Stochastic discount factors and the optimal timing of irreversible investments

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Abstract
By using a general semimartingale framework, we show how the transformation of an optimal stopping problem under the objective probability measure into an optimal stopping problem under the risk-neutral probability measure looks like. We also note that the difference between equivalent and a locally equivalent are important when considering infinite time horizons (i.e., when considering perpetual options).

Keywords: optimal stopping, stochastic discount factors, irreversible investments

JEL Classification G11, G13
1 Introduction

In Thijssen [9], an approach to optimal investment problems using stochastic discount factors in a geometric Brownian motion setting is analyzed. It is shown that by using a change of measure technique, the solution to certain optimal stopping problems can be completely characterized. In this note we use a more general semimartingale framework, and show how the transformation from an optimal stopping problem under the objective probability measure $P$ transforms into an optimal stopping problem under the risk-neutral probability measure $Q$. We specifically note that when considering infinite time horizons (i.e. when considering perpetual options), care must be taken regarding the difference between equivalent and locally equivalent measures. This difference does not exist when considering options with a finite time to maturity.

The rest of this note is organized as follows. The modelling framework is introduced in Section 2 and the problem of optimal timing of irreversible investments is discussed in Section 3. We end with a summary in Section 4.

2 The modelling framework

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a complete filtered probability space, where the filtration $(\mathcal{F}_t)$ is assumed to satisfy the usual assumptions of right-continuity and $\mathcal{F}_0$ containing all null sets of $\mathcal{F}$. We recall that a stopping time (with respect to the filtration $(\mathcal{F}_t)$) is a random time $\tau: \Omega \to [0, \infty]$ such that

\[ \{ \tau \leq t \} \in \mathcal{F}_t \]

for every $t \geq 0$.

We assume the existence of a stochastic discount factor $\Lambda = (\Lambda_t)$, i.e. a stochastic process such that the value $V_t(Y)$ at time $t \in [0, T]$ of a cash flow $Y$ paid out at time $T$ is given by

\[ V_t(Y) = E \left[ \frac{\Lambda_T Y}{\Lambda_t} \mid \mathcal{F}_t \right]. \]

In order to get a reasonable model from an economic point of view, the stochastic discount factor $\Lambda$ should be strictly positive. We also add some more ‘technical’ requirements on the stochastic discount factor.

Assumption 2.1 In addition to the stochastic discount factor $\Lambda$ being strictly positive, we also assume that it is a special semimartingale satisfying $\Lambda_- > 0$ and $\Lambda_0 = 1$.

Note that in the assumption of $\Lambda$ being a special semimartingale is included the facts that $\Lambda$ is adapted and càdlàg.
Remark 2.2 In some definitions of, or assumptions on, the stochastic discount factor, there is a requirement on the finiteness of some moment of the SDF. Qin & Linetsky [8] requires that an SDF should satisfy $E \left[ \frac{\Lambda_T}{\Lambda_t} \right] < \infty$ for $0 \leq t \leq T$, and Munk [7] that $E \left[ \Lambda_t^2 \right] < \infty$ for $t \geq 0$. In practice, we will often have finiteness of moments, but we do not make these requirements explicit.

Under Assumption 2.1 we can use the multiplicative decomposition given in Theorem 6.19 in Jacod [4] (see also Proposition 2 in Döberlein & Schweizer [3]) and write

$$\Lambda_t = D_t M_t,$$

where $D$ is a strictly positive predictable càdlàg process with finite variation satisfying $D_0 = 1$, and $M$ is a strictly positive càdlàg local martingale with $M_0 = 1$. This decomposition is unique in the sense that if there exists two decompositions, $\Lambda_t = D_t M_t = D'_t M'_t$, then $D$ and $D'$, and $M$ and $M'$ are indistinguishable.

