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Probabilistic Assessment of Pile Drivability in Swedish Soils

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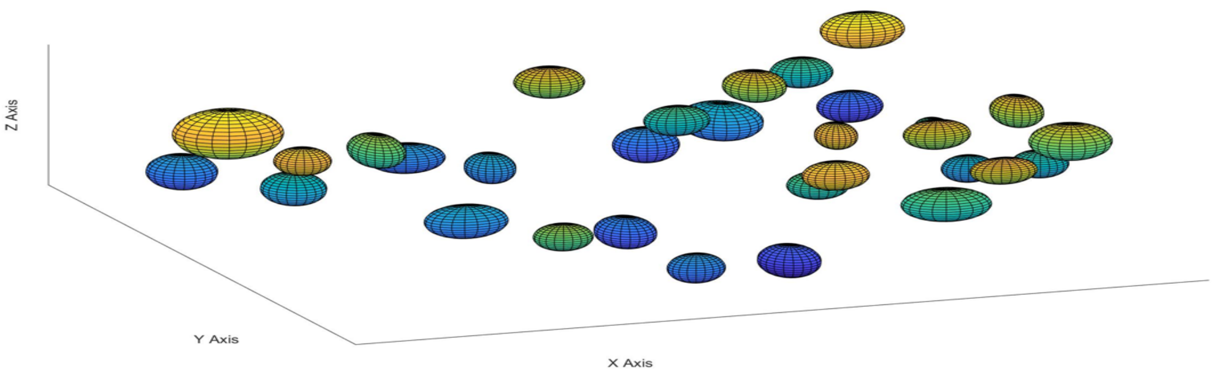

KTH ROYAL INSTITUTE OF TECHNOLOGY
SCHOOL OF ARCHITECTURE AND THE BUILT ENVIRONMENT

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Master Thesis, 2020
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Abstract

Site investigations are often performed prior to the design of pile foundations with the aim to collect data regarding soil properties including boulder content. The obtained data is typically limited due to non-homogeneous characteristic of the soil. The geological conditions of the Mälardalen region are characterized by glacial and post glacial clay overlaying on the layer of moraine containing boulders on a bedrock. Thus, pile refusal in results of encountering boulders is a common issue during pile driving in this region. The current methods to assess the pile refusal risk are mostly qualitative and relies on the expertise gained from experience.

This thesis aims to develop a numerical model to quantify the boulder content in a soil strata with a confidence interval based on the site investigation results. Furthermore, this study estimates the probability and the consequences of hitting boulders while installing piles.

The model simulations show that an increase in the boulder content raises the probability of hitting a boulder, but it is not proportional to the diameter of piles. It means that even in low rate of boulder content with small piles there is a high probability to hit a boulder. Results from simulations using simplified quantitative method show that slender piles have substantial consequences in case they encounter boulders. Thus, these piles are more prone to pile refusal or breakage. It is suggested that in projects where small piles are preferred due to the superstructure design, enough tests be performed to have an estimate of boulder content with tighter interval and more precise design. On the other hand, in projects with low pile counts, it is recommended to have a conservative design with large diameter piles instead of performing numerous in-situ tests. This is because the price for piling becomes relatively low compared to site investigations' cost.

Keywords

Pile refusal, Monte Carlo method, probabilistic model, boulder.

Sammanfattning

Markundersökningar utförs innan projektering av pålgrundläggningar, där ett av målen är att uppskatta blockhalten i marken. Den information som framgår av normala markundersökningar är oftast begränsad, bland annat på grund att jordlagrens naturliga variation. Jordlagerföljden i Mälardalsregionen kännetecknas av glacial och postglacial lera som överlagrar morän, oftast innehållande block, som sedan vilar på berg. En typisk risk vid slagning av pålar till fast botten i området är därför bortslagning av pålar på block. De metoder som idag används är till största delen kvalitativa och bygger på erfarenhet från tidigare projekt.

Detta examensarbete syftar till att utveckla en numerisk modell för att kunna kvantifiera utsträckningen av block i en jordmassa samt påverkan av blockhalt i en typisk jordlagerföljd utifrån resultat av markundersökningar. Sannolikheten av följden av att träffa på block vid påldrivning uppskattas även.

Simuleringar med den framtagna modellen visar att en ökning av blockhalten i en jord ökar sannolikheten att påträffa block vid slagning av pålar, men att ökningen i är inte proportionerlig med pålens diameter. Detta innebär att även en liten blockhalt ger en förhållandevis stor sannolikhet att påträffa block vid slagning. En förenklad kvantitativ modell visar att slanka pålar har relativt större risk att slås bort jämfört med pålar av större diameter.

Det föreslås att vid projekt där slanka pålar föredras på grund av överbyggnadens typ, ska markundersökningen utökas för kunna bestämma blockhalten med större säkerhet. För projekt där ett begränsat antal pålar används är det troligtvis mer ekonomiskt att använda pålar med större diameter för att minska risken för bortslagning.

Nyckelord

Bortslagning av pålar, Monte Carlo metod, probabilistisk modell, block.

Preface

This thesis has been performed as the final part within the Master of Science program in Civil and Architectural Engineering at KTH Royal Institute of Technology. This thesis has been written for the division of Soil and Rock Mechanics at the School of Architecture and Built Environment in collaboration with ELU Konsult AB.

I would like to express my greatest gratitude toward Anders Beijer Lundberg for his exceptional cooperation during the whole process, from understanding the problem till the final report.

I would also like to thank Gary Axelsson at ELU Konsult AB and Stefan Larsson at KTH for their valuable comments and input.

Further acknowledgement is directed to my friend, Saujanya Shah for his help in debugging MATLAB codes and keeping the motivation high during this work.

Finally, I would like to thank my family for their endless support throughout my journey. Words are powerless to express my gratitude to them.

Stockholm, September 2020
Maedeh Alinejad Kordmahalleh

Nomenclature

Abbreviations

CPT	Cone Penetration Test
GEO	Geotechnical capacity
HFA	Dynamic probing test
Jb	Soil-rock probing
MCS	Monte Carlo Simulation
SE	Standard Deviation
STR	Structural capacity
VBC	Volumetric Boulder Content
$\widehat{\mathrm{VBC}}$	Volumetric Boulder Content estimator
Vim	Weight sounding test
WBC	Weight Boulder Content
WOR	Weighted Overlap Ratio
$\mathrm{WOR_b}$	Normalized WOR by pile bending stiffness
$\mathrm{WOR_m}$	Normalized WOR by pile bending moment capacity

Latin Symbols

A	Numerical coefficient of the quadratic equation
B	Numerical coefficient of the quadratic equation
b	Breadth of the domain [m]
C	Numerical coefficient of the quadratic equation
d_x	Boulder diameter along the X-axis [m]
d_y	Boulder diameter along the Y-axis [m]
d_z	Boulder diameter along the Z-axis [m]
f_y	Yield strength of steel [MPa]
h	Height of the domain [m]
$L_{boulder}$	Sum of boulder penetration length in all boreholes [m]
L_{total}	Total length of boreholes [m]
l	Length of the domain [m]
n	Sample size
r_b	Penetration length ratio
r_e	Uniformly distributed random numbers
r_n	Normally distributed random numbers
r_u	Uniformly distributed random numbers
t	Critical value from student distribution
u_1	Intersection point of line and ellipsoid

u_2	Intersection point of line and ellipsoid

Greek Symbols

$\propto$	Significance level
β	Scale parameter of exponential distribution
β_0	Partial regression coefficient
β_1	Partial regression coefficient
μ	Mean of the normal distribution
σ	Standard deviation of the normal distribution

Contents

1. Introduction

Prior to design of pile foundations, a site investigation is often carried out to obtain physical and engineering properties of the ground. This information is typically limited due to heterogeneity of the soil. The soil strata in Sweden, specifically in the Mälardalen region, is composed mainly of soft clay with a layer of moraine containing cobbles and boulders lying on the bedrock. Hence, piles installed in this region are prone to premature refusal and/or being damaged due to boulder occurrence. The problems associated with pile installation can bring financial burden and delay to the project since the foundation needs to be re-designed and damaged or broken piles be replaced.

Commission on Pile Research (2013) proposed a method to classify the boulder occurrence rate and corresponding risk of encountering boulders while pile driving qualitatively. Currently, the available methods to predict the potential risk in a piling process are mostly qualitative, thus making the industry dependent on knowledge harnessed through experience.

The lack of quantitative measure describing the soil properties in relation to boulder contents is pronounced. Site investigation results are unique and are not comparable by simple metrics, thereby inhibiting extrapolation of knowledge harnessed through experience across multiple projects. This lack of quantitative metric also restricts correct documentation of the relationship between site investigation and pile refusal reports. The author identifies the lack of numerical metric describing soil-boulder properties as the primary cause for the lack of correlation between site investigation results and pile refusal data. This thesis is an attempt to translate site investigation results into quantitative measure to describe the soil-boulder properties.

It should also be addressed that, analogous to experience throughout the years, the method proposed by this thesis work needs to be extended to incorporate data from historical projects. This would allow a quantitative correlation between site investigation results and pile refusal rate. Limited by time, this thesis focuses on proposing a quantitative method to aid engineers in predicting boulder content in soil strata and selection of suitable pile type.

1.1. Objectives

The main objectives of this thesis revolve around the following:

- Estimation of statistical properties of boulder content in soil, based on site investigation results.
- Quantification of probability of encountering boulders during installation of piles in soil with different boulder content.
- Estimation of consequences of encountering boulders semi-qualitatively while installing piles.

Additionally, a comparison between the accounting risk and difference in cost of piling is presented. This thesis aims to give recommendations to help pile engineers in pile selection.

