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Some local–global phenomena in locally finite graphs

Armen S. Asratian∗, Jonas B. Granholm†, Nikolay K. Khachatryan‡

Abstract

In this paper we present some results for a connected infinite graph $G$ with finite degrees where the properties of balls of small radii guarantee the existence of some Hamiltonian and connectivity properties of $G$. (For a vertex $w$ of a graph $G$ the ball of radius $r$ centered at $w$ is the subgraph of $G$ induced by the set $M_r(w)$ of vertices whose distance from $w$ does not exceed $r$.) In particular, we prove that if every ball of radius 2 in $G$ is 2-connected and $G$ satisfies the condition $d_G(u) + d_G(v) \geq |M_2(w)| - 1$ for each path $uwv$ in $G$, where $u$ and $v$ are non-adjacent vertices, then $G$ has a Hamiltonian curve, introduced by Kündgen, Li and Thomassen (2017). Furthermore, we prove that if every ball of radius 1 in $G$ satisfies Ore’s condition (1960) then all balls of any radius in $G$ are Hamiltonian.

Keywords: Hamilton cycle, local conditions, infinite graphs, Hamilton curve

1 Introduction

Interconnection between local and global properties of mathematical objects has always been a subject of investigations in different areas of mathematics. Usually by local properties of a mathematical object, for example a function, we mean its properties in balls with small radii. A general question is the following: How well can global properties of a mathematical object be inferred from the local properties?

If the mathematical object under consideration is a graph, balls of radius $r$ are defined only for integers $r \geq 0$. For a vertex $u$ of a graph $G$ the ball of radius $r$ centered at $u$ is the subgraph of $G$ induced by the set $M_r(u)$ of vertices whose distance from $u$ does not exceed $r$. In the present paper we consider graphs without loops and multiple edges. The following problem arises naturally:

Problem 1. What can we say about global properties of a graph using balls of small radii?

A number of existing results in graph theory give strong interconnections between local and global properties of a graph. Consider the following example:

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Example 1.1. A graph $G$ is $k$-edge colorable if its edges can be colored with $k$ colors so that no pair of adjacent edges have the same color. Vizing’s theorem [40] on $k$-edge colorings can be formulated as follows: A graph $G$ has a $k$-edge coloring if the degree of every vertex of $G$ is strictly less than $k$.

Thus a local property (every vertex degree is strictly less than $k$) implies that $G$ has a global property ($G$ is $k$-edge colorable).

In contrast with Example 1.1, the property of a graph of being connected cannot be recognized using balls of small radii only, because in any graph (connected or disconnected) all balls of any radius are connected. However note that any result on a connected graph $G$ concerning a global property can be reformulated in terms of components of $G$ without mentioning the connectedness of $G$.

Consider an example:

Example 1.2. An Euler tour of a graph $G$ is a walk in $G$ that starts and finishes at the same vertex and traverses each edge exactly once. Euler’s theorem (see, for example [17]) says that a connected graph $G$ has an Euler tour if and only if every vertex of $G$ has even degree. This theorem can be reformulated as follows: Every non-trivial component of a graph $G$ has an Euler tour if and only if every vertex of $G$ has even degree.

Some other global properties of graphs were investigated in [32] by using the properties of balls of small radii. In this paper we consider mostly Hamiltonian properties of graphs. A finite graph $G$ is called Hamiltonian if it has a Hamilton cycle, that is, a cycle containing all the vertices of $G$. There is a vast literature in graph theory devoted to obtaining sufficient conditions for Hamiltonicity (see, for example, the surveys [23,24]).

Almost all of the existing sufficient conditions for a finite graph $G$ to be Hamiltonian contain some global parameters of $G$ (e.g., the number of vertices) and only apply to graphs with large edge density ($|E(G)| \geq \text{constant} \cdot |V(G)|^2$) and/or small diameter ($o(|V(G)|)$). The following two classical theorems are examples of such results:

**Theorem 1.1** (Ore [37]). A finite graph $G = (V(G), E(G))$ with $|V(G)| \geq 3$ is Hamiltonian if $d_G(u) + d_G(v) \geq |V(G)|$ for each pair of non-adjacent vertices $u$ and $v$ of $G$, where $d_G(u)$ denotes the degree of $u$. (A graph satisfying this condition is called an Ore graph.)

**Theorem 1.2** (Jung [29], Nara [34]). Let $G = (V(G), E(G))$ be a finite 2-connected graph such that $d_G(u) + d_G(v) \geq |V(G)| - 1$ for each pair of non-adjacent vertices $u, v$. Then either $G$ is Hamiltonian or $G \in \mathcal{K}$ where $\mathcal{K} = \{G : K_{p,p+1} \subseteq G \subseteq K_p \lor K_{p+1} \text{ for some } p \geq 2 \}$ ($\lor$ denotes the join operation).

Asratian and Khachatryan [1,3–5] showed that many of the global sufficient conditions for Hamiltonicity of a finite graph $G$ have local analogues where every global parameter of $G$ is replaced by a parameter of a ball with small radius. Such results are called localization theorems and give a possibility to extend known classes of Hamiltonian graphs. For example, the following generalization of Ore’s theorem was obtained in [4] (see also [17, Thm. 10.1.3]):

**Theorem 1.3** (Asratian and Khachatryan [4]). Let $G$ be a connected finite graph on at least 3 vertices where for every vertex $w \in V(G)$ the condition $d_G(u) + d_G(v) \geq |N(u) \cup N(v) \cup N(w)|$ holds for every path $uvw$ with $uv \notin E(G)$, where $N(w)$ denotes the set of neighbors of $w$. Then $G$ is Hamiltonian.
A generalization of Theorem 1.2 was obtained in [1]:

**Theorem 1.4** (Asratian [1]). Let $G$ be a connected finite graph with $|V(G)| \geq 3$ where all balls of radius 2 in $G$ are 2-connected and $d_G(u) + d_G(v) \geq |M_2(w)| - 1$ for every path $uvw$ with $uv \notin E(G)$. Then either $G$ is Hamiltonian or $G \in \mathcal{K}$.

Note some phenomena related to these results:

1) Although Theorem 1.2 is a generalization of Ore’s theorem, the localizations of these two theorems (Theorem 1.3 and Theorem 1.4) are incomparable to each other in the sense that neither theorem implies the other. For example, the graph on the left hand side in Fig. 1 satisfies the condition of Theorem 1.3 and does not satisfy the condition of Theorem 1.4, and the graph on the right hand side satisfies the condition of Theorem 1.4 and does not satisfy the condition of Theorem 1.3.

2) All graphs satisfying the conditions of Theorem 1.1 or Theorem 1.2 have diameter at most two and large edge density. In contrast with this, Theorem 1.3 and Theorem 1.4 apply to infinite classes of finite graphs $G$ with large diameter ($\geq \text{constant} \cdot |V(G)|$) and small edge density ($|E(G)| \leq \text{constant} \cdot |V(G)|$). For example, the graphs in Fig. 1 can be extended to graphs with any diameter.