If there exists a traded asset which makes no payments, such as dividends or coupons, to its owner, and has price $S_t$ at time $t \geq 0$, then the value at time $t$ of getting the cash flow $S_T$ at time $T \geq t$ must be equal to $S_t$:

$$S_t = E \left[ \frac{\Lambda_T}{\Lambda_t} S_T \bigg| \mathcal{F}_t \right].$$

This can be written

$$\Lambda_t S_t = E \left[ \Lambda_T S_T | \mathcal{F}_t \right].$$

With $t = 0$ this yields

$$E \left[ \Lambda_T S_T \right] = S_0 < \infty,$$

so if $S = (S_t)$ is a positive stochastic process, then $\Lambda S$ is a martingale. A special case of a traded asset is the zero-coupon bond. This asset gives the owner 1 unit of currency at a given time, known as the time of maturity. We let $p(t,T)$ denote the price at time $t \in [0,T]$ of a zero-coupon bond with face value 1 maturing at time $T \geq 0$:

$$p(t,T) = E \left[ \frac{\Lambda_T}{\Lambda_t} p(T,T) \bigg| \mathcal{F}_t \right] = E \left[ \frac{\Lambda_T}{\Lambda_t} \bigg| \mathcal{F}_t \right].$$

In order to be able to connect the stochastic discount factor with the risk-neutral measure $Q$ (see below for details), the local martingale $M$ in the multiplicative decomposition above needs to be a true martingale. When the stochastic discount factor $\Lambda$ is a special semimartingale and $M$ in the decomposition given in Equation (1) is a true martingale, then the pair $(P, \Lambda)$ is called good in the terminology of Döberlein & Schweizer [3].

We now present some modelling situations that gives sufficient conditions for $(P, \Lambda)$ to be good. First, recall that a bank account with, possibly stochastic, interest rate is a financial asset with price process $B$ satisfying

$$dB_t = r_t B_t dt \quad \text{and} \quad B_0 = 1,$$
where \( r = (r_t) \) is an adapted stochastic process. The value of the bank account at time \( t \geq 0 \) is given by
\[
B_t = e^{\int_0^t r_s \, ds}.
\]

**Proposition 2.3** If there exists a bank account, then \((P, \Lambda)\) is good.

**Proof.** Since \( B \) is the price process of a traded asset that does not pay out any dividends, we know that \( M_t^B := \Lambda_t B_t \) is a martingale. Thus, we can write
\[
\Lambda_t = B_t^{-1} M_t^B = e^{-\int_0^t r_s \, ds} M_t^B.
\]

Uniqueness of the decomposition in Equation (1) implies that
\[
D_t = e^{-\int_0^t r_s \, ds} \text{ and } M_t = M_t^B,
\]
which shows that \( M \) is a true martingale in this case. \( \square \)

In many cases we often directly assume a stochastic discount factor on the form \( \Lambda_t = e^{-\rho t} M_t \), where \( \rho \in \mathbb{R} \) and \( M \) is a strictly positive càdlàg martingale satisfying \( M_0 = 1 \). In this case, we have the following result.

**Proposition 2.4** Assume that the stochastic discount factor can be written \( \Lambda_t = e^{-\rho t} M_t \), where \( \rho \in \mathbb{R} \) and \( M \) is a strictly positive càdlàg martingale satisfying \( M_0 = 1 \). In this case, not only is \( \Lambda \) strictly positive and satisfies \( \Lambda_0 = 1 \), it also holds that \( \Lambda_- > 0 \) and \((P, \Lambda)\) is good.

**Proof.** With \( \Lambda_t = e^{-\rho t} M_t \), the function \( e^{-\rho t} \) is the finite variation part in the multiplicative decomposition of \( \Lambda \), and \( M \) is the local martingale part. Since \( M \) is assumed to be a true martingale, \((P, \Lambda)\) is good. Since \( \Lambda_t = e^{-\rho t} M_t \) and every strictly positive martingale satisfies \( M_- > 0 \), it follows that \( \Lambda_- > 0 \). \( \square \)

Possible interpretations of the parameter \( \rho \) include:

(a) As a constant rate of return on a bank account (see above).

