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# Representations of Finite-Dimensional Algebras and Gabriel's Theorem

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# REPRESENTATIONS OF FINITE-DIMENSIONAL ALGEBRAS AND GABRIEL'S THEOREM

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ABSTRACT. In representations theory we try to understand an algebra $A$ by studying the modules over $A$ instead. One class of algebras that are easy to study are path algebras, because of their combinatorial structure. From the module-theoretic point of view, representation-finite algebras are also easy to study. Gabriel's theorem classifies all path algebras that are representation-finite by Dynkin graphs, giving a class of algebras which is very well understood. In this thesis we will give a description of the above and present a part of a proof of Gabriel's theorem.

## Contents

# 1. Introduction

Representation theory is the study of algebraic structures by representing their elements as linear transformations of vector spaces. This enables us to translate problems in abstract algebra to problems in linear algebra, which is a well understood subject [2]. One algebraic structure of interest are algebras. An algebra is a ring with a compatible vector space structure, for example the field of real numbers and the ring of quaternions are algebras over $\mathbb{R}$.

Quivers, which are simply directed graphs, give rise to a class of algebras called path algebras. Path algebras are the algebras this thesis will focus on. Another algebraic structure that is of interest are modules. A module can be thought of as a vector space over an algebra, being the generalisation of a vector space over a field. One can view an algebra as a module by letting the underlying vector space of the algebra act on itself. From this we can study algebras with the tools of modules. In this thesis we will focus on finite-dimensional modules. An important class of modules are indecomposable modules which are the most basic modules, in the sense that each other module can be constructed using indecomposable modules. If an algebra admits finitely many indecomposable modules up to isomorphism, then we say that it is representation-finite.

Gabriel's theorem is a theorem that classifies representaion-finite path algebras. The classification is done through Dynkin graphs, which is a specific class of graphs that appears in many areas of mathematics (see also Definition 6.1). First, we assign to every quiver $Q$ a quadratic form $\mathbb{Z}^n \to \mathbb{Z}$. Then, a positive root of $Q$ is a nonzero vector in $\mathbb{Z}^n$ with nonnegative coordinates such that the evaluation of the quadratic form at it is equal to one. Gabriel's theorem firstly tells us that if the underlying graph of $Q$ is Dynkin, then there exists a bijection between indecomposable modules over that path algebra and positive roots of $Q$. Secondly it tells us that representation-finite path algebras are precisely those quivers whose underlying graphs are Dynkin [1].

The aim of this thesis is to provide relevant definitions and prove some necessary results to be able to understand a part of a proof of Gabriel's theorem. The target audience for this thesis are those who are familiar with abstract algebra, linear algebra, elementary graph theory and some topology.

This thesis is based [1] and lecture notes by William Crawley-Boevey [4]. For more on the subject I would recommend [2] and [3].

**Acknowledgments.** I want to thank my supervisor Laertis Vaso for proposing an interesting subject for me to work with. I am truly grateful for all the time you spent clarifying and explaining things to me in a such a way that I could comprehend it and proceed with learning new things.

## 2. Algebras

We begin this section by fixing an algebraically closed field $K$.

**Definition 2.1.** Let $A$ be an associative ring with identity. We say that $A$ is an associative ***K*-algebra** if $A$ is a vector space over $K$ with compatible scalar multiplication, meaning that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $a, b \in A$ and $\lambda \in K$.

From now on we will simply write *algebra* when we mean some associative $K$-algebra.

**Definition 2.2.** Let $A$ be an algebra. The **dimension** of $A$ is the dimension of $A$ as a $K$-vector space, denoted by $\dim_K A$. We say that $A$ is **finite-dimensional** if $\dim_K A$ is finite.

In algebra, we are interested in studying some objects with some structure as well as maps between these objects that preserve that structure. The following class of maps is the relevant one for algebras.

**Definition 2.3.** Let $A$ and $B$ be two algebras, then a ring homomorphism $f : A \to B$ is a ***K*-algebra homomorphism** if $f$ is a $K$-linear map. If $f$ is bijective we say that $f$ is a ***K*-algebra isomorphism**, $A$ and $B$ are **isomorphic**, and we denote this by $A \cong B$.

**Definition 2.4.** Let $A$ be an algebra and let $e \in A$. We say that $e$ is **idempotent** if $e^2 = e$. If $ea = ae$ for all $a \in A$ we say that $e$ is **central**. Two idempotents $e_1, e_2 \in A$ are **orthogonal** if $e_1e_2 = e_2e_1 = 0$. If $e$ can not be written as a sum of nonzero orthogonal idempotents of $A$ we say that $e$ is **primitive**.

The idempotents of an algebra $A$ are important, among other things, in the study of indecomposable modules. We will define what an indecomposable module is in the next section.

## 3. Modules

We begin this section by fixing a $K$-algebra $A$. Similarly to defining vector spaces over some field $K$, we can define similar structure over an algebra $A$ called an $A$-module.

**Definition 3.1.** A **right *A*-module** over $A$ is a pair $(M, \cdot)$ where $M$ is a $K$-vector space with a binary operation $\cdot : M \times A \to M$, $(m, a) \mapsto ma$ such that the following conditions hold:

i) $(x + y)a = xa + ya$
ii) $x(a + b) = xa + xb$
iii) $x(ab) = (xa)b$
iv) $x1 = x$
v) $(x\lambda)a = x(a\lambda) = (xa)\lambda$

for all $x, y \in M$, $a, b \in A$ and $\lambda \in K$.

The definition and notation of left modules are analogous. We will denote right modules by $M_A$ or sometimes simply by $M$ if we have already stated whether it is a right or left module. From now on we will assume, unless specified, that a module $M$ is a right $A$-module.

If $A$ is a $K$-algebra, we can view $A$ as a right $A$-module with the underlying $K$-vector space of $A$ as vector space and the underlying ring multiplication as the binary operation. Because of this we can use results about modules to study the algebra $A$.

**Definition 3.2.** Let $M$ be an $A$-module. The **dimension** of $M$ is the dimension of the underlying $K$-vector space of $M$, denoted $\dim_K M$. We say that $M$ is **finite-dimensional** if $\dim_K M$ is finite.

**Definition 3.3.** Let $M$ be an $A$-module. A **submodule** $N$ of $M$ is a subspace of $M$ such that $na \in N$ for all $n \in N$ and $a \in A$.

**Definition 3.4.** Let $M$ and $N$ be $A$-modules. An ***A*-module homomorphism** is a $K$-linear map $h : M \to N$ such that $h(ma) = h(m)a$ for all $m \in M$ and $a \in A$. The set of all such homomorphisms $h$ is denoted by $\mathbf{Hom}_{\boldsymbol{A}}\boldsymbol{(M,N)}$. The homomorphisms of $\mathrm{Hom}_A(M, M)$ are called **endomorphisms** of $M$ and the set of endomorphisms of $M$ is denoted $\mathbf{End}_{\boldsymbol{A}}\boldsymbol{(M)}$.

The set $\mathrm{Hom}_A(M, N)$ is a vector space with pointwise addition and scalar multiplication. Moreover, the vector space $\mathrm{End}_A(M)$ is an algebra with respect to composition of maps and the identity map on $M$ as identity.

**Definition 3.5.** Let $M$ and $N$ be two $A$-modules and $h : M \to N$ an $A$-module homomorphism. If $h$ is surjective or injective we say that $h$ is an **epimorphism** or **monomorphism** respectively. If $h$ is bijective we say that $h$ is an **isomorphism**. In this case we say that $M$ and $N$ are **isomorphic** and we denote this by $M \cong N$. If $M$ and $N$ are isomorphic, then we say that they belong to the same **isomorphism class**.

