REFINED CATALAN AND NARAYANA CYCLIC SIEVING

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Abstract. We prove several new instances of the cyclic sieving phenomenon (CSP) on Catalan objects of type A and type B. Moreover, we refine many of the known instances of the CSP on Catalan objects. For example, we consider triangulations refined by the number of “ears”, non-crossing matchings with a fixed number of short edges, and non-crossing configurations with a fixed number of loops and edges.

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1. Introduction

The original inspiration for this paper is a natural interpolation between type A and type B Catalan numbers. For \( n \geq 0 \) consider the expression

\[
\binom{2n}{n} - \binom{2n}{n-s-1}.
\]

For \( s = 0 \), we recover the \( n \)th Catalan number and for \( s = 1 \), we recover the \((n+1)\)th Catalan number. When \( s = n \), we obtain the central binomial coefficient \( \binom{2n}{n} \), which is known as the \( n \)th type B Catalan number, see [Arm09]. There are several combinatorial families of objects which are counted by the expression in (1), certain standard Young tableaux and lattice paths to name a few. The expression in (1)
has the $q$-analog given by the difference of $q$-binomials
\[ \binom{2n}{n}_q - q^{s+1} \binom{2n}{n-s-1}_q. \]  
(2)

For $s \in \{0, 1, n\}$, the polynomials in (2) appear in instances of the cyclic sieving phenomenon. Furthermore, it follows from [APRU20, Theorem 46] that there exist group actions such that the polynomials in (2) exhibit cyclic sieving for all $s \in \{0, 1, \ldots, n\}$.

**Definition 1** (Cyclic sieving, [RSW04]). Let $X$ be a set and $C_n$ be the cyclic group of order $n$ acting on $X$. Let $f(q) \in \mathbb{N}[q]$. We say that the triple $(X, C_n, f(q))$ exhibits the cyclic sieving phenomenon (CSP) if for all $d \in \mathbb{Z}$,
\[ |\{x \in X : q^d \cdot x = x\}| = f(\xi^d) \]  
(3)

where $\xi$ is a primitive $n^{th}$ root of unity.

Note that it follows immediately from the definition that $|X| = f(1)$. In the study of cyclic sieving, it is mainly the case that the $C_n$-action and the polynomial $f(q)$ are natural in some sense. The group action could be some form of rotation or cyclic shift of the elements of $X$. The polynomial usually has a closed form and is also typically the generating polynomial for some combinatorial statistic defined on $X$. See B. Sagan’s article [Sag11] for a survey of various types of CSP instances.

Many known instances of the cyclic sieving phenomenon involve a set $X$ whose size is a Catalan number. Once such a CSP triple is obtained, one can ask if $X$ can be partitioned $X = \sqcup_j X_j$ in such a way that the group action on $X$ induces a group action on $X_j$ for all $j$, and, in that case, also ask if there is a refinement of the CSP triple in question.

**Definition 2** (Refinement of cyclic sieving). The family $\{(X_j, C_n, f_j(q))\}_j$ of CSP triples is said to refine the CSP triple $(X, C_n, f(q))$ if
\begin{itemize}
  \item $\bigcup_j X_j = X$,
  \item $\sum_j f_j(q) = f(q)$ and
  \item the $C_n$-action on $X_j$ coincides with the $C_n$-action on $X$ restricted to $X_j$, for all $j$.
\end{itemize}

Typically, the sets $X_j$ are of the form $X_j = \{x \in X : \text{st}(x) = j\}$ for some statistic $\text{st} : X \to \mathbb{N}$ that is preserved by the group action. Examples of such statistics are the number of cyclic descents of a word, the number of blocks of a partition, and the number of ears of a triangulation of an $n$-gon—all with the group action being (clockwise) cyclic rotation. Throughout the paper, we shall consistently use the order of the group (or group generator) as subscript. For example, rotation by $2\pi/n$ is denoted $\text{rot}_n$.

For $s \in \{0, n\}$, the $q$-analog in (2) admits a natural refinement, so that the type $A$ and type $B$ $q$-Narayana polynomials are recovered. The $q$-Narayana polynomials can be used to refine the aforementioned instances of the CSP. It is therefore natural to ask if there is a $q$-analog of (1) for arbitrary $s \in \{0, 1, \ldots, n\}$ which also exhibits similar combinatorial properties as the type $A$ and type $B$ $q$-Narayana polynomials. We discuss partial results and motivations behind this problem in Section 3.

In the process of analyzing this intriguing question, we discovered several new instances of the cyclic sieving phenomenon. Some concern new $q$-analogs of Catalan numbers, while others refine known instances. In the tables in Section 2.5 we present...
a comprehensive (but most likely incomplete) overview of the current state-of-the-art regarding the cyclic sieving phenomenon involving Catalan and Narayana objects of type $A$ and $B$.

### 1.1. Overview of our results

We only highlight some of the results in our paper; in addition we also prove several other results which fill gaps in the literature. In Section 4, the main result is the following theorem, which is a new refined CSP instance on Catalan objects. It can be stated either in terms of promotion (denoted $\partial_{2n}$) on two-row standard Young tableaux with $k$ cyclic descents, \( \text{SYT}_{cdes}(n^2, k) \), or non-crossing perfect matchings with $k$ short edges, \( \text{NCM}_{sh}(n, k) \).

**Theorem 3 (Theorem 27).** Let \( k, n \geq 2 \) be natural numbers and let

\[
\text{Syt}(n, k; q) := \frac{q^{k(k-2)}(1 + q^n)^{n+1}}{[n+1]_q} \begin{bmatrix} n \end{bmatrix}_q \begin{bmatrix} n - 2 \end{bmatrix}_q.
\]

Then

\[
\sum_k \text{Syt}(n, k; q) = \text{Cat}(n; q),
\]

and the triples

\( (\text{SYT}_{cdes}(n^2, k), \langle \partial_{2n} \rangle, \text{Syt}(n, k; q)) \)

and

\( (\text{NCM}_{sh}(n, k), \langle \text{rot}_{2n} \rangle, \text{Syt}(n, k; q)) \)

exhibit the cyclic sieving phenomenon.

In Section 5, we study the set of so-called non-crossing (1,2)-configurations on $n$ vertices, which we denote by \( \text{NCC}(n+1) \). The cardinality of this set is the Catalan number \( \text{Cat}(n+1) = \binom{2n}{n} - \binom{2n}{n-2} \). We define a simple “rotate-and-flip” action on \( \text{NCC}(n+1) \) which has order $2n$ and is reminiscent of promotion.

**Theorem 4 (Theorem 33).** The triple

\[
\left( \text{NCC}(n+1), \langle \text{twist}_{2n} \rangle, \begin{bmatrix} 2n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n \end{bmatrix}_q \right)
\]

exhibits the cyclic sieving phenomenon.

Note that we use a quite non-standard $q$-analog of the Catalan numbers here, which has not appeared in the context of cyclic sieving before. Cyclic sieving on non-crossing (1,2)-configurations was studied earlier by M. Thiel [Thi17], with rotation as the group action. In Theorem 38 and Corollary 39, we refine Thiel’s result. In particular, we obtain a new CSP instance involving the $q$-Narayana polynomial \( \text{Nar}(n+1, k; q) \).

In Section 6, we study various instances of cyclic sieving involving the type $B$ Catalan numbers, \( \binom{2n}{n} \). Some results have more or less appeared in earlier works, but we make some of the results more explicit. One novel result is a type $B$ version of Theorem 4 where we consider the twist action on type $B$ non-crossing (1,2)-configurations. Briefly, such objects are obtained from elements in \( \text{NCC}(n) \) by choosing to mark one edge.
Theorem 5 (Theorem 46). The triple
\[
\left( \text{NCC}^B(n+1), \langle \text{twist}_{2n}^2 \rangle, \left[ \frac{2n}{n} \right]_q \right)
\]
exhibits the cyclic sieving phenomenon.

As in type A, we also obtain a refined cyclic sieving result in Theorem 50 where we consider rotation instead.

In Section 7, we briefly consider two-column semistandard Young tableaux, and note in Theorem 54 that \((\text{SSYT}(2^k, n), \langle \hat{\partial}_n \rangle, \text{Nar}(n+1, k+1; q))\) is a CSP triple, where \(\hat{\partial}_n\) denotes the so-called \(k\)-promotion and \(\text{SSYT}(2^k, n)\) is the set of semistandard Young tableaux of the rectangular shape \(2^k\) whose maximal entry is at most \(n\).

In Section 8, we refine the classical CSP triple on triangulations of an \(n\)-gon by taking ears into consideration. An ear in a triangulation is a triangle formed by three cyclically consecutive vertices. We let \(\text{TRI}_{\text{ear}}(n, k)\) denote the set of triangulations of an \(n\)-gon with \(k\) ears.

Theorem 6 (Theorem 57 and Theorem 58). Let \(2 \leq k \leq \frac{n}{2}\) and let
\[
\text{Tri}(n, k; q) := q^{(k-2)} \left[ \frac{n}{k} \right]_q \left[ \frac{n-4}{2k-4} \right]_q \text{Cat}(k-2; q) \left( \sum_{j=0}^{n-2k} q^{j(n-2)} \left[ \frac{n-2k}{j} \right]_q \right).
\]
Then
\[
\sum_k \text{Tri}(n, k; q) = \text{Cat}(n-2; q),
\]
and
\[
\left( \text{TRI}_{\text{ear}}(n, k), \langle \text{rot}_n \rangle, \text{Tri}(n, k; q) \right)
\]
exhibits the cyclic sieving phenomenon.

In the last section, we consider another natural interpolation between type A and type B Catalan objects and prove a cyclic sieving result using standard methods.

Finally, a word about the proofs in this paper. There are traditionally two different approaches to proving instances of the cyclic sieving phenomenon — combinatorial or representation-theoretical (using vector spaces and diagonalization). In this paper we exclusively use the combinatorial approach, meaning that we need to explicitly evaluate the CSP-polynomials at roots of unity and also count the number fixed points of the sets under the group actions. It may also involve the use of equivariant bijections to derive new CSP triples from the previously known ones.

2. Preliminaries

We shall use standard notation in the area of combinatorics, see the go-to references [Sta01 Mac95]. In particular, \([n] := \{1, 2, \ldots, n\}\) and it should not be confused with the \(q\)-analog \([n]_q\) defined further down.

\(^1\)Or “brute-force.”
2.1. **Words and paths.** Given a word \( w = w_1 \cdots w_n \in [k]^n \), a **descent** is an index \( i \in [n-1] \) such that \( w_i > w_{i+1} \). We let the **major index**, denoted \( \text{maj}(w) \), be the sum of the descents of \( w \). An **inversion** in \( w \) is a pair of indices \( i, j \in [n] \) such that \( i < j \) and \( w_i > w_j \). We let \( \text{inv}(w) \) be the number of inversions of \( w \). Let \( \text{BW}(n, k) \) denote the set of binary words of length \( n \) with exactly \( k \) ones.

Let \( \text{PATH}(n) \) be the set of paths from \((0,0)\) to \((n,n)\) using north, \((1,0)\), and east, \((0,1)\), steps. A **peak** is a north step followed by an east step, and a **valley** is an east step followed by a north step. We have an obvious bijection \( \text{PATH}(n) \leftrightarrow \text{BW}(2n, n) \) where we identify north steps with zeros. Given \( P \in \text{PATH}(n) \), we let \( \text{maj}(P) \) be defined as the sum of the positions of the valleys of the path \( P \). Observe that this coincides with the major index of the corresponding binary word, as valleys correspond to descents. We shall also let \( \text{pmaj}(P) \) denote the sum of the positions of the peaks. For a path \( P \in \text{PATH}(n) \), we let the **depth**, \( \text{depth}(P) \) be the largest value of \( r \geq 0 \) such that the path touches the line \( y = x - r \). Let us define \( \text{PATH}_s(n) \subseteq \text{PATH}(n) \) as the set of paths with \( \text{depth}(P) \leq s \). We set \( \text{DYCK}(n) := \text{PATH}_0(n) \).

2.2. **\( q \)-analogs.** Roughly, a \( q \)-analogue of a certain expression is a rational function in the variable \( q \) from which we can obtain the original expression in the limit \( q \to 1 \).

**Definition 7.** Let \( n \in \mathbb{N} \). Define the **\( q \)-analogue** of \( n \) as \( [n]_q := 1 + q + \cdots + q^{n-1} \). Furthermore, define the **\( q \)-factorial** of \( n \) as \( [n]_q! := [n]_q[n-1]_q \cdots [1]_q \). Lastly, the **\( q \)-binomial coefficient** is defined as

\[
\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q![k]_q!} = \sum_{b \in \text{BW}(n,k)} q^{\text{inv}(b)} = \sum_{b \in \text{BW}(n,k)} q^{\text{maj}(b)}
\]

if \( n \geq k \geq 0 \), and \( [n]_q := 0 \) otherwise. Note that the \( q \)-binomial coefficients are polynomials in the variable \( q \), see [Sta11] for more background. The **\( q \)-multinomial coefficients** are defined in a similar manner.

**Theorem 8 (\( q \)-Vandermonde identity).** The \( q \)-Vandermonde identity states that for non-negative integers \( a, b, c \), we have that

\[
\binom{a+b}{c}_q = \sum_j q^{j(a-c+j)} \binom{a}{c-j}_q \binom{b}{j}_q.
\] (4)

**Theorem 9 (\( q \)-Lucas theorem, see e.g. [Sag92]).** Let \( n, k \in \mathbb{N} \). Let \( n_1, n_0, k_1, k_0 \) be the unique natural numbers satisfying \( 0 \leq n_0, k_0 \leq d - 1 \) and \( n = n_1 d + n_0 \), \( k = k_1 d + k_0 \). Then

\[
\binom{n}{k}_q \equiv \binom{n_1}{k_1}_q \binom{n_0}{k_0}_q \pmod{\Phi_d(q)}
\]

where \( \Phi_d(q) \) is the \( d \)-th cyclotomic polynomial. In particular, we have

\[
\binom{n}{k}_\xi = \binom{n_1}{k_1}_\xi \binom{n_0}{k_0}_\xi
\] (5)

if \( \xi \) is a primitive \( d \)-th root of unity.

When \( \xi \) is a root of unity, let \( o(\xi) \) denote the smallest positive integer with the property that \( \xi^{o(\xi)} = 1 \). The following is a standard lemma that should not need a proof.
Lemma 10. Let $n, k, d \in \mathbb{N}$ and let $\xi$ be a primitive $n^{th}$ root of unity. Then

$$\lim_{q \to \xi^d} \frac{[n]_q}{[k]_q} = \begin{cases} \frac{n}{k} & \text{if } o(\xi^d) \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

We will use Theorem 9 and Lemma 10 in later sections.

Lemma 11. Let $\xi$ be a primitive $n^{th}$ root of unity, and suppose that $f \in \mathbb{N}[q]$ is such that $f(\xi^j) \in \mathbb{Z}$ for all $j \in \mathbb{Z}$. Then for all $j \in \mathbb{Z}$, $f(\xi^j) = f(\xi^{\gcd(j,n)})$.

Proof. In [AA19, Lem. 2.2], it is proved that $f$ (up to mod $q^n - 1$) is a linear combination of

$$h_d(q) := \sum_{i=0}^{n/d-1} q^{di} = \frac{[n]_q}{[d]_q} \quad \text{where } d \mid n.$$ 

It then suffices to verify that

$$h_d(\xi^j) = h_d(\xi^{\gcd(j,n)}) = \begin{cases} \frac{n}{d} & \text{if } \frac{n}{\gcd(j,n)} \mid d, \\ 0 & \text{otherwise} \end{cases}$$ 

for all $d \mid n$, $j \in \mathbb{Z}$, which is straightforward by using Lemma 10. \hfill \Box

Hence, if we know that $f(\xi^d) \in \mathbb{Z}$ for all $d \in \mathbb{Z}$, it suffices to verify (3) for all $d \mid n$. There is a related result about computing the number of fixed points.

Lemma 12. Suppose that $C_n = \langle g \rangle$ acts on the set $X$. If $d \in \mathbb{Z}$, then

$$|\{x \in X : g^d \cdot x = x\}| = |\{x \in X : g^{\gcd(n,d)} \cdot x = x\}|$$

Proof. Note that all elements of $C_n$ with order $o$ generate the same subgroup $S \subseteq C_n$. If $h, h'$ are both of order $o$, then $\langle h \rangle = \langle h' \rangle = S$, and $h \cdot x = x$ implies that $h' \cdot x = h^e \cdot x = x$, for some $e \in \mathbb{Z}$. \hfill \Box

Lemma 11 and Lemma 12 are useful facts and are used implicitly in many papers. We shall use them without further mention throughout the paper.

2.3. Catalan and Narayana numbers. The Catalan numbers $\text{Cat}(n) := \frac{1}{n+1} \binom{2n}{n}$ are indexed by natural numbers. These numbers occur frequently in combinatorics, see [OEIS A000108] in the OEIS, and give the cardinalities of many families of combinatorial objects. For the purpose of this paper, we note that the following sets all have cardinality $\text{Cat}(n)$.

- $\text{DYCK}(n)$: the set of Dyck paths of size $n$, that is, the subset of paths in $\text{PATH}(n)$ which never touch the line $y = x - 1$,
- $\text{SYT}(n^2)$: the set of standard Young tableaux with two rows of length $n$,
- $\text{NCP}(n)$: the set of non-crossing partitions on $n$ vertices,
- $\text{NCM}(n)$: the set of non-crossing matchings on $2n$ vertices,
- $\text{TRI}(n)$: the set of triangulations of an $(n+2)$-gon,
- $\text{NCC}(n)$: the set of non-crossing $(1,2)$-configurations on $n-1$ vertices.