3) The set of Ore graphs and the set of graphs satisfying Theorem 1.3 have similar cyclic properties. For example, every Ore graph $G$ with $|V(G)| \geq 4$ is pancyclic (i.e. contains cycles of all length from 3 to $|V(G)|$), unless $G = K_{n,n}$ for some $n \geq 2$ (see Bondy [10]). Moreover each vertex of an Ore graph $G$ with $|V(G)| \geq 4$ lies on a cycle of every length from 4 to $|V(G)|$ (see Cai Xiao-Tao [12]). Asratian and Sarkisian [7] showed that every graph $G$ satisfying the condition of Theorem 1.3 has the same properties.

Localization theorems were also found (see [1–5]) for results of Dirac [20], Bondy [9], Nash-Williams [35], Bauer et al. [8], Häggkvist and Nicoghossian [28], Moon and Moser [33]. A general method for localization of global criteria for Hamiltonicity of finite graphs was suggested by the authors in [2].

A large part of the results of local nature in Hamiltonian graph theory is devoted to **claw-free graphs**, that is, graphs that have no induced subgraph isomorphic to $K_{1,3}$ [21]. The following well-known result was obtained in [36].

**Theorem 1.5** (Oberly and Sumner [36]). A finite, connected, claw-free graph $G$ on at least 3 vertices is Hamiltonian if for each vertex $u$ of $G$ the subgraph induced by the set of neighbors of $u$ is connected.
In 2004-2017 some Hamiltonian properties of finite graphs were extended to infinite locally finite graphs, that is, infinite graphs where all vertices have finite degrees. There are two important notions for a locally finite graph $G$ related to this topic. The first one, called a Hamilton circle of $G$, was introduced by Diestel and and Kühn [19], and the other one, called a Hamiltonian curve of $G$, was introduced by Kündgen, Li and Thomassen [31] (see the definitions of these two concepts in section 2). Some results on the existence of Hamilton circles in infinite locally finite graphs were obtained in [11,22,25–27].

The next result on Hamiltonian curves was proved in [31].

**Theorem 1.6** ([31]). The following are equivalent for a locally finite graph $G$.

(i) For every finite vertex set $S$, $G$ has a cycle containing $S$.

(ii) $G$ has a Hamiltonian curve.

This theorem gives possibilities to extend some results on finite graphs to infinite graphs. For example, the following result was noted in [31]:

**Theorem 1.7.** Let $G$ be a connected, locally finite, infinite graph where $d_G(u) + d_G(v) \geq |N(u) \cup N(v) \cup N(w)|$ for each path $uvw$ with $uv \not\in E(G)$. Then $G$ has a Hamiltonian curve.

In this paper we present some results for a connected infinite locally finite graph $G$ where the properties of balls of small radii guarantee the existence of some Hamiltonian and connectivity properties of $G$. In particular, we prove that if all balls of radius 2 in $G$ are 2-connected and $d_G(u) + d_G(v) \geq |M_2(w)| - 1$ for each path $uvw$ with $uv \not\in E(G)$, then $G$ has a Hamiltonian curve.

Theorem 1.6 implies that a connected infinite locally finite graph $G$ has a Hamiltonian curve if any ball of any radius in $G$ is Hamiltonian. In contrast with this we show that the Hamiltonicity of all balls is not sufficient for $G$ to have a Hamilton circle. We obtain a similar result for infinite graphs: For any integer $d \geq 3$ there exists a connected non-Hamiltonian finite graph of diameter $d$ where all balls of $G$, except $G$ itself, are Hamiltonian. In contrast with this we show that if every ball of radius 1 in a connected locally finite graph $G$ (finite or infinite) is an Ore graph, then every ball of any radius in $G$ is Hamiltonian. We also show that the $k$-connectedness of all balls of radius $r$ in a locally finite graph $G$, where $r$ is an integer, implies the $k$-connectedness of all balls in $G$ with radius bigger than $r$. This is a generalization of a result of Chartrand and Pippert [13]. We finish the paper with a conjecture concerning Hamilton circles.

## 2 Definitions and notations

We use [17] for terminology and notation not defined here and consider graphs without loops and multiple edges only. A graph $G$ is called locally finite if every vertex of $G$ has finite degree. A graph $G$ is finite or infinite according to the number of vertices in $G$.

Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph $G$, and let $d_G(u, v)$ denote the distance between vertices $u$ and $v$ in $G$. The greatest distance between any two vertices in $G$ is the diameter of $G$.

For each integer $r \geq 0$ and each $u \in V(G)$ we denote by $N_r(u)$ and $M_r(u)$ the set of all $v \in V(G)$ with $d_G(u, v) = r$ and $d_G(u, v) \leq r$, respectively. The set
$N_1(u)$ is usually denoted by $N(u)$. The subgraph induced by the set $M_r(u)$ is denoted by $G_r(u)$ and called the ball of radius $r$ centered at $u$. In fact, for each vertex $u$ of a connected finite graph $G$ there is an integer $r(u)$ such that $G$ is a ball of radius $r(u)$ centered at $u$.

Let $G$ be a connected graph and $v$ a vertex in a ball $G_r(u), r \geq 1$. We call $v$ an interior vertex of $G_r(u)$ if $M_1(v) \subseteq M_r(u)$. Clearly, every vertex in $G_{r-1}(u)$ is interior for $G_r(u)$, and if $G = G_r(u)$ then all vertices in $G$ are interior vertices of $G_r(u)$.

Let $C$ be a cycle of a graph. We denote by $\bar{C}$ the cycle $C$ with a given orientation, and by $\bar{C}$ the cycle $C$ with the reverse orientation. If $u, v \in V(C)$ then $u\bar{C}v$ denote the consecutive vertices of $C$ from $u$ to $v$ in the direction specified by $\bar{C}$. The same vertices, in reverse order, are given by $v\bar{C}u$. We use $u^+$ to denote the successor of $u$ on $\bar{C}$ and $u^+$ to denote its predecessor. Analogous notation is used with respect to paths instead of cycles.

A path containing all vertices of a graph $H$ is called a Hamilton path of $H$.

A graph is $k$-connected if the removal of fewer than $k$ vertices results in neither a disconnected graph nor the trivial graph consisting of a single vertex. The greatest integer $k$ such that $G$ is $k$-connected is the connectivity $\kappa(G)$ of $G$.

Let $G$ be an infinite locally finite graph. A one-way infinite path in $G$ is called a ray, and a two-way infinite path is called a double ray. Two rays are equivalent if no finite set of vertices separate them in $G$, which means that for every finite vertex set $S \subseteq V(G)$ both rays have a tail (subray) in the same component of $G - S$. This is an equivalence relation whose equivalence classes are called ends of $G$. Every end can be viewed as a particular “point of infinity”.

