Time-Optimal Charging Processes for Quantum Batteries

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Abstract

We examine time-optimal processes of charging a quantum battery from an initial state to a maximally energetic state through unitary dynamics. We assume that the dynamics are restricted by one of two constraints, the bounded bandwidth or bounded variance constraints. We calculate lower bounds on the charging time for both constraints in the form of quantum speed limits. For the bounded bandwidth constraint we also find the minimal charging time for a large class of systems. In the bounded variance case we present current results in terms of properties of the dynamics. Lastly we examine how time-optimal processes are influenced when multiple batteries are allowed to interact and correlate. We explicitly calculate the charging time for a certain class of systems, and we find that it is decreased if we allow correlation between batteries.

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Introduction

In modern society, energy is one of the most valuable resources. A crucial part of its usage is the existence of the battery. The ability to store and successively extract energy from a system is necessary for everything wireless, be it cellphones, cars or pacemakers. While there is much to be said about the battery itself, it is of great importance to consider the devices which we use to charge it. Of course, the energy which we store must be provided from an external charger, and it is natural to question the efficiency of this charging process. We might for example have a limited time in which we want to fully charge the battery, in which case we require the charger to be sufficiently efficient to reach our goal. In other words, we require the time-duration of the charging process to be optimal.

When discussing cellphone and car batteries it is sufficient to discuss classical physics, and increasing charger efficiency is an engineering problem more than anything else. However now, with the possibility of nanodevices, there is much interest in smaller systems, and similar questions can be asked at a quantum mechanical level. The field of quantum thermodynamics aims to formulate thermodynamical laws and properties in the quantum regime [1], while quantum information theory discusses the information contents in quantum systems [2]. The field of quantum batteries lies in the intersection of these two fields and concerns quantum systems which we can charge and store with energy [3]. Furthermore, the above question of optimal charging is naturally extended to the quantum case because of the large amount of research done on time-optimality in quantum information theory. Since we do not usually want a battery to spontaneously interact with an external environment, it is reasonable to assume that it is isolated with the exception of when a controlled charging device is applied. We therefore assume that the quantum battery evolves unitarily and that the dynamics is generated by a Hamiltonian which is supplied by and represents a certain charging device.

When charging a classical battery it is valid to quantify various constraints on the charging device. For example, a charger of a certain size might only be able to supply the battery with a certain, bounded current. This will restrict how small we can make the duration of the charging process, and it needs to be taken into consideration for all realistic chargers and batteries. For quantum batteries such constraints are not as obvious. However, it is possible to put quantitative bounds on the Hamiltonian responsible for the dynamics of the quantum state. These constraints in conjunction with the problem of time-optimal charging processes give us a natural question, which is the leading topic of this thesis.

Main Topic of Thesis

Consider a quantum battery represented by a quantum state with arbitrary energy-content. If we impose a quantitative constraint on the Hamiltonian responsible for the dynamics of the battery, what is the minimal duration required to charge the battery to a state of maximal energy?

When answering these questions we will consider two possible constraints separately, the *bounded bandwidth constraint* and the *bounded variance constraint*. Furthermore we will examine how the minimal time is affected if we charge multiple batteries simultaneously, allowing them to interact through quantum correlations.

This thesis is based upon an article written by me and my collaborators [4].

Overview

This thesis is heavily reliant on the use of quantum operators. However it should — Section 1 in particular — be accessible to all physicists with a fundamental understanding of quantum mechanics. To this end we will presume that the reader is familiar with concepts covered in [2, 5], such as density operators, unitary operators, entropy and the von Neumann equation. While we at later stages must introduce mathematical tools which might be new to some, we will not provide their full detail, but instead present them to the extent that they can be used with relative comfort. For further information on these mathematical frameworks the readers will be referred to [6, 7]. We will now summarize each section. The whole thesis is summarized with a flow chart in Fig. 1. We advice the reader to consult this flow chart throughout the thesis.

In Section 1 we impose the properties we require of a quantum battery represented by a quantum state ρ. We define the energy content of the battery as the expectation value of a Hamiltonian H_0, and we assert which dynamics are allowed in the considered context. We impose a necessary assumption on the initial state of the battery, and discuss a way of preparing quantum batteries that satisfy this assumption. Lastly we present a method of imposing constraints on the charging process.

Section 2 presents a method of determining the minimal duration given the properties and constraints defined in the preceding section. The method consists of considering the time-dependent unitary operators which charge the battery as curves in the unitary group. This turns the problem into a time-optimization problem. Subsequently, by equipping the unitary group with certain constraint-dependent metrics, the problem is transferred onto finding shortest curves. We find that the method of determining the minimal time is heavily dependent on the spectra of ρ and H_0 and must be split into various cases. We then find a beneficial way of representing these cases in terms of permutations. Lastly we find our first result as two "quantum speed limits", lower bounds on the minimal duration. We find one such for each considered constraint, and these are shown to be sometimes reachable.

In Section 3 we attempt to calculate the minimal duration given the bounded bandwidth constraint. We determine a distance formula and apply it to two different cases; non-degenerate and degenerate spectra. We find an exact solution for all batteries belonging to the former case. In the latter case the problem becomes more complicated, and we have to make further assumptions to determine the minimal duration.

Section 4 is dedicated to emphasizing the subtle differences between the bounded variance constraint and the bounded bandwidth constraint. We present the geometry of the problem and describe some properties of time-optimal control Hamiltonians. We further compare these properties to those for the bounded bandwidth constraint.

Section 5 examines how the minimal duration of a charging process is affected if the battery is allowed to correlate with other identical batteries. We quantify a decrease in minimal duration by means of a quantity which we call the *quantum advantage*. This section is solely dependent on the contents of Section 2, and with certain assumptions we find that allowing correlations is beneficial to time-optimal charging processes. Furthermore, we find an interesting dependency on the parity of the number of possibly interacting batteries.

Finally Section 6 concludes the thesis, pointing out results and notions which are important to remember. We further present an outlook on future potential research topics, some of which are already under progress in conjunction with this thesis.

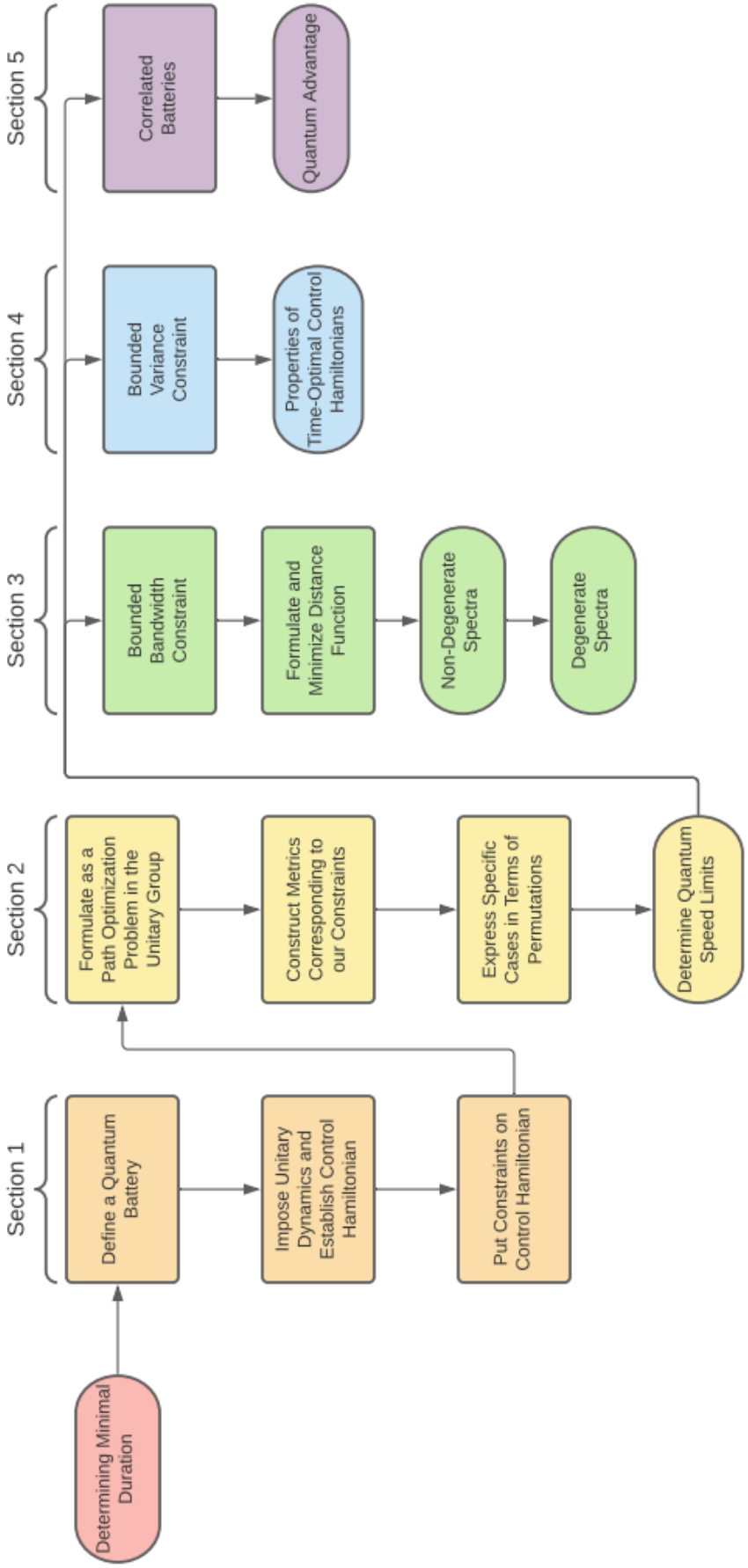


Figure 1: A flow chart depicting the general structure of the thesis. Rounded boxes signalize main results.

Contents

1 Quantum Batteries

Main Topics of Section

- Definitions of the main properties of a quantum battery, energy storage and energy insertion/extraction.
- Detailing dynamics and necessary assumptions for initial states.
- Presentation of two constraints on controllable dynamics.

A battery is a physical system which we require to have two main properties. The first property is that it should be able to store energy with high reliability. By this we mean that we should have made sure that there is a minimal risk of "leakage", i.e. energy being spontaneously exchanged with an external device. Classically, such uncertainties may be the result of a heat transfer to the device or an environment, and to achieve an ideal battery we must reduce this to the point of omission. Secondly, we require a method of inserting energy into it or, conversely, extracting energy from it. An ideal battery requires that this method is fully controlled, by which we mean that energy is neither lost nor unaccounted for during its implementation. Such a method usually makes use of an external device referred to as a *charger* or *discharger* which interacts with the battery.

For a quantum battery, these two main properties need to be defined in the quantum mechanical framework. This is what the remainder of this section is dedicated to.

1.1 Energy Storage

To establish energy storage we first need to define the physical system which represents the quantum battery. Consider a complex Hilbert space $\mathcal{H}$ of finite dimension d and model a quantum state as a density operator ρ on $\mathcal{H}$. This quantum system represents the battery, and we say that ρ is the *battery state*. To assign to this state an energy we must supply the battery with an internal Hamiltonian H_0. It is natural to define the energy content of ρ as the energy expectation value

$$E(\rho) := \mathrm{Tr}[H_0\rho]. \tag{1}$$

Energy storage is achieved if the energy expectation value is conserved, i.e. independent of time. The first step towards this is to examine internal properties of the battery, i.e. consequences of ρ and H_0. To begin with we want to make sure that H_0 is time-independent, since if not, the energy content might fluctuate. Note that as a consequence of being an internal Hamiltonian, H_0 does cause ρ to evolve unitarily [2] such that

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \tag{2}$$

where $U(t) = \exp(-iH_0t)$. However this time evolution keeps the energy content Eq. (1) invariant since $U(t)$ and H_0 commute. Hence it does not conflict with our idea of energy storage.

The battery could however still spontaneously interact with external systems, in which case the time-dependency of the battery becomes more general and stops being governed by Eq. (2). To rid ourselves of this inconvenience we could further define the battery as

a *closed* system [2]. Hence all changes in dynamics are fully characterized by alterations of the battery's internal Hamiltonian, i.e. by the change

$$H_0 \to H(t) = H_0 + H_c(t). \tag{3}$$

We say that $H_c(t)$ is the possibly time-dependent *deviation* from the initial observable H_0. To describe the battery as a closed system can be further motivated by observing that alterations to the internal Hamiltonian do not change the von Neumann entropy of the battery [2]. This allows us to draw a parallel to thermodynamics where heat transfer is accompanied by a change in entropy.

We have now laid down the conditions under which energy storage can be imposed on the battery. This allows us to move on to the second required property of a quantum battery, namely clear definitions for how to insert or extract energy into/from it. As we will find, the assumption that the battery is a closed system will greatly assist us in this regard.

1.2 Energy Insertion and Extraction

Since the battery is a closed system, energy insertion or extraction can happen only by a change in the internal Hamiltonian. With $H(t)$ in Eq. (3) as our new Hamiltonian we exchange the dynamics for storage, Eq. (2), with

$$\rho(t) = U(t)\rho(0)U^\dagger(t), \qquad U(t) = \mathcal{T}\exp\left(-i\int_0^t H(t')dt'\right). \tag{4}$$

Here $\mathcal{T}$ is the time-ordering operator. This is equivalently described by imposing the von Neumann equation

$$\dot{\rho}(t) = -i[H(t), \rho(t)]. \tag{5}$$

Note that if $H_c(t) = 0$, the internal Hamiltonian is once again given by H_0 and the battery dynamics by Eq. (2). Hence $H_c(t)$ is fully responsible for controlling insertion and extraction of energy in the battery. It then becomes suitable for us to name it the *control Hamiltonian*, or *control* for short. Henceforth we will let the charging/discharging device be fully represented by $H_c(t)$, and the setting can be visualized in Fig. 2.

We have now fully established a method of inserting or extracting energy in a controlled manner which befits a battery. Since the theory is identical for charging or discharging the battery, we will restrict ourselves to the charging process.

1.2.1 Quantum Quench

We required that we have complete control over the energy that is transferred during charging. Since a quantum battery can lose energy when charged too long, we want a method of instantaneously turning $H_c(t)$ on or off. We refer to this property of instantaneously activating or deactivating $H_c(t)$ as a *quantum quench*.

Assume that $H_c(t)$ can fully charge the battery and let τ be the time required for this process. We refer to τ as the process' duration. Let ϵ be an arbitrary positive number

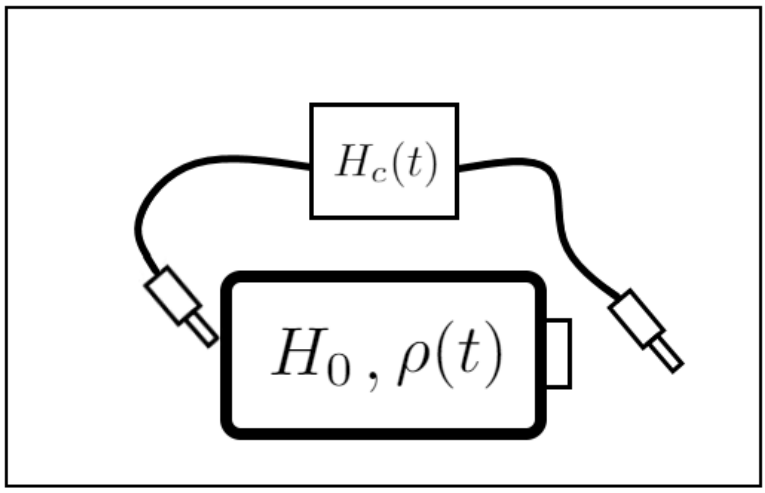


Figure 2: A quantum battery with state $\rho(t)$ and fixed internal Hamiltonian H_0. We can control the charging of this battery by applying a charging device that changes the Hamiltonian of the battery to $H(t) = H_0 + H_c(t)$. The unitary dynamics of the battery will then be governed by $H(t)$.

and define a smooth function $q_\epsilon(t)$ that satisfies

$$q_\epsilon(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \tau, & \text{if } t \geq \tau + 2\epsilon, \end{cases} \tag{6}$$

$$\dot{q}_\epsilon(t) \in [0, 1] \tag{7}$$

Define the *quenched* control Hamiltonian by $H'_c(\epsilon, t) = \dot{q}_\epsilon(t) H_c(q_\epsilon(t))$. On account of $q_\epsilon(t)$ being smooth, this alternative choice of control is equal to the zero operator for times $t \leq 0$ and $t \geq \tau + 2\epsilon$. Hence it is quenched at these times. We now formulate the following proposition.

Proposition 1.1. *The quenched control $H'_c(\epsilon, t)$ fully charges the battery. Furthermore, by decreasing ϵ the duration of this process can be made arbitrarily close to τ.*

We prove this proposition in Appendix A.1. Note that this proof uses notation and methods developed later in this thesis. This proposition allows us to always replace a fully charging control with a a version which is quenched before and after the process. The price we pay is that the duration of the process becomes infinitesimally longer than τ. The lower bounds we examine are still important, however, especially since we must determine τ before being able to apply the above quench.

1.3 Accessible Battery States

Having established the evolution dynamics of the battery it becomes relevant to ask which states are accessible given a certain initial state $\rho_i = \rho(0)$. Unitary dynamics preserves the spectrum of our state [2]. Therefore, let $\mathbf{p} = \{p_1, p_2, \cdots, p_d\}$ be the spectrum of ρ_i and define $\mathcal{D}(\mathbf{p})$ as the manifold of states with this spectrum. The unitary group acts transitively on $\mathcal{D}(\mathbf{p})$. That is, any state in $\mathcal{D}(\mathbf{p})$ can be transformed into every other

state in it through a unitary action. Because of this, $\mathcal{D}(\mathbf{p})$ is the full space of accessible battery states and will thus be referred to as the *battery state space*. Henceforth, a "state" will refer to an element in $\mathcal{D}(\mathbf{p})$. We will now examine some important states and their properties.

1.3.1 Incoherent and Extremal States

Having assigned $\mathcal{D}(\mathbf{p})$ as the battery state space, it is important to identify which states represent a "full" or "empty" battery. In terms of the energy content, these are those which maximize or minimize the energy expectation value, Eq. (1). We borrow some terminology from [8, 9, 10] and refer to a full battery state as an *active state*[1] ρ_a and an empty battery state as a *passive state* ρ_p.

Remark 1. Note that if either ρ_i or H_0 can be written as the identity times a constant, then ρ_i is simultaneously active and passive. We consider this as a trivial example, and for the remainder of this thesis we will assume that such is not the case.

The active and passive states are fully dependent on the spectrum of the initial state ρ_i, and it is natural to ask how ρ_i is prepared. It is convenient to prepare it in such a way that it is held fixed before we activate the control $H_c(t)$ through the quantum quench. That is, the initial state is represented by a single element in $\mathcal{D}(\mathbf{p})$ for all times before $t = 0$. We remind ourselves that before we attach a control the system is governed by the dynamics of H_0, Eq. (2). This dynamics will evolve all states except those with which H_0 commutes. We refer to those states as *incoherent states*. Hence, it is useful to prepare ρ_i as an incoherent state. Henceforth, all initial states are assumed to be incoherent. The following proposition extends incoherence to our active and passive sets.

Proposition 1.2. *Active and passive states are incoherent.*

The proof is postponed until Appendix A.1. While this result is not intuitively useful, it is similarly appropriate to keep the battery "fixed" after the charging is completed. Predominately, however, the incoherence of both initial and active state will prove essential for the results of this thesis.

Remark 2. In fact, the general theory of this thesis is fully applicable to any set of isoenergetic final states, assuming that they are incoherent. Hence the results also apply if we consider the minimal duration required to fully discharge a battery, i.e. transforming ρ_i to a passive state ρ_p. This can be of interest during reverse processes where we want to use the battery to power an external device.

Remark 3. It is important to realize that there can exist multiple active and passive states in $\mathcal{D}(\mathbf{p})$. This observation will be essential in Section 2.1.2, where we seek to minimize the charging duration τ.

1.3.2 Preparation of Incoherent States

There are multiple ways to prepare incoherent states, one of them being to perform a *non-selective von Neumann measurement* with respect to H_0 [2]. Let $E_1, E_2, \cdots, E_m$ be the distinct eigenvalues of H_0. The Hilbert space $\mathcal{H}$ can be decomposed into m mutually

[1] In thermodynamics ρ_a is usually called a *maximally active* state.

orthogonal Hilbert spaces $\mathcal{H}_1, \cdots, \mathcal{H}_m$. Each such $\mathcal{H}_j$ is the eigenspace of H_0 corresponding to the eigenvalue E_j. Thus, $\mathcal{H}_j$ is spanned by all eigenvectors with eigenvalue E_j, and has dimension equal to the multiplicity of E_j. Let Π_j be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j$. Then H_0 can be written as

$$H_0 = \bigoplus_{j=1}^{m} E_j \Pi_j. \tag{8}$$

A selective measurement of H_0 on ρ with outcome E_j would cause ρ to transform as

$$\rho \to \rho' = \frac{\Pi_j \rho \Pi_j}{\mathrm{Tr}[\rho \Pi_j]}. \tag{9}$$

The denominator $\mathrm{Tr}[\rho\Pi_j]$ is the probability of measuring the energy E_j. A non-selective measurement is made by disregarding the selection. Hence, a non-selective measurement of H_0 prepares the state

$$\rho \to \rho_i = \bigoplus_{j=1}^{m} \mathrm{Tr}[\rho\Pi_j] \frac{\Pi_j \rho \Pi_j}{\mathrm{Tr}[\rho \Pi_j]} = \bigoplus_{j=1}^{m} \Pi_j \rho \Pi_j. \tag{10}$$

This state commutes with H_0. Given sufficient information of our observable, this is a viable method of preparing an initial incoherent state. This will be the assumed preparation method in this thesis, since it allows for initial states of arbitrary spectra.

