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Boundedness of the Riesz potential in central Morrey–Orlicz spaces

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Abstract

Boundedness of the maximal operator and the Calderón–Zygmund singular integral operators in central Morrey–Orlicz spaces were proved in papers (Maligranda et al. in Colloq Math 138:165–181, 2015; Maligranda et al. in Tohoku Math J 72:235–259, 2020) by the second and third authors. The weak-type estimates have also been proven. Here we show boundedness of the Riesz potential in central Morrey–Orlicz spaces and the corresponding weak-type version.

Keywords Riesz potential · Orlicz functions · Orlicz spaces · Morrey–Orlicz spaces · Central Morrey–Orlicz spaces · Weak central Morrey–Orlicz spaces

Mathematics Subject Classification Primary 46E30 · 42B20; Secondary 42B35

1 Orlicz spaces and central Morrey–Orlicz spaces

First of all, we recall the definition of Orlicz spaces on $\mathbb{R}^n$ and some of their properties to be used later on (see [24] and [26] for details).

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A function $\Phi\colon [0,\infty) \to [0,\infty)$ is called an *Orlicz function*, if it is an increasing continuous and convex function with $\Phi(0) = 0$. Each such a function Φ has an integral representation $\Phi(u) = \int_0^u \Phi'_+(t)\,dt$, where the right-derivative Φ'_+ is a nondecreasing right-continuous function (see [24, Theorem 1.1]). We will write below estimates for everywhere differentiable Orlicz function Φ, but then using the above integral representation, these estimates will be true for almost all $u > 0$ with its right-derivative Φ'_+ instead of derivative Φ'. Of course, we have estimates

$$\Phi(u) \le u\,\Phi'(u) \le \Phi(2u) \ \text{ for all } \ u > 0. \tag{1}$$

If we want to include in the Orlicz spaces, for example, spaces $L^\infty(\mathbb{R}^n)$, $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for $1 \le p < \infty$, then we need to consider a broader class of functions than Orlicz functions, the so-called Young functions. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is called a *Young function*, if it is a nondecreasing convex function with $\lim_{u\to 0^+}\Phi(u) = \Phi(0) = 0$, and not identically 0 or ∞ in $(0,\infty)$. It may have jump up to ∞ at some point $u > 0$, but then it should be left continuous at u.

Let (Ω, Σ, μ) be a σ-finite complete nonatomic measure space and $L^0(\Omega)$ be the space of all μ-equivalent classes of real-valued and Σ-measurable functions defined on Ω.

For any Young function Φ, the *Orlicz space* $L^\Phi(\Omega)$, which contains all $f \in L^0(\Omega)$ such that $\int_\Omega \Phi(\varepsilon|f(x)|)\,d\mu(x) < \infty$ for some $\varepsilon = \varepsilon(f) > 0$ with the *Luxemburg–Nakano norm*

$$\|f\|_{L^\Phi} = \inf\left\{\varepsilon > 0\colon \int_\Omega \Phi\Big(\frac{|f(x)|}{\varepsilon}\Big)\,d\mu(x) \le 1\right\}, \tag{2}$$

is a Banach space (cf. [24, pp. 70–71], [26, pp. 15–16], [27, pp. 125–127] and [38, pp. 67–68]). The *fundamental function* of the Orlicz space $L^\Phi(\Omega)$ is

$$\varphi_{L^\Phi(\Omega)}(t) = \|\chi_A\|_{L^\Phi(\Omega)} = \|\chi_{[0,\mu(A)]}\|_{L^\Phi([0,\infty))} = 1/\Phi^{-1}(1/t),$$

where χ_A is the characteristic function of the set $A \subset \Omega$, $t = \mu(A)$ and Φ^{-1} is the right-continuous inverse of Φ defined by $\Phi^{-1}(v) = \inf\{u \ge 0\colon \Phi(u) > v\}$ with $\inf \emptyset = \infty$.

To each Young function Φ one can associate another convex function Φ^*, i.e., the *complementary function* to Φ, which is defined by

$$\Phi^*(v) = \sup_{u>0}\,[uv - \Phi(u)] \ \text{ for } \ v \ge 0.$$

Then Φ^* is also a Young function and $\Phi^{**} = \Phi$. Note that $u \le \Phi^{-1}(u){\Phi^*}^{-1}(u) \le 2u$ for all $u > 0$.

We say that a Young function Φ satisfies the Δ_2-*condition* and we write shortly $\Phi \in \Delta_2$, if $0 < \Phi(u) < \infty$ for $u > 0$ and there exists a constant $D_2 \ge 1$ such that

$$\Phi(2u) \le D_2\Phi(u) \ \text{ for all } \ u > 0. \tag{3}$$

In this paper we consider Orlicz spaces $L^{\Phi}(\mathbb{R}^n)$ on $\mathbb{R}^n$ with the Lebesgue measure. Then we define the Morrey–Orlicz spaces $M^{\Phi,\lambda}(\mathbb{R}^n)$ and central Morrey–Orlicz spaces $M^{\Phi,\lambda}(0)$. In the 2000s, several authors (for example, F. Deringoz, V. S. Guliyev, J. J. Hasanov, T. Mizuhara, E. Nakai, S. Samko, Y. Sawano, H. Tanaka and others) defined Orlicz versions of the Morrey space, i.e., Morrey–Orlicz spaces, and investigated the boundedness for the Hardy–Littlewood maximal operator and other operators on them (see, for example, [11, 19, 20, 34, 39] and the references therein). The Orlicz version of central Morrey spaces, i.e., central Morrey–Orlicz spaces were defined in papers by the second and third authors. They investigated boundedness on central Morrey–Orlicz spaces of the Hardy–Littlewood maximal operator in paper [28] and also boundedness of the Calderón–Zygmund singular integral operators on them in paper [29]. In this paper we present conditions under which the Riesz potential is bounded on central Morrey–Orlicz spaces.

For any Young function Φ, number $\lambda \in \mathbb{R}$, a set $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$ and for $f \in L^0(\mathbb{R}^n)$ let

$$\|f\|_{\Phi,\lambda,A} = \inf \Big\{\varepsilon > 0\colon \frac{1}{|A|^{\lambda}} \int_A \Phi\Big(\frac{|f(x)|}{\varepsilon}\Big)\, dx \le 1\Big\},$$

and the corresponding (smaller) expression

$$\|f\|_{\Phi,\lambda,A,\infty} = \inf \Big\{\varepsilon > 0\colon \sup_{u>0} \Phi\big(\frac{u}{\varepsilon}\big)\, \frac{1}{|A|^{\lambda}}\, d(f\chi_A, u) \le 1\Big\},$$

where $d(f,u) = |\{x \in \mathbb{R}^n\colon |f(x)| > u\}|$. Note that $\|f\|_{\Phi,\lambda,A,\infty} \le \|f\|_{\Phi,\lambda,A}$ provided that the expression on the right is finite. In fact, if $\|f\|_{\Phi,\lambda,A} < c$, then for arbitrary $u > 0$ we have

$$\begin{aligned} 1 &\ge \frac{1}{|A|^{\lambda}} \int_A \Phi\left(\frac{|f(x)|}{c}\right) dx \ge \frac{1}{|A|^{\lambda}} \int_{\{x\in A\colon |f(x)|>u\}} \Phi\left(\frac{|f(x)|}{c}\right) dx \\ &\ge \frac{1}{|A|^{\lambda}}\, \Phi\left(\frac{u}{c}\right) d(f\chi_A, u), \end{aligned}$$

and $\|f\|_{\Phi,\lambda,A,\infty} \le c$. Hence, $\|f\|_{\Phi,\lambda,A,\infty} \le \|f\|_{\Phi,\lambda,A}$.

Using these notions and considering open balls $B(x_0,r)$ with a center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$, i.e. $B(x_0,r) = \{x \in \mathbb{R}^n\colon |x - x_0| < r\}$, and also open balls $B(0,r) = B_r$ with a center at 0 we can define *Morrey–Orlicz spaces* $M^{\Phi,\lambda}(\mathbb{R}^n)$ and *weak Morrey–Orlicz spaces* $WM^{\Phi,\lambda}(\mathbb{R}^n)$:

$$M^{\Phi,\lambda}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n)\colon \|f\|_{M^{\Phi,\lambda}} = \sup_{x_0\in\mathbb{R}^n, r>0} \|f\|_{\Phi,\lambda,B(x_0,r)} < \infty \right\} \tag{4}$$

and

$$WM^{\Phi,\lambda}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n)\colon \|f\|_{WM^{\Phi,\lambda}} = \sup_{x_0\in\mathbb{R}^n, r>0} \|f\|_{\Phi,\lambda,B(x_0,r),\infty} < \infty \right\}. \tag{5}$$

Similarly, we can define *central Morrey–Orlicz spaces* $M^{\Phi,\lambda}(0)$ and *weak central Morrey–Orlicz spaces* $WM^{\Phi,\lambda}(0)$:

$$M^{\Phi,\lambda}(0) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) \colon \|f\|_{M^{\Phi,\lambda}(0)} = \sup_{r>0} \|f\|_{\Phi,\lambda,B_r} < \infty \right\} \tag{6}$$

and

$$WM^{\Phi,\lambda}(0) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) \colon \|f\|_{WM^{\Phi,\lambda}(0)} = \sup_{r>0} \|f\|_{\Phi,\lambda,B_r,\infty} < \infty \right\}. \tag{7}$$

All these spaces are Banach ideal spaces on $\mathbb{R}^n$ (sometimes they are $\{0\}$, that is, they contain only all functions equivalent to 0 on $\mathbb{R}^n$). Moreover, we have continuous embeddings $M^{\Phi,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WM^{\Phi,\lambda}(\mathbb{R}^n)$, $M^{\Phi,\lambda}(0) \overset{1}{\hookrightarrow} WM^{\Phi,\lambda}(0)$ and also $M^{\Phi,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} M^{\Phi,\lambda}(0)$, $WM^{\Phi,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} WM^{\Phi,\lambda}(0)$.

Let us recall that the normed subspace $X = (X, \|\cdot\|_X)$ of $L^0(\Omega)$ is an *ideal space* on Ω: if $f, g \in X$ with $|f(x)| \le |g(x)|$ for μ-almost all $x \in \Omega$, and $g \in X$, then $f \in X$ and $\|f\|_X \le \|g\|_X$. Here and further, for two Banach ideal spaces X and Y, we use the symbol $X \hookrightarrow Y$ rather than $X \subset Y$ for continuous embedding. Moreover, the symbol $X \overset{C}{\hookrightarrow} Y$ indicates that $X \hookrightarrow Y$ with the norm of the embedding operator not bigger than C, i.e., $\|f\|_Y \le C\, \|f\|_X$ for all $f \in X$.