**Definition 3.6.** The **direct sum** of $A$-modules $M_1, \ldots M_n$ is the $K$-vector space direct sum $M_1 \oplus \cdots \oplus M_n$ equipped with a module structure defined by

$$(m_1, \ldots, m_n)a = (m_1 a, \ldots, m_n a)$$

for $a \in A$ and $m_i \in M_i$, $i = 1, \ldots, n$. If $M$ is a nonzero $A$-module which can not be written as a direct sum of nonzero $A$-modules, we say that $M$ is **indecomposable**.

**Lemma 3.7** (Fitting's Lemma)**.** *Let $A$ be an algebra and $M$ a finite-dimensional $A$-module. Let $\theta : M \to M$ be an $A$-module homomorphism. Then there exist $n \geq 1$ such that*

$$M = \ker(\theta^n) \oplus \mathrm{im}(\theta^n) \tag{1}$$

*as a direct sum of $A$-submodules of $M$.*

*Proof.* First we want to show that (1) holds for the $K$-vector space structure of $M$. Let $\theta : M \to M$ be an $A$-module homomorphism. Let $x \in \ker(\theta^i)$ for some positive integer i. Then we have that

$$\theta^{i+1}(x) = \theta(\theta^i(x)) = \theta(0) = 0$$

so $\ker(\theta^i) \subseteq \ker(\theta^{i+1})$. Now let $x \in \mathrm{im}(\theta^{i+1})$, then there exist $y \in M$ such that

$$x = \theta^{i+1}(y) = \theta^i(\theta(y))$$

and $\theta(y) \in M$ so we have that $x \in \mathrm{im}(\theta^i)$ and $\mathrm{im}(\theta^{i+1}) \subseteq \mathrm{im}(\theta^i)$.

Since $M$ is finite-dimensional the inclusions $\ker(\theta) \subseteq \ker(\theta^2) \subseteq \ker(\theta^2) \cdots \subseteq M$ and $\mathrm{im}(\theta) \supseteq \mathrm{im}(\theta^2) \supseteq \mathrm{im}(\theta^3) \cdots \supseteq M$ eventually becomes equalities, meaning that

$$\ker(\theta^{n_1}) = \ker(\theta^{n_1+1}) = \ker(\theta^{n_1+3}) = \dots$$
$$\mathrm{im}(\theta^{n_2}) = \mathrm{im}(\theta^{n_2+1}) = \mathrm{im}(\theta^{n_2+3}) = \dots$$

for some $n_1, n_2 \in \mathbb{N}_{>0}$.

Let $n = \max(n_1, n_2)$ and now we want to show that $\mathrm{im}(\theta^n) \cap \ker(\theta^n) = \{0\}$. Let $x \in \mathrm{im}(\theta^n) \cap \ker(\theta^n)$. Then we have that $\theta^n(x) = 0$ and $\theta^n(y) = x$ for some $y \in M$. Then

$$\theta^n(x) = \theta^n(\theta^n(y)) = \theta^{2n}(y) = 0$$

so that $y \in \ker(\theta^{2n})$. Since $\ker(\theta^{2n}) = \ker(\theta^n)$, we have $y \in \ker(\theta^n)$. Hence we have that $\theta^n(y) = \theta^{2n}(y) = 0$ and thus $x = 0$. By viewing $\theta : M \to M$ as a $K$-linear map, the Rank-Nullity Theorem tells us that $\dim_K(\ker(\theta^n)) + \dim_K(\mathrm{im}(\theta^n)) = \dim_K(M)$ and since $\ker(\theta^n) \cap \mathrm{im}(\theta^n) = \{0\}$, we have that

$$\begin{aligned}
\dim(\ker(\theta^n) + \mathrm{im}(\theta^n)) &= \dim_K(\ker(\theta^n)) + \dim_K(\mathrm{im}(\theta^n)) - \dim_K(\ker(\theta^n) \cap \mathrm{im}(\theta^n)) \\
&= \dim_K(\ker(\theta^n)) + \dim_K(\mathrm{im}(\theta^n)) - 0 \\
&= \dim_K(M).
\end{aligned}$$

Hence $M = \ker(\theta^n) \oplus \mathrm{im}(\theta^n)$ as $K$-vector spaces.

Lastly we need to check that $\ker(\theta^n)$ and $\mathrm{im}(\theta^n)$ are $A$-submodules. If we view $\theta : M \to M$ as a $K$-linear map then clearly $\theta^n : M \to M$ is an $A$-module homomorphism and $\ker(\theta^n)$ and $\mathrm{im}(\theta^n)$ are subspaces. Let $x \in \ker(\theta^n)$ and $a \in A$. Then we have that

$$0 = 0 \cdot a = \theta^n(x) \cdot a = \theta^n(xa) \in \ker(\theta^n)$$

and so $\ker(\theta^n) \subseteq M$. Similarly let $x \in \mathrm{im}(\theta^n)$. Then there exist some $y \in M$ such that $\theta^n(y) = x$ and we get that

$$xa = \theta^n(y)a = \theta^n(ya) \in \mathrm{im}(\theta^n).$$

So $\mathrm{im}(\theta^n) \subseteq M$. Hence we get that $M = \ker(\theta^n) \oplus \mathrm{im}(\theta^n)$ is a direct sum of $A$-submodules.

□

**Lemma 3.8.** *Let $A$ be a finite-dimensional algebra. The following are equivalent.*

- i) *The only idempotents of $A$ are 0 and 1.*
- ii) *If $a \in A$, then $a$ or $1 - a$ are invertible.*

*Proof.* Let $e \in A$ be an idempotent and we will show that $e = 1$ or $e = 0$. Assume first that for all $a \in A$ we have that $a$ or $1 - a$ is invertible. Then we have that $e = e^2$ which we can rewrite as $0 = e(1 - e)$. If $e$ is invertible, then we have that $(1 - e) = 0$ and so $e = 1$. Otherwise we have that $1 - e$ in invertible, and so $e = 0$.

Assume now that the only idempotents of $A$ are 0 and 1. We define the map

$$\begin{aligned}
\varphi_a \colon A &\to A, \\
x &\mapsto ax,
\end{aligned}$$

where $a \in A$. If we view $A$ as a right $A$-module and let $x, y \in A$ and $r, s \in K$. We have that

$$\varphi_a(x)r + \varphi_a(y)s = (ax)r + (ay)s = a(xr) + a(ys) = a(xr + ys) = \varphi_a(xr + ys),$$

and for all $b \in A$,

$$\varphi_a(x)b = (ax)b = a(xb) = \varphi_a(xb).$$

We have now shown that $\varphi_a$ is an $A$-module homomorphism and by Fitting's Lemma we have that,

$$A = \text{im}(\varphi_a^n) \oplus \ker(\varphi_a^n)$$

for some integer $n \geq 1$. We know that $1 \in \ker(\varphi_a^n) \oplus \text{im}(\varphi_a^n)$ which we can write as $1 = \varepsilon_1 + \varepsilon_2$ where $\varepsilon_1 \in \ker(\varphi_a^n)$ and $\varepsilon_2 \in \ker(\varphi_a^n)$. Multiplying $1 = \varepsilon_1 + \varepsilon_2$ with $\varepsilon_1$ from the right we get that $\varepsilon_1 = \varepsilon_1^2 + \varepsilon_2\varepsilon_1$. Rewritting this gives us $\varepsilon_1 - \varepsilon_1^2 = \varepsilon_2\varepsilon_1$. We then have that $\varepsilon_1 - \varepsilon_1^2 \in \ker(\varphi_a^n)$ and $\varepsilon_2\varepsilon_1 \in \text{im}(\varphi_a^n)$, which means that $\varepsilon_1 = \varepsilon_1^2$ and $\varepsilon_2\varepsilon_1 = 0$. Similarly multiplying $1 = \varepsilon_1 + \varepsilon_2$ with $\varepsilon_2$ from the right yeilds that $\varepsilon_2 = \varepsilon_2^2$ and $\varepsilon_1\varepsilon_2 = 0$. Hence $\varepsilon_1$ and $\varepsilon_2$ are idempotents. Then by assumption $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$ or vice versa.