1.4 Constraints

While charging a battery it is natural to encounter some constraints, e.g., the wires we use to charge a device can only handle a certain maximum current. Such a constraint restricts the time needed to fully charge the battery, and the number of possible constraints are countless.

In the quantum case these constraints are not as apparent, but one could consider constraints that limit what charges the battery, i.e., the control $H_c(t)$. In this thesis we analyze two such constraints, the *bounded bandwidth* and the *bounded variance* constraints. We will for convenience often omit the time-dependency from the notation and simply write H_c for the control.

1.4.1 Bounded Bandwidth Constraint

Let ω be some constant positive number. The bounded bandwidth constraint is

$$\mathrm{Tr}[H_c^2] \leq \omega^2. \tag{11}$$

This constraint is considered in various time-optimization problems [11, 12, 13] and bounds the absolute values of the eigenvalues of H_c from above.

1.4.2 Bounded Variance Constraint

The bounded variance is, as the name suggest, a restriction of the variance

$$\mathrm{Var}(H_c, \rho(t)) = \mathrm{Tr}[H_c^2 \rho(t)] - \mathrm{Tr}[H_c \rho(t)]^2 \leq \omega^2. \tag{12}$$

Here, ω is again some fixed positive number. We remind the reader that if we would perform measurements of H_c on an infinite number of copies of $\rho(t)$, the variance is a measure of the deviation from the expectation value. Variance is generally important in the field of quantum information and is relevant in various problems related to time-optimization [3, 14]. Furthermore, there are interesting and important differences between Eqs. (11) and (12) which are important to point out in the context of more general constraints. We will introduce these in Section 2.2.

Remark 4. For various geometrical reasons we will only consider ρ_i with full rank when considering the bounded variance constraint. While the results of this thesis are applicable even when ρ_i does not have full rank, the proofs become significantly more complicated. For more on this subject read [15].

1.4.3 Saturation of Constraints

The above constraints both have the property that they are homogeneous of second order in H_c. That is, if f is our constraint function and λ is a constant, then

$$f(\lambda H_c, \rho(t)) \leq \lambda^2 f(H_c, \rho(t)). \tag{13}$$

The following proposition proves something very powerful for our two constraints Eqs. (11) and (12).

Proposition 1.3. *If $f(H_c, \rho(t)) \leq \omega^2$ is the only imposed constraint, then it is saturated for all time-optimal charging processes.*

We postpone the proof until Appendix A.1. This proposition tells us that f can be considered as a restriction of some resource. From this point of view it then becomes reasonable that optimal time is achieved if we use all resources available. For the remainder of this thesis we will assume that both constraints Eqs. (11) and (12) are saturated.

Remark 5. The reader might ask if constraints or their saturation conflicts with the quenching of H_c in Section 1.2.1. We prove that such isn't the case in the proof of Proposition 1.1 in Appendix A.1.

Remark 6. It might be of interest to consider the case where multiple different constraints are imposed simultaneously, e.g. the two arbitrary constraints $f \leq \omega_1^2$ and $g \leq \omega_2^2$. In this case we cannot prove that Proposition 1.3 applies, and these constraints may or may not be individually saturated at different times. For more on this see [16].

2 The Charging Process

Main Topics of Section

- Formulating the time-optimization problem as a path-optimization problem
- Propose metrics which allows us to more easily handle the constraints.
- Formulate cycle-notation which assists us in categorizing various solutions.

We charge the battery by evolving an initial state ρ_i to an active state ρ_a, letting τ be the duration of this process. The objective is to optimize the charging process in the sense that we minimize this duration. However, due to the imposed constraints on H_c, we cannot make this duration arbitrarily small, and our problem develops into a time optimization problem: Given our constraints, what is the minimal duration $\tau_{\min}$ such that $\rho(\tau_{\min})$ in Eq. (4) is an active state? We will begin this section by establishing how to solve this problem by means of a related subject; path optimization.

2.1 Time Optimization as a Path Optimization Problem

We are interested in finding the minimal duration $\tau_{\min}$ required for an evolving battery $\rho(t)$ in $\mathcal{D}(\mathbf{p})$ to become active. This is a time optimization problem of a non-static quantum state, something which in general is a very difficult problem [11, 12, 13, 16, 17, 18]. In our case there exists a convenient method as a consequence of the unitary dynamics, namely time-optimal curves in the unitary group. We will start off by making an important simplification by expressing everything in the interaction picture, the motivation for which will be apparent in Section 2.2.

2.1.1 Quantum Drift and the Interaction Picture

So far the charging process of the battery is governed by the Hamiltonian $H = H_0 + H_c$ while our constraints restrict the latter term, H_c. It would however be convenient to express the problem in a frame where the constraints are turned into limitations on the full dynamics of the system. The interaction picture will do just this.

Define the *interaction picture equivalents* of ρ and H_c as

$$\begin{aligned} \rho_I &:= e^{iH_0t}\rho e^{-iH_0t}, \\ H_{cI} &:= e^{iH_0t}H_c e^{-iH_0t}. \end{aligned} \tag{14}$$

We remind the reader that the interaction picture represents that the whole state space $\mathcal{D}(\mathbf{p})$ is rotating in time. Hence all coherent states get an explicit time-dependency. In contrast, all incoherent states are fixed points in this frame, which is why it is beneficial to impose that our initial and final states are incoherent. We now present a proposition regarding how ρ_I evolves.

Proposition 2.1. *If ρ obeys Eq. (5) with $H = H_0 + H_c$, then*

$$\dot{\rho}_I(t) = -i[H_{cI}(t), \rho_I(t)]. \tag{15}$$

Proof. Using Eq. (14),

$$\begin{aligned}\dot{\rho}_I &= i[H_0, \rho_I] + e^{iH_0 t}\dot{\rho}e^{-iH_0 t} \\ &= i[H_0, \rho_I] - ie^{iH_0 t}[H_0 + H_c, \rho]e^{-iH_0 t} \\ &= -ie^{iH_0 t}[H_c, \rho]e^{-iH_0 t} \\ &= -i[H_{cI}, \rho_I].\end{aligned} \tag{16}$$

This concludes the proof. □

Proposition 2.1 shows that the interaction picture allows us to instead consider unitaries of the form

$$U(t) = \mathcal{T}\exp\left(-i\int_0^t H_{cI}(t')dt'\right) \tag{17}$$

Furthermore, both constraints Eqs. (11) and (12) are invariant under this change of frame: If $\mathrm{Tr}\,[H_c^2] \leq \omega^2$ or $\mathrm{Var}\,(H_c, \rho(t))$, then $\mathrm{Tr}\,[H_{cI}^2] \leq \omega^2$ or $\mathrm{Var}\,(H_{cI}, \rho_I(t)) \leq \omega^2$. Hence, by means of expressing the problem in the interaction picture we have managed to transform the constraints from limitations on the partial dynamics to the full charging dynamics. For the remainder of the thesis we assume that everything is expressed in the interaction picture unless otherwise stated. When we write H_c and ρ these will refer to the interaction picture equivalents.

2.1.2 Unitary Evolution as Curves

We want to find the minimal duration $\tau_{\min}$ such that $\rho(\tau_{\min})$ is an active state. If we remind ourselves of Eq. (4) we can equivalently ask ourselves: for a unitary operator $U(t)$, what is the minimal time $\tau_{\min}$ required for $U(\tau_{\min})\rho_i U^\dagger(\tau_{\min})$ to be an active state? This is a different outlook on the same problem. Instead of focusing on the battery itself we ask how much time we require to implement our unitary dynamics[2].

Let $\mathcal{U}(\mathcal{H})$ be the unitary group, i.e. the group of unitary operators on $\mathcal{H}$. The unitary operator $U(t)$ in Eq. (17) can be regarded as a continuous curve in $\mathcal{U}(\mathcal{H})$. Its time-derivative is given by

$$\dot{U}(t) = -iH_c(t)U(t) \tag{18}$$

and can be regarded as the tangent vector or velocity of the curve at time t. We can choose $U(0) = \mathbb{1}$ such that this curve emanates from the identity, and we require that it activates our state at time τ, i.e., that $U(\tau)\rho_i U^\dagger(\tau)$ is an active state. We now remind ourselves of Remark 3 which stated that there might exist multiple active states. Similarly there might be multiple distinct unitaries which activates our state. To this end we let $\mathcal{A}(\rho_i)$ be the "activating set" of "activating" unitary operators A for which $A\rho_i A^\dagger$ is active. In Fig. 3 we visualize these concepts.

Since we want to achieve minimal duration τ we must choose a curve $U(t)$ which hits any point in $\mathcal{A}(\rho_i)$ as fast as possible. Hence, it becomes essential to determine how $\mathcal{A}(\rho_i)$ looks. To this end let $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ be the groups of unitary operators that commute with H_0 and ρ_i, respectively. We call these the *isotropy groups* of H_0 and ρ_i. The following proposition is proven in Appendix A.2.

[2]This is in fact a default approach in the field of quantum computation, where one often wants to know the minimal time required to implement different gates, see [11, 12, 13, 16, 17, 16].

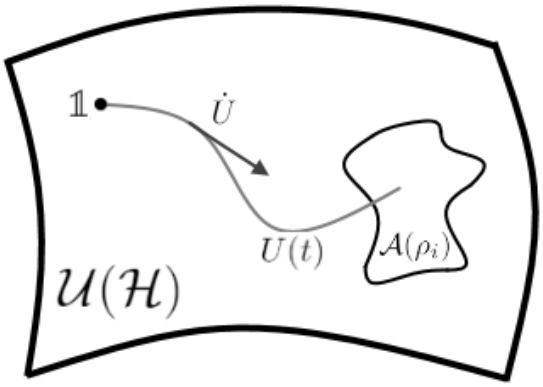


Figure 3: A curve of unitary operators $U(t)$ with tangent vector $\dot{U}$ in the unitary group $\mathcal{U}(\mathcal{H})$. The curve connects identity $\mathbb{1}$ and the activating set $\mathcal{A}(\rho_i)$, hence it activates the battery.

Proposition 2.2. *Given an arbitrary unitary $A \in \mathcal{A}(\rho_i)$,*

$$\mathcal{A}(\rho_i) = \{UAV : U \in \mathcal{U}(\mathcal{H})_{H_0}, V \in \mathcal{U}(\mathcal{H})_{\rho_i}\}. \tag{19}$$

We mentioned earlier that $\dot{U}(t)$ could be interpreted as the velocity of the curve $U(t)$ at time t. In order to quantify the speed of $U(t)$, i.e., assign a size to the velocity $\dot{U}(t)$, we must first equip the unitary group with a *Riemannian* metric. Recall that a Riemann metric is a smoothly varying field of inner products on the tangent spaces of $\mathcal{U}(\mathcal{H})$, see [6, 7]. Such a metric also allows us to measure distances on $\mathcal{U}(\mathcal{H})$. While the choice of metric is fully up to us, we want it to suit the problem at hand. In Proposition 1.3 we proved that the constraints are saturated, hence equal to some constant value ω^2 for all times t. If we manage to find a metric g for which the squared speed of $U(t)$ agrees with the constraint function f, then the squared speed in this metric is bounded from above by the constraint ω^2. Due to Proposition 1.3 this implies that time-optimal control Hamiltonians generate unitary curves whose speed is constant. Furthermore, if the speed of a curve is constant, then minimal time is achieved by letting it be a shortest curve. Hence, given a "suitable" metric according to our definition, the problem boils down to finding the length of the shortest curve connecting $\mathbb{1}$ and $\mathcal{A}(\rho_i)$. Finding metrics which allows us to do this is the topic of the subsequent section.

2.2 Constraint-Induced Metrics

We want to choose a Riemannian metric g on $\mathcal{U}(\mathcal{H})$ such that the squared speed v^2 of time-optimal unitary curves equals the imposed constraint function. In other words, if $U(t)$ is generated by a time-optimal control H_c satisfying our constraint, then g should satisfy

$$v^2 = g(\dot{U}, \dot{U}) = f(H_c, \rho) = \omega^2. \tag{20}$$

The latter equality is a consequence of Proposition 1.3. This then implies that $v = \omega$ such that the speed is constant and optimal paths become shortest curves. Before defining metrics for either constraint we must discuss some algebraic and geometric properties of the unitary group.

The unitary group $\mathcal{U}(\mathcal{H})$ is a *Lie group*, a smooth manifold where the group law and inverse are smooth maps. Since it is a smooth manifold there belongs to each point U in $\mathcal{U}(\mathcal{H})$ a tangent vector space which we denote by $T_U\mathcal{U}$. The tangent space at identity is the Lie algebra of $\mathcal{U}(\mathcal{H})$, which will be denoted by $\mathfrak{u}(\mathcal{H})$. The Lie algebra is spanned by the generators of its Lie group through the exponential map. That is, for any U in $\mathcal{U}(\mathcal{H})$ there exists a vector ξ in $\mathfrak{u}(\mathcal{H})$ such that $U = \exp(\xi)$. In the case of the unitary group, $\mathfrak{u}(\mathcal{H})$ consists of the skew-Hermitian operators on $\mathcal{H}$, i.e. those operators for which $\xi^\dagger = -\xi$. If ξ is time-dependent it will instead generate a curve in the unitary group. For example, if we let $\xi(t) = -itH$ where H is a Hamiltonian, then $\exp(\xi(t))$ is the time-evolution operator of H.

For every curve $U(t)$ generated by $\xi(t)$ we can define the initial speed as the length of the tangent vector at identity, i.e. the "length" of $\xi(0)$ in $\mathfrak{u}(\mathcal{H})$. To do this we must endow the Lie algebra with an inner product. Let ξ, η be elements in $\mathfrak{u}(\mathcal{H})$ and consider the inner products

$$\langle \xi, \eta \rangle_{\text{HS}} = \frac{1}{2} \operatorname{Tr}\left[\xi^\dagger \eta + \eta^\dagger \xi\right], \tag{21}$$

$$\langle \xi, \eta \rangle_{\text{Var}} = \frac{1}{2} \operatorname{Tr}\left[\rho_i \left(\xi^\dagger \eta + \eta^\dagger \xi\right)\right]. \tag{22}$$

We must now argue that these inner products imply two metrics respectively. In Lie theory, there is a one-to-one correspondence between the inner products on the Lie algebra and what is called the *left-invariant metrics* on the Lie group [7]. Let W be a unitary operator and define the map $L_W(U) = WU$. The map L_W maps $\mathcal{U}(\mathcal{H})$ onto itself and is referred to as the *left action* of the Lie group. Furthermore, given L_W there exists a *push forward* or *differential* map denoted by dL_W which further describes how $T_U\mathcal{U}$ is mapped onto $T_{WU}\mathcal{U}$, [6, 7]. This differential map is in turn given by

$$dL_W : \dot{U} \mapsto (\dot{WU}) = W\dot{U}. \tag{23}$$

A left-invariant metric is a metric which satisfies

$$g(\dot{U}_1, \dot{U}_2) = g(dL_W(\dot{U}_1), dL_W(\dot{U}_2)). \tag{24}$$

We now observe that the inner products constructed in Eqs. (21) and (22) uniquely imply two left-invariant metrics, g_{HS} and g_{Var}. Metrics defined through inner products in this manner are examples of *Riemannian metrics* [6, 7].

Note that for any unitary operator U there exists a left action $L_{U^\dagger}$ which maps U to the identity. The corresponding differential maps any tangent space $T_U\mathcal{U}$ to the Lie algebra. Due to left-invariance of the metrics, Eq. (24), this allows us to use the same expression for the inner product on both $T_U\mathcal{U}$ and the Lie algebra. Hence,

$$g_{\text{HS}}(\dot{U}_1, \dot{U}_2) = \frac{1}{2} \operatorname{Tr}\left[\dot{U}_1^\dagger \dot{U}_2 + \dot{U}_2^\dagger \dot{U}_1\right], \tag{25}$$

$$g_{\text{Var}}(\dot{U}_1, \dot{U}_2) = \frac{1}{2} \operatorname{Tr}\left[\rho_i \left(\dot{U}_1^\dagger \dot{U}_2 + \dot{U}_2^\dagger \dot{U}_1\right)\right]. \tag{26}$$

This allows us to evaluate the length of any tangent $\dot{U}(t)$ of $U(t)$. From both metrics and Eq. (18) we find

$$g_{\text{HS}}\left(\dot{U}(t), \dot{U}(t)\right) = \operatorname{Tr}\left[H_c^2(t)\right], \tag{27}$$

$$g_{\text{Var}}\left(\dot{U}(t), \dot{U}(t)\right) = \operatorname{Tr}\left[\rho(t) H_c^2(t)\right]. \tag{28}$$

By comparing Eq. (27) with Eq. (12) we can observe that the speed of $U(t)$ with respect to the first inner product equals the bounded bandwidth condition, hence we have guaranteed that Eq. (20) is fulfilled. It is however not immediately clear that Eq. (28) corresponds to the variance Eq. (12), we affirm that such is indeed the case if $U(t)$ is a *horizontal lift* of $\rho(t)$. We postpone what this entails to Section 4, where the variance case is treated in detail. Hence we have found two constraint-induced metrics which allow us to consider shortest curves.

Remark 7. While subtle, there are consequential geometrical differences between the above metrics. The reason is that, just as it is invariant under the left action as defined above, g_{HS} is similarly invariant under the *right action* $R_W(U) = UW$. If a metric is simultaneously left and right-invariant it is commonly referred to as *bi-invariant*, [7]. Hence, g_{HS} is bi-invariant while g_{Var} is only left-invariant. These geometrical differences will prove very important later in the thesis when determining the geodesic distance. More on this topic can be read in [19].

Remark 8. Both metrics Eqs. (25) and (26) are invariant under the right action of $\mathcal{U}(\mathcal{H})_{\rho_i}$. This will be of great use in Section 4.

2.3 General Shortest Curves

We found metrics which keep the squared speed constant and equal to our constraint, the remaining problem is to find the length of the shortest curve[3] for which the battery ends up activated. Since the shortest curve traversed at constant speed is necessarily a geodesic, we will henceforth refer to this length as the *geodesic distance*. Between any two operators U and V we denote it by $\text{dist}(U, V)$. We must now mention a very important concept to keep in mind when we look for the geodesic distance.

2.3.1 Geodesic Distance to $\mathcal{A}(\rho_i)$

When considering two points in the unitary group, the geodesic distance between them is unique. There may of course be multiple shortest curves between the points, but all these shortest curves have the same length. In our case, however, we want to connect the identity operator not to a point, but to the whole set $\mathcal{A}(\rho_i)$. This introduces a difficulty since some points in $\mathcal{A}(\rho_i)$ may be closer to identity than others.

This introduces one more step in our calculations. Since we want to minimize the length of the curve, we must also minimize over all geodesic distances between the identity and $\mathcal{A}(\rho_i)$. Due to Proposition 2.2, we thus have

$$\text{dist}(\mathbb{1}, \mathcal{A}(\rho_i)) = \min_{U,V} \Big\{ \text{dist}(\mathbb{1}, UAV) \Big\}, \tag{29}$$

where the right-hand side is minimized over all Us in $\mathcal{U}(\mathcal{H})_{H_0}$ and all Vs in $\mathcal{U}(\mathcal{H})_{\rho_i}$.

2.4 The Cycle Representation

For completely general dimensions and eigenspectra of H_0 and ρ_i the number of cases is infinite, and while we have introduced useful mathematical tools we can only hope to find

[3]The existence of a shortest curve is guaranteed on account of that the unitary group is compact and without boundary [6].

$\tau_{\min}$ for some of them. It is therefore useful to find a way to distinguish between solvable and non-solvable cases. As we will show in later sections, the *cycle representation* of A allows us to do this. In multiple cases it will also assist us in calculating $\tau_{\min}$. Moreover, it allows us to determine a lower bound on $\tau_{\min}$ for all cases and tells us when this bound can be reached.

The representation we seek requires that we choose a basis in which we express our states. We choose the basis $|k\rangle$ for which

$$\rho_i = \sum_{k=1}^{d} p_k \left|k\right\rangle \left\langle k\right| . \tag{30}$$

We index the eigenvectors such that the energy eigenspaces are in increasing order, $\langle k|H_0|k\rangle \leq \langle k+1|H_0|k+1\rangle$. Let σ denote a permutation of the eigenvectors of ρ_i. Further write $\sigma(k)$ for the permutation of the kth eigenvector. If every eigenvector is permuted by σ in this way, then ρ_i is evolved into $\rho_\sigma := A_\sigma \rho_i A_\sigma^\dagger$, where

$$A_\sigma = \sum_{k=1}^{d} \left|\sigma(k)\right\rangle \left\langle k\right| . \tag{31}$$

We refer to A_σ as the *permutation operator* of σ. The following proposition is an observation which is proven in Appendix A.2.

Proposition 2.3. *The target set $\mathcal{A}(\rho_i)$ contains at least one permutation operator A_σ of the form Eq.* (31)*. That is, ρ_σ is active.*

There can of course exist multiple distinct permutations σ for which ρ_σ is active. Henceforth, A_σ will refer to any operator that can be written as a permutation operator in the basis of Eq. (30), assuming it activates our state. Remember that A_σ is not necessarily the activating unitary closest to identity, but a good candidate for the arbitrary activating operator A in Eq. (29). We now present a useful way of expressing this permutation using a *cycle decomposition*.

