mathcal{H}) = P_{a_1} \cdots P_{a_L}(a \otimes h_1 \otimes \cdots \otimes h_L) = h_1 \otimes \cdots \otimes h_L \otimes a \hspace{1cm} (6.45)$$

giving us

$$t_0 \mathcal{H} = \lambda^L \text{Tr}_a \sum_{i=1}^{L} P_{ai}(a \otimes \mathcal{H}) = \lambda^L \text{Tr}_a \left( \prod_{i=1}^{L} P_{ai} \right) (a \otimes h_1 \otimes \cdots \otimes h_L) \hspace{1cm} (6.46)$$

where we have marked out the position of the $a$ space which stays the same although we have shuffled its content. We can now take the trace of the operator $t_0$ when we know how it act by using the following relation, for a $2 \times 2$ matrix $A$

$$\text{Tr} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hspace{1cm} (6.47)$$

to get

$$t_0 \mathcal{H} = \lambda^L \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \frac{a}{h_1 \otimes h_2 \otimes \cdots \otimes h_L} \\ 0 \end{pmatrix} \right) + \lambda^L \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \frac{a}{h_1 \otimes h_2 \otimes \cdots \otimes h_L} \\ 0 \end{pmatrix} \right)$$

$$= \lambda^L \left( \prod_{i=1}^{L} h_i \otimes \cdots \otimes h_L \otimes h_1 \right). \hspace{1cm} (6.48)$$
We see that $t_0$ is simply the shift operator and we can now redirect an operator, $X_l$, acting on $h_l$ to work on $h_{l+n}$ by

$$X_{l+n} = (i^{-L}t_0)^{-n}X_l(i^{-L}t_0)^n = t_0^{-n}X_l t_0^n.$$  \hspace{1cm} (6.49)

This is very similar to the translation operator in ordinary quantum mechanics. Defining a total momentum

$$P = -i\log(t(i/2))$$  \hspace{1cm} (6.50)

we can write 6.49 on the standard form

$$X_{l+n} = e^{-inP}X_l e^{inP}.$$ \hspace{1cm} (6.51)

The second term in the expansion 6.43 will be the linear term

$$t_1 = t'(i/2) = \text{Tr}_{a}T'_a(i/2) = \sum_{k=1}^{L} \text{Tr}_{a} \prod_{l \neq k}^{L} P_{a,l}.$$  \hspace{1cm} (6.52)

Every term will then have the same structure as for $t_0$ but will omit the $l$th element and, hence, shift all elements one slot to the left apart from the $l$th element which will stay, and the $(l+1)$th element which will be shifted two slots since it is passing the $l$th element. In other words, elements $l$ and $l+1$ are exchanged and the whole product space is shifted to the left giving

$$t_1(h_1 \otimes \cdots \otimes h_l \otimes h_{l+1} \otimes \cdots \otimes h_L)$$

$$= i^{L-1} \sum_{l=1}^{L} (h_2 \otimes \cdots \otimes h_{l+1} \otimes h_l \otimes \cdots \otimes h_L \otimes h_1).$$ \hspace{1cm} (6.53)

Shifting it back with $t_0^{-1}$ will leave us with a simple exchange of elements $l$ and $l+1$ times the constant factors $i^{-L}$ from $t_0^{-1}$ and $i^{L-1}$ from $t_1$. Using this we see that

$$t_0^{-1}t_1 = \sum_{k=1}^{L} t_0^{-1} \text{Tr}_{a} \prod_{l \neq k}^{L} P_{a,l} = \sum_{k=1}^{L} i^{-L} t_0^{-1} P_{k,k+1} = -i \sum_{l=1}^{L} P_{l,l+1}$$ \hspace{1cm} (6.54)

which can neatly express our Hamiltonian 6.15 with our neighbour Hamiltonian 6.27 as

$$H = \frac{J_0}{2}(i t_0^{-1}t_1 - L1) = \frac{J_0}{2} \left( i \frac{d}{d\lambda} \left( \log(t(\lambda)) \right) \right)_{\lambda=i/2} - L1.$$ \hspace{1cm} (6.55)

So, now we know that if we can diagonalise our $t(\lambda)$ we have also diagonalised our Hamiltonian, since it can be expressed in terms of $t(\lambda)$ and the unit, and have an exact solution to our system.
Higher terms of $t(\lambda)$ will then be more involved permutations in the chain from when the derivatives will make more holes in the chain of permutations, but we have already reached our partial goal here to express our Hamiltonian in terms of the transfer matrix.

6.2 The Bethe Ansatz for SU(2)

It turns out to be possible to diagonalise the $t(\lambda)$ operator and, thus, also the Hamiltonian. This was first done by H. Bethe in 1931 [8]. His method of doing so is referred to as the Bethe ansatz. The idea of the Bethe ansatz is to use the auxiliary spaces introduced in the last section and study the elements of the operator expressed as matrices in this auxiliary space. The transfer matrix in our auxiliary space has four elements

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a,$$

each element being an operator on the full spin chain Hilbert space, $\mathcal{H}$. The $a$ indicates that the matrix is acting on space $a$. The $\text{Tr}_a$ is then only the ordinary trace giving that the operator whose eigenvector we seek is written $t(\lambda) = A(\lambda) + D(\lambda)$.