(b) As a subjective discount factor. See Dixit & Pindyck [2].

(c) As induced by a consumption model. See e.g. Thijssen [9], Cochrane [1] or Munk [7].

We end this section by considering the case when \( \Lambda \) is a supermartingale. In this case, for every \( 0 \leq t \leq S \leq T \)
\[
p(t, S) = E \left[ \frac{\Lambda_S}{\Lambda_t} \bigg| \mathcal{F}_t \right] \leq E \left[ \frac{\Lambda_S}{\Lambda_t} \bigg| \mathcal{F}_t \right] = p(t, S),
\]
so in this case we have decreasing zero-coupon bond prices (which we essentially can interpret as having a non-negative interest rate); see Döberlein & Schweizer [3] and references therein for more on this. Assuming that \( \Lambda \) is a supermartingale implies not only that interest rates are non-negative, but also that several of the ‘technical’ conditions we put on \( \Lambda \) in Assumption 2.1 are automatically satisfied.

The following result follows from combining Theorem 6.19 and Proposition 6.10 in Jacod [4] (see also Proposition 2 in Döberlein & Schweizer [3]).
Proposition 2.5 Let $X$ be a strictly positive and càdlàg supermartingale with $X_0 = 1$. Then $X$ can be written $X_t = D_t M_t$, where $D$ is a strictly positive, decreasing, predictable, càdlàg process of finite variation satisfying $D_0 = 1$, and $M$ is a strictly positive càdlàg local martingale with $M_0 = 1$. Furthermore $X_\tau > 0$ and $X$ is also a special semimartingale.

In many cases, we assume that there exists a bank account with a non-negative and constant interest rate. In this case, we can use the previous proposition to show the following result.

Proposition 2.6 Let $\Lambda$ be a stochastic discount factor, and assume that there exists a bank account with constant rate $r \geq 0$. Then $\Lambda$ is a supermartingale (and hence also a special semimartingale) satisfying $\Lambda_\tau > 0$ and $(P, \Lambda)$ is good. Furthermore $D$ in the multiplicative decomposition of $\Lambda$ is given by $D_t = e^{-\rho t}$.

3 The optimal investment timing problem

We now turn to the problem of valuing an investment in which we are allowed to optimally choose the time of the investment. From now on we assume that the pair $(P, \Lambda)$ is good and that $D$ in the multiplicative decomposition of $\Lambda$ is given by $D_t = e^{-\rho t}$ for some $\rho \in \mathbb{R}$.

A cash flow process $Y = (Y_t)$ represents the cash flow we are given if we choose to invest at time $t \geq 0$, and in this case we are given the amount $Y_t$ at time $t$. One example of this is the cash flow given by the opportunity to pay the investment cost $I > 0$ in order to get the cash flow $X_t$. In this case $Y_t = \max(X_t - I, 0)$. We assume that every cash flow process $Y$ that we consider is a special semimartingale satisfying $E[\Lambda_t Y_t] < \infty$ for every $t \geq 0$.

Now fix a cash flow process $Y$, and consider choosing the time $\tau$, a stopping time, at which the cash flow $Y_\tau$ is received. In order to value this investment opportunity at time $t = 0$, we want to find the stopping time $\tau$ that maximizes $E[\Lambda_\tau Y_\tau]$. To analyse this problem, we start by fixing $T > 0$ and consider the value of the investment opportunity when the admissible stopping times are the ones bounded by $T$. First of all, by using the optional stopping theorem, we can write

$$
E[\Lambda_\tau Y_\tau] = E[e^{-\rho \tau} M_\tau Y_\tau] = E[e^{-\rho \tau} E[M_T | \mathcal{F}_\tau] Y_\tau] = E[M_T e^{-\rho \tau} Y_\tau] = E^Q [e^{-\rho \tau} Y_\tau],
$$