Examples of such objects are listed in Appendix A.
Throughout this paper, we use MacMahon’s $q$-analog of the Catalan numbers. For any natural number $n$, the $n^{th}$ $q$-Catalan number is defined by

$$\text{Cat}(n; q) := \frac{1}{[n+1]_q} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q = \left[ \begin{array}{c} 2n \\ n \end{array} \right] - q \left[ \begin{array}{c} 2n \\ n-1 \end{array} \right]_q,$$

$$= \sum_{P \in \text{DYCK}(n)} q^{\text{maj}(P)} = \sum_{T \in \text{SYT}(n^2)} q^{\text{maj}(T)-n}.$$  (6) (7)

A definition of $\text{maj}$ on standard Young tableaux can be found in the next section.

The Narayana numbers $\text{Nar}(n,k) := \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$, indexed by two natural numbers $n$ and $k$ such that $1 \leq k \leq n$, are also well-known and have many applications, see the OEIS entry A001263. The Narayana numbers refine the Catalan numbers in the sense that $\sum_{k} \text{Nar}(n,k) = \text{Cat}(n)$. For our purposes, it suffices to know that the following sets all have cardinality $\text{Nar}(n,k)$.

- $\text{DYCK}(n,k)$: the set of paths in $\text{DYCK}(n)$ with exactly $k$ peaks,
- $\text{SYT}(n^2,k)$: the set of tableaux in $\text{SYT}(n^2)$ with exactly $k$ descents,
- $\text{NCP}(n,k)$: the set of partitions in $\text{NCP}(n)$ with exactly $k$ blocks,
- $\text{NCC}(n,k)$: the set of non-crossing $(1,2)$-configurations in $\text{NCC}(n)$ such that the numbers of proper edges plus the number of loops is equal to $k-1$,
- $\text{NCM}(n,k-1)$: the set of non-crossing matchings in $\text{NCM}(n)$ with $k-1$ even edges.

The $q$-Narayana numbers are defined as the $q$-analog

$$\text{Nar}(n,k; q) := q^{k(k-1)} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\text{maj}(P)} = \sum_{P \in \text{DYCK}(n,k)} q^{\text{maj}(P)} = \sum_{T \in \text{SYT}(n^2,k)} q^{\text{maj}(T)-n}.$$ (8)

The $q$-Narayana numbers refine the $q$-Catalan numbers, that is, $\sum_{k} \text{Nar}(n,k; q)$ is equal to $\text{Cat}(n,q)$. We also mention that there is a bijection $\text{NCPtoDYCK}$ from $\text{NCP}(n,k)$ to $\text{DYCK}(n,k)$ described in Bijection 8. Thus,

$$\text{Nar}(n,k; q) = \sum_{\pi \in \text{NCP}(n,k)} q^{\text{maj}(\text{NCPtoDYCK}(\pi))}. $$

For more background, see [Sim94] and [ZZ11].

2.4. Type B Catalan numbers. We shall now describe the type $B$ analogs of the combinatorial objects we saw in the previous section. The type $B$ Catalan numbers $\text{Cat}^B(n)$ are defined as

$$\text{Cat}^B(n) := \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k}.$$ (9)

The type $B$ Narayana numbers $\text{Nar}^B(n,k)$ are defined as

$$\text{Nar}^B(n,k) := \binom{n}{k}. $$ (10)

The type $B$ Narayana numbers clearly refine the $B$ Catalan numbers, as can be seen from (9). Among other things, they count the number of elements in $\text{PATH}(n)$ with $k$ valleys. For a more comprehensive list, see A008459 in the OEIS, and also
the reference \cite{Arm09} for more background. The \textit{q-analogs of the type B Catalan numbers} and the \textit{type B q-Narayana numbers} are defined as

\[
\text{Cat}_B(n; q) := \binom{2n}{n}_q, \quad \text{Nar}_B(n, k; q) := q^{k^2} \binom{n}{k}_q \binom{n}{k}_q.
\] (11)

It is straightforward to verify that \(\text{Cat}_B(n; q) = \sum_{k=0}^{n} \text{Nar}_B(n, k; q)\). Moreover, one can show that

\[
\sum_{k=0}^{n} t^k \text{Nar}_B(n, k; q) = \sum_{T \in \text{SYT}((2n,n)/(n))} t^{\text{Des}(T)} q^{\text{maj}(T)}
\] (12)

\[
\sum_{P \in \text{PATH}(n)} t^{\text{valleys}(P)} q^{\text{maj}(P)},
\] (13)

see \cite{Sul98,Sul02}.

The following combinatorial interpretation of the type B q-Narayana numbers is mentioned in I. Macdonald’s book \cite[p. 400]{Mac95}. Let \(V\) be a \(2n\)-dimensional vector space over \(F_q\), and let \(U\) be an \(n\)-dimensional subspace of \(V\). Then \(\text{Nar}_B(n, k; q)\) is the number of \(n\)-dimensional subspaces \(U'\) of \(V\) such that \(\dim(U \cap U') = n - k\).

2.5. \textbf{Overview of the CSP on Catalan and Narayana objects.} Table 1 and Table 2 list the state-of-the-art of the CSP on Catalan-type objects of type A and B respectively, including the results proven in the present paper. Examples of such objects can be found in Appendix A. We use several bijections (described in Appendix B) between Catalan and Narayana objects, see Figure 1.

The bijections in Figure 1 respect a Narayana refinement and so, for example, \(\text{SYT}(n^2, k) \rightarrow \text{NCM}(n)\) (Bijection 6) and \(\text{NCM}(n) \rightarrow \text{DYCK}(n)\) (Bijection 4), but they do not respect the particular Narayana refinements.

The bijections in Figure 1 respect a Narayana refinement and so, for example, \(\text{SYT}(n^2, k)\), \(\text{NCM}(n, k - 1)\) and \(\text{NCP}(n, k)\) are all equinumerous. Furthermore, composing the natural bijections Bijection 6 and the inverse of Bijection 7, we get that promotion on SYT corresponds to rotation on non-crossing matchings.
Table 1. The current state-of-the-art regarding cyclic sieving on type A Catalan objects, including the new results presented in this article.

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<th>Group &amp; polynomial</th>
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<tr>
<td>Proposition 20</td>
<td>$\text{Nar}(n, k; q)$</td>
</tr>
<tr>
<td>Non-cross (1,2)-config.</td>
<td>Rotation $\text{rot}_n$</td>
</tr>
<tr>
<td>[Thi17]</td>
<td>$\text{Cat}(n+1; q)$</td>
</tr>
<tr>
<td>Non-cross. (1,2)-config. with $l$ loops and $e$ edges</td>
<td>Rotation $\text{rot}_n$</td>
</tr>
<tr>
<td>Theorem 38</td>
<td>$q^{e(e+1)+e(n+1)} [e, e, l, n-2e-l]_q$</td>
</tr>
<tr>
<td>Non-cross. (1,2)-config. with $k$ edges or loops</td>
<td>Rotation $\text{rot}_n$</td>
</tr>
<tr>
<td>Corollary 39</td>
<td>$\text{Nar}(n+1, k; q)$</td>
</tr>
<tr>
<td>Two-column SSYT, SSYT($2^k, n$)</td>
<td>$k$-promotion $\tilde{\partial}_n$</td>
</tr>
<tr>
<td>Theorem 54</td>
<td>$\text{Nar}(n+1, k+1; q)$</td>
</tr>
<tr>
<td>Non-cross. (1,2)-config.</td>
<td>Twisted rotation $\text{twist}_{2n}$</td>
</tr>
<tr>
<td>Theorem 33</td>
<td>$2n \overline{n} - q^{2n} \overline{n-2}_q$</td>
</tr>
</tbody>
</table>

However, promotion on standard Young tableaux does not preserve the number of descents, but rotation preserves the number of even edges of matchings. It follows that the cyclic sieving phenomenon on non-crossing matchings with a specified number of even edges does not correspond to one on SYT($n^2$) with a fixed number of descents with promotion as the action. In general, a specific Narayana refinement might be incompatible with a cyclic group action. By Bijection 6, here the compatible statistic one should use for SYT($n^2$) is the number of even entries in the first row.

A note on the general philosophy of the paper. Having many different sets of objects and well-behaved bijections between these sets turns out to be a very fruitful
approach to proving instances of the CSP. In this context, well-behaved often times means that the bijection is equivariant. If a group action looks complicated on a certain set, it can perhaps be made easier if one first applies an equivariant bijection and then studies the image. For example, promotion on SYT($n^2$) is complicated while rotation on NCM($n$) is easier. There is a type of converse of the above. If one has two different CSP triples with identical CSP-polynomials and whose cyclic groups have the same order, then there exists an equivariant bijection between these two sets (by sending orbits to orbits of the same size).

<table>
<thead>
<tr>
<th>Type $B$ set &amp; reference</th>
<th>Group &amp; polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary words BW($2n,n$)</td>
<td>Cyclic shift $\text{shift}_{2n}$</td>
</tr>
<tr>
<td>[RSW04 Prop. 4.4]</td>
<td>$[2n]_q$</td>
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<tr>
<td>Skew two-row SYT, SYT((2n,n)/(n))</td>
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<td>[SW12 Section 3.1]</td>
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</tr>
<tr>
<td>Type $B$ root poset order ideals OI$_B(n)$</td>
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<tr>
<td>[AST13 Thm. 1.5] and [SW12]</td>
<td>$[2n]_q$</td>
</tr>
<tr>
<td>Type $B$ non-crossing partitions NCP$^B(n)$</td>
<td>Rotation $\text{rot}_{2n}$</td>
</tr>
<tr>
<td>[AST13 Thm. 1.5]</td>
<td>$[2n]_q$</td>
</tr>
<tr>
<td>Marked (1,2)-configs.</td>
<td>Twisted rotation $\text{twist}^2_{2n}$</td>
</tr>
<tr>
<td>Theorem 46</td>
<td></td>
</tr>
<tr>
<td>Binary words BW($2n,n$)</td>
<td>Cyclic shift $\text{shift}_{2n}$</td>
</tr>
<tr>
<td>Proposition 44 [AST18 Thm. 1.5]</td>
<td>$q^{k(k-1)}(1+q^n)[n]_q[n-1]_q$</td>
</tr>
</tbody>
</table>

| Marked (1,2)-configs. with $e$ edges and $l$ loops | Rotation $\text{rot}_n$ |
| Theorem 50                  |                    |
| Marked (1,2)-configs. with $k$ edges and loops | Rotation $\text{rot}_n$ |
| Corollary 51                | Complicated        |
| Marked (1,2)-configs.      | Rotation $\text{rot}_n$ |
| Corollary 51                | Complicated        |
| Type $B$ NCP with $2k$ or $2k+1$ blocks | Rotation $\text{rot}_n$ |
| (30)                        | $q^{k^2}[n]_q^2$   |
| Type $B$ triangulations on $2n+2$ vertices | Rotation $\text{rot}_{n+1}$ |
| See [EF08 Thm. 4.1]         | $[2n]_q$           |

**Table 2.** The state-of-the-art regarding cyclic sieving on type $B$ Catalan objects, including the new results presented in this article.

3. **TYPE A/B-NARAYANA NUMBERS AND A QUEST FOR A $q$-ANALOG**

We shall now discuss a natural interpolation between type $A$ and type $B$ Catalan numbers. The following observation illustrates this interpolation. For any $s \geq 0$, the sets below are equinumerous:

1. the set of skew standard Young tableaux SYT((n + s, n)/(s)),
2. the set of lattice paths, PATH$_s(n)$,
the set of order ideals in the type $B$ root poset with at most $s$ elements on
the top diagonal.
Note that for $s = 0$, we recover sets of cardinality $\text{Cat}(n)$, and for $s = n$, we recover
sets of cardinality $\text{Cat}^B(n)$. Bijective arguments are given below in Proposition 13
and Proposition 14.
Let $T \in \text{SYT}(\lambda/\mu)$ where the diagram of $\lambda/\mu$ has $n$ boxes. A descent
of $T$ is an integer $j \in \{1, \ldots, n-1\}$ such that $j+1$ appears in a row below $j$. The major
index of $T$ is the sum of the descents. The major-index generating function for skew
standard Young tableaux is defined as
$$f^{\lambda/\mu}(q) := \sum_{T \in \text{SYT}(\lambda/\mu)} q^{\text{maj}(T)}$$
when $\lambda/\mu$ is a skew shape. Our motivation for studying this polynomial is [APRU20, Thm. 46] which states that for any skew shape $\lambda/\mu$ where each row contains a
multiple of $m$ boxes, there must exist some cyclic group action $C_m$ of order $m$ such
that
$$(\text{SYT}(\lambda/\mu), C_m, f^{\lambda/\mu}(q))$$
is a CSP triple. We do not know how such a group action looks like except in the case $m = 2$. In that case one can use evacuation, defined by Schützenberger [Sch63].

3.1. Skew standard Young tableaux with two rows. We now describe a bi-
jection between skew SYT with two rows and certain lattice paths.

Proposition 13. Given $s \in \{0, \ldots, n\}$, there is a bijection
$$\text{SYT}((n + s, n)/(s)) \rightarrow \text{PATH}_s(n)$$
which sends descents in the tableau to peaks in the path.

Proof. A natural generalization of the standard bijection works: an $i$ in the upper
or lower row corresponds to the $i$th step in the path being north or east, respectively.
Evidently, a descent in the tableau is sent to a peak in the path. ∎

Recall that for a SYT $T$ the statistic $\text{maj}(T)$ is the sum of the position of the
descents, which is then sent to $\text{pmaj}(P)$ which is the sum of the positions of the
peaks in the corresponding path $P$.

Let
$$X_{n,s}(q) := \sum_{P \in \text{PATH}_s(n)} q^{\text{pmaj}(P)} \quad \text{and} \quad Y_{n,s}(q) := \sum_{P \in \text{PATH}_s(n)} q^{\text{maj}(P)}.$$ 
By Proposition 13 we also have $X_{n,s}(q) = f^{(n+s,n)/(s)}(q)$.

Proposition 14 ([Kra89, Thm. 7]). For $n \geq 1$ and $n \geq s \geq 0$,
$$X_{n,s}(q) = \binom{2n}{n}_q - \binom{2n}{n-s-1}_q \quad \text{and} \quad Y_{n,s}(q) = \binom{2n}{n}_q - q^{s+1} \binom{2n}{n-s-1}_q.$$ (16)

In particular,
$$q^{-n}X_{n,0}(q) = Y_{n,0}(q) = \text{Cat}(n; q) \quad \text{and} \quad X_{n,n}(q) = Y_{n,n}(q) = \text{Cat}^B(n; q).$$
3.2. Root lattices in type $A/B$. The following illustrate the root ideals of $B_n$ where $n = 3$. There are in total $\binom{2n}{3} = 20$ such ideals. A root ideal is simply a lower set in the root poset—marked as shaded boxes in the diagrams below. Root ideals are also called non-nesting partitions of type $W$, where $W$ is the Weyl group of some root system.

An explicit bijection from the set of skew standard Young tableaux $\text{SYT}((n + s, n)/(s))$ to the root ideals of $B_n$ with at most $s$ elements on the top diagonal is described below. First, let $\text{OI}(n, s)$ be the set of root ideals with at most $s$ elements on the top diagonal.

**Bijection 1.** Let $a_1, a_2, \ldots, a_n$ be the top row of the skew tableau. We identify this top row using the bijection in Proposition 13 with a path $\alpha \in \text{PATH}_s(n)$ and get that $\text{depth}(\alpha) = \max_i \{a_i - 2i + 1\}$. Let $j$ be the smallest value for which the maximum is obtained, so $\text{depth}(\alpha) = a_j - 2j + 1$. We then define the map $\phi$ as changing the step $a_j - 1$, just before reaching maximal depth for the first time, from an east step to a north step. That is, $\phi(\alpha) = a_1, \ldots, a_{j-1}, a_j - 1, a_j, \ldots, a_n$. This new path ends at $(n - 1, n + 1)$ and has depth one less than $\alpha$. We repeat $\text{depth}(\alpha)$ times and get $\phi^{\text{depth}(\alpha)}(\alpha)$ which ends in $(n - \text{depth}(\alpha), n + \text{depth}(\alpha))$ and has depth zero. This path always starts with a north step, and the boxes below and to the right of it make up a root ideal $o$ in $B_n$ with $\text{depth}(\alpha)$ elements in the top diagonal. Since $\text{depth}(\alpha) \leq s$ this gives $o \in \text{OI}(n, s)$ and the desired map. See Figure 2 for an example. The inverse $\phi^{-1}$ is easily obtained as follows. Given a root ideal $o$, let $\beta(o)$ be the north-east path along its boundary, starting with an extra north step. Now, change the north step of $\beta(o)$ after the last time the path has reached maximum depth to an east step. The inverse of the bijection is obtained by iterating $\phi^{-1}$ until the path ends in $(n, n)$. The map $\phi$ has been used many times before, see e.g. [ALP19].

We naturally define $\text{maj}(o) := \text{maj}(\beta(o))$ and $\text{pmaj}(o) := \text{pmaj}(\beta(o))$ for $o \in \text{OI}(n, s)$.