Note that Morrey–Orlicz spaces and central Morrey–Orlicz spaces are generalizations of Orlicz spaces and Morrey spaces (on $\mathbb{R}^n$). In particular, we can obtain the following spaces (see [28] for more details):

(i) (Orlicz and weak Orlicz spaces) If $\lambda = 0$, then

$$M^{\Phi,0}(\mathbb{R}^n) = M^{\Phi,0}(0) = L^{\Phi}(\mathbb{R}^n) \text{ and}$$
$$WM^{\Phi,0}(\mathbb{R}^n) = WM^{\Phi,0}(0) = WL^{\Phi}(\mathbb{R}^n).$$

(ii) (Beurling–Orlicz and weak Beurling–Orlicz spaces) If $\lambda = 1$, then

$$M^{\Phi,1}(\mathbb{R}^n) = B^{\Phi}(\mathbb{R}^n) \text{ and } WM^{\Phi,1}(\mathbb{R}^n) = WB^{\Phi}(\mathbb{R}^n).$$

As for $B^{\Phi}(\mathbb{R}^n)$ and $WB^{\Phi}(\mathbb{R}^n)$, see [28].

(iii) (Classical Morrey, weak Morrey, central Morrey and weak central Morrey spaces) If $\Phi(u) = u^p, 1 \le p < \infty$ and $\lambda \in \mathbb{R}$, then $M^{\Phi,\lambda}(\mathbb{R}^n) = M^{p,\lambda}(\mathbb{R}^n)$, $WM^{\Phi,\lambda}(\mathbb{R}^n) = WM^{p,\lambda}(\mathbb{R}^n)$ and $M^{\Phi,\lambda}(0) = M^{p,\lambda}(0)$, $WM^{\Phi,\lambda}(0) = WM^{p,\lambda}(0)$.
Here $M^{p,\lambda}(\mathbb{R}^n)$, $WM^{p,\lambda}(\mathbb{R}^n)$, $M^{p,\lambda}(0)$, $WM^{p,\lambda}(0)$ are the classical Morrey, weak Morrey, central Morrey and weak central Morrey spaces, respectively.

We want to note that $M^{p,\lambda}(\mathbb{R}^n) \ne \{0\}$ if and only if $0 \le \lambda \le 1$ (see [6, Lemma 1]) and $M^{p,\lambda}(0) \ne \{0\}$ if and only if $\lambda \ge 0$ (see [4, 6, 7]). Moreover, $M^{p,0}(\mathbb{R}^n) = M^{p,0}(0) = L^p(\mathbb{R}^n)$ and $M^{p,1}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$ (see [25, Theorem 4.3.6]). However, $L^{\infty}(\mathbb{R}^n) \overset{1}{\hookrightarrow} M^{p,1}(0)$ and the inclusion is strict. For example, in one-dimensional case

$f(x) = \sum_{n=0}^{\infty} 2^{n/p} \chi_{[n,n+2^{-n}]}(|x|) \in M^{p,1}(0) \setminus L^{\infty}(\mathbb{R}^1)$. Of course, for $0 \le \lambda \le 1$ the inclusion $M^{p,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} M^{p,\lambda}(0)$ holds and is strict for $0 < \lambda \le 1$ (a suitable example we can find in [22, p. 156]). It is also true that if $1 \le p < q < \infty, 0 \le \mu < \lambda < 1$ and $\frac{1-\lambda}{p} = \frac{1-\mu}{q}$, then

$$M^{q,\mu}(\mathbb{R}^n) \overset{1}{\hookrightarrow} M^{p,\lambda}(\mathbb{R}^n) \quad \text{and} \quad M^{q,\mu}(0) \overset{1}{\hookrightarrow} M^{p,\lambda}(0). \tag{8}$$

Both inclusions are proper (see, for example, [21]); the second embedding in (8) is also true for $1 < \lambda < \mu$. The embeddings (8) follow by the Hölder–Rogers inequality with $\frac{q}{p} > 1$, since for any $x_0 \in \mathbb{R}^n$ we have

$$\begin{aligned}
\int_{B(x_0,r)} |f(x)|^p \, dx &\le \left(\int_{B(x_0,r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{1-p/q} \\
&= \left(\frac{1}{|B(x_0, r)|^{\mu}} \int_{B(x_0,r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{1-p/q+\mu p/q} \\
&= \left(\frac{1}{|B(x_0, r)|^{\mu}} \int_{B(x_0,r)} |f(x)|^q \, dx \right)^{p/q} |B(x_0, r)|^{\lambda},
\end{aligned}$$

and from the fact that $1 - p/q + \mu p/q = (\mu - 1)\, p/q + 1 = -(1 - \lambda) + 1 = \lambda$.

If the supremum in definitions (4)–(7) is taken over all $r > 1$, then we will have corresponding definitions of non-homogeneous Morrey–Orlicz spaces, non-homogeneous weak Morrey–Orlicz spaces, non-homogeneous central Morrey–Orlicz spaces and non-homogeneous weak central Morrey–Orlicz spaces.

2 The Riesz potential in Lebesgue, Orlicz and Morrey spaces

The *Riesz potential* of order $\alpha \in (0, n)$ of a locally integrable function $f \in L^0(\mathbb{R}^n)$ is defined as

$$I_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad \text{for } x \in \mathbb{R}^n. \tag{9}$$

The linear operator I_{α} plays a role in various branches of analysis, including potential theory, harmonic analysis, Sobolev spaces and partial differential equations. Therefore, investigations of the boundedness of the operator I_{α} between different spaces are important.

The classical *Hardy–Littlewood–Sobolev theorem* states that if $1 < p < q < \infty$, then a Riesz potential I_{α} is of strong-type (p, q), that is, bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $1/q = 1/p - \alpha/n$. For $p = 1 < q < \infty$ Zygmund proved that I_{α} is of weak-type $(1, q)$, that is, bounded from $L^1(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$, where $1/q = 1 - \alpha/n$. The weak-L^q space $WL^q(\mathbb{R}^n) = L^{q,\infty}(\mathbb{R}^n)$, called also the Marcinkiewicz space, consists of all $f \in L^0(\mathbb{R}^n)$ such that the quasi-norm $\|f\|_{q,\infty} = \sup_{t>0} t\, |\{x \in \mathbb{R}^n : |f(x)| > t\}|^{1/q}$ is finite. The proofs of these results we can find in the books [15, pp. 125–127], [16, pp. 2–5], [40, pp. 117–121], [41, pp. 150–154] and [42, pp. 86–87].

The boundedness of I_α from an Orlicz space $L^\Phi(\mathbb{R}^n)$ to another Orlicz space $L^\Psi(\mathbb{R}^n)$ was studied by Simonenko (1964), O'Neil (1965) and Torchinsky (1976) under some restrictions on the Orlicz functions Φ and Ψ. In 1999 Cianchi [10] gave a necessary and sufficient condition for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to weak Orlicz space $WL^\Psi(\mathbb{R}^n)$. Another sufficient conditions for boundedness of the Riesz operator I_α (and even for a generalized fractional operator I_ρ) were given in 2001 by Nakai [32, 33]. Then in 2017, Guliyev–Deringoz–Hasanov in [20, Theorem 3.3], gave more readable necessary and sufficient conditions for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Results concerning boundedness of the Riesz potential between Morrey spaces were first obtained by Spanne with the Sobolev exponent $1/q = 1/p - \alpha/n$, and this result was published in 1969 by Peetre [36]: if $0 < \alpha < n$, $1 < p < n(1-\lambda)/\alpha$, $0 < \lambda < 1$, $1/q = 1/p - \alpha/n$ and $\lambda/p = \mu/q$, then the Riesz potential I_α is bounded from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q,\mu}(\mathbb{R}^n)$. Then in 1975 a stronger result was obtained by Adams [1], and reproved by Chiarenza–Frasca [9]. Adams proved boundedness of I_α from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q_1,\lambda}(\mathbb{R}^n)$ with a better exponent q_1, namely $1/q_1 = 1/p - \alpha/[n(1-\lambda)]$. Adams result is stronger than the Peetre–Spanne theorem because $q < q_1$ and $(1-\mu)/q = (1-\lambda)/q_1$, from which follows the embedding $M^{q_1,\lambda}(\mathbb{R}^n) \overset{1}{\hookrightarrow} M^{q,\mu}(\mathbb{R}^n)$ and this means that the target space $M^{q_1,\lambda}(\mathbb{R}^n)$ is smaller than target space $M^{q,\mu}(\mathbb{R}^n)$ in the Peetre–Spanne result. Central Morrey spaces $M^{p,\lambda}(0)$ were first introduced in [14, p. 607] and in [2, p. 5] (see also [8, p. 257] and [13, p. 500] for $\lambda = 1$). Further studies of the central Morrey spaces and their generalizations were investigated, for example, in [6, 17, 18, 37].

Result on the boundedness of the Riesz potential in these spaces was proved by Fu–Lin–Lu [12, Proposition 1.1]: if $1 < p < n(1-\lambda)/\alpha$, $0 < \lambda < 1$, $1/q = 1/p - \alpha/n$ and $\lambda/p = \mu/q$, then the Riesz potential I_α is bounded from $M^{p,\lambda}(0)$ to $M^{q,\mu}(0)$ (see also [5, 17, 18]). Komori-Furuya and Sato [23, Proposition 1] showed that Adams type result on boundedness in central Morrey spaces does not hold. They showed that if $\frac{1-\mu}{q} = \frac{1-\lambda}{p} - \frac{\alpha}{n}$ and $\alpha/n < 1/p - 1/q < \alpha/[n(1-\lambda)]$, then I_α is not bounded from $M^{p,\lambda}(0)$ to $M^{q,\mu}(0)$ because $\mu/q = \lambda/p - (1/p - \alpha/n - 1/q) < \lambda/p$.

We will generalize the last results to central Morrey–Orlicz spaces. In Theorem 2, the necessary conditions for boundedness of I_α are given, and in Theorem 3 – sufficient conditions are presented.

In the proof of boundedness of the Riesz potential in the central Morrey-Orlicz spaces we will need some necessary estimates. We will present them in the next section.