The Freudenthal compactification $|G|$ of $G$ is a topological space obtained by taking $G$, seen as 1-complex, and adding the ends of $G$ as additional points. For the precise definition of $|G|$ see [17]. It should be pointed out that, inside $|G|$, every ray of $G$ converges to the end of $G$ it is contained in.

Extending the notion of cycles, Diestel and Kühn [19] defined circles in $|G|$ as the image of a homeomorphism which maps the unit circle $S^1$ in $\mathbb{R}^2$ to $|G|$. The graph $G$ is called Hamiltonian if there exists a circle in $|G|$ which contains all vertices and ends of $G$. Such a circle is called a Hamilton circle of $G$.

A closed curve in $|G|$ is the image of an continuous map from the unit circle $S^1$ in $\mathbb{R}^2$ to $|G|$. A closed curve in the Freudenthal compactification $|G|$ is called a Hamiltonian curve of $G$ if it meets every vertex of $G$ exactly once (and hence it meets every end at least once).

### 3 Locally finite graphs with Hamiltonian balls

A graph $G$ is called locally Hamiltonian (locally traceable) if for every vertex $u$ of $G$ the subgraph induced by the set $N(u)$ is Hamiltonian (has a Hamilton path). In fact, a graph $G$ is locally traceable if and only if every ball of radius 1 in $G$ is Hamiltonian. Some cyclic properties of locally Hamiltonian and locally traceable graphs were found in [6, 14, 16, 38, 39].

We call a locally finite graph $G$ uniformly Hamiltonian if every ball of finite radius in $G$ is Hamiltonian. This concept was defined for finite graphs in [6], where some classes of uniformly Hamiltonian graphs were found. Here we prove the following result which was conjectured in [6].
Theorem 3.1. Let $G$ be a connected locally finite graph on at least 3 vertices
(finite or infinite), where every ball of radius 1 is an Ore graph. Then $G$
is uniformly Hamiltonian.

Proof. By the hypothesis of the theorem $d_{G_1(w)}(u) + d_{G_1(w)}(v) \geq |M_1(w)|$
for each $w \in V(G)$ and each pair of non-adjacent vertices $u, v \in N(w)$. Clearly,

$$d_{G_1(w)}(u) + d_{G_1(w)}(v) = |M_1(w) \cap N(u) \cap N(v)| + |M_1(w) \cap (N(u) \cup N(v))|.$$ 

Then Ore’s condition for the ball $G_1(w)$ is equivalent to the condition

$$|M_1(w) \cap N(u) \cap N(v)| \geq |M_1(w) \setminus (N(u) \cup N(v))|$$

(1)

for each pair of non-adjacent vertices $u, v \in N(w)$.

Suppose that for some integer $r \geq 1$ one of the balls of radius $r$ in $G$
say $G_r(x)$, is not Hamiltonian. Among all cycles in $G_r(x)$ which contain $x$, let
$C$ be one of maximum length. Consider a vertex $a \in V(G_r(x)) \setminus V(C)$. Let
d_2(x,a) = t$ and $P$ be a shortest path between $x$ and $a$. Then there are two
consecutive vertices $w_1$ and $v$ on $P$ such that $d(w_1) = t - 1$ and
$v \notin V(C)$. Clearly, $M_1(w_1) \subseteq V(G_r(x))$, since $t \leq r$. Let $C$ be the cycle $C$
with a given orientation, and let $w_1, \ldots, w_k$ be the vertices of $W = N(v) \cap V(C)$
occuring on $\bar{C}$ in the order of their indices.

Set $W^+ = \{w_1^+, \ldots, w_k^+\}$. Clearly, for each $i, 1 \leq i \leq k$, the vertices $v$
and $w_i^+$ have no common neighbor in $G_1(w_i)$ outside $C$, because if $vz, zw_i^+ \in E(G)$
for some $i$ and $z \notin M_1(w_i) \setminus V(C)$, the cycle $w_i vz w_i^+ \bar{C} w_i$ in $G_r(x)$ contains $x$
and is longer than $C$. Therefore $N(w_i^+) \cap N(v) \cap M_1(w_i) \subseteq V(C)$, for $i = 1, \ldots, k$.

This implies that $k \geq 2$, since $w_i^+, v \in M_1(w_i) \setminus (N(w_i^+) \cup N(v))$ and therefore,
by (1), $|M_1(w_i) \cap N(w_i^+) \cap N(v)| \geq 2$.

Clearly, $w_i \neq w_{i-1}$ for $i = 1, \ldots, k - 1$, because if $w_i = w_{i-1}$ for some $i$, the
cycle $w_{i-1}vw_i \bar{C} w_{i-1}$ in $G_r(x)$ contains $x$ and is longer than $C$. Furthermore,
$w_i^+ w_j^+ \notin E(G)$ for $1 \leq i < j \leq k$ because otherwise the cycle $w_i w_j w_i^+ w_j^+ \bar{C} w_i$
in $G_r(x)$ contains $x$ and is longer than $C$. Thus, the set $W^+ \cup \{v\}$ is an
independent set.

Now we count the number of edges $e(W,W^+)$ between $W$ and $W^+$. Since
d(v, w_i^+) = 2 for each $i = 1, \ldots, k$, we obtain from (1) that

$$\sum_{i=1}^{k} |M_1(w_i) \cap N(w_i^+) \cap N(v)| \geq \sum_{i=1}^{k} |M_1(w_i) \setminus (N(w_i^+) \cup N(v))|.$$ 

(2)

Furthermore

$$e(W,W^+) \geq \sum_{i=1}^{k} |M_1(w_i) \cap N(w_i^+) \cap N(v)|$$

(3)

and

$$\sum_{i=1}^{k} |M_1(w_i) \setminus (N(w_i^+) \cup N(v))| \geq e(W,W^+) + k$$

(4)

because $M_1(w_i) \cap N(w_i^+) \cap N(v) \subseteq W \cap N(w_i^+)$ and $v \in M_1(w_i) \setminus (N(w_i^+) \cup N(v))$
for each $i = 1, \ldots, k$. But (3) and (4) contradict (2). Therefore, $C$ contains all
vertices of the ball $G_r(x)$, that is, $G_r(x)$ is Hamiltonian. \(\square\)
Note that Theorem 1.5 can be formulated in terms of balls as follows: A finite connected graph $G$ on at least 3 vertices is Hamiltonian if for every vertex $u \in V(G)$ the ball $G_1(u)$ satisfies the condition $\kappa(G_1(u)) \geq 2 \geq \alpha(G_1(u))$, where $\kappa(G_1(u))$ is the connectivity of $G_1(u)$ and $\alpha(G_1(u))$ is the maximum number of mutually non-adjacent vertices of $G_1(u)$.