If $\varepsilon_1 = 1$ then $1 \in \ker(\varphi_a^n)$, and so $\varphi_a^n(1) = a^n = 0$. We have that $(1 - a)(1 + a + \cdots + a^{n-1}) = 1 - a^n = 1$ and so $1 - a$ is invertible.

Now if $\varepsilon_2 = 1$, then $1 \in \text{im}(\varphi_a^n)$, so there exist some $b \in A$ such that $\varphi_a^n(b) = 1$. Then we have that $1 = a^n b = a(a^{n-1}b)$ so $a$ has a right inverse. If we define a map $\psi_a : A \to A$ by $\psi_a(x) = xa$ and view $A$ as a left $A$-module we similarly see that either $1 - a$ has an inverse or $a$ has a left inverse. Hence if $1 - a$ has no inverse, then $a$ has a left and right inverse. Let $c$ be the left inverse of $a$ and $d$ the right inverse, then

$$c = c \cdot 1 = c(ad) = (ca)d = 1 \cdot d = d$$

and thus $c = d$, being the inverse of $a$. Hence either $a$ or $1 - a$ is invertible.

□

As stated earlier, we can view an algebra $A$ as a module. Lemma 3.8 is only concerned about finite-dimensional algebras. However in the proof we view a finite-dimensional algebra $A$ as an $A$-module and we use the tools of modules to give results of the algebra structure of $A$.

The following definition comes from Lemma I.4.6 in [1]. The reason for this adjusted definition is that it is more convenient for us to use.

**Definition 3.9.** Let $A$ be a finite-dimensional algebra. We say that $A$ is **local** if $A$ satisfies any of the conditions of Lemma 3.8.

**Lemma 3.10.** *Let $A$ be a finite-dimensional algebra. An idempotent $e \in A$ is primitive if and only if $eAe$ is local.*

*Proof.* Let $e$ be an idempotent and assume that $eAe$ is local. Assume that $e$ is not primitive, that is for some nonzero orthogonal idempotents $e_1$ and $e_2$ of $A$ we can write $e = e_1 + e_2$. Then we have that,

$$e_1 e = e_1(e_1 + e_2) = e_1$$
$$ee_1 = (e_1 + e_2)e_1 = e_1$$
$$\Rightarrow e_1 = ee_1e$$

In a similar manner we get that $e_2 = ee_2e$. Both $e_1$ and $e_2$ are idempotents of $eAe$ but since we assumed $eAe$ to be local we have that $e_1 = 1$ and and $e_2 = 0$ or vice versa. This is a contradiction since we assumed $e_1$ and $e_2$ to be nonzero, so $e$ is primitive.

Let $e$ be a primitive idempotent of $A$. Assume that $a \in eAe$ is an idempotent. The identity of $eAe$ is $e$ since any element $b \in eAe$ can be written as $b = ebe$ and we have that $eb = e(ebe) = (e^2)be = ebe$ and similaly if we multiply from the right. Let $f = e - a$, then $f$ is an idempotent since,

$$f^2 = (e - a)^2 = e^2 - ea - ae + a^2 = e - 2a + a = e - a = f.$$

Similarly, $a = e - f$ is an idempotent. We then have that

$$af = a(e-a) = ae - a^2 = a - a = 0,$$
$$fa = (e-a)a = ea - a^2 = a - a = 0.$$

Hence $f$ and $a$ are orthogonal idempotents. We rewrite $f = e - a$ to $e = f + a$. Because $e$ is primitive then either $f = 0$ and $a = e$ or vice versa. Thus $eAe$ is local. □

**Definition 3.11.** Let $A$ be a finite-dimensional algebra and $\{e_1, \ldots, e_n\}$ a set of primitive orthogonal idempotents of A. We say that the set $\{e_1, \ldots, e_n\}$ is **complete** if

$$1 = \sum_{i=1}^{n} e_i$$

is the identity element of $A$.

**Lemma 3.12.** *Let $A$ be a finite-dimensional algebra and $M$ an $A$-module. If the algebra $End_A(M)$ is local, then $M$ is indecomposable.*

*Proof.* Assume that $\mathrm{End}_A(M)$ is local and that $M$ decomposes as $M \cong M_1 \oplus M_2$, where $M_1$ and $M_2$ are nonzero $A$-modules. Then we have the algebra

$$\begin{aligned} \mathrm{End}_A(M) &\cong \mathrm{End}_A(M_1 \oplus M_2) \\ &= \mathrm{Hom}_A(M_1 \oplus M_2, M_1 \oplus M_2) \\ &= \left\{ \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} : \varphi_{ij} \in \mathrm{Hom}_A(M_i, M_j) \right\} \end{aligned}$$

The identity of $\mathrm{End}_A(M)$ is

$$\begin{bmatrix} \mathrm{Id}_{M_1} & 0 \\ 0 & \mathrm{Id}_{M_2} \end{bmatrix} = \begin{bmatrix} \mathrm{Id}_{M_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{M_2} \end{bmatrix}$$

and we have that,

$$\begin{bmatrix} \mathrm{Id}_{M_1} & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} \mathrm{Id}_{M_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{M_2} \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{M_2} \end{bmatrix}. \tag{3}$$

Hence (1) and (2) are nonzero idempotents of $\mathrm{End}_A(M)$. This contradicts our assumption that $\mathrm{End}_A(M)$ is local and $M$ is therefore indecomposable. □

**Definition 3.13.** We say that a nonzero module $S$ is **simple** if any submodule of $S$ is either 0 or $S$.

**Proposition 3.14.** *If $S$ is a finite-dimensional simple module, then $S$ is indecomposable.*

*Proof.* Let $S$ be a finite-dimensional simple $A$-module. Assume that we can decompose $S$ as the direct sum $M = M_1 \oplus M_2$. Since $M$ is simple the only submodules of $S$ are $S$ or 0. If $M_1 = S$ then $S = S \oplus M_2$ and since $S$ is finite-dimensional it must be that $M_2 = 0$. Similarly if $M_2 = S$ then $M_1 = 0$. Therefore it must be that $M_1 = S$ or $M_2 = S$ and $S$ is hence indecomposable. □

**Lemma 3.15.** *Let $M$ be an $A$-module and* $\dim_A(M) = 1$*, then $M$ is simple.*

*Proof.* Let $M$ be a one dimensional $A$-module. Assume that $M$ can be written as the direct sum $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ are nonzero. Then $\dim_A(M_1) \geq 1$ and $\dim_A(M_2) \geq 1$. We then have that $\dim_A(M) = \dim_A(M_1) + \dim_A(M_2) \geq 2$. A contradiction since we assumed $M$ to be one dimensional. □

**Theorem 3.16** (Unique Decomposition Theorem)**.** *Let $A$ be a finite-dimensional algebra.*

i) *Every finite-dimensional $A$-module $M$ has a decomposition $M \cong \oplus_{i=1}^n M_i$, where each $M_i$ is an indecomposable module and the endomorphism algebra $End(M_i)$ is local for every $M_i$.*

ii) *If* $\oplus_{i=1}^{n} M_i \cong \oplus_{j=1}^{m} N_j$, *where* $M_i$ *and* $N_j$ *are indecomposable, then* $m = n$ *and there exists a permutation* $\sigma$ *of* $\{1, \dots n\}$ *such that* $M_i \cong N_{\sigma(i)}$ *for each* $i = 1, \dots, n$.