3 Some technical results

To prove the main results of this paper, we need some technical calculations. In order not to hide the main ideas in proofs of the main results we collect such calculations in Lemma 1 below.

Lemma 1 *Let Φ be a Young function, Φ^* its complementary function, $0 \le \lambda \le 1$ and $r > 0$. Then*

(i) $\int_{B_r} |f(x)g(x)|\,dx \le 2\,|B_r|^\lambda\, \|f\|_{\Phi,\lambda,B_r} \|g\|_{\Phi^*,\lambda,B_r}$.

(ii) $\|\chi_{B(x_0,r_0)}\|_{\Phi^*,\lambda,B_r} \le \frac{|B_r \cap B(x_0,r_0)|}{|B_r|^\lambda}\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|}\right)$, *where $B_r \cap B(x_0,r_0) \ne \emptyset$ for $x_0 \in \mathbb{R}^n$ and $r_0 > 0$.*
In particular, $\|\chi_{B_r}\|_{\Phi^*,\lambda,B_r} \le \dfrac{\Phi^{-1}\left(|B_r|^{\lambda-1}\right)}{|B_r|^{\lambda-1}}$.

(iii) $\|\chi_{B_t}\|_{\Phi,\lambda,B_r} = 1/\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B_t|}\right)$ *and* $\|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)} = \dfrac{1}{\Phi^{-1}(|B_t|^{\lambda-1})}$ *for any $t > 0$.*

(iv) $\|\chi_{B_t}\|_{\Phi,\lambda,B_r,\infty} = 1/\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B_t|}\right)$ *and* $\|\chi_{B_t}\|_{WM^{\Phi,\lambda}(0)} = \dfrac{1}{\Phi^{-1}(|B_t|^{\lambda-1})}$ *for any $t > 0$.*

Proof (i) This estimate was proved in [29, Lemma 2.6].

(ii) Since for $u > 0$ we have $\Phi^*\left(\frac{u}{\Phi^{-1}(u)}\right) \le u$ (cf. Lemma 2.6 in [29]) it follows for $u = \frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|}$ that

$$\begin{aligned}
&\int_{B_r} \Phi^*\left(\frac{\chi_{B(x_0,r_0)}(x)|B_r|^\lambda}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|}\right)|B_r\cap B(x_0,r_0)|}\right)dx \\
&= \int_{B_r\cap B(x_0,r_0)} \Phi^*\left(\frac{|B_r|^\lambda}{\Phi^{-1}(\frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|})|B_r\cap B(x_0,r_0)|}\right)dx \\
&\le \frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|}\int_{B_r\cap B(x_0,r_0)} dx = |B_r|^\lambda.
\end{aligned}$$

Hence, $\|\chi_{B(x_0,r_0)}\|_{\Phi^*,\lambda,B_r} \le \Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B(x_0,r_0)|}\right)\frac{|B_r\cap B(x_0,r_0)|}{|B_r|^\lambda}$, and (ii) follows.

(iii) Let $t > 0$. Since $\Phi(\Phi^{-1}(u)) \le u$ for any $u > 0$ it follows that

$$\begin{aligned}
&\int_{B_r} \Phi\left(\chi_{B_t}(x)\,\Phi^{-1}(\frac{|B_r|^\lambda}{|B_r\cap B_t|})\right)dx = \int_{B_r\cap B_t} \Phi\left(\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B_t|}\right)\right)dx \\
&\le \int_{B_r\cap B_t} \frac{|B_r|^\lambda}{|B_r\cap B_t|}dx = |B_r|^\lambda,
\end{aligned}$$

and so $\|\chi_{B_t}\|_{\Phi,\lambda,B_r} \le 1/\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r\cap B_t|}\right)$. On the other hand,

$$1 \ge \frac{1}{|B_r|^\lambda}\int_{B_r} \Phi\left(\frac{\chi_{B_t}(x)}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r}}\right)dx = \Phi\left(\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r}}\right)\frac{|B_r\cap B_t|}{|B_r|^\lambda},$$

or

$$\frac{|B_r|^\lambda}{|B_r \cap B_t|} \geq \Phi(\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r}}).$$

Since $u \leq \Phi^{-1}(\Phi(u))$ for any $u > 0$ such that $\Phi(u) < \infty$ we obtain

$$\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right) \geq \Phi^{-1}\left(\Phi(\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r}})\right) \geq \frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r}},$$

which together with the previous estimate gives equality $\|\chi_{B_t}\|_{\Phi,\lambda,B_r} = 1/\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right)$. Thus,

$$\begin{aligned}
\|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)} &= \sup_{r>0} \|\chi_{B_t}\|_{\Phi,\lambda,B_r} = \sup_{r>0} \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right)} \\
&= \max\left[\sup_{r\leq t} \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right)}, \sup_{r\geq t} \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right)}\right] \\
&= \max\left[\sup_{r\leq t} \frac{1}{\Phi^{-1}\left(|B_r|^{\lambda-1}\right)}, \sup_{r\geq t} \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_t|}\right)}\right] = \frac{1}{\Phi^{-1}(|B_t|^{\lambda-1})},
\end{aligned}$$

and point (iii) of the lemma has been proved.

(iv) For $t > 0$ we have

$$\begin{aligned}
&\sup_{u>0} \Phi(\frac{u}{\varepsilon}) \frac{1}{|B_r|^\lambda} |\{x \in B_r : \chi_{B_t}(x) > u\}| \\
&\quad = \sup_{0<u<1} \Phi(\frac{u}{\varepsilon}) \frac{|B_r \cap B_t|}{|B_r|^\lambda} = \Phi(\frac{1}{\varepsilon}) \frac{|B_r \cap B_t|}{|B_r|^\lambda}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\chi_{B_t}\|_{\Phi,\lambda,B_r,\infty} &= \inf\left\{\varepsilon > 0 \colon \Phi\left(\frac{1}{\varepsilon}\right) \frac{|B_r \cap B_t|}{|B_r|^\lambda} \leq 1\right\} \\
&\leq \inf\left\{\varepsilon > 0 \colon \frac{1}{\varepsilon} \leq \Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right)\right\} \leq 1/\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right),
\end{aligned}$$

because $1/\varepsilon \leq \Phi^{-1}(\Phi(1/\varepsilon))$. On the other hand, since $1 \geq \Phi(\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r,\infty}}) \frac{|B_r \cap B_t|}{|B_r|^\lambda}$ it follows that

$$\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r,\infty}} \leq \Phi^{-1}\left(\Phi(\frac{1}{\|\chi_{B_t}\|_{\Phi,\lambda,B_r,\infty}})\right) \leq \Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B_t|}\right),$$

which together gives the first equality in (iv). The second equality in (iv) has the same proof as the second equality in (iii). □

4 On the norm of the dilation operator in central Morrey–Orlicz spaces

For any $a > 0$ and $x \in \mathbb{R}^n$ we define the *dilation operator* D_a by

$$D_a f(x) = f(ax), \quad f \in L^0(\mathbb{R}^n).$$

The dilation operator is bounded in central Morrey–Orlicz spaces $M^{\Phi,\lambda}(0)$ and we will calculate its norm. For this purpose quantity $s_{\Phi^{-1}}$ is needed for the Orlicz function Φ:

$$s_{\Phi^{-1}}(t) = \sup_{s>0} \frac{\Phi^{-1}(st)}{\Phi^{-1}(s)}, \qquad t > 0. \tag{10}$$

Theorem 1 *If Φ is an Orlicz function, $0 \le \lambda \le 1$ and $a > 0$, then the operator norm of D_a is*

$$\|D_a\|_{M^{\Phi,\lambda}(0) \to M^{\Phi,\lambda}(0)} = s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right). \tag{11}$$

Proof By definition of $s_{\Phi^{-1}}$, for any $s > 0$, $a > 0$, we have

$$\Phi^{-1}\left(a^{n(\lambda-1)} s\right) \le s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right) \Phi^{-1}(s),$$

and so

$$\Phi\left(\frac{\Phi^{-1}\left(a^{n(\lambda-1)} s\right)}{s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right)}\right) \le \Phi\left(\Phi^{-1}(s)\right) = s.$$

For $a^{n(\lambda-1)} s = \Phi(u)$ we have $u = \Phi^{-1}\left(a^{n(\lambda-1)} s\right)$ and

$$\Phi\left(\frac{u}{s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right)}\right) \le a^{n(1-\lambda)} \Phi(u), \quad \text{for any } u > 0. \tag{12}$$

Therefore, from (12) it follows that for any $f \in M^{\Phi,\lambda}(0)$ and $r > 0$,

$$\begin{aligned}
&\int_{B_r} \Phi\left(\frac{|D_a f(x)|}{s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx \\
&= \int_{B_r} \Phi\left(\frac{|f(ax)|}{s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx \\
&= a^{-n} \int_{B_{ar}} \Phi\left(\frac{|f(y)|}{s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right) \|f\|_{M^{\Phi,\lambda}(0)}}\right) dy
\end{aligned}$$

$$\leq a^{-n}a^{n(1-\lambda)}\int\limits_{B_{ar}}\Phi\left(\frac{|f(y)|}{\|f\|_{M^{\Phi,\lambda}(0)}}\right)dy \leq a^{-\lambda n}|B_{ar}|^{\lambda}$$
$$= a^{-\lambda n}v_n^{\lambda}(ar)^{\lambda n} = |B_r|^{\lambda},$$

which means that $\|D_a f\|_{M^{\Phi,\lambda}(0)} \leq s_{\Phi^{-1}}\left(a^{n(\lambda-1)}\right)\|f\|_{M^{\Phi,\lambda}(0)}$. Here, $v_n = |B_1|$.

To show that (11) holds we consider the characteristic function $\chi_{B_t}(x)$ of the ball $B_t,\ t > 0$. Note that $D_a\chi_{B_t}(x) = \chi_{B_{t/a}}(x)$. Moreover, by Lemma 1(iii) we get

$$\sup_{t>0}\frac{\|D_a\chi_{B_t}\|_{M^{\Phi,\lambda}(0)}}{\|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)}} = \sup_{t>0}\frac{\Phi^{-1}(|B_t|^{\lambda-1})}{\Phi^{-1}(|B_{t/a}|^{\lambda-1})} = \sup_{t>0}\frac{\Phi^{-1}(v_n^{\lambda-1}t^{n(\lambda-1)})}{\Phi^{-1}\left(v_n^{\lambda-1}(\frac{t}{a})^{n(\lambda-1)}\right)}$$
$$= \sup_{s>0}\frac{\Phi^{-1}(s)}{\Phi^{-1}(a^{n(1-\lambda)}s)} = \sup_{s>0}\frac{\Phi^{-1}(sa^{n(\lambda-1)})}{\Phi^{-1}(s)} = s_{\Phi^{-1}}(a^{n(\lambda-1)}).$$

This brings us to (11). □

5 The Riesz potential in central Morrey–Orlicz spaces—necessary conditions

We begin to study the boundedness of the Riesz potential, first finding the necessary conditions for its boundedness.