Looking at the ferromagnetic vacuum, that is the state with all spins up,

$$|0\rangle_+ = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

we can see that it is an eigenstate to $t(\lambda)$. To start with,

$$A(\lambda) |0\rangle_+ = \begin{pmatrix} 1 & 0 \end{pmatrix}_a T_a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where now all the exchange operators will act like identities since all elements are equal so that $L_{ij}(\lambda) = ((\lambda - i/2) + i)\mathbf{1}$, giving

$$A(\lambda) |0\rangle_+ = (\lambda + i/2)^L |0\rangle_+.$$  

(6.59)

Similarly we have

$$D(\lambda) |0\rangle_+ = \begin{pmatrix} 0 & 1 \end{pmatrix}_a T_a \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(6.60)

where any exchange operator $P_{at}$ will move a spin up state to the $a$ state giving
zero. Thus, only the $(\lambda - i/2)1$ terms in the $L$s will give non zero contribution:

$$D(\lambda) |0\rangle_+ = (\lambda - i/2)^L |0\rangle_+ .$$

(6.61)

So, the vacuum is an eigenstate for $t(\lambda)$ with eigenvalue $(\lambda + i/2)^L + (\lambda - i/2)^L$. In fact, $|0\rangle_+$ is an eigenstate of $C(\lambda)$ as well. Repeating the routine above one find that the $C(\lambda)$ annihilates our vacuum: $C(\lambda) |0\rangle_+ = 0$. Observe that

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

(6.62)

### 6.2.1 Bethe States

We now observe that $B(\lambda)$ acting on our vacuum will result in a sum of states with one spin up flipped to a spin down:

$$B(\lambda) |0\rangle_+ = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_a T_a \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right)_a \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right)_a .$$

(6.63)

This will be true for any state constructed of spin up/down states. Acting on the vacuum twice will result in a sum of states with two spins down and so on. One would assume that we would be able to operate $L$ times on the vacuum state until all spins are flipped, but this is not the case. Bethe states with $J > L/2$ has zero norm, that is $\langle 0 | B^\dagger(\lambda_J) \cdots B^\dagger(\lambda_1) B(\lambda_1) \cdots B(\lambda_J) | 0 \rangle = 0$ for $J > L/2$.

States constructed by letting products of $B(\lambda_i)$s act on our vacuum are called Bethe states:

$$|\lambda_1, \ldots, \lambda_J\rangle = B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+$$

(6.64)

with $J \leq L/2$. Bethe states will be eigenstates of $t(\lambda)$ for certain values of $\{\lambda_i\}$. To see that we need to know what happens when we rearrange the order of $t(\lambda) = A(\lambda) + D(\lambda)$ and $B(\mu)$.

To do so, we express the FCR from 6.38 in matrices in the auxiliary spaces $a \otimes b$ by using the definitions on our operators,

$$P_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{a \otimes b}$$

(6.65)
The way to do it is to let the $A(\lambda)$ and the $D(\lambda)$ pass through the row of $B(\lambda)$s.
to act on the vacuum state. Starting with the first term we have
\[
A(\lambda) B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+ = \left( \prod_{j=1}^{J} a(\lambda - \lambda_j) \right) (\lambda + i/2)^L B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+ \tag{6.74}
\]
\[
+ \sum_{j=1}^{J} \Omega_j(\lambda, \lambda_1, \ldots, \lambda_J) B(\lambda_1) \cdots B(\lambda_{j-1}) B(\lambda_j) B(\lambda_{j+1}) \cdots B(\lambda_J) |0\rangle_+ .
\]

The first term of 6.74, which is the part containing only the first term in 6.70, is already diagonal, but not the second term, which is all the other contributions and can be put on this form since all \(B(\lambda_i)\)s commute. The commutativity of the \(B(\lambda_i)\)s also assures that all \(\Omega_i\) will have the same structure so that it is sufficient only to look at \(\Omega_1\). To find \(\Omega_1\) we start off as in 6.74 but look only at the terms containing no \(B(\lambda_1)\) after passing through with \(A(\lambda)\):
\[
A(\lambda) B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+ = a(\lambda - \lambda_1) B(\lambda_1) A(\lambda) B(\lambda_2) \cdots B(\lambda_J) |0\rangle_+ \\
+ b(\lambda - \lambda_1) B(\lambda) A(\lambda_1) B(\lambda_2) \cdots B(\lambda_J) |0\rangle_+ \\
= b(\lambda - \lambda_1) a(\lambda_1 - \lambda_2) B(\lambda) B(\lambda_2) A(\lambda_1) B(\lambda_3) \cdots B(\lambda_J) |0\rangle_+ \\
+ \text{Terms containing } B(\lambda_1) \\
= b(\lambda - \lambda_1) \left( \prod_{i \neq 1} a(\lambda_1 - \lambda_i) \right) B(\lambda) B(\lambda_2) \cdots B(\lambda_J) A(\lambda_1) |0\rangle_+ \\
+ \text{Terms containing } B(\lambda_1) \\
= (\lambda_1 + i/2)^L b(\lambda - \lambda_1) \left( \prod_{i \neq 1} a(\lambda_1 - \lambda_i) \right) B(\lambda) B(\lambda_2) \cdots B(\lambda_J) |0\rangle_+ \\
+ \text{Terms containing } B(\lambda_1) \tag{6.75}
\]
giving that
\[
\Omega_1(\lambda, \lambda_1, \ldots, \lambda_J) = (\lambda_1 + i/2)^L b(\lambda - \lambda_1) \prod_{i \neq 1} a(\lambda_1 - \lambda_i) \tag{6.76}
\]
and then also
\[
\Omega_j(\lambda, \lambda_1, \ldots, \lambda_J) = (\lambda_j + i/2)^L b(\lambda - \lambda_j) \prod_{i \neq j} a(\lambda_j - \lambda_i) . \tag{6.77}
\]
Exactly the same method can be used to see that we have