*Proof.* See pages 23-24 in [1]. □

**Definition 3.17.** Let $A$ be a finite-dimensional algebra, then $A$ is said to be **of finite representation type** or **representation-finite** if there are finitely many indecomposable finite-dimensional $A$-modules up to isomorphism.

Given an $A$-module $M$ over a finite-dimensional algebra $A$, the Unique Decomposition Theorem tells us that $M$ can be decomposed to a direct sum of indecomposable modules. If we then are given another $A$-module $N$, finding $\mathrm{Hom}_A(M, N)$ corresponds to finding all $A$-module homomorphisms between their indecomposable summands. Hence finding all $\mathrm{Hom}_A(M, N)$ comes down to find all homomorphisms between all indecomposable $A$-modules. If the algebra is representation-finite, there are finitely many indecomposable modules. Finding all such homomorphisms can be done in a finite amount of time. From this point of view, representation-finite algebras are very well understood and easy to study.

## 4. Quivers

4.1. **Basic quiver structure.** In general, being able to visualise objects makes it easier to understand them. This motivates the definition of quivers, which we now will use to represent algebras. Later on we will also use quivers to visualise how one can represent an $A$-module by $K$-vector spaces and linear maps between them.

**Definition 4.1.** An **undirected graph** $G$ is a triplet $(N, E, r)$ where $N$ is the set of **nodes**, $E$ is the set of **edges** and $r$ is a **relation** such that every edge $e \in E$ is associated to an unordered pair of nodes $(n, n') \in N \times N$ by $r$.

**Definition 4.2.** A **directed graph** $G$ is a triplet $(N, E, r)$ where $N$ is the set of **nodes**, $E$ is the set of **edges** and $r$ is a **relation** such that every edge $e \in E$ is associated to an ordered pair of nodes $(s(e), t(e)) \in N \times N$ for $e \in E$, where we call $s(e)$ and $t(e)$ for **source** node and **target** node of $e$ respectively. We say that the **direction** of $e$ is from node $s(e)$ to node $t(e)$.

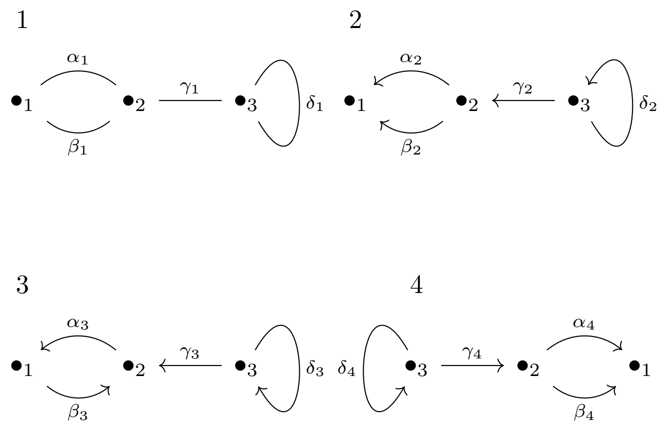
For a graph $G$, we note that the pair of nodes that is associated to an edge do not necessarily need to be distinct. There is no limitation to how many nodes a graph can have and how many edges there can be between a pair of nodes.

**Definition 4.3.** For a directed graph $G$ the **underlying graph** $\overline{G}$ is the undirected graph given by removing the directions of the arrows in $G$.

**Definition 4.4.** Let $G$ be an undirected graph and $a, b \in N$. A **walk** between nodes $a$ and $b$ is a sequence of edges $e_i$ for $i = 1, \ldots, \ell$ of $G$ such that $a$ and $b$ are associated by $r$ to $e_1$ and $e_\ell$ respectively and there is a node $m_i$ that is associated to $e_i$ and $e_{i+1}$ by r, for $i = 1, \ldots, \ell - 1$.

**Definition 4.5.** An undirected graph $G$ is **connected** if there exist a walk between any two nodes of $G$. A directed graph is **connected** if its underlying graph is connected.

**Example 4.6.** Here is an example of four different graphs.




Graphs 2, 3 and 4 are directed graphs with graph 1 as their underlying graph. Graphs 2 and 3 only differ in one arrow, namely $\beta_2$ and $\beta_3$. It might seem that arrows $\delta_2$ and $\delta_3$ differ, however this is not the case since they have the same source node and target node. Graphs 2 and 4 are the same since all arrows have the same source node and target node.

**Definition 4.7.** A **quiver** $Q = (Q_0, Q_1, s, t)$ is a directed graph where $Q_0$ is the set of nodes, $Q_1$ is the set of edges called **arrows** and $s, t$ are functions $Q_1 \to Q_0$ that maps an arrow to its source and target node. We say that $Q$ is **finite** if the sets of nodes and arrows are finite.

Note that there is no difference between directed graphs and quivers, besides the fact that we refer to edges as arrows in quivers. The reason for using the word quiver is to indicate that the work is related to representation theory, since directed graphs are related to many other areas in mathematics.

From now on all quivers will be finite and we will denote $Q_0 = \{1, 2, \dots, n\}$.

**Definition 4.8.** Let $Q$ be a quiver and $a, b \in Q_0$. A **path** of length $\ell \geq 1$ is a sequence of arrows $\alpha_i$ for $i = 1, \dots, \ell$ from source node $a$ to target node $b$ such that $t(\alpha_i) = s(\alpha_{i+1})$, $s(\alpha_1) = a$ and $t(\alpha_\ell) = b$. We denote such a path by $\alpha_1\alpha_2 \dots \alpha_\ell$. A path can be visualised as

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_\ell} a_\ell = b$$

For each node $a \in Q_0$ we associate a path of length 0 which we call the **trivial path** of $a$, denoted by $\varepsilon_a$. We will let $Q_\ell$ denote the set of paths in a quiver $Q$ with length $\ell$.

The notation $Q_\ell$ is compatible with the set of nodes $Q_0$ and the set of arrows $Q_1$ since all paths with length 0 are the nodes and all paths with length 1 are the set of all arrows.

**Definition 4.9.** A quiver $Q$ is **acyclic** if the only path for which the target node and source node coincide is the trivial path for all paths in $Q$.

**Remark 4.10.** An undirected graph is acyclic if there exist no walk with the same source node and target node. When talking about acyclic graphs it is an important distinction whether we mean quivers or undirected graphs. For example the underlying graph for an acyclic quiver can be cyclic, as shown below.

4.2. **Path algebras.**

**Definition 4.11.** Let $Q$ be a quiver. The **path algebra** $KQ$ of $Q$ is the algebra with the $K$-vector space whose basis is given by the set $Q_\ell$, for $\ell \geq 0$. We define the product of two basis vectors as the concatenation of two paths in $Q$. That is for the vectors with corresponding paths $(a|\alpha_1, \dots, \alpha_\ell|b)$ and $(c|\beta_1, \dots, \beta_k|d)$ their product is,

$$(a|\alpha_1, \dots, \alpha_\ell|b)(c|\beta_1, \dots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_k|d)$$

where $\delta_{bc}$ denotes the Kronecker delta.

Path algebras are defined combinatorially and so are easy to study. They provide many examples of algebras. If $KQ$ is the path algebra of a quiver $Q$, then $\varepsilon_i KQ \varepsilon_j$ is the set of elements in $KQ$ generated by the paths in $Q$ with source node $i$ and target node $j$. Notice that $\varepsilon_i KQ \varepsilon_j$ is a subspace of $KQ$ and if $i = j$ it is also an $K$-algebra.