**Proposition 15.** The map in Bijection 1 is a bijection so

$$|\text{OI}(n, s)| = \binom{2n}{n} - \binom{2n}{n - 1 - s}.$$ 

Furthermore,

$$\sum_{o \in \text{OI}(n, s)} q^{\text{pmaj}(o)} = \left[\binom{2n}{n}_q - \binom{2n}{n - s - 1}_q\right].$$
and
\[ \sum_{o \in \text{OI}(n,s)} q^{\text{maj}(o)} = {2n \choose n}_q - q \left[ \frac{2n}{n-s-1} \right]_q + \sum_{d=1}^s (1-q) \left[ \frac{2n}{n-d} \right]_q. \]

Proof. The map is clearly a bijection and the first formula follows. For the second statement note that the map \( \phi \) does not change the peaks and thus not pmaj, but it changes the position of one valley and decreases maj by 1. Thus the \( q \)-polynomial for pmaj is identical to \( X_{n,s}(q) \) in Proposition 14. The maj-generating polynomial for paths with depth at most \( s \) is \( Y_{n,s}(q) \) by Proposition 14. Thus, for paths having depth exactly \( d \), it is \( q^d \left[ \frac{2n}{n-d} \right]_q - q^{d+1} \left[ \frac{2n}{n-d-1} \right]_q \). A path with depth \( d \) is mapped by \( \phi^d \) to a root ideal with exactly \( d \) elements in the top diagonal. This gives the sum
\[ \sum_{o \in \text{OI}(n,s)} q^{\text{maj}(o)} = \sum_{d=0}^s \left( q^d \left[ \frac{2n}{n-d} \right]_q - q^{d+1} \left[ \frac{2n}{n-d-1} \right]_q \right) q^{-d}, \]
which simplifies to the formula given. \( \square \)

Remark 16. There is a notion of rowmotion as an action on order ideals. Unfortunately, this action does not seem to have a nice order when restricted to \( \text{OI}(n,s) \) for general \( s \), see [SW12]. For example, for \( n = 3 \) and \( s = 1 \) we have the following orbit of length 4, implying that the action does not have the order we are looking for (which is \( n = 3 \)).

Lemma 17. Let \( \xi \) be a primitive \((2n)^{th}\) root of unity. For all integers \( n > s \geq 0 \) and \( d \mid 2n \), we have
\[ Y_{n,s}(\xi^d) = \chi(d,2) \left( \frac{d}{2} \right) - (-1)^d \chi(n-s-1,2n/d) \left( \frac{d}{2n} \right). \]
where \( \chi(a,b) \) is equal to 1 if \( b \) divides \( a \) and 0 otherwise.

Proof. The evaluation follows from the \( q \)-Lucas theorem, Theorem [9]. \( \square \)

In light of [15], it would be of great interest to explicitly describe a group action \( C_n \) so that \( X_{n,s}(q) \) or \( Y_{n,s}(q) \) is the corresponding polynomial in a CSP triple (which
must exist due to \([15]\)). In Section 5 we give an explicit action in the case \(s = 1\), which gives a new CSP triple involving the Catalan numbers.

### 3.3. Narayana connection.

We discuss an open problem regarding the interpolation between type \(A\) and type \(B\) \(q\)-Narayana numbers. This problem is part of a broader set of questions regarding the interplay of cyclic sieving and characters in the symmetric group, see \([APRU20]\). We argue that the small special case discussed below is interesting in its own right.

Recall that

\[
\sum_{D \in \text{DYCK}(n)} q^{\text{maj}(D)} t^{\text{peaks}(D)} = \sum_{k=1}^{n} t^{k} \text{Nar}(n, k; q),
\]

where the sum ranges over Dyck paths of size \(n\). Hence, \(\sum_{k=1}^{n} \text{Nar}(n, k; q) = X_{n,0}(q)\). Note that the set of Dyck paths with \(k\) peaks is in bijection with the set of non-crossing set partitions of \([n]\) with \(k\) blocks.

**Problem 18 (Main Narayana problem).** Refine the expression

\[
Y_{n,s}(q) = \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q - q^{s+1} \left[ \begin{array}{c} 2n \\ n-s-1 \end{array} \right]_q
\]

for all \(s \geq 0\) in the same way as the \(q\)-Narayana numbers \(\text{Nar}(n, k; q)\) refine the case \(s = 0\).

This problem is not really interesting unless we impose some additional requirements. In Problem [18], we are hoping to find a family of polynomials, \(N(s, n, k; q) \in \mathbb{N}[q]\) with some of the following properties:

**Specializes to \(\text{Nar}(n, k; q)\):** For \(s = 0\), we have

\[
N(0, n, k; q) = \text{Nar}(n, k; q).
\]

**Refines the \(Y_{n,s}(q)\) in \([16]\):** We want that for all \(s \geq 0\), we have the identity

\[
\sum_{k=1}^{n} N(s, n, k; q) = \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q - q^{s+1} \left[ \begin{array}{c} 2n \\ n-s-1 \end{array} \right]_q.
\]

**Is given by some generalization of the the peak statistic:** We hope for some statistic \(\text{peaks}_s(P)\) such that \(\text{peaks}_0(P)\) is the usual number of peaks of a Dyck path and

\[
\sum_{P \in \text{PATH}_s(n)} q^{\text{maj}(P)} t^{\text{peaks}_s(P)} = \sum_{k=1}^{n} t^{k} N(s, n, k; q).
\]

(17)

We can alternatively consider some other family of combinatorial objects mentioned in \([3]\), such as type \(B\) root ideals with at most \(s\) elements on the top diagonal, or standard Young tableaux in \(\text{SYT}((n+s, n)/(s))\) with some type of generalized descents.

**Refines \(\text{Cat}^B(n)\) at \(s = n\):** For \(s = n\), we have a natural candidate

\[
N(n, n, k; q) = q^{k(k-1)} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_q \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q = [n+1]_q \text{Nar}(n, k; q).
\]

(18)
Note that $N(n, n, k; q)$ is not equal to $\text{Nar}^B(n, k; q)$ that appear in Section 2.4. The combinatorial interpretation in this case is as follows:

$$\sum_{P \in \text{PATH}(n)} q^{\text{maj}(P)} t^{\text{modpeaks}(P)} = \sum_{k=1}^{n} t^k q^{k(k-1)} \left[ \frac{n-1}{k-1} \right] q^{\left[ \frac{n+1}{k} \right]}$$

where a modified peak is any occurrence of 01 (north-east) in the path, plus 1 if the path ends with a north step.

**Palindromicity:** The Narayana numbers have quite nice properties. First of all, $\sum_{k=1}^{n} t^k \text{Nar}(n, k; q)$ is a palindromic polynomial (in $t$). For example, for $n = 5$, this sum is $t^5 + 10t^4 + 20t^3 + 10t^2 + t$. One would therefore hope that for fixed $s$ the sum $\sum_{k=1}^{n} t^k N(s, n, k; q)$ is palindromic. The $s = n$ candidate given by $[n+1]_q \text{Nar}(n, k; q)$ is also palindromic.

**Palindromicity II:** Each $N(s, n, k; q)$ is a palindromic polynomial (in $q$). This is true for $\text{Nar}(n, k; q)$ and the expression in (18).

**Gamma-positivity:** The sum $\sum_{k=1}^{n} t^k N(s, n, k; 0)$ is $\gamma(t)$-positive (see the survey [Ath18] for the definition). The corresponding statement seems to hold for the expression in (18). One might hope that the general expression $\sum_{k=1}^{n} t^k N(s, n, k; 0)$ also has $\gamma(t)$-positivity.

**Values at roots of unity and cyclic sieving:** We require that $N(s, n, k; \xi)$ is a non-negative integer whenever $\xi$ is an $n$th root of unity. This resonates well with the palindromicity properties, and cyclic sieving for (15). Taking (17) into account, we would like that for every $k \geq 0$,

$$\{P \in \text{PATH}_s(n) : \text{peaks}_s(P) = k\}, \langle \beta_n \rangle, N(s, n, k; q)$$

is a CSP triple for some action $\beta_n$ of order $n$. Note that such a refinement is known in the case $s = 0$, as shown in the table below. Note also that there cannot be a cyclic group action of order $2n$ that fits together with $N(0, n, k; q)$ in a CSP triple: for example, at $n = 4, k = 2$ this is not an integer at a primitive $24^{th}$ root of unity.

<table>
<thead>
<tr>
<th>Set</th>
<th>Group action</th>
<th>$q$-statistic</th>
<th>peak-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dyck paths</td>
<td>—</td>
<td>maj</td>
<td>peaks</td>
</tr>
<tr>
<td>Non-crossing partitions†</td>
<td>Rotation</td>
<td>maj</td>
<td>blocks</td>
</tr>
<tr>
<td>SYT($n^2$)</td>
<td>$\partial^2$</td>
<td>maj</td>
<td>—</td>
</tr>
<tr>
<td>Non-crossing matchings†</td>
<td>Rotation</td>
<td>maj</td>
<td>—</td>
</tr>
</tbody>
</table>

*Table 3.* We only have the full Narayana refinement picture for the non-crossing partition family. That is, there is a “peak”-statistic and a group action of order $n$ preserving the peak-statistic. Note that promotion on Dyck paths does not preserve the number of peaks. †For these sets maj is computed via a bijection to paths.

**Example 19.** For $n = 2, s = 1$, we have that $Y_{2,1}(q) = q^4 + q^3 + q^2 + q + 1$. We want to refine this into two polynomials corresponding to $k = 1, 2$. The criteria to have non-negative evaluations at roots at unity, here $-1$, tells us that $q^3$ and $q$ must be together with at least one other term each. By palindromicity II there are five
possibilities for $N(1, 2, 2; q)$: $q^4 + q^3 + q + 1, q^4 + q^3 + q^2 + q, q^4 + q^3 + q^2, q^4 + q^3$ and $q^4$.

4. Case $s = 0$ and non-crossing matchings

The goal of this section is to prove two Narayana-refinements of cyclic sieving on non-crossing perfect matchings by considering the number of even edges and short edges. The second result corresponds to a refinement of the CSP on $\text{SYT}(n^2)$ under promotion, where we refine the set by the number of cyclic descents.

4.1. Even edge refinement. Given a non-crossing perfect matching, let $\text{even}(M)$ denote the number of edges $\{i, j\}$ where $i < j$ and $i$ is even. We refer to them as even edges, and all non-even edges are called odd. Let $\text{NCM}(n, k)$ be the set of $M \in \text{NCM}(n)$ such that $\text{even}(M) = k$.

Note that for parity reasons an edge $\{i, j\}$ must have $i + j$ odd. Thus the set of non-crossing perfect matchings on $2n$ vertices with $k$ even edges is invariant under rotation by $\text{rot}_n$ since

- any odd edge $(i, 2n)$ is mapped to the even edge $(2, i + 2)$;
- any even edge $(j, 2n - 1)$ is mapped to the odd edge $(1, j + 2)$.

The first result is essentially just a restatement of [RSW04 Thm. 7.2].

**Proposition 20.** For $0 \leq k \leq n$, the triple $(\text{NCM}(n, k), \text{rot}_n, \text{Nar}(n, k + 1; q))$ exhibits the cyclic sieving phenomenon.

**Proof.** Mapping non-crossing matchings to non-crossing partitions via the inverse of $\text{NCPtoNCM}$ takes matchings with $k$ even edges to partitions with $k + 1$ blocks, see Bijection [□]. This CSP result was proven already in [RSW04 Thm. 7.2].

4.2. Short edge refinement.

**Definition 21.** We define promotion $\partial_{2n} : \text{SYT}(n^2) \to \text{SYT}(n^2)$ as the following composition of bijections:

$$\partial_{2n} := \text{SYTtoNCM}^{-1} \circ \text{rot}_{2n} \circ \text{SYTtoNCM}.$$ 

If $T \in \text{SYT}(n^2)$, we use the shorthand $\partial_{2n}(T)$ to mean $\partial_{2n}(T)$.

Promotion is originally defined for Young tableaux of all shapes using the so-called jeu-de-taquin. The notion has been generalized to arbitrary posets by R. Stanley, see [Sta09].

**Definition 22.** Let $T \in \text{SYT}(n^2)$. Define the cyclic descent set $c\text{Des}(T)$ as follows. We have $\text{Des}(T) = c\text{Des}(T) \cap [1, 2n - 1]$ and let $2n \in c\text{Des}(T)$ if and only if $1 \in c\text{Des}(\partial_{2n} T)$. Denote the number of cyclic descents of $T$ by $\text{cdes}(T) := |c\text{Des}(T)|$ and denote $\text{SYT}_{\text{cdes}}(n^2, k)$ the set of $T \in \text{SYT}(n^2)$ such that $\text{cdes}(T) = k$.

The above definition of cyclic descent set can be generalized in a straightforward manner to all rectangular standard Young tableaux—that is, tableaux of shape $\lambda = (a^b)$. In [Hua20], an explicit construction is given, where it is shown that all shapes which are not connected ribbons admit a type of cyclic descent statistic. It follows that one can define the set $\text{SYT}_{\text{cdes}}(\lambda, k)$ for all such shapes $\lambda$ as well.

The set $\text{SYT}_{\text{cdes}}(n^2, k)$ is in bijection with a certain subset of $\text{DYCK}(n)$ which we shall now describe. We first recall the standard bijection $\text{SYTtoDYCK}$ between
SYT\((n^2)\) and DYCK\((n)\): given a \(T \in \text{SYT}(n^2)\), let \(\text{SYTtoDYCK}(T) = w_1w_2 \cdots w_{2n}\) be the Dyck path where \(w_i = 0\) if \(i\) is in the top row and \(w_i = 1\) otherwise.

Call a Dyck path \(w_1w_2w_3 \cdots w_{2n}\) elevated if \(w_2w_3 \cdots w_{2n-1}\) is also a Dyck path. A Dyck path which is not elevated is called non-elevated. Elevated Dyck paths of size \(n\) are in natural bijection with Dyck paths of size \(n - 1\). The next lemma now easily follows.

**Lemma 23.** Let \(T \in \text{SYT}_{\text{cdes}}(n^2, k)\). Then \(2n \in \text{cDes}(T)\) if and only if the Dyck path \(\text{SYTtoDYCK}(T)\) is elevated.

It follows from Lemma 23 that the restriction of \(\text{SYTtoDYCK}\) to \(\text{SYT}_{\text{cdes}}(n^2, k)\) is a bijection to the set of \(D \in \text{DYCK}(n)\) such that \(D\) either is non-elevated and has \(k\) peaks or is elevated and has \(k - 1\) peaks. Hence, we get (using an argument by T. Došlić [Doš10, Prop. 2.1]),

\[
|\text{SYT}_{\text{cdes}}(n^2, k)| = \text{Nar}(n, k) - \text{Nar}(n - 1, k) + \text{Nar}(n - 1, k - 1)
= \frac{2}{n + 1} \binom{n + 1}{k} \binom{n - 2}{k - 2}.
\]  

(20)

These numbers are a shifted variant of the OEIS entry A108838. Define the following \(q\)-analog of these numbers. For any two natural numbers \(n\) and \(k\), let

\[
\text{Syt}(n, k; q) := \sum_{T \in \text{SYT}_{\text{cdes}}(n^2, k)} q^{\text{maj}(T) - n} = \sum_{D} q^{\text{maj}(D)}
\]  

(21)

where the second sum is taken over all \(D \in \text{DYCK}(n)\) that are either non-elevated with \(k\) peaks or elevated with \(k - 1\) peaks. For integers \(k, n \geq 1\) we claim that

\[
\text{Syt}(n, k; q) = \text{Nar}(n, k; q) - q^{k-1}\text{Nar}(n - 1, k; q) + q^{k-2}\text{Nar}(n - 1, k - 1; q).
\]  

(22)

To see this, consider the restriction of \(\text{SYTtoDYCK}\) to \(\text{SYT}_{\text{cdes}}(n^2, k)\). If \(D\) is an elevated Dyck path of size \(n\) with \(k\) peaks and \(D'\) is the corresponding Dyck path of size \(n - 1\), then \(\text{maj}(D) - \text{maj}(D') = k - 1\), as each of the \(k - 1\) valleys contribute one less to the major index in \(D'\) compared to in \(D\).

The polynomials \(\text{Syt}(n, k; q)\) refine the \(q\)-Catalan numbers, which is easily seen by comparing their definition with (6).

**Proposition 24.** For all integers \(n\),

\[
\sum_{k} \text{Syt}(n, k; q) = \text{Cat}(n; q).
\]

It is easy to see that \(\text{Syt}(0, 0; q) = \text{Syt}(1, 1; q) = 1\) and \(\text{Syt}(n, k; q) = 0\) for all other pairs of natural numbers \(n, k\) such that either \(n \leq 1\) or \(k \leq 1\). For larger \(n\) and \(k\), we have the following closed form for \(\text{Syt}(n, k; q)\).