Let $c_k = (k_1, k_2, \cdots, k_l)$ be a cycle of length l which permutes integers $k_1, \cdots, k_l$ according to $k_1 \to k_2 \to \cdots \to k_l \to k_1$. For our purposes, these integers will correspond to the indices of the chosen basis $|k\rangle$. Every permutation σ can be uniquely decomposed into a series of m such disjoint cycles [20],

$$\sigma = c_1 c_2 \cdots c_{m-1} c_m. \tag{32}$$

Each cycle c_r represents how the permutation A_σ operates on a certain subspace $\mathcal{H}_{c_r}$ of $\mathcal{H}$. Each such $\mathcal{H}_{c_r}$ is spanned by the eigenvectors whose indices are permuted by c_r. Furthermore, these subspaces are held invariant by A_σ such that each eigenvector is permuted onto the subspace it started in. The lengths of the cycles represents the dimensions of these cycle-invariant subspaces and will turn out to have major relevance when improving the charging duration τ. Cycles of length one represent fixed eigenvectors and will henceforth be referred as *trivial cycles*. Cycles of length two will turn out to be important, and we will refer to these as *transpositions*. All other cycles are referred to as *non-trivial cycles*. We will now present an example which visualizes a cycle representation, and how to construct it. For many examples in this thesis we will let $\text{spec}\{A\} = (A_1, A_2, \cdots, A_d)$ denote the *ordered* spectrum of an operator A. By this we mean that the eigenvalues are ordered according to the basis $|k\rangle$ of Eq. (30), such that $A_k = \langle k|A|k\rangle$.

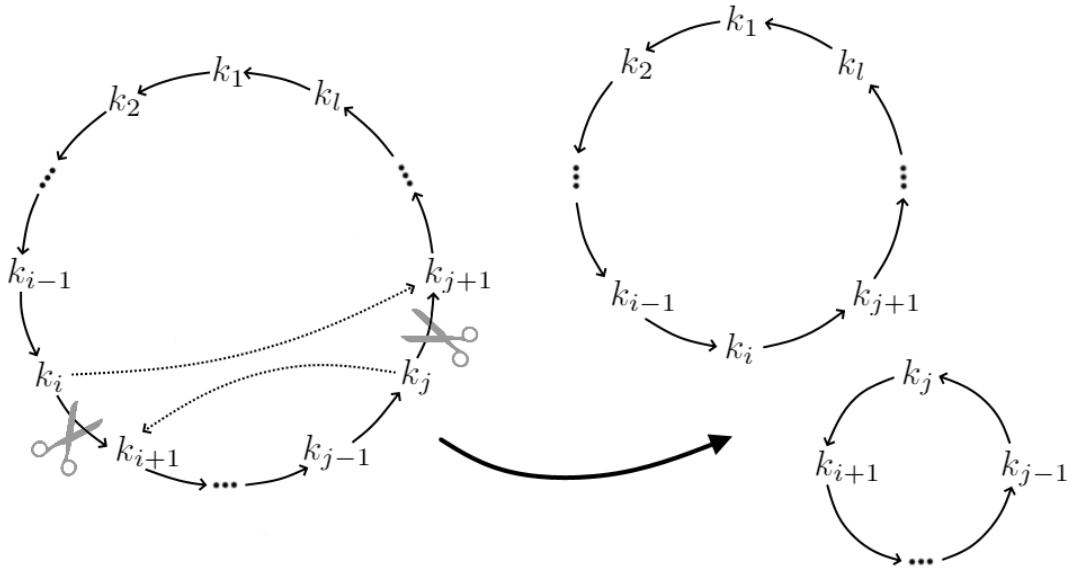


Figure 4: A general example of a cycle reduction. A cycle of the form $(k_1, k_2, \cdots, k_l)$ is split at the position of the scissors and reduced into the two shorter cycles of the form $(k_1, \cdots, k_i, k_{j+1}, \cdots, k_l)$ and $(k_{i+1}, \cdots, k_j)$.

Example 1. Let the spectrum of H_0, ρ_i and ρ_a be

$$\begin{aligned} \text{spec}\{H_0\} &= (E_1, E_1, E_2, E_2, E_2, E_3, E_3, E_4, E_4), \\ \text{spec}\{\rho_i\} &= (p_4, p_1, p_6, p_2, p_5, p_5, p_3, p_8, p_7), \\ \text{spec}\{\rho_a\} &= (p_1, p_2, p_3, p_4, p_5, p_5, p_6, p_7, p_8), \end{aligned} \tag{33}$$

where all indices represent distinct eigenvalues and $p_k \leq p_{k+1}$ to ensure that ρ_a is an active state. A permutation σ which performs this permutation can be written as

$$\sigma = (1, 4, 2)(3, 7)(5)(6)(8, 9), \tag{34}$$

where each cycle is found by comparing the spectra of ρ_i and ρ_a.

2.4.1 Cycle Reductions

In Example 1, we note that the fifth cycle $(8, 9)$ can be replaced by two trivial cycles, $(8)(9)$, such that

$$\sigma' = (1, 4, 2)(3, 7)(5)(6)(8)(9). \tag{35}$$

While σ' is a different permutation, $A_{\sigma'}$ is still an activating unitary. As we will find later, increasing the number of distinct cycles in this manner will in many cases strictly decrease the charging duration τ. Hence, σ' would be a more optimal permutation than the one provided in Example 1. Splitting a cycle into multiple, shorter cycles will be referred to as a *cycle reduction*. A general cycle reduction is presented in Fig. 4.

We say that a permutation is admissible if it activates our battery. A permutation σ which cannot be subjected to further cycle reduction while remaining admissible will be referred to as a *fully reduced permutation*. This concept will be important in later sections, and it is often useful to consider fully reduced permutations.

Remark 9. Note that for complicated σ there could potentially exist multiple distinct methods of obtaining a permutation which is fully reduced. Hence we do not generally know whether a fully reduced permutation is unique.

2.5 Quantum Speed Limits

When considering time optimization processes in quantum information we want to consider lower bounds on the time that is required. Such a lower bound is what in quantum information theory is referred to as a *quantum speed limit*[4] [14, 21]. While the lower bound is not always possible to reach, the quantum speed limit still gives insight into the problem. In Propositions 2.4 and 2.5 we present two quantum speed limits, $\tau_{\text{qsl}}^{\text{B}}$ and $\tau_{\text{qsl}}^{\text{V}}$, one for each constraint. However, we must first introduce some notation.

Let an activating permutation σ transform an eigenvector $|k\rangle$ to $|\sigma(k)\rangle$. Consider now the two possibilities

$$p_k = p_{\sigma(k)}, \tag{36}$$

$$\langle k|H_0|k\rangle = \langle \sigma(k)|H_0|\sigma(k)\rangle\,. \tag{37}$$

Note that if *either* of the above equalities hold, then the permutation of the kth eigenvector conserves the energy expectation value, Eq. (1). Define now the *overlap* κ as the number of indices for which either of the above equalities hold. Contrary to the overlap we define the *discrepancy* δ as the number of indices for which neither equality holds. The sum of κ and δ is the dimension of the system. We further define the parameter P by

$$P = \sum_j p_j. \tag{38}$$

The sum is over the indices j that *do not* satisfy any of Eqs. (36) and (37).

Remark 10. It is important to realize that κ, δ and P are invariant of the choice of permutation σ, assuming that it activates the battery. The reason is that Eqs. (36) and (37) can be interpreted as conditions for when an eigenvector *can* be held fixed without altering the activation of the battery.

We can now formulate two quantum speed limits as propositions.

Proposition 2.4. *For the bounded bandwidth constraint,* $\tau_{\min}$ *is bounded from below by*

$$\tau_{\text{qsl}}^{\text{B}} = \frac{\pi\sqrt{\delta}}{2\omega}. \tag{39}$$

Proposition 2.5. *For the bounded variance constraint,* $\tau_{\min}$ *is bounded from below by*

$$\tau_{\text{qsl}}^{\text{V}} = \frac{\pi\sqrt{P}}{2\omega}. \tag{40}$$

The proofs for these quantum speed limits are postponed to Appendix A.2. We now ask whether there are any special cases where these bounds can be reached. The proposition below gives us our first result.

[4]Contrary to its name, a quantum speed limit does not actually limit the *speed* of a quantum process, but bounds the minimal time required to implement it.

Proposition 2.6. *If the permutation σ can be decomposed into solely transpositions and trivial cycles, then the minimal duration is equal to the quantum speed limit.*

The proof is postponed until Appendix A.2. This first result not only answers the question for multiple more simple cases, but also further proves the usefulness of the cycle decomposition. If it can be decomposed into cycles of length no greater than 2 we can determine the charging time $\tau_{\min}$ without discussing shortest curves. We now present such an example which will become useful later in the thesis.

Example 2. Let ρ_i be a non-degenerate, d-dimensional passive state with non-degenerate observable H_0. Then the ordered spectra are

$$\begin{aligned} \operatorname{spec}\{H_0\} &= (E_1, E_2, \cdots, E_{d-1}, E_d), \\ \operatorname{spec}\{\rho_i\} &= (p_d, p_{d-1}, \cdots, p_2, p_1), \\ \operatorname{spec}\{\rho_a\} &= (p_1, p_2, \cdots, p_{d-1}, p_d). \end{aligned} \tag{41}$$

We find that $A_\sigma \rho_i A_\sigma^\dagger = \rho_a$, where A_σ is characterized by the permutation

$$\sigma = \begin{cases} (1,d)(2,d-1), \cdots, \left(\frac{d}{2}, \frac{d+2}{2}\right) & \text{if } d \text{ is even}, \\ (1,d)(2,d-1), \cdots, \left(\frac{d-1}{2}, \frac{d+3}{2}\right)\left(\frac{d+1}{2}\right) & \text{if } d \text{ is odd}, \end{cases} \tag{42}$$

This permutation decomposition solely consists of transpositions or trivial cycles such that Proposition 2.6 applies, making this an example for when $\tau_{\min}$ is equal to the quantum speed limit. If we assume that the control H_c satisfies the bounded bandwidth constraint we find using Eq. (39) that

$$\tau_{\min} = \begin{cases} \frac{\pi\sqrt{d}}{2\omega} & \text{if } d \text{ is even}, \\ \frac{\pi\sqrt{d-1}}{2\omega} & \text{if } d \text{ is odd}. \end{cases} \tag{43}$$

If instead H_c satisfies the bounded variance constraint, $\tau_{\min}$ is found by applying Eq. (40),

$$\tau_{\min} = \begin{cases} \frac{\pi}{2\omega} & \text{if } d \text{ is even}, \\ \frac{\pi\sqrt{1-p_{(d+1)/2}}}{2\omega} & \text{if } d \text{ is odd}. \end{cases} \tag{44}$$

If Proposition 2.6 is not satisfied, however, the problem immediately becomes more complicated. In those cases we have to solve Eq. (29), which is dependent on the imposed constraint. Hence we proceed by considering them individually.

3 Optimal Time for Bounded Bandwidth

Main Topics of Section

- Formulate length of shortest curves.
- Determine the minimal duration for non-degenerate spectra.
- Develop a method to determine the minimal duration for degenerate spectra and establish when it is applicable.

The metric introduced for the bounded bandwidth implies a certain geodesic distance. In order to formulate it we must first present the consequences of Eq. (25) being bi-invariant. This property will in fact imply the following Proposition.

Proposition 3.1. *Time-optimal control Hamiltonians which satisfy the bounded bandwidth constraint, Eq.* (12)*, are necessarily time-independent.*

The proof of Proposition 3.1 is presented in Appendix A.3. We remind ourselves that we are in the interaction picture, hence the above proposition allows us to simplify Eq. (17) and write $U(t) = \exp(-iH_c t)$. Having made this observation, let Log be the principal logarithm on the unitary group and $\|\cdot\|$ the Hilbert-Schmidt norm. The logarithm of a unitary operator U is the skew-Hermitian ξ for which $U = \exp(\xi)$. Since this is a multi-valued function we choose the principal logarithm which guarantees that the eigenvalues z_i of $\mathrm{Log}(U)$ all satisfy $\mathrm{Im}(z_i) \in [-\pi, \pi)$. We prove in Appendix A.3 the following proposition.

Proposition 3.2. *If we constrain the control bandwidth, the geodesic distance between U and V is given by*

$$\mathrm{dist}(U, V) = \|\mathrm{Log} U^\dagger V\|. \tag{45}$$

This geodesic distance in conjunction with Eq. (29) then shows that the smallest distance to the activating set $\mathcal{A}(\rho_i)$ is

$$\mathrm{dist}(\mathbb{1}, \mathcal{A}(\rho_i)) = \min_{U,V} \|\mathrm{Log}(UAV)\|. \tag{46}$$

This is generally difficult to minimize, however. As we will see Eq. (46) is always possible to evaluate if both H_0 and ρ_i are non-degenerate. Instead, if either is degenerate, it can be determined only for some very special kinds of spectra. This forces us to split the following section into multiple parts, each considering a special set of state and energy spectra.

3.1 Non-Degenerate Cases

Suppose that H_0 and ρ_i are non-degenerate, i.e., the eigenspaces of H_0 and ρ_i are one-dimensional. Then, all operators which commute with them are necessarily diagonal in the chosen basis $|k\rangle$. Hence the isotropy groups $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ are identical and consists of operators represented by diagonal unitary matrices. The eigenvalues of these operators are still arbitrary, however. Let this common isotropy group of diagonal unitaries be written as $\mathcal{U}(\mathcal{H})_{\mathrm{diag}}$.

In the non-degenerate case there only exists a single active state ρ_a. This stems from that if the energy eigenvalues are non-degenerate and indexed in increasing order, $E_k < E_{k+1}$,

then so must the eigenvalues of ρ_a be, $p_k < p_{k+1}$, in order for Eq. (1) to be maximized. As a direct result the permutation σ which activates ρ_i is unique. It is essential to note, however, that while the active state ρ_a is unique the activating unitary is not. Hence we are still required to minimize over the isotropy groups in Eq. (46).

Recall that every cycle c_r represents how A_σ operates on the subspace $\mathcal{H}_{c_r}$ of $\mathcal{H}$. Hence, the permutation operator can always be written as a direct sum of sub-permutations on these subspaces, $A_\sigma = \bigoplus_r A_{c_r}$. Similarly, since the isotropy groups now consist of diagonal unitaries U, V on $\mathcal{H}$, these can be written as a direct sum of operators U_{c_r}, V_{c_r} on $\mathcal{H}_{c_r}$. If we consider the geodesic distance of Eq. (46) this allows us to write

$$\|\mathrm{Log}\left(\bigoplus_{r=1}^{m} U_{c_r} A_{c_r} V_{c_r}\right)\|^2 = \sum_{r=1}^{m} \|\mathrm{Log}(U_{c_r} A_{c_r} V_{c_r})\|^2, \tag{47}$$

where the square is necessary for equality to hold. If we minimize this over U, V the following proposition is proven.

Proposition 3.3. *Let $\sigma = c_1 \cdots c_m$ be our cycle decomposition. If H_c satisfies the bounded bandwidth condition and H_0, ρ_i are non-degenerate, then*

$$\tau_{\min} = \frac{\pi}{\sqrt{3}\omega}\sqrt{d - \sum_{r=1}^{m} \frac{1}{l_r}}, \tag{48}$$

l_r being the length of cycle c_r.

The proof is presented in Appendix A.3. Proposition 3.3 shows that the unique permutation σ fully determines Eq. (46) in the non-degenerate case, hence the minimal charging time $\tau_{\min}$. Another consequence of this observation is that the corresponding quantum speed limit, Eq. (39), can be reached for non-degenerate cases if and *only* if each $l_r \leq 2$. If such is the case Eqs. (39) and (48) are equal, and if not Eq. (48) must be larger. We will now conclude the non-degenerate cases by presenting two examples where we apply Proposition 3.3.

Example 3. Consider the $d = 3$ dimensional case where we have the spectra

$$\begin{aligned} \mathrm{spec}\{H_0\} &= (E_1, E_2, E_3), \\ \mathrm{spec}\{\rho_i\} &= (p_2, p_1, p_3), \\ \mathrm{spec}\{\rho_a\} &= (p_1, p_2, p_3). \end{aligned} \tag{49}$$

In this case we can activate the battery with the (fully reduced) permutation $\sigma = (1, 2)(3)$. Since both ρ_i and H_0 are non-degenerate we can apply Eq. (48) and find

$$\tau_{\min} = \frac{\pi}{\sqrt{3}\omega}\sqrt{3 - \frac{1}{2} - 1} = \frac{\pi}{\sqrt{2}\omega}. \tag{50}$$

This result coincides with the quantum speed limit Eq. (39) given that $\delta = 2$. This was in turn was predicted by Proposition 2.6 since σ can be decomposed into transpositions and trivial cycles.

Example 4. Consider the same dimension and spectra as in Example 3, the only difference being the ordering of the initial battery spectrum,

$$\begin{aligned} \mathrm{spec}\{H_0\} &= (E_1, E_2, E_3), \\ \mathrm{spec}\{\rho_i\} &= (p_3, p_1, p_2), \\ \mathrm{spec}\{\rho_a\} &= (p_1, p_2, p_3). \end{aligned} \tag{51}$$

In this case $\sigma = (3, 2, 1)$, and by using Eq. (48) we find

$$\tau_{\min} = \frac{\pi}{\sqrt{3}\omega}\sqrt{3 - \frac{1}{3}} = \frac{\pi}{\omega}\frac{\sqrt{8}}{3}. \tag{52}$$

This is larger than the quantum speed limit Eq. (39) given that $\delta = 3$, which reinforces our statement that $\tau_{\min} = \tau^{\mathrm{B}}_{\mathrm{qsl}}$ *if and only if* the length $l_r \leq 2$ for each cycle c_r in σ.

3.2 Degenerate Cases

When any or both of H_0 and ρ_i have degenerate spectra the isotropy groups $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ consist of more arbitrary unitaries compared to the non-degenerate cases, and the permutation is no longer unique. As a result, Eq. (48) need no longer hold.

But the right-hand side of Eq. (48) has another role. Given a certain permuting operator A_σ, it always acts as an upper bound on the minimal duration $\tau_{\min}$, even for arbitrary degeneracies. This is a consequence of the fact that the isotropy group $\mathcal{U}(\mathcal{H})_{\mathrm{diag}}$ of the non-degenerate case is a subgroup of all possible general isotropy groups $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$. Consequently, by choosing U and V in Eq. (46) from $\mathcal{U}(\mathcal{H})_{\mathrm{diag}}$, the process duration τ can be made equal to the right-hand side of Eq. (48). However, since we restricted ourselves to $\mathcal{U}(\mathcal{H})_{\mathrm{diag}}$, we do not know if there are other unitaries in $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ which decrease the geodesic distance further. Thus,

$$\tau_{\min} \leq \frac{\pi}{\sqrt{3}\omega}\sqrt{d - \sum_{r=1}^{m}\frac{1}{l_r}}. \tag{53}$$

Notice that this bound is minimal if the permutation σ is fully reduced. This is the case since if any cycle of length l_r is replaced by two shorter cycles of lengths $l_{r'}$ and $l_{r''}$ the sum above increases and, consequently, the right-hand side of Eq. (53) decreases.

Generally, we cannot guarantee that Eq. (53) is saturated. In fact, we have not developed a general formula for $\tau_{\min}$ for degenerate cases. As we will show, however, there exists special sets of degenerate spectra where $\tau_{\min}$ can be determined. In the following section we present two main assumptions which, if upheld, allows a method that determines $\tau_{\min}$ for some degenerate cases.

3.2.1 The Decomposition Method

The decomposition method is a procedure which will allow us to divide the problem of the degenerate case into smaller parts. These parts will then be possible to examine and calculate independently. We will find that if each part has any of three mathematical characteristics, then $\tau_{\min}$ can be determined. This method is only applicable if we impose two assumptions on H_0 and ρ_i, however.

Consider the permutation σ and its cycle decomposition $c_1 c_2 \cdots c_m$. The cycles can be arbitrarily grouped together into $n \leq m$ sub-permutations σ_l, such that

$$\sigma = \underbrace{c_1 \cdots c_i}_{\sigma_1}\underbrace{c_{i+1} \cdots c_j}_{\sigma_2} \cdots \underbrace{c_k \cdots c_m}_{\sigma_n} = \sigma_1 \sigma_2 \cdots \sigma_n. \tag{54}$$

Each sub-permutation σ_l operates on a Hilbert subspace $\mathcal{H}_{\sigma_l}$. These $\mathcal{H}_{\sigma_l}$ are spanned by the eigenvectors whose indices are permuted by the corresponding σ_l. This allows us to present the following proposition which is tied to our first required assumption.

Proposition 3.4. *If each eigenspace of H_0 and ρ_i is contained within a $\mathcal{H}_{\sigma_l}$, then these $\mathcal{H}_{\sigma_l}$ are all invariant under action of unitaries in the activating set $\mathcal{A}(\rho_i)$. Furthermore, there exists a shortest curve from $\mathbb{1}$ to $\mathcal{A}(\rho_i)$ which preserves each $\mathcal{H}_{\sigma_l}$.*

The proof is provided in Appendix A.3. The first assumption we make is that the premises of this proposition are satisfied, i.e., that each eigenspace of H_0 and ρ_i is contained in a $\mathcal{H}_{\sigma_l}$. This assumption implies that the elements of the isotropy groups $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ can be decomposed into direct sums of unitary operators U_{σ_l} and V_{σ_l} on $\mathcal{H}_{\sigma_l}$. Simultaneously, A_σ can be written as a direct sum of the permutations A_{σ_l} on $\mathcal{H}_{\sigma_l}$. This is reminiscent of the non-degenerate case, and similarly to Eq. (47) we find

$$\mathrm{dist}(\mathbb{1}, \mathcal{A}(\rho_i))^2 = \sum_{l=1}^{n} \min_{U_{\sigma_l}, V_{\sigma_l}} \|\mathrm{Log}(U_{\sigma_l} A_{\sigma_l} V_{\sigma_l})\|^2. \tag{55}$$

Hence the first assumption has allowed us to split the problem of minimizing over U and V into multiple parts.