**Theorem 4.12.** *Let $Q$ be a finite quiver and $KQ$ the corresponding path algebra. Assume that $KQ$ is finite-dimensional. The set $\{\varepsilon_i \mid i \in Q_0\}$ of trivial paths in $Q$ is a complete set of primitive orthogonal idempotents of $KQ$ and the identity element of $KQ$ is given by*

$$1 = \sum_{i=1}^{n} \varepsilon_i. \tag{4}$$

*Proof.* Let us first show that the elements in the set $\{\varepsilon_i \mid i \in Q_0\}$ are idempotents and orthogonal. Let $i$ and $j$ be two nodes in $Q$ with corresponding trivial paths $\varepsilon_i$ and $\varepsilon_j$. Since the product of elements in $KQ$ is defined as concatenation of paths in $Q$ we have that $\varepsilon_i^2 = \varepsilon_i$ and $\varepsilon_i\varepsilon_j = 0$ if $i \neq j$. Hence $\varepsilon_i$ and $\varepsilon_j$ are orthogonal idempotent of $KQ$.

Let us now show that the elements in $\{\varepsilon_i \mid i \in Q_0\}$ are primitive. Let $\varepsilon_i$ be an arbitrary idempotent of the set $\{\varepsilon_i \mid i \in Q_0\}$. By Lemma 3.10 it is enough to show that $\varepsilon_i KQ \varepsilon_i$ is local. Assume that $e$ is an idempotent of $\varepsilon_i KQ \varepsilon_i$. Then we have that $e^2 = e = e\varepsilon_i$ which gives us that $e(e - \varepsilon_i) = 0$. Since $e$ and $e - \varepsilon_i$ are linear combinations of elements in $\varepsilon_i KQ \varepsilon_i$ we can express them as sums

$$e = \sum_{\rho \in \varepsilon_i KQ \varepsilon_i} \lambda_\rho \rho,$$
$$e - \varepsilon_i = \sum_{\gamma \in \varepsilon_i KQ \varepsilon_i} \lambda_\gamma \gamma.$$

Take the longest nonzero path $\rho'$ and $\gamma'$ appearing in $e$ and $e - \varepsilon_i$ respectively. In particular we have that $\lambda_{\rho'} \neq 0$ and $\lambda_{\gamma'} \neq 0$. We can express $e(e - \varepsilon_i)$ as,

$$0 = e(e - \varepsilon_i) = \left( \sum_{\rho \neq \rho'} \lambda_\rho \rho \right) \left( \sum_{\gamma \neq \gamma'} \lambda_\gamma \gamma \right) + \sum_{\rho \neq \rho'} \lambda_\rho \rho \lambda_{\gamma'} \gamma' + \sum_{\gamma \neq \gamma'} \lambda_\gamma \gamma \lambda_{\rho'} \rho' \tag{5}$$
$$+ \lambda_{\rho'} \lambda_{\gamma'} \rho' \gamma'. \tag{6}$$

Since $\rho'\gamma'$ does not appear as a summand on the right-hand side of row (5), we have that $\lambda_{\rho'}\lambda_{\gamma'} = 0$, a contradiction since we assumed $\lambda_{\gamma'}\gamma'$ and $\lambda_{\rho'}\rho'$ to be nonzero and so $\varepsilon_i KQ \varepsilon_i$ is local.

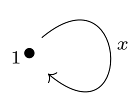
Lastly let us show (4). Assume that $\varepsilon = \sum_{i=1}^n \varepsilon_i$ and $\rho$ a path in $KQ$.

$$\begin{aligned} \rho\varepsilon &= \rho \sum_{i=1}^{n} \varepsilon_i \\ &= \rho \varepsilon_{t(\rho)} + \rho \sum_{i \neq t(\rho)} \varepsilon_i & \left[ \rho \sum_{i \neq t(\rho)} \varepsilon_i = 0 \right] \\ &= \rho \end{aligned}$$

Showing that $\varepsilon\rho = \rho$ is done in a similar manner. Since $x \in KQ$ is a linear combination of paths, we have that $1 = \sum_{i=1}^n \varepsilon_i$ is the identity of $KQ$.

□

**Example 4.13.** The quiver $Q$ below is a quiver consisting of a single node with a looped arrow. Let $\varepsilon_1$ denote the trivial path of the single node in the quiver. Note that $x\varepsilon_1 = \varepsilon_1 x = x$. Then the set of all paths of this quiver is $\{\varepsilon_1, x, x^2, \dots, x^\ell, \dots\}$ which is the basis of the underlying vector space of $KQ$. This vector space is infinite-dimensional and so $KQ$ is an infinite-dimensional algebra. In fact $KQ \cong K[t]$, that is $KQ$ is isomorphic to the polynomial vector space with coefficients in $K$. The isomorphism is given by mapping $\varepsilon_1 \mapsto 1$ and $x^i \mapsto t^i$.




**Example 4.14.** Let $Q$ be the quiver pictured below and let the set $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be the set of trivial paths of $Q$. The set of arrows of $Q$ are $\{\alpha, \beta\}$ and by concatenation we can get the set of all possible paths for this quiver. The set of all possible paths is $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \beta, \alpha\beta\}$ given by the multiplication table shown below. The path algebra $KQ$ of this quiver has as vector space basis the set of paths of $Q$, i.e. $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \beta, \alpha\beta\}$ is the basis set of the underlying vector space of $KQ$. Since the basis set is finite, this is a finite-dimensional algebra.

$$\bullet_1 \xleftarrow{\beta} \bullet_2 \xleftarrow{\alpha} \bullet_3$$

| $*$ | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\alpha$ | $\beta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|
| $\varepsilon_1$ | $\varepsilon_1$ | 0 | 0 | 0 | 0 | 0 |
| $\varepsilon_2$ | 0 | $\varepsilon_2$ | 0 | 0 | $\beta$ | 0 |
| $\varepsilon_3$ | 0 | 0 | $\varepsilon_3$ | $\alpha$ | 0 | $\alpha\beta$ |
| $\alpha$ | 0 | $\alpha$ | 0 | 0 | $\alpha\beta$ | 0 |
| $\beta$ | $\beta$ | 0 | 0 | 0 | 0 | 0 |
| $\alpha\beta$ | $\alpha\beta$ | 0 | 0 | 0 | 0 | 0 |

In fact $KQ$ is isomorphic to the lower triangular $3 \times 3$ matrix algebra

$$KQ \cong \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix},$$

and the isomorphism is given by the mappings shown below.

$$\varepsilon_1 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_2 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon_3 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \alpha\beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

**Theorem 4.15.** *Let $Q$ be a finite quiver and $KQ$ the corresponding path algebra, then $Q$ is acyclic if and only if $KQ$ is finite-dimensional.*

*Proof.* Let $Q$ be a cyclic quiver and assume to a contradiction that $KQ$ is finite-dimensional. Then there exist a path $\rho$ in $Q$ such that $s(\rho) = t(\rho)$. Then the set of all paths contains the set of paths $\{\rho, \rho^2, \rho^3, \dots, \rho^l, \dots\}$ which is an infinite set. Since this set is a subset of paths of all paths, then the set of paths is also infinite. Since the basis of the vector space of $KQ$ is given by the set of all paths in $Q$, it follows that $KQ$ is infinite-dimensional.