Theorem 2 *Let* $0 < \alpha < n$, Φ, Ψ *be Orlicz functions and* $0 \leq \lambda, \mu < 1$.

(i) *If the Riesz potential* I_α *is bounded from* $M^{\Phi,\lambda}(0)$ *to* $M^{\Psi,\mu}(0)$, *then there are positive constants* C_1, C_2 *such that*

(a) $u^{\frac{\alpha}{n}}\,\Phi^{-1}(u^{\lambda-1}) \leq C_1\,\Psi^{-1}(u^{\mu-1})$ *for any* $u > 0$.
(b) $s_{\Psi^{-1}}(u^{\mu-1}) \leq C_2\,u^{\frac{\alpha}{n}}\,s_{\Phi^{-1}}(u^{\lambda-1})$ *for any* $u > 0$.

(ii) *If there exists a small constant* $c > 0$ *such that* $c \leq \frac{v_n^{\lambda/\mu}}{v_{n-1}}$ *with* $v_0 = 1$ *and*

$$\liminf_{t\to\infty}\frac{\Phi^{-1}(ct^{\lambda})}{\Psi^{-1}(t^{\mu})} = \infty,$$

then I_α *is not bounded from* $M^{\Phi,\lambda}(0)$ *to* $M^{\Psi,\mu}(0)$.

Proof (i) (a) Let $t > 0$ and $x \in B_t$. In this case we have

$$I_\alpha\chi_{B_t}(x) = \int\limits_{B_t}|x-y|^{\alpha-n}\,dy \geq (2t)^{\alpha-n}|B_t| = v_n 2^{\alpha-n}t^{\alpha}$$

or

$$t^{\alpha}\,\chi_{B_t}(x) \leq \frac{2^{n-\alpha}}{v_n}\,I_\alpha\chi_{B_t}(x)\,\chi_{B_t}(x).$$

Then

$$\|t^{\alpha}\,\chi_{B_t}\|_{M^{\Psi,\mu}(0)} \le \frac{2^{n-\alpha}}{v_n}\,\|I_\alpha \chi_{B_t}\|_{M^{\Psi,\mu}(0)} \le \frac{2^{n-\alpha}}{v_n}\,C\,\|\chi_{B_t}\|_{M^{\Phi,\lambda}(0)},$$

and by the Lemma 1 (iii) we obtain

$$\frac{t^{\alpha}}{\Psi^{-1}(|B_t|^{\mu-1})} \le \frac{2^{n-\alpha}}{v_n}\,C\,\frac{1}{\Phi^{-1}(|B_t|^{\lambda-1})},$$

which means

$$\frac{t^{\alpha}}{\Psi^{-1}(v_n^{\mu-1}t^{(\mu-1)n})} \le \frac{2^{n-\alpha}}{v_n}\,\frac{C}{\Phi^{-1}(v_n^{\lambda-1}t^{(\lambda-1)n})}.$$

Thus,

$$t^{\alpha/n}\,\Phi^{-1}(v_n^{\lambda-1}t^{\lambda-1}) \le \frac{2^{n-\alpha}}{v_n}\,C\,\Psi^{-1}(v_n^{\mu-1}t^{\mu-1}),$$

which by a simple change of variables can be rewritten as

$$u^{\alpha/n}\,\Phi^{-1}(u^{\lambda-1}) \le C_1\,\Psi^{-1}(u^{\mu-1}) \text{ for any } u > 0,$$

where $C_1 = 2^{n-\alpha}v_n^{\alpha/n-1}C$.

(i) (b) First, note that we have identity

$$I_\alpha(D_t f)(x) = t^{-\alpha}\,D_t(I_\alpha f)(x) \;\text{ for any } t > 0.$$

In fact,

$$I_\alpha(D_t f)(x) = \int_{\mathbb{R}^n} \frac{f(ty)}{|x-y|^{n-\alpha}}\,dy = t^{-\alpha}\int_{\mathbb{R}^n} \frac{f(y)}{|y-tx|^{n-\alpha}}\,dy = t^{-\alpha}\,D_t(I_\alpha f)(x).$$

Now, let $f \in M^{\Phi,\lambda}(0)$. Using the above identity and applying Theorem 1 we obtain

$$\|I_\alpha(D_t f)\|_{M^{\Psi,\mu}(0)} = t^{-\alpha}\,\|D_t(I_\alpha f)\|_{M^{\Psi,\mu}(0)} = t^{-\alpha}\,s_{\Psi^{-1}}(t^{n(\mu-1)})\,\|I_\alpha f\|_{M^{\Psi,\mu}(0)}.$$

Assumption of boundedness of I_α and reuse of Theorem 1 gives

$$\begin{aligned}
\|I_\alpha f\|_{M^{\Psi,\mu}(0)} &= \frac{t^{\alpha}}{s_{\Psi^{-1}}(t^{n(\mu-1)})}\|I_\alpha(D_t f)\|_{M^{\Psi,\mu}(0)} \\
&\le \frac{t^{\alpha}}{s_{\Psi^{-1}}(t^{n(\mu-1)})}\,C\,\|D_t f\|_{M^{\Phi,\lambda}(0)} \\
&= C\,\frac{t^{\alpha}}{s_{\Psi^{-1}}(t^{n(\mu-1)})} s_{\Phi^{-1}}(t^{n(\lambda-1)})\,\|f\|_{M^{\Phi,\lambda}(0)},
\end{aligned}$$

or

$$\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \leq C\, \frac{u^{\alpha/n} s_{\Phi^{-1}}(u^{\lambda-1})}{s_{\Psi^{-1}}(u^{\mu-1})}\, \|f\|_{M^{\Phi,\lambda}(0)} \ \text{ for any } u > 0.$$

Thus,

$$\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \leq C\, \inf_{u>0} \frac{u^{\alpha/n} s_{\Phi^{-1}}(u^{\lambda-1})}{s_{\Psi^{-1}}(u^{\mu-1})}\, \|f\|_{M^{\Phi,\lambda}(0)}.$$

We must have that $\inf_{u>0} \frac{u^{\alpha/n} s_{\Phi^{-1}}(u^{\lambda-1})}{s_{\Psi^{-1}}(u^{\mu-1})} = c > 0$ since otherwise $I_\alpha f = 0$ and we get a contradiction. Therefore,

$$s_{\Psi^{-1}}(u^{\mu-1}) \leq \frac{C}{c}\, u^{\alpha/n} s_{\Phi^{-1}}(u^{\lambda-1}) \ \text{ for any } u > 0.$$

(ii) We follow the same argument as in [23, Proposition 1]. Let $R \geq 1$, $x_R = (R, 0, ..., 0) \in \mathbb{R}^n$ and $f_R(x) = \chi_{B(x_R,1)}(x)$. Then

$$\begin{aligned}
\|f_R\|_{M^{\Phi,\lambda}(0)} &= \sup_{r>0} \inf\left\{\varepsilon > 0\colon \frac{1}{|B_r|^\lambda} \int_{B_r} \Phi\left(\frac{\chi_{B(x_R,1)(x)}}{\varepsilon}\right) dx \leq 1\right\} \\
&= \sup_{r>0} \inf\left\{\varepsilon > 0\colon \frac{1}{|B_r|^\lambda} \int_{B_r \cap B(x_R,1)} \Phi\left(\frac{1}{\varepsilon}\right) dx \leq 1\right\} \\
&= \sup_{r>0} \inf\left\{\varepsilon > 0\colon \frac{|B_r \cap B(x_R,1)|}{|B_r|^\lambda} \Phi\left(\frac{1}{\varepsilon}\right) \leq 1\right\} \\
&= \sup_{r>R-1} \inf\left\{\varepsilon > 0\colon \frac{|B_r \cap B(x_R,1)|}{|B_r|^\lambda} \Phi\left(\frac{1}{\varepsilon}\right) \leq 1\right\},
\end{aligned}$$

because if $0 < r \leq R-1$ then $|B_r \cap B(x_R, 1)| = 0$. Thus,

$$\|f_R\|_{M^{\Phi,\lambda}(0)} = \sup_{r>R-1} \frac{1}{\Phi^{-1}\left(\frac{|B_r|^\lambda}{|B_r \cap B(x_R,1)|}\right)}.$$

We will consider two cases: $R-1 < r < R$ and $r \geq R$. In the first case, using calculations from [6, p. 161], we can prove that for $n \geq 2$

$$|B_r \cap B(x_R, 1)| \leq 2^{\frac{n}{2}} v_{n-1} \left(\frac{r}{R}\right)^n,$$

and so

$$\frac{|B_r|^\lambda}{|B_r \cap B(x_R,1)|} \geq \frac{v_n^\lambda r^{\lambda n} R^n}{2^{\frac{n}{2}} v_{n-1} r^n} \geq \frac{v_n^\lambda}{2^{\frac{n}{2}} v_{n-1}} R^{\lambda n} > \frac{v_n^\lambda}{2^n v_{n-1}} R^{\lambda n}.$$

For $n = 1$ and $R - 1 < r < R$ with $v_0 = 1$ we have

$$\frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|} = \frac{(2r)^\lambda}{r - R + 1} = \frac{2^\lambda r^{\lambda-1}}{1 - \frac{R-1}{r}} > \frac{2^\lambda R^{\lambda-1}}{1 - \frac{R-1}{r}}$$
$$= \frac{2^\lambda R^\lambda}{R - \frac{R(R-1)}{r}} > 2^\lambda R^\lambda = v_1^\lambda R^\lambda > \frac{v_1^\lambda}{2\, v_0} R^\lambda.$$

In the second case, $|B_r \cap B(x_R, 1)| \le |B(x_R, 1)| = v_n$ and

$$\frac{|B_r|^\lambda}{|B_r \cap B(x_R, 1)|} \ge \frac{v_n^\lambda r^{\lambda n}}{v_n} \ge v_n^{\lambda-1} R^{\lambda n}.$$