1.2. Methodology

To achieve the objectives, a statistical tool in the program MATLAB is established to simulate site investigation and subsequent piling in Holocene soil strata. Monte Carlo simulation is carried out on the analysis owing to stochastic characteristics of this simulation.

1.3. Scope and delimitations

To allow the necessary work to be done in the area of interest and the numerical model be developed, some limitations are considered as listed below:

- Soil is assumed to be homogenous. This vital assumption gives the opportunity to extend the simulation result within the modeled domain.
- Depth of soil layer is assumed to be constant and equal to 5 meters.
- Borehole columns of in-situ testing are assumed to be drilled vertically.

- The diameter of borehole columns are neglected, and boreholes are modeled by lines.

Note that the mechanical behavior of piles encountering boulders is not seen in the developed numerical model. Simulating this behavior requires an enhanced model to account for influential factors such as surrounding soil, strength of boulders and mechanical characteristics of pile which is deemed out of the scope of this thesis.

1.4. Contents

This thesis begins with an introductory chapter, which gives initial information on the problem and why this is interest of study. Second chapter consists of a background review of topics that are relevant to work performed in the subsequent chapters. This chapter includes a short review of Sweden's geology, Site investigation methods, piling, Monte Carlo simulation, and lastly reviews of the papers which form the fundament of this thesis and the considered case studies. Third chapter elaborates on the numerical modeling developed to simulate boulders within soil mass, in-situ testing and subsequent piling process. Fourth chapter presents the obtained results for estimating the boulder content in soil strata and risk of encountering boulders for various pile type and size. This chapter also consist results from case studies for validation of the simulation and illustrating the issue of pile refusal. Finally, Chapter 5 concludes the work in this thesis and describes future work that can be performed to follow up on this thesis work.

2. Background

2.1. Soil conditions in Sweden

Soil in Sweden was largely formed during the last glaciation and post-glaciation period. During these periods, the ice sheets began retreating and picked up formerly sedimented soil material as well as eroded bedrock and deposited them as moraine. During melting phase, meltwater beneath the ice sheets deposited the well-sorted sediment of gravel and cobbles as esker and deltas. Fine-grained materials were banked further away from meltwater rivers. Decrease in overburden pressure from ice sheets led to land rise and the material formerly sedimented at sea bottom reworked and created new deposits, clay and sit. These deposits consist of glacial and post-glacial sediments. These processes are the fundamental geological processes driving soil formation in Sweden (Bergdhal, et al., 2003).

Figure 1 illustrates a typical soil sequence in the Mälardalen region in eastern Sweden consisting of glacial and post glacial clay with a layer of till on bedrock and large amounts of coarse-grained sediments such as sand, gravel, cobbles and boulders (Johansson, 1984). It should be noted that the measurements given above the see level are merely an apporximation to provide an estimation for ticknessess of the layers.

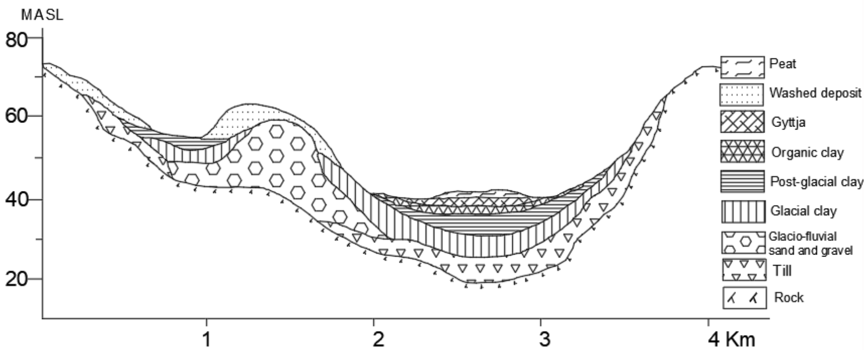

Figure 1: Typical soil layer sequence in the Mälardalen region (Handboken Bygg Geoteknik)

According to the soil boundaries describied in SS-EN ISO 14688-1, rock fragments larger than 200 mm are referred as boulders.

2.2. Site investigations

While designing a pile foundation, it is common to perform a site investigation in order to obtain physical and engineering characteristics of the soil and bed rock (Feleming, et al., 2009).

Commonly used sounding methods according to Swedish geotechnical society handbook of geotechnical site investigations (2013) are:

- Cone penetration test, (CPT)
- Mechanical statistical sounding
- Weight sounding test, (Vim)
- Dynamic probing test, (HFA)
- Soil-rock probing, (Jb)

The sounding method is chosen based on the existing soil type. It is conventional to use CPT in clay and soil with limited content of cobbles and stones. Dynamic probing test is more favorable for silty soil, sand and gravel, since CPT cannot be pushed down enough. Due to high content of boulders and pebbles in moraine, soil-rock probing is usually performed in this type of soil. Note that the drilling for Jb probing normally continues for 3-5 meters to ensure that the bed rock is reached and the resistance is not created by boulders (Bergdhal, et al., 2003).

2.2.1. Soil-rock probing

As applicable to the objectives of this thesis regarding boulder's impact, and the geology of interest, soil-rock probing is the chosen method to collect data about boulders in soil mass.

Figure 2 illustrates an example of the site investigation result for soil-rock probing. The highlighted rectangles represent boulders drilled by this method. One of the numerical parameters that can be obtained from the site investigation result is "penetration length" which represents the length drilled into the boulder during soil-rock probing.

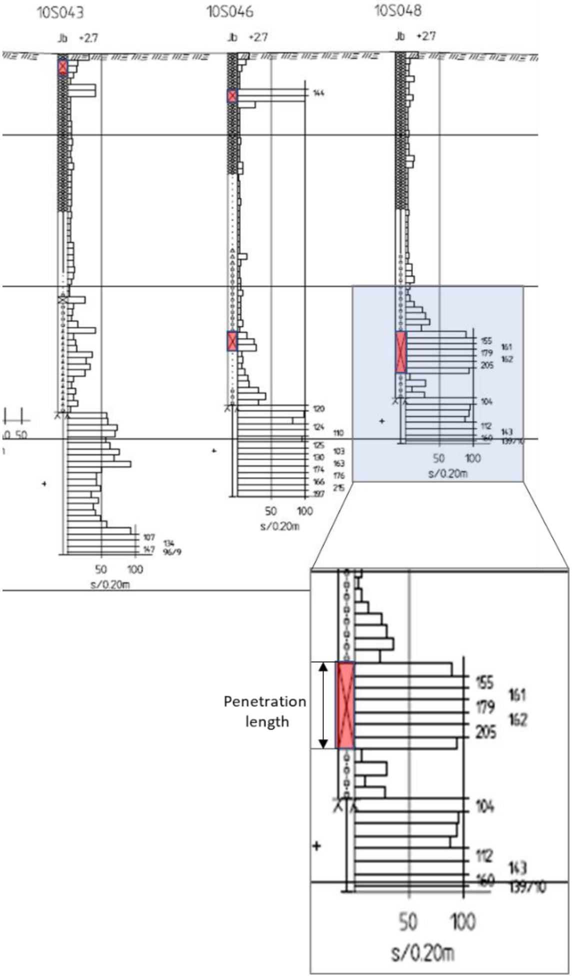

Figure 2: An example of the site investigation result with illustration of the penetration length

2.3. Piling in Sweden

2.3.1. Pile types and methods

Piles can be divided into two groups based on the load transformation method, end bearing pile and friction/cohesion pile. End bearing piles transfer the load to a firm stratum located at a considerable depth while in the case of friction/cohesion piles, carrying capacity is derived from friction or cohesion of soil in contact with the pile shaft. Applicable to the geology of the region focused on this thesis, end bearing piles are more common due to the low carrying capacity of soft clay.

Further, piles can be categorized based on the method of installation, namely driven piles and drilled piles. In Sweden, driven pre-cast concrete piles with dimensions of 235 mm, 270 mm and 350 mm are widely used. Drilled steel pipe piles with diameter varying between 76 mm and 800 mm are also available in this region (Axelsson, 2016).

2.3.2. Pile Design in STR/GEO

To calculate the structural bearing capacity of a pile, buckling and yielding of material are considered in the calculation. This calculation is performed according to 2nd order theory, wherein pile deflection during loading is accounted for. Due to high bedding modulus of friction soil, there is a low chance that buckling occurs. However, possible material yielding due to additional moment and pile imperfection is checked (Axelsson, 2016).

It is common to verify the geotechnical bearing capacity by dynamic load testing. Bearing capacity is evaluated in a closed form using Case method or/and rigorous method of numerical modeling such as using CAPWAP software. In these methods, measurement of force and velocity of piles during driving are required. These parameters are achieved by frequently measuring strain and acceleration with the Pile Driving Analyzer (Likins, 2000).

2.3.3. Design assessment, including boulders in the ground

Premature refusal or breakage of piles encountering boulders is a typical issue in driven piles which does not affect drilled piles. However, driven concrete piles are widely used due to their lower cost. Selection of suitable pile type is a balance between the cost and refusal rate.

In the risk analysis, the risk can be defined as the function of probability and consequence. The probability is the likelihood of the occurrence of an event and consequence is the impact of the event on the object(s). The risk level is equivalent to probability multiplied by consequence (Deloitte & Touche LLP, et al., 2012). To assess the risk of pile breakage, the probability of encountering a boulder and its impact are required. The consequence of encountering boulders depends on boulders' characteristics such as size, strength, depth and how centrically a pile hits a boulder. Pile characteristics such as bending stiffness and bending moment capacity are influential as well.

2.4. Monte Carlo method

Monte Carlo simulation is a process/method which relies on the generation of random objects or processes by means of a computer (Kroese, et al., 2014). These objects represent the natural objects as part of the modelling of a real-life system. The idea surrounding Monte Carlo techniques is to repeat the experiment many times (or use a sufficiently long simulation run) to obtain many quantities of interest using the Law of Large Numbers and other methods of statistical inference.