The next theorem is an extension of Theorem 1.5 and a result obtained in [6] for finite claw-free graphs. The proof for infinite locally finite graphs is the same as in [6].

**Theorem 3.2.** Let $G$ be a connected, locally finite, claw-free graph on at least three vertices (finite or infinite) where for each vertex $u$ the subgraph induced by the set of neighbors of $u$ is connected. Then $G$ is uniformly Hamiltonian.

Note that Theorem 3.1 and Theorem 3.2 are incomparable in the sense that neither theorem implies the other. For example, the graph on the right hand side in Fig. 2 is not claw-free and satisfies the condition of Theorem 3.1, and the graph on the left hand side satisfies the condition of Theorem 3.2, but does not satisfy the condition of Theorem 3.1.

In Theorems 3.1 and 3.2, the Hamiltonicity of balls of radius 1 of a connected, finite graph $G$ induces Hamiltonicity of $G$ and all balls in $G$. This is not true in general. For example, it is known that there exist finite connected non-Hamiltonian graphs where all balls of radius 1 are Hamiltonian (see, for example, [38]). We will prove a stronger result.

**Theorem 3.3.** For any integer $d \geq 3$ there exists a non-Hamiltonian finite graph $G$ of diameter $d$ such that every ball of $G$, except $G$ itself, is Hamiltonian.

**Proof.** Let $H_1, H_2, H_3, \ldots$ be a sequence of graphs where

- $V(H_1) = \{a_1, a_2, b_1, b_2, b_3, b_4\}$,
  $E(H_1) = \{a_1b_1, a_1b_2, a_2b_3, a_2b_4\}$,

- and, for $i \geq 2$,
  $V(H_i) = \{b_{4i-7}, \ldots, b_{4i}, c_{4i-7}, \ldots, c_{4i-4}\}$,
  $E(H_i) = \{b_jb_k : 4i-7 \leq j < k \leq 4i\} \cup \{c_jb_j, c_jb_{j+4} : 4i-7 \leq j \leq 4i-4\}$.

Consider a graph $G(d) = \bigcup_{i=1}^{d-1} H_i \cup F_d$ where $F_d$ is a graph with

- $V(F_d) = \{a_3, a_4, b_{4d-7}, b_{4d-6}, b_{4d-5}, b_{4d-4}\}$
- $E(F_d) = \{a_3b_{4d-7}, a_3b_{4d-6}, a_3b_{4d-5}, a_4b_{4d-4}\}$. 

Figure 2
The graph $G(4)$ can be seen in Fig. 3. Clearly the diameter of $G(d)$ equals $d$. Furthermore, $G(d)$ is not Hamiltonian because the edges incident with vertices of degree 2 induce two disjoint cycles.

Put $G = G(d)$. For any $S \subset V(G)$ we denote by $\langle S \rangle$ the subgraph of $G$ induced by the vertices in $S$. Now we will show that for every vertex $v \in V(G)$ the ball $G_r(v)$ is Hamiltonian if $G_r(v) \neq G$.

It is not difficult to verify that any such ball with $r \geq 3$ is isomorphic, for some $s \geq r$, to one of the graphs $\bigcup_{i=1}^s H_i$ and $\bigcup_{i=2}^s H_i$. Both graphs are Hamiltonian: the first one has a Hamilton cycle $C$ consisting of all edges incident with vertices of degree 2, and the edges $b_4 - 3b_4$ and $b_4 - 2b_4$, and the second one has a Hamilton cycle $C'$ obtained from $C$ by deleting the edges $b_1a_1, a_1b_2, b_3b_2, a_2b_4$ and adding the edges $b_1b_2$ and $b_3b_4$.

The ball $G_1(a_i), i = 1, 2, 3, 4,$ and $G_1(c_j), j = 1, \ldots, 4d - 8$, are triangles and thus Hamiltonian. The ball $G_1(b_i)$ is isomorphic to the graph $\langle \{b_1, \ldots, b_{12}, c_1, c_3\} \rangle$ if $5 \leq i \leq 4d - 8$ and to $\langle \{a_1, b_1, \ldots, b_8, c_1\} \rangle$ otherwise. The Hamiltonicity of these graphs is evident.

Finally consider the balls of radius 2 which differ from $G$:

- $G_2(a_i)$ is isomorphic to $\langle \{a_1, b_1, \ldots, b_8, c_1, c_2\} \rangle$, $i = 1, 2, 3, 4$.
- $G_2(b_i)$ is isomorphic to $\bigcup_{j=2}^i H_j$ if $9 \leq i \leq 4d - 12$, to $\bigcup_{j=1}^i H_j$ if $5 \leq i \leq 8$ or $4d - 11 \leq i \leq 4d - 8$, and to $\bigcup_{j=1}^3 H_j$ if $1 \leq i \leq 4$ or $4d - 7 \leq i \leq 4d - 4$.
- $G_2(c_i)$ is isomorphic to $\langle \{b_1, \ldots, b_{16}, c_1, c_3\} \rangle$ if $5 \leq i \leq 4d - 12$ and to $\langle \{a_1, b_1, \ldots, b_{12}, c_1, c_3\} \rangle$ if $i \leq 4$ or $i \geq 4d - 11$, unless $d = 3$ in which case $G_2(c_i)$ is isomorphic to $\langle \{a_1, a_3, b_1, \ldots, b_8, c_1\} \rangle$ for $i = 1, 2, 3, 4$.

It is not difficult to verify that these graphs are Hamiltonian. Therefore all balls of $G$, except $G$ itself, are Hamiltonian.

Now we consider the ability of infinite uniformly Hamiltonian graphs to have Hamilton circles and Hamiltonian curves.

**Proposition 3.4.** If all balls of all radii in a connected, locally finite, infinite graph $G$ are Hamiltonian, then $G$ has a Hamiltonian curve.

**Proof.** Let $S$ be a finite subset of $V(G)$ and let $r$ be the maximum distance between any two vertices in $S$. Choose a vertex $a \in S$. Then all vertices of $S$ are
in the ball $G_r(a)$. Since $G_r(a)$ is Hamiltonian, there is a cycle $C$ containing $S$. Therefore, by Theorem 1.6, $G$ has a Hamiltonian curve.

In contrast with this result we have the following:

**Proposition 3.5.** There exists an infinite, connected, locally finite graph $H$ such that every ball of finite radius in $H$ is Hamiltonian, but $H$ has no Hamilton circle.

**Proof.** Consider the infinite graph $H = \bigcup_{i=1}^{\infty} H_i$ where $H_1, H_2, H_3, \ldots$ is the sequence of graphs defined in the proof of Theorem 3.3. The graph $H$ can be seen in Fig. 4. One can prove that every ball of finite radius is Hamiltonian using the same argument as in the proof of Theorem 3.3.