Let $KQ$ be finite-dimensional, then the underlying vector space has a finite basis given by all possible paths in $Q$. Then no path $\gamma$ in $Q$ with length $\ell \geq 1$ can have a source node and target node which coincide since if that was the case then the paths $\gamma, \gamma^2, \dots$ would be paths in $Q$ and thus basis vectors of the underlying vector space of $KQ$. But the set $\{\gamma, \gamma^2, \dots\}$ is infinite while $KQ$ is finite-dimensional, so $\gamma, \gamma^2, \dots$ can not be linearly independent, contradicting the assumption that $\gamma$ has length $\ell \geq 1$. □

We want to be able to classify the finite-dimensional path algebras that are representation-finite. Theorem 4.15 to study path algebras with quivers that are acyclic.

## 5. Representations

In this section we define representations of a quiver $Q$, which we will later see that it can be seen as modules over the path algebra $KQ$.

**Definition 5.1.** Let $Q$ be a quiver. A **representation** $M$ of $Q$ is defined by the following data:

i) Each node $i \in Q_0$ is associated to a $K$-vector space $M_i$.
ii) Each arrow $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$ in $Q_0$, is associated to a $K$-linear map $\varphi_\alpha : M_i \to M_j$.

We denote the representation by $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and we say that $M$ is **finite-dimensional** if each vector space $M_i$ associated to each $i \in Q_0$ is finite-dimensional.

**Definition 5.2.** Let $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of a quiver $Q$. A **morphism of representations** is a family of $K$-linear maps $f = (f_i)_{i \in Q_0}$ that is compatible with the structure maps $\varphi_\alpha$, i.e. for each $f_i$ we have that $f_i : M_i \to N_i$ is a $K$-linear map and $\psi_\alpha f_i = f_j \varphi_\alpha$. This can be visualised by the following commutative diagram.

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_j} \\ N_i & \xrightarrow{\psi_\alpha} & N_j \end{array}$$

**Definition 5.3.** Let $Q$ be a quiver and $M$ a finite-dimensional representation of $Q$. We define the **dimension vector** of $M$ to be the vector,

$$\underline{\dim} M = \big(M_1, \dots, M_n\big)^t$$

in $\mathbb{Z}^n$.

**Example 5.4.** Let $Q$ be the quiver

$$\bullet_1 \longleftarrow \bullet_2 \longleftarrow \bullet_3.$$

Some representations of $Q$ with corresponding dimension vector are the following.

$$M_1 = K \xleftarrow{0} 0 \xleftarrow{0} 0 \qquad \underline{\dim}(M_1) = (1,0,0)$$

$$M_2 = 0 \xleftarrow{0} K \xleftarrow{0} 0 \qquad \underline{\dim}(M_2) = (0,1,0)$$

$$M_3 = 0 \xleftarrow{0} 0 \xleftarrow{0} K \qquad \underline{\dim}(M_3) = (0,0,1)$$

$$M_4 = K \xleftarrow{1} K \xleftarrow{0} 0 \qquad \underline{\dim}(M_4) = (1,1,0)$$

$$M_5 = 0 \xleftarrow{0} K \xleftarrow{1} K \qquad \underline{\dim}(M_5) = (0,1,1)$$

$$M_6 = K \xleftarrow{1} K \xleftarrow{1} K \qquad \underline{\dim}(M_6) = (1,1,1).$$

**Theorem 5.5.** *Let $Q$ be a finite, connected and acyclic quiver. Then there exists a bijection between finite-dimensional representations of $Q$ and finite-dimensional $KQ$ modules up to isomorphism. Moreover, there also exist a bijection between their homomorphisms that respects composition of homomorphisms.*

*Proof.* Proving Theorem 5.5 is out of the scope for this thesis. The bijection on modules and representations is defined in the following way.

Let $Q$ be any finite quiver with vertices $Q_0 = \{1, \dots, n\}$
Let $\mathcal{M}$ be a representation of $Q$. We want to define a $KQ$-module $M$ corresponding to $\mathcal{M}$. As a vector space, we set

$$M = \oplus_{i=1}^{n} M_i.$$

For the $KQ$-multiplication, we first denote the canonical inclusion and projective mappings

$$M_i \xrightarrow{\iota_i} M \xrightarrow{\pi_i} M_i.$$

The binary operation is given by

$$x\rho = \iota_{t(\rho)}\varphi_\rho\pi_{s(\rho)}(x),$$

where $\rho$ is a path in $Q$, moreover we let $\varphi_{\varepsilon_i} = 1$ for any node $i$ in $Q_0$. We extend the above definition linearly to elements of $KQ$.

Given a $KQ$-module $M$ we define a representation $\mathcal{M}$ by

$$\begin{aligned} &M_i = M\varepsilon_i \\ &\varphi_\alpha(x) = x\alpha = x\alpha e_t(\alpha) \in M_{t(\alpha)} \end{aligned}$$

for all $i \in Q_0$ and $\alpha \in Q_1$.

For a detailed proof, see pages 72-74 in [1]. □

Theorem 5.5 is important because given a finite, connected and acyclic quiver $Q$ it allows us to talk about finite-dimensional representation of a quiver $Q$ and finite-dimensional modules of the path algebra $KQ$ as if they are the same. This means that we are able to translate problems given in the world of finite-dimensional $KQ$-modules to the world of finite-dimensional representations of Q. For example we can decide if a representation $\mathcal{M}$ is indecomposable by determining if the corresponding module $M$ is indecomposable. Let us give an example of Theorem 5.5.

**Example 5.6.** Let $Q$ be the quiver

$$\bullet_1 \xleftarrow{\beta} \bullet_2 \xleftarrow{\alpha} \bullet_3.$$

Let $\mathcal{M}$ be a representation of $Q$ given by $M_1 = M_2 = M_3 = K$, $\varphi_\alpha = 1$ and $\varphi_\beta = -1$, that is

$$\mathcal{M} = K \xleftarrow{-1} K \xleftarrow{1} K.$$

Given the representation $\mathcal{M}$ we have that the vector space for the corresponding $KQ$-module $M$ is $K^3 = K \oplus K \oplus K$. A basis of $M$ is given by the vectors $x = (1, 0, 0)$, $y = (0, 1, 0)$ and $z = (0, 0, 1)$. Since elements in $M$ are $K$-linear combinations of $x$, $y$ and $z$, it is enough to show how the elements of $KQ$ act on the basis elements. From $\mathcal{M}$ we have that $\varphi_\alpha = 1$ and $\varphi_\beta = -1$. If we let $\alpha$ act on $z$ we get,

$$\begin{aligned} z\alpha &= \iota_2 \circ \varphi_\alpha \circ \pi_3(z) = \iota_2 \circ \varphi_\alpha \circ \pi_3((0, 0, 1)) \\ &= \iota_2 \circ \varphi_\alpha(1) = \iota_2(1) = (0, 1, 0) = y. \end{aligned}$$

The actions on the elements is given by the table.

| * | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\alpha$ | $\beta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|
| $x$ | $x$ | 0 | 0 | 0 | 0 | 0 |
| $y$ | 0 | $y$ | 0 | 0 | $-x$ | 0 |
| $z$ | 0 | 0 | $z$ | $y$ | 0 | $-x$ |

Now let $M = KQ$ and we will find the corresponding representation $\mathcal{M}$. Then the vector spaces of $\mathcal{M}$ are

$$M\varepsilon_1 \longleftarrow M\varepsilon_2 \longleftarrow M\varepsilon_3$$

for which the basis of each vector space $M\varepsilon_i$ is

$$\begin{aligned} M_1 &= KQ\varepsilon_1 = \operatorname{span}\{\varepsilon_1, \beta, \alpha\beta\} \\ M_2 &= KQ\varepsilon_2 = \operatorname{span}\{\varepsilon_2, \alpha\} \\ M_3 &= KQ\varepsilon_3 = \operatorname{span}\{\varepsilon_3\}. \end{aligned}$$

The linear maps $\varphi_\alpha$ and $\varphi_\beta$ for $\mathcal{M}$ are defined by

$$\begin{aligned} \varphi_\alpha(\varepsilon_3) &= \varepsilon_3\alpha = \alpha \\ \varphi_\beta(\varepsilon_2) &= \varepsilon_2\beta = \beta \\ \varphi_\beta(\alpha) &= \alpha\beta. \end{aligned}$$

We then that the representation $\mathcal{M}$ is given by

$$K^3 \xleftarrow[\varphi_\beta]{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} K^2 \xleftarrow[\varphi_\alpha]{\begin{bmatrix} 0 & 1 \end{bmatrix}} K.$$

We will give one example how Theorem 5.5 is used in the translation of solving problems about finite-dimensional modules over path algebras to their corresponding quiver representations.