Thus,

$$\| f_R \|_{M^{\Phi,\lambda}(0)} \le \max\left[\frac{1}{\Phi^{-1}\left(\frac{v_n^\lambda}{2^n\, v_{n-1}} R^{\lambda n}\right)}, \frac{1}{\Phi^{-1}(v_n^{\lambda-1} R^{\lambda n})} \right].$$

Since $\frac{v_{n-1}}{v_n} \ge \sqrt{\frac{n}{2\pi}}$ with $v_0 = 1$ (see [3, Theorem 2]), it follows that $\frac{2^n\, v_{n-1}}{v_n} \ge 1$ and then $\frac{v_n^\lambda}{2^n\, v_{n-1}} \le v_n^{\lambda-1}$, which gives

$$\| f_R \|_{M^{\Phi,\lambda}(0)} \le \frac{1}{\Phi^{-1}\left(\frac{v_n^\lambda}{2^n\, v_{n-1}} R^{\lambda n}\right)}.$$

Next, we will estimate $I_\alpha f_R$. If $x, y \in B(x_R, 1)$ then $|x - y| \le 2$ and we obtain

$$I_\alpha f_R(x) = \int_{\mathbb{R}^n} \frac{\chi_{B(x_R,1)}(y)}{|x - y|^{n-\alpha}}\, dy = \int_{B(x_R,1)} |x - y|^{\alpha - n}\, dy$$
$$\ge 2^{\alpha-n} |B(x_R, 1)|\, \chi_{B(x_R,1)}(x) = 2^{\alpha-n} v_n\, \chi_{B(x_R,1)}(x).$$

Thus,

$$\| I_\alpha f_R \|_{M^{\Psi,\mu}(0)} = \sup_{r>0} \| I_\alpha f_R \|_{\Psi,\mu,B_r} \ge \| I_\alpha f_R \|_{\Psi,\mu,B_{R+1}}$$
$$= \inf\left\{ \varepsilon > 0\colon \int_{B_{R+1}} \Psi\left(\frac{|I_\alpha f_R(x)|}{\varepsilon} \right) dx \le |B_{R+1}|^\mu \right\}.$$

Since $x \in B(x_R, 1)$ and $B_{R+1} \cap B(x_R, 1) = B(x_R, 1)$ it follows that

$$\| I_\alpha f_R \|_{M^{\Psi,\mu}(0)} \ge \inf\left\{ \varepsilon > 0\colon \int_{B(x_R,1)\cap B_{R+1}} \Psi\left(2^{\alpha-n} v_n / \varepsilon \right) dx \le |B_{R+1}|^\mu \right\}$$

$$= \frac{2^{\alpha-n} v_n}{\Psi^{-1}\left(\frac{|B_{R+1}|^{\mu}}{|B(x_R,1)|}\right)} = \frac{2^{\alpha-n} v_n}{\Psi^{-1}\left(v_n^{\mu-1}(R+1)^{\mu n}\right)}$$
$$\geq \frac{2^{\alpha-n} v_n}{\Psi^{-1}\left(v_n^{\mu-1} 2^{\mu n} R^{\mu n}\right)}.$$

Making the substitution $t^{\mu} = v_n^{\mu-1} 2^{\mu n} R^{\mu n}$ we obtain

$$\frac{\|I_\alpha f_R\|_{M^{\Psi,\mu}(0)}}{\|f_R\|_{M^{\Phi,\lambda}(0)}} \geq 2^{\alpha-n}\, v_n \frac{\Phi^{-1}\left(\frac{v_n^{\lambda}}{2^n v_{n-1}} R^{\lambda n}\right)}{\Psi^{-1}(v_n^{\mu-1} 2^{\mu n} R^{\mu n})} = 2^{\alpha-n}\, v_n \frac{\Phi^{-1}\left(\frac{v_n^{\frac{\lambda}{\mu}}}{2^{n+\lambda n}\, v_{n-1}}\, t^{\lambda}\right)}{\Psi^{-1}(t^{\mu})}$$
$$\geq \frac{2^{\alpha-n}\, v_n}{2^{n+\lambda n}} \frac{\Phi^{-1}\left(\frac{v_n^{\frac{\lambda}{\mu}}}{v_{n-1}}\, t^{\lambda}\right)}{\Psi^{-1}(t^{\mu})} \geq 2^{\alpha-2n-\lambda n}\, v_n \frac{\Phi^{-1}\left(c\, t^{\lambda}\right)}{\Psi^{-1}(t^{\mu})},$$

and

$$\liminf_{R\to\infty} \frac{\|I_\alpha f_R\|_{M^{\Psi,\mu}(0)}}{\|f_R\|_{M^{\Phi,\lambda}(0)}} \geq 2^{\alpha-2n-\lambda n}\, v_n \liminf_{t\to\infty} \frac{\Phi^{-1}(ct^{\lambda})}{\Psi^{-1}(t^{\mu})} = \infty.$$

Thus, the operator I_α is not bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0)$. □

6 The Riesz potential in central Morrey–Orlicz spaces – sufficient conditions

We want to prove boundedness of the Riesz potential I_α between two different central Morrey–Orlicz spaces. The following lemmas are important for proving the main result.

Lemma 2 *Let $0 < \alpha < n$, Φ be an Orlicz function and $0 \leq \lambda < 1$. If $f \in M^{\Phi,\lambda}(0)$, then there exists a constant $C_3 > 0$ such that*

$$\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y)|}{|y|^{n-\alpha}}\, dy \leq C_3\, \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_r|}^{\infty} t^{\alpha/n}\, \Phi^{-1}(t^{\lambda-1})\, \frac{dt}{t}$$

for all $r > 0$.

Proof We prove this lemma using the same arguments as in the proof of Theorem 7.1 in [35] and Lemma 2.5 in [29]. From the Lemma 1 (i) and (ii) it follows that

$$\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y)|}{|y|^{n-\alpha}}\, dy = \sum_{j=1}^{\infty} \int_{B_{2^j r}\setminus B_{2^{j-1} r}} \frac{|f(y)|}{|y|^{n-\alpha}}\, dy \leq \sum_{j=1}^{\infty} \frac{1}{(2^{j-1} r)^{n-\alpha}} \int_{B_{2^j r}} |f(y)|\, dy$$

$$
\begin{aligned}
&= 2^{n-\alpha} v_n^{1-\alpha/n} \sum_{j=1}^{\infty} \frac{1}{|B_{2^j r}|^{1-\alpha/n}} \int_{B_{2^j r}} |f(y)|\,dy \\
&\leq 2^{n-\alpha+1} v_n^{1-\alpha/n} \sum_{j=1}^{\infty} |B_{2^j r}|^{\lambda-1+\alpha/n} \|f\|_{\Phi,\lambda,B_{2^j r}} \|1\|_{\Phi^*,\lambda,B_{2^j r}} \\
&\leq C_3' \sum_{j=1}^{\infty} |B_{2^j r}|^{\lambda-1+\alpha/n} \|f\|_{\Phi,\lambda,B_{2^j r}} \frac{\Phi^{-1}(|B_{2^j r}|^{\lambda-1})}{|B_{2^j r}|^{\lambda-1}} \\
&= \frac{C_3'}{n\ln 2} \|f\|_{M^{\Phi,\lambda}(0)} \sum_{j=1}^{\infty} |B_{2^j r}|^{\alpha/n} \Phi^{-1}(|B_{2^j r}|^{\lambda-1}) \int_{|B_{2^{j-1} r}|}^{|B_{2^j r}|} \frac{dt}{t} \\
&\leq \frac{C_3'}{n\ln 2} 2^{\alpha} \|f\|_{M^{\Phi,\lambda}(0)} \sum_{j=1}^{\infty} \int_{|B_{2^{j-1} r}|}^{|B_{2^j r}|} t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \\
&\leq C_3 \|f\|_{M^{\Phi,\lambda}(0)} \int_{|B_r|}^{\infty} t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t},
\end{aligned}
$$

where $C_3' = 2^{n-\alpha+1} v_n^{1-\alpha/n}$ and $C_3 = \frac{2^{\alpha}}{n\ln 2} C_3'$. Thus, we arrive to the assertion of Lemma 2. □

Next, we show the following well-definedness of $I_\alpha f$ when $f \in M^{\Phi,\lambda}(0)$.

Lemma 3 *Let $0 < \alpha < n$, Φ be an Orlicz function and $0 \leq \lambda < 1$. If the integral $\int_{|B_r|}^{\infty} t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t}$ is convergent for any $r > 0$ and $f \in M^{\Phi,\lambda}(0)$ then the Riesz potential $I_\alpha f$ is well-defined.*

Proof We will prove this lemma using the same arguments that were presented in the proof in [30, Theorem 2.1]. Let $f \in M^{\Phi,\lambda}(0)$, $r > 0$ and $x \in B_r$, and let

$$
I_\alpha f(x) = I_\alpha(f\chi_{B_{2r}})(x) + I_\alpha(f(1-\chi_{B_{2r}}))(x). \tag{13}
$$

Since $f\chi_{B_{2r}} \in L^1(\mathbb{R}^n)$, the first term is well-defined. Indeed, in view of [31, Theorem 1.1, Chapter 2] the requirement $I_\alpha |f\chi_{B_{2r}}| \not\equiv \infty$ for any $f \in M^{\Phi,\lambda}(0)$ and $r > 0$ is equivalent to

$$
\int_{B_{2r}} (1+|y|)^{\alpha-n} |f(y)|\,dy < \infty.
$$

The last inequality is true since $\|(1+|y|)^{\alpha-n}\|_{\Phi^*,\lambda,B_{2r}} \leq \frac{(1+2r)^{\alpha-n}}{(\Phi^*)^{-1}(|B_{2r}|^{\lambda-1})}$ and by Lemma 1 (i) we obtain

$$
\begin{aligned}
\int_{B_{2r}} (1+|y|)^{\alpha-n} |f(y)|\,dy &\leq 2|B_{2r}|^{\lambda} \|f\|_{\Phi,\lambda,B_{2r}} \|(1+|y|)^{\alpha-n}\|_{\Phi^*,\lambda,B_{2r}} \\
&\leq 2|B_{2r}|^{\lambda} \frac{(1+2r)^{\alpha-n}}{(\Phi^*)^{-1}(|B_{2r}|^{\lambda-1})} \|f\|_{M^{\Phi,\lambda}(0)} < \infty.
\end{aligned}
$$

For the second term for any $x \in B_r$ we have

$$|I_\alpha(f(1-\chi_{B_{2r}}))(x)| \le \int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \le 2^{n-\alpha}\int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}}\,dy.$$

Since the integral $\int_{|B_r|}^\infty t^{\alpha/n}\Phi^{-1}(t^{\lambda-1})\frac{dt}{t}$ is convergent for any $r>0$ and $f \in M^{\Phi,\lambda}(0)$ it follows from Lemma 2 that $I_\alpha(f(1-\chi_{B_{2r}}))(x)$ is well-defined for all $x \in B_r$.