**Lemma 25.** For all integers \(k, n \geq 2\),

\[
\text{Syt}(n, k; q) = \frac{q^{k-2}(1 + q^n) \binom{n + 1}{k} \binom{n - 2}{k - 2}}{[n + 1]_q [k]_q}.
\]  

(23)

Proof. We may restrict ourselves to the case when \(n \geq k\) as both sides of (23) are identically zero otherwise. We write \(\text{Syt}(n, k; q)\) using the expression in (22) and
expand the \( q \)-Narayana numbers to obtain
\[
\frac{q^{k(k-1)}}{[n]_q} \binom{n}{k} \frac{n}{[k]_q} - q^{k-1} \frac{q^{k(k-1)}}{[n-1]_q} \binom{n-1}{k} \frac{n-1}{[k-1]_q} \\
+ q^{k-2} \frac{q^{(k-1)(k-2)}}{[n-1]_q} \binom{n-1}{k-1} \frac{n-1}{[k-1]_q} \\
= \frac{q^{k(k-2)}}{[n+1]_q} \binom{n+1}{k} \frac{n-2}{[k-2]_q} \left( q^k \frac{[n-1]_q}{[k-1]_q} - q^{2k-1} \frac{[n-k+1]_q[n-k]_q}{[n]_q[k-1]_q} + \frac{[k]_q}{[n]_q} \right).
\]

The expression in the parentheses is then rewritten as
\[
\frac{q^k [n]_q [n-1]_q - q^{2k-1} [n-k+1]_q [n-k]_q + [k]_q [k-1]_q}{[n]_q [k-1]_q}.
\]

We must now show that this is equal to \( 1 + q^n \) or, equivalently, that the following identity holds:
\[
q^k [n]_q [n-1]_q - q^{2k-1} [n-k+1]_q [n-k]_q + [k]_q [k-1]_q = (1 + q^n) [n]_q [k-1]_q. \quad (24)
\]

If \( n = k \), the identity is clearly true, so we may assume that \( n > k \). By using \([j]_q = (1 - q^j)/(1 - q)\) for integers \( j \geq 1 \), we get the equivalent equation
\[
\frac{q^k (1 - q^n)(1 - q^{n-1})}{(1 - q)^2} + \frac{(1 - q^k)(1 - q^{k-1})}{(1 - q)^2} + \frac{q^{2k-1}(1 - q^{n-k+1})(1 - q^{n-k})}{(1 - q)^2}.
\]

Clearing denominators and expanding the products gives
\[
(q^{2n+k-1} + q^k - q^{n+k-1} - q^{n+k}) + (q^{2k-1} - q^{k-1} - q^k + 1) = \]
\[
(q^{k+2n-1} - q^{k-1} - q^{2n} + 1) + (q^{2n} - q^{k+n-1} - q^{k+n} + q^{2k-1})
\]
which evidently holds for all integers \( n, k \). \( \square \)

The edge \( xy \) in a non-crossing perfect matching is said to be short if either \( x = i \) and \( y = i + 1 \) for some \( i \) or if \( x = 1 \) and \( y = 2n \). If \( M \in \text{NCM}(n) \), then we denote \( \text{short}(M) \) its number of short edges and \( \text{NCM}_{sb}(n, k) \) the set of \( M \in \text{NCM}(n) \) such that \( \text{short}(M) = k \). The set \( \text{SYT}_{cdes}(n^2, k) \) is in a natural bijection with \( \text{NCM}_{sb}(n, k) \). To see this, we use the standard bijection \( \text{SYTtoNCM} \) between \( \text{SYT}(n^2) \) and \( \text{NCM}(n) \), see Bijection 3 in Appendix B.1

\[
\begin{array}{ccccccc}
1 & 2 & 5 & 6 & 8 \\
3 & 4 & 7 & 9 & 10
\end{array}
\]

\[
\xrightarrow{\text{SYTtoNCM}}
\]

It follows from our definition of promotion that \( \text{SYTtoNCM} \) is an equivariant bijection in the sense that
\[
\text{SYTtoNCM} (\partial_{2n} T) = \text{rot}_{2n} (\text{SYTtoNCM}(T)). \quad (25)
\]

From the definition of \( \text{SYTtoNCM} \) and (25), one can prove the following lemma.
Lemma 26. Let \( T \in \text{SYT}(n^2) \). Then \( x \in \text{cDes}(T) \) if and only if \( xy \), where \( x < y \), is a short edge in \( \text{SYT} \to \text{NCM}(T) \).

Theorem 27. Let \( n, k \) be natural numbers. The triple

\[
(\text{NCM}_{sh}(n, k), \{\text{rot}_{2n}\}, \text{Syt}(n, k; q))
\]

exhibits the cyclic sieving phenomenon.

Proof. Let \( \xi \) be a primitive \((2n)^{th}\) root of unity. Write \( k = k_1 o(\xi^d) + k_0 \) for the unique natural numbers \( k_1 \) and \( k_0 \) such that \( 0 \leq k_0 < o(\xi^d) \). Then, by dividing into cases and applying Theorem 9 (the \( q \)-Lucas theorem) twice, we get

\[
\text{Syt}(n, k; \xi^d) = \begin{cases} 
2 \binom{n+1}{k_1} \binom{n-2}{k-2} & \text{if } d = 2n, \\
2 \binom{n/o(\xi^d)}{k_1} \binom{n/o(\xi^d)-1}{k-1} & \text{if } o(\xi^d) \mid n \text{ and } k_0 = 0, \\
2 \binom{2n}{k+1} \binom{(n-3)/2}{k-1} & \text{if } o(\xi^d) = 2, n \text{ odd and } 2 \mid k, \\
0 & \text{otherwise.}
\end{cases}
\]

We prove that these evaluations agree with the number of fixed points in \( \text{NCM}_{sh}(n, k) \) under \( \text{rot}_{2n} \) on a case-by-case basis.

Case \( d = 2n \): Trivial.

Case \( o(\xi^d) \mid n \) and \( k_0 = 0 \): By using Bijection 3, we see that such rotationally symmetric perfect matchings are in bijection with the set \( \text{BW}^{k_1}((n/o(\xi^d)) \). To see that this set has the desired cardinality, we equate the two expressions in [32] and [33] and then take \( q = 1 \).

Case \( o(\xi^d) = 2, n \text{ odd and } 2 \mid k \): It is easy to check that the assertion holds in the case \( n = 3 \) and \( k = 2 \). It thus remains to show the assertion for \( n > 3 \). Such a non-crossing perfect matching must have a diagonal (an edge that connects two vertices \( i \) and \( i + n \) (mod \( 2n \)) that divides the matching into two halves. The diagonal can be chosen in \( n \) ways. The matching is now determined uniquely by one of its two halves. To choose one half, we choose a non-crossing matching on \((n-1)/2 \) vertices with \( k/2 \) short edges, not including a potential short edge between the vertices closest to the diagonal. Such a matching is either i) an element of \( \text{NCM}_{sh}((n-1)/2, k/2) \) which does not have an edge between the two vertices that lie closest to the diagonal or ii) an element of \( \text{NCM}_{sh}((n-1)/2, k/2 + 1) \) which has a short edge between the two vertices that lie closest to the diagonal.

Let us note that, in general, the fraction of elements in \( \text{NCM}_{sh}(n, k) \) that have a short edge adjacent to a given side is equal to \( k/2n \). This is easily seen by considering rotations of such a non-crossing perfect matching. Hence, in our case the number of matchings fixed by \( \text{rot}_{2n} \) is equal to

\[
n \left( \frac{n-1-k/2}{n-1} \right) | \text{NCM}_{sh} \left( \frac{n-1}{2}, \frac{k}{2} \right) | + \frac{k/2 + 1}{n-1} | \text{NCM}_{sh} \left( \frac{n-1}{2}, \frac{k}{2} + 1 \right) |
\]

Substituting the values from [20] (recall that \( |\text{NCM}_{sh}(a, b)| = |\text{Syt}_{cdes}(a^2, b)| \)), it remains to show that this expression is identical to the one given by \( \text{Syt}(n, k; -1) \). This can now be verified with a computer algebra system, such as Sage [Sag20].

The remaining cases: We need to show that, in all the remaining cases, there are no rotationally symmetric non-crossing perfect matchings. Suppose first that \( o(\xi^d) = 2, n \) is odd and \( 2 \nmid k \). It is clear that such a non-crossing perfect matching


must have a diagonal dividing the matching into two halves. The two halves are identical up to a rotation of $\pi$ radians and so, in particular, they must have the same number of short edges. In other words, the number of short edges must be even, contradicting $2 \nmid k$. Suppose next that $o(\xi^d) \mid n$ and $k_0 \neq 0$. Such a matching is completely determined by how the vertices $1, 2, \ldots, d$ are paired up. It follows that the number of short edges must be a multiple of $2n/d$, contradicting $k_0 \neq 0$.

Suppose lastly that $o(\xi^d) \mid 2n$ but $o(\xi^d) \nmid n$. To analyze this case, we first prove the following.

**Claim:** If $o(\xi^d) \mid 2n$ and $o(\xi^d) \nmid n$, then $d$ is odd.

**Proof of Claim.** The hypothesis implies that the number of 2’s in the prime factorization of $2n$ is equal to the number of 2’s in the prime factorization of $o(\xi^d)$. Hence, the number $2n/o(\xi^d)$ is odd. Combining this with the fact that $o(\xi^d) = 2n/gcd(2n, d)$ yields that $gcd(2n, d)$ is odd. But this implies that $d$ is odd, so we are done. □

We now use the claim and note that there cannot be any non-crossing perfect matchings that are fixed under rotation by an odd number of steps, except in the case when $o(\xi^d) = 2$ and $n$ is odd, i.e. when the matching has a diagonal. This exhausts all possibilities and thus the proof is complete. □

**Theorem 27** can be stated in an alternative way as follows. Since SYTtoNCM maps cyclic descents to short edges, we see that SYTcdes($\lambda, k$) is closed under promotion for all rectangular $\lambda$. Recall that $(\text{SYT}(\lambda), \langle \partial \rangle, \text{Cat}(n; q))$ exhibits the cyclic sieving phenomenon, for rectangular $\lambda$. In the case when $\lambda = (n, n)$, we have the following refinement with regards to the number of cyclic descents.

**Corollary 28.** Let $n, k$ be natural numbers. The triple

$$(\text{SYT}_{\text{cdes}}(n^2, k), \langle \partial_{2n} \rangle, \text{Syt}(n, k; q))$$

exhibits the cyclic sieving phenomenon.

A related result is alluded to by C. Ahlbach, B. Rhoades and J. Swanson in the presentation slides [Swa18]. They claim to have proven a refinement of the cyclic sieving phenomenon on standard Young tableaux with the group action being promotion in the Catalan case. It is not clear from the slides if they refine by cyclic descents or in some other way. Therefore, we cannot tell if their result is identical to Theorem 28 or not.

It follows from [Rho10] Lemma 3.3 that the number of cyclic descents remains fixed under promotion of rectangular standard Young tableaux. Experiments suggests that Corollary 28 generalizes to all rectangular standard Young tableaux. This would be a refinement of the famous CSP result on rectangular tableaux, see [Rho10] Theorem 1.3]. More precisely, we denote $f^\lambda_k(q) := \sum_T q^{maj(T)}$ where the sum is taken over all standard Young tableaux of shape $\lambda$ with exactly $k$ cyclic descents.

**Conjecture 29.** Let $n, m, k$ be natural numbers and put $\lambda = (n^m)$. The triple

$$\left(\text{SYT}_{\text{cdes}}(\lambda, k), \langle \partial_{nm} \rangle, q^{-\kappa(\lambda)} f^\lambda_k(q) \right)$$

exhibits the cyclic sieving phenomenon. Here, $\kappa(\lambda) := \sum_i (i - 1)\lambda_i$. 


5. Case $s = 1$ and non-crossing $(1, 2)$-configurations

For $s = 1$, there is a nice Catalan family, given by non-crossing $(1, 2)$-configurations described in [Sta15, Family 60]. In the first subsection, we introduce a twisted rotation action on such configurations, and prove a new instance of Catalan CSP. In the second subsection, we refine a CSP result of Thiel, where the group action is given by rotation.

5.1. A new Catalan CSP under twisted rotation. A non-crossing $(1, 2)$-configuration of size $n$ is constructed by placing vertices $1, \ldots, n-1$ around a circle, and then drawing some non-intersecting edges between the vertices. Here, we allow vertices to have a loop, which is counted as an edge. There are $\text{Cat}(n)$ elements in this family. Let $\text{NCC}(n)$ be the set of such objects of size $n$, and let $\text{NCC}(n, k)$ be the subset of those with $k-1$ edges, loops included. See Appendix A for a figure when $n = 3$.

Bijection 2 (Laser construction). See Figure 3 for an example. Let $P \in \text{DP}(n)$. Define the non-crossing $(1, 2)$-configuration $\text{DYCKtoNCC}(P)$ as follows. First, number the east-steps with $1, 2, \ldots, n-1$. Secondly, if there is a valley at $(i, j)$, draw a line (a laser) from $(i, j)$ to $(i + \Delta, j + \Delta)$, where $\Delta$ is the smallest positive integer such that $(i + \Delta, j + \Delta)$ lies on $P$. Now, consider an east-step ending in $(i_1, j_1)$ on $P$. If there is a laser drawn from $(i_1, j_1)$, then let $(i_2, j_2)$ be the vertex of $P$ where this laser ends. Then there is an edge between $j_1$ and $j_2 - 1$ in $\text{DYCKtoNCC}(P)$ (this can be a loop). The remaining vertices in $\text{DYCKtoNCC}(P)$ will be unmarked, that is, unpaired and without a loop.

Proposition 30. The map $\text{DYCKtoNCC}$ is a bijection $\text{DP}(n) \rightarrow \text{NCC}(n)$.

A proof of Proposition 30 can essentially be found in [Bod19, Prop. 6.5]. M. Bodnar studies so called $n+1, n$-Dyck paths and shows that these are in bijection with $\text{NCC}(n)$. It is not hard, however, to see that the set of $n+1, n$-Dyck paths is in bijection with $\text{DYCK}(n)$ by removing the first north-step.

Note that there is a natural correspondence between Dyck paths of size $n$ and paths of size $n-1$ that stay weakly above the diagonal $y = x-1$. If $P = w_1 w_2 \cdots w_{2n-1} w_{2n}$ is a Dyck path of size $n$, then let $P' = w_2 \cdots w_{2n-1} \in \text{PATH}_1(n-1)$. Furthermore, $P$ and $P'$ have the same number of valleys.

![Figure 3. An example of the bijection DYCKtoNCC in Bijection 2.](image)

Lemma 31. We have that $|\text{NCC}(n, k)| = \text{Nar}(n, k)$, the Narayana numbers.
Proof. The bijection \( \text{DYCKtoNCC} \) maps Dyck paths with \( k - 1 \) valleys to non-crossing \((1,2)\)-configurations with \( k - 1 \) edges. It remains to note that a Dyck path with \( k - 1 \) valleys has \( k \) peaks. \( \square \)

**Remark 32.** Recall that the Motzkin numbers \( M_i \) count the number of ways to draw non-intersecting chords on \( i \) vertices arranged around a circle, see A001006 in the OEIS. The set \( \{ C \in \text{NCC}(n + 1) : \text{loops}(C) = l \} \) has cardinality \( \binom{n}{l} M_{n-l} \) since we can first choose the \( l \) vertices which have loops, and then proceed by choosing one of the \( M_{n-l} \) possible arrangements of non-intersecting chords on the remaining \( n - l \) vertices.

Let \( \text{rot}_n \) denote rotation by one step, acting on \( \text{NCC}(n + 1) \). Furthermore, let \( \gamma \) denote the the action of removing the mark on vertex 1 if it is marked, and marking it if it is unmarked. It does not do anything if 1 is connected to an edge. We refer to this as a flip.

Let the twist action be defined as \( \text{twist}_{2n} := \text{rot}_n \circ \gamma \). It is straightforward to see that \( \text{twist}_{2n} \) generates a cyclic group of order \( 2n \) acting on \( \text{NCC}(n + 1) \). Alternatively, we can act by \( (\text{rot}_n \circ \gamma)^{n-1} \), which closely resembles promotion. Recall that promotion on SYT may be defined as a sequence of swaps, for \( i = 1, 2, \ldots, n-1 \), where swap \( i \) interchanges the labels \( i \) and \( i + 1 \) if possible.

![Figure 4. The result of \( \text{rot}_n \circ \gamma \) on an element in \( \text{NCC}(13) \).](image)

**Theorem 33** (A new cyclic sieving on Catalan objects). The triple

\[
\left( \text{NCC}(n + 1), (\text{twist}_{2n}), \binom{2n}{n} - q^2 \binom{2n}{n-2} \right)
\]

exhibits the cyclic sieving phenomenon. Note that

\[
\binom{2n}{n} - q^2 \binom{2n}{n-2} = \frac{[2]_q}{[n+2]_q} [2n+1]_q.
\]

Proof. We compute the number of fixed points under \( \text{twist}_{2n}^m \), where we may without loss of generality assume \( m \mid 2n \). There are two cases to consider, \( m \) odd and \( m \) even. In the first, we must, according to Lemma 17 show that the number of fixed points under \( \text{twist}_{2n}^m \) is

\[
\begin{cases} 
\binom{m}{(m-1)/2} & \text{if } m = n/2 \text{ is odd,} \\
0 & \text{otherwise.}
\end{cases}
\]

For the first expression we reason as follows. Since \( m \) is odd, any fixed point for such \( m \) must consist of a diagonal (an edge from \( i \) to \( i + m \)) and two rotationally
symmetrical halves, both consisting of \( m - 1 \) vertices. In such a non-crossing configuration, no vertex can be isolated. To see this, note that if vertex \( j \) is isolated, then so are \( j + km \pmod{n} \), \( k \in \mathbb{Z} \), but the \( j + km \pmod{n} \) would need to be both marked and unmarked, a contradiction. Thus, all vertices in the non-crossing configuration are incident to an edge and it is in fact a non-crossing matching. The diagonal can be chosen in \( m \) ways and a non-crossing matching on one of the two halves can be chosen in \( \text{Cat}((m-1)/2) \) ways, so there are \( m\text{Cat}((m-1)/2) = \binom{m}{(m-1)/2} \) fixed points.

In the second case above, if \( n = m \) then at least one vertex has to be isolated since \( m \) is odd, which implies there can be no fixed points. For \( n/m > 2 \), we use a “parity” argument. Since any isolated vertices among \( S = \{ n - m + 2, n - m + 3, \ldots, n, 1 \} \) change from unmarked to marked and vice versa under \( \text{twist}^m_{2n} \), the number of isolated vertices has to be even. Since \( m \) is odd, this implies there must be an odd number of edges from \( S \) to \( [n] \setminus S \) in a fixed point. However, note that \( S \) and the edges out of \( S \) completely determine the configuration. Hence the edges must have their other endpoints in the two neighboring intervals of length \( m \). But this violates being rotationally symmetric under rotations of \( m \) steps since the number of edges is odd.