\[
D(\lambda)B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+ \\
= \left( \prod_{j=1}^J c(\lambda - \lambda_j) \right) (\lambda - i/2)^L B(\lambda_1) \cdots B(\lambda_J) |0\rangle_+ \\
+ \sum_{j=1}^J \Xi_j(\lambda, \lambda_1, \ldots, \lambda_J) B(\lambda_1) \cdots B(\lambda_{j-1}) B(\lambda_j) B(\lambda_{j+1}) \cdots B(\lambda_J) |0\rangle_+ 
\]

with

\[
\Xi_j(\lambda, \lambda_1, \ldots, \lambda_J) = (\lambda_j - i/2)^L d(\lambda - \lambda_j) \prod_{i \neq j} c(\lambda_j - \lambda_i). 
\]

We can now express the \( t(\lambda) \) acting on an arbitrary Bethe state

\[
t(\lambda) |\lambda_1, \ldots, \lambda_J \rangle \\
= \left( (\lambda + i/2)^L \prod_{j=1}^J a(\lambda - \lambda_j) + (\lambda - i/2)^L \prod_{j=1}^J c(\lambda - \lambda_j) \right) |\lambda_1, \ldots, \lambda_J \rangle \\
+ \sum_{j=1}^J \left( \left( \Omega_j(\lambda, \lambda_1, \ldots, \lambda_J) + \Xi_j(\lambda, \lambda_1, \ldots, \lambda_J) \right) \\
\times B(\lambda_1) \cdots B(\lambda_{j-1}) B(\lambda_j) B(\lambda_{j+1}) \cdots B(\lambda_J) |0\rangle_+ \right) 
\]

and now we can see that if only the second term is zero, we will have an eigenstate of \( t(\lambda) \). The condition then for the Bethe state 6.64 to be diagonal is that \( \Omega_j(\lambda, \lambda_1, \ldots, \lambda_J) + \Xi_j(\lambda, \lambda_1, \ldots, \lambda_J) = 0 \). These conditions are the Bethe equations which using 6.77 and 6.79 will be on the form

\[
\left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^L = \prod_{i \neq j} \frac{\lambda_j - \lambda_i + i}{\lambda_j - \lambda_i - i}. 
\]

For the Bethe state \( |\lambda_1, \ldots, \lambda_J \rangle \) the eigenvalue of \( t(\lambda) \) is

\[
\Lambda(\lambda) = (\lambda + i/2)^L \prod_{j=1}^J \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + (\lambda - i/2)^L \prod_{j=1}^J \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} 
\]

So, to sum up what was done in this section: We split our transfer matrix into matrix elements. We found a vacuum state which was an eigenstate to the trace of the transfer matrix. We constructed \( L/2 \) orthogonal states which were also eigenstates to the transfer matrix, if the Bethe equation was valid for the set of spectral parameters.
6.2.2 Total Momentum and Energy

Having now a large set of states we can calculate their total momentum and energy. For this we have the relations 6.50 and 6.55. Remembering the eigenvalues of $t(\lambda)$ in 6.82 we find

$$P|\lambda_1,\ldots,\lambda_J\rangle = -i \sum_{j=1}^{J} \log \frac{\lambda_j + i/2}{\lambda_j - i/2} |\lambda_1,\ldots,\lambda_J\rangle$$

(6.83)

and

$$H|\lambda_1,\ldots,\lambda_J\rangle = \frac{J_0}{2} \left( \frac{N(i/2)}{\Lambda(i/2)} - L \right) |\lambda_1,\ldots,\lambda_J\rangle$$

(6.84)

6.3 The Bethe Ansatz for $SO(6)$

As was already mentioned the matrix of anomalous dimensions is diagonalisable using the Bethe ansatz. The matrix we want to diagonalise is

$$\Gamma = \frac{\hat{\lambda}}{16\pi^2} \sum_{l=1}^{L} \left( K_{l,l+1} + 2I_{l,l+1} - 2P_{l,l+1} \right).$$

(6.85)

The identity and the exchange operator are the same as before and the trace operator, $K_{ij}$, is defined as

$$K_{ab}(a \otimes b) = (a \cdot b) \sum_{i=1}^{6} (\hat{e}_i \otimes \hat{e}_i)$$

(6.86)

where $a \cdot b$ is the $\mathbb{R}^6$ scalar product of the elements in $a$ and $b$ and $\hat{e}_i$ are six orthonormal base vectors in $\mathbb{R}^6$.

For a $SO(6)$ spin chain the $R$ matrix is of the form

$$R_{ij}(\lambda) = \frac{1}{2} \left( \lambda (\lambda - 2) I_{ij} - (\lambda - 2) P_{ij} + \lambda K_{ij} \right)$$

(6.87)

which also satisfies the Yang-Baxter equation 6.34. To show this is not hard, but demands some bookkeeping skills and, more importantly, patience\(^1\).