Further, since for $0 < s < r$,

$$f\chi_{B_{2s}} + f(1-\chi_{B_{2s}}) = f\chi_{B_{2r}} + f(1-\chi_{B_{2r}}),$$

it follows that for $x \in B_s \subset B_r$,

$$I_\alpha(f\chi_{B_{2s}})(x) + I_\alpha(f(1-\chi_{B_{2s}}))(x) = I_\alpha(f\chi_{B_{2r}})(x) + I_\alpha(f(1-\chi_{B_{2r}}))(x).$$

This shows that $I_\alpha f$ is independent of B_r containing x. Thus, $I_\alpha f$ is well-defined on $\mathbb{R}^n$. □

Now we will present sufficient conditions on spaces so that the operator I_α is bounded between distinct central Morrey–Orlicz spaces. In the proofs of these estimates we will use estimates from [28] for the Hardy–Littlewood maximal operator. The *Hardy–Littlewood maximal operator M* is defined for $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy.$$

Then, for an Orlicz function Φ and $0 \le \lambda \le 1$, this operator M is bounded on $M^{\Phi,\lambda}(0)$, provided $\Phi^* \in \Delta_2$, that is, there exists a constant $C_0 > 1$ such that

$$\|Mf\|_{M^{\Phi,\lambda}(0)} \le C_0\,\|f\|_{M^{\Phi,\lambda}(0)} \quad \text{for all } f \in M^{\Phi,\lambda}(0) \tag{14}$$

(see [28, Theorem 6(i)]). Moreover, M is bounded from $M^{\Phi,\lambda}(0)$ to $WM^{\Phi,\lambda}(0)$, that is, there exists a constant $c_0 > 1$ such that $\|Mf\|_{WM^{\Phi,\lambda}(0)} \le c_0\,\|f\|_{M^{\Phi,\lambda}(0)}$ for all $f \in M^{\Phi,\lambda}(0)$ (see [28, Theorem 6(ii)]).

Theorem 3 *Let $0 < \alpha < n$, Φ, Ψ be Orlicz functions and either $0 < \lambda, \mu < 1, \lambda \ne \mu$ or $\lambda = \mu = 0$. Assume that there exist constants $C_4, C_5 \ge 1$ such that*

$$\int_u^\infty t^{\frac{\alpha}{n}}\,\Phi^{-1}(t^{\lambda-1})\,\frac{dt}{t} \le C_4\,\Psi^{-1}(u^{\mu-1}) \quad \textit{for all } u > 0 \tag{15}$$

and

$$\int_u^\infty t^{\frac{\alpha}{n}}\,\Phi^{-1}(\frac{r^\lambda}{t})\,\frac{dt}{t} \le C_5\,\Psi^{-1}(\frac{r^\mu}{u}) \quad \textit{for all } u > 0 \textit{ and for all } r > 0. \tag{16}$$

(i) *If* $\Phi^* \in \Delta_2$, *then* I_α *is bounded from* $M^{\Phi,\lambda}(0)$ *to* $M^{\Psi,\mu}(0)$, *that is, there exists a constant* $C_6 \geq 1$ *such that* $\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \leq C_6 \|f\|_{M^{\Phi,\lambda}(0)}$ *for all* $f \in M^{\Phi,\lambda}(0)$.

(ii) *The operator* I_α *is bounded from* $M^{\Phi,\lambda}(0)$ *to* $WM^{\Psi,\mu}(0)$, *that is, there exists a constant* $c_6 \geq 1$ *such that* $\|I_\alpha f\|_{WM^{\Psi,\mu}(0)} \leq c_6 \|f\|_{M^{\Phi,\lambda}(0)}$ *for all* $f \in M^{\Phi,\lambda}(0)$.

Remark 1 The same conclusions hold for non-homogeneous versions of $M^{\Phi,\lambda}(0)$ and $M^{\Psi,\mu}(0)$.

Remark 2 From the estimate (15) we get the inequality (a) in Theorem 2(i). Namely, using the concavity of the function Φ^{-1} we get

$$\int_u^\infty t^{\frac{\alpha}{n}} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \geq \int_u^{2u} t^{\frac{\alpha}{n}} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} \geq u^{\frac{\alpha}{n}} \Phi^{-1}((2u)^{\lambda-1}) \ln 2$$
$$\geq \frac{\ln 2}{2^{1-\lambda}} u^{\frac{\alpha}{n}} \Phi^{-1}(u^{\lambda-1}).$$

Remark 3 Note that if either $\lambda = \mu > 0$ or $\lambda = 0$ and $\mu > 0$, then estimate (16) doesn't hold.

Remark 4 If $\lambda = \mu = 0$, then inequalities (15) and (16) are the same. Moreover, condition (15) in this case is a sufficient condition for boundedness of I_α from Orlicz space $L^\Phi(\mathbb{R}^n)$ to weak Orlicz space $WL^\Psi(\mathbb{R}^n)$, and if additionally $\Phi^* \in \Delta_2$ then I_α is bounded from Orlicz space $L^\Phi(\mathbb{R}^n)$ to Orlicz space $L^\Psi(\mathbb{R}^n)$ (proof we can find, for example, in [20, Theorem 3.3]).

In the proof of Theorem 3 the following lemma plays a crucial role.

Lemma 4 *Let* $0 < \alpha < n$, Φ, Ψ *be Orlicz functions,* $\Phi^* \in \Delta_2$ *and either* $0 < \lambda, \mu < 1, \lambda \neq \mu$ *or* $\lambda = \mu = 0$. *If the estimate* (16) *holds, then there exists a constant* $C_7 \geq 1$ *such that*

$$\int_{B_r} \Psi\left(\frac{\int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy}{C_7 \|f\|_{M^{\Phi,\lambda}(0)}}\right) dx \leq |B_r|^\mu, \quad \textit{for all } f \in M^{\Phi,\lambda}(0) \textit{ and } r > 0.$$

Proof Let $f \in M^{\Phi,\lambda}(0)$. We write $I_\alpha(f\chi_{B_{2r}})$ as follows

$$I_\alpha(f\chi_{B_{2r}})(x) = \int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy = \int_{|x-y|\leq\delta} \frac{|f(y)\chi_{B_{2r}}(y)|}{|x-y|^{n-\alpha}}\,dy$$
$$+ \int_{|x-y|>\delta} \frac{|f(y)\chi_{B_{2r}}(y)|}{|x-y|^{n-\alpha}}\,dy =: J_1 f(x) + J_2 f(x),$$

where $\delta > 0$ will be defined later on. It is known that

$$J_1 f(x) \leq C_8 \, |B_\delta|^{\frac{\alpha}{n}} M(f \chi_{B_{2r}})(x),$$

where $C_8 = \frac{2^\alpha}{2^\alpha - 1} C_3'$. Note that for any parameters $u > 0$ and $r > 0$ we have

$$\begin{aligned}\int_u^\infty t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{t}\right) \frac{dt}{t} &\geq \int_u^{2u} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{t}\right) \frac{dt}{t} \\ &\geq \ln 2 \, u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{2u}\right) \geq \frac{\ln 2}{2} \, u^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{u}\right).\end{aligned}$$

Thus, applying (16) we obtain

$$J_1 f(x) \leq \frac{2}{\ln 2} C_5 \, C_8 \, \frac{\Psi^{-1}\left(\frac{|B_{2r}|^\mu}{|B_\delta|}\right)}{\Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B_\delta|}\right)} M(f \chi_{B_{2r}})(x).$$

Following Hedberg's method we get for $J_2 f(x)$

$$\begin{aligned}J_2 f(x) &= \sum_{k=1}^\infty \int\limits_{2^{k-1}\delta < |x-y| \leq 2^k \delta} \frac{|f(y)\chi_{B_{2r}}(y)|}{|x-y|^{n-\alpha}} \, dy \\ &\leq \sum_{k=1}^\infty (2^{k-1}\delta)^{\alpha-n} \int\limits_{|x-y| \leq 2^k \delta} |f(y)\chi_{B_{2r}}(y)| \, dy \\ &= \sum_{k=1}^\infty (2^{k-1}\delta)^{\alpha-n} \int\limits_{B_{2r}} |f(y)\chi_{B(x,2^k\delta)}(y)| \, dy.\end{aligned}$$

From Lemma 1 (i) and (ii) it follows that

$$\begin{aligned}J_2 f(x) &\leq 2 \, |B_{2r}|^\lambda \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^{k-1}\delta)^{\alpha-n} \|\chi_{B(x,2^k\delta)}\|_{\Phi^*,\lambda,B_{2r}} \\ &\leq 2^{n-\alpha+1} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^\infty (2^k \delta)^{\alpha-n} |B_{2r} \cap B(x, 2^k \delta)| \\ &\quad \cdot \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B_{2r} \cap B(x, 2^k\delta)|}\right).\end{aligned}$$

Taking into account that $u\Phi^{-1}(1/u)$ is increasing and $|B_{2r} \cap B(x, 2^k\delta)| \le |B(x, 2^k\delta)|$ we obtain

$$
\begin{aligned}
J_2 f(x) &\le 2^{n-\alpha+1} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^{\infty} (2^k\delta)^{\alpha-n} \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B(x,2^k\delta)|}\right) |B(x,2^k\delta)| \\
&= 2^{n-\alpha+1} v_n \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^{\infty} (2^k\delta)^{\alpha} \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B_{2^k\delta}|}\right) \\
&= \frac{C_3'}{n\ln 2} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^{\infty} |B_{2^k\delta}|^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B_{2^k\delta}|}\right) \int_{|B_{2^{k-1}\delta}|}^{|B_{2^k\delta}|} \frac{dt}{t} \\
&\le \frac{C_3'}{n\ln 2} \|f\|_{\Phi,\lambda,B_{2r}} \sum_{k=1}^{\infty} |B_{2^k\delta}|^{\frac{\alpha}{n}} \int_{|B_{2^{k-1}\delta}|}^{|B_{2^k\delta}|} \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{t}\right) \frac{dt}{t} \\
&\le C_3 \|f\|_{\Phi,\lambda,B_{2r}} \int_{|B_\delta|}^{\infty} t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{t}\right) \frac{dt}{t} \\
&\le C_5\, C_3 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\left(\frac{|B_{2r}|^\mu}{|B_\delta|}\right).
\end{aligned}
$$