In the case \( 2 \mid m \), according to Lemma 17 we must show that the number of fixed points under \( \text{twist}^m_{2n} \) is equal to
\[
\begin{cases}
\binom{2n}{n} - \binom{2n}{n+2} & \text{if } m = 2n, \\
\text{Cat}(n/2) & \text{if } m = n \text{ is even}, \\
\binom{m}{m/2} & \text{otherwise}.
\end{cases}
\]

Counting fixpoints for the first case is trivial. For the second expression, note that \( \text{twist}^m_{2n} \) is simply the action of flipping the markings on all vertices. A fixed point can therefore not have any isolated vertices. What remains are non-crossing matchings, which there are \( \text{Cat}(n/2) \) many of.

It remains to prove the third expression. Let \( m = 2d \).

**Case \( n \) and \( n/d \) are even.** In this case, the only possible invariant configurations are non-crossing matchings that are rotationally symmetrical when rotating \( 2d \) steps. Recall from Bijection 9 that such matchings are in bijection with \( \text{BW}(2d, d) \) and this set clearly has cardinality \( \binom{2d}{d} \).

**Case \( n/d \) is odd.** Here we can have fixed points under \( \text{twist}^m_{2n} \) with unpaired vertices, for examples see Figure 5. The orbit of a vertex \( j \) under the operation \( \text{twist}^m_{2n} \) is \( \{ j + dk \pmod{n} \} \) \( k \in \mathbb{Z} \) and if \( 1 \leq j \leq d \) is an unpaired, unmarked (that is, without a loop) vertex, the vertices \( \{ j + 2dk \pmod{n} \} \) \( 0 \leq k < n/2d \) must be unmarked whereas \( \{ j + 2dk \pmod{n} \} \) \( n/2d < k < n/d \) will be marked. Note that in the latter case \( j + 2dk \pmod{n} = j + d + 2dr \) for \( r = k - (n/d + 1)/2 \). Thus it suffices to understand the vertices from 1 to \( d \). We claim that for every \( 0 \leq i \leq \lfloor d/2 \rfloor \) we get a valid fixed point by choosing \( i \) left vertices and \( i \) right vertices and matching them in a non-crossing manner as in the previous case. Then we can choose to put a loop at any subset of the remaining \( d - 2i \) unpaired vertices. Every fixed point is now constructed exactly once. This gives a total of \( \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{i} \binom{d-i}{i} 2^{d-2i} \) fixed points. Finally we need to prove that this sum is equal to \( \binom{2d}{d} \). We will use a bijection to all possible subsets \( A \) of size \( d \) from two rows with numbers 1 to \( d \), the numbers in the top row being blue and the bottom row red. For a given \( i \) we choose \( i \) numbers and let both the red and the blue belong to \( A \) and then \( i \) numbers such that neither
blue nor red belong to $A$. Finally the term $2^{d-2i}$ corresponds to choosing any subset of the remaining $d-2i$ numbers such that the red numbers in that subset belong to $A$ and the blue in the complement are in $A$.

![Figure 5. A fixed point under twist$_{2n}^m$, $n = 12$, $m = 8$ and $2n/m = 3$, and below an orbit under twist$_{2n}^2$ of size 3.](image)

**Problem 34.** It would be nice to refine the CSP triple in Theorem 33 to the hypothetical Narayana case discussed in Section 3. That is, we want the following equality to hold:

$$\left[\frac{2n}{n}\right]_q - q^{2} \left[\frac{2n}{n-2}\right]_q = \sum_{k=1}^{n} N(1, n, k; q)$$

where $N(1, n, k; 1)$ is the number of NCC on $n$ vertices with $(k-1)$ edges. This is a natural consideration, as indicated by Lemma 31.

5.2. **A refinement of Thiel’s result.** Recall that rot$_n$ acts on non-crossing $(1, 2)$-configurations of size $(n + 1)$ via a $2\pi/n$-rotation. M. Thiel proved the following.

**Proposition 35** (See [Thi17]). Let $n \in \mathbb{N}$. The triple

$$(\text{NCC}(n+1), \langle\text{rot}_n\rangle, \text{Cat}(n+1; q))$$

exhibits the cyclic sieving phenomenon.

Denote $\text{NCC}(n + 1, e, l)$ the number of non-crossing $(1, 2)$-configurations with $n$ vertices, $e$ proper edges and $l$ loops. We determine an element in $\text{NCC}(n + 1, e, l)$ by first choosing the $2e$ vertices that are incident to some proper edge in $\binom{n}{2e}$ ways, then choosing a non-crossing matching among these $2e$ vertices in $\text{Cat}(e)$ ways and finally choosing $l$ of the remaining $n-2e$ vertices to be loops. Hence,

$$|\text{NCC}(n + 1, e, l)| = \binom{n}{2e} \text{Cat}(e) \binom{n-2e}{l}.$$  \hfill (26)
For any $e, l \in \mathbb{N}$, define the following $q$-analog of the above the expression:

\[
\text{Ncc}(n, e, l; q) := q^{c(e+1)+(n+1)}\left[\begin{array}{c} n \\ 2e \end{array}\right]_q \text{Cat}(e; q) \left[\begin{array}{c} n - 2e \\ l \end{array}\right]_q
\]

\[
= q^{c(e+1)+(n+1)}\left[\begin{array}{c} 1 \\ e + 1 \end{array}\right]_q \left[\begin{array}{c} n \\ e, l, n - 2e - l \end{array}\right]_q.
\]

**Example 36.** Consider the case with $n = 4$ and $k = 2$ proper edges and loops. We have

\[
\text{Ncc}(4, 2, 0; q) = q^6(1 + q^2)
\]

\[
\text{Ncc}(4, 1, 1; q) = q^7(1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5)
\]

\[
\text{Ncc}(4, 0, 2; q) = q^{10}(1 + q + 2q^2 + q^3 + q^4)
\]

It is easily verified that these polynomials refine $\text{Nar}(5, 3; q)$. That is,

\[
\text{Nar}(5, 3; q) = q^6(1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8)
\]

\[
= \text{Ncc}(4, 2, 0; q) + \text{Ncc}(4, 1, 1; q) + \text{Ncc}(4, 0, 2; q).
\]

**Lemma 37.** For every $k \geq 0$ we have the identity

\[
\text{Nar}(n + 1, k + 1; q) = \sum_{e+l = k} \text{Ncc}(n, e, l; q).
\]

**Proof.** Unraveling the definitions, it suffices to show that

\[
q^{k(k+1)} \frac{[n + 1]}{[n + 1]} \frac{[n + 1]}{[k + 1]} \frac{[n - 2e]}{[k - e]} \frac{[2e]}{[e + 1]}
\]

\[
= \sum_{0 \leq e \leq k} q^{c(e+1)+(n+1)(k-e)} [n - 2e]^{[k - e]} [e + 1]^{[e]}
\]

\[
\text{where we have omitted the } q\text{-subscripts for brevity. We start doing cancellations,}
\]

\[
q^{k(k+1)} \frac{[n]!}{[k + 1] [n - k] [n - k + 1]} = [n]! \sum_{0 \leq e \leq k} \frac{1}{[e + 1]} [k - e]^{[n - k - e]} [e]^{[e]}
\]

\[
\text{and additional cancellations and some rewriting gives}
\]

\[
q^{k(k+1)} \frac{[n + 1]}{[k + 1] [n - k] [n - k + 1]} = \sum_{0 \leq e \leq k} q^{c(e+1)+(n+1)(k-e)} [k - e]^{[n - k - e]} [e]^{[e]}
\]

\[
\text{Further rewriting now gives}
\]

\[
q^{k(k+1)} \frac{[n + 1]}{[n - k] [k + 1]} = \sum_{0 \leq e \leq k} q^{c(e+1)+(n+1)(k-e)} [k]^{[k - e]} [n - k + 1]^{[n - k - e]} [e]^{[e]}
\]

Thus, the identity we wish to prove is equivalent to showing that

\[
\left[\begin{array}{c} n + 1 \\ k + 1 \end{array}\right]_q = \sum_{0 \leq e \leq k} q^{c(e+1)+(n+1)(k-e)} \left[\begin{array}{c} k - e \\ e + 1 \end{array}\right]_q \left[\begin{array}{c} n - k + 1 \\ e \end{array}\right]_q.
\]
However, this follows from the $q$-Vandermonde identity (Theorem 8) by substituting $a = n + 1 - k$, $b = k$, $c = k + 1$ and $j = k - e$. $\square$

It is clear that one can restrict the action of $\text{rot}_a$ to $\text{NCC}(n+1, e, l)$. The following result is a refinement of Proposition 35.

**Theorem 38.** Let $n, e, l \in \mathbb{N}$. The triple

\[(\text{NCC}(n + 1, e, l), \langle \text{rot}_a \rangle, \text{Ncc}(n, e, l; q))\]

exhibits the cyclic sieving phenomenon.

**Proof.** Let $\xi$ be a primitive $n^{th}$ root of unity and let $d \mid n$. Write $e = e_1(n/d) + e_0$ and $l = l_1(n/d) + l_0$ for the unique natural numbers $e_1, e_0, l_1, l_0$ such that $0 \leq e_0 < n/d$ and $0 \leq l_0 < n/d$. Using Theorem 9 (the $q$-Lucas theorem) repeatedly, we get

\[
\text{NCC}(n, e, l; \xi^d) = \begin{cases} 
\binom{n}{e_0} \text{Cat}(e) \binom{n-2e}{l} & \text{if } d = n, \\
\binom{d}{2e_1} \binom{d-2e_1}{e_1} \binom{d-e_1}{l_1} & \text{if } e_0 = 0 \text{ and } l_0 = 0, \\
\binom{d}{e} \binom{d-e}{l_1} & \text{if } d = n/2, e_0 = 1 \text{ and } l_0 = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

We prove that these evaluations agree with the number of fixed points in $\text{NCC}(n + 1, e, l)$ under $\text{rot}_a$ on a case-by-case basis.

**Case** $d = n$: Trivial.

**Case** $e_0 = 0$ and $l_0 = 0$: A $(1, 2)$-configuration that is fixed by $\text{rot}_a$ is completely determined by its first $d$ vertices. Among these $d$ vertices, there must $2e(d/n) = 2e_1$ vertices that are incident to an edge and $l(d/n) = l_1$ loops. There are $\binom{d}{2e_1}$ ways to choose $2e_1$ from the first $d$ vertices. The number of ways to arrange these edges in an admissible way is equal to the number of perfect matchings that are invariant when rotating $2e_1$ steps. By Bijection 5 we know that there are $\binom{d}{e}$ such matchings. Lastly, choose $l_1$ loops among the remaining $d - 2e_1$ vertices in $\binom{d-2e_1}{l_1}$ ways. These choices are all independent and the desired result follows.

**Case** $d = n/2$, $e_0 = 1$ and $l_0 = 0$: Such a $(1, 2)$-configuration must have a diagonal (an edge from $i$ to $i + d$) that splits the $(1, 2)$-configuration into two halves. The diagonal can be chosen in $d$ ways. The $(1, 2)$-configuration is now determined uniquely by one of its halves. Such a half must have $d - 1$ vertices with $(e-1)/2 = e_1$ edges and $l_1$ loops. Choose the $2e_1$ vertices that are incident to an edge from the $d - 1$ vertices in $\binom{d-2e_1}{l_1}$ ways. The number of the ways to arrange these edges in an admissible way is equal to the number of non-crossing perfect matchings on $2e_1$ vertices, namely $\text{Cat}(e_1)$. Finally, choose $l_1$ loops from the remaining $d - e$ vertices in $\binom{d-e}{l_1}$ ways. Since these choices are independent, the number of fixed points is given by

\[
d\binom{d-1}{2e_1} \binom{d-e}{l_1} \binom{d}{e} \binom{d-e}{l_1} = \binom{d}{e} \binom{d-e}{l_1}
\]

where equality follows from some simple manipulations of binomial coefficients.

**The remaining cases:** Suppose that $P \in \text{NCC}(n + 1, e, l)$ is invariant under $\text{rot}_a$, where $d \neq n$. There are $l/(n/d)$ loops among the first $d$ vertices, so if $l_0 \neq 0$, there cannot be such a $P$. Hence assume that $l_0 = 0$. If $d \neq n/2$ and $e_0 \neq 0$, then for each edge $ij$ in $P$ there must be edges $(i+d)(j+d), (i+2d)(j+2d), \ldots, (i+n-d)(j+n-d)$.
in $P$ (where addition is taken modulo $n$). Hence the number of edges must be a multiple of $n/d$ which cannot be the case if $e_0 \neq 0$.

This exhausts all possibilities and thus the proof is complete. □

Recall that $\text{NCC}(n+1,k)$ is the set of non-crossing $(1,2)$-configurations $P$ on $n$ vertices such that the number of loops plus proper edges of $P$ is equal to $k-1$. In other words,

$$\text{NCC}(n+1,k) = \bigcup_{i=0}^{k-1} \text{NCC}(n+1,i,k-1-i).$$

By applying Lemma 37, we obtain the following result.

**Corollary 39.** For every $n,k \in \mathbb{N}$ such that $0 \leq k \leq n+1$,

$$\langle \text{NCC}(n+1,k), \langle \text{rot}_n \rangle, \text{Nar}(n+1,k; q) \rangle$$

exhibits the cyclic sieving phenomenon.

There is already a known instance of the cyclic sieving phenomenon with the $q$-Narayana numbers as the polynomial, namely that of non-crossing partitions with a fixed number of blocks and where the group action is rotation [RSW04, Thm 7.2]. Note, however, that in Corollary 39 the cyclic group has a different order than the one with non-crossing partitions.

**Remark 40.** We cannot hope to find a refinement of the above CSP result involving the Kreweras numbers as in [RS18]. For example, consider $n = 4$ and $k = 2$. There are two partitions of $n$ into $k$ parts, namely $(3,1)$ and $(2,2)$. There are 4 non-crossing partitions with parts given by $(3,1)$ and 2 non-crossing partitions with parts given by $(2,2)$. But $\text{NCC}(4,2)$ has two orbits under rotation, both of size 3.

6. Case $s = n$ and Type B Catalan numbers

In this section, we prove several instances of the CSP, related to type $B$ Catalan numbers. We first consider a $q$-Narayana refinement on non-crossing matchings. In the subsequent subsection, we consider a cyclic descent refinement on binary words. Finally, in the last subsection we prove a type $B$ analog of Theorem 38.

6.1. Type B Narayana CSP. A type $B$ non-crossing partition of size $n$ is a non-crossing partition of $\{1, \ldots, n, n+1, \ldots, 2n\}$ which is preserved under a half-turn rotation. These were first defined by Reiner in [Rei97]. We let this set be denoted $\text{NCP}^B(n)$ and let $\text{rot}_B$ denote the action on $\text{NCP}^B(n)$ by rotation of $\pi/n$. Note that we only need to make a half-turn before arriving at the initial position.

**Proposition 41.** The triple

$$\left( \text{NCP}^B(n), \langle \text{rot}_B \rangle, \binom{2n}{n}_q \right)$$

is a CSP triple.

**Proof.** There are many ways to prove this. For example, $\text{NCP}^B(n)$ can first be put in bijection with type $B$ non-crossing matchings, which are non-crossing matchings on $4n$ vertices that are symmetric under a half-turn, by using Bijection 7.

We then consider the first $2n$ new vertices, and for each vertex $u$, we record a 1 if the edge $u \to v$ is oriented clockwise, and 0 otherwise. This is a binary word of
length $2n$ with $n$ ones. Furthermore, $\text{rot}B_n$ of the non-crossing partition corresponds to $\text{shift}^2_{2n}$ on the binary word. The triple $(\text{BW}(2n, n), \langle \text{shift}^2_{2n}, \alpha \rangle_n^q)$ exhibits the CSP (see [RSW04] Prop. 4.4), so it is direct from the definition of cyclic sieving that replacing $\text{shift}^2_{2n}$ by $\text{shift}^2_{2n}$ also gives a CSP triple.

We shall now consider Narayana refinements of type $B$ non-crossing partitions and non-crossing matchings. First, we introduce the following polynomial:

$$\Pi_n(q; t) := \sum_{j=0}^{n} q^{j^2} \left( \frac{n}{j} \right)_q \left( t^{j^2} q^{n-j} \left[ \frac{n-1}{j-1} \right]_q + t^{j+1} \left[ \frac{n-1}{j} \right]_q \right).$$  \hspace{1cm} (29)

Note that by using the $q$-Pascal identity, $\Pi_n(q; 1) = \sum_j q^{j^2} \left[ \frac{n^2}{j} \right]_q = \left[ \frac{2n}{n} \right]_q$, so the sum of the polynomials

$$[t^0]\Pi_n(q; t), \quad [t^1]\Pi_n(q; t), \quad \ldots, \quad [t^{2n}]\Pi_n(q; t)$$

refines the type $B$ $q$-Catalan numbers. With the polynomial formulated, cyclic sieving is easy to prove by following the proof of [RSW04] Thm 7.2. As a side note, the coefficients of the polynomial at $q = 1$ are given by the OEIS entry [A088855].