We still have no simple method of minimizing each term, however. To this end we require a second assumption followed by some well-known geometrical results. Let $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ be the group of unitaries on $\mathcal{H}_{\sigma_l}$ which commute with the projection of H_0 onto $\mathcal{H}_{\sigma_l}$. Similarly define $\mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$. We can now propose a second proposition, whose proof we postpone until Appendix A.3.

Proposition 3.5. *If $\mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$ is a subgroup of $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ for every l, then Eq. (55) can be rewritten as*

$$\mathrm{dist}(\mathbb{1}, \mathcal{A}(\rho_i)) = \sqrt{\sum_{l=1}^{n} \min_{U_{\sigma_l}} \|\mathrm{Log}(A_{\sigma_l} U_{\sigma_l})\|^2}, \tag{56}$$

where we only minimize over all U_{σ_l} in the larger group $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$. Similarly, if $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0} \subseteq \mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$, we have

$$\mathrm{dist}(\mathbb{1}, \mathcal{A}(\rho_i)) = \sqrt{\sum_{l=1}^{n} \min_{V_{\sigma_l}} \|\mathrm{Log}(A_{\sigma_l} V_{\sigma_l})\|^2}. \tag{57}$$

This proposition leads to our second assumption, namely that $\mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i} \subseteq \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ and, consequently, that Eq. (56) applies. While a seemingly restrictive assumption at first, this implies that each eigenspace of ρ_i is contained within an eigenspace of H_0. This means that when we prepare ρ_i through a measurement, the probabilities of measuring two non-equal energy eigenvalues are necessarily different. This is a special case, but one of importance.

Interestingly, each individual term in the sum of Eq. (56) is in fact a well-known geodesic distance of a Riemannian metric on a certain kind of manifold characterized by the corresponding $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$. These manifolds are referred to in mathematics as *flag manifolds* [7]. Hence, if the two mentioned assumptions hold we can decompose the squared total geodesic distance of Eq. (45) into a sum of geodesic distances, each dependent on the characteristics of $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$. This is the decomposition method.

3.2.2 Geodesic Distances on Flag Manifolds

Let $\mathcal{U}(n)$ be the group of n-dimensional unitary matrices in the chosen basis Eq. (30) and let d_l be the dimension of $\mathcal{H}_{\sigma_l}$ such that $\mathcal{U}(\mathcal{H}_{\sigma_l}) = \mathcal{U}(d_l)$. For every pair W_1 and W_2 in

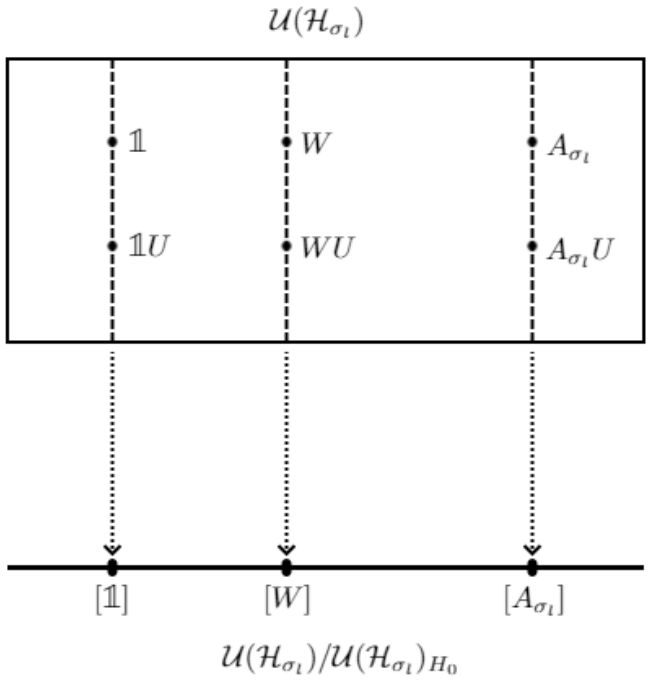


Figure 5: A visualization of how the equivalence relations "groups together" various subsets of $\mathcal{U}(\mathcal{H}_{\sigma_l})$. Each dashed line represents a subset which is gained by considering a single element in $\mathcal{U}(\mathcal{H}_{\sigma_l})$ and multiplying it from the right by an operator in $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$. Each such dashed line is then "projected" onto a space where each subset is considered a single element.

$\mathcal{U}(d_l)$ we now define the equivalence relation

$$W_1 \sim W_2 \Leftrightarrow W_2^\dagger W_1 \in \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}. \tag{58}$$

We denote the equivalence class of a W in $\mathcal{U}(d_l)$ by $[W]$, such that

$$[W] = \{WU : U \in \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}\}. \tag{59}$$

An image which visualizes this is given in Fig. 5.

With the equivalence relation Eq. (58), all unitaries of the form $A_{\sigma_l}U_{\sigma_l}$ in Eq. (56) belong to the same equivalence class $[A_{\sigma_l}]$. In fact, Eq. (58) partitions $\mathcal{U}(\mathcal{H}_{\sigma_l})$ into subsets which are invariant under the right action of the isotropy group $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$, and each such subset is represented by an equivalence class. As it happens, each class $[W]$ can be considered as an element of a *quotient manifold* denoted by $\mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ [6, 7].[5] Furthermore, if we let $\mathcal{H}_1^l, \mathcal{H}_2^l, \cdots, \mathcal{H}_{m_l}^l$ denote the m_l energy eigenspaces contained in $\mathcal{H}_{\sigma_l}$ we can write $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ as a product of the m unitary groups $\mathcal{U}(\mathcal{H}_k^l)$. For conveniences sake let d_l be the dimension of $\mathcal{H}_{\sigma_l}$ and d_l^k be the dimensions of the respective $\mathcal{H}_k^l$. If we consider the chosen basis Eq. (30) we can then let $\mathcal{U}(n)$ denote the group of n-dimensional unitary matrices and, consequently, the quotient manifold can be written in the form

$$\mathcal{U}(d_l)/\big(\mathcal{U}(d_1^l) \times \mathcal{U}(d_2^l) \times \cdots \times \mathcal{U}(d_{m_l}^l)\big)\,. \tag{60}$$

[5] That it is a quotient *manifold* and not a quotient *space* stems from that $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H}_{\sigma_l})$. In fact, $\mathcal{U}(d_l)/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ is also a Lie group [6].

Since the sum of all d_k^l equals d_l, this quotient space has the structure of what in mathematics is referred to as a *flag manifold* [7].

Proposition 3.6. *If we equip $\mathcal{U}(d_l)$ with the bi-invariant metric which agree with the Hilbert-Schmidt inner product on the Lie algebra of $\mathcal{U}(d_l)$, then the geodesic distance between the elements $[\mathbb{1}]$ and $[A_{\sigma_l}]$ in the flag manifold Eq.* (60) *is given by*

$$\text{dist}\Big([\mathbb{1}],[A_{\sigma_l}]\Big) = \min_{U_{\sigma_l}} \|\text{Log}(A_{\sigma_l}U_{\sigma_l})\|. \tag{61}$$

The proof is postponed until Appendix A.3. Proposition 3.6 allows us to express Eq. (56) in terms of geodesic distances on flag manifolds;

$$\text{dist}(\mathbb{1},\mathcal{A}(\rho_i)) = \sqrt{\sum_{l=1}^{n} \text{dist}\Big([\mathbb{1}],[A_{\sigma_l}]\Big)^2}. \tag{62}$$

For some special examples of flag manifolds the geodesic distance is known, in which case each individual term in Eq. (62) can be determined.

First consider the case where each $\mathcal{H}_{\sigma_l}$ only contains two energy eigenspaces $\mathcal{H}_1^l$ and $\mathcal{H}_2^l$. In this case the flag manifold is of the form

$$\mathcal{U}(d_l)/\big(\mathcal{U}(d_1^l)\times\mathcal{U}(d_l-d_1^l)\big)\,. \tag{63}$$

Flag manifolds with this structure are also referred to as *Grassmannians* or *Grassmann manifolds* and are very important in certain fields of research, e.g. quantum computing [22]. If we let Π_1 be the orthogonal projection of $\mathcal{H}_{\sigma_l}$ onto $\mathcal{H}_1^l$, and $s_1, s_2, \cdots, s_{d_1^l}$ be the singular values of $\Pi_1^\dagger A_{\sigma_l}\Pi_1$, then we have

$$\text{dist}\Big([\mathbb{1}],[A_{\sigma_l}]\Big) = \sqrt{2\sum_{i=1}^{d_1^l}(\arccos(\sqrt{s_i}))^2}. \tag{64}$$

The derivation of this distance is quite complicated. For details we refer to [23]. Note that since A_{σ_l} is a permutation matrix, the singular values of the projection $\Pi_1^\dagger A_{\sigma_l}\Pi_1$ onto $\mathcal{H}_1^l$ are either 0 or 1. Furthermore the number of 1s equals the number of trivial cycles contained in $\mathcal{H}_1^l$, which we refer to as the partial overlap κ_l. From Eq. (64) we find that

$$\text{dist}\Big([\mathbb{1}],[A_{\sigma_l}]\Big) = \frac{\pi\sqrt{2(d_1^l-\kappa_l)}}{2}. \tag{65}$$

Remark 11. While Eq. (65) is sufficient for energy eigenspaces of arbitrary dimensions, it might be interesting for the reader to comment on the special case where one of them is one dimensional, e.g. $\mathcal{H}_1^l$. Then the flag manifold has the structure

$$\mathcal{U}(d_l)/(\mathcal{U}(d_l-1)\times\mathcal{U}(1)) \tag{66}$$

which is referred to as a *projective space*. In this case κ_l can only be equal to d_1^l or d_1^l-1 and Eq. (65) reduces to

$$\text{dist}\Big([\mathbb{1}],[A_{\sigma_l}]\Big) = \begin{cases} \frac{\pi}{\sqrt{2}}, & \kappa_l = d_1^l - 1. \\ 0 & \kappa_l = d_1^l. \end{cases} \tag{67}$$

Readers familiar with quantum information might notice that this is the Fubini-Study distance on projective Hilbert space between two orthogonal eigenvectors. This connection is no happenstance. In fact, there exists a so called *diffeomorphism* between the projective space Eq. (66) and projective Hilbert space. What this means and further details on this subject can be found in [6, 7].

Consider the case where each energy eigenspace contained in $\mathcal{H}_{\sigma_l}$ has dimension 1. We have

$$\mathcal{U}(d_l)/(\mathcal{U}(1)\times\mathcal{U}(1)\times\cdots\times\mathcal{U}(1)) \tag{68}$$

where $\mathcal{U}(1)$ is repeated d_l times. This is referred to as a *full flag manifold*. Let $l_1, l_2, \cdots, l_m$ be the lengths of the m cycles contained in σ_l, then

$$\mathrm{dist}\Big([\mathbb{1}], [A_{\sigma_l}]\Big) = \sqrt{\frac{\pi^2}{3}\sum_{i=1}^{m}\frac{l_i^2-1}{l_i}}. \tag{69}$$

The proof is highly analogous to that of Proposition 3.3 in Appendix A.3, the only difference being that we exchange the permutation σ for the sub-permutation σ_l. This is a natural case to consider, since it occurs when we have some "partial non-degeneracy". If all $\mathcal{H}_{\sigma_l}$ has this structure then the total state is non-degenerate, and through Eq. (62) we regain Eq. (48).

Remark 12. We could instead make the opposite assumption that $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0} \subseteq \mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$ and apply Eq. (57). The results are fully analogous to the ones above with the exception that we'd have to examine the state eigenvalues when determining which flag manifolds each quotient space on the various $\mathcal{H}_{\sigma_l}$ are.

For completely general spectra some $\mathcal{H}_{\sigma_l}$ might not be of any of the three types described above, i.e. it contains more than two eigenspaces of dimension larger than 1. In this case the flag manifold is referred to as a *generalized flag manifold* and the geodesic distance is currently unknown[6]. Hence, the decomposition method does not always work. However it adds a large number of spectra to our list of solvable cases. We have to first examine whether the eigenspaces of ρ_i are contained in those of H_0, and then if those are contained in a collection of cycles, i.e. a sub-permutation. The method works if the eigenspaces of H_0 has the appropriate dimensions in their respective sub-permutation. We will now consider some examples where the decomposition method can be applied.

Example 5. Let the spectrum of H_0 and ρ_i be

$$\begin{aligned}
\mathrm{spec}\{H_0\} &= (E_1, E_1, E_2, E_2, E_3, E_4, E_5, E_6, E_6, E_7)\\
\mathrm{spec}\{\rho_i\} &= (p_3, p_4, p_2, p_1, p_6, p_7, p_5, p_9, p_{10}, p_8)\\
\mathrm{spec}\{\rho_a\} &= (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10})
\end{aligned} \tag{70}$$

If we assume that ρ_i is non-degenerate, i.e. $p_k \neq p_j$ for all $k \neq j$, then the maximally reduced permutation for which $A_\sigma \rho_i A_\sigma^\dagger = \rho_a$ is

$$\sigma = (1,3)(2,4)(5,6,7)(8,10)(9). \tag{71}$$

[6] A numerical method of determining the geodesic distance on generalized flag manifolds is proposed in [24].

We now want to choose sub-permutations such that Proposition 3.4 is satisfied, i.e., each eigenspace of H_0 and ρ_i is contained within a σ_l. This is guaranteed by the choices

$$\begin{aligned}\sigma_1 &= (1,3)(2,4),\\ \sigma_2 &= (5,6,7),\\ \sigma_3 &= (8,10)(9).\end{aligned} \tag{72}$$

Furthermore Proposition 3.5 is trivially satisfied since ρ_i is non-degenerate, and we can apply the decomposition method. We will now independently examine the Hilbert spaces $\mathcal{H}_{\sigma_l}$ belonging to each σ_l. For $\mathcal{H}_{\sigma_1}$ the quotient manifold is a Grassmann manifold since it contains two energy eigenspaces, therefore Eq. (65) gives us

$$\mathrm{dist}\Big([\mathbb{1}],[A_{\sigma_1}]\Big) = \pi. \tag{73}$$

Subspace $\mathcal{H}_{\sigma_2}$ contains three energy eigenspaces, however since each is one-dimensional the quotient is a full flag manifold. From Eq. (69) we find

$$\mathrm{dist}\Big([\mathbb{1}],[A_{\sigma_2}]\Big) = \frac{\sqrt{8}\pi}{3}. \tag{74}$$

Finally $\mathcal{H}_{\sigma_3}$ is a Grassmann manifold since it contains two eigenspaces. However in this case one of them is one-dimensional, and due to Remark 11 we can apply Eq. (66) and find

$$\mathrm{dist}\Big([\mathbb{1}],[A_{\sigma_3}]\Big) = \frac{\pi}{\sqrt{2}}. \tag{75}$$

Consequently, according to Eq. (62) we find

$$\tau_{\min} = \frac{1}{\omega}\mathrm{dist}(\mathbb{1},\mathcal{A}(\rho_i)) = \frac{1}{\omega}\sqrt{\pi^2+\frac{8\pi^2}{9}+\frac{\pi^2}{2}} = \sqrt{\frac{43}{18}}\frac{\pi}{\omega}. \tag{76}$$

This method concludes the extent to which we can determine the minimal duration $\tau_{\min}$, given that the control H_c obeys the bounded bandwidth constraint Eq. (11).

4 The Geometry of Bounded Variance

Main Topics of Section

- Clarify the theory which supports the chosen metric.
- Provide current results in terms of properties of time-optimal controls.

When considering the bounded variance cases the problem is vastly different than that of Section 3. While subtle, this is a direct consequence of that the metric Eq. (26) is not bi-invariant. For example we can no longer guarantee that time-optimal controls are time-independent, since Proposition 3.1 requires bi-invariance of the metric. Hence we cannot produce a geodesic distance function similar to the one in Proposition 3.2.

While this complication makes us currently unable to determine minimal durations of charging processes, we can instead discuss properties of time-optimal controls. This section will be dedicated to describing the geometry of the problem in more detail, which will prove that the inner product Eq. (26) truly guarantees that the speed of time-optimal curves equal the variance constraint Eq. (11). We will also be able to show that some processes require time-dependent controls in order to be time-optimal.

4.1 Projections and Horizontal Lifts

We began Section 2 by shifting focus and "lifting" time-optimal $\rho(t)$ to curves $U(t)$ in the unitary group $\mathcal{U}(\mathcal{H})$. We did this since we had convenient methods of equipping $\mathcal{U}(\mathcal{H})$ with suitable metrics, transforming the problem into a path optimization problem. We now ask ourselves if, given the convenient metrics on $\mathcal{U}(\mathcal{H})$, we can "go back" to the state space $\mathcal{D}(\mathbf{p})$ and consider activation of ρ_i as a path optimization problem there instead. In order to do this let us define a map π from the unitary group $\mathcal{U}(\mathcal{H})$ to $\mathcal{D}(\mathbf{p})$ by

$$\pi : U \mapsto U\rho_i U^\dagger. \tag{77}$$

Note that if U is multiplied from the right by an operator V in the isotropy group $\mathcal{U}(\mathcal{H})_{\rho_i}$ we find

$$UV \mapsto UV\rho_i V^\dagger U^\dagger = U\rho_i U^\dagger. \tag{78}$$

Hence the map π is a many-to-one map, and the pre-image of each element in $\mathcal{D}(\mathbf{p})$ is a copy of $\mathcal{U}(\mathcal{H})_{\rho_i}$ in $\mathcal{U}(\mathcal{H})$. Henceforth π will be called the *projection* of $\mathcal{U}(\mathcal{H})$ onto $\mathcal{D}(\mathbf{p})$. Furthermore we will borrow some terminology from the mathematical theory of *fibre bundles*, [6], and refer to the the copy of $\mathcal{U}(\mathcal{H})_{\rho_i}$ which is mapped onto $\rho = U\rho_i U^\dagger$ through Eq. (77) as the *fibre* over ρ. This is visualized in Fig. 6.

In order to see how $U(t)$ is projected onto $\rho(t)$ we now take a look at how tangent vectors are mapped given Eq. (77). We recall the theory in Section 2.2 and denote the differential of π by $d\pi$. Let $U(t)$ be generated by a skew-Hermitian $\xi(t)$ in the Lie algebra $\mathfrak{u}(\mathcal{H})$, that is

$$U(t) = \mathcal{T}\left(\int_0^t dt' \xi(t')\right). \tag{79}$$

Then we find

$$d\pi : \dot{U}(t) = \xi(t)U(t) \mapsto \dot{\rho}(t) = [\xi(t), \rho(t)]. \tag{80}$$

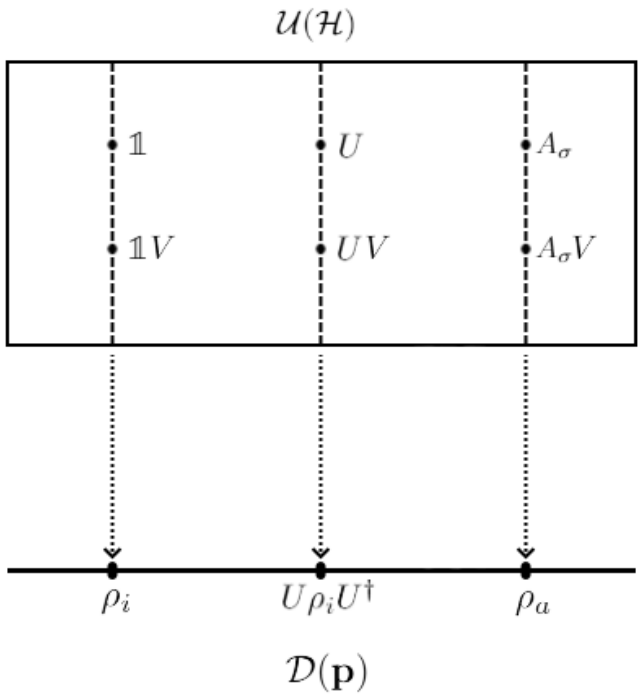


Figure 6: A visualization of the projection π Eq. (77) from the unitary group $\mathcal{U}(\mathcal{H})$ to the state space $\mathcal{D}(\mathbf{p})$. Each dashed line in $\mathcal{U}(\mathcal{H})$ denotes a copy of $\mathcal{U}(\mathcal{H})_{\rho_i}$.

We now make a crucial geometrical argument whose support is presented in Appendix A.4: Given a metric g on $\mathcal{U}(\mathcal{H})$ every $\dot{U}(t)$ can be uniquely decomposed into a sum of orthogonal components

$$\dot{U}(t) = \xi^v(t)U(t) + \xi^h(t)U(t) = \xi(t)U(t), \tag{81}$$

such that $\xi^v(t)$ commutes with $\rho(t)$. By orthogonality we mean that

$$g_{\text{Var}}\left(\xi^v(t)U(t), \xi^h(t)U(t)\right) = 0. \tag{82}$$

We refer to $\xi^v(t)$ and $\xi^h(t)$ as the *vertical* and *horizontal* components of $\xi(t)$. The following proposition is a consequence.

Proposition 4.1. *The component $\xi^v(t)U(t)$ belongs to the kernel of $d\pi$ at $U(t)$:*

$$d\pi\left(\xi^v(t)U(t)\right) = 0. \tag{83}$$

Due to Proposition 4.1 there exists multiple different $U(t)$ emanating from identity which all project onto the same $\rho(t)$, on account of that these $U(t)$ are generated by skew-Hermitians with various vertical components. One of these unitary curves has no vertical component and is fully generated by $\xi^h(t)$, and we refer to this unique[7] unitary curve $U^h(t)$ as the *horizontal lift* of $\rho(t)$, see Fig. 7. We further refer to a $\xi(t)$ as *parallel transporting* if it only consists of a horizontal component.