Simple Monte Carlo algorithms can be applied to study complex physical systems to encode them to a set of basic events. Monte Carlo method eliminates or minimizes reliance on analytical methods, which oftentimes are tedious or impossible to efficiently implement on a computer system (Kroese, et al., 2014). In this study the Monte Carlo technique is similarly applied to calculations due to the stochastic characteristic of the simulations.

2.4.1. Convergence

Convergence in its simplest essence is when the number of samples are sufficiently large to represent the properties of the population. Monte Carlo iterations are terminated when convergence criteria are fulfilled. In a stochastic process, various parameters can contribute in checking convergence of an experiment. The weakest convergence can be checked in relation to the first moment of resulting distribution namely, mean. Convergence is said to be achieved for mean when a sample of an iterated process does not change the average of all samples. The average of output parameters is plotted to find a suitable number of iterations, and it is maintained throughout observations.

In certain situations, as applicable to the objectives of this thesis, the output parameters may still be allowed to bear a certain range of error, to compensate for the extensive computational time required by a high number of iterations. Additionally, the assumption of homogenous soil profile precludes the necessity of a fully converged result.

2.5. Literature review

Pile refusal on boulders is mentioned in the most piling texts but there are very few studies on this phenomenon. Currently, the industry mostly relies on experience gained by professionals in this field, to predict the rate of pile refusal. A qualitative method to predict boulder content rate and the corresponding risk is discussed in the article “Driven friction piles/Report 103” (Commission on Pile Research, 2013). Additionally, Stuyts, et al. (2017) offers a methodology to evaluate pile refusal risk quantitatively. The aim of this thesis is to merge the gist of these reports and provide a concise numerical estimate according to Sweden’s geology to aid in solving the related problems faced during projects.

Commission on Pile Research (2013) has proposed the following table (Table 1) to facilitate the estimation of boulder occurrence in a soil layer. This qualitative classification is based on the penetration length per meter of soil-rock probing columns.

Table 1: Classification of the boulder density in a soil layer

Boulder density	Penetration length per meter borehole
Very low	0-0.015
Low	0.015-0.050
Medium	0.050-0.150
High	0.150-0.300
Very high	>0.300

Complementary to the above table, a qualitative classification for risk of encountering boulders in the soil layer down to bedrock or expected pile stop level is given in Table 2. This classification is associated with average number of captured boulders in borehole columns.

Table 2: Classification for risk of encountering boulders

Boulder risk	Number of boulders per borehole
Very small	<0.02
Low	0.02-0.05
Medium	0.05-0.20
High	0.20-0.50
Very high	>0.50

Finally, a formula is proposed by Commission on pile research to calculate the boulder content in weight percentage. This formula is built on the simplified ratio between the boulder penetration length in all boreholes ($L_{boulder}$) and total length of boreholes (L_{total}) referring to Jb probing results. Note that constant value of 1.4 refers to an assumed ratio between density of the rock and friction soil.

$$\begin{matrix} Boulder\ content \\ (weight\ percentage) \end{matrix} = \frac{1.4}{0.4 + {L_{total}}/{L_{boulder}}} \times 100 \qquad (1)$$

The report further classifies soil according to the weight percentage of boulder content as following:

- soil is referred as *blocky* if boulder content is between 5% to 20%.
- Soil is referred as *very blocky* if boulder content is above 20%.
- Soil layer with boulder content exceeding 40% is categorized as *boulder-soil* or *stone-soil* depending on which fraction is dominant.

Stuyts, et al. (2017) presents a methodology for quantitative assessment of pile refusal risk using probabilistic methods. This paper illustrates the use of the probabilistic methods at a UK offshore wind farm. Their methodology includes identifying the obstruction using site data and computing enhanced end pile resistance associated with encountering a boulder. According to this methodology, a pile refusal occurs if a pile driving hammer with a certain energy cannot overcome the enhanced resistance. Monte Carlo is applied to provide random generation of boulders. Each generated boulder is checked if it can lead to pile refusal and subsequently the percentage of pile refusal is calculated. However, this paper assumes that the distribution of boulders in the ground is previously estimated from core samples and does not explicitly describe how to assess the properties of the soil strata.

2.6. Case study

To illustrate the issue of pile refusal, two case studies are elaborated, Solna Simhall and Veddesta bridge. The data of these two cases including results of site investigation and piling reports are used in the analysis.

The Solna Simhall project is a mixed commercial development and swimming hall in the municipality of Solna, North of Stockholm. The foundation work consisted of initial excavations inside a retaining wall, followed by pile driving.

The Veddesta bridge is a road bridge which connects the areas of Barkarby and Veddesta. The bridge crosses the Stockholm-Västerås railway and the E18-highway. The bridge consists of several different segments, and the selected case study involves the road bridge and parking garage, East of the highway E18.

3. Methodology

The main objectives of this thesis, mentioned in section 1.1, are accomplished by developing a numerical code containing three parts: numerical random generation of boulders with various geometrical properties, simulation of site investigation process and modeling piling process. Monte Carlo simulation (MCS) is incorporated into the analysis to enable the random generation of boulders based upon the size distribution in a domain.

The methodology and the assumptions underlying the numerical model are elaborated in sections 3.1, 3.2 and 3.3. Further, section 3.4 demonstrates a concise workflow for usage of results from this thesis work in projects.

3.1. Generation of boulders in soil strata

As explained in the first chapter, the moraine/gravel strata frequently contain boulders. Some assumptions regarding the modelling of this strata are discussed to build a simplified numerical model.

The boulder content in soil strata is controlled by a dimensionless parameter in the model, herein introduced as Volumetric Boulder Content (VBC). The definition of this parameter is given in equation (2). Random generations of boulders stop when the volume of generated boulders reaches the assigned threshold VBC.

$$VBC = \frac{Volume\ of\ boulders}{Total\ \ volume\ of\ soil\ strata} \tag{2}$$

3.1.1. Statistical model for boulder generation

The simulation process is initiated by generating a 3-D domain of single-layer soil with length (l), breadth (b) and height (h) of 25, 25 and 5 meters, respectively. Height of 5 meters is chosen to represent a typical soil stratum. Boulders within the domain are modelled by ellipsoids with axes parallel to the coordinate axes. Each boulder is defined with 6 components that represent its center coordinates and diameters along the axes. An example of boulders' distribution in the domain is illustrated in Figure 3.

It is assumed that boulder size correlates well with exponential distribution. The boulder's diameter in the Z-axis (d_z) is generated by the following equation:

$$d_z = -\beta \times \ln(r_e) \tag{3}$$

where β is the scale parameter of exponential distribution and r_e is uniformly distributed random numbers generated by the MATLAB function, `rand` (MATLAB documentation, 2020). The ratio between the boulder diameter in X and Y axes (d_x and d_y) with boulder diameter in the Z-axis (d_z) is presumed to follow a normal distribution with a distributed mean of 1.5 and standard deviation 0.25. Normally distributed random numbers are generated with the aid of the MATLAB function, `randn` (MATLAB documentation, 2020) where numbers are drawn from the standard normal distribution. Equation (4) scales these random numbers to the normal distribution of interest, N (1.5,0.25), and generates d_x and d_y. It is noted that parameters d_x and d_y are independent. To avoid a negative value for boulder diameters and only keep problematic boulders in the

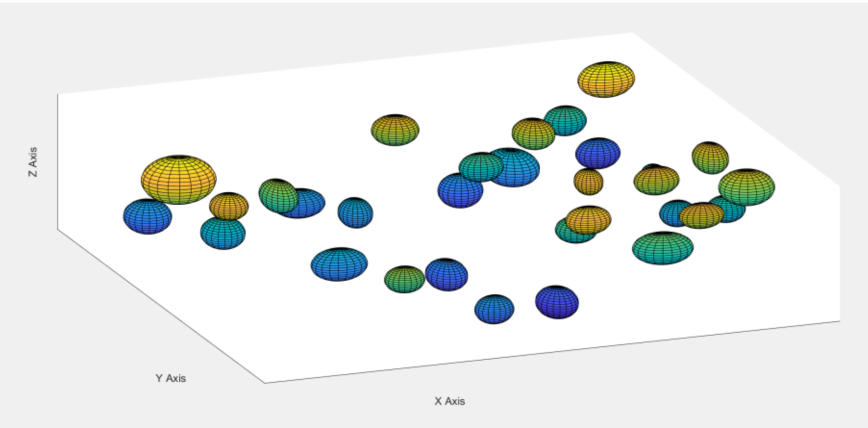

Figure 3: An example of distributed boulders in the domain

domain, the generated d_x and d_y below 0.2 m are discarded and regenerated.

$$d_x or\ d_y = d_z \times (\mu + \sigma \times r_n) \quad (4)$$

where μ and σ are the mean and standard deviation of the normal distribution, respectively, and r_n is a normally distributed random number.

The boulder's centroid is distributed uniformly in the domain, generated using the subsequent equations:

$$x_b = l \times r_u \quad (5)$$

$$y_b = b \times r_u \quad (6)$$

$$z_b = h \times r_u \quad (7)$$

where x_b, y_b and z_b are coordinates of the boulder centroid and r_u stands for uniformly distributed random numbers generated by `rand` (MATLAB documentation, 2020) .

While generating boulders, it is essential to check that generated boulders do not overlap with each other. This aim is accomplished using a custom MATLAB function which checks the overlap between a generated ellipsoid and existing ones and disregard the ellipsoid in case of collision. The code for checking overlap between ellipsoids is credited to Al Rumaithi (2020).

3.2. In-situ testing

In order to correlate the site investigation results with the boulder content within the soil mass, a methodology is formulated in two steps: forward analysis and inverse analysis.