**Proposition 42.** Let $n, k \geq 0$ be integers. Then

$$\left( \{ P \in \text{NCP}^B(n) : \text{blocks}(P) = k \}, \langle \text{rot}B_n \rangle, [t^k]\Pi_n(q; t) \right),$$

and

$$\left( \{ P \in \text{NCP}^B(n) : 2k \leq \text{blocks}(P) \leq 2k+1 \}, \langle \text{rot}B_n \rangle, q^{k^2} \left[ \frac{n^2}{k} \right]_q \right)$$

exhibit the cyclic sieving phenomenon.

Moreover, for every $n \geq 1$ and $k$, $0 \leq k \leq n$,

$$\left( \{ M \in \text{NCM}^B(n) : \text{even}(M) = k-1 \}, \langle \text{rot}B_n \rangle, [t^k]\Pi_n(q; t) \right)$$

and

$$\left( \{ M \in \text{NCM}^B(n) : 2k-1 \leq \text{even}(M) \leq 2k \}, \langle \text{rot}B_n \rangle, q^{k^2} \left[ \frac{n^2}{k} \right]_q \right)$$

exhibit the cyclic sieving phenomenon.

**Proof.** Everything is trivial unless $1 \leq k \leq n$, so we assume this holds. Using Bijection [7], the first two statements are equivalent to the last two, so we only need to prove the former. The number of half-turn symmetric non-crossing partitions with $2k$ blocks is $\frac{k}{2} \left( \begin{pmatrix} n \end{pmatrix} \right)^2$ and with $2k+1$ blocks $\frac{n-k}{n} \left( \begin{pmatrix} n \end{pmatrix} \right)^2$. This can be proven in different ways, but it suffices to refer to [AR04] Lem. 4.4.

Divide the numbers into $2d$ intervals $t \frac{n}{2} + 1, \ldots, (t + 1) \frac{n}{2}$, $t \in \{0, \ldots, 2d-1\}$. If a partition $P$ satisfies $\text{rot}B_n^{n/d}(P) = P$, a block only contains numbers from two adjacent intervals or it is a central block with numbers from every interval. Let $r$ be the number of blocks that contain numbers from the intervals with $t = 0$ and $t = 1$, but no other. Then the total number of blocks is $2dr$ or $2dr+1$, the latter if there is also a central block. In the proof of [RSW04] Thm 7.2, they show that the number of partitions $P \in \text{NCP}^B(n)$ invariant under $\text{rot}B_n^{n/d}$ with $2dr$ and $2dr+1$ blocks are

$$\frac{dr}{n} \left( \frac{n}{r} \right)^2 \text{ and } \frac{n-dr}{n} \left( \frac{n}{r} \right)^2,$$

respectively.
We now evaluate \([t^k]\Pi_n(q; t)\) at a primitive \(d\text{th}\) root of unity. The case \(d = 1\) is trivial. If \(k \neq 0, 1 \pmod{d}\), \(d \geq 2\), then it is clearly zero. For \(k = 2dr\), we get

\[
[t^{2dr}]\Pi_n(q; t) = q^{(dr)^2+n-dr} \left[\binom{n}{dr} q \left(\frac{n-1}{dr-1} q\right)^{n-dr}\right]
\]

which by Theorem 9 (the \(q\)-Lucas theorem) becomes \(\binom{n/d}{r} (\frac{n/d-1}{r-1}) = \left(\frac{n/d}{r}\right)^2 \frac{r}{n/d}\), which is what we want. A similar calculation gives the case \(k = 2dr + 1\). The expression \(q^{k^2} \left[\frac{n}{k}\right] q\) in (30) is just the sum of the two cases. \(\Box\)

A cyclic sieving result involving type \(B\) Kreweras numbers (and thus type \(B\) Catalan numbers) was proven in [RS18 Thm. 1.7]. The downside is that the Kreweras numbers in type \(B\) are not indexed by usual partitions, but partitions of \(2n + 1\), where each even part has even multiplicity.

6.2. A second refinement of the type \(B\) Catalan numbers. Let \(BW(2n, n)\) be the set of binary words of length \(2n\) with exactly \(n\) ones. Define a cyclic descent of a binary word \(b = b_1b_2\cdots b_{2n}\) as an index \(i\) such that \(b_i > b_{i+1}\), where the indices are taken modulo \(2n\). The number of cyclic descents of \(b\) is denoted \(cdes(b)\). As an example, if \(b = 01101011\), then \(cdes(b) = 3\). For any two natural numbers \(n\) and \(k\), let \(BW^k(n) \subset BW(2n, n)\) consist of all \(b \in BW(2n, n)\) such that \(cdes(b) = k\). Define

\[
BW(n, k; q) := \sum_{b \in BW^k(n)} q^{\maj(b)}.
\]

At \(q = 1\), this is \([A335340]\) in the OEIS and two times \([A103371]\). Note that we have \(BW(0, 0; q) = 1\) and \(BW(n, k; q) = 0\) if \(k > n\).

**Lemma 43.** For all integers \(1 \leq k \leq n\),

\[
BW(n, k; q) = q^{k(k-1)} (1 + q^n) \left[\binom{n}{k} q \left(\frac{n-1}{k-1} q\right)^{n-1}\right].
\]

*Proof.* The set \(BW^k(n)\) is in bijection with a certain subset of \(PATH(n)\) which we shall now describe. Call binary words of the form \(b = 0b_2b_3\cdots b_{2n-1}\) *elevated* and call binary words that are not elevated *non-elevated* (so a binary word is elevated if \(cdes(b) = des(B) + 1\)). Elevated binary words in \(PATH(n)\) are in natural bijection with paths in \(PATH(n - 1)\) by letting the elevated binary word \(0b_2b_3\cdots b_{2n-1}\) correspond to the binary word \(b_2b_3\cdots b_{2n-1-1}\).

It follows that a word in \(BW^k(n)\) corresponds either to a non-elevated path in \(PATH(n)\) with \(k\) valleys or to an elevated path in \(PATH(n)\) with \(k - 1\) valleys. Using this correspondence and (12), one gets that

\[
BW(n, k; q) = q^{k^2} \left[\frac{n}{k} q\right] - q^k \cdot q^{k^2} \left[\frac{n-1}{k} q\right] + q^{k-1} \cdot q^{(k-1)^2} \left[\frac{n-1}{k-1} q\right].
\]

Here, the factors \(q^k\) and \(q^{k-1}\) appear since by translating a binary word \(c \in BW(2(n-1), n-1)\) with \(k\) descents into its corresponding elevated binary word \(c'\) in \(BW(2n, n)\), we have \(\maj(P') - \maj(P) = k\) as each descent of \(c'\) contributes one more to \(\maj\) than in \(c\).
It remains to show that the expression in (34) coincides with the one in (33). To do this, we rewrite

\[
q^k \frac{[n]^2}{[k]_q} - q^n \cdot q^{k_2} \left( \frac{n - 1}{k} \right)^2_q + q^{k_1} \cdot q^{(k-1)^2} \left( \frac{n - 1}{k - 1} \right)^2_q \\
= q^{k(k-1)} \frac{n}{[k]_q} \frac{n - 1}{[k - 1]_q} \left( q^k \frac{[n]^2}{[k]_q} - q^{2k} \frac{[n - k]^2}{[n]_q[k]_q} + [k]_q \right) \\
= q^{k(k-1)} \frac{n}{[k]_q} \frac{n - 1}{[k - 1]_q} \left( q^k \frac{[n]^2}{[k]_q} - q^{2k} \frac{[n - k]^2}{[n]_q[k]_q} + [k]_q \right).
\]

It is therefore sufficient to show that the expression inside the parentheses is equal to \(1 + q^n\) or, equivalently, that the following equation holds:

\[q^k [n]_q^2 - q^{2k} [n - k]_q^2 + [k]_q^2 = (1 + q^n)[n]_q[k]_q.\]

This equation can be derived from (24) by adding \(q^{k+n-1}[n]_q + q^{2k-1}[n - k]_q + q^{k-1}[k]_q = q^{k-1}(1 + q^n)[n]_q\) to each side of the equation. This concludes the proof. \(\square\)

The number of cyclic descents of a binary word is clearly invariant under cyclic shifts of the word so one has a group action of \(\text{rot}_{2n}\) on \(\text{BW}^k(n)\). The following proposition follows from [AS18 Cor. 1.6], although they do not compute the closed-form expression of Equation (32).

**Proposition 44.** For all \(n, k \in \mathbb{N}\) such that \(1 \leq k \leq n\), the triple

\[\left(\text{BW}^k(n), \text{shift}_{2n}, \text{Bw}(n, k; q)\right)\]

exhibits the cyclic sieving phenomenon.

### 6.3. Type B non-crossing configurations with a twist

Recall that \(\text{NCC}(n+1)\) denotes the set of non-crossing \((1,2)\)-configurations on \(n\) vertices. We shall now modify this family slightly.

**Definition 45.** Let \(\text{NCC}^B(n)\) be the set of non-crossing \((1,2)\)-configurations on \(n - 1\) vertices, with the extra option that one of the proper edges may be marked. We let \(\text{NCC}^B(n, e, l) \subset \text{NCC}^B(n)\) be the subset with exactly \(e\) proper edges, and \(l\) loops. Finally, let \(\text{NCC}^B(n, k)\) be the subset of \(\text{NCC}^B(n)\) with \(k\) edges and loops, i.e.

\[\text{NCC}^B(n, k) := \bigcup_{e+l=k} \text{NCC}^B(n, e, l).\]

It follows directly from the definition that \(|\text{NCC}^B(n, e, l)| = (e + 1)|\text{NCC}(n, e, l)|\) and it is not difficult to sum over all possible \(e, l\) to prove that \(|\text{NCC}^B(n + 1, k)| = \text{Nar}^B(n, k) = \binom{n}{k}^2\).

**Theorem 46.** We let \(\text{twist}_{2n}\) act on \(\text{NCC}^B(n + 1)\) as before (the marked edge is also rotated), which gives an action of order \(2n\). Then

\[\left(\text{NCC}^B(n + 1), \langle \text{twist}_{2n}^2 \rangle, \binom{2n}{n}_q\right)
\]

is a CSP triple.
Proof. We compute the number of fixed points under \((\text{twist}_{2n}^2)^d\) where we can without loss of generality assume \(d \mid n\). Write \(n = md\). By Theorem 9
\[
\binom{2n}{n}_q = \binom{2d}{d}_q
\]
at a primitive \(m\)th root of unity. The claim follows from Theorem 33 except in the cases where a marked edge can appear in a fixed point. Note that in the case \(4 \mid n\) and \(2d = n/2 \text{ or } 3n/2\) a marked edge would have to split the configuration into two non-crossing matchings on an odd number of vertices. Hence there cannot be a marked edge in a fixed point in this case.

The only case left is \(n \mid 2d\). First, \(2d = 2n\) is trivial. Second, if \(2d = n\), no fixed point can have marked vertices, as is noted in the proof of Theorem 33. Hence we only have non-crossing matchings on \(2d\) vertices with one edge possibly marked, the number of which is \((d + 1)\text{Cat}(d) = \binom{2d}{d}_q\). □

It should be possible to prove Theorem 46 bijectively.

Problem 47. Find an equivariant bijection between \(\text{NCC}^B(n+1)\) and \(\text{BW}(2n,n)\) sending \(\text{twist}_{2n}^2\) to \(\text{shift}_{2n}^2\).

Note that the triple in Theorem 46 exhibits the so-called Lyndon-like cyclic sieving [ALP19], which is not intuitively clear (as it is for \(\text{BW}(2n,n)\)).

Remark 48. Theorem 46 does not hold when only considering \(\text{twist}_{2n}^2\). For \(n = 2\), \(\binom{2n}{n}_q\) evaluated at a primitive 4th root of unity gives 0. However, there are 6 elements in \(\text{NCC}^B(3)\), two of which are fixed under \(\text{twist}_4\); consider an edge between vertices 1 and 2, which may or may not be unmarked. Since there are no loops or isolated vertices, these two elements are fixed. Can one modify the \(q\)-analog of \(\binom{2n}{n}_q\) so that it is compatible with \(\text{twist}_{2n}^2\)?

Problem 49. Is it possible to define a refinement \(P(n,e,l; q)\) of \(\binom{2n}{n}_q\) so that
\[
\left( \bigcup_{l=0}^{n} \text{NCC}^B(n+1,e,l), (\text{twist}_{2n}^2), \sum_{l \geq 0} P(n,e,l; q) \right)
\]
is a CSP triple?

Unfortunately the polynomials
\[
\text{Ncc}^B(n,e,l; q) := q^{e^2 + n_l(e + 1)} \left[ \frac{n}{2e} \right]_q \text{Cat}(e; q) \left[ \frac{n - 2e}{l} \right]_q
\]
do not serve this purpose even though they do satisfy the identities (proof omitted)
\[
\text{Ncc}^B(n,e,l; 1) = |\text{NCC}^B(n+1,e,l)|,
\text{Nar}^B(n,k; q) = \sum_{e+l=k} \text{Ncc}^B(n,e,l; q).
\]

6.4. Thiel’s CSP for type B.

Theorem 50. We let rotation \(\text{rot}_n\) act on \(\text{NCC}^B(n+1,e,l)\), and let
\[
\text{Ncc}(n,e,l; q) := q^{e(e+1) + (n+1)l} \left[ \frac{n}{2e} \right]_q \text{Cat}(e; q) \left[ \frac{n - 2e}{l} \right]_q
\]
Then
\[
\left( \text{NCC}^B(n + 1, e, l), \langle \text{rot}_n \rangle, (1 + [e]_q)\text{Ncc}(n, e, l; q) \right)
\]
(36)
is a CSP triple.

**Proof.** We can split \( \text{NCC}^B(n + 1, e, l) \) into two sets, the first set \( A \) being the case without a marked edge, and the second set \( B \) the case with a marked edge. Then it suffices to prove that
\[
(A, \langle \text{rot}_n \rangle, \text{Ncc}(n, e, l; q)) \quad \text{and} \quad (B, \langle \text{rot}_n \rangle, [e]_q \text{Ncc}(n, e, l; q))
\]
are CSP triples. The first one is already proved in Theorem 38.

For the second, consider \( \text{rot}_d^n \), and without loss of generality write \( n = kd \). A single marked edge can only appear in a fixed point if \( d = n \) or \( d = n/2 \). The former is trivial. Now, rewrite the polynomial as
\[
q^e(e+1)+(n+1)l \begin{bmatrix} n \\ 2e \end{bmatrix} _q \begin{bmatrix} 2e \\ e-1 \end{bmatrix} _q \begin{bmatrix} n-2e \\ l \end{bmatrix} _q,
\]
and apply Theorem [9]. At a primitive \( k^{th} \) root of unity, this evaluates to 0 unless \( k \mid 2e \), \( k \mid e - 1 \) and \( k \mid l \). The second implies \( \gcd(k, e) = 1 \), so by the first \( k \mid 2 \). If \( k = 2 \), that is \( d = n/2 \), we get that the number of fixed points should be
\[
\left( \binom{n}{2} e \right) \left( \binom{e-1}{2} \right) \left( \binom{n-2e}{2} \right).
\]
This is indeed the case. The marked edge has to split the configuration into two symmetric parts, and connects \( i \) to \( i + n/2 \) for some \( 1 \leq i \leq n/2 \). The symmetric configurations are on \( n/2 \) - 1 vertices, and have \( (e - 1)/2 \) edges and \( l/2 \) marked vertices each. The number of fixed points is hence
\[
\frac{n}{2} \left( \binom{n}{2} e \right) \left( \binom{e-1}{2} \right) \left( \binom{n-2e}{2} \right) = \left( \frac{n}{2} e \right) \left( \binom{e-1}{2} \right) \left( \frac{n-2e}{2} \right).
\]
\( \square \)

By summing over the cases when \( e + l = k \), we get the following corollary:

**Corollary 51.** We have a \( q \)-analog of the type B Narayana numbers, which admits the CSP triple
\[
\left( \text{NCC}^B(n + 1, k), \langle \text{rot}_n \rangle, U_{n,k}(q) \right),
\]
where \( U_{n,k}(q) = \sum_{e=0}^{k} q^{e(e+1)+(n+1)(k-e)} (1 + [e]_q) [n]_{2e} \text{Cat}(e; q) [\frac{n-2e}{k-e}]_q \).

**Proof.** As in the discussion before Theorem 46, it is not difficult to prove that when \( q = 1 \), we do indeed obtain \( \binom{n}{2} \), so this is a \( q \)-Narayana refinement. \( \square \)

Now, summing over all \( k \) gives cyclic sieving on \( \text{NCC}^B(n + 1) \) under rotation. We leave it as an open problem to find a nice expression for \( \sum_k U_{n,k}(q) \).
7. Two-column semistandard Young tableaux

The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is defined as the sum

$$s_\lambda(x_1, \ldots, x_n) := \sum_{T \in \text{SSYT}(\lambda, n)} \prod_{j \in \lambda} x_{T(j)}$$

where SSYT$(\lambda, n)$ is the set of semi-standard Young tableaux of shape $\lambda$ with maximal entry at most $n$. The product is taken over all labels in $T$.

P. Brändén gave the following interpretation of $q$-Narayana numbers.

**Theorem 52** (See [Brä04, Thm. 6]). For $0 \leq k \leq n - 1$,

$$\text{Nar}(n, k + 1; q) = s_{2k}(q, q^2, \ldots, q^{n-1}).$$

(37)

There is a type $B$ analog of Theorem 52.