In order to assert lengths of the tangents of $\rho(t)$, Eq. (83), we now want to equip the state space $\mathcal{D}(\mathbf{p})$ with a metric. In our context there exists a good way of doing this, namely

[7]This horizontal lift is in truth unique only with the assumption that $U(t)$ emanates from identity. Recall that we made this choice in Section 2.1.2.

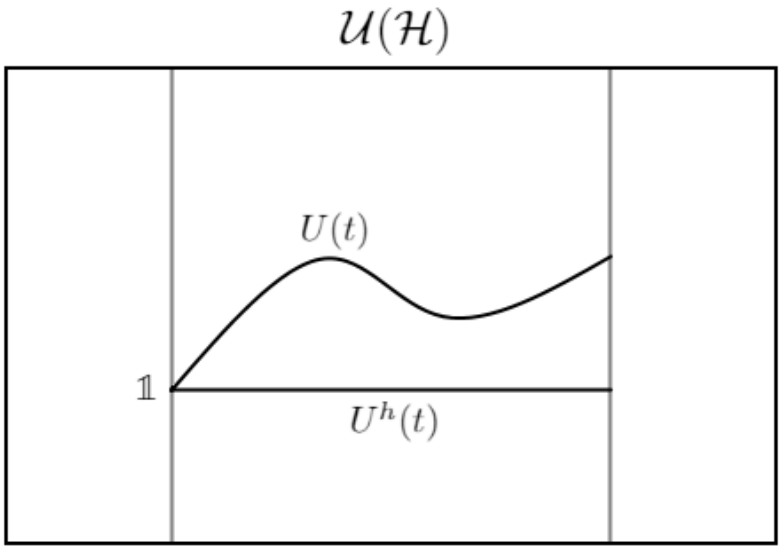


Figure 7: Two curves in the unitary group $\mathcal{U}(\mathcal{H})$, emanating from identity and reaching the same fibre (grey line). $U(t)$ is generated by a skew-Hermitian with both vertical and horizontal components. $U^h(t)$ has no vertical component and is the horizontal lift of $\rho(t)$.

such that the map of the horizontal lift $U^h(t)$ onto $\rho(t)$ is an isometry. In other words, if we equip $\mathcal{U}(\mathcal{H})$ and $\mathcal{D}(\mathbf{p})$ with the metrics g_{Var} and $g_{\mathcal{D}}$, respectively, then

$$g_{\mathcal{D}}\Big(\dot{\rho}(t), \dot{\rho}(t)\Big) = g_{\text{Var}}\Big(\dot{U}^h(t), \dot{U}^h(t)\Big). \tag{84}$$

In this way we can use an appropriate metric on $\mathcal{U}(\mathcal{H})$ to infer a metric on the state space. This specific way of inferring metrics through an isometry with the horizontal lift is referred to as a *Riemannian submersion* [25], and becomes useful since we already equipped the unitary group with the metric g_{Var}, Eq. (26). We remind ourselves of the assumption made in Remark 4 of Section 1.4.2, namely that ρ_i is of full rank.

4.2 Time-Optimal Controls for Bounded Variance

In the previous section we presented a way of perceiving the geometry of the problem by projecting unitary curves onto a $\rho(t)$ in the state space. We then defined one of these unitary curves as the horizontal lift of $\rho(t)$ and inferred a metric on $\mathcal{D}(\mathbf{p})$ through an isometry with the horizontal lift. We will now make use of this in an effort to determine some properties of time-optimal controls. In tandem with this we will be able to prove that the currently utilized inner product, Eq. (26), truly respects the bounded variance constraint.

4.2.1 Almost Parallel Transporting Controls

To calculate the speed squared of $\rho(t)$ we now calculate Eq. (84) given that the projected unitary curve $U(t)$ is generated by the control $H_c(t) = H_c^v(t) + H_c^h(t)$. Here $H_c^v(t)$ and $H_c^h(t)$ are the control components corresponding to how we decomposed $\xi(t)$,

$$H_c^v(t) = -i\xi^v(t), \quad H_c^h(t) = -i\xi^h(t). \tag{85}$$

Let $\Pi_j(t)$ be the orthogonal projection of $\mathcal{H}$ onto the jth eigenspace of $\rho(t)$ and assume that it has m distinct eigenvalues, then

$$H_c^v(t) = \sum_{j=1}^{m} \Pi_j(t) H_c(t) \Pi_j(t). \tag{86}$$

In the ordered eigenbasis of $\rho(t)$ these are represented by block diagonal matrices. We consequently find the horizontal component $H_c^h(t)$ as $H_c(t) - H_c^v(t)$, which in turn is block *off*-diagonal in the same basis. The following proposition follows from Eqs. (26), (84) and (86).

Proposition 4.2. *The squared speed of $\rho(t)$ and its horizontal lift $U^h(t)$ is bounded from above such that*

$$g_{\mathcal{D}}(\dot{\rho}(t), \dot{\rho}(t)) \leq \mathrm{Var}(H_c(t), \rho(t)). \tag{87}$$

Furthermore, this bound is saturated if and only if $H_c^v(t) = E(t)\mathbb{1}$, $E(t)$ being a possibly time-dependent scalar.

We prove this in Appendix A.4. Since we in Proposition 4.1 already proved that the dynamics are invariant of the vertical component $H_c^v(t)$, any such choice will not alter the curves $\rho(t)$. Since optimal time is achieved if the speed along a curve is maximal this proves that time-optimal controls are of the form $H_c(t) = E(t)\mathbb{1} + H_c^h(t)$. If on this form we say that the control is *almost parallel transporting*, and they become parallel transporting if and only if $E(t) = 0$ for all t. This is a large restriction on available optimal controls, and one should keep this in mind when looking for optimal durations in the bounded variance case.

Remark 13. The same theory applied above can be done in the bounded bandwidth case, given that we replace the metric g_{Var} with the metric g_{HS}, Eq. (25). In this case we find that time-optimal controls are always parallel transporting, i.e., the vertical component is always the zero operator. These differences are ultimately due to the bi-invariance of g_{HS}.

4.2.2 A Geodesic Equation

We are interested in shortest curves in the unitary group, given that they are traversed with constant speed. In other words we are interested in shortest geodesics in the unitary group, hence they should satisfy some geodesic equation.

Let $\mathcal{B}(\mathcal{H})$ be the space of all linear operators on $\mathcal{H}$. The unitary group $\mathcal{U}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$. We extend the metric g_{Var} to the surrounding space $\mathcal{B}(\mathcal{H})$ such that, for any vectors X and Y in $\mathcal{B}(\mathcal{H})$,

$$g_{\mathcal{B}}(X, Y) = \frac{1}{2} \mathrm{Tr}\left[\rho_i \left(X^\dagger Y + Y^\dagger X\right)\right]. \tag{88}$$

The curve $U(t)$ can be regarded as a curve in $\mathcal{B}(\mathcal{H})$, and its acceleration in this surrounding space is the second temporal derivative, $\ddot{U}(t)$. In order for $U(t)$ to be a geodesic in $\mathcal{U}(\mathcal{H})$ the acceleration $\ddot{U}(t)$ must belong to the kernel of the orthogonal projection onto $T_{U(t)}\mathcal{U}(\mathcal{H})$, the tangent space of $\mathcal{U}(\mathcal{H})$ at $U(t)$. Equivalently, $\ddot{U}(t)$ must be perpendicular to $T_{U(t)}\mathcal{U}(\mathcal{H})$. This is a main result in differential geometry, and for more details see [26]. With the help of the metric g_{Var} we can determine when these are perpendicular,

and we formulate the corresponding geodesic equation as a proposition which we prove in Appendix A.4.

Proposition 4.3. *A unitary curve $U(t)$ is a geodesic with respects to the metric g_{Var} if it satisfies the geodesic equation*

$$i\left\{\dot{H}_c(t), \rho(t)\right\} + \left[H_c^2(t), \rho(t)\right] = 0. \tag{89}$$

Remark 14. With similar theory we can determine the geodesic equation for the bounded bandwidth case. And by performing the same calculation which proves Proposition 4.3 except with the metric g_{HS} we find the geodesic equation $\dot{H}_c(t) = 0$, which is the time-independency of optimal controls which was stated in Proposition 3.1. Once again this difference is a consequence of bi-invariance.

The usefulness of the geodesic equation Eq. (89) is immediate if we consider the following example, Example 6, where we will see that there exists cases where time-independent controls H_c cannot charge the battery in optimal time.

Example 6. Consider the non-degenerate 3-dimensional battery with spectra

$$\begin{aligned} \text{spec}\{H_0\} &= (E_1, E_2, E_3), \\ \text{spec}\{\rho_i\} &= (p_3, p_1, p_2), \\ \text{spec}\{\rho_a\} &= (p_1, p_2, p_3). \end{aligned} \tag{90}$$

We want to find a time-optimal control H_c which performs this activation. If we further assume that H_c is time-independent, then by applying Proposition 4.3 we find that it must satisfy

$$i\left\{\dot{H}_c, \rho(t)\right\} + \left[H_c^2, \rho(t)\right] = \left[{H_c^h}^2, \rho_i\right] = 0. \tag{91}$$

Note that we have made the choice $H_c^v = 0$, which is possible due to Proposition 4.1. Hence we are looking for a H_c which is off-diagonal in the basis of ρ_i, and whose square is diagonal. In this basis we have

$$H_c = \alpha \left|2\right\rangle \left\langle 1\right| + \beta \left|3\right\rangle \left\langle 1\right| + \gamma \left|3\right\rangle \left\langle 2\right| + \text{h.c.}, \tag{92}$$

where h.c. denotes the Hermitian conjugate. In order for H_c^2 to be diagonal we require that either two out of α, β and γ are zero. If we assume that $\beta = \gamma = 0$, then the unitary generated by H_c is given by

$$U = \exp(-itH_c) = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} \left(\alpha \left|2\right\rangle \left\langle 1\right| + \alpha^* \left|1\right\rangle \left\langle 2\right|\right). \tag{93}$$

Such a unitary can only interchange the first and second eigenvalue of ρ_i in some duration τ. The result is similar for non-zero β or γ. Consequently a time-optimal, time-independent control H_c cannot implement an activating unitary.

As seen above the properties of time-optimal controls declared in Proposition 4.1 and 4.3 have helped us show that we cannot solve the bounded variance case in such a a straight forward manner as we could in the bounded bandwidth case. These differences are consequences of the bi-invariance of the Hilbert-Schmidt metric, which is a subtle, mathematical detail. The main take-away of this section is thus to keep such things in mind when examining different constraints, and ask whether the corresponding metric is left, right or bi-invariant.

5 Charging an Ensemble of Batteries

Main Topics of Section

- Charging multiple batteries simultaneously, allowing correlations during the process.
- Defining a quantum advantage which quantifies the decrease of minimal charging duration as a consequence of allowing correlations.
- Calculating the quantum advantage for some simple examples and remarking on results.

So far we have considered the process of charging a single battery. While we defined the battery as a quantum system, the problem has had the characteristics of a path optimization problem in the unitary group, and unique quantum properties have not made a distinct appearance. The role of such properties become very important in the situation where we have multiple batteries which we want to charge simultaneously. The problem is then converted into that of a multi-partite quantum state, where quantum effects such as entanglement gets a predominant role. The implications of quantum entanglement and similar effects are some of the main things which discerns quantum mechanics from its classical counterpart, hence they are widely examined in modern research. However we want to remind the reader that entanglement is not the only form of quantum correlation. Quantum correlation is the umbrella term for general, non-local properties of quantum systems. How to discern various quantum correlations from others in the context of mixed states is still under discussion [27, 28].

The question we would like to ask now is; if we consider multiple identical quantum batteries and allow them to correlate, how is the minimal duration of the charging process for each individual battery affected? As we will find, results seem to indicate that the minimal duration decreases if we allow correlation between batteries. Multiple papers have considered various advantages of making use of quantum correlations [3, 29], and we borrow their terminology by expressing a decrease in the minimal duration in terms of a *quantum advantage*.

5.1 Multi-Battery States

In Fig. 8 we present an image which encompasses everything in this and the subsequent section, and we encourage the reader to consult it when reading. Consider an ensemble of N individual quantum batteries of dimension d, all which may or may not be correlated. If they are, then they may interchange information, hence they do not individually evolve according to Eqs. (4) and (5). This makes the former results non-applicable for each individual battery in the presence of correlations. To maneuver this issue let $\rho_i^{(N)}$ be a multi-partite system consisting of N identically prepared batteries ρ_i. The process of charging each battery can then be represented by evolving $\rho_i^{(N)}$ to another multi-partite state $\rho_f^{(N)}$ consisting of N activated batteries ρ_a. We now make a powerful assumption by writing

$$\rho_i^{(N)} = \bigotimes_{j=1}^{N} \rho_i, \quad \rho_f^{(N)} = \bigotimes_{j=1}^{N} \rho_a. \tag{94}$$

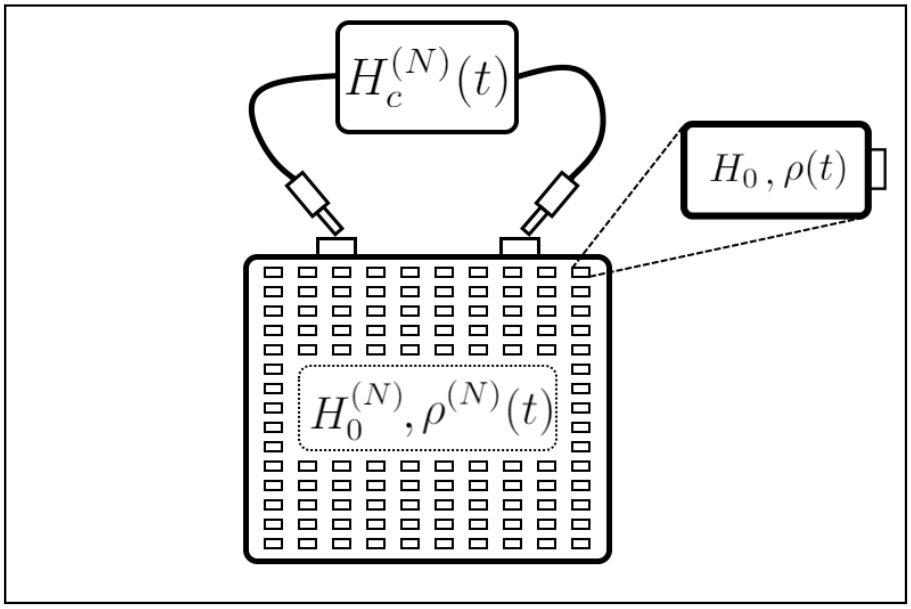


Figure 8: A joint quantum state $\rho^{(N)}$ with Hamiltonian $H_0^{(N)}$. It is composed of N identical marginal batteries ρ with Hamiltonian H_0. It is driven by a joint control Hamiltonian $H_c^{(N)}$ which charges each marginal battery simultaneously, possibly allowing them to interact during this process.

We remind the reader that if a multi-partite state is a product state, then the marginal states are necessarily non-correlated. Eq. (94) is actually a necessary assumption since we want the batteries to be non-correlated during storage. If not we would never be able to pick out and extract the resources of a single battery without inadvertently altering the others, and the batteries could not be independently prepared.

Remark 15. Note that the theory presented in this section technically does not require that each battery is in the same initial state. Hence, it would make sense to replace each ρ_i by a ρ_i^k, k being an index which labels each battery. While this would be more general, the focus of this section is only to shed light on the advantage of quantum correlations.

Let $\rho^{(N)}(t)$ be a time-dependent multi-partite state that evolves $\rho_i^{(N)}$ into $\rho_f^{(N)}$ in duration τ_N. We will refer to $\rho^{(N)}(t)$ as the *joint* state of N simultaneously charged *marginal* batteries. Since τ_N is the duration required to charge all the batteries simultaneously we want to minimize this time. In order to do so we want to be able to apply the results of Sections 2-4 to $\rho^{(N)}(t)$, which requires it to satisfy all the requirements we imposed on a single battery in Section 1.

Firstly we require that $\rho^{(N)}(t)$ is the state of a closed, hence unitarily evolving system,

$$\dot{\rho}^{(N)}(t) = -i[H^{(N)}(t), \rho^{(N)}(t)]. \tag{95}$$

Above, $H^{(N)}(t)$ is the *joint* total Hamiltonian of the joint state. Similar to Section 1.2 we want write this as a sum of a constant *joint observable* $H_0^{(N)}$ and a *joint control* $H_c^{(N)}(t)$. We can determine an expression for the former by arguing that if we measure the joint energy of non-correlated batteries, the result must be the sum of the energy of each

individual battery. This is achieved if we write

$$H_0^{(N)} = \sum_{l=1}^{N} \bigotimes_{k=1}^{N} H_0^{\delta_{kl}}, \tag{96}$$

where δ_{kl} is Kronecker's delta such that $H_0^{\delta_{kl}}$ is the identity operator if $k \neq l$. Furthermore we note that the states $\rho_i^{(N)}$ and $\rho_f^{(N)}$ are incoherent states with respect to $H_0^{(N)}$, which solidifies a second requirement. The third and last requirement is that we have a well defined restriction Ω^2 on the resource of $H_c^{(N)}(t)$. That is, if f is a constraint function with the properties of Eq. (13), then

$$f\left(H_c^{(N)}(t), \rho^{(N)}(t)\right) \leq \Omega^2. \tag{97}$$

The value Ω should be the same regardless of which joint control we choose. Hence we consider a joint control of the form

$$H_c^{(N)} = \sum_{l=1}^{N} \bigotimes_{k=1}^{N} H_c^{\delta_{kl}}. \tag{98}$$

Such a $H_c^{(N)}$ would act locally on each battery with control H_c, i.e., each battery would evolve according to Section 1. If we consider the bandwidth and variance constraints, Eqs. (11) and (12), and let ω^2 be the constraint on each marginal battery we find that

$$\mathrm{Tr}\left[{H_c^{(N)}}^2(t)\right] \leq N d^{N-1}\omega^2 = \Omega^2, \tag{99}$$

$$\mathrm{Var}(H_c^{(N)}(t), \rho^{(N)}(t)) \leq N\omega^2 = \Omega^2 \tag{100}$$

are valid joint constraints. Consequently these are the constraints which we choose to impose on *all* joint controls $H_c^{(N)}$, i.e., those not necessarily on the form Eq. (98). Further comments and details on these calculations are provided in Appendix A.5.

We have now assured ourselves that $\rho^{(N)}(t)$ fulfills all requirements necessary for the results of Sections 2-4 to be applicable to it. It is worth remarking that $\rho_f^{(N)}$ may not be an active state itself, i.e., that it maximizes the expectation value of $H_0^{(N)}$. This is not a requirement, however, as we recall Remark 2 in Section 1.3. Consequently we should never consider the joint state as a battery on its own, but rather an ensemble of batteries whose dynamics obey the precedent theory. This will be crucial to keep in mind when considering examples, since this distinction will put heavy restrictions on the unitary operators which evolves the joint state.

5.2 The Quantum Advantage

Having determined the properties of the joint state and found them suitable to our needs, we can now define the quantum advantage. Let $\tau_{N\min}$ be the minimal duration in which the joint state evolves from $\rho_i^{(N)}$ to $\rho_f^{(N)}$, i.e., the minimal time required for all batteries to be activated with a joint process whose control satisfies Eqs. (99) and (100). Then define the quantum advantage B as

$$B = 1 - \frac{\tau_{N\min}}{\tau_{\min}}. \tag{101}$$

The denominator $\tau_{\min}$ is the minimal time required to activate a marginal battery independently. Hence it is given by the former results of this thesis. The quantum advantage

B is necessarily a value in the range $[0, 1)$, where an increasing, non-zero value implies a decreasing process duration due to allowing correlations. We want to maximize the quantum advantage B.

Note that B cannot be negative since $\tau_{N\min}$ always is less than $\tau_{\min}$. This is a consequence of the fact that we always can charge each battery individually by choosing the joint control in Eq. (98). If the local control H_c is time-optimal, the duration of this process is $\tau_{\min}$.

5.3 Multi-Battery Quantum Speed Limits

When attempting to determine the minimal joint duration $\tau_{N\min}$ it is natural to start by finding quantum speed limits for the joint state, corresponding to those in Proposition 2.4 and 2.5. We refer to these as *joint quantum speed limit*, and for the bounded bandwidth and bounded variance cases we denote them by $\tau^{\mathrm{B}}_{N\mathrm{qsl}}$ and $\tau^{\mathrm{V}}_{N\mathrm{qsl}}$, respectively. Proposition 2.4 and 2.5 allows us to write

$$\tau^{\mathrm{B}}_{N\mathrm{qsl}} = \frac{\pi\sqrt{\Delta}}{2\Omega}, \quad \tau^{\mathrm{V}}_{N\mathrm{qsl}} = \frac{\pi\sqrt{\mathcal{P}}}{2\Omega}. \tag{102}$$

Here Δ and $\mathcal{P}$ are the joint state correspondents of δ and P in Eqs. (39) and (40). We define these in a manner similar to Section 2.5: Let $\boldsymbol{\sigma}$ be a permutation of the eigenvectors of $\rho_i^{(N)}$ such that $\rho_f^{(N)} = A_{\boldsymbol{\sigma}}\rho_i^{(N)}A_{\boldsymbol{\sigma}}^\dagger$. We define κ as the number of eigenvectors $|\mathbf{j}\rangle$ for which $\boldsymbol{\sigma}(\mathbf{j})$ preserve the energy expectation value of $H_0^{(N)}$. Then $\Delta = d^N - \kappa$ where d^N is the dimension of the joint state. If $q_{\mathbf{j}}$ is the eigenvalue corresponding to $|\mathbf{j}\rangle$, then the parameter $\mathcal{P}$ is defined as

$$\mathcal{P} = \sum_{\mathbf{j}} q_{\mathbf{j}}. \tag{103}$$

The sum is over those $\mathbf{j}$ for which $\sigma(\mathbf{j})$ preserve the energy expectation value.

