On the Existence of Electrodynamics on Manifold-like Polyfolds

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Abstract
This essay examines the question whether the classical theory of electrodynamics can be extended to a spacetime which locally changes dimension and if such an endeavour is mathematically possible. Recent research has developed a new generalisation of smooth manifolds, the so-called M-polyfolds, which constitutes a sufficient foundation to make this endeavour a physical plausibility. These M-polyfolds then facilitate the capability to define the velocity of a curve going through a dimensionally shifting spacetime. Moreover, necessary extensions to the theory of M-polyfolds is developed in order to tailor the theory to a more physically focused framework. Concluding the essay, Maxwell’s equations on M-polyfolds are defined.

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CHAPTER 1

Introduction

Recent discussions in theoretical physics, see [24, 41, 54, 81], treat classical electrodynamics on \( n \)-dimensional spacetime rather than the usual 4-dimensional spacetime. In \( n \)-dimensional spacetime various interesting exceptions to the classical theory of electrodynamics arise for other than four dimensions. For example, Huygens’s principle, which states that every point in a wavefront can be viewed as a point source of the propagating wave, does not hold in odd-dimensional spacetime. In particular, this impacts the discussion related to refraction. Furthermore, for higher than four spacetime dimensions, the divergence of the proper Coulomb potential for a charge is stronger, and therefore the self-force on the charge becomes way more involved. Next, a natural question to ask is whether electrodynamics is possible on a spacetime that changes dimension locally. This essay shall give an exposition on a way to formulate a mathematical model to make electrodynamics on a dimensionally changing spacetime. Note that classical differentiable manifolds have a locally fixed dimension, therefore manifolds are not a good model to describe a spacetime with a locally changing dimension. Instead, we are forced to use something else.

Conveniently, a recent 21st century theory for generalised differential geometry shall be useful. This new theory developed in [32–34] uses global models called M-polyfolds which allows for a smooth local change of dimensions, see Figure 7.2 below for an example of a M-polyfold. Furthermore, M-polyfolds are in of themselves interesting as mathematical objects. In principle, M-polyfolds are modified versions of Banach manifolds, though it is not a generalisation in the true sense of the word. Indeed, it is false that every Banach manifold is a M-polyfold. Moreover, after Eells and Elworthy proved in 1970 that infinite dimensional Banach manifolds are open subsets of the modelling space itself [13], some of the interest in research on Banach manifolds fell out of fashion. In contrast, as it is most certainly not true for the theory of M-polyfolds, the theory of M-polyfolds is a potential well of research.

Furthermore, the original motivation for the theory of M-polyfolds is the generalisation of M-polyfolds, polyfolds. In short, the theory of polyfolds is to M-polyfolds what the theory of orbifolds is to manifolds. These polyfolds conveniently describe moduli spaces for pseudoholomorphic curves
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[28]. Most importantly, polyfold theory solves transversality issues for said curves, that is its first and foremost applications are to symplectic field theory, see the surveys [15,16] on the subject. For some general applications to symplectic field theory, see [19,38,72]. Furthermore, the theory of polyfolds have seen some success in certain subfields of symplectic field theory, in applications to Gromov-Witten theory [31,62] and to Floer theory [3,82,83]. Additionally, there has been some development in proving its correspondence to other recent similar theories such as the Kuranishi theory [80]. For further reading see the lecture notes [8,21,22,78] on the subject. However, it is important to mention that [36] is the main resource and quite extensive reference text for the theory of polyfolds.

Therefore, the goal of the essay is to provide a sufficient background and introduce new concepts so that electrodynamics can be defined on a spacetime that is modelled as a M-polyfold. For example one might imagine a spacetime where space looks like the M-polyfolds depicted in Figure 7.2. In order for electrodynamics to make sense on such M-polyfolds, one need to ask whether Maxwell’s equations

\[ d\mathcal{F} = 0 \quad \text{and} \quad * d * \mathcal{F} = \mathcal{J} \]

makes sense on M-polyfolds. Here \( \mathcal{F} \) is the electromagnetic field tensor and \( \mathcal{J} \) is the covariant current. Furthermore, \( d \) should correspond to a version of the exterior derivative, and \( * \) is a generalised Hodge operator.