We can use roughly the same route as for the $SU(2)$ case using only the $R$ matrix, we will need no Lax operator. As in section 6.1.3 we define a transfer

\(^1\)To show it use the properties $P_{12}P_{23} = P_{23}P_{13}$, $P_{12}K_{23} = K_{13}P_{12}$, $K_{12}K_{23} = K_{13}K_{12}$, $K_{12}^2 = K_{13}$, $K_{23}^2 = K_{23}$ and their permutations.
matrix acting on $a \otimes \mathcal{H}$ but this time express it in terms of $R$

$$T_a(\lambda) = \prod_{l=1}^{L} R_{a,l}(\lambda)$$

(6.88)

with $t(\lambda)$ being its trace and $t_0$ being $t(\lambda)$s Taylor coefficients around 0 now instead. Again we see how the first two terms act on a state in our Hilbert space:

$$t_0 = t(0) = \text{Tr}_a \prod_{l=1}^{L} R_{a,l}(0) = \text{Tr}_a \prod_{l=1}^{L} P_{al}$$

(6.89)

where we see that it (up to $t^L$) is the same shift operator as for the XXX$_{1/2}$ case.

The second term is a bit more involved:

$$t_1 = t'(0) = \text{Tr}_a \frac{d}{d\lambda} \prod_{l=1}^{L} R_{a,l}(\lambda) \bigg|_{\lambda=0}$$

$$= \sum_{l=1}^{L} \text{Tr}_a R_{a1}(0) \cdots R_{a,l-1}(0) R'_{a,l}(0) R_{a,l+1}(0) \cdots R_{aL}(0)$$

(6.90)

$$= \sum_{l=1}^{L} \text{Tr}_a P_{a1} \cdots P_{a,l-1} \left(-I_{al} - \frac{1}{2} P_{al} + \frac{1}{2} K_{al}\right) P_{a,l+1} \cdots P_{aL}.$$

Looking at the first term in the bracket we see that it, as in the sections before 6.54, is exchanging $h_l$ and $h_{l+1}$ and then shifting the content in all spaces to the left. The second term is simply shifting to the left. The third is applying the trace to $h_l$ and $h_{l+1}$, then shifting the content of all spaces one slot left. Shifting it back with $t^{-1}(\lambda)$ as before leaves us with an exchange, an identity and a trace of $h_l$ and $h_{l+1}$:

$$t_0^{-1} t_1 = \sum_{l=1}^{L} \left(-P_{l,l+1} - \frac{1}{2} I_{l,l+1} + \frac{1}{2} K_{l,l+1}\right)$$

(6.91)

which is close to 6.85 but not identical. We see though, that what differs from the renormalisation operator is an overall constant and a term proportional to the identity. Adding a term proportional to the identity, just as we did for the $SU(2)$ case, to an operator will render the same eigenvectors giving that

$$\Gamma = \frac{\lambda}{16\pi^2} \left(2t_0^{-1}t_1 + 3L1\right)$$

(6.92)

will have the same eigenstates as $t(\lambda)$.

Now we need the eigenvalues, $\Lambda(\lambda)$, of $t(\lambda)$ as in pages 49–51 but the process for this case is much more involved. It is however possible to go
through the Bethe ansatz much in the same fashion as was done in 6.2 but now the $T_a(\lambda)$ is a $6 \times 6$ real matrix in $a$-space so there will not be only one operator to construct the states with but 15 (the upper diagonal elements).

To find the eigenvalues we will instead use another method developed by N. Yu. Reshetikhin [22, 23] where one finds a relation of the eigenvalues on the form $\Lambda(\lambda)\bar{\Lambda}(-\lambda) = f(\lambda)$, solves this equation and enforce the solution to be polynomial in $\lambda$ as it should by construction.

To find the relation one take advantage of the symmetry in the $R$ operator:

$$R_{12}(2 - \bar{\lambda}) = (R_{12}(\lambda))^T$$

(6.93)

Here $T_2$ is the transpose of the second product operator (it works for the first as well) and, by their construction, $P^T_2 = K$.

This implies for the $t(\lambda)$s

$$(t(\lambda))^\dagger = \left( \text{Tr}_a \prod_{l=1}^L R_{al}(\lambda) \right)^\dagger = \text{Tr}_a \prod_{l=1}^L R_{al}(2 - \bar{\lambda})
= t(2 - \bar{\lambda})$$

(6.94)

where the bar is complex conjugate. Observe here that the dagger acts on the Hilbert space but not on the auxiliary space. Assuming that $\lambda$ is real we get the relation for $\Lambda(\lambda)$ being the eigenvalues of $t(\lambda)$

$$\Lambda(\lambda)\bar{\Lambda}(-\lambda) = \Lambda(\lambda)\Lambda(\lambda + 2) = t(\lambda)t(\lambda + 2) = t(\lambda + 2)t(\lambda)$$

(6.95)

We use this now to express

$$\Lambda(\lambda)\bar{\Lambda}(-\lambda) = \Lambda(\lambda)\Lambda(\lambda + 2) = t(\lambda)t(\lambda + 2) = t(\lambda + 2)t(\lambda)$$

(6.96)

where we assume we act on an eigenstate and we remember that the $t(\lambda)$s commute. We see that all elements in the product of 6.96 are on the form $R_{ai}(\lambda + 2)R_{bi}(\lambda) = R_{ai}(\lambda + 2)R_{bi}(\lambda)$ which has a simplifying property. From 6.34 we see that

$$R_{ab}(2)R_{ai}(\lambda + 2)R_{bi}(\lambda) = R_{ab}(\lambda)R_{ai}(\lambda + 2)R_{ab}(2)$$