where we have changed measure on $\mathcal{F}_T$ to a a new measure $Q$ by using the Radon-Nikodym derivative $dQ/dP = M_T$. To find the optimal value of the investment, we maximize this expression over $\tau \in \mathcal{S}_T$, where for $T \geq 0$ we let $\mathcal{S}_T$ denote the set of stopping times $\tau \leq T$. The previous calculation shows that

$$
\sup_{\tau \in \mathcal{S}_T} E[\Lambda_\tau Y_\tau] = \sup_{\tau \in \mathcal{S}_T} E^Q [e^{-\rho \tau} Y_\tau].
$$
This change of measure technique works since $\tau$ is a stopping time bounded by $T$. In the case of taking the supremum over a set of general, i.e. not necessarily bounded, stopping times, the martingale $M$ needs to be uniformly integrable for the above change of measure technique to work. It turns out that even if $(P, \Lambda)$ is good, the martingale $M$ in the multiplicative decomposition of $\Lambda$ need not be uniformly integrable.

**Example 3.1** The following model is used by Thijssen [9] (see also Cochrane [1] and Munk [7]). The stochastic discount factor is assumed to have the form

$$d\Lambda_t = -\mu_\Lambda \Lambda_t dt - \sigma_\Lambda \Lambda_t dW_t,$$

where $W$ is a standard $P$-Brownian motion and $\mu_\Lambda, \sigma_\Lambda > 0$ are two constants. Hence, it is assumed that the stochastic discount factor $\Lambda$ is a geometric Brownian motion. We have

$$\Lambda_t = \exp \left( - \left( \mu_\Lambda + \frac{\sigma_\Lambda^2}{2} \right) t - \sigma_\Lambda W_t \right),$$

so in this case the decomposition in Equation (1) is given by

$$D_t = e^{-\mu_\Lambda t} \quad \text{and} \quad M_t = \exp \left( -\sigma_\Lambda W_t - \frac{\sigma_\Lambda^2 t}{2} \right),$$

and the local martingale $M$ is in this case a true martingale, and $(P, \Lambda)$ is good. Using the fact that $W_t/t \to 0$ $P$-a.s., we see that when $\sigma_\Lambda > 0$ it holds that $M_t \to M_\infty = 0$ $P$-a.s. If $M$ was a uniformly integrable martingale, then

$$M_0 = E[M_\infty],$$

while in this case

$$M_0 = 1 \neq 0 = E[M_\infty],$$

so $M$ can not be uniformly integrable.

As this examples shows, $M$ in the multiplicative decomposition of $\Lambda$ need not be a uniformly integrable martingale even if $(P, \Lambda)$ is good. Hence, in order to use the change-of-measure technique to solve problems of the type

$$\sup_{\tau \in \mathcal{S}} E[\Lambda_\tau Y_\tau 1(\tau < \infty)],$$

where $\mathcal{S}$ is the set of all stopping times, i.e. to allow the optimal investment time to be any stopping time (we are also allowed to choose $\tau = \infty$, i.e. to choose never to stop), we need to modify the previous approach.

Recall that two measures $P$ and $Q$ are *locally equivalent* if

$$P|_{\mathcal{F}_t} \sim Q|_{\mathcal{F}_t},$$

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for every $t \geq 0$. We let $L_t$ denote the Radon-Nikodym derivative on $\mathcal{F}_t$:

$$L_t = \frac{dP}{dQ} \bigg|_{\mathcal{F}_t}.$$ 

If the martingale $(L_t)$ is a uniformly integrable martingale, then $P$ and $Q$ are equivalent. By defining

$$\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = M_t,$$

where $M$ is the strictly positive martingale in the multiplicative decomposition of $\Lambda$, we get a measure $Q$ that is at least locally equivalent to $P$. Since we only need to consider measure changes on $\mathcal{F}_\tau$, where $\tau$ is finite, we in fact get

$$E[\Lambda_\tau Y_\tau 1(\tau < \infty)] = E\left[e^{-\rho \tau} M_\tau Y_\tau 1(\tau < \infty)\right] = E^Q \left[e^{-\rho \tau} Y_\tau 1(\tau < \infty)\right].$$