First, we need to make sure that differentiability is well-defined on a point even if all neighbourhoods around the point contains points of different dimension. In particular, a purely physical argument demands differentiability everywhere on spacetime. The theory of M-polyfolds that we summarise in Chapters 3 to 5 solves this issue directly for finite-dimensional

\[ \text{Figure 7.2. – A finite-dimensional M-polyfold embedded in } \mathbb{R}^3. \]
M-polyfolds. Next, since the definition of the Hodge operator requires a notion of a volume form, we need a new notion of differential forms that allows for a local change in dimension. In Chapter 6 we not only summarise some results of sc-differential \( k \)-forms see [36, Section 4.4], we also invent a completely new mathematical object, generalised sc-differential forms see Definition 6.1. In a nutshell, it is a sc-differential form that allows for a local change of its order, i.e. sometimes it may be a \( k \)-form and sometimes an \( n \)-form. Moreover, the theory of tensors are generalised to the setting of M-polyfolds. To the knowledge of the author this has not been done before to this date. In particular, we are interested to define the Lorentz metric on M-polyfolds. In particular, Chapter 6 and Chapter 7 introduce these concepts and give a class of M-polyfolds where the exterior derivative of generalised sc-differential forms is well-defined.

The essay is structured as follows: In Chapter 2, we remind the reader on classical subjects central to the theory of M-polyfolds. In the following chapter, we introduce the concept of scale-calculus and we prove that the chain rule is satisfied. Next, i.e. Chapter 4, the global models called M-polyfolds are defined, then we discuss issues related to the boundary and sc-smooth partition of unity. In Chapter 5, we discuss the tangent spaces on M-polyfolds and some of its consequences related to a version of the implicit function theorem. In the subsequent chapter, tensors and a new generalisation of differential forms on M-polyfolds are treated. Thereafter, some local models and M-polyfolds are explicitly constructed in Chapter 7. We finish the essay with Chapter 8, a brief discussion on electrodynamics on M-polyfolds.
CHAPTER 2

Preliminaries

Here, we shall introduce and remind the reader about a few basic subjects that are fundamental to the essay. The reader is expected to know basic topology, integration theory, and real analysis. Though a reader that is well-versed in differential geometry only, should have little problem with the subjects that shall be discussed here. First, we shall treat functional analysis. Since functional analysis is particularly useful when dealing with infinite dimensional analysis, it is central to the rest of the essay. Subsequently, we shall go through some important topological results, and thereafter some multilinear algebra. We shall finish the chapter by lightly touching on a result in the theory of Bochner integrals, it shall not be as central as the other subjects but needed nonetheless.

1. Functional Analysis

We shall start by reminding the reader about the fundamental object in functional analysis, Banach spaces.

**Definition 2.1.** Let $(X, \|\cdot\|)$ be a normed vector space. If every Cauchy sequence on $X$ converges in the norm, then $(X, \|\cdot\|)$ is called a Banach space.

Of course, a Hilbert space is a special case of a Banach space. This is the case as a Hilbert space is an inner product space such that every Cauchy sequence converges in the induced norm. The following subclass of linear operators shall be of most importance.

**Definition 2.2.** Let $X$, $Y$ be Banach spaces and let $L : X \to Y$ be a linear operator. If for every bounded set $U \subseteq X$, the set $L(U)$ is relatively compact in $Y$, then $L$ is called a compact linear operator.

**Remark 2.3.** There are plenty of other equivalent definitions of compact linear operators. The definition given above follows that of [43], for other ways to define compact linear operators see e.g. [11, 12, 14].

Following the notation in [58], we shall have need of direct sums of Banach spaces. Note that as long as we take the direct sum of finitely many Banach spaces, the underlying set is the same as the Cartesian product. As a matter of fact, the underlying set for the direct sum of infinitely many
Banach spaces can not be the same as the Cartesian product. This is the case since addition over groups is ill-defined when infinitely many elements are added together.

**Definition 2.4.** Let \((U, +_U, \cdot_U)\) and \((V, +_V, \cdot_V)\) be vector spaces over a field \(K\). We define the (exterior) direct sum of \(U\) and \(V\) to be

\[
U \oplus V := \{(u, v) : u \in U \text{ and } v \in V\},
\]

together with the addition \(+ : U \oplus V \times U \oplus V \to U \oplus V\) and the scalar multiplication \(\cdot : K \times U \oplus V \to U \oplus V\) defined as

\[
(u, v) + (u', v') = (u +_U u', v +_V v'),
\]

respectively

\[
k \cdot (u, v) = (k \cdot_U u, k \cdot_V v),
\]

where \((u, v), (u', v') \in U \oplus V\) and \(k \in K\).