(6.97)

where $R_{ab}(2) = K_{ab}$ is in fact proportional to a projector since $KK = 6K$ by construction. $K$ projects down to the one dimensional space spanned by $\sum_i \hat{e}_i \otimes \hat{e}_i$ and has the orthogonal projector $K^\perp = 6I - K$. Using this we see that

$$K_{ab}R_{ai}(\lambda + 2)R_{bi}(\lambda)K_{ab}^\perp = 0.$$  

(6.98)
This tells us that in a basis of the product space \( a \otimes b \) with \( 6^{-1/2} \sum_i \hat{e}_i \otimes \hat{e}_i \) as the first base vector our operator expressed in \( a \otimes b \) will look like

\[
R_{ia}(\lambda + 2)R_{ib}(\lambda) = \begin{pmatrix}
R_i & 0 & \cdots & 0 \\
* & \ddots & \ddots & \\
* & \ddots & R_i \\
* & & & \\
\end{pmatrix}
\]  
(6.99)

now \( R_i \) acts on \( i \) space and \( \tilde{R}_i \) is a \( 35 \times 35 \) matrix with elements acting on \( i \) space. In these coordinates we can express 6.96 as

\[
\Lambda(\lambda)\bar{\Lambda}(-\lambda) = \operatorname{Tr}_{a} \operatorname{Tr}_{b} \begin{pmatrix}
R_1 \cdots R_L & 0 & \cdots & 0 \\
* & \ddots & \ddots & \\
* & \ddots & R_1 \cdots \tilde{R}_L \\
* & & & \\
\end{pmatrix}
\]  
(6.100)

where we used the fact that the trace over \( a \otimes b \) is invariant under coordinate transformation. The trace in the last row is the trace of the \( 35 \times 35 \) matrix.

Now, the first term in 6.100 one can find from the definition of \( R(\lambda) \) sandwiched between the first base vector \( 6^{-1/2} \sum_i \hat{e}_i \otimes \hat{e}_i \) (as was done on page 47 to find matrix elements):

\[
R_i = \frac{1}{6} \sum_j (\hat{e}_j \otimes \hat{e}_j)^T R_{ia}(\lambda + 2)R_{ib}(\lambda) \sum_k \hat{e}_k \otimes \hat{e}_k \\
= \frac{1}{4}(\lambda^2 - 1)(\lambda^2 - 4)1_i
\]  
(6.101)

This is surprisingly simple and lets us express 6.100 as

\[
\Lambda(\lambda)\bar{\Lambda}(-\lambda) = \left( \frac{(\lambda^2 - 1)(\lambda^2 - 4)}{4} \right)^L + \lambda^L \Lambda_R(\lambda)
\]  
(6.102)

where the term \( \lambda^L \Lambda_R(\lambda) \) is the eigenvalue of \( \operatorname{Tr} \tilde{R}_1 \cdots \tilde{R}_L \). The factor \( \lambda^L \) can be pulled out of the term from the construction of \( R(\lambda) \).

N. Yu. Reshetikhin shows in his paper that the solution for equation 6.102 in order to also satisfy 6.95 must be of the form

\[
\Lambda(\lambda) = \frac{1}{4^L} \left[ (\lambda - 1)^L (2\lambda - 4)^L H(\lambda) + \lambda^L (2\lambda - 2)^L F(\lambda) + \lambda^L (2\lambda - 4)^L G(\lambda) \right]
\]  
(6.103)
where the introduced functions \( H(\lambda) \), \( F(\lambda) \) and \( G(\lambda) \) satisfies
\[
1 = H(\lambda)H(-\lambda), \quad \text{(6.104)}
\]
\[
H(\lambda) = F(2-\lambda), \quad \text{(6.105)}
\]
\[
G(\lambda) = G(2-\lambda). \quad \text{(6.106)}
\]

These three functions can in turn be expressed in terms of three groups of parameters \( \lambda_{1,i} \), \( \lambda_{2,j} \) and \( \lambda_{3,k} \) with \( i = 1 \ldots J_1 \), \( j = 1 \ldots J_2 \) and \( k = 1 \ldots J_3 \). In order for the eigenvalues to be polynomial in \( \lambda \), which they have to be by construction, the groups of parameters must obey
\[
\left( \frac{\lambda_{1,i} + i/2}{\lambda_{1,i} - i/2} \right)^L = \prod_{j \neq i} \frac{\lambda_{1,i} - \lambda_{1,j}}{\lambda_{1,i} + \lambda_{1,j}} \prod_{j} \frac{\lambda_{1,i} - \lambda_{2,j} - i/2}{\lambda_{1,i} + \lambda_{2,j} + i/2} \prod_{j} \frac{\lambda_{1,i} - \lambda_{3,j} - i/2}{\lambda_{1,i} + \lambda_{3,j} + i/2}
\]
\[
1 = \prod_{j \neq i} \frac{\lambda_{2,i} - \lambda_{2,j} + i/2}{\lambda_{2,i} + \lambda_{2,j} - i/2} \prod_{j} \frac{\lambda_{2,i} - \lambda_{1,j} - i/2}{\lambda_{2,i} + \lambda_{1,j} + i/2}
\]
\[
1 = \prod_{j \neq i} \frac{\lambda_{3,i} - \lambda_{3,j} - i/2}{\lambda_{3,i} + \lambda_{3,j} + i/2} \prod_{j} \frac{\lambda_{3,i} - \lambda_{1,j} + i/2}{\lambda_{3,i} + \lambda_{1,j} - i/2} \quad \text{(6.107)}
\]

where in the first equation \( i = 1 \ldots J_1 \), the second \( i = 1 \ldots J_2 \) and in the third \( i = 1 \ldots J_3 \).