Any Hamilton circle in $H$ would need to pass through all vertices of degree 2, and thus all edges incident with those vertices. But these edges induce two disjoint double-rays, which means that the unique end of $H$ would be traversed twice, a contradiction. Thus $H$ has no Hamilton circle.

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4 A phenomenon related to ball connectivity

A graph $G$ is called *locally $k$-connected* if, for each vertex $u \in V(G)$, the subgraph induced by the set $N(u)$ is $k$-connected. Chartrand and Pippert [13] showed that if $G$ is a locally $(k-1)$-connected finite graph, $k \geq 2$, then every component of $G$ is $k$-connected. This means that if every ball of radius 1 is $k$-connected, then every component of $G$ is also $k$-connected. We will prove a stronger result:

**Theorem 4.1.** Let $G$ be a locally finite graph and $r$ a positive integer. If all balls of radius $r$ in $G$ are $k$-connected, $k \geq 2$, then all balls in $G$ of radius bigger than $r$ are $k$-connected.

Theorem 4.1 follows from the following theorem:

**Theorem 4.2.** Let $G$ be a locally finite graph and $r \geq 1$ be an integer. If all balls of radius $r$ in $G$ are $k$-connected, $k \geq 2$, then all balls of radius $r + 1$ in $G$ are $k$-connected too.

**Proof.** We will consider two cases.
Case 1. $r = 1$.

It is clear that $k$-connectedness of $G_1(x)$ implies that $|M_1(x)| \geq k + 1$ for each $x \in V(G)$. Suppose that for a vertex $x$ the ball $G_2(x)$ is not $k$-connected. Then there exists a subset $S \subset M_2(x)$ such that $|S| < k$ and $G_2(x) - S$ is disconnected. Among all such sets $S$, let $S_0$ be one of minimum cardinality. Clearly, $S_0$ contains an interior vertex $v$ of $G_2(x)$ because otherwise $G_2(x) - S_0$ is connected. Thus $M_1(v) \subset M_2(x)$. The minimality of $S_0$ implies that there are two neighbors $w_1$ and $w_2$ of $v$ such that $w_1$ and $w_2$ belong to different components in $G_2(x) - S_0$. Then the set $S_0 \cap M_1(v)$ separates $w_1$ and $w_2$ in $G_1(v)$ and $|S_0 \cap M_1(v)| < k$. This contradicts the $k$-connectedness of $G_1(v)$. Hence all balls of radius 2 in $G$ are $k$-connected.

Case 2. $r \geq 2$.

We will show that the graph $G_{r+1}(x) - S$ is connected for each vertex $x$ and each subset $S \subset M_{r+1}(x)$ with $|S| < k$. Since $G_r(x)$ is $k$-connected, the graph $G_r(x) - S$ is connected. Furthermore, $N(x) \setminus S \neq \emptyset$, since otherwise the set $N(x)$ is a cutset of size strictly less than $k$ in $G_r(x)$. Choose a vertex $y \in N(x) \setminus S$. Then $y \in M_r(u) \setminus S$ for each $u \in M_1(x)$, since $r \geq 2$ and $y \notin S$. Now for each vertex $z \in M_{r+1}(x) \setminus S$ there is a vertex $u \in M_1(x)$ such that $z \in M_r(u) \setminus S$, because $M_{r+1}(x) = \bigcup_{u \in M_1(x)} M_r(u)$.

Since $G_r(u)$ is $k$-connected and $|S| < k$, there is a path between $y$ and $z$ in $G_r(u) - S$ and, therefore, in $G_{r+1}(x) - S$ too. Thus $G_{r+1}(x) - S$ is connected, so we can conclude that $G_{r+1}(x)$ is $k$-connected.

Corollary 4.3. If every ball of radius 1 in a connected graph $G$ is $k$-connected, $k \geq 2$, then $G$ and all balls of any radius in $G$ are $k$-connected.

5 Two classes of infinite graphs with Hamiltonian curves

The proofs of many local sufficient conditions for the existence of a Hamilton cycle in a finite graph work by starting with an arbitrary cycle and iteratively extending it until it covers all vertices of the graph. If the extensions of the cycles are chosen carefully enough, such a proof can be used to prove the existence of Hamiltonian curves in infinite locally finite graphs by applying Theorem 1.6. In this section we give two examples of such an approach.

Our first result concerns the following theorem:

Theorem 5.1 (Chvátal and Erdős [15]). A finite graph $G$ on at least 3 vertices is Hamiltonian if $\kappa(G) \geq \alpha(G)$, where $\kappa(G)$ is the connectivity of $G$ and $\alpha(G)$ is the maximum number of mutually non-adjacent vertices of $G$.

Khachatrian [30] noted that the proof of this theorem given in [15], can be used to prove the following result:

Theorem 5.2 (Khachatrian [30]). Let $r$ be a positive integer and $G$ a connected finite graph on at least three vertices, where $\kappa(G_r(u)) \geq \alpha(G_{r+1}(u))$ for every $u \in V(G)$. Then $G$ is Hamiltonian.
We extend this result to infinite graphs by slightly changing the proof in [30].

**Theorem 5.3.** Let \( r \) be a positive integer and \( G \) a connected, infinite, locally finite graph such that \( \kappa(G_r(u)) \geq \alpha(G_{r+1}(u)) \) for each vertex \( u \) of \( G \). Then \( G \) has a Hamiltonian curve.

**Proof.** We will show that for any finite vertex set \( S \subset V(G) \) there is a cycle in \( G \) containing \( S \). Let \( q \) be the maximum distance between any two vertices in \( S \). Choose a vertex \( a \in S \) and an integer \( n = q + r + 1 \). Then the ball \( G_n(a) \) contains the set \( S \) and, moreover, for every \( u \in S \) the vertices of \( G_r(u) \) are interior vertices of the ball \( G_n(a) \). Among all cycles in \( G_n(a) \) which contain \( a \), let \( C \) be one of maximum length. Suppose to the contrary that \( S \setminus V(C) \neq \emptyset \). Consider a vertex \( y \in S \setminus V(C) \) and a shortest \((a, y)\)-path in \( G_n(a) \). Clearly, there are two adjacent vertices \( v \) and \( u \) on this path such that \( v \notin V(C) \), \( u \in V(C) \) and \( u \) is an interior vertex of \( G_y(u) \). Let \( \hat{C} \) be the cycle \( C \) with a given orientation. We have that \( 2 \leq \kappa(G_{r+1}(u)) \leq \kappa(G_r(u)) \) since \( vu \notin E(G) \). Then, by Menger's theorem [17], in \( G_r(u) \) there are \( k \) internally disjoint \((v, u^+)\)-paths \( Q_1, \ldots, Q_k \), where \( k = \kappa(G_y(u)) \). Maximality of \( C \) implies that each \( Q_i \) has at least one common vertex with \( C \). This means that there are paths \( P_1, \ldots, P_k \) having initial vertex \( v \) that are pairwise disjoint, apart from \( v \), and that share with \( C \) only their terminal vertices \( v_1, \ldots, v_k \), respectively. Furthermore, maximality of \( C \) implies that \( vu^+ \notin E(G) \) for each \( i = 1, \ldots, k \). Then there is a pair \( i, j \) such that \( 1 \leq i < j \leq k \) and \( v_i^+v_j^+ \in E(G) \), as otherwise there are \( k \) mutually non-adjacent vertices \( v, v_1^+, \ldots, v_k^+ \) in \( G_{r+1}(v) \) which contradicts the condition \( \alpha(G_{r+1}(v)) \leq \kappa(G_r(v)) \). Since the vertices of \( G_r(u) \) are interior vertices of the ball \( G_n(a) \), the paths \( P_1, \ldots, P_k \) lie in \( G_n(a) \).