Now we choose $\delta > 0$ such that

$$
\frac{Mf(x)}{C_0 \|f\|_{M^{\Phi,\lambda}(0)}} = \Phi^{-1}\left(\frac{|B_{2r}|^\lambda}{|B_\delta|}\right),
$$

where the constant C_0 is from (14). Then

$$
J_1 f(x) \le \frac{2}{\ln 2} C_5\, C_8\, C_0 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\left(\frac{|B_{2r}|^\mu}{|B_\delta|}\right),
$$

and

$$
\begin{aligned}
&\int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy = J_1 f(x) + J_2 f(x) \\
&\quad\le \left(\frac{2}{\ln 2} C_5\, C_8\, C_0 + C_5 C_3\right) \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\left(\frac{|B_{2r}|^\mu}{|B_\delta|}\right)
\end{aligned}
$$

Thus, with $C_9 = 2\, C_5 \max\left(\frac{2}{\ln 2} C_0 C_8,\ C_3\right)$ we obtain

$$
\int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le C_9 \|f\|_{M^{\Phi,\lambda}(0)} \Psi^{-1}\left(|B_{2r}|^{\mu-\lambda} \Phi\left(\frac{Mf(x)}{C_0 \|f\|_{M^{\Phi,\lambda}(0)}}\right)\right).
$$

Then

$$\Psi\left(\frac{\int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy}{C_9\|f\|_{M^{\Phi,\lambda}(0)}}\right) \le |B_{2r}|^{\mu-\lambda}\,\Phi\left(\frac{Mf(x)}{\|Mf\|_{M^{\Phi,\lambda}(0)}}\right)$$
$$= 2^{n(\mu-\lambda)}\,|B_r|^{\mu-\lambda}\,\Phi\left(\frac{Mf(x)}{\|Mf\|_{M^{\Phi,\lambda}(0)}}\right).$$

Finally, with $C_7 = 2^{n(\mu-\lambda)}C_9$ we get

$$\frac{1}{|B_r|^{\mu}}\int\limits_{B_r}\Psi\left(\frac{\int_{B_{2r}} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy}{C_7\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)dx \le \frac{1}{|B_r|^{\lambda}}\int\limits_{B_r}\Phi\left(\frac{Mf(x)}{\|Mf\|_{M^{\Phi,\lambda}(0)}}\right)dx \le 1$$

and we arrive to the statement of this lemma. □

Proof of Theorem 3 (i) Let $0<\alpha<n$ and $0<\lambda<1$, $0<\mu<1$. Let also $f\in M^{\Phi,\lambda}(0)$ and $r>0$. Since $I_\alpha f$ is well-defined by Lemma 3, we prove only that

$$\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \le C_6\,\|f\|_{M^{\Phi,\lambda}(0)}.$$

Now, by (13), for $C_6 = 2\,\max(C_7,\ 2^{n-\alpha}C_3\,C_4)$, it follows that

$$\begin{aligned}
&\int_{B_r}\Psi\left(\frac{|I_\alpha f(x)|}{C_6\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)dx\\
&\quad\le \frac{1}{2}\int_{B_r}\Psi\left(\frac{|I_\alpha(f\chi_{B_{2r}})(x)|}{C_7\|f\|_{M^{\Phi,\lambda}(0)}}\right)dx + \frac{1}{2}\int_{B_r}\Psi\left(\frac{|I_\alpha(f(1-\chi_{B_{2r}}))(x)|}{2^{n-\alpha}\,C_3\,C_4\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)dx\\
&\quad=: \frac{1}{2}\,(I_1+I_2)\,.
\end{aligned}$$

From Lemma 4 we get that $I_1 \le |B_r|^{\mu}$ for all $r>0$.

Next, we estimate I_2. Since for $x\in B_r$ and $|y|\ge 2r$ we have $|x|<r\le\frac{|y|}{2}$ and $|x-y|\ge|y|-|x|>\frac{|y|}{2}$, it follows that

$$|I_\alpha(f(1-\chi_{B_{2r}}))(x)| \le \int_{\mathbb{R}^n\setminus B_{2r}}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \le 2^{n-\alpha}\int_{\mathbb{R}^n\setminus B_{2r}}\frac{|f(y)|}{|y|^{n-\alpha}}\,dy. \tag{17}$$

By Lemma 2 and the estimate (15) we obtain

$$\Psi\left(\frac{\int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}}\,dy}{C_3\,C_4\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)dx \le \Psi\left(\frac{1}{C_4}\int_{|B_{2r}|} t^{\alpha/n}\,\Phi^{-1}(t^{\lambda-1})\frac{dt}{t}\right)$$

$$\leq \Psi\left(\Psi^{-1}(|B_{2r}|^{\mu-1})\right) \leq |B_{2r}|^{\mu-1}.$$

Thus, for $x \in B_r$

$$I_2 \leq \int_{B_r} \Psi\left(\frac{\int_{\mathbb{R}^n\setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}}\,dy}{C_3\,C_4\,\|f\|_{M^{\Phi,\lambda}(0)}}\right) dx \leq |B_{2r}|^{\mu-1}\cdot|B_r| < |B_r|^{\mu}.$$

Hence,

$$\frac{1}{|B_r|^{\mu}}\int_{B_r}\Psi\left(\frac{|I_\alpha f(x)|}{C_6\,\|f\|_{M^{\Phi,\lambda}(0)}}\right) dx < 1,$$

and so

$$\|I_\alpha f\|_{M^{\Psi,\mu}(0)} \leq C_6\,\|f\|_{M^{\Phi,\lambda}(0)}.$$

(ii) Similarly to the previous case, by (13), we obtain for $u > 0$

$$\begin{aligned}
&\Psi\left(\frac{|I_\alpha f(x)|}{c_6\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)\\
&\quad\leq \frac{1}{2}\Psi\left(\frac{|I_\alpha(f\chi_{B_{2r}})(x)|}{c_7\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)+\frac{1}{2}\Psi\left(\frac{|I_\alpha(f(1-\chi_{B_{2r}}))(x)|}{2^{n-\alpha+1}\,C_3\,C_4\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)\\
&\quad=: \frac{1}{2}\,(I_3+I_4)\,,
\end{aligned}$$

with $c_6 = 2\max(c_7,\, 2^{n-\alpha+1}\,C_3\,C_4)$, $c_7 = 2^{n(\mu-\lambda)+1}c_9$ and $c_9 = 2\,C_5\max\left(\frac{2}{\ln 2}c_0\right.$ $\left.C_8, C_3\right)$.

Since $\Psi(u)\,d(g,u) = v\,d(g,\Psi^{-1}(v)) = v\,d(\Psi(g),v)$ for any $u > 0$ with $v = \Psi(u)$ and

$$d\left(\Psi\left(\frac{|I_\alpha f(x)|}{c_6\|f\|_{M^{\Phi,\lambda}(0)}}\right), u\right) \leq d\,(I_3,u)+d\,(I_4,u)\,,$$

it follows that

$$\sup_{u>0}\frac{\Psi(u)}{|B_r|^{\mu}}\,d\left(\frac{|I_\alpha f(x)|}{c_6\|f\|_{M^{\Phi,\lambda}(0)}}, u\right) \leq \sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\,(I_3,u)+\sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\,(I_4,u)\,.$$

From the proof of Lemma 4 for all $r > 0$

$$I_3 = \Psi\left(\frac{|I_\alpha(f\chi_{B_{2r}})(x)|}{c_7\,\|f\|_{M^{\Phi,\lambda}(0)}}\right) \leq \frac{1}{2}|B_r|^{\mu-\lambda}\,\Phi\left(\frac{Mf(x)}{\|Mf\|_{WM^{\Phi,\lambda}(0)}}\right)$$

and

$$\begin{aligned}\sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\,(I_1,u) &\le \sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\left(\frac{1}{2}|B_r|^{\mu-\lambda}\Phi\left(\frac{Mf(x)}{\|Mf\|_{WM^{\Phi,\lambda}(0)}}\right),u\right)\\ &=\frac{1}{2}\sup_{u>0}\frac{u}{|B_r|^{\lambda}}\,d\left(\Phi\left(\frac{Mf(x)}{\|Mf\|_{WM^{\Phi,\lambda}(0)}}\right),u\right)\\ &=\frac{1}{2}\sup_{u>0}\frac{\Phi(u)}{|B_r|^{\lambda}}\,d\left(\frac{Mf(x)}{\|Mf\|_{WM^{\Phi,\lambda}(0)}},u\right)\le\frac{1}{2}.\end{aligned}$$

For I_4, using Lemma 2 we obtain

$$I_4=\Psi\left(\frac{|I_\alpha(f(1-\chi_{B_{2r}}))(x)|}{2^{n-\alpha+1}\,C_3\,C_4\,\|f\|_{M^{\Phi,\lambda}(0)}}\right)\le\frac{1}{2}|B_r|^{\mu-1}$$

and

$$\sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\,(I_4,u)\le\sup_{u>0}\frac{u}{|B_r|^{\mu}}\,d\left(\frac{1}{2}|B_r|^{\mu-1},u\right)=\frac{1}{2}\sup_{u>0}u\,d\left(\frac{1}{|B_r|},u\right)\le\frac{1}{2}.$$