Given a certain ρ_i and H_0 it was relatively straight forward to determine δ and P, hence the single battery quantum speed limits. This simplicity vanish when attempting to calculate Δ and $\mathcal{P}$, since the spectra of $\rho_i^{(N)}$ and $H_0^{(N)}$ are heavily degenerate and depend on N. Consequently Eq. (102) becomes difficult to calculate in full generality. In the next section we will consider non-degenerate and passive batteries in which case we will be able to develop a method which gives us Δ and $\mathcal{P}$ as functions of N.

5.4 Non-Degenerate and Passive Batteries

In this section we will consider some examples of correlated batteries and calculate the corresponding quantum advantage, Eq. (101). We will do this by means of examining the spectra of $\rho_i^{(N)}$ and $H_0^{(N)}$. Let $\mathbf{p} = (p_1, p_2, \cdots, p_d)$ be the spectrum of ρ_i and $\mathbf{E} = (E_1, E_2, \cdots, E_d)$ be that of H_0. Let also $\mathbf{j} = (j_1, j_2, \cdots, j_d)$ be a sequence of non-negative integers which sum to N. Then, according to Eqs. (94) and (96) we find that

$$q_{\mathbf{j}} = p_1^{j_1}p_2^{j_2}\cdots p_d^{j_d}, \quad \mathcal{E}_{\mathbf{j}} = j_1E_1 + j_2E_2 + \cdots + j_dE_d \tag{104}$$

are eigenvalues of $\rho_i^{(N)}$ and $H_0^{(N)}$, respectively. The multiplicities of these eigenvalues are bounded from below by the multinomial coefficient

$$M_{\mathbf{j}} = \binom{N}{j_1, j_2, \cdots, j_d} = \frac{N!}{j_1!\, j_2!\, , \cdots, j_d!}. \tag{105}$$

This can be seen by rewriting the trace of $\rho_i^{(N)}$,

$$\mathrm{Tr}\left[\rho_i^{(N)}\right] = \left(\sum_{k=1}^{d} p_k\right)^N = \sum_{j_1,j_2,\cdots,j_d} \binom{N}{j_1, j_2, \cdots, j_d} p_1^{j_1} p_2^{j_2} \cdots p_d^{j_d} = \sum_{\mathbf{j}} M_{\mathbf{j}} q_{\mathbf{j}}. \tag{106}$$

The latter sum is over all different sequences $\mathbf{j}$. The same results are found if we replace $\rho_i^{(N)}$ by $H_0^{(N)}$. Note that the multiplicity can be higher than Eq. (105) if some eigenvalues with different sequences $\mathbf{j}$ happen to be equal.

Let us now assume that ρ_i is simultaneously non-degenerate and passive, i.e. it minimizes the energy expectation value Eq. (1). Such a battery can be individually charged according to Example 2. This implies that the permutation σ which actives ρ_i is unique and of the form Eq. (42), and that it can be implemented in the quantum speed limits, Eqs. (39) and (40). We now make the important notion that for every $\mathbf{j}$ there exists a *reversed* sequence $\mathbf{j}^\star = (j_d, j_{d-1}, \cdots, j_1)$. This sequence give rise to the eigenvalues

$$q_{\mathbf{j}^\star} = p_1^{j_d} p_2^{j_d - 1} \cdots p_d^{j_1}, \quad \mathcal{E}_{\mathbf{j}^\star} = j_d E_1 + j_{d-1} E_2 + \cdots + j_1 E_d. \tag{107}$$

If compared to Eq. (104) we find that if $q_{\mathbf{j}}$ and $q_{\mathbf{j}^\star}$ are interchanged, then each individual battery is activated according to the permutation Eq. (42). Hence the joint state $\rho_i^{(N)}$ can be transformed into $\rho_f^{(N)}$ with a permutation $\boldsymbol{\sigma}$ whose cycles are transpositions of the form $(\mathbf{j}, \mathbf{j}^\star)$. This implies that the joint quantum speed limits, Eq. (102), become applicable. This allows us to rewrite the quantum advantage, Eq. (101), as

$$B = 1 - \sqrt{\frac{\Delta}{N d^{N-1} \delta}}, \quad B = 1 - \sqrt{\frac{\mathcal{P}}{NP}}, \tag{108}$$

where we made use of Eqs. (39), (40), (99), (100) and (102). The problem of calculating the quantum advantage for N batteries has been fully reduced to determining Δ and $\mathcal{P}$ along with their single battery correspondents.

While we in Example 2 could determine the overlap immediately on account of ρ_i and H_0 being non-degenerate, such is not the case for $\rho_i^{(N)}$ and $H_0^{(N)}$. Because of this, we must return to the original definition of of the overlap. Consider the conditions $q_{\mathbf{j}} = q_{\mathbf{j}^\star}$ and $\mathcal{E}_{\mathbf{j}} = \mathcal{E}_{\mathbf{j}^\star}$. If either of these hold the energy expectation value of $H_0^{(N)}$ is invariant both under $\boldsymbol{\sigma}(\mathbf{j})$ and $\boldsymbol{\sigma}(\mathbf{j}^\star)$. Then, per definition, κ is necessarily equal to the number of eigenvectors for which $q_{\mathbf{j}} = q_{\mathbf{j}^\star}$ or $\mathcal{E}_{\mathbf{j}} = \mathcal{E}_{\mathbf{j}^\star}$. These conditions are now what we seek to solve.

5.4.1 Overlap Conditions for Full-Rank Spectra

Assume that each ρ_i has full rank. From Eqs. (104) and (107) we find that we can write the conditions for when $q_{\mathbf{j}} = q_{\mathbf{j}^\star}$ and $\mathcal{E}_{\mathbf{j}} = \mathcal{E}_{\mathbf{j}^\star}$ as

$$\prod_{i=1}^{\lfloor d/2 \rfloor} \left(\frac{p_i}{p_{d-i+1}}\right)^{j_i - j_{d-i+1}} = 1, \tag{109}$$

$$\sum_{i=1}^{\lfloor d/2 \rfloor} (j_i - j_{d-i+1})(E_i - E_{d-i+1}) = 0. \tag{110}$$

Above, $\lfloor \cdot \rfloor$ is the floor function which rounds down to the closest integer. The solutions to any of these conditions are the sequences $\mathbf{j}$. Hence we let $\{\mathbf{j}\}_{\mathbf{p}}$ and $\{\mathbf{j}\}_{\mathbf{E}}$ denote the sets of sequences which solve Eqs. (109) and (110), respectively. This allows us to formulate the union of these,

$$\{\mathbf{j}\}_{\mathbf{p}} \cup \{\mathbf{j}\}_{\mathbf{E}}, \tag{111}$$

which then is the set of sequences solving any of Eqs. (109) and (110). We will refer to the elements of this union as the *overlap sequences* of full-rank spectra. We now remind ourselves of that the degeneracy of the eigenvalue which correspond to each $\mathbf{j}$ is bounded from below by the corresponding multinomial coefficient $M_{\mathbf{j}}$, Eq. (105). Consequently the number of eigenvalues part of a redundant transposition equals the sum of all $M_{\mathbf{j}}$ which corresponds to overlap sequences. If we reduce all these redundant cycles to trivial cycles we find

$$\Delta \geq d^N - \sum_{\mathbf{j}} M_{\mathbf{j}}, \tag{112}$$

where the sum is over all $\mathbf{j}$ in the union Eq. (111). We can similarly calculate $\mathcal{P}$ since each eigenvalue q_j belonging to a trivial cycle has multiplicity $M_{\mathbf{j}}$, and we find that

$$\mathcal{P} \geq 1 - \sum_{\mathbf{j}} M_{\mathbf{j}} q_j. \tag{113}$$

While Eqs. (112) and (113) are straight forward results, determining Eq. (111) is generally difficult. We will now examine this method for 2 and 3-dimensional batteries, which are more commonly referred to as *qubit* and *qutrit* batteries. For these cases $\{\mathbf{j}\}_{\mathbf{p}}$ and $\{\mathbf{j}\}_{\mathbf{E}}$ are always equal, which simplifies the problem to the extent that we will be able to express Δ as a function of N. We will also discuss larger dimensions and present to which extent the union becomes problematic. To this end we will cover a third case more explicitly, the 4-dimensional *ququart* batteries.

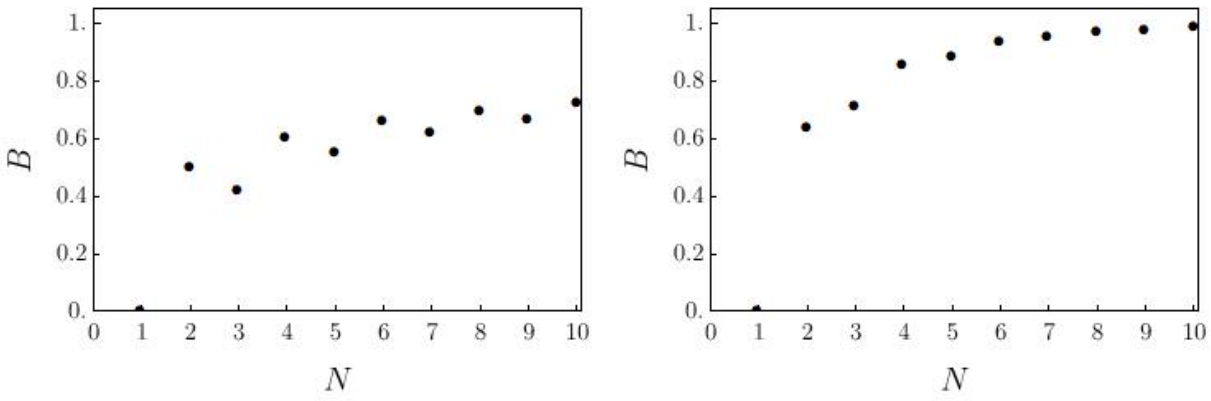
Remark 16. Note that if we do not assume that ρ_i has full rank, then Eq. (109) is no longer valid. However in this case the zero eigenvalues increases the multiplicity of some eigenvalues, hence it decreases Δ and $\tau^{\mathrm{B}}_{N\mathrm{qsl}}$. The assumption of full rank thus gives us a lower bound on the quantum advantage, which is satisfactory when examining quantum effects. A similar argument applies to $\mathcal{P}$ and $\tau^{\mathrm{V}}_{N\mathrm{qsl}}$.

5.4.2 Qubits

Qubits, or quantum bits, are 2-dimensional quantum systems with spectrum $\mathbf{p} = \{p_1, p_2\}$. The only non-trivial case in accordance with Remark 1 in Section 1.3.1 is the non-degenerate one. For this case the initial state is necessarily passive, which implies that the proposed method is viable for arbitrary, non-trivial qubit batteries.

For the qubit the sequences has two elements, $\mathbf{j} = (i, j)$. From both Eqs. (109) and (110) we find the same solutions, namely that $i = j$. Consequently $\{\mathbf{j}\}_{\mathbf{p}}$ and $\{\mathbf{j}\}_{\mathbf{E}}$ are identical and contain the sequences of the form (i, i). Furthermore, since i, j sum to N we find $i = j = \frac{N}{2}$, which never holds for odd N. Consequently, Eq. (112) gives us the two cases

$$\Delta = \begin{cases} 2^N, & \text{if } N \text{ is odd,} \\ 2^N - \frac{N!}{\left(\frac{N}{2}!\right)^2}, & \text{if } N \text{ is even.} \end{cases} \tag{114}$$



(a) (b)

Figure 9: (a) The quantum advantage B, Eq. (108), for an ensemble of N qubits given the bounded bandwidth constraint. The advantage is not strictly increasing due to a parity-dependency. (b) The advantage B for an ensemble of N qubits given the bounded variance constraint. The spectrum of ρ_i is $\mathbf{p} = (0.6, 0.4)$. In contrast to Fig. 9a it is strictly increasing, despite the parity-dependency.

And similarly from Eq. (113) we find

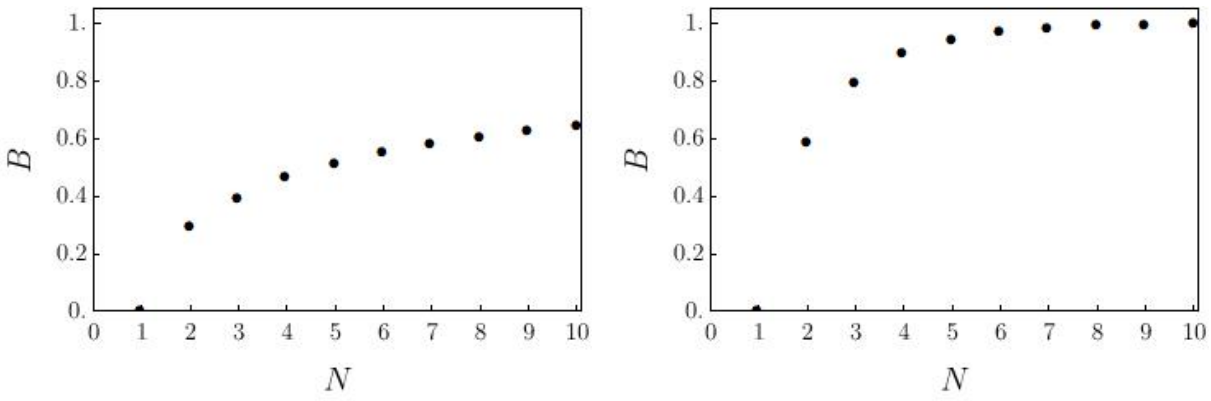
$$\mathcal{P} = \begin{cases} 1, & \text{if } N \text{ is odd,} \\ 1 - \frac{N!}{\left(\frac{N}{2}!\right)^2}(p_1 p_2)^{\frac{N}{2}}, & \text{if } N \text{ is even.} \end{cases} \tag{115}$$

For qubits we have an interesting parity-dependence of the number of batteries N. The implications of this are not immediately obvious from Eqs. (114) and (115), however. To get a better understanding of the consequences we plot the quantum advantage Eq. (108) as a function of N qubits in Fig. 9. Since the quantum advantage Eq. (108) quantifies to which extent quantum correlation between our batteries reduces their minimal charging duration, it might seem natural that a larger number N of batteries allow more complicated correlations, hence increasing the advantage. Such is however not the case, and Fig. 9a demonstrates that odd numbers N of qubits satisfying the bounded bandwidth condition produces a lower advantage compared to both closest even numbers of qubits. On the other hand, despite the fluctuating behaviour of Fig. 9a, the advantage still converges towards 1 for large N. Meanwhile the plot of the bounded variance case, Fig. 9b, also present consequences of the parity-dependence. The same fluctuations do not appear in this case, however.

Remark 17. From a geometrical standpoint this parity-dependency on the number N of batteries might not be surprising. By applying another battery the geometry of the joint state space changes considerably, and it is reasonable to assume that the time required to evolve $\rho_i^{(N)}$ into $\rho_f^{(N)}$ changes in an erratic, possibly fluctuating manner. Hence the quantum advantage fluctuates. Finding physical interpretation of these geometrical effects would be of great interest.

5.4.3 Qutrits

Qutrits are, as the name proposes, 3-dimensional quantum systems. For this dimension and above, we can no longer guarantee that the non-trivial cases are passive, hence we must assume that such is the case to apply the above method. However the simplicities of



(a) (b)

Figure 10: (a) The advantage B of an ensemble of N qutrits given the bounded bandwidth constraint. (b) The advantage B of an ensemble of N qutrits given the bounded variance constraint. The spectrum of ρ_i is $\mathbf{p} = (0.5, 0.3, 0.2)$.

the qubit case are still present. We still assume that the battery is non-trivial, i.e., that the battery cannot be simultaneously passive and active.

Let $\mathbf{j} = (i, k, j)$ be the sequences for qutrits. Similar to qubits, Eqs. (109) and (110) both return $i = j$. Hence the sequence sets $\{\mathbf{j}\}_\mathbf{p}$ and $\{\mathbf{j}\}_\mathbf{E}$ are identical and composed of sequences of the form (i, k, i). The integer k is free and we find that $i = j = \frac{N-k}{2}$. Now there is no distinct difference between odd and even N, hence there is no parity-dependence. We find

$$\Delta = 3^N - \sum_{k=0}^{N} \frac{N!}{\left(\frac{N-k}{2}!\right)^2 k!}, \tag{116}$$

$$\mathcal{P} = 1 - \sum_{k=0}^{N} \frac{N!}{\left(\frac{N-k}{2}!\right)^2 k!} (p_1 p_3)^{\frac{N-k}{2}} p_2^k. \tag{117}$$

To compare this to Fig. 9 we plot the advantage Eq. (108) for qutrits in Fig. 10. Without the parity-dependence on N the fluctuations of Fig. 9 vanish, even for the bounded bandwidth constraint.

5.4.4 Higher Dimensions

For qubits and qutrits we noticed that the solutions to Eqs. (109) and (110) coincided, a simplicity that vanishes for higher dimensions where the conditions become spectrum-dependent. Consequently Eqs. (112) and (113) can become complicated to calculate. To make this clear we examine the above method for the simplest spectrum-dependent case; the non-degenerate 4-dimensional ququart battery.

Let $\mathbf{j} = (i, j, k, l)$ be the sequences for ququarts. From Eqs. (109) and (110) we find

$$k - j = (i - l)\alpha, \quad \alpha = \log_{p_2/p_3}\left(\frac{p_1}{p_4}\right), \tag{118}$$

$$k - j = (i - l)\beta, \quad \beta = \frac{E_4 - E_1}{E_3 - E_2}. \tag{119}$$

If both α and β are rational, then the above equations may have different solutions. Hence the union of $\{\mathbf{j}\}_\mathbf{p}$ and $\{\mathbf{j}\}_\mathbf{E}$ becomes difficult. There exists solvable cases, however:

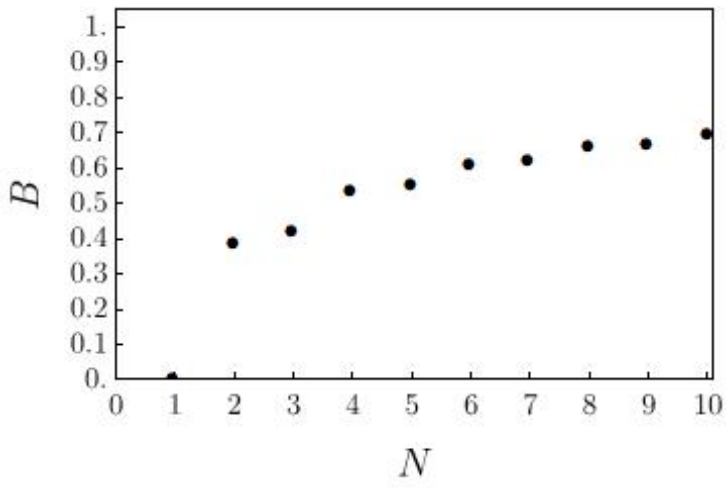


Figure 11: The quantum advantage B of an ensemble of N ququarts, assuming that α and β are irrational numbers. It has a parity-dependency on N, however in contrast to Fig. 9a it is strictly increasing.

Assume that both α and β are irrational. Then $\{\mathbf{j}\}_{\mathbf{p}}$ and $\{\mathbf{j}\}_{\mathbf{E}}$ are the same and contain sequences which satisfy

$$i - l = k - j = 0. \tag{120}$$

In this case we find that

$$\Delta = \begin{cases} 4^N, & \text{if } N \text{ is odd,} \\ 4^N - \sum_{i=0}^{N/2} \frac{N!}{\left(i!\left(\frac{N}{2}-i\right)!\right)^2}, & \text{if } N \text{ is even.} \end{cases} \tag{121}$$

$$\mathcal{P} = \begin{cases} 1, & \text{if } N \text{ is odd,} \\ 1 - \sum_{i=0}^{N/2} \frac{N!}{\left(i!\left(\frac{N}{2}-i\right)!\right)^2} (p_1 p_4)^i (p_2 p_3)^{\frac{N}{2}-i}, & \text{if } N \text{ is even.} \end{cases} \tag{122}$$

Once again we encounter a parity-dependence. This is possibly a property of even-dimensional batteries. In Figure Fig. 11 we plot the quantum advantage Eq. (108) for the bounded bandwidth. While the parity-dependence is somewhat apparent, the quantum advantage remains strictly increasing with N.

For higher dimensions determining the union of the solution sequences of Eqs. (109) and (110) becomes increasingly difficult due to an increase in spectrum dependent parameters similar to α and β. On the other hand, a lower bound on the quantum advantage is still found for the cases where $\{\mathbf{j}\}_{\mathbf{p}}$ and $\{\mathbf{j}\}_{\mathbf{E}}$ are identical, and Eqs. (112) and (113) become saturated. The solution for such cases is a straight forward generalization of Eq. (120), namely

$$j_i = j_{d-i+1}. \tag{123}$$

Remark 18. It can in fact be shown that there exists special ququart spectra for which the quantum advantage is not strictly increasing. Instead it displays a fluctuating behaviour similar to that of the qubit case. While interesting, we will not delve further into this topic.

6 Conclusions and Future Research

In this thesis we have studied the minimal duration required to fully charge a quantum battery. The battery is represented by a quantum state whose energy content is defined by the expectation value relative to an internal Hamiltonian. We assumed that the battery satisfied a von Neumann equation and that it was prepared in a state which commuted with the observable. Furthermore, we assumed that the control Hamiltonian responsible for the increase in energy content satisfied either of two constraints, the bounded bandwidth or the bounded variance constraints. We found that for time-optimal processes the problem could be solved by determining the length of shortest geodesic curves in the unitary group. The nature of shortest curves were heavily dependent on both the initial state of the battery and the imposed constraint. This split the problem into multiple different cases. To differentiate between these we found it useful to use cycle decompositions of the possible permutation operators that fully charges the battery.