Now by deleting the edges \( v_iv_i^+ \) and \( v_jv_j^+ \) from \( C \) and adding the edge \( v_i^+v_j^+ \) together with the paths \( P_i \) and \( P_j \), we obtain in \( G_n(a) \) a cycle that is longer than \( C \) and contains \( a \); a contradiction. Therefore, \( C \) contains the set \( S \). Then, by Theorem 1.6, \( G \) has a Hamiltonian curve.

Note that for any integer \( r \geq 1 \) there is an infinite locally finite graph that satisfies the conditions of Theorem 5.3. Consider, for example, the graph \( G = G(r) \) which is defined as follows: its vertex set is \( \bigcup_{i=0}^{\infty} V_i \), where \( V_0, V_1, V_2, \ldots \) are pairwise disjoint finite sets with \( |V_0| = 2 \) and \( |V_i| \geq r + 3 \) for each \( i \geq 1 \), and two vertices \( x, y \) of \( G \) are adjacent if and only if either \( x, y \in V_i \) for some \( i \geq 1 \), or \( x \in V_i, y \in V_{i+1} \) for some \( i \geq 0 \). It is not difficult to verify that the graph \( G = G(r) \) is not claw-free and satisfies the conditions of Theorem 5.3.

Our second result is an extension of Theorem 1.4.

**Theorem 5.4.** Let \( G \) be a connected, infinite, locally finite graph where every ball of radius 2 is 2-connected and \( d_G(u) + d_G(v) \geq |M_2(w)| - 1 \) for each path \( uwv \) with \( uv \notin E(G) \). Then \( G \) has a Hamiltonian curve.

Note the following two simple properties of a graph \( G \) satisfying the condition of Theorem 5.4:

**Property 5.1.** If \( uwv \) be a path in \( G \) with \( uv \notin E(G) \), then the condition \( d_G(u) + d_G(v) \geq |M_2(w)| - 1 \) is equivalent to the condition \( |N(u) \cap N(v)| \geq |M_2(w)| \setminus (N(u) \cup N(v))| - 1 \).

**Property 5.2.** Let \( G \) be a graph satisfying the conditions of Theorem 5.4, \( \hat{C} \) a cycle in \( G \) with a given orientation, and let \( v, w \) be vertices such that
Claim 2.
Proof. The proof is by contradiction. Suppose that 
\[ v \in V(G) \setminus V(\overline{C}), w \in N(v) \cap V(\overline{C}), N(v) \cap N(w^+) = N(v) \cap N(w^-) = \{w\}, \]
and \(w^+, w^- \notin N(v)\). Then \(w^+\) is adjacent to every vertex in \(M_2(w) \setminus (M_1(w) \cup \{w^+\})\) and \(w^-\) is adjacent to every vertex in \(M_2(w) \setminus (M_1(w) \cup \{w^-\})\). In particular, \(w^+w^- \in E(G)\).

**Proof of Theorem 5.4.** Let \(G\) be a graph satisfying the conditions of Theorem 5.4, and let \(q\) be the maximum distance between any two vertices in \(S\). Choose a vertex \(a \in S\) and let \(n = q + 5\). Then the ball \(G_n(a)\) contains the set \(S\) and, moreover, \(M_2(u) \subset M_n(a)\) for every \(u \in S\). Among all cycles in \(G_n(a)\) which contain the vertex \(a\), let \(C\) be one of maximum length. We will show that \(C\) contains \(S\). Suppose to the contrary that \(S \setminus V(C) \neq \emptyset\). We will show that this leads to a contradiction, by constructing in \(G_n(a)\) a longer cycle \(C'\) containing the vertex \(a\). Let \(\overline{C}\) be the cycle \(C\) with a given orientation.

Further we follow the proof of Theorem 1.4 given in [1] with some changes. The main changes are that all transformations of \(C\) will be done inside the ball \(G_n(a)\), and therefore we need more technical details on the structures of the considered balls.

**Claim 1.** There is a vertex \(v\) in \(G_n(a) - V(C)\) such that \(M_2(v) \subset M_n(a)\) and \(v\) has at least two neighbors on \(C\).

**Proof.** Assume that the claim is false, that is, any vertex \(x\) in \(G_n(a) - V(C)\) with \(M_2(x) \subset M_n(a)\) has at most one neighbor on \(C\).

Consider a vertex \(y \in S \setminus V(C)\) and a shortest \((a,y)\)-path in \(G_n(a)\). Clearly, there are two adjacent vertices \(v\) and \(w\) on this path such that \(v \notin V(G)\), \(w \in V(C)\). Since \(d_G(a,v) \leq d_G(a,y) \leq q = n - 5\), the vertex \(v\) is an interior vertex of \(G_n(a)\) and \(M_5(v) \subset M_n(a)\).

Let \(N(v) \cap V(C) = \{w\}\). Then, by Property 5.2, \(w^-w^+ \in E(G)\). Since the ball \(G_2(w)\) is 2-connected, there is a path \(P = z_1z_2\ldots z_p\) in \(G_2(w) - w\) with \(z_1 = w^+\) and \(z_p = v\). Clearly, \(P\) has an internal vertex which lies on \(\overline{C}\). Let \(z_k\) be a vertex on \(V(P) \cap V(C)\), \(k > 1\), such that all other vertices on the path \(z_k \overline{P} v\) do not belong to \(C\). Clearly, \(k < p - 1\) because \(N(v) \cap V(C) = \{w\}\). Furthermore, \(M_2(z_{k+1}) \subset M_n(a)\) since \(z_{k+1} \in M_3(v)\) and \(M_5(v) \subset M_n(a)\). Then, by our assumption, \(N(z_{k+1}) \cap V(C) = \{z_k\}\) which implies, by Property 5.2, that \(z_{k}z_{k}^+ \in E(G)\). Clearly, \(z_{k} \neq w^+\) since otherwise \(G_n(a)\) contains a longer cycle \(C' = w^-w^+z_{k}^+z_{k}z_{k}^+w^+\) with \(a \in V(C')\); a contradiction.