**Theorem 53.** For $0 \leq k \leq n$,

$$q^{k(k+1)} \left[ \begin{array}{c} n \\ k \end{array} \right]_q = s_{2k+1}(q, q^2, \ldots, q^n).$$

(38)

**Proof.** We first note that $s_{2k+1}(q, q^2, \ldots, q^n) = (s_1(q, q^2, \ldots, q^n))^2$. To compute $s_1(q)$, we simply sum over all $k$-subsets of $[n]$. This gives immediately that

$$s_1(q, q^2, \ldots, q^n) = q^{k(k+1)/2} \left[ \begin{array}{c} n \\ k \end{array} \right]_q,$$

and the theorem above follows. □

It is then reasonable to interpret

$$s_{2k+1}(q, q^2, \ldots, q^n)$$

for $0 \leq s \leq k$ as an interpolation between type $A$ ($s = 0$) and type $B$ ($s = k$) $q$-Narayana polynomials. Note that this approach is different from what is sought after in Section 3. The expression in (39) can easily be computed by the dual Jacobi–Trudi identity, see [Mac95]. We find that (39) is equal to

$$q^{k(k+1)} \left[ \begin{array}{c} n \\ k \end{array} \right] - q^{(s+1)^2} \left[ \begin{array}{c} n \\ k-s-1 \end{array} \right] q^{s} \left[ \begin{array}{c} n \\ k+s+1 \end{array} \right]_q.$$

The first part of the theorem below follows from combining [Rho10, Thm. 1.4] and Theorem 52.

**Theorem 54.** Assume $1 \leq k < n$ and let $\hat{\partial}_{n-1}$ act on SSYT$(2^k, n-1)$ via so-called $k$-promotion, so that $\hat{\partial}_{n-1}$ has order $n-1$. Then

$$\left( \text{SSYT}(2^k, n-1), \langle \hat{\partial}_{n-1} \rangle, \text{Nar}(n, k+1; q) \right)$$

is a CSP triple. Moreover, there is a cyclic group $\langle \varphi \rangle$ of order $n$ acting on SSYT$(2^k, n-1)$ such that

$$\left( \text{SSYT}(2^k, n-1), \langle \varphi \rangle, \text{Nar}(n, k+1; q) \right)$$

is a CSP triple.
Proof. We can define the action $\varphi$ as follows. Given $T \in \text{SSYT}(2^k, n-1)$, define

$$Q = \{j : 1 \leq j \leq n-1, |\{i : T_{i,1} \geq j\}| = |\{i : T_{i,2} \geq j\}|\}$$

and let $P$ be the subset of $Q$ containing numbers not occurring as entries in $T$. Further, define $\ell = \max P$ and $b = \max Q$. If $P = \emptyset$, let $\ell = 0$. Now $\varphi(T)$ is defined as follows. If $\ell = n-1$, then we just add one to every entry in $T$ and are done. If $\ell \neq n-1$, then we add one to every entry in $T$ and

- in the first column: remove $b+1$, add 1 in increasing order;
- in the second column: remove $n$, add $\ell + 1$ in increasing order.

This can be seen to be an action with the desired properties by referring to Bijection $3$ from $\text{SSYT}(2^k, n-1)$ to $\text{NCP}(n, k+1)$. Then rotation one step of the latter set corresponds to $\varphi$ where $P$ are the other elements in the same block as $n$, $b$ is the smallest element in the block of $n-1$ (if $n-1 \not\in P$), and $\ell$ is the largest element in the block of $n$ other than $n$ itself. Clearly, $b+1$ must be removed from the first column since it will be the smallest element in the block of $n$ in $\varphi(T)$, and $\ell + 1$ must be added to the second since it will be the largest element in the block containing 1 in $\varphi(T)$. $\square$

Bijection 3. Let $T \in \text{SSYT}(2^k, n-1)$. Starting from $i = 1$, consider $x_i = T_{i,2}$. Find the largest $y \in T_{i,1}, y \leq x_i$, which is not in

$$P_{i-1} := \bigcup_{j \in [i-1]} p_j,$$

and set $y_i = y$. Let $p_i = \{z \in \mathbb{N} : y_i \leq z \leq x_i, z \not\in P_{i-1}\}$. Repeat for $i < k + 1$. Finally, let $p_{k+1} = [n] \setminus P_k$. Then the blocks $p_1, \ldots, p_{k+1}$ form a non-crossing partition in $\text{NCP}(n, k+1)$ by construction. Note that exactly one element from each of $T_{i,1}$ and $T_{i,2}$ is contained in $p_i$. Note also that this is in fact the unique non-crossing partition in $\text{NCP}(n, k+1)$ having parts whose smallest and largest elements are $y_i$ and $x_i$ respectively.

The inverse of the above bijection can be described as follows. Let $p_1[p_2] \cdots [p_{k+1}] \in \text{NCP}(n, k+1)$ and assume $n \in p_{k+1}$. Let the first column of $T$ consist of the smallest elements from $p_i, 1 \leq i \leq k$, in increasing order from top to bottom, and the second column of $T$ of the largest elements from the same blocks, also in increasing order. To see why $T \in \text{SSYT}(2^k, n-1)$, it suffices to suppose that we on some row $i$ have $T_{i,1} > T_{i,2}$, so the smallest element in the block containing $T_{i,1}$ must be $T_{j,1}$ for some $j < i$. Then the smallest element in the block containing $T_{j,2}$ has to be $T_{k,1}$ for some $j < k < i$, the smallest element in the block containing $T_{j+1,2}$ some $T_{k',1}$ for $j < k' < i$, and so on. We match all elements $T_{j,2}, \ldots, T_{i-1,2}$ to elements in the first column between $T_{i,1}$ and $T_{i,1}$, but this yields a contradiction. Note that $T$ is the unique element of $\text{SSYT}(2^k, n-1)$ whose first column consists of the smallest elements of $p_1, \ldots, p_k$ and whose second column consists of the largest elements of these parts. Hence it is clear that this indeed is the inverse.

Below is an example of Bijection $3$ and $\varphi$, when $n = 8$, $k = 4$.

$$T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 7 & 7 \end{bmatrix} \quad \leftrightarrow \quad \{\{1,4\}, \{2\}, \{3\}, \{7\}, \{5,6,8\}\}, \quad \varphi(T) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}$$
Remark 55. The bijection in the proof of \[\text{[Bra04 Thm. 5]}\] together with Bijection 8 provides a different bijection between SSYT\((2^k, n - 1)\) and NCP\((n, k + 1)\) where, if \(T \in \text{SSYT}(2^k, n - 1)\), then \(T_{1,1}, T_{2,1} - T_{1,1}, \ldots, T_{k,1} - T_{k-1,1}, n - T_{k,1}\) are the sizes of the blocks with \(T_{1,2}, T_{2,2}, \ldots, T_{k,2}\) as the largest elements.

There is a type \(B\) version of Theorem 54.

Theorem 56. Let \(\hat{\partial}_n\) act on \(\text{SSYT}(2^k 1^k/1^k, n)\) via \(k\)-promotion, then

\[
\left(\text{SSYT}(2^k 1^k/1^k, n), \langle \hat{\partial}_n \rangle, q^{k(k+1)} \binom{n}{k}^2 \right)
\]

is a CSP triple.

Proof. We describe a bijection from \(\text{SSYT}(2^k 1^k/1^k, n)\) to \(\text{BW}(n, k) \times \text{BW}(n, k)\). Let \(T \in \text{SSYT}(2^k 1^k/1^k, n)\). The corresponding pair of binary words \((b_1, b_2)\) is constructed as follows. Write \(b_1 = b_{11} \ldots b_{1n}\) and let \(b_{1i} = 1\) if \(T\) has an \(i\) in the left column, and otherwise, let \(b_{1i} = 0\). In an analogous way, let \(b_2\) be determined by the entries in the right column of \(T\). It is easy to see that this bijection is equivariant with the corresponding group action being \(\text{rot}_n\) on the pair of binary words. Here, \(\text{rot}_n\) acts by cyclically shifting both of the words one step. It follows from \[\text{[RSW04 Thm. 1.1]}\] that

\[
\left(\text{BW}(n, k) \times \text{BW}(n, k), \langle \text{rot}_n \rangle, \binom{n}{k}^2 \right)
\]

exhibits the cyclic sieving phenomenon. It is not hard to see that the two different CSP-polynomials agree at \(n\)th roots of unity. This completes the proof. \(\square\)

It is natural to ask if the first part of Theorem 54 and Theorem 56 generalize to the intermediate skew shapes. We would then hope that \(k\)-promotion acting on \(\text{SSYT}(2^k 1^s/1^s, n)\), for \(1 \leq s \leq k - 1\), has order \(n\). However, this is not the case, as for \(n = 4, k = 2, s = 1\), the tableaux

\[
\begin{array}{ccc}
1 & 4 & 1 \\
2 & 3 & 4 \\
\end{array}
\]

form an orbit under \(k\)-promotion, but we want a group action of order 4.

Perhaps some other group action gives a CSP triple with \(m\) as the CSP-polynomial. In a recent preprint, Y.-T. Oh and E. Park \[\text{[OP20]}\] the authors show some closely related results, regarding cyclic sieving on SSYT.

8. Triangulations of \(n\)-gons with \(k\)-ears

We shall now consider type \(A\) triangulations of an \(n\)-gon. The main result in this section is a refinement of the CSP instance on triangulations of \(n\)-gons which are counted by \(\text{Cat}(n - 2)\), see \[\text{[RSW04 Thm. 7.1]}\]. We also mention a cyclic sieving result on type \(B\) triangulations.
8.1. Refined CSP on triangulations by considering ears. Let $\text{TRI}(n)$ denote
the set of triangulations of a regular $n$-gon. If the vertices $i, i+1, i+2 \pmod{n}$ are
pairwise adjacent for $T \in \text{TRI}(n)$, we say they form an ear of $T$. Let $\text{TRI}_{\text{ear}}(n, k)$
denote the set of triangulations of a regular $n$-gon with exactly $k$ ears, and let $\text{Tri}(n, k)$
be the cardinality of this set. It was shown by F. Hurtado and M. Noy [HN96 Thm. 1] that
\[
\text{Tri}(n, k) = \frac{n}{k} \binom{n-4}{2k-4} \text{Cat}(k-2) \cdot 2^{n-2k}
\quad \text{whenever} \quad 2 \leq k \leq \frac{n}{2}.
\] (40)

We now introduce the following $q$-analog of the expression in (40). For integers $n$
and $k$ satisfying $2 \leq k \leq \frac{n}{2}$, let
\[
\text{Tri}(n, k; q) := q^{k-2} \frac{[n]_q}{[k]_q} \binom{n-4}{2k-4} q^{\text{Cat}(k-2; q)} \left( \sum_{j=0}^{n-2k} q^{j(n-2)} \binom{n-2k}{j}_q \right).
\]

At first glance, one might hope that there is an easier expression for $\text{Tri}(n, k; q)$. However,
note that $\text{Tri}(n, k; q)$ is not palindromic in general. As an example, consider $\text{Tri}(6, 2; q) = 1 + q^4 + q^5 + q^8$. This means that one cannot hope to find a
formula for $\text{Tri}(n, k; q)$ which only involves products of palindromic polynomials.
In particular, it cannot be a product of $q$-binomial coefficients.

The choice of $\text{Tri}(n, k; q)$ is motivated by the following theorem.

Theorem 57. For all integers $n \geq 4$, we have that
\[
\sum_k \text{Tri}(n, k; q) = \text{Cat}(n-2; q).
\] (41)

In other words, the polynomials $\text{Tri}(n, k; q)$ refine the $q$-Catalan numbers.

Proof. We first recall some notation from $q$-hypergeometric series, where we use
[GR04 Appendix I-II] as the main reference. We set $(a; q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1})$ so that
\[
\binom{m}{r}_q = \frac{(q; q)_m}{(q; q)_r(q; q)_{m-r}} \quad \text{and} \quad [m]_q = \frac{(q; q)_m}{(1-q)(q; q)_{m-1}}.
\]

We have, [GR04 I.7–I.26], that for all $a,$
\[
(a; q)_{n+k} = (a; q)_k (aq^k; q)_n, \quad (a; q)_{n-k} = (a; q)_n \frac{(1-a; q)_k}{(a^2-a; q)_k} \left( \frac{q}{a} \right)^{k-n-k},
\]
\[
(aq^{-n}; q)_n = q^{-\binom{2}{1}} \left( -\frac{a}{q} \right)^n (q/a; q)_n, \quad (aq^{-n}; q)_k = q^{-\binom{2}{1}} \left( -\frac{a}{q} \right)^n \frac{(q/a; q)_n (a; q)_k}{(q^{1-k}/a; q)_n},
\]
\[
(q^{-n}; q)_k = (-1)^k q^{\binom{2}{1}-nk} \frac{(q; q)_n}{(q; q)_{n-k}}.
\]
\[
(aq^n; q)_n = \frac{(a; q)_{2n}}{(a; q)_n}.
\]
Moreover, we let
\[ 2\phi_1 \left[ \frac{a}{c} \right. \left. b ; q ; 2 \right] := \sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n. \]

The $q$-Chu–Vandermonde identity [GRO4, I.6, II.7], can be stated in the following two ways:
\[ 2\phi_1 \left[ \frac{a}{c} \right. \left. q^{-n} \right] = \left( \frac{c/a; q}{c; q} \right)_n q^n \quad \text{and} \quad 2\phi_1 \left[ \frac{a}{c} \right. \left. q^{-n} q^n / a \right] = \left( \frac{c/a; q}{c; q} \right)_n \quad (42) \]

We are now ready to prove Theorem 57, which is equivalent to proving that for all $n \geq 4$,
\[ \sum_{k \geq 2} q^{k(k-2)} \frac{[n][n-1]}{[k][k-1]} \frac{2k-4}{2} q^2 \sum_{j=0}^{n-2k} q^{j(n-2)} \left[ \begin{array}{c} n-2k \\ j \end{array} \right]_q = \left[ \begin{array}{c} 2n-4 \\ n-2 \end{array} \right]_q. \quad (43) \]

After shifting the $k$-indices by 2, and the $n$-indices by 4, and multiplying both sides with $(1 + q)$, we must show that
\[ \sum_{k,j} R(k, j) = \frac{(1 + q) \left[ \begin{array}{c} 2n+4 \\ n+2 \end{array} \right]_q}{[n+3][n+4]}, \quad (44) \]
where
\[ R(k, j) = \frac{(1 + q)q^{k(k+2)} + j(n+2)}{[k+1][k+2]} \left[ \begin{array}{c} n \\ 2k \end{array} \right]_q \left[ \begin{array}{c} n-2k \\ j \end{array} \right]_q q^{k(k+2)} q^{j(n+2)} (q; q)_n \]
\[ = \frac{(q; q)_k (q^n; q)_k (q; q)_{n-2k-j} (q; q)_j}{(q; q)_k (q^n; q)_k (q; q)_j} \]
\[ = (-1)^j q^{n(2k+j) - \frac{(2k+1)(2k+2)}{2}} q^{k(k+2)} q^{j(n+2)} (q^{-n}; q)_{2k+j}. \]

The right hand side of (44) simplifies to $\frac{(q^n; q)_{2n}}{(q^n; q)_n (q^n; q)_n}$. There is no issue with extending the summation index in (44) so that $k$, $j$ ranges over all integers since $R(k, j)$ vanishes unless $0 \leq j \leq n - 2k$. By shifting the indexing, so that $r := k$, $s := k + j$, it suffices to prove that
\[ \sum_{r, s} S(r, s) = \frac{(q^n; q)_{2n}}{(q^n; q)_n (q^n; q)_n} \quad (45) \]
where
\[ S(r, s) := R(r, s - r) = (-1)^{s+r} q^{2ns+2s-\left( \frac{s+1}{2} \right)} q^{r^2} \frac{(q^{-n}; q)_{s+r}}{(q; q)_r (q^n; q)_r (q; q)_{s-r}}. \]

By using the identities
\[ (q^{-n}; q)_{s+r} = (q^{n-s}; q)_r (q^{-n}; q)_s \quad \text{and} \quad (q; q)_s-r = \frac{(q; q)_s}{(q^{-s}; q)_r} (-1)^r q^{\left( \frac{s}{2} \right) - r}, \]

we have
\[ S(r, s) = (-1)^{s+r} q^{2ns+2s-\left( \frac{s+1}{2} \right)} q^{r^2} \frac{(q^{-n}; q)_r (q^{-n}; q)_s (q^{-s}; q)_r (q^{-s}; q)_s}{(q; q)_r (q^n; q)_r (q; q)_s} \]
\[ = (-1)^{s+r} q^{\left( \frac{s}{2} \right)} (q^{-n}; q)_s (q^{-n}; q)_s (q^{-s}; q)_r (q^{-s}; q)_r q^r. \]
Thus, (43) is equivalent to

\[
\sum_{r,s} (-q^{2n+2})^s q^{-\binom{s}{2}} (q^{-n}; q)_s (q^{s-n}; q)_r (q^{-s}; q)_r \cdot q^r = \frac{(q^5; q)_{2n}}{(q^5; q)_n (q^5; q)_n}.
\]

But this follows from substituting \( a = q^2 q^n \) and \( c = q^3 \) in the following claim and then expanding the \( q \)-hypergeometric series, together with using the fact that \((q^5; q)_{2n} = (q^5; q)_n (q^n q^5; q)_n\).