Note now that for a set of the introduced parameters where we have \( \lambda_{2,j} \) nor \( \lambda_{3,k} \), that is when \( J_2 = J_3 = 0 \), the first equation for \( \lambda_{1,i} \) exactly reduces to the Bethe equations for the \( SU(2) \) case in 6.81 and the following two are then trivially satisfied. This reflects the \( SU(2) \) sector of \( SO(6) \) and signals that the equations 6.107 are generalisations of 6.81. Later on the Bethe equations were generalised to arbitrary groups by E. Ogievetsky and P. Wiegman [19].

Using now the decomposition of the eigenvalues of \( \Lambda(\lambda) \) the total momentum \( P \) and the Hamiltonian \( H \) can be worked out
\[
P = -i \log(\Lambda(0)) = -i \sum_{i} \log \frac{\lambda_{1,i} + i/2}{\lambda_{1,i} - i/2} \quad \text{(6.108)}
\]
\[
H = 2 \sum_{i} \frac{1}{\omega_{1,i}^2 + 1/4} \quad \text{(6.109)}
\]

Remember now that \( \Lambda(\lambda) \) is defined slightly different so that the pure translation is at \( \Lambda(0) \) and not at \( \Lambda(i/2) \) as in Section 6.2. The fact that we assumed \( \lambda \) to be real will not be a problem. What we are interested in is the expansion at the point \( \lambda = 0 \).
6.4 Finite Size Corrections

In the last sections we learned that the problem of diagonalising our $\Gamma$ was now transformed into the problem of solving the Bethe equations for the Bethe roots and then plugging them into the expression for the anomalous dimension. For large $L$ it is indeed simpler than diagonalising $\Gamma$, which was a $6^L \times 6^L$ matrix for the $SO(6)$ case, but it is still tricky. What can be done for the BMN operators in Section 4.3.2 is to use the fact that $L \to \infty$ observing that the sums can now be expressed as integrals.

We start by recalling the Bethe equation

$$ \left( \frac{\lambda_j + i/2}{\lambda_j - i/2} \right)^L = \prod_{i \neq j} \frac{\lambda_j - \lambda_i + i}{\lambda_j - \lambda_i - i} $$

in where we look at the rescaled variables $\lambda_i = Lu_i$ so that we rewrite the Bethe equation

$$ \left( \frac{1 + i/2}{1 - i/2} \right)^L = \prod_{i \neq j} \frac{1 + \frac{i}{L(u_j - u_i)}}{1 - \frac{i}{L(u_j - u_i)}}. $$

Using now that $L$ is large and $\log \frac{1 + \delta}{1 - \delta} = 2\delta + O(\delta^3)$ we find that taking the logarithm of 6.111

$$ \frac{1}{u_j} + 2\pi n_j = \frac{1}{L} \sum_{i \neq j} \frac{1}{u_j - u_i}. $$

The $n_j$ reflects which branch of the logarithm the root is located in. These sums can now be expressed as integrals and solved using the Riemann-Hilbert method.

In [III] in this thesis we look for the $1/L$ corrections to the anomalous dimensions for operator in the $SU(2)$ sector. This is done in two different ways: First we do it by using B. S. Shastry’s and A. Dahr’s observation [25] that the Bethe roots in 6.112 are the roots of the associated Laguerre polynomials giving us a way to calculate the anomalous dimension for $L \to \infty$ with $1/L$ corrections. In the appendix we also show that it can be done in a more straightforward fashion which is probably easier to generalise to the full $SO(6)$ case or hopefully even to the full supersymmetric group $SU(2,2|4)$.

6.5 Further Corrections and Generalisations

The method above has been extended to include not only the scalar fields (the $SO(6)$ sector) but also other operators of the SYM theory. N. Beisert and M. Staudacher [5] generalized this to include also the field strength and the fermionic spectrum. They argue that the complete set of one loop contribu-
tions to the anomalous dimension of single trace operators can be calculated by studying the Bethe ansatz of a $SU(2, 2|4)$ supersymmetric spin chain.

Methods to expand the reasoning above to include also higher loop contributions to the Feynman diagrams has been studied. The case for up to two loop contributions was examined in [1, 4]. To express higher loop contributions in terms of spin chains will include not only closest neighbour interaction but also planar contributions mixing operators further apart. Different sub-sectors of the full system has been expressed up to third loop contributions in terms of spin chains [2, 3]. At three loop problem arise, the two sides in the correspondence do not appear to match. The source of this is not yet known but it has been suggested that it might come from taking the limits $\lambda' \to 0$ and $L \to \infty$ in different order [3, 13].
7 Numerical investigations of $SU(2)$ Yang-Mills

The work done in [I] is not closely related to the other two papers but there are some concepts that should be explained closer. For that I use this, the last chapter of the comprehensive summary.