**Example 5.7.** We claim that all representations of Example 5.4 are indecomposable. We will first look at the representations $M_1$, $M_2$ and $M_3$. These representations are one dimensional and thus indecomposable, by Lemma 3.15.

For the rest, we will only show $M_6$. Showing the others is done in a similar manner. By Lemma 3.12 and Theorem 5.5 it is enough to show that the endomorphism algebra of each representation is local. We will first compute the endomorphism algebra of $M_6$:

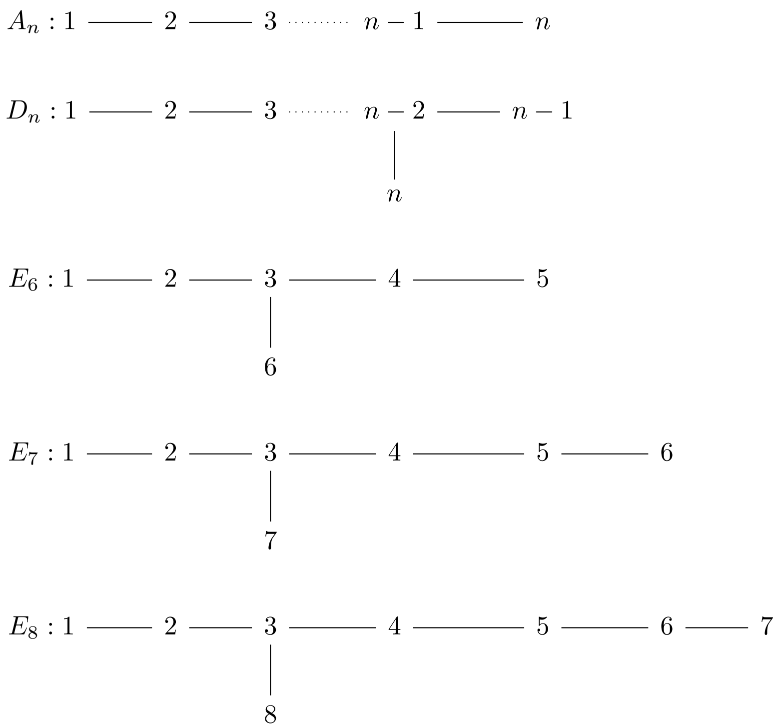
$$\begin{array}{ccccc} K & \xleftarrow{1} & K & \xleftarrow{1} & K \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ K & \xleftarrow{1} & K & \xleftarrow{1} & K \end{array}$$

We have that $f_1 \circ 1 = 1 \circ f_2$ and $f_2 \circ 1 = 1 \circ f_3$ which implies that $f_1 = f_2 = f_3$. But $f_1$ is a linear map $f_1 : K \to K$, hence $f_1 = \lambda_1$ is a scalar in $K$. It follows that $\operatorname{End}_A(M_6) \cong K$ which is a local algebra since it is field.

## 6. Dynkin

### 6.1. Dynkin graphs.

**Definition 6.1.** The ADE Dynkin graphs are the following:




where $n \geq 1$ for $A_n$ and $n \geq 4$ for $D_n$.

From now on we we will simply refer to the ADE Dynkin graphs as Dynkin graphs.

### 6.2. Roots and quadratic forms.

**Definition 6.2.** The **Tits form** of a quiver $Q$ is a quadratic form on $\mathbb{Z}^n$ defined by

$$q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

If it is clear from the context which quiver $Q$ we mean we will simply write $q$ instead of $q_Q$. Note that the Tits form $q$ of a given quiver $Q$ does not depend on the orientation of the arrows. Thus $q$ is equal for all quivers $Q$ which have the same underlying graph $\overline{Q}$. Note also that if we extend $q_Q$ to $\mathbb{R}$ then if $\lambda$ is some scalar in $\mathbb{R}$ and $x \in \mathbb{Z}^n$, then $q(\lambda x) = \lambda^2 q(x)$ which one can see directly from the definition.

**Definition 6.3.** Let $Q$ be a quiver with Tits form $q$. A **root** of $q$ is a vector $x \in \mathbb{Z}^n$ such that $q(x) = 1$. We say that a root $x$ is **positive** if $x \neq 0$ and $x_i \geq 0$ for $i = 1, \ldots, n$.

**Example 6.4.** Let $Q$ be the quiver pictured below.

$$\bullet_1 \longleftarrow \bullet_2 \longleftarrow \bullet_3$$

Then the Tits form of $Q$ is given by

$$\begin{aligned} q_Q(x) &= x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 \\ &= \frac{x_1^2 + x_3^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2}{2} \end{aligned}$$

We can clearly see that the positive roots are $(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)$ and $(0, 1, 1)$.

The observant reader will realise that the quiver in Example 5.4 and 6.4 are the same and that the dimension vectors of the representations in 5.4 coincide with the positive roots. Moreover, all representations of Example 5.4 are indecomposable by Example 5.7. There does indeed exist a one to one correspondence between the dimension vectors of indecomposable representations of $Q$ and positive roots of quivers with underlying graph Dynkin. This is what is stated by Gabriel's Theorem I [4, Theorem 1 page 19].

**Theorem 6.5** (Gabriel's Theorem I)**.** *Let $Q$ be a finite, connected and acyclic quiver; and $\overline{Q}$ is the underlying graph of $Q$ If $\overline{Q}$ is Dynkin, then the mapping $\underline{\dim} : M \mapsto \underline{\dim} M$ induces a bijection between the set of isoclasses of indecomposable modules and the positive roots of $q$.*

**Definition 6.6.** Let $q$ be the Tits form of a quiver $Q$. We call $q$ **weakly positive** if $q(x) > 0$ for all $x > 0$. We call $q$ **positive definite** if $q(x) > 0$ for all $x \neq 0$.