**Claim 2.** There is an interior vertex \(u\) in \(G_n(a)\) outside of \(C\) and a vertex \(w \in N(u) \cap V(C)\) such that either \(|N(u) \cap N(w^+)| \geq 2\) or \(|N(u) \cap N(w^-)| \geq 2\).

**Proof.** The proof is by contradiction. Suppose that
\[
N(u) \cap N(w^+) = N(u) \cap N(w^-) = \{w\},
\] (5)
for each pair \(u, w\) where \(u\) is an interior vertex in \(G_n(a)\) outside of \(C\) and \(w \in N(u) \cap V(C)\).

Choose an interior vertex \(v\) in \(G_n(a)\) outside of \(C\) such that \(M_2(v) \subset M_n(a)\) and \(v\) has at least two neighbors on \(C\) (such a vertex exists by Claim 1). Set
Suppose to the contrary that $v \in W$. Let $w_1, \ldots, w_k$ be the vertices of $W$, occurring on $\overline{C}$ in the order of their indices. By (5), we have

$$N(v) \cap N(w^+_i) = N(v) \cap N(w^-_i) = \{w_i\}, \quad (i = 1, \ldots, k). \quad (6)$$

By the condition of Theorem 5.4, the ball $G_2(v)$ is a 2-connected graph.

Consider in $G_2(v) - w_1$ a shortest path $P = u_0u_1 \cdots u_i$ where $u_0 = w^+_1$ and $u_i \in \{w_2, \ldots, w_k\}$. By (6), $u_i \notin N(v)$. Then $d(v, u_i) = 2$ and there is a vertex $v_1 \in N(v)$ which is adjacent to $u_i$. We will show that $v_1 = w_1$. Since $M_2(v) \subset M_n(a)$, $v_1$ is an interior vertex in $G_n(a)$. Clearly, $v_1 \in N(v) \cap V(C) = \{w_1, \ldots, w_k\}$ because otherwise $G_n(a)$ contains a cycle $C'$ with $a \in V(C')$ which is longer than $C$. (For example, $C' = w_1v_1w_1w^+_1\overline{C}w_1$ if $u_1 \notin V(C)$, and $C' = w_1v_1w_1w^+_1\overline{C}w_1$ if $u_1 \in V(C)$, where $w^+_1 \in E(G)$ by Property 5.2.) Suppose that $v_1 \in \{w_2, \ldots, w_k\}$, say $v_1 = w_2$. Then, $w^+_1 \subset M_2(w_2) - (M_1(v) \cup \{w^+_2\})$. Therefore, by (6) and Property 5.2, $w^+_1$ is adjacent to $w_1$. But then the cycle $w_1v_2w^+_2\overline{C}w^+_2\overline{C}w_2$ is longer than $C$, a contradiction. Therefore, $v_1 \notin \{w_2, \ldots, w_k\}$ and $v_1 = w_1$, that is,

$$u_1w_1 \notin E(G), \quad u_1w_i \notin E(G), \quad (i = 2, \ldots, k). \quad (7)$$

Since $u_1$ is adjacent to the consecutive vertices $w_1$ and $w^+_1$ on $\overline{C}$, and $C$ is a longest cycle of $G$,

$$u_1 \in V(C). \quad (8)$$

We will now show that

$$vu_2 \notin E(G). \quad (9)$$

Suppose to the contrary that $vu_2 \notin E(G)$. Then (7) and $u_1 \notin V(P)$ imply that $u_2 \notin N(v) \setminus V(C)$. Also, since $M_2(v) \subset M_n(a)$, $u_1$ is an interior vertex in $G_n(a)$. We have $u_1u_2 \notin E(G)$, $u_1 \in V(C)$ and $u_2 \notin N(v) \setminus V(C)$. Therefore, by (5) and Property 5.2, $u_1w^+_1 \in E(G)$. But then the cycle $w_1vu_2u_1w^+_1\overline{C}u_1w^+_1\overline{C}w_1$ is longer than $C$.

Thus $vu_2 \notin E(G)$. Clearly, $u_2 \in M_2(w_1)$ because $w_1u_1, u_2 \in E(G)$. Then by (9) and Property 5.2, $w^+_1u_2 \in E(G)$. But this contradicts the assumption that $u_1u_2 \cdots u_k$ is a shortest path with origin $w^+_1$ and terminus in $\{w_2, \ldots, w_k\}$.

The proof of Claim 2 is completed.

We continue to prove the theorem. By Claim 2, there is an interior vertex $v \in V(G)$ outside of $C$ and $w_1 \in V(C)$ such that either $|N(v) \cap N(w^+_1)| \geq 2$ or $|N(v) \cap N(w^-_1)| \geq 2$. Without loss of generality we assume that $|N(v) \cap N(w^+_1)| \geq 2$. The choice of $C$ implies that $N(v) \cap N(w^+_1) \subset V(C)$. Set $W = N(v) \cap V(C)$ and $k = |W|$. Let $w_1, \ldots, w_k$ be the vertices of $W$, occurring on $\overline{C}$ in the order of their indices, $k \geq 2$. Set $W^+ = \{w^+_1, \ldots, w^+_k\}$.

We will count the number of edges $e(W^+, W)$ between $W^+$ and $W$. The choice of $C$ implies that $W^+ \cup \{v\}$ is an independent set, and $N(w^+_1) \cap N(v) \cap (V(G) \setminus V(C)) = 0$, for $1 \leq i \leq k$. Moreover, for each path $vu_1w^+_1$, $1 \leq i \leq k$, we have, by the hypothesis of the theorem and by Property 5.1,

$$|N(v) \cap N(w^+_i)| \geq |M_2(w_i) \setminus (N(v) \cup N(w^+_i))| - 1. \quad (10)$$

Obviously, $N(w_i) \cap W^+ \subset N(w_i) \setminus (N(v) \cup N(w^+_i)) \cup \{v\}$. Thus,

$$|N(w_i) \cap W^+| \leq |N(w_i) \setminus (N(v) \cup N(w^+_i))| - 1 \leq |M_2(w_i) \setminus (N(v) \cup N(w^+_i))| - 1.$$
This and (10) imply that \(|N(w_i) \cap W^+| \leq |N(v) \cap N(w_i^+)|\). Hence,
\[
e(W^+, W) = \sum_{i=1}^{k} |N(w_i) \cap W^+| \leq \sum_{i=1}^{k} |N(v) \cap N(w_i^+)| = e(W^+, W).
\]

It follows, for each \(i, 1 \leq i \leq k\), that
\[
N(w_i) \setminus (N(v) \cup N(w_i^+) \cup \{v\}) = N(w_i) \cap W^+ \subseteq W^+.
\] (11)

Noting that \(k \geq 2\) and the fact that \(|N(w_i^+) \cap N(v)| \geq 2\), we now prove by contradiction that \(w_i^+ = w_{i+1}^+\) for each \(i = 1, \ldots, k\). (We consider \(w_{k+1} = w_1\).