3.2.1. Forward analysis

In this step, for known VBC values and number of boreholes, site investigation simulation is performed and its results, namely the penetration lengths, are obtained. Further, a parameter called penetration length ratio (r_b) is calculated based on the result as given by equation (8).

$$r_b = \frac{\sum Penetration\ length}{Total\ length\ of\ borehole\ \text{columns}} \tag{8}$$

MCS is applied on this process and values of r_b are stored. MCS is terminated after 20,000 iterations as the convergence criterion mentioned in section 2.4.1 is fulfilled. As an example, the variation of r_b with iteration count is illustrated in Figure 4. As it is marked with red dashed lines, mean of r_b tends to converge after 20,000 iterations and its fluctuation is not significant. Therefore, 20,000 iterations are chosen for MCS termination.

Note that the resolution of soil-rock probing is assumed to be about 0.15 meter. It means that the penetration length less than this amount is not registered by the method. Therefore, the simulation is adjusted accordingly and penetration length below 0.15 meter are removed from the results.

In section 3.2.1.1 the implementation of the statistical model utilized to simulate the site investigation is described.

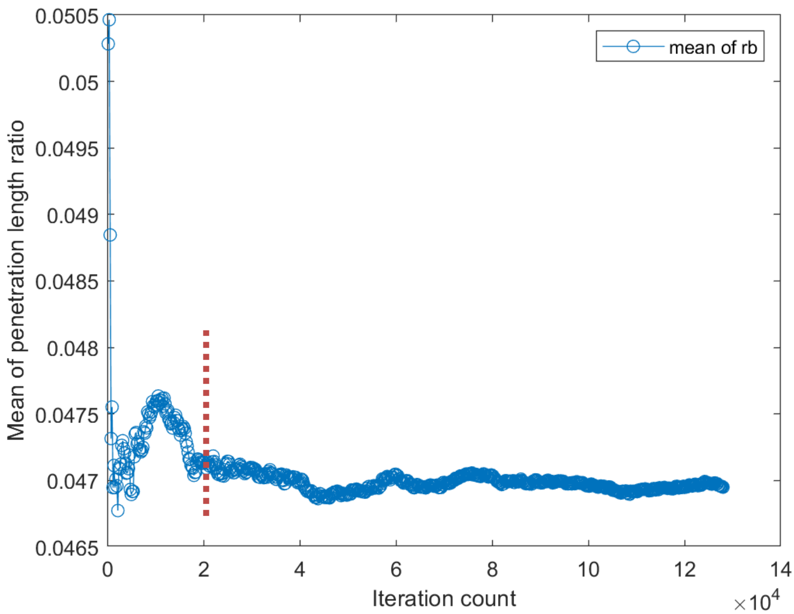

Figure 4: Variation of mean of penetration ratio with iteration count for input VBC 1%.

3.2.1.1. Statistical model for In-situ testing

Borehole columns for in-situ testing are simulated by vertical lines where their length is equal to h. Since boulders are randomly distributed in the domain, there is no necessity to have randomly located boreholes. Therefore, a 4 by 4 grid is chosen to model the boreholes' location in the domain. As it is clear in Figure 5, this grid size provides minimum 5 meter spacing on the X-Y plane where it is a realistic implementation. Note that if required number of boreholes in the simulation is less than 16, the extra generated boreholes are discarded prior to further analysis.

Intersection points between the lines and ellipsoids, representing boreholes and boulders, are found to calculate the penetration length. For this purpose, simultaneous equations are solved for the lines and ellipsoids as shown in the following equations. The derivation of this formula is given in Appendix A.

$$\text{Penetration length} = |u_1 - u_2| \tag{9}$$

where,

$$u_1, u_2 = h \times \left(\frac{-B \pm \sqrt{B^2 - 4 \times A \times C}}{2 \times A}\right) \tag{10}$$

and

$$A = \left(\frac{2 \times h}{d_z}\right)^2$$

$$B = \frac{-2 \times h \times z_t}{(0.5 \times d_z)^2} \tag{11}$$

$$C = \left(\frac{x_t - x_b}{0.5 \times d_x}\right)^2 + \left(\frac{y_t - y_b}{0.5 \times d_y}\right)^2 + \left(\frac{z}{0.5 \times d_z}\right)^2 - 1$$

where u_1 and u_2 are intersection points. A, B and C are numerical coefficients of the quadratic equation and x_t, y_t and z_t are coordinates of the borehole columns.

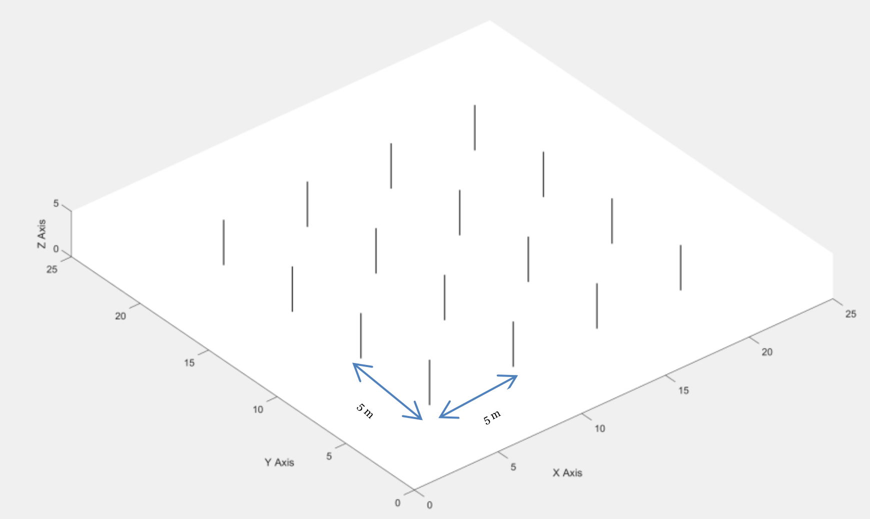

Figure 5: Arrangement of systematic boreholes in the domain

3.2.2. Inverse analysis

In this step, functional dependence of VBC to r_b is addressed. The relationship between these parameters are modeled using linear regression. This relationship can be expressed by equation (12) which represents a line that best fits the data.

$$y_i = \beta_0 + \beta_1 \times x_i \tag{12}$$

where,

- y_i are random variables that refers to VBC,
- x_i is the set of observations that refers to r_b,
- β_0 and β_1 are partial regression coefficients that represent the intercept and slope of fitted line.

Least squares approach is used to obtain the partial regression coefficients (β_0 and β_1) and the standard error. This is performed with the aid of LINEST command in Excel (Office Support, 2020).

When regression coefficients for various number of borehole columns are known, VBC estimator ($\widehat{VBC}$) can be written as:

$$\widehat{VBC} = \beta_0 + \beta_1 \times r_b \pm margin\ of\ error \tag{13}$$

The margin of error depends on level of confidence and sample size and is calculated by following formula:

$$margin\ of\ error = t_{\propto/2,n-2} \times SE \tag{14}$$

where,

- $t_{\propto/2,n-2}$ is critical value from student distribution,
- $\propto$ is significance level,
- n is sample size,
- SE stands for standard error.

Assuming a confidence interval of 95% and using 20,000 samples, the critical value (t) is given as 1.645.

3.3. Simulation of the piling process

Referring to definition of risk in section 2.3.3, probability of encountering a boulder and its consequence should be studied. In this study, the probability is found by simulating the piling process in the soil mass containing boulders. MCS is applied in this simulation for various VBC, pile sizes and types.

The consequences of a pile hitting a boulder can be categorized into several groups. A pile might push the boulder away or deflect and be slightly damaged. Significant pile deflection and breakage or stopping on top of a boulder are other possible behaviors. These mechanical behaviors of pile are under influence of various factors such as bending stiffness and moment capacity of the pile, the properties of surrounding soil, overlap between piles and boulders etc.

In this thesis, studying the consequences is limited to defining a parameter to help evaluation of the influence of overlap between pile and boulders, bending stiffness and moment of piles. It provides an opportunity to semi-qualitatively estimate the risk of pile loss. However, comprehensive analysis of the pile behavior encountering boulders requires an enhanced framework modeling all influential factors mentioned above, which is beyond the scope of this thesis.

Section 3.3.1 gives the implementation of statistical model of simulating the piling process.

3.3.1. Statistical model for piling

Piling simulation is carried out for two pile types, driven concrete pile and drilled steel pile. Owing to randomly distributed boulders, consistent locations for the piles are assumed. It is enforced that no pile is located on the boundary and piles are distributed evenly. The selection of a suitable number of piles is a tradeoff between computational time and uniqueness of encountered boulders. As it is illustrated in Figure 6, a grid of 5 by 5 is chosen for piles location within the domain.

Concrete and steel piles are modeled by cuboid and cylinders, respectively. A 3-D overview of these piles are given in Figure 7.

The overlap between piles and boulders are investigated for further analysis. To avoid complex computation for the overlap of 3-D objects, piles and boulders are projected on the X-Y plane. Projections of piles and boulders are represented by polygons using `polyshape` (MATLAB documentation, 2020). Concrete piles are represented by polygons with four vertices, steel piles and boulders with 50 vertices. Figure 8 illustrates an overview of a pile encountering a boulder in the domain, and the projection of the pile and the bolder is shown in Figure 9.