For the direct sum between two Banach spaces \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) we define the induced norm \(\|\cdot\|_{\oplus}\) on \(X \oplus Y\) to be

\[
\|x \oplus y\|_{\oplus} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}
\]

for all \(x \in X\) and all \(y \in Y\). Note that we choose this definition in order for this to overlap with the definition of the direct sum of Hilbert spaces as seen in [12].

**Remark 2.5.** We could as well have chosen the norm to be

\[
\|x \oplus y\|_{\oplus} = \|x\|_X + \|y\|_Y,
\]

or

\[
\|x \oplus y\|_{\oplus} = \max\{\|x\|_X, \|y\|_Y\}.
\]

As a matter of fact these two choices for a norm are norm-equivalent.

We shall denote open balls centered at a point \(p \in X\) with radius \(r > 0\) by \(B(p, r)\), that is

\[
B(p, r) = \{x \in X : \|p - x\| < r\}.
\]

The property that open balls are convex for normed spaces is important, but take care observing that this is not true for general metric spaces. The proof is considered standard and well-known, see e.g. [4, Chapter 6.1].

**Proposition 2.6.** Let \(X\) be a normed space, then \(B(x, r) \subseteq X\) is convex.

**Proof.** Let \(y, z \in B(x, r)\) be two arbitrary points. The set

\[
S = \{y + (z - y)t : t \in [0, 1]\}
\]

is contained in an open ball centered at \(x\) with radius \(r\) since

\[
\|S_0\|_X^2 + \|S_1\|_Y^2 = \|y\|_X^2 + \|z\|_Y^2 > \|x\|_X^2 + \|y\|_Y^2 = \|S\|_{\oplus}^2
\]

and

\[
\|S\|_{\oplus} = \sqrt{\|S_0\|_X^2 + \|S_1\|_Y^2} < \|x\|_X + \|y\|_Y = \|x \oplus y\|_{\oplus}.
\]
is the line segment that joins $y$ and $z$. Left to prove is that $S \subseteq B(x, r)$. It holds that
\[
\|y + (z - y)t - x\| = \|(1 - t)(y - x) + t(z - x)\|
\leq \|(1 - t)(y - x)\| + \|t(z - x)\|
= (1 - t)\|y - x\| + t\|z - x\|
< (1 - t)r + tr
= r.
\]
Hence, for all $s \in S$ it holds that $s \in B(x, r)$, and we can conclude that $B(x, r)$ is convex. \qed

It is a standard fact that continuous bilinear maps on Banach spaces are smooth. As inner products are a special case of continuous bilinear maps, we shall use this fact on Hilbert spaces. This is also why we can guarantee that the norm is smooth on Hilbert spaces but the norm is not smooth on general Banach spaces.

**Proposition 2.7.** Let $E, F, G$ be Banach spaces, and let $\beta : E \times F \to G$ be a continuous bilinear map. Then, $\beta$ is infinitely Fréchet differentiable.

**Proof.** The proof follows from a few observations. By definition, for $x, h_1 \in E$, and $y, h_2 \in F$, we have that
\[
\beta(x + h_1, y + h_2) - \beta(x, y) = \beta(x, h_2) + \beta(h_1, y) + \beta(h_1, h_2).
\]
For $x \in E$ and $y \in F$, let $A : E \times F \to L(E \times F, G)$ be defined by
\[
A(x, y)(h_1, h_2) = \beta(x, h_2) + \beta(h_1, y)
\]
where $(h_1, h_2) \in E \times F$. It follows that $A$ is linear over both $(x, y)$ and $(h_1, h_2)$ in $E \times F$. It indeed holds that $A(x, y) = D\beta(x, y)$, this is the case since
\[
\frac{\|\beta(x + h_1, y + h_2) - \beta(x, y) - A(x, y)(h_1, h_2)\|}{\|(h_1, h_2)\|} = \frac{\|\beta(x, h_2) + \beta(h_1, y) + \beta(h_1, h_2) - \beta(x, h_2) - \beta(h_1, y)\|}{\|(h_1, h_2)\|}
\]
\[
= \frac{\|\beta(h_1, h_2)\|}{\|(h_1, h_2)\|} \leq \frac{\|\beta\|\|h_1\|\|h_2\|}{\sqrt{\|h_1\|^2 + \|h_2\|^2}} \leq \|\beta\||\|h_2\|
\]
and as $\beta$ is continuous