Assume without loss of generality that \(w_1^+ \neq w_2^+\), whence \(w_2^+ \notin W^+\). Observe that \(w_2^+ \in N(w_1^+)\), otherwise from (11), \(w_2^+ \notin W^+\). Since \(C\) is a longest cycle, \(w_2^+ \notin E(G)\). Hence \(w_2^+ \neq w_3^-\). Repetition of this argument shows that \(w_i^- \neq w_{i+1}^+\) and \(w_i^+ w_i^- \in E(G)\) for all \(i \in \{1, \ldots, k\}\). By assumption, \(N(w_i^+) \cap N(v)\) contains a vertex \(x \neq w_i\). Since \(C\) is a longest cycle, \(x \in V(C)\), say that \(x = w_i\). But then the cycle \(w_1^+w_i^+w_i^-\bar{C}w_i^+\bar{C}w_i\) is longer than \(C\). This contradiction proves that \(w_i^+ = w_{i+1}^+\) for each \(i = 1, \ldots, k\), where \(w_{k+1} = w_1\).

Thus \(V(C) = W \cup W^+\), that is, \(v\) is adjacent to each second vertex of \(V(C)\).

Since \(G\) is infinite, \(V(G) \setminus \{V(C) \cup \{v\}\} \neq \emptyset\). Furthermore, since all balls of radius 2 in \(G\) are 2-connected, the graph \(G\) is 2-connected itself. Thus there is a vertex \(v_1\) outside \(C\) that has a neighbor on \(C\).

**Case 1.** \(N(v_1) \cap W \neq \emptyset\).

Then, by (11), \(v_1v_1 \in E(G)\). This and \(a \in V(C)\) imply that \(|a, v_1| \leq 3\) and \(M_2(v_1) \subseteq M_n(a)\). Without loss of generality we assume that \(v_1w_1 \in E(G)\). Then either \(|N(v_1) \cup N(w_1^-)| \geq 2\) or \(|N(v_1) \cap N(w_1^+)| \geq 2\). (Otherwise, by Property 5.2, \(w_1^- w_1^+ \in E(G)\) and \(G_n(a)\) contains a longer cycle \(w_i^- w_i^+ v_1v_1w_1\).) Using for \(v_1\) the same argument as for \(v\), we can conclude that \(v_1\) is adjacent to each second vertex of \(V(C)\), that is, \(v_1w_1 \in E(G)\), \(i = 1, \ldots, k\). Clearly, \(w_i^+ \neq a\) for some \(t, 1 \leq t \leq k\). But then there is a longer cycle \(w_1v_1w_1v_1\bar{C}w_1\) containing \(a\), a contradiction.

**Case 2.** \(N(v_1) \cap W^+ \neq \emptyset\) and \(N(u) \cap W = \emptyset\) for every \(u \in V(G) \setminus \{V(C) \cup \{v\}\}\).

Without loss of generality we assume that \(v_1w_1^+ \in E(G)\). Then \(M_2(w_1^+) \subseteq M_n(a)\) and \(v_1 \in M_n(a)\) because \(|a, w_1^+| \leq 4\) and \(n = q + 6 \geq 6\). Since \(G_3(w_1^+)\) is 2-connected, there is a \((v_1, w_1^-)\)-path \(Q\) in \(G_2(w_1^+) - w_1^-\). Let \(z\) be a vertex on \(V(Q) \cap V(C)\) such that all other vertices on the path \(v_1Qz\) do not belong to \(C\). Clearly, \(z = w_j^+\) for some \(2 \leq j \leq k\) because, by our assumption, \(N(u) \cap W = \emptyset\) for every \(u \in V(G) \setminus (V(C) \cup \{v\})\). Then \(G_n(a)\) contains a longer cycle \(C' = w_1v_1w_j\bar{C}w_1v_1Qw_j\bar{C}w_1\) with \(a \in V(C')\).

This final contradiction shows that \(S \subseteq V(C)\). Thus for any finite set \(S \subseteq V(G)\) there is a cycle in \(G\) containing \(C\). Then, by Theorem 1.6, \(G\) has a Hamiltonian curve. The proof of Theorem 5.4 is completed.

The class of graphs satisfying the conditions of Theorem 5.4 contains some claw-free graphs (for example, the graph at the top of Fig. 5), as well as graphs that are not claw-free and do not satisfy the conditions of Theorem 1.7 (for example, the graph at the bottom of Fig. 5).

Theorem 5.4 implies the following result:
Corollary 5.5. A connected, infinite, \(k\)-regular graph \(G\) has a Hamiltonian curve if every ball of radius 2 in \(G\) is 2-connected and \(2k \geq |M_2(w)| - 1\) for every vertex \(w \in V(G)\).

Note that the graph at the top of Fig. 5 is an infinite 4-regular graph satisfying the conditions of Corollary 5.5.

Corollary 5.5 is an extension of the following theorem of Nash-Williams [35]: A 2-connected finite \(k\)-regular graph \(G\) is Hamiltonian if \(2k \geq |V(G)| - 1\).

The condition \(2k \geq |M_2(x)| - 1\) in Corollary 5.5 can be rewritten as

\[
2k \geq |M_2(x)| - 1 = (1 + k + |N_2(x)|) - 1,
\]

which is equivalent to \(|N_2(x)| \leq k\), where \(N_2(x)\) denotes the set of vertices at distance 2 from \(x\). Therefore Corollary 5.5 can be reformulated as follows:

Corollary 5.6. A connected, infinite, \(k\)-regular graph \(G\) has a Hamiltonian curve if every ball of radius 2 in \(G\) is 2-connected and the number of vertices at distance 2 from any vertex in \(G\) is at most \(k\).

Diestel [18] conjectured that the condition of Asratian and Khachatryan for finite graphs (see Theorem 1.3) guarantees the existence of Hamilton circles in an infinite locally finite graph \(G\):

Conjecture 1 (Diestel [18]). A connected, infinite, locally finite graph \(G\) has a Hamilton circle if \(d_G(u) + d_G(v) \geq |N(u) \cup N(v) \cup N(w)|\) for each path \(uwv\) with \(uv \notin E(G)\).

We believe that the following conjecture is true:

Conjecture 2. A connected, infinite, locally finite graph \(G\) has a Hamilton circle if all balls of radius 2 in \(G\) are 2-connected and \(d_G(u) + d_G(v) \geq |M_2(w)| - 1\) for each path \(uwv\) with \(uv \notin E(G)\).

Finally, note that for each integer \(n \geq 1\) there are infinite locally finite graphs with \(n\) ends satisfying the conditions of these conjectures. A graph with three ends satisfying the conditions of both conjectures can be seen in Fig. 6.
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