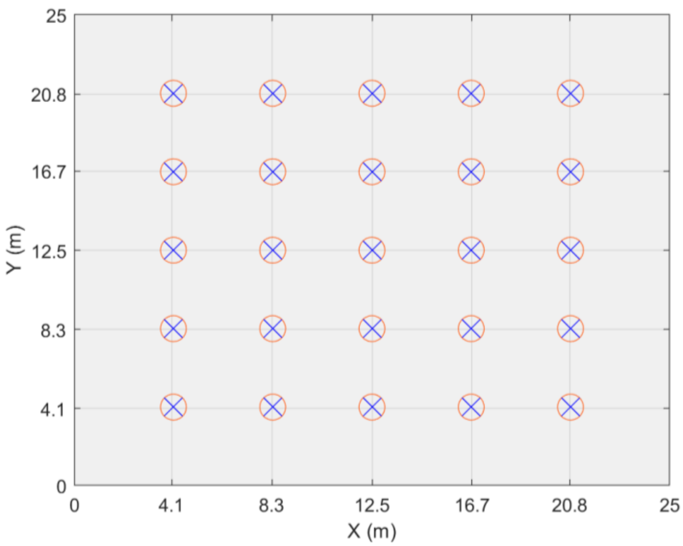

Figure 6: Arrangement of piles in the domain

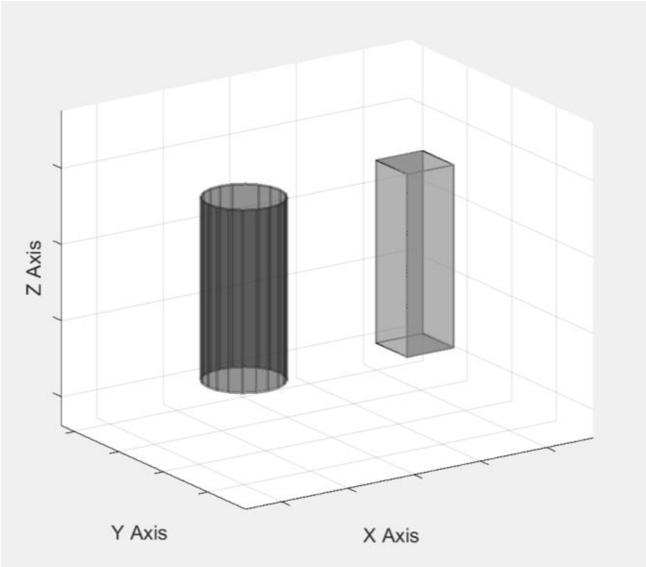

Figure 7: 3-D overview of a steel pile (left) and concrete pile (right)

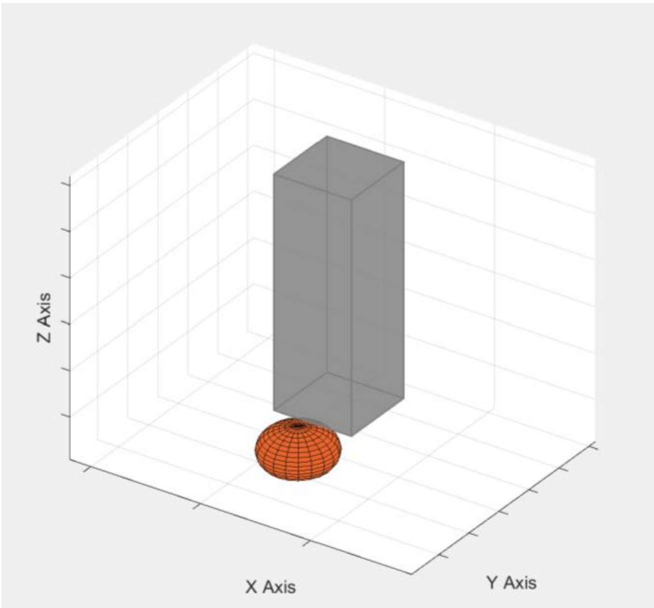

Figure 8: 3-D representation of a concrete pile encountering a boulder

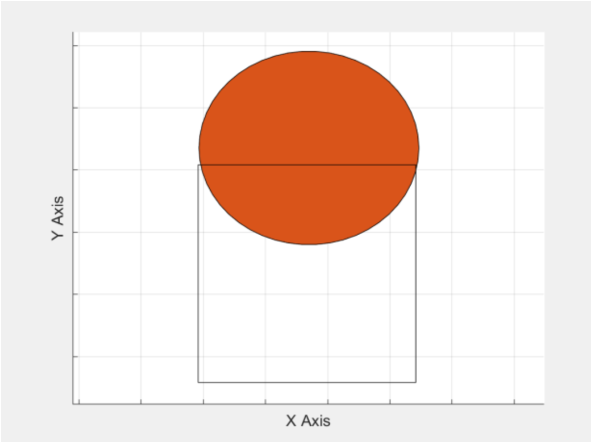

Figure 9: 2-D representation of a concrete pile encountering a boulder

The intersection between polygons are checked in order to find encountering boulders and is carried out with help of the `intersect` function (MATLAB documentation, 2020). This function returns a polygon with area describing the geometric intersection of two given polygons. Therefore, the area of the resultant polygon represents the overlap area between piles and boulders.

The piling simulation provides two main outputs, namely *Hit ratio* and *Weighted overlap ratio* (WOR). The definitions of mentioned parameters are given in following equations:

$$Hit\ ratio = \frac{number\ of\ piles\ encountering\ boulders}{total\ number\ of\ piles} \tag{15}$$

$$Weighted\ overlap\ ratio = \frac{overlap\ area}{pile\ cross\ section} \times \frac{boulder\ volume}{pile\ volume} \tag{16}$$

It should be noted that the pile volume in equation (16) is the volume of a pile of unit length. The mean of WOR is used for further analysis.

Hit ratio and WOR represent the probability and the consequences mentioned in section 3.3. WOR is not a fundamental parameter but rather a non-dimensional parameter showing how big the boulder volume is compared to a representative volume of the pile. In order to investigate the pile sensitivity toward encountering boulders, the aforementioned parameter is normalized with pile characteristics namely, its bending stiffness and moment capacity as given in equations (17) and (18). Note that for concrete piles only normalization by bending stiffness is applied. Because moment capacity of this pile type depends on the compressive stress on the pile and there is no direct formula to calculate the maximum moment.

$$WOR_b = \frac{WOR}{E \times I} \tag{17}$$

where,

- WOR_b stands for normalized WOR by pile bending stiffness,
- E stands for young's modulus and is assumed to be 35 GPa and 210 GPa for concrete and steel, respectively,
- I is the moment of inertia in m^4.

$$WOR_m = \frac{WOR}{F_y \times w} \tag{18}$$

where,

- WOR_m is normalized WOR by pile bending moment capacity,
- F_y is yield strength of steel and is assumed to be equal to 460 MPa,
- w is first moment of area in m^3.

In the piling simulation, MCS is terminated after 2,000 iterations when the convergence criterion mentioned in section 2.4.1 is fulfilled. As an example, the convergence for *Hit ratio* and WOR of a steel pile with diameter 0.09 m are illustrated in Figure 10 and Figure 11, respectively.

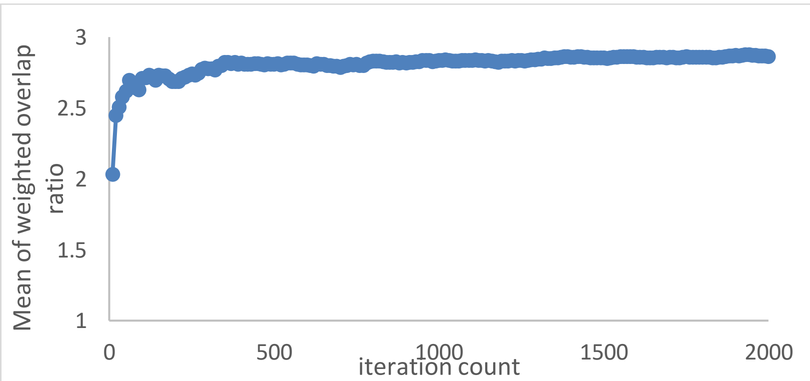

Figure 10: Convergence of WOR for steel piles with diameter 0.09 m

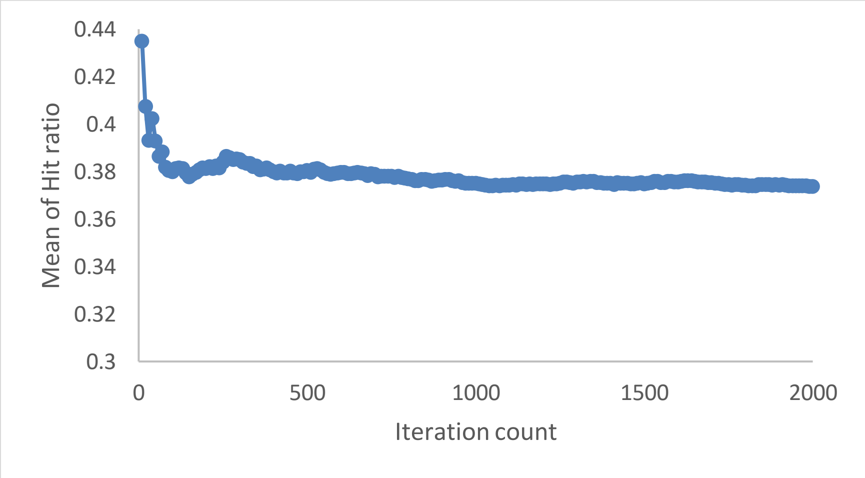

Figure 11: Convergence of *Hit ratio* for steel piles with diameter 0.09 m

3.4. Workflow for practical usage

Figure 12 provides a concise workflow for application of the results of this thesis work to ongoing projects/ projects in the future. The workflow starts with collection of site investigation results to gather information on the project site. This information from site investigation results is interpreted to quantitative measure in the form of penetration lengths. Then, rb is calculated from the penetration lengths. This thesis proposes an equation for translating site investigation results to rb and further provides equations to calculate a VBC from this value. Calculation of VBC is a crucial step in the quantification of site condition in relation to boulder contents.