**Proposition 6.7.** *Let $Q$ be a quiver. If $\overline{Q}$ is Dynkin, then $q$ is positive definite.*

*Proof.* Assume $\overline{Q}$ is Dynkin, then the following are the quadratic forms $q$ of all Dynkin diagrams:

$$\begin{aligned} q_{A_n}(x) =& \frac{x_1^2 + x_n^2 + \sum_{i=1}^{n-1}(x_i - x_{i+1})^2}{2} \\ q_{D_n}(x) =& \frac{2x_1^2 + (x_{n-2} - 2x_n)^2 + (x_{n-2} - 2x_{n-1})^2 + 2\sum_{i=1}^{n-3}(x_i - x_{i+1})^2}{4} \\ q_{E_6}(x) =& \frac{(3x_6)^2 + (6x_1 - 3x_2)^2 + (6x_5 - 3x_4)^2 + 3\sum_{i=1}^{3}(3x_{2i} - 2x_3)^2}{36} \\ q_{E_7}(x) =& \frac{6(x_1^2 + (2x_6 - x_5)^2 + (2x_7 - x_3)^2) + (4x_2 - 3x_3)^2}{24} \\ &+ \frac{(4x_4 - 3x_3)^2 + 2((3x_1 - 2x_2)^2 + (3x_5 - 2x_4)^2)}{24} \\ q_{E_8}(x) =& \frac{30(x_7^2 + (2x_1 - x_2)^2 + (2x_8 - x_3)^2) + 2(6x_4 - 5x_3)^2}{120} \\ &+ \frac{10((3x_2 - 2x_3)^2 + (3x_7 - 2x_6)^2) + 3(5x_5 - 4x_4)^2}{120} \\ &+ \frac{5(4x_6 - 3x_5)^2}{120}. \end{aligned}$$

For each $q$, every summand is positive since they are a sum of squares. Moreover, $q = 0$ if and only if $x = 0$. This can be seen if one realises that every monomial $x_i$ must be equal to 0, and from there see how each other squared term must be equal to 0. Hence $q$ is positive definite. □

We will now proceed by giving an idea how to get each Tits form of every Dynkin graph as in the proof above.

**Example 6.8.** To get the Tits form of the Dynkin graphs as in the proof of Proposition 6.7 one can view the graphs $D_n$,$E_6$, $E_7$ and $E_8$ as modified versions of $A_n$. We will only show how to find $q_{D_n}$ by viewing the graph $D_n$ as the graph $A_{n-1}$ with a node $n$ associated to node $n - 2$ by an edge.

Notice that in the proof of Proposition 6.7 each term in $\sum_{\alpha \in Q_1} x_{s(\alpha)}x_{t(\alpha)}$ is contained in squared terms. So the monomial corresponding to the edge between node $n - 2$ and $n$ will be contained in a squared term of the form $(ax_{n-2} - bx_n)^2$ for some integers $a$ and $b$.

Firstly we want to expose the monomials associated to node $n-2$ in $q_{A_{n-1}}$.

$$\begin{aligned}
q_{A_{n-1}}(x) =& \frac{x_1^2 + x_{n-1}^2 + \sum_{i=1}^{n-2}(x_i - x_{i+1})^2}{2} \\
=& \frac{x_1^2 + x_{n-1}^2 + (x_{n-3} - x_{n-2})^2 + (x_{n-2} - x_{n-1})^2}{2} \\
&+ \frac{\sum_{i=1}^{n-4}(x_i - x_{i+1})^2}{2} \\
=& \frac{x_1^2 + 2x_{n-1}^2 + 2x_{n-2}^2 + x_{n-3}^2 - 2x_{n-3}x_{n-2} - 2x_{n-2}x_{n-1}}{2} \\
&+ \frac{\sum_{i=1}^{n-4}(x_i - x_{i+1})^2}{2}
\end{aligned}$$

Secondly we would want to have $x_{n-2}^2$ exposed but the other exposed terms to be squared again. This is not possible by the form of $q_{A_{n-1}}$ given above. However if we multiply the numerator and denominator of $q_{A_n-1}$ by 2 and then we complete the squares, we have

$$\begin{aligned}
q_{A_{n-1}}(x) =& \frac{2x_1^2 + 4x_{n-1}^2 + 4x_{n-2}^2 + 2x_{n-3}^2 - 4x_{n-3}x_{n-2} - 4x_{n-2}x_{n-1}}{4} \\
&+ \frac{2\sum_{i=1}^{n-4}(x_i - x_{i+1})^2}{4} \\
=& \frac{2x_1^2 + x_{n-2}^2 + (x_{n-2} - 2x_{n-1})^2 + 2(x_{n-3} - x_{n-2})^2}{4} \\
&+ \frac{2\sum_{i=1}^{n-4}(x_i - x_{i+1})^2}{4} \\
=& \frac{2x_1^2 + x_{n-2}^2 + (x_{n-2} - 2x_{n-1})^2 + 2\sum_{i=1}^{n-3}(x_i - x_{i+1})^2}{4}
\end{aligned}$$

We have now succeeded in exposing node $n-2$ as the monomial $x_{n-2}$. This is the node where we we want to extend $A_{n-1}$ to $D_n$. Notice that $q_{A_{n-1}} + x_n^2 - x_{n-2}x_n = q_{D_n}$ and so adding $x_n^2 - x_{n-2}x_n$ gives

$$\begin{aligned}
q_{D_n}(x) =& \frac{2x_1^2 + x_{n-2}^2 + (x_{n-2} - 2x_{n-1})^2 + 2\sum_{i=1}^{n-3}(x_i - x_{i+1})^2}{4} + x_n^2 - x_{n-2}x_n \\
=& \frac{2x_1^2 + (x_{n-2} - 2x_n)^2 + (x_{n-2} - 2x_{n-1})^2 + 2\sum_{i=1}^{n-3}(x_i - x_{i+1})^2}{4}
\end{aligned}$$

which is the correct Tits form of Dynkin graph $D_n$.

**Proposition 6.9.** *Let $q$ be the Tits form of a Dynkin graph $\overline{Q}$. Then $q$ has only a finite number of positive roots.*

*Proof.* Let $||-|| : \mathbb{R}^n \to \mathbb{R}$ be the Euclidean norm. Extend $q : \mathbb{Z}^n \to \mathbb{Z}$ to $q : \mathbb{R}^n \to \mathbb{R}$. By Proposition 6.7 it is clear that $q(x) > 0$ for all $x > 0$. Define a set

$$\mathcal{Q} = \{x \in \mathbb{R}^n \mid x > 0, ||x|| = 1\}.$$

Since $\mathcal{Q}$ is a closed and bounded subset of $\mathbb{R}^n$, it is compact. By the extreme value theorem and since $q$ is continuous we have that $q|_{\mathcal{Q}}$ will attain a minimum value $\mu > 0$ in $\mathcal{Q}$. Thus we have for all $x > 0$ the inequality,

$$0 < \mu \leq q\left(\frac{x}{||x||}\right) = \frac{1}{||x||^2}q(x)$$

Recall that $x$ is a root if $q(x) = 1$. For each positive root $y \in \mathbb{Z}^n$ we have that $\mu \leq \frac{1}{||y||^2}q(y)$ which implies that $\mu \leq \frac{1}{||y||^2}$ and we get that $||y|| \leq \frac{1}{\sqrt{\mu}}$. Hence $q$ only has finitely many roots. □

Now we will state the main theorem of this thesis.

## 7. Gabriel's theorem

**Theorem 7.1** (Gabriel's Theorem II)**.** *Let $Q$ be a finite, connected and acyclic quiver; and $\overline{Q}$ is the underlying graph of $Q$. The the following statements are equivalent:*

i) *If $\overline{Q}$ is Dynkin, then $KQ$ is representation-finite.*
ii) *If $KQ$ is representation-finite, then $\overline{Q}$ is Dynkin.*

*Proof of i).* Let $\overline{Q}$ be Dynkin, then the indecomposable representations correspond to the positive roots by Gabriel's Theorem I, and there are only finite number of roots by Proposition 6.9.

For proof of ii), see [4, Theorem 2 page 20]. □

If we know the quiver $Q$, then we can decide whether $A = KQ$ is representaion-finite or not. If $A$ is representation-finite, then by Gabriel's Theorem I we know that each indecomposable $A$-module is given by a of positive roots of the Tits form for the given $Q$.

**Example 7.2.** We can now see that the roots of Dynkin graph $A_3$ corresponds the dimension vectors in example 5.4 and so $M_1, \ldots, M_6$ are a complete list of isomorphism classes of indecomposable $KQ$-modules.

## References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