In [I] an effective action of non-supersymmetric $SU(2)$ Yang-Mills was examined numerically. To explain the procedure we start with by explaining the concept of Poincaré maps. In Section 7.2 we start by presenting the unperturbed, tree-level effective action to then in Section 7.3 present the one-loop effective action.

7.1 Poincaré Maps

For a dynamical system with bound motion it is often interesting to study a Poincaré map. To do so we must first define a surface of section which is a surface, $S$, in the phase space, $M$, of the dynamical system. The Poincaré map, $P(\vec{x})$, is now the map from this surface onto itself such that

$$P(\vec{x}(0)) = \vec{x}(T)$$

(7.1)

where $\vec{x}(t)$ is a trajectory on $M$ and

$$\vec{x}(0), \vec{x}(T) \in S \quad \text{and} \quad \vec{x}(t) \notin S \quad \text{for} \quad 0 < t < T.$$  

(7.2)

In words this means that $P$ takes a point on the surface of section, $\vec{x}_0$, lets this point flow in $M$ until it intersects the surface again and maps $\vec{x}_0$ to that point.

Later in this section “study the Poincaré map” really means that we iterate $P(\vec{x})$ plotting points on the surface of section. That is, we plot all the points where a very long trajectory intersects our surface of section.

This method is very convenient when studying chaotic systems. A central property to look for is the existence of conserved quantities. If a flow has a conserved quantity the trajectories will be bound to surfaces in phase space. The cross sections of these surfaces will make out lines of dots in the iterated Poincaré map. So, basically, if we see space filling dots on the plot we have no conserved quantity and chaotic motion.
7.2 The Unperturbed Effective Action

A tree-level effective action of the low energy limit of $SU(2)$ Yang-Mills has been proposed. One way to find it is to use the Faddeev-Niemi decomposition \cite{11} where the off diagonal contribution to $A_\mu$ are decomposed as follows:

$$A_\mu^1 + iA_\mu^2 = i\psi_1 e_\mu + i\psi_2 e_\mu$$  \hspace{1cm} (7.3)

and $A_\mu^3$ is left intact. This can be seen in 6.17 where $\sigma_x$ and $\sigma_y$ are the off diagonal generators of $SU(2)$ and $\sigma_z$ is the diagonal. The basis vector is defined as

$$e_\mu = \frac{1}{\sqrt{2}}(e^1_\mu + ie^2_\mu) \quad \text{where} \quad g^{\mu\nu}e_\mu^a e_\nu^b = S^{ab}. \hspace{1cm} (7.4)$$

Noting now that the left hand side has $4 + 4 = 8$ degrees of freedom but the right hand side has two complex scalars $(2 + 2 = 4)$, and 2 orthonormal vectors $(3 + 2 = 5)$ giving 9 degrees of freedom we are left with one extra symmetry: The rotation mixing $e^1_\mu$ and $e^2_\mu$.

To find the classical effective potential we shift our $A_\mu^a$ with $\tilde{A}_\mu^a$ which is our classical background field:

$$A_\mu^a \rightarrow \tilde{A}_\mu^a + A_\mu^a.$$  \hspace{1cm} (7.5)

The part of the Lagrangian containing only the classical background field with no derivatives will be our effective potential. We expand our gauge field part of the action in 3.14 using the definition 3.13

$$-\frac{1}{4}(F_{\mu\nu}^a)^2 = -\frac{g^2}{4}f^{abc}f^{ade}A_\mu^b \tilde{A}_\nu^c A_\mu^d A_\nu^e + \text{derivative terms.} \hspace{1cm} (7.6)$$

With our shift we now have the effective potential

$$V_{\text{eff}} = \frac{g^2}{4}f^{abc}f^{ade}\tilde{A}_\mu^b \tilde{A}_\nu^c \tilde{A}_\mu^d \tilde{A}_\nu^e. \hspace{1cm} (7.7)$$

The real shift of $A_\mu^a$ is reproduced in the decomposed variables to a similar real shift of $\psi_1$ and $\psi_2$:

$$\psi_1 \rightarrow x_1 + \psi_1 \hspace{1cm} (7.8)$$
$$\psi_2 \rightarrow x_2 + \psi_2 \hspace{1cm} (7.9)$$

such that

$$\tilde{A}_\mu^1 + i\tilde{A}_\mu^2 = ix_1 e_\mu + ix_2 \bar{e}_\mu = \frac{1}{\sqrt{2}}(x_2 - x_1)e^2_\mu + \frac{i}{\sqrt{2}}(x_2 + x_1)e^1_\mu. \hspace{1cm} (7.10)$$
Since the shifts are all real we can identify

\[ A_1^\mu = \frac{1}{\sqrt{2}} (x_2 - x_1) e_\mu \]  \hspace{1cm} (7.11)

\[ A_2^\mu = \frac{1}{\sqrt{2}} (x_2 + x_1) e_\mu \]  \hspace{1cm} (7.12)

and use that to express our classical effective potential in our decomposed variables:

\[ V_{\text{eff}} = \frac{g^2}{8} (x_1^2 - x_2^2)^2. \]  \hspace{1cm} (7.13)

The motion with this effective potential is then essentially

\[ S = \int dt \, \frac{1}{2} [\dot{x}_1^2 + \dot{x}_2^2 - (x_1^2 - x_2^2)^2]. \]  \hspace{1cm} (7.14)

The potential \( U(x_1, x_2) = \frac{1}{2} (x_1^2 - x_2^2)^2 \) is maybe more easily recognised if we look at the coordinates \( x = x_1 + x_2 \) and \( y = x_1 - x_2 \) where it assumes the form \( U = \frac{1}{4} x^2 y^2 \). This is a potential that has come up in several fields of physics, for instance in Matrix theory.