VBC is cross referred to graphs given in sections 4.3.1 and 4.3.2 to infer *Hit ratio* and WORs which represent the probability and consequence of a pile encountering boulders, respectively. Analogous to expertise and knowledge gained by experience, the information correlating VBC to pile refusal rate can be further extended by including information from historic projects as discussed in chapter 4.4 and represented in Figure 20.

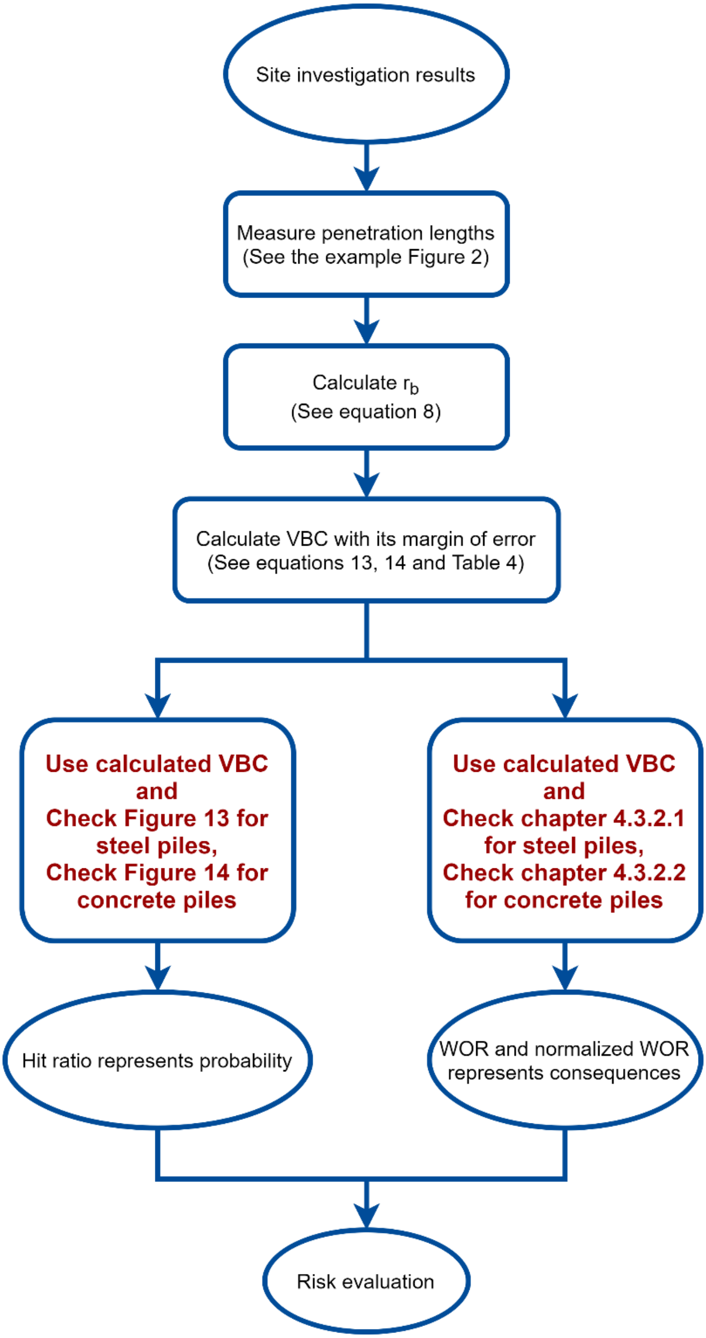

Figure 12: Generalized workflow for practical usage

4. Results

In this chapter, the results obtained from in-situ testing and piling simulation are presented. The first section illustrates the variation of parameters used in Monte Carlo Simulation. In sections 4.2 and 4.3, results for unknown parameters of the VBC estimators and MC results for piling simulation are elaborated, respectively. Finally, the data obtained from two case studies are given in the last section.

4.1. Configuration of parameters

This section gives the variation of parameters used in the simulations:

- Number of Jb probing:

  In-situ testing simulation is carried out for number of boreholes varied between 1 to 10 tests.

- VBC:

  The size of VBC in In-situ testing and piling process is variated between one to ten percent with increments of one percent.

- Pile type and size:

  Piling simulation is performed for two pile types: pre-cast square concrete piles and steel pipe piles. The cross-section dimensions of simulated piles are as presented in the following table:

Table 3: Variation of parameter in piling simulation

<table>
<tr><td>Cross section of Concrete pile's (m)</td><td colspan="2">0.235 ×0.235</td><td colspan="2">0.270×0.270</td><td colspan="3">0.350×0.350</td></tr>
<tr><td>Steel pipe pile's diameter (m)</td><td>0.09</td><td>0.13</td><td>0.17</td><td>0.22</td><td>0.32</td><td>0.4</td><td>0.8</td></tr>
</table>

4.2. In-situ testing

In this section, the unknown constants for VBC estimators as mentioned in 3.2.2 are presented. VBC estimators can be computed using equation (13) for which the partial regression coefficients (β_0 and β_1) are extracted from the simulation model, given in Table 4. It should be noted that the margin of error is computed based on 95% confidence interval. To utilize the obtained result for projects with higher number of performed tests, the parameters required for VBC estimator can be extrapolated from given table.

Table 4: VBC estimators' parameters for various Jb probing

Number of Jb probing	β_0	β_1	Margin of error
1	4.29	25.49	±4.15
2	3.49	41.49	±3.72
3	2.97	52.03	±3.42
4	2.56	60.32	±3.16
5	2.28	66.13	±2.98
6	2.05	71.12	±2.81
7	1.85	75.15	±2.66
8	1.71	78.01	±2.56
9	1.59	80.66	±2.46
10	1.49	82.72	±2.38

4.3. Piling process

4.3.1. Hit ratio

Hit ratio represents how probable it is, for a pile of a certain size, to hit a boulder as mentioned in section 3.3.1. The VBC of domain increases towards the top left of the chart. The graphs demonstrate that the increase in *Hit ratio* is not commensurate with increase in pile diameter, instead, the *Hit ratio* depends largely on the VBC for lower VBC with lower pile sizes. For larger VBC and larger pile sizes, this dependency diminishes. Figure 13 and Figure 14 represent this relation for steel and concrete pile, respectively.

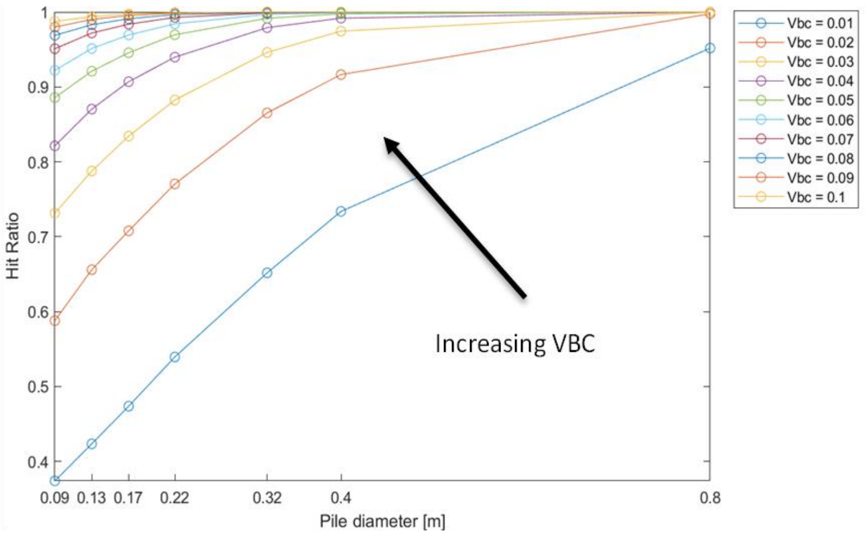

Figure 13: Variation of *Hit ratio* with VBC for the steel pile

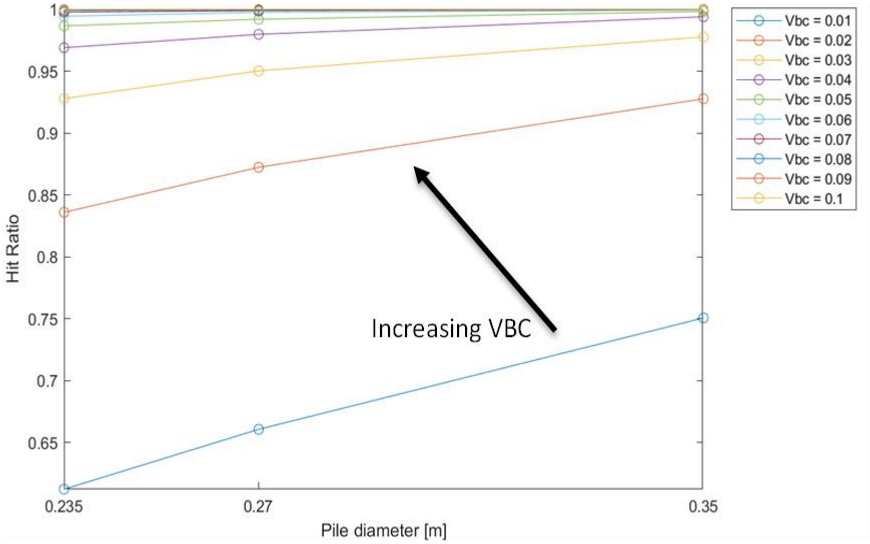

Figure 14: Variation of *Hit ratio* with VBC for the concrete pile

4.3.2. Weighted overlap ratio

WOR and normalized WOR, described in section 3.3.1, depends on VBC and pile sizes, as given in this section. Section 4.3.2.1 covers the obtained results of steel piles in terms of WOR and its normalized data with the bending stiffness and bending moment capacity. Additionally, the graphs illustrating the variation of WOR and the normalized value with bending stiffness for concrete piles are given in section 4.3.2.2.