**Claim:** For non-negative integers \( n \), we have the identity

\[
\sum_{k \geq 0} (-aq^n)^k q^{-\binom{k}{2}} (q^{-n}; q)_k (aq/q^k; q)_k (cq/q^k)! = \frac{(ac; q)_n}{(c; q)_n}. \tag{46}
\]

Applying the first \( q \)-Chu-Vandermonde identity, the left-hand side becomes

\[
\sum_{k \geq 0} (-aq^n)^k q^{-\binom{k}{2}} (q^{-n}; q)_k (aq/q^k; q)_k (cq/q^k)! \left( \frac{cq/k}{aq} \right)^k.
\]

Now, using the identity \((aq/q^k; q)_k = q^{-\binom{k}{2}} (-a)^k (1/a; q)_k\) we see that the left-hand side of (46) is equal to

\[
\sum_{k \geq 0} (-aq^n)^k q^{-\binom{k}{2}} (q^{-n}; q)_k q^{-\binom{k}{2}} (-a)^k (1/a; q)_k (cq/k)! \left( \frac{cq/k}{aq} \right)^k.
\]

Simplification gives

\[
\sum_{k \geq 0} (acq^n)^k (q^{-n}; q)_k (1/a; q)_k (c; q)_k (cq/k)! = 2\phi_1 \left[ \frac{1}{c} q^{-n}; \frac{cq}{a}, \frac{cq^n}{a} \right].
\]

This is now a special case of the second \( q \)-Chu-Vandermonde identity and we are done. \( \square \)

A curious observation is that Theorem 57 refines the \( q \)-Catalan numbers in the same spirit as the following \( q \)-analog of Touchard’s identity [And10 Thm. 1], which states that

\[
\text{Cat}(n+1; q) = \sum_{r \geq 0} q^{2r^2+2r} \binom{n}{2r} \frac{\text{Cat}(r; q) (\frac{-q^{r+2}}{q})_{n-r}}{(-q; q)_r}.
\]

We let \( \text{rot}_n \) act on \( \text{TRI}(n) \) by rotating a triangulation one step clockwise. As \( \text{rot}_n \) also preserves the set \( \text{TRI}_{\text{cat}}(n,k) \), we have a group action of \( \langle \text{rot}_n \rangle \) on \( \text{TRI}_{\text{cat}}(n,k) \).

**Theorem 58.** For all integers \( 2 \leq k \leq \frac{n}{2} \), the triple

\[(\text{TRI}_{\text{cat}}(n,k), \langle \text{rot}_n \rangle, \text{Tri}(n,k; q)) \]

exhibits the cyclic sieving phenomenon.
Proof. Let $\xi$ be a primitive $n^{th}$ root of unity and let $d$ be an integer. Repeatedly using Theorem 9 and Lemma 10 yields

$$\text{Tri}(n, k; \xi^d) = \begin{cases} \frac{n}{\pi} a^{n-2k} \binom{n-4}{2k-4} \text{Cat}(k-2) & \text{if } d = n \\ \frac{n}{\pi} a^{n/2-k} \binom{n/2-2}{k-2} (k/2-1) & \text{if } o(\xi^d) = 2 \text{ and } 2 \mid k \\ \frac{n}{\pi} a^{n/3-2k/3} \binom{n/3-2}{2k/3-2} (k/3-1) & \text{if } o(\xi^d) = 3 \text{ and } 3 \mid k \\ 0 & \text{otherwise.} \end{cases}$$

We prove that these evaluations agree with the number of fixed points in $\text{TRI}_{ear}(n, k)$ under $\text{rot}_n^d$ on a case-by-case basis.

Case $d = n$: Trivial.

Case $o(\xi^d) = 2$ and $2 \mid k$: Such a triangulation must have a diagonal (an edge from some $i$ to $i + d$) that divides the triangulation into two halves. The diagonal can be chosen in $n/2$ ways. The triangulation is determined uniquely by one of its halves. To choose one half, we choose a triangulation of a $(n/2 + 1)$-gon with $k/2$ ears whose sides do not coincide with the diagonal. Such a triangulation is either (i) an element of $\text{TRI}_{ear}(n/2 + 1, k/2)$ which does not have an ear with an edge coinciding with the diagonal or (ii) an element of $\text{TRI}_{ear}(n/2 + 1, k/2 + 1)$ which has an ear coinciding with the diagonal. Based on the rotational symmetry, we note that $2k/n$ of the elements in $\text{TRI}_{ear}(n, k)$ have an ear that has an edge adjacent to a given side. Thus, the number of triangulations fixed by $\text{rot}_n^{n/2}$ is equal to

$$\frac{n}{2} \left( \frac{n/2 + 1 - k}{n/2 + 1} \text{Tri}(n/2 + 1, k/2) + \frac{2(k/2 + 1)}{n/2 + 1} \text{Tri}(n/2 + 1, k/2 + 1) \right).$$

Sage [Sag20] confirms that this symbolic expression agrees with the one given by evaluating $\text{Tri}(n, k; \xi^d)$.

Case $o(\xi^d) = 3$ and $3 \mid k$: Similar to the above case. Such a triangulation must have a central triangle (a triangle with vertices $i$, $i + n/3$ and $i + 2n/3$ (mod $n$)) that divides the triangulation into three parts. The central triangle can be chosen in $n/3$ ways. The triangulation is determined uniquely by one of its three parts. Choosing one part is done with a similar argument as above, so the number of triangulations fixed by $\text{rot}_n^{n/3}$ is equal to

$$\frac{n}{3} \left( \frac{n/3 + 1 - 2k/3}{n/3 + 1} \text{Tri}(n/3 + 1, k/3) + \frac{2(k/3 + 1)}{n/3 + 1} \text{Tri}(n/3 + 1, k/3 + 1) \right).$$

Sage confirms that this symbolic expression agrees with the one given by evaluating $\text{Tri}(n, k; \xi^d)$.

The remaining cases: If $o(\xi^d) = 2$ (or 3), it is clear that any triangulation fixed by $\text{rot}_n^d$ must have a diagonal (or, respectively, a central triangle). Thus each of the two halves (or, respectively, three parts) must have the same number of ears and hence $2 \mid k$ (or $3 \mid k$). If $o(\xi^d) > 3$, then it is clear that there are no fixed triangulations under the action of $\text{rot}_n^d$. This exhausts all possibilities and thus the proof is done.

Problem 59. One might ask if there are further refinements. However, note that

$$\frac{n}{k} \binom{n-2k}{2k-4} \text{Cat}(k-2), \quad 0 \leq j \leq n - 2k,$$

do not refine $\frac{n}{k} 2^{n-2k} \binom{n-4}{2k-4} \text{Cat}(k-2)$ since the former is not always an integer, for example, at $n = 5, k = 2, j = 1.$
Remark 60. Unfortunately, there is no Narayana refinement of rotation acting on triangulations. To see this, observe that $\text{Nar}(2, 2; q) = q^2$ but at a $4^{th}$ root of unity $\xi$, we have $\text{Nar}(2, 2; \xi) = -1$.

8.2. Triangulations of type $B$. Let us define type $B$ triangulations $\text{TRI}^B(n)$, as the set of elements in $\text{TRI}(2n)$ which are invariant under rotation by a half-turn. In such a triangulation, there is always an edge through the center. There are $n$ choices of such an edge, and then we need to choose a triangulation on one half of the $2n$-gon. This gives $n \cdot \frac{1}{n} \binom{2(n-1)}{n-1} = \binom{2n-2}{n-1}$ such type $B$ triangulations. The following result is straightforward to prove but also follows from [EF08, Thm. 4.1].

Proposition 61 (See [EF08, Thm. 4.1]). The triple

$$\left(\text{TRI}^B(n), \langle \text{rot}_n \rangle, \left\lceil \frac{2n-2}{n-1} \right\rceil_q \right)$$

exhibits the cyclic sieving phenomenon.

The polynomial $\left\lfloor n \right\rfloor_q \text{Tri}(n+1, k; q)$ does not seem to give a refinement of Proposition 61.

9. Marked non-crossing matchings

A marked non-crossing matching is a non-crossing matching where some of the regions have been marked. Let $\text{NCM}^{(r)}(n)$ denote the set of marked non-crossing matchings with exactly $r$ marked regions. Since every non-crossing matching in $\text{NCM}(n)$ has $n + 1$ regions, it follows that $|\text{NCM}^{(r)}(n)| = \binom{n+1}{r} |\text{NCM}(n)|$.

In particular, for $r = 1$ we can think of our objects as non-crossing matchings of vertices on the outer boundary of an annulus rather than on a disk.

This model is reminiscent of the non-crossing permutations considered in [Kim13], where points on the boundary of an annulus are matched in a non-crossing fashion, but with some other technicalities imposed.

The following generalizes Proposition 20.

Theorem 62. Let $1 \leq k \leq n$ and $0 \leq r \leq n + 1$. Then

$$\left\{ M \in \text{NCM}^{(r)}(n) : \text{even}(M) = k \}, \langle \text{rot}_n \rangle, \text{Nar}(n, k + 1; q) \left\lceil \frac{n+1}{r} \right\rceil_q \right\}$$

is a CSP triple.
Figure 6. Partitioning a non-crossing perfect matching of size $2n = 18$ with $d$-fold rotational symmetry, $d = 3$, into segments of length $2d$. Each of the three edges from $[2d]$ to vertices with bigger labels has a unique region to its left.

Proof. Consider elements of $NCM(n)$ with $k$ even edges, fixed by $\text{rot}_n^d$, where we may without loss of generality assume $d \mid n$. As noted in Section 4.1, $\text{even}(M)$ is invariant under $\text{rot}_n$. Write $n = md$. By the $q$-Lucas theorem (Theorem 9),

$$\left[\frac{n+1}{r}\right]_q = \begin{cases} \left(\frac{d}{r/m}\right) & \text{if } m \mid r, \\ \left(\frac{d}{(r-1)/m}\right) & \text{if } m \mid r - 1, \\ 0 & \text{otherwise} \end{cases}$$

at a primitive $m$th root of unity.

Divide the $2n$ vertices of a non-crossing matching with $d$-fold rotational symmetry (remember that $\text{rot}_n$ rotates the vertices two steps) into $m$ segments of length $2d$, say $[2d], [4d] \setminus [2d]$, and so on. The $d$ edges going from $[2d]$ to higher vertices (in $[4d]$ including $[2d]$) each have a unique region to their left. This means that the number of marked regions $r$ must be $jm$, or $jm + 1$ if there is a central region which is marked, see Figure 6. We can thus get a fixed point of $\text{rot}_n^d$ by choosing to mark $j = r/m$ (or, in the latter case, $(r - 1)/m$) of the $d$ regions associated to the edges from $[2d]$ to vertices with larger labels, which gives $\left(\frac{d}{r/m}\right)$ and $\left(\frac{d}{(r-1)/m}\right)$ respectively. Now, we combine this with Proposition 20 and the theorem follows.

\[\square\]

Appendix A. Catalan objects

A.1. Type A objects. Below is an overview of the Catalan objects we consider for $n = 3$. Recall that $\text{Cat}(3) = 5$. 
A.2. Type B objects. Below is an overview of the considered type B Catalan objects for \( n = 2 \). Recall that \( \text{Cat}^B(2) = 6 \).
Appendix B. Bijections

Here we recall several bijections on Catalan objects which have appeared earlier in the literature. We have tried to find the earliest reference for each.

B.1. NCM and binary words. Suppose $xy$ is an edge in a non-crossing perfect matching, with $x < y$. We say that $x$ is the starting vertex and $y$ is the ending vertex. Further, denote $\text{NCM}_n(k)$ the subset of all $N \in \text{NCM}(n)$ such that $\text{short}(N) = k$.

We now describe a well-known bijection $\text{NCMtoDYCK}$ from $\text{NCM}(n)$ to $\text{DYCK}(n)$.

**Bijection 4.** Take $M \in \text{NCM}(n)$ and construct the Dyck path $\text{NCMtoDYCK}(M) = b_1b_2 \cdots b_{2n}$ as follows. For vertices $i = 1, 2, \ldots, 2n$ in $M$, let $b_i = 0$ if $i$ is a starting vertex and let $b_i = 1$ if $i$ is an end vertex. It is not hard to see that this procedure ensures that the resulting binary word is a Dyck path.

Let $d$ be a natural number such that $d \mid n$. If a matching $M \in \text{NCM}(n)$ has $d$-fold rotational symmetry, it is sufficient to understand how the vertices $1, 2, \ldots, d$ are matched up. Here, we restrict ourselves to the case when $M$ does not have a diagonal—for the case when $M$ has a diagonal, see the third case in the proof of Theorem 27. In this case, there is a bijection $\text{BWtoNCM}$ between $\text{BW}(2d, d)$ and rotationally symmetric non-crossing perfect matchings.

**Bijection 5.** Let $c = c_1c_2 \cdots c_{2d} \in \text{BW}(2d, d)$. We show how to construct the corresponding $\text{BWtoNCM}(c) \in \text{NCM}(n)$. Think of the vertices $1, 2, \ldots, 2d$ being arranged on a line. For all vertices $i = 1, 2, \ldots, 2d$, we let vertex $i$ be a “left” vertex if $c_i = 0$ and a “right” vertex if $c_i = 1$. Then we match every left vertex with the closest available right vertex to its right without creating a crossing of edges. There is a unique way of doing this so that no left vertex remains unpaired between a matched pair of vertices. However, there might be some left vertices which are not matched because there are not sufficiently many right vertices to their right. There must also be equally many unpaired right vertices because there are not enough left vertices to their left. Since this will be the same in every interval of vertices $[2dk + 1, 2d(k + 1)]$ there is a unique way to pair the remaining left vertices with the remaining right vertices of the interval to the right and vice versa.

We can prove a stronger statement about rotationally symmetric NCMs. We study $\text{BWtoNCM}$ restricted to $\text{NCM}_n(n, k)$. Once again, such a matching is completely determined by how the vertices $1, 2, \ldots, 2d$ are paired up. Among these $2d$ vertices, there are two possible cases to consider.

**Case 1:** Exactly $dk/n$ short edges where either vertex 1 is a left vertex or vertex 2 is a right vertex.

**Case 2:** Exactly $dk/n - 1$ short edges where vertex 1 is a right vertex and vertex 2 is a left vertex.

If one applies $\text{BWtoNCM}^{-1}$ (that is, left vertex corresponds to 0 and right vertex corresponds to 1) to the non-crossing perfect matchings in the two above cases, one gets the image $\text{BW}^{dk/n}(d)$.

B.2. NCM and SYT. We recall the definition of this bijection.

**Bijection 6.** Let $T \in \text{SYT}(n^2)$ and construct $\text{SYTtoNCM}(T)$ as follows. For $i \in \{1, 2, \ldots, 2n\}$, let vertex $i$ be a starting vertex if $i$ is in the first row and an end vertex otherwise. It is not hard to show that determining the starting and ending vertices uniquely determines a non-crossing matching.
B.3. **NCP and NCM.** There is a bijection between non-crossing partitions and non-crossing matchings, \(\text{NCPtoNCM} : \text{NCP}(n) \rightarrow \text{NCM}(n)\), that directly restricts to a bijection between \(\text{NCP}^B\) and \(\text{NCM}^B\). This bijection has another nice property; it is equivariant with regards to the Kreweras complement on the non-crossing perfect matchings and rotation on the non-crossing perfect matchings [Hei07, Thm. 1].

**Bijection 7.** Consider \(\pi \in \text{NCP}(n)\), and for every vertex \(j \in \{1, \ldots, n\}\), insert a new vertex \((2j - 1)'\) immediately after \(j\), and \((2j - 2)'\) immediately before \(j\), where we insert \((2n)'\) immediately before \(1\). There we match \((2j)'\) to the closest point \((2k - 1)'\), \(j < k\), such that the edge between those vertices does not cross any of the blocks in \(\pi\). If no such \(k\) exists, we match to the smallest number \(k\), \(0 < k \leq j\).

Since \(j = k\) is always possible the map is well-defined. This gives a perfect matching \(\sigma\) on \(1', 2', \ldots, (2n)'\) which is non-crossing, and in addition, does not cross any of the blocks of \(\pi\). We can get the inverse of the map \(\text{NCPtoNCM}\) by putting back the vertices \(1, 2, \ldots, n\) and letting all vertices that can have an edge between them without crossing any edge of the perfect matching \(\sigma\) belong to the same block.

The following is an example of this bijection.

\[
\text{NCPtoNCM} : \begin{array}{c}
1 & 2 & 3 \\
6 & 5 & 4 \\
\end{array} \rightarrow \begin{array}{c}
1 & 2 & 3 \\
10 & 9 & 8 \\
\end{array}
\]

\(\text{NCPtoNCM} : (47)\)

One can also illustrate the bijection as follows. For each singleton block, add an edge between two copies of the vertex. For each block of size two, split the edge into two non-crossing edges and hence each vertex into two copies. Finally, for each block with \(m \geq 3\) elements, push apart the edges at the vertices so that we have \(m\) non-crossing edges on \(2m\) vertices. See Figure 7.
Note that, by definition, every block except the one whose minimum element is 1 corresponds to an even edge under the bijection. If \( j > 1 \) is the smallest element of a block and \( k \) is the largest, then \( (2j - 2) \) is matched to \( (2k - 1) \). The remaining even vertices are matched to smaller odd vertices. Hence Bijection 7 in fact gives a bijection between the sets \( \text{NCP}(n, \ell) \) and \( \text{NCM}(n, \ell - 1) \).

B.4. NCP and Dyck paths. We briefly describe a bijection between non-crossing partitions and Dyck paths, \( \text{NCPtoDYCK} \), with the property that the number of parts is sent to the number of peaks. This bijection is often attributed to P. Biane [Bia97].

Bijection 8. Let \( \pi = B_1|B_2|\cdots|B_k \) be a non-crossing partition of size \( n \), where the blocks are ordered increasingly according to maximal member. By convention, we let \( B_0 \) be a block of size 0, where the maximal member is also 0. We construct a Dyck path \( D \in \text{DYCK}(n) \) from \( \pi \) as follows. For each \( j = 1, \ldots, k \), we have a sequence of \( \max(B_j) - \max(B_{j-1}) \) north steps, immediately followed by \(|B_j| \) east steps.

References


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