The potential \( U = \frac{1}{4} x^2 y^2 \) will for fix non zero values of \( y \) be more and more narrow parabolas in the \( x \)-direction as \( y \) increase and vice versa. It can be viewed as two crossed valleys getting more and more narrow further away from their intersection. See left image of Figure 1 in [I] remembering that we rotated the system by \( 45^\circ \).

The motion of 7.14 is very hard to predict. As the trajectory goes off into any of the four valleys it will oscillate in the valley and eventually return out of it. After returning out it again goes off into any of the four valleys. Trajectories look very chaotic. Studying the Poincaré map a trajectory will show area filling and it was long assumed that the motion of 7.14 was fully chaotic. This was proved wrong by P. Dahlqvist and G. Russberg [9] when they found an island of stability around a closed orbit in the system.

This island of stability is quite a peculiar thing. Studying the motion one can easily find myriads of closed orbits but they all turn out to be unstable. The closed orbit P. Dahlqvist and G. Russberg (the DR orbit) found was not unstable, nor was it stable. A trajectory of the same energy as DR orbit starting close to it will stay on a shell around it. On a Poincaré map of a suitable surface of section this will show up as rings around the DR orbit (being a point on the Poincaré map). Each nearby trajectory marks out a ring after many revolutions. Such a pattern in a Poincaré map is depicted in Figure 3 in [I]. Hence, the Liapunov exponents around the fixed orbit are essentially equal to zero. Starting too far from the DR orbit the motion will eventually become chaotic.
Furthermore, the system possesses a scaling invariance
\[ q_i \to c q_i, \]
\[ p_i \to c^2 p_i, \]
\[ t \to \frac{1}{c^2} t \]
leaving all trajectories unaltered.

7.3 The Perturbed Effective Action

The corrected version of the effective action introduced in last section was proposed by G. K. Savidy [24] and later on also by L. Freyhult [12]. In the second reference the same decomposition as used before was again used but now one loop corrections were added to the effective potential. The corrected effective potential is
\[ V_{\text{eff}} = \frac{g^2}{8} (x_1^2 - x_2^2)^2 + \frac{22}{3} \frac{1}{16\pi^2} \frac{g^4}{8} (x_1^2 - x_2^2)^2 \left( \log \frac{|x_1^2 - x_2^2|}{M^2} - \frac{25}{6} \right) \]
where now \( M \) is a dimensionful parameter. The dynamics of this potential is described in the action
\[ S = \int dt \left[ \frac{1}{2} \left( x_1^2 + x_2^2 - (x_1^2 - x_2^2)^2 \right) \left( 1 + \frac{\lambda^2}{2} \log \frac{|x_1^2 - x_2^2|}{M^2} \right) \right] \]
which for \( \lambda = 0 \) coincide with 7.14. With this correction the potential has a new minimum. Since the function \( f(\phi) = \frac{1}{2} \phi (1 + \frac{1}{2} \log |\phi|) \) has a unique positive minimum \( \phi_{\text{min}} = e^{-1-2/\lambda} \) and \( f(\phi_{\text{min}}) = -\frac{\lambda}{4} e^{-1-2/\lambda} \) we find that the minimum of our potential is
\[ U_{\text{min}} = -\frac{\lambda}{4} e^{-1-2/\lambda} \text{ for } |x_1^2 - x_2^2| = e^{-1-2/\lambda}. \]
From this we see that the minimum potential degenerates from the two intersecting diagonals into four separate hyperbolas. This corrected potential is seen in the left image of Figure 1 in [I].

This system also possesses a scaling invariance
\[ q_i \to c q_i, \]
\[ p_i \to \frac{c^2}{\sqrt{1 - \lambda \log c^2}} p_i, \]
\[ t \to \frac{\sqrt{1 - \lambda \log c^2}}{c} t \]
but in order to make it work we must rescale the parameter $\lambda$ as well

$$
\lambda \rightarrow \frac{\lambda}{1 - \lambda \log c^2}.
$$

(7.24)

For this case we find that the rescaling parameter is bounded, $c < \exp \left( \frac{1}{2\lambda} \right)$, in order for the scaling not to be singular nor complex.

What we found and presented in [I] was a closed orbit with similar properties to the trajectory P. Dahlqvist and G. Russberg found for the unperturbed potential. The orbit found has energy $E < 0$ showing that it oscillates in one of the four hyperbolas since it cannot pass the diagonals where $U = 0$. In [I] this orbit is depicted in Figure 2 and its Poincaré map in Figure 3.

Again we have nearby trajectories of same energy making out rings around the closed orbit. The difference now is that we have a parameter $\lambda$. Adjusting $\lambda$ it was possible to find cases where the nearby trajectories were actually closed. From energy conservation we know that the trajectory lives in a three dimensional hyperplane. Since throughout the trajectory $x_1 > 0$ giving that the trajectory is fully described in coordinates $x_2, p_1$ and $p_2$ and can not intersect with itself (can not untie knots).
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