4.3.2.1. Steel pile

VBC of the curves, Figure 15, Figure 16 and Figure 17, increase towards the top right of the graphs. Figure 15 shows that even with a significant variation of VBC, WOR does not change considerably. On the other hand, WOR drops significantly with an increase in pile diameter. It should be recalled that WOR is the parameter representing the consequences of a pile-boulder interaction.

Figure 16 and Figure 17 show the increasing insensitivity of piles towards pile-boulder interaction. At the same time, sensitivity of slender piles is underscored in these graphs, in which the consequences are dramatically high.

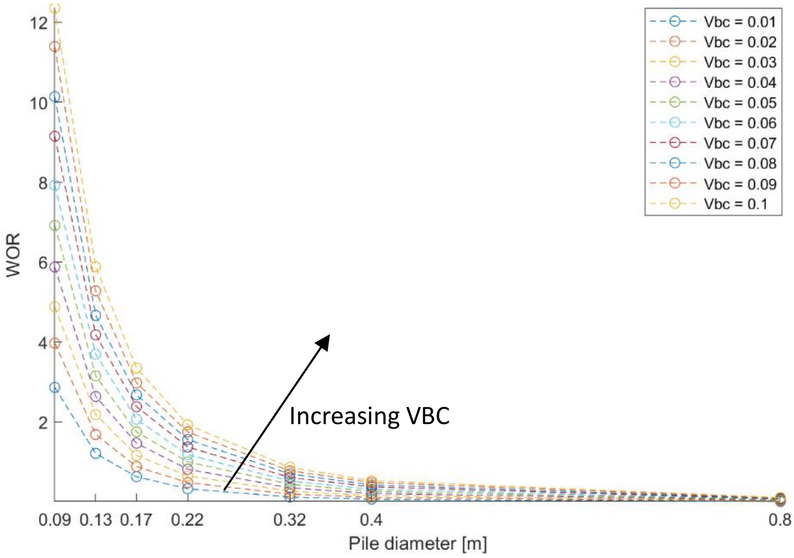

Figure 15: Variation of WOR with VBC for steel piles

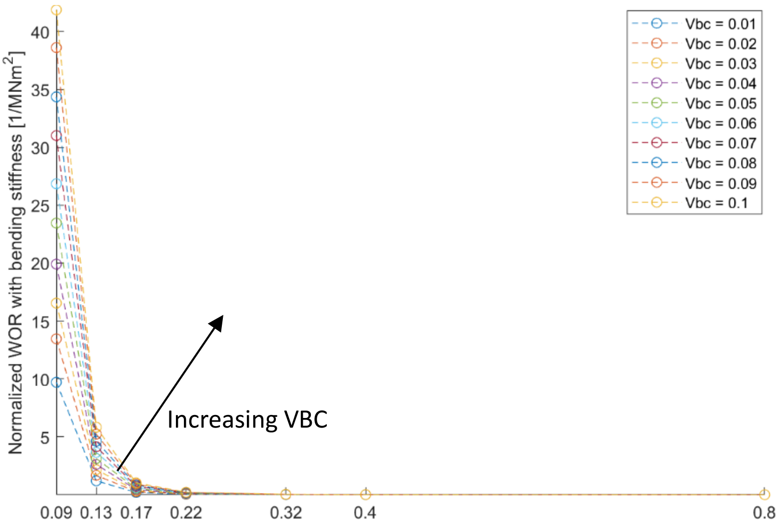

Figure 16: Variation of WOR_b with VBC for steel piles

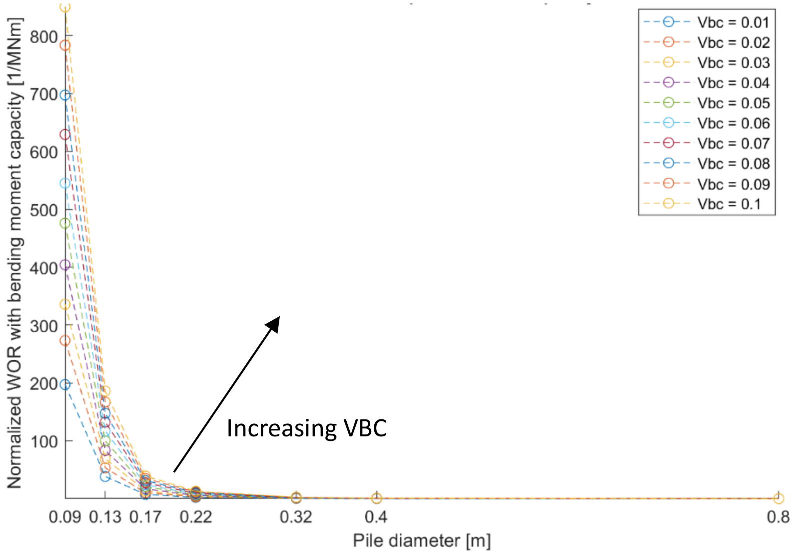

Figure 17: Variation of WOR_m with VBC for steel piles

4.3.2.2. Concrete pile

VBC of the curves in Figure 18 and Figure 19 increase towards the top right. The results are similar as described in section 4.3.2.1. Increase in pile diameter drastically reduces consequences of a pile-boulder interaction.

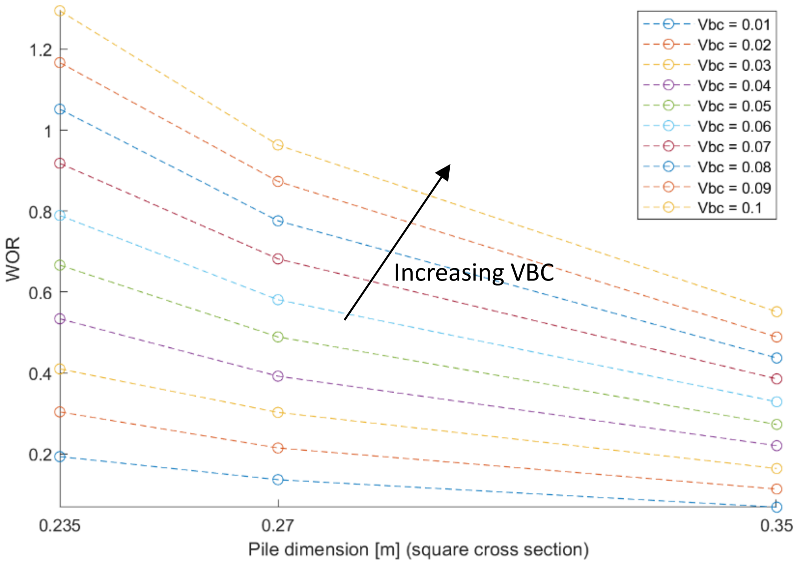

Figure 18: Variation of WOR with VBC for concrete piles

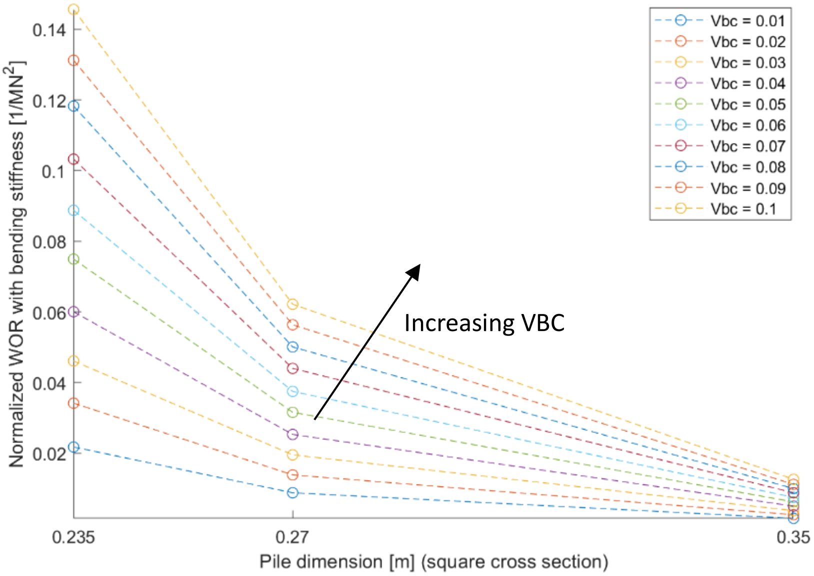

Figure 19: Variation of WOR_b with VBC for concrete piles

4.4. Case study

In this chapter, the evaluation on two case studies is given. In this evaluation, penetration lengths are extracted from the Jb probing results and is used to calculate the $\widehat{vbc}$ with its margin of error. Afterward, the information from the site regarding the piling process is studied. This data is used to investigate the type and number of piles used in the case studies and calculate the rate of pile loss.

4.4.1. Solna Simhall

According to the results of Jb probing, number of borehole columns, total borehole columns' length, total penetration length and corresponding r_b are given in the following table.

Table 5: Obtained data from results of Jb probing

Borehole columns' count	18
Total borehole length (m)	328.15
Total penetration length (m)	4.70
$\boldsymbol{r_b}$	0.01432

$\widehat{VBC}$ is calculated according to equation (13) where the unknown parameters are taken from Table 4 and Table 5. The mentioned equation gives $\widehat{VBC}$ equal to 2.75% with lower and upper bounds of 0.29% and 5.22%, respectively.

According to the piling report, 900 piles are used in this project where 625 piles are pre-cast concrete piles and the remaining are steel piles, see Table 6. In this project, 27 concrete piles were stopped on the obstacles while there was only one steel pile refused during installation. The loss rate for two pile types is shown in Figure 20.