In Section 2.5 we found the first result in the form of two quantum speed limits, one for each constraint. These act as lower bounds on the minimal duration for arbitrary batteries. This bound is not necessarily saturated. On the other hand we managed to prove that the minimal duration is equal to the quantum speed limit for certain choices of initial battery states. The case where the battery is initially fully discharged is one such important example.

A larger extent of cases were covered for the bounded bandwidth constraint in Section 3. For this constraint we first managed to determine the minimal duration for the complete set of cases where both battery state and its observable Hamiltonian had non-degenerate spectra. If, instead, either was degenerate the issue became more complicated. Although the non-degenerate solution was found to be an upper bound on the minimal duration. While not able to find the exact minimal duration for arbitrary degenerate cases, we developed a decomposition method which does so for some special, yet useful cases. The method works by decomposing the problem into more easily solved sub-problems where we can utilize well-known properties of flag manifolds. Here there is much room for further investigation. Developing the knowledge of geodesic distances on generalized flag manifolds could improve the extent to which the method is applicable.

When we in Section 4 considered the bounded variance constraint we detailed some complications which arise as consequences of the metric not being bi-invariant. Instead of determining minimal durations we established two criteria on time-optimal controls. The former states that time-optimal controls are almost parallel transporting, while the latter is a geodesic equation which has to be satisfied for shortest curves. We compared these to the bounded bandwidth case, where the geodesic equation becomes a requirement for time-optimal controls to be time-independent. We concluded the section with an example which shows that some batteries cannot be activated with control Hamiltonians that are both time-optimal and time-independent.

Finally we dedicated Section 5 to examining the consequences of charging multiple batteries simultaneously, assuming we allow correlations during, but not before nor after the charging process. Since the correlations can be of quantum nature this allowed us to define a quantum advantage Eq. (101), a parameter which quantifies to which extent allowing correlations is beneficial for the minimal duration. By assuming that each individual battery is identical, passive and non-degenerate we managed to find an expression for the

quantum advantage. When calculating the advantage for some examples we manage to show that allowing correlations can be beneficial for certain systems. We also find that that the advantage sometimes has a dependency on the parity of the number of batteries. We discuss the interest of this behaviour briefly. While difficult, it would be of use to give physical interpretations to the changes in geometry when we add another battery to the simultaneously charged ensemble. This could possibly be done by examining various hypothesized measures of correlation, [27, 28], and comparing it to the parity-dependency in the quantum advantage. This could possibly give hints to the nature of quantum correlations for mixed states.

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References

[1] S. Vinjanampathy, J. Anders, Quantum Thermodynamics, arXiv:1508.06099.

[2] J. Audretsch, Entangled Systems, Wiley-VCH Verlag, 2007.

[3] S. Julià-Farré, T. Salamon, A. Riera, M. N. Bera, and M. Lewenstein, Bounds on Capacity and Power of Quantum Batteries, Phys. Rev. Research **2**, 023113 (2020).

[4] D. Allan, O. Andersson, and N. Hörnedal, Time-Optimal Quantum Transformations with Bounded Bandwidth, arXiv:2011.11963.

[5] J. J. Sakurai, and J. Napolitano, Modern Quantum Mechanics, Second Ed., Addison-Wesley Publishing Company 2011.

[6] M. Nakahara, Geometry, Topology and Physics, Second Ed., Intitute of Physics Publishing, 2003.

[7] A. Arvanitoyeorgos, An Introduction to Lie Groups and the Geometry of Homogeneous Spaces, Student Mathematical Library Vol. 22, (American Mathematical Society 2003).

[8] W. Pusz, and S. L. Woronowicz, Passive states and KMS states for general quantum systems, Comm. Math. Phys. **58**, 273 (1978).

[9] A. Lenard, Thermodynamical proof of the Gibbs formula for elementary quantum system, J. Stat. Phys. **19**, 6 (1978).

[10] P. Skrzypczyk, R. Silva, and N. Brunner, Passivity, complete passivity, and virtual temperatures, Phys. Rev. E **91**, 052133 (2015).

[11] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Quantum Brachistochrone, arXiv:quant-ph/0511039.

[12] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Quantum Brachistochrone for Mixed States, arXiv:1910.08140.

[13] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Time-Optimal Unitary Operations, Phys. Rev. A **75**, 042308 (2007).

[14] S. Deffner and S. Campbell, Quantum speed limits: from Heisenberg's uncertainty principle to optimal quantum control, J. Phys. A: Math. Theor. **50**, 453001 (2017).

[15] O. Andersson, Holonomy in Quantum Information Geometry, Thesis, arXiv:1910.08140.

[16] X. Wang, M. Allegra, K. Jacobs, S. Lloyd, C. Lupo, and M. Mohseni, Quantum Brachistochrone Curves as Geodesics: Obtaining Accurate Minimum-Time Protocols for the Control of Quantum Systems, Phys. Rev. Lett. **114**, 170501 (2015).

[17] D. C. Brody, G. W. Gibbons, and D. M. Meier, Time-Optimal Navigation Through Quantum Wind, New J. Phys. **17**, 033048 (2015).

[18] D. C. Brody, and D. M. Meier, Solution to the Quantum Zermelo Navigation Problem, Phys. Rev. Lett. **114**, 100502 (2015).

[19] J. Milnor, Curvatures of Left Invariant Metrics on Lie Groups, Adv. in Math. **21**, 293-329 (1976).

[20] S. Lang, Algebra, Third Ed., Addison-Wesley Publishing Company 1993.

[21] M. R. Frey, Quantum speed limits–primer, perspectives, and potential future directions, Quantum Inf Process **15**, 3919-3950 (2016).

[22] K. Fujii, Introduction to Grassmann Manifolds and Quantum Computation, J. of App. Math. 2:8, 371–405 (2002).

[23] A. Edelman, T. A. Arias, and S. T. Smith, The geometry of algorithms with orthogonality constraints, Siam J. Matrix Anal. Appl. **20**, 303 (1998).

[24] X. Ma, M. Kirby, and C. Peterson, The flag manifold as a tool for analyzing and comparing data sets, arXiv:2006.14086.

[25] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications, Mathematical Surveys and Monographs Vol. 91 (2002).

[26] J. M. Lee, Riemannian Manifolds: An Introduction to Curvature, Springer-Verlag, 1997.

[27] H. Ollivier and W. H. Zurek, Quantum Discord: A Measure of the Quantumness of Correlations, Phys. Rev. Lett. **88**, 017901 (2001).

[28] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[29] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the Charging Power of Quantum Batteries, Phys. Rev. Lett. **118**, 150601 (2017).

[30] L. Mandelstam and Ig. Tamm The Uncertainty Relation Between Energy and Time in Non-relativistic Quantum Mechanics, J. Phys. USSR **9**, 249-254 (1945).

A Appendix

A.1 Quantum Batteries

Proof of Proposition 1.1

Assume that the control $H_c(t)$ fully charges the battery in duration τ, i.e., $U(\tau)\rho_i U^\dagger(\tau)$ is an active state. In the interaction picture the battery state and the control are given by

$$\rho_I(t) = e^{itH_0}\rho(t)e^{-itH_0}, \tag{124}$$
$$H_{cI}(t) = e^{itH_0}H_c(t)e^{-itH_0}. \tag{125}$$

The unitary dynamics of $\rho_I(t)$ in this frame are given by (see Section 2.1.1)

$$U_I(t) = \mathcal{T}\exp\left(-i\int_0^t dt' H_{cI}(t')\right). \tag{126}$$

Consider now the curve $W_I(t) = U_I(q_\epsilon(t))$ where $q_\epsilon(t)$ is a smooth function satisfying Eq. (6). This curve satisfies

$$W(0) = U_I(q_\epsilon(0)) = U_I(0) = \mathbb{1}, \tag{127}$$
$$W(\tau + 2\epsilon) = U_I(q_\epsilon(\tau + 2\epsilon)) = U_I(\tau) \tag{128}$$

and is generated by $H'_{cI}(\epsilon, t) := \dot{q}_\epsilon(t)H_{cI}(q_\epsilon(t))$, since

$$\dot{W}_I(t) = \dot{q}_\epsilon(t)\dot{U}_I(q_\epsilon(t)) = -i\dot{q}_\epsilon(t)H_{cI}(q_\epsilon(t))U_I(q_\epsilon(t)) = -iH'_{cI}(\epsilon, t)W_I(t). \tag{129}$$

If we let $\rho'_I(t) = W_I(t)\rho_i W_I^\dagger(t)$ we find, using Eq. (127), that

$$\rho'_I(\tau + 2\epsilon) = U_I(\tau)\rho_i U_I^\dagger(\tau) \tag{130}$$

Note that the right-hand side is an active state per assumption. Hence the control $H'_{cI}(\epsilon, t)$ fully charges the battery in time $\tau + 2\epsilon$. The physics are independent on frame, and we can thus equivalently say that the control

$$H'_c(\epsilon, t) = \dot{q}_\epsilon(t)e^{-itH_0}H_{cI}(q_\epsilon(t))e^{itH_0} = \dot{q}_\epsilon(t)H_c(q_\epsilon(t)) \tag{131}$$

induces a unitary $U(t)$ such that $\rho(t)$ is active at time $\tau + 2\epsilon$. If we let $\epsilon \to 0$, then this time goes towards the duration τ. This concludes our proof.

We will now prove that the quenched control $H'_c(\epsilon, t)$ satisfy both constraints introduce in Section 1.4. Note that since the physics are independent on frame, it is enough to prove it in the interaction picture. In both calculations we use that the constraint is homogeneous of second order in the first argument, the fact that $\dot{q}_\epsilon^2(t) \in [0, 1]$, and that $H_{cI}(q_\epsilon(t))$ is a time-reparametrization of $H_{cI}(t)$. For the bounded bandwidth constraint, Eq. (11), we have

$$\mathrm{Tr}\left[{H'_{cI}}^2(\epsilon, t)\right] = \dot{q}_\epsilon^2(t)\,\mathrm{Tr}\left[H_{cI}^2(q_\epsilon(t))\right] \leq \omega^2. \tag{132}$$

Similarly, for the bounded variance constraint, Eq. (12),

$$\mathrm{Var}\left(H'_{cI}(\epsilon, t), \rho_I(t)\right) = \dot{q}_\epsilon^2(t)\mathrm{Var}\left(H_{cI}(q_\epsilon(t)), \rho_I(t)\right) \leq \omega^2 \tag{133}$$

Proof of Proposition 1.2

Let ρ_e be an extremal battery state in $\mathcal{D}(\mathbf{p})$. Since $\mathcal{D}(\mathbf{p})$ is a compact manifold and the energy function $E(\rho)$ is smooth, ρ_e is necessarily a stationary with respects to $E(\rho)$. Consequently, for any tangent vector $-[A, \rho_e]$ to $\mathcal{D}(\mathbf{p})$ at ρ_e the differential map $dE(-i[A, \rho_i])$ vanishes. We find

$$\begin{aligned} 0 &= dE(-i[A, \rho_e]) \\ &= \mathrm{Tr}\left[H_0(-i[A, \rho_e])\right] \\ &= \mathrm{Tr}\left[iA[AH_0, \rho_e]\right]. \end{aligned} \tag{134}$$

Since this must apply for all Hermitian operators A we deduce that $[H_0, \rho_e] = 0$ and that extremal states are incoherent.

Proof of Proposition 1.3

Consider a constraint function $F(t) = f(H_c(t), \rho(t))$ which is homogeneous of second order in $H_c(t)$. Assume that $H_c(t)$ is time-optimal, i.e. the curve it generates in $\mathcal{D}(\mathbf{p})$ reaches an active state in duration $\tau_{\min}$. Let $g(t)$ be a continuous and everywhere positive function for which $F(t) \leq g(t) \leq \omega^2$. Furthermore define the function

$$s(t) = \frac{1}{\omega}\int_0^t dt' \sqrt{g(t')} \tag{135}$$

on the domain $[0, \tau_{\min}]$. Since $g(t)$ is continuous and positive, $s(t)$ is necessarily smooth strictly increasing function. Consequently it has an inverse $t(s)$ which satisfies

$$0 < \dot{t}(s) = \frac{\omega}{\sqrt{g(t(s))}} \leq \frac{\omega}{\sqrt{F(t(s))}}. \tag{136}$$

If $f(H_c)$ happen to be zero, replace the right-hand side by positive infinity. Let $\rho'(s) = \rho(t(s))$ be a curve in $\mathcal{D}(\mathbf{p})$ parametrized by s which reaches an active state in time $s = s(\tau_{\min})$. This curve is generated by the Hermitian operator $H_c'(s) = H_c(t)\dot{t}(s)$,

$$\begin{aligned} \dot{\rho}'(s) &= \dot{\rho}(t(s))\dot{t}(s) \\ &= -i\Big[H_c(t(s)), \rho(t(s))\Big]\dot{t}(s) \\ &= -i\Big[H_c'(s), \rho'(s)\Big]. \end{aligned} \tag{137}$$

Furthermore since the constraint function F is homogeneous of second order Eq. (136) guarantees that $H_c'(s)$ satisfies the constraint, $f\left(H_c'(s), \rho(s)\right) \leq \omega^2$.

Assume that the constraint is not saturated for H_c, i.e. $F(t) < \omega^2$ at some t. Then $g(t(s))$ can be chosen strictly less than ω^2 in a neighbourhood of that t. We find

$$s(\tau_{\min}) = \frac{1}{\omega}\int_0^t dt' \sqrt{g(t')} < \tau_{\min}. \tag{138}$$

However this contradicts that $\tau_{\min}$ is a lower bound on the duration. Consequently, by proof of contradiction, $F(t) = \omega^2$ for all time-optimal processes.

A.2 The Charging Process

Proof of Proposition 2.2

Let W be any unitary operator which activates our state, $W\rho_i W^\dagger = \rho_a$. The energy isotropy group $\mathcal{U}(\mathcal{H})_{H_0}$ acts transitively on all active states from the left, i.e. for each $A \in \mathcal{A}(\rho_i)$ there exists a $U \in \mathcal{U}(\mathcal{H})_{H_0}$ such that $UA\rho_i A^\dagger U^\dagger = W\rho_i W^\dagger$. However from this we find

$$W^\dagger UA\rho_i = \rho_i W^\dagger UA. \tag{139}$$

Thus, ρ_i commutes with the unitary operator $V = W^\dagger UA$, and V is an element in the state isotropy group $\mathcal{U}(\mathcal{H})_{\rho_i}$. Hence W can always be written of the form $W = UAV$ where A is an activating unitary, $U \in \mathcal{U}(\mathcal{H})_{H_0}$ and $V \in \mathcal{U}(\mathcal{H})_{\rho_i}$.

The reverse follows straight forward from that Eq. (1) is invariant under U, V when acting on ρ_i from the left with UAV.

Proof of Proposition 2.3

In the basis Eq. (30) we have

$$\rho_i = \sum_{k=1}^{d} p_k \left|k\right\rangle. \tag{140}$$

where the indices are ordered such that $\langle k|H_0|k\rangle \leq \langle k+1|H_0|k+1\rangle$. This ordering is possible due to Proposition 1.2. Consider the state

$$\rho_a = \sum_{k=1}^{d} p_{\sigma(k)} \left|\sigma(k)\right\rangle. \tag{141}$$

where σ is some permutation. If $p_{\sigma(k)} \leq p_{\sigma(k+1)}$, then this is an active state, and ρ_i can be activated with the unitary

$$U = \sum_{k=1}^{d} \left|\sigma(k)\right\rangle\left\langle k\right|. \tag{142}$$

This is of the form of a permutation operator, which concludes the proof.

Proof of Proposition 2.4

Let $|k\rangle$ be an eigenvector of ρ_i and define $|k(t)\rangle = \exp(-itH_c)\,|k\rangle$ as a curve on the unit sphere. Each vector $|k\rangle$ must either evolve to an orthogonal eigenspace or remain fixed. The number of vectors that must evolve in order to activate ρ_i is δ, and the length of these curves is equal to the spherical distance between two orthogonal states, $\pi/2$. Furthermore the squared speed of each curve equals $\langle k|H_c^2|k\rangle$ and Proposition 1.3 guarantees that these sum to ω^2. We find

$$\omega^2\tau_{\min}^2 = \sum_{k=1}^{d} \langle k(t)|H_c^2|k(t)\rangle\,\tau_{\min}^2 \geq \frac{\pi^2\delta}{4}. \tag{143}$$

This allows us to write

$$\tau_{\min} \geq \frac{\pi\sqrt{\delta}}{2\omega} \tag{144}$$

which proves the proposition.

Proof of Proposition 2.5

Let $|k(t)\rangle$ be the set of curves characterized by $|\dot{k}(t)\rangle = -iH_c(t)\,|k(t)\rangle$ and $|k(0)\rangle = |k\rangle$. Then the spectral decomposition of $\rho(t)$ can be written as

$$\rho(t) = \sum_k p_k \,|k(t)\rangle\,\langle k(t)|\,. \tag{145}$$

We now define the dispersions $\Delta H_c(t)$ and $\Delta_k H_c(t)$ by

$$\Delta H_c(t) = \sqrt{\mathrm{Tr}\left[H_c^2(t)\rho(t)\right] - \mathrm{Tr}\left[H_c(t)\rho(t)\right]^2}, \tag{146}$$

$$\Delta_k H_c(t) = \sqrt{\langle k(t)|H_c^2(t)|k(t)\rangle - \langle k(t)|H_c(t)|k(t)\rangle^2}. \tag{147}$$

Note that due to convexity,

$$\begin{aligned}\Delta^2 H_c(t) &= \sum_k p_k \,\langle k(t)|H_c^2(t)|k(t)\rangle - \left(\sum_k p_k \,\langle k(t)|H_c(t)|k(t)\rangle\right)^2 \\ &\geq \sum_k p_k \,\langle k(t)|H_c^2(t)|k(t)\rangle - \sum_k p_k \,\langle k(t)|H_c(t)|k(t)\rangle^2 = \sum_k p_k \Delta_k^2 H_c(t).\end{aligned} \tag{148}$$

Let τ be the duration of the process and define the time average of the dispersion $\Delta H_c(t)$ by

$$\langle \Delta H_c(t)\rangle = \frac{1}{\tau}\int_0^\tau dt \Delta H_c(t). \tag{149}$$

Jensen's inequality and Eq. (148) then implies that

$$\langle \Delta^2 H_c(t)\rangle \geq \sum_k p_k \,\langle \Delta_k^2 H_c(t)\rangle \geq \sum_k p_k \,\langle \Delta_k H_c(t)\rangle^2\,. \tag{150}$$

There exists a well-known quantum speed limit which is known as the Mandelstam-Tamm quantum speed limit (MTQSL), see [30, 15]. Let τ_k be the time required for each eigenvector $|k(t)\rangle$ to reach its destination, and let S be the set of eigenvectors required to evolve into a perpendicular eigenspace in order for $\rho(\tau)$ to be active, i.e., $S = \{|k(t)\rangle : \langle k|k(\tau)\rangle = 0\}$. If $k \notin S$, then $\tau_k = 0$. If instead $k \in S$ the MTQSL states that

$$\tau_k \geq \frac{\pi}{2\,\langle \Delta_k H_c(t)\rangle}. \tag{151}$$

Note that $\tau_{\min} \geq \max_k\{\tau_k\}$. Due to Eq. (12) and Proposition 1.3 we have that $\langle \Delta^2 H_c(t)\rangle = \omega^2$, hence

$$\omega^2 \tau_{\min}^2 = \langle \Delta^2 H_c(t)\rangle \max_k\{\tau_k\}^2 \geq \sum_k p_k \,\langle \Delta_k H_c(t)\rangle^2 \,\tau_k^2 \geq \frac{\pi^2}{4}\sum_{k\in S} p_k. \tag{152}$$

If we define $P = \sum_{k\in S} p_k$ we find that

$$\tau_{\min} \geq \frac{\pi\sqrt{P}}{2\omega}. \tag{153}$$

This concludes the proof.

Proof of Proposition 2.6

Let $\sigma = c_1 c_2 \cdots c_m c_{m+1} \cdots c_n$ be the fully reduced cycle decomposition where the first m cycles are of the form $c_k = (k_1, k_2)$ and the latter are trivial. If we first consider the bounded bandwidth constraint we can construct the control Hamiltonian

$$H_c = \frac{\omega}{\sqrt{2m}} \sum_{k=1}^{m} \left(|k_1\rangle \langle k_2| + |k_2\rangle \langle k_1| \right). \tag{154}$$

This choice of control satisfies Eq. (11) and generates the unitary $-iA_\sigma$ in time $\tau = \frac{\pi\sqrt{2m}}{2\omega}$. Furthermore, the discrepancy of ρ_i is $\delta = 2m$, and we prove

$$\tau = \frac{\pi\sqrt{\delta}}{2\omega} = \tau^{\mathrm{B}}_{\mathrm{qsl}}. \tag{155}$$

For the bounded variance similarly impose the control

$$H_c = \frac{\omega}{\sqrt{\sum_{k=1}^{m}(p_{k_1} + p_{k_2})}} \sum_{k=1}^{m} \left(|k_1\rangle \langle k_2| + |k_2\rangle \langle k_1| \right). \tag{156}$$

It satisfies Eq. (12) and generates the same unitary in the time

$$\tau = \frac{\pi\sqrt{\sum_{k=1}^{m}(p_{k_1} + p_{k_2})}}{2\omega} := \frac{\pi\sqrt{P}}{2\omega} = \tau^{\mathrm{V}}_{\mathrm{qsl}}. \tag{157}$$

Remark 19. The reader might wonder why we choose to generate $-iA_\sigma$ and not A_σ. In fact, the generating control of A_σ cannot achieve the above times for neither constraint. This shows that not all unitaries in the activating set $\mathcal{A}(\rho_i)$ are equidistant and that this has to be taken into consideration when examining more complicated systems.