In the second equality we used that

$$\frac{dQ}{dP} \bigg|_{\mathcal{F}_\tau} = M_\tau \text{ on } \{\tau < \infty\}$$

then the fact that $e^{-\rho \tau} Y_\tau 1(\tau < \infty)$ is an $\mathcal{F}_\tau$-measurable random variable. We can now conclude that

$$\sup_{\tau \in S} E[\Lambda_\tau Y_\tau 1(\tau < \infty)] = \sup_{\tau \in S} E^Q \left[e^{-\rho \tau} Y_\tau 1(\tau < \infty)\right],$$

where the measure $Q$ is locally equivalent to $P$ with Radon-Nikodym derivative $M_t$ on $\mathcal{F}_t$, $t \geq 0$.

**Remark 3.2** We have introduced the indicator function $1(\tau < \infty)$ in order to give the payoff 0 to any strategy where $\tau = \infty$. More generally, we could let the payoff we get when $\tau = \infty$ be defined as $\lim \sup_{t \to \infty} \Lambda_t Y_t$. In some applications this limit is equal to 0, in which case the indicator function can be dropped from the expected value.

In order to be able to calculate

$$\sup_{\tau \in S} E^Q \left[e^{-\rho \tau} Y_\tau 1(\tau < \infty)\right],$$

we need the dynamics of $Y$ under $Q$. Here we very briefly describe how the dynamics of $Y$ changes as we move from $P$ to $Q$. We refer to e.g. Jeanblanc et. al. [6] or Jacod & Shiryaev [5] for more on this, and especially for the definition of the brackets $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$. We let $Z$ be the negative of the stochastic logarithm of $M$: $Z = -\mathcal{L}(M)$. It follows from Corollary 8.7 b) in Jacod & Shiryaev [5] that

$$M = \mathcal{E}(-Z).$$
Since $M$ is a local martingale and $Z$ is defined as a stochastic integral with respect to $M$, $Z$ is also a local martingale. The cash flow process $Y$ is assumed to be a special semimartingale and we let the semimartingale decomposition be

$$Y_t = Y_0 + A_t + N_t,$$

where $A$ is a predictable process of finite variation and $N$ is a local martingale, and such that $[Y, M]$ is locally integrable under $P$, then the decomposition under $Q$ is given by

$$Y_t = Y_0 + A_t + \int_0^t \frac{d[Y, M]_s}{M_s} + N^Q_t,$$

where $N^Q$ is a $Q$-local martingale. Since $dM_t = -M_t - dZ_t$, we can write

$$Y_t = Y_0 + A_t - \langle Y, Z \rangle_t + N^Q_t.$$

**Remark 3.3** The conditions that $Y$ is a special semimartingale, rather than only a semimartingale, and that $[Y, M]$ is locally integrable, in order for $\langle Y, M \rangle$ to exist, is always fulfilled if $Y$ is continuous. Hence, these technical conditions are imposed in order to derive the dynamics when we allow for jumps in $Y$.

### 4 Summary

In order to be able to use a martingale to change measure on $\mathcal{F}_\infty$, the martingale needs to be uniformly integrable. To get the equality

$$E [\Lambda_t Y_\tau 1(\tau < \infty)] = E^Q [e^{-\rho \tau} Y_\tau 1(\tau < \infty)],$$

however, it is enough for $P$ and $Q$ to be locally equivalent, which they are when $M$ is strictly positive. The distinction between equivalent and locally equivalent measures is important, since in many financial model the martingale $M$ in the multiplicative decomposition of a stochastic discount factor is not be uniformly integrable even when $(P, \Lambda)$ is good.

### References