Table 6: Data pertaining to piles in Solna Simhall

Pile type	Pile size(mm)	Pile count
Driven concrete	270*270	625
Drilled steel	Φ 220	275

4.4.2. Veddesta bridge

Repeating the procedure as in the previous case study, results from Jb probing conducted at the site are obtained as given in the following table:

Table 7: Data obtained from results of Jb probing

Borehole columns' count	16
Total borehole length (m)	149.30
Total penetration length (m)	3.65
r_b	0.02445

With the aid of Table 4 and Table 7 , $\widehat{VBC}$ is given as 3.65% with lower and upper bounds of 1.05 % and 6.24%, respectively.

According to the piling report, a total of 475 piles are used for the bridge supports where the majority, 398 piles, are driven concrete piles and the remaining 58 and 19 piles are driven and drilled steel pile, respectively, see Table 8. Among the piles installed in the bridge supports, 51 piles faced premature refusal, all being concrete piles. It should be noted that prior to piling process, two steel piles with diameter of 170 mm were installed to examine the pile drivability and both piles were refused.

Table 8: Data pertaining to piles in Veddesta bridge

Pile type	Pile size(mm)	Pile count
Driven concrete	270*270	398
Drilled steel	Φ 270	58
Driven steel	Φ 220	19

4.4.3. Evaluation of case studies

The results of two elaborated case studies are summarized in Figure 20. In this graph, the Y-axis represents rate of pile refusal on boulders and the X-axis represents the calculated VBC using penetration lengths from site investigation of the same project. The increase of concrete pile refusal is proportional to the estimated VBC, as predicted by the model developed in this thesis work. The empirical data collected from case studies can also be used to calibrate WOR values to realize a quantitative refusal rate. As an example, the results from Solna Simhall show that the mean VBC is 2.75%, representing a WOR of 0.275. The results from Veddesta bridge show that the mean VBC is 3.65%, corresponding to a WOR of 0.36. These correspond to a pile refusal rate of 4.25% and 10.75%.

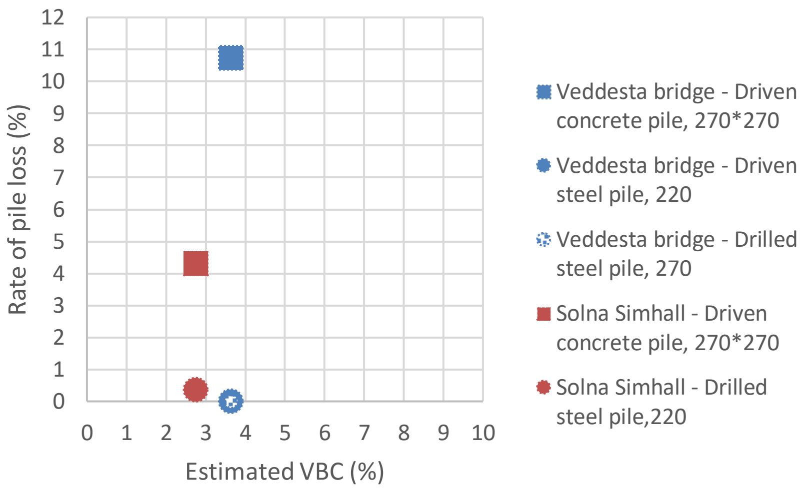

Figure 20: Pile loss rate for concrete and steel piles used in the case studies

As it was mentioned in the literature study, Commission on Pile Research also proposed a formula to estimate the boulder content in soil strata. Here, it is aimed to compare the boulder content results from both methods and have a comparison. It should be noted that the formula proposed by Commission on Pile Research is based on Weight boulder content (WBC). Therefore, VBC values are converted to WBC by multiplying into 1,4 which is assumed ratio between density of rock and friction soil according to Commission on Pile. WBC computed from both methods are represented in following table. Table 9 shows relatively lower estimation of boulder content compared to the proposed method in this study using MCS.

Table 9: WBC values from proposed methods in this study and Pile Commission

Case study	WBC [%] (This study)	WBC [%] (Pile Commission)
Solna Simhall	3.85	1.99
Veddesta bridge	5.11	3.39

5. Discussion and conclusion

A novel numerical model has been implemented in MATLAB to study pile refusal during pile driving. An assumed five-meter strata of moraine has been modeled and three different programs have been developed to simulate the soil strata, to simulate the site investigation process and to simulate the pile driving.

The simulation of the site investigation process enables evaluation of volumetric boulder content (VBC) from standard tests. Increasing the number of performed tests results in higher certainty of VBC and a shrink in the margin of error, as can be seen in Table 4. In practical design, it is crucial to gain an accurate estimate of VBC since the selection of suitable pile is under influence of this parameter.

The risk of damaging piles associated with pile installation in soil containing boulders is qualitatively assessed using results of piling simulation. The risk is a function of probability and consequences of encountering a boulder while installing piles. The probability and consequences are here estimated by parameters *Hit ratio* and WOR as given in sections 4.3.1 and 4.3.2, respectively. Although it is out of scope of this thesis to compute a quantitative consequence, a semi-qualitative consequence can be inferred from the results of piling simulation.

It is concluded from Figure 13 and Figure 14 that increasing VBC raises the probability of hitting a boulder during pile installation. However, this increment is not proportional to diameter of the pile. It means that even with low VBC and small pile diameter, there is a high probability to hit boulders.

The results of piling simulation, specifically normalized WOR, as given in section 4.3.2, indicate that the consequences of encountering boulders are significant in small-diameter piles due to lower bending stiffness and moment capacity compared to large-diameter piles. As expected, slender piles are more prone to be damaged during installation. In other words, there is a higher risk in using slender piles compared to piles with larger diameter. Empirical evidence confirms this fact, but the numerical model demonstrates why the risk is much higher for small-diameter piles.

If there is an interest of using small piles in a project due to the superstructure design, it is recommended to perform enough number of site investigation tests to ascertain VBC and lower unpredictable risk. On the other hand, if a project requires low number of piles, it is suggested to have over-capacity piles with larger diameters instead of performing considerable number of tests to overcome limitation from a wide range in predicted VBC. Since the cost of performing probing tests to estimate VBC with a low margin of error will be relatively large compared to the cost of piles.

Figure 20 shows the results obtained from the case studies, which elaborates that a small increase in VBC among two historical projects results in significant increase in the rate of pile refusal on boulders for concrete piles.

In section 4.4.3 the boulder content of the case studies are estimated according to the method suggested in this thesis work and the equation proposed by Commission on Pile Research. According to Table 9 the method suggested in Report 103 (2013) appears to be underestimating the boulder content in a soil strata and it might need to be calibrated with the results of quantitative methods.

As mentioned in section 2.5, the paper written by Stuyts, et al. (2017), where they studied the risk of damaging a pile encountering boulders, can be considered the predecessor to this thesis work. This thesis was conducted since the piling method and typical pile types used in Sweden are dissimilar to the ones considered by Stuyts et al. (2017). Further, this thesis proposes a method to estimate VBC from Jb probing and simulate piling process to estimate the risk of damaging piles encountering boulders, semi-qualitatively. These results can assist a piling engineer to have a qualitative judgement of which piles to use depending on the quantity of piles and the site investigation.

5.1. Suggestion and future work

Studying quantitative consequences of hitting boulders requires modeling the mechanical behavior of piles while encountering boulders. This simulation is deemed out of scope of this thesis since there are various influential factors that needs to be simulated such as properties of surrounding soil, boulders and piles. For future work, it is suggested to extend the numerical model developed in this thesis to include simulation of the mechanical behavior of piles as well.

In this thesis, the empirical data are extracted from only two case studies due to time consuming analysis process and difficulties to gain access to performed projects. More empirical data can be collected to determine the influence of encountering boulders for various pile size under a certain VBC and assist in the prediction of pile refusal risk corresponding to certain pile type and size. These empirical data can be used to calibrate WOR values to realize a quantitative refusal rate.

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Appendix A

An ellipsoid is defined by centroid coordinates (x_c, y_c , z_c) and semi-axes (r_x, r_y, r_z). The equation for the ellipsoid is:

$$\frac{(x-x_c)^2}{r_x^{\,2}}+\frac{(y-y_c)^2}{r_y^{\,2}}+\frac{(z-z_c)^2}{r_z^{\,2}}=1$$

A vertical line is defined to represent a borehole column. If point 1 is located on the ground surface and point 2 is on the bottom of domain, properties of these points are: $P_1(x_t, y_t, 0)$ and $P_2(x_t, y_t, 5)$.

General equation for a line in 3-D:

$$x = x_1 + (x_2 - x_1)\times u$$

$$y = y_1 + (y_2 - y_1)\times u$$

$$z = z_1 + (z_2 - z_1)\times u$$

Applying P1 and P2 to the above line equation:

$$x = x_t$$

$$y = y_t$$

$$z = 5\times u$$

To find the intersection between the line and the ellipsoid:

$$\frac{(x_t-x_c)^2}{r_x^{\,2}}+\frac{(y_t-y_c)^2}{r_y^{\,2}}+\frac{(5u-z_c)^2}{r_z^{\,2}}=1$$

Rearranging the equation in form of $Ax^2 + Bx + C = 0$
where,

$$A=\frac{25}{r_z^{\,2}}$$

$$B=\frac{-10\times z_c}{r_z^{\,2}}$$

$$C=(\frac{x_t-x_c}{r_x})^2+(\frac{y_t-y_c}{r_y})^2+(\frac{z_c}{r_z})^2-1$$

Intersection points (u1 and u2) are equal to:

$$u_1, u_2 = 5\times\frac{-B\pm\sqrt{B^2-4\times A\times C}}{2\times A}$$

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