A.3 Optimal Time for Bounded Bandwidth

Proof of Proposition 3.1

According to Proposition 3.10 in [7], all geodesics of a bi-invariant metric emanating from the identity element in a Lie group G are one-parameter subgroups of G. A one-parameter subgroup $\phi(t)$ of the unitary group is given by a unique skew-Hermitian ξ through the map $\phi : t \to \exp(t\xi)$. The shortest curve between two points in a manifold is a geodesic if the curve is traversed at constant speed. Since the metric in Eq. (25) conserves speed along time-optimal curves and is bi-invariant it follows that it is generated by a unique, i.e. time-independent control H_c.

Proof of Proposition 3.2

Since the metric g_{HS} is bi-invariant, so is the corresponding geodesic distance. Hence if $U^\dagger V = W$ we find

$$\mathrm{dist}(U, V) = \mathrm{dist}(\mathbb{1}, W) = \|\mathrm{Log} W\|, \tag{158}$$

and it becomes sufficient to prove the latter inequality. To this end let $\exp(-itH_c)$ be a shortest geodesic connecting $\mathbb{1}$ and W. Being a shortest geodesic, the time required to

reach W is $\tau_{\min}$, hence $\mathrm{Log}W = \tau_{\min} B$ for any B that shares eigenvalues with H_c. Since the Hilbert-Schmidt norm only depends on the eigenvalues of a matrix, we find

$$\|\mathrm{Log}W\| = \tau_{\min}\|H_c\| = \tau_{\min}\omega, \tag{159}$$

where the second equality follows from Proposition 1.3. Since the speed in the chosen metric is ω the length of $\exp(-itH_c)$ is equal to $\|\mathrm{Log}W\|$, hence it is the geodesic distance.

Proof of Proposition 3.3

Write $c_r = (r_1, r_2, \cdots, r_{l_r})$ and let $e^{i\alpha_{r_j}}$ and $e^{i\beta_{r_j}}$ be the jth eigenvalues of U_{c_r} and V_{c_r}, respectively. We then find that the characteristic equation of $U_{c_r}A_{c_r}V_{c_r}$ is $(e^{i\theta_r} - \lambda^{l_r}) = 0$, where

$$\theta_r = \sum_{j=1}^{l_r}(\alpha_{r_j} + \beta_{r_j}) \bmod 2\pi. \tag{160}$$

Therefore we find that the eigenvalues of $U_{c_r}A_{c_r}V_{c_r}$ are given by $\lambda_k = e^{i(\theta_r + 2\pi k)/l_r}$ where k ranges from 0 to $l_r - 1$. Each term in Eq. (47) then becomes

$$\begin{aligned}\|\mathrm{Log}(U_{c_r}A_{c_r}V_{c_r})\|^2 &= \frac{1}{l_r^2}\sum_{k=0}^{l_r-1}(\theta_r + 2\pi k)^2 \\ &= \frac{1}{l_r}\left(\theta_r^2 + 2\pi\theta_r(l_r - 1) + \frac{2}{3}\pi^2(l_r - 1)(2l_r - 1)\right).\end{aligned} \tag{161}$$

Minimizing this over all U, V corresponds to minimizing over θ_r. Consequently we find

$$\begin{aligned}\min_{U,V}\|\mathrm{Log}(UA_\sigma V)\|^2 &= \sum_{r=1}^{m}\min_{\theta_r}\|\mathrm{Log}(U_{c_r}A_{c_r}V_{c_r})\|^2 \\ &= \frac{\pi^2}{3}\sum_{r=1}^{m}\frac{l_r^2 + 1}{l_r} = \frac{\pi^2}{3}\left(d - \sum_{r=1}^{m}\frac{1}{l_r}\right).\end{aligned} \tag{162}$$

The latter equality follows from that the sum of all cycle lengths l_r equals the batteries dimension. Hence by taking the square root we find the geodesic distance for all non-degenerate cases, and the minimal duration is given by Proposition 3.3.

Proof of Proposition 3.4

Foremost all $\mathcal{H}_{\sigma_l}$ are invariant under A_σ. Meanwhile the isotropy groups $\mathcal{U}(\mathcal{H})_{H_0}$ and $\mathcal{U}(\mathcal{H})_{\rho_i}$ keep the eigenspaces of H_0 and ρ_i invariant per construction. Hence, if each $\mathcal{H}_{\sigma_l}$ is spanned by various contained eigenspaces it must also be invariant. Proposition 2.2 then guarantees that all unitaries in the activating set $\mathcal{A}(\rho_i)$ keeps each $\mathcal{H}_{\sigma_l}$ invariant.

To prove that $\mathcal{H}_{\sigma_l}$ is invariant during traversal of a shortest geodesic, let W be a closest activating unitary and decompose it as $W = \bigoplus_l W_{\sigma_l}$ where W_{σ_l} is an operator on $\mathcal{H}_{\sigma_l}$. If we define on $\mathcal{H}_{\sigma_l}$ the Hamiltonian

$$H_l = \frac{\mathrm{Log}W_{\sigma_l}}{i\tau_{\min}} \tag{163}$$

it will generate the operator W_{σ_l} in duration $\tau_{\min}$. The direct sum of all those H_l is a Hamiltonian H_c on $\mathcal{H}$ for which the bandwidth is

$$\mathrm{Tr}[H_c H_c^\dagger] = \frac{\|\mathrm{Log} W\|^2}{\tau_{\min}^2} = \omega^2. \tag{164}$$

The latter inequality since $\|\mathrm{Log} W\|$ is the geodesic distance between $\mathbb{1}$ and W according to Proposition 46. Hence H_c is a valid control which generates geodesic curves in each $\mathcal{H}_{\sigma_l}$, all with duration $\tau_{\min}$. Consequently there exists a shortest geodesic between $\mathbb{1}$ and W generated by H_c keeps each $\mathcal{H}_{\sigma_l}$ invariant.

Proof of Proposition 3.5

Consider each individual term in Eq. (55), we find

$$\min_{U_{\sigma_l}, V_{\sigma_l}} \|\mathrm{Log}(U_{\sigma_l} A_{\sigma_l} V_{\sigma_l})\|^2 = \min_{U_{\sigma_l}, V_{\sigma_l}} \|U_{\sigma_l} \mathrm{Log}(A_{\sigma_l} W_{\sigma_l}) U_{\sigma_l}^\dagger\|^2 \tag{165}$$

where $W_{\sigma_l} = V_{\sigma_l} U_{\sigma_l}$. If we assume that $\mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i} \subseteq \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$, then W_{σ_l} is an unitary in $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$. Furthermore, the Hilbert-Schmidt norm is base independent, hence

$$\mathrm{dist}(\mathbb{1}, \mathcal{A}(\rho_i))^2 = \sum_{l=1}^{m} \min_{W_{\sigma_l}} \|\mathrm{Log}(A_{\sigma_l} W_{\sigma_l})\|^2 \tag{166}$$

from which we find Eq. (56). If instead $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0} \subseteq \mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$ then W_{σ_l} is instead an unitary in $\mathcal{U}(\mathcal{H}_{\sigma_l})_{\rho_i}$ and we analogously find Eq. (57).

Proof of Proposition 3.6

Let the structure (E, B, π, F) be a fibre bundle with total space E, base space B, fibre F and (surjective) submersion map $\pi : E \to B$ [6, 7]. Since $\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ is a closed subgroup of $\mathcal{U}(\mathcal{H}_{\sigma_l})$, where $\mathcal{U}(\mathcal{H}_{\sigma_l})$ is a Lie group, the quotient space $\mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ can be regarded as the base space of the fibre bundle

$$\Big(\mathcal{U}(\mathcal{H}_{\sigma_l}), \mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}, \pi, \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}\Big) \tag{167}$$

with submersion

$$\begin{aligned} &\pi : \mathcal{U}(\mathcal{H}_{\sigma_l}) \to \mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}, \\ &\pi : WU \mapsto [W], \quad U \in \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}. \end{aligned} \tag{168}$$

Since the metric we chose on the unitary group, hence $\mathcal{U}(\mathcal{H}_{\sigma_l})$, is a collection of inner products on the tangent spaces of the Lie group Eqs. (21) and (22), $\mathcal{U}(\mathcal{H}_{\sigma_l})$ is a Riemannian Manifold. Furthermore if $\mathcal{U}(\mathcal{H}_{\sigma_l})$ is Riemannian then there exists a metric g on the quotient Lie group $\mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ such that the submersion Eq. (168) is an isometry, see Section 4.1[8]. This is also referred to as a Riemannian submersion [7]. Due to the isometry we find

$$\begin{aligned} \min_U \mathrm{dist}(\mathbb{1}, WU) &= \mathrm{dist}\Big(\pi(\mathbb{1}), \pi(WU)\Big) \\ &= \mathrm{dist}\Big([\mathbb{1}], [W]\Big) = \min_U \|\mathrm{Log} WU\| \end{aligned} \tag{169}$$

where the latter inequality follows from Proposition 3.2. If we exchange the set WU, $U \in \mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$, with the set of operators of the form $A_{\sigma_l} U_{\sigma_l}$ with which we are concerned the proof of Proposition 3.6 is complete.

[8]Note that we in that section replace $\mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$ with $\mathcal{D}(\mathbf{p})$, which in turn is diffeomorphism of $\mathcal{U}(\mathcal{H})/\mathcal{U}(\mathcal{H})_{\rho_i}$. This theory is identical to that of $\mathcal{U}(\mathcal{H}_{\sigma_l})/\mathcal{U}(\mathcal{H}_{\sigma_l})_{H_0}$.

A.4 The Geometry of Bounded Variance

On Vertical and Horizontal Tangent Spaces

We remind ourselves that we consider the map $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{D}(\mathbf{p})$. Let T_W be the tangent space at a point W in $\mathcal{U}(\mathcal{H})$ and decompose it into two orthogonal spaces T^v_W and T^h_W such that

$$T_W = T^v_W \oplus T^h_W. \tag{170}$$

We refer to T^v_W and T^h_W as the *vertical* and *horizontal* tangent spaces at W and identify the former with the space of all vectors at W which are tangent to the fibre over $W\rho_i W^\dagger$. Let η^v be any member of $\mathfrak{u}(\mathcal{H})_{\rho_i}$. It will generate a curve $\exp(t\eta^v)$ in $\mathcal{U}(\mathcal{H})_{\rho_i}$. The fibre can be written as the left action of W on the isotropy group $\mathcal{U}(\mathcal{H})_{\rho_i}$, hence

$$T^v_W = \left\{ \frac{d}{dt} \left(W \exp\left(t\eta^v\right)\right) \Big|_{t=0} : \eta^v \in \mathfrak{u}(\mathcal{H})_{\rho_i} \right\} = \left\{ W\eta^v : \eta^v \in \mathfrak{u}(\mathcal{H})_{\rho_i} \right\}. \tag{171}$$

Let us choose the metric g_{Var}, Eq. (26). We can then characterize T^h_W by those vectors $X \in T_W$ for which $g_{\mathrm{Var}}\left(W\eta^v, X\right) = 0$. Since the metric is left-invariant we can pull it back to the Lie algebra through the differential left action $dL_{W^\dagger}$. Hence if we define $\eta^h = W^\dagger X$ we find that

$$T^h_W = \left\{ W\eta^h : g_{\mathrm{Var}}\left(\eta^v, \eta^h\right) = 0 \,\forall\, \eta^v \in \mathfrak{u}(\mathcal{H})_{\rho_i} \right\}. \tag{172}$$

Thus the decomposition of T_W into the components T^v_W and T^h_W was uniquely determined through the fibre structure and the considered metric.

Let $U(t)$ be a curve generated by a $\xi(t) \in \mathfrak{u}(\mathcal{H})$ such that $\dot{U}(t) = \xi(t)U(t)$. Furthermore let $\dot{U}^v(t) = \xi^v(t)U(t)$ be the component of $\dot{U}(t)$ which is contained in $T^v_{U(t)}$. Then, by Eq. (171) we find that

$$\dot{U}^v(t) = \xi^v(t)U(t) = U(t)\eta^v \quad \Rightarrow \quad \xi^v(t) = U(t)\eta^v U^\dagger(t), \tag{173}$$

where $\eta^v \in \mathfrak{u}(\mathcal{H})^v$. Similarly, if $\dot{U}^h(t) = \xi^h(t)U(t)$ is the component contained in $T^h_{U(t)}$ and we find that

$$\xi^h(t) = U(t)\eta^h U^\dagger(t), \tag{174}$$

where η^h is perpendicular to η^v. In order for $\dot{U}(t) = \dot{U}^v(t) + \dot{U}^h(t)$ it follows that $\xi(t) = \xi^v(t) + \xi^h(t)$. Eq. (173) now allows us to prove Proposition 4.1 since

$$\left[\xi^v(t), \rho(t)\right] = U(t)\left[\eta^v, \rho_i\right]U^\dagger(t) = 0, \tag{175}$$

which follows from that η^v generates an operator in $\mathcal{U}(\mathcal{H})_{\rho_i}$. Hence,

$$\begin{aligned} \dot{\rho}(t) &= \left[\xi(t), \rho(t)\right] \\ &= \left[\xi^v(t) + \xi^h(t), \rho(t)\right] = \left[\xi^h(t), \rho(t)\right] \end{aligned} \tag{176}$$

which concludes the proof of Proposition 4.1.

Proof of Proposition 4.2

Due to the isometry in Eq. (84) we calculate the squared speed of $\rho(t)$ to be

$$\begin{aligned} g_{\mathcal{D}}\Big(\dot{\rho}(t),\dot{\rho}(t)\Big) &= g_{\mathrm{Var}}\left(-iH_c^h(t)U^h(t),-iH_c^h(t)U^h(t)\right) \\ &= \mathrm{Tr}\left[\rho_i\left(U^{h\dagger}(t)H_c^{h^2}(t)U^h(t)\right)\right] \\ &= \mathrm{Tr}\left[\rho(t)\left(H_c(t)-H_c^v(t)\right)^2\right] \\ &= \mathrm{Tr}\left[\rho(t)H_c^2(t)\right]-\mathrm{Tr}\left[\rho(t)H_c^{v2}(t)\right]. \end{aligned} \tag{177}$$

Note that $\rho(t)=U^h(t)\rho_i U^{h\dagger}(t)$ as a consequence of Proposition 4.1. Furthermore we in the last equality made use of that $H_c^h(t)H_c^v(t)$ is traceless as a consequence of Eq. (86).

The variance of $\rho(t)$ with respects to $H_c^v(t)$ is given by

$$\mathrm{Var}\left(H_c^v(t),\rho(t)\right)=\mathrm{Tr}\left[\rho(t)H_c^{v2}(t)\right]-\mathrm{Tr}\left[\rho(t)H_c^v(t)\right]^2\geq 0 \tag{178}$$

Consequently we can rewrite Eq. (177) as

$$\begin{aligned} g_{\mathcal{D}}\Big(\dot{\rho}(t),\dot{\rho}(t)\Big) &\leq \mathrm{Tr}\left[\rho(t)H_c^2(t)\right]-\mathrm{Tr}\left[\rho(t)H_c^v(t)\right]^2 \\ &= \mathrm{Tr}\left[\rho(t)H_c^2(t)\right]-\mathrm{Tr}\left[\rho(t)H_c(t)\right]^2=\mathrm{Var}(H_c(t),\rho(t)). \end{aligned} \tag{179}$$

The latter equality follows from that $\mathrm{Tr}[\rho(t)H_c^h(t)]=0$. This shows that the squared speed is bounded from above by the variance. Furthermore equality holds if and only if Eq. (178) is equal to zero, i.e. the variance of $\rho(t)$ relative $H_c^v(t)$ is zero for all t. This only holds if $H_c^v(t)=E(t)\mathbb{1}$, i.e. it is evenly distributed. Such a choice of vertical component satisfies the orthogonality requirement $g_{\mathrm{Var}}(-iH_c^v(t)U(t),-iH_c^hU(t))=0$, and is thus valid. This concludes our proof.

Proof of Proposition 4.3

The tangent space at $U(t)$ is spanned by all operators of the form $\eta U(t)$, $\eta\in\mathfrak{u}(\mathcal{H})$. Given a unitary curve $U(t)$ with tangent vector $\dot{U}(t)=-iH_c(t)U(t)$, its acceleration vector $\ddot{U}(t)$ in $\mathcal{B}(\mathcal{H})$ is given by

$$\ddot{U}(t)=-i\dot{H}_c(t)U(t)-H_c^2(t)U(t). \tag{180}$$

Hence, for this to be perpendicular to $\eta U(t)$ with respects to g_{Var} we find that

$$\begin{aligned} g_{\mathrm{Var}}(\ddot{U}(t),\eta U(t)) &= \frac{1}{2}\mathrm{Tr}\left[\rho_i\left(\ddot{U}^\dagger\eta U-U^\dagger\eta\ddot{U}\right)\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\rho(t)\left(i\dot{H}_c(t)\eta-H_c^2(t)\eta+i\eta\dot{H}_c(t)+\eta H_c^2(t)\right)\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\eta\left(i\left\{\dot{H}_c(t),\rho(t)\right\}+\left[H_c^2(t),\rho(t)\right]\right)\right]=0. \end{aligned} \tag{181}$$

For this to hold for every $\eta\in\mathfrak{u}(\mathcal{H})$ it follows that

$$i\{\dot{H}_c(t),\rho(t)\}+[H_c^2(t),\rho(t)]=0, \tag{182}$$

which concludes our proof.

A.5 Charging Correlated Batteries

Calculation of Eqs. (99) and (100)

First consider the bounded bandwidth case. Let $\mathbb{1}_d$ be the identity operator of dimension d. By taking the Hilbert-Schmidt norm of Eq. (98) we find

$$\begin{aligned}\operatorname{Tr}\left[\left(H_c^{(N)}\right)^2\right] &= N \operatorname{Tr}\left[H_c^2 \otimes \bigotimes_{k=1}^{N-1} \mathbb{1}_d\right] + \binom{N}{2} \operatorname{Tr}\left[H_c \otimes H_c \otimes \bigotimes_{k=1}^{N-2} \mathbb{1}_d\right] \\ &= N d^{N-1} \operatorname{Tr}\left[H_c^2\right] + d^{N-2}\binom{N}{2} \operatorname{Tr}\left[H_c\right]^2,\end{aligned} \tag{183}$$

where $\binom{N}{2}$ is the binomial coefficient. We want to choose a suitable value Ω^2 that bounds Eq. (183) from above. Denote the latter term of the right-hand side by $\epsilon \geq 0$, which vanishes for time-optimal processes due to Remark 13. This allows us to write

$$N d^{N-1} \operatorname{Tr}\left[H_c^2\right] \leq \Omega^2 - \epsilon. \tag{184}$$

Due to Eq. (11) the left-hand side is bounded from above by $N d^{N-1}\omega^2$, hence

$$N d^{N-1}\omega^2 = \Omega^2 - \epsilon \leq \Omega^2. \tag{185}$$

If $\Omega^2 > N d^{N-1}\omega^2$ then $\epsilon > 0$, hence Eq. (183) can never saturate the bound on the resource with local time-optimal processes. Since we will want to compare local and global time-optimal processes we instead choose $\Omega^2 = N d^{N-1}\omega^2$.

Consider now the bounded variance case. The variance of $\rho^{(N)}$ relative the joint control Eq. (98) is

$$\operatorname{Var}(H_c^{(N)}, \rho^{(N)}) = \operatorname{Tr}\left[\left(H_c^{(N)}\right)^2 \rho^{(N)}\right] - \operatorname{Tr}\left[H_c^{(N)}\rho^{(N)}\right]^2. \tag{186}$$

We calculate each term individually and find that

$$\begin{aligned}\operatorname{Tr}\left[\left(H_c^{(N)}\right)^2 \rho^{(N)}\right] &= N \operatorname{Tr}\left[H_c^2\rho \otimes \bigotimes_{k=1}^{N-1} \rho\right] + \binom{N}{2} \operatorname{Tr}\left[H_c\rho \otimes H_c\rho \otimes \bigotimes_{k=1}^{N-2} \rho\right] \\ &= N \operatorname{Tr}\left[H_c^2\rho\right] + \binom{N}{2} \operatorname{Tr}\left[H_c\rho\right]^2,\end{aligned} \tag{187}$$

and

$$\operatorname{Tr}\left[H_c^{(N)}\rho^{(N)}\right]^2 = \left(N \operatorname{Tr}\left[H_c\rho \otimes \bigotimes_{k=1}^{N-1} \rho\right]\right)^2 = N^2 \operatorname{Tr}\left[H_c\rho\right]^2. \tag{188}$$

After adding Eqs. (187) and (188) together and simplifying we find

$$\operatorname{Var}(H_c^{(N)}, \rho^{(N)}) = N\operatorname{Var}(H_c, \rho) - \binom{N}{2} \operatorname{Tr}\left[H_c\right]^2. \tag{189}$$

Again we want to bound Eq. (189) from above by some suitable value Ω^2. Denote the second term on the right-hand side by $\epsilon \geq 0$, with equality for horizontal single battery controls $H_c = H_c^h$, see Section 4. We find that

$$N\operatorname{Var}(H_c, \rho) \leq \Omega^2 + \epsilon. \tag{190}$$

and with Eq. (12)

$$N\omega^2 = \Omega^2 + \epsilon \geq \Omega^2. \tag{191}$$

We want the resource Ω^2 to allow all choices of single battery controls H_c. If $\Omega^2 < N\omega^2$ then $\epsilon > 0$, and Eq. (189) can never be saturated with horizontal controls. Again, since we will want to compare time-optimal processes we need to allow all time-optimal processes. Therefore we make the choice $\Omega^2 = N\omega^2$.