Thus,

$$\sup_{u>0}\frac{\Psi(u)}{|B_r|^{\mu}}\,d\left(\frac{|I_\alpha f(x)|}{c_6\|f\|_{M^{\Phi,\lambda}(0)}},u\right)\le 1$$

and $\|I_\alpha f\|_{WM^{\Psi,\mu}(0)}\le c_6\,\|f\|_{M^{\Phi,\lambda}(0)}$. □

Example 1 Let $0<\alpha<n$, $1<p<\frac{n(1-\lambda)}{\alpha}$, $0\le\lambda<1$, and

$$\Phi(u)=u^p,\quad \Psi(u)=u^q\quad\text{with}\quad 1<p<q<\infty.$$

Then $\Phi^*(u)=(p-1)\,p^{-p'}u^{p'}$, where $1/p+1/p'=1$ and $\Phi^*(2u)=2^{p'}\Phi^*(u)$, that is, $\Phi^*\in\Delta_2$. The estimate (15) holds since

$$\int_u^\infty t^{\alpha/n}\Phi^{-1}(t^{\lambda-1})\,\frac{dt}{t}=\int_u^\infty t^{\frac{\alpha}{n}+\frac{\lambda-1}{p}}\,\frac{dt}{t}=\frac{1}{\frac{1-\lambda}{p}-\frac{\alpha}{n}}\,u^{\frac{\alpha}{n}+\frac{\lambda-1}{p}}$$

for all $u>0$, where the last integral is convergent because $p<\frac{n(1-\lambda)}{\alpha}$. If $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $\frac{\lambda}{p}=\frac{\mu}{q}$, then $\frac{\alpha}{n}+\frac{\lambda-1}{p}=\frac{\lambda}{p}-(\frac{1}{p}-\frac{\alpha}{n})=\frac{\mu}{q}-\frac{1}{q}$ and

$$\int_u^\infty t^{\alpha/n}\Phi^{-1}(t^{\lambda-1})\,\frac{dt}{t}=\frac{q}{1-\mu}\,u^{\frac{\mu-1}{q}}=\frac{q}{1-\mu}\,\Psi^{-1}(u^{\mu-1}),$$

that is, the estimate (15) holds. Also estimate (16) holds since for all $u, r > 0$

$$\int_u^\infty t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{t}\right) \frac{dt}{t} = r^{\frac{\lambda}{p}} \int_u^\infty t^{\frac{\alpha}{n}-\frac{1}{p}} \frac{dt}{t} = \frac{r^{\frac{\lambda}{p}}}{\frac{1}{p}-\frac{\alpha}{n}} u^{\frac{\alpha}{n}-\frac{1}{p}}$$
$$= q\, r^{\frac{\mu}{q}} u^{-1/q} = q\, \Psi^{-1}\left(\frac{r^\mu}{u}\right).$$

From the Theorem 3 we get the Spanne–Peetre type result proved in [12, Proposition 1.1], that is, the Riesz potential I_α is bounded from $M^{p,\lambda}(0)$ to $M^{q,\mu}(0)$ under the conditions $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \le \lambda < 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$.

Remark 5 It is easy to see that for $0 \le \lambda < 1$ if Φ_1, Φ_2 are two Orlicz functions and there exists a constant $k > 0$ such that $\Phi_2(u) \le \Phi_1(ku)$ for all $u > 0$, then $\|f\|_{\Phi_2,\lambda,A} \le k\, \|f\|_{\Phi_1,\lambda,A}$ provided the right side is finite. Furthemore, $M^{\Phi_1,\lambda}(\mathbb{R}^n) \overset{k}{\hookrightarrow} M^{\Phi_2,\lambda}(\mathbb{R}^n)$ and $M^{\Phi_1,\lambda}(0) \overset{k}{\hookrightarrow} M^{\Phi_2,\lambda}(0)$. Hence it follows that if two Orlicz functions Φ_1, Φ_2 are equivalent, i.e. there exist positive constants k_1, k_2 such that $\Phi_1(k_1 u) \le \Phi_2(u) \le \Phi_1(k_2 u)$ for all $u > 0$, then $M^{\Phi_1,\lambda}(\mathbb{R}^n) = M^{\Phi_2,\lambda}(\mathbb{R}^n)$ and $M^{\Phi_1,\lambda}(0) = M^{\Phi_2,\lambda}(0)$ with equivalent norms.

Example 2 Let $0 < \alpha < n$, $0 \le \lambda < 1$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $a > 0$ and

$$\Phi^{-1}(u) = \begin{cases} u^{\frac{1}{p}} & \text{for } 0 \le u \le 1, \\ u^{\frac{1}{p}}\,(1+\ln u)^{-a} & \text{for } u \ge 1, \end{cases} \qquad \Psi^{-1}(u) = u^{\frac{1}{q}} \text{ with } 1 < p < q < \infty.$$

If $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{\lambda}{p} = \frac{\mu}{q}$, then condition (15) is satisfied. Really, for $u \ge 1$ we have equality as in the Example 1. If $0 < u < 1$, then using the fact that function $\left(1+\ln t^{\lambda-1}\right)^{-a}$ is strictly increasing of variable t on $(0, 1]$, we get $\left(1+\ln t^{\lambda-1}\right)^{-a} \le 1$ for $0 < t \le 1$ and so

$$\int_u^\infty t^{\alpha/n} \Phi^{-1}(t^{\lambda-1}) \frac{dt}{t} = \int_u^1 t^{\frac{\alpha}{n}+\frac{\lambda-1}{p}} (1+\ln t^{\lambda-1})^{-a} \frac{dt}{t} + \int_1^\infty t^{\frac{\alpha}{n}+\frac{\lambda-1}{p}} \frac{dt}{t}$$
$$\le \int_u^\infty t^{\frac{\alpha}{n}+\frac{\lambda-1}{p}} \frac{dt}{t} = \frac{u^{\frac{\alpha}{n}+\frac{\lambda-1}{p}}}{\frac{1-\lambda}{p}-\frac{\alpha}{n}} = \frac{q}{1-\mu} u^{\frac{\mu-1}{q}}$$
$$= \frac{q}{1-\mu} \Psi^{-1}(u^{\mu-1}),$$

that is, the estimate (15) holds. Next, we consider condition (16). If $u \ge r^\lambda$, then

$$\int_u^\infty t^{\frac{\alpha}{n}} \Phi^{-1}\left(\frac{r^\lambda}{t}\right) \frac{dt}{t} = r^{\frac{\lambda}{p}} \int_u^\infty t^{\frac{\alpha}{n}-\frac{1}{p}} \frac{dt}{t} = \frac{r^{\frac{\lambda}{p}}}{\frac{1}{p}-\frac{\alpha}{n}} u^{\frac{\alpha}{n}-\frac{1}{p}} = q\, r^{\frac{\mu}{q}} u^{-\frac{1}{q}} = q\, \Psi^{-1}\left(\frac{r^\mu}{u}\right).$$

Let now $0 < u < r^\lambda$. Then, $(1+\ln\frac{r^\lambda}{t})^{-a} \le 1$ as an increasing function of t on $(0, r^\lambda]$ and since $u < t \le r^\lambda$, we have

$$\begin{aligned}\int_u^\infty t^{\frac{\alpha}{n}}\,\Phi^{-1}\left(\frac{r^\lambda}{t}\right)\frac{dt}{t} &= r^{\frac{\lambda}{p}}\int_u^{r^\lambda} t^{\frac{\alpha}{n}-\frac{1}{p}}\left(1+\ln\frac{r^\lambda}{t}\right)^{-a}\frac{dt}{t} + r^{\frac{\lambda}{p}}\int_{r^\lambda}^\infty t^{\frac{\alpha}{n}-\frac{1}{p}}\frac{dt}{t}\\ &\le r^{\frac{\lambda}{p}}\int_u^{r^\lambda} t^{\frac{\alpha}{n}-\frac{1}{p}}\frac{dt}{t} + r^{\frac{\lambda}{p}}\int_{r^\lambda}^\infty t^{\frac{\alpha}{n}-\frac{1}{p}}\frac{dt}{t}\\ &= r^{\frac{\lambda}{p}}\int_u^\infty t^{\frac{\alpha}{n}-\frac{1}{p}}\frac{dt}{t} = \frac{r^{\frac{\lambda}{p}}}{\frac{1}{p}-\frac{\alpha}{n}}\,u^{\frac{\alpha}{n}-\frac{1}{p}} = q\,r^{\frac{\mu}{q}}\,u^{-\frac{1}{q}} = q\,\Psi^{-1}\left(\frac{r^\mu}{u}\right),\end{aligned}$$

that is, the estimate (16) holds. The function Φ^{-1} is increasing, unbounded, obviously concave on $(0, 1)$ and concave for large u. Therefore, there exists a concave function on $(0, \infty)$ which is equivalent to Φ^{-1} and so Φ is equivalent to an Orlicz function. Also we have equivalence

$$\Phi(u) \approx \begin{cases} u^p & \text{for } 0 \le u \le 1,\\ u^p\,(1+\ln u)^{ap} & \text{for } u \ge 1.\end{cases}$$

Moreover, since

$$s_{\Phi^{-1}}(t) = \begin{cases} t^{1/p}(1-\ln t)^a & \text{for } 0 < t \le 1,\\ t^{1/p} & \text{for } t \ge 1,\end{cases}$$

it follows that the Matuszewska–Orlicz index $\beta_{\Phi^{-1}} = \frac{1}{p}$ and so $1 = \frac{1}{\beta_{\Phi^*}} + \frac{1}{\alpha_\Phi} = \frac{1}{\beta_{\Phi^*}} + \beta_{\Phi^{-1}} = \frac{1}{\beta_{\Phi^*}} + \frac{1}{p}$ or $\beta_{\Phi^*} = \frac{p}{p-1} < \infty$, which means that $\Phi^* \in \Delta_2$ (for definitions and properties of indices – see [26, pp. 87–89]). Thus, by Remark 5, the space $M^{\Phi,\lambda}(0)$ is a Banach space and by Theorem 3 the Riesz potential I_α is bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0) = M^{q,\mu}(0)$.

Example 3 Let $0 < \alpha < n$, $0 \le \lambda < 1$, $1 < p < \frac{n(1-\lambda)}{\alpha}$, $0 \le b \le a$ and

$$\Phi^{-1}(u) = u^{\frac{1}{p}}\,(1+|\ln u|)^{-a} \quad\text{and}\quad \Psi^{-1}(u) = u^{\frac{1}{q}}\,(1+|\ln u|)^{b} \quad\text{for}\quad u > 0.$$

If $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{\lambda}{p} = \frac{\mu}{q}$, then conditions (a), (b) of Theorem 2(i) and (15), (16) are satisfied. The calculations are similar to those in Example 2 so we will omit them here. Observe only that

$$s_{\Phi^{-1}}(t) = t^{1/p}(1+|\ln t|)^a,\ s_{\Psi^{-1}}(t) = t^{1/q}(1+|\ln t|)^b.$$

Then, the functions Φ^{-1}, Ψ^{-1} are increasing, unbounded and concave near 0 and for large u, and so the inverses Φ, Ψ are equivalent to Orlicz functions. Thus, by Remark 5, the spaces $M^{\Phi,\lambda}(0)$, $M^{\Psi,\mu}(0)$ are Banach spaces and by Theorem 3 the Riesz potential I_α is bounded from $M^{\Phi,\lambda}(0)$ to $M^{\Psi,\mu}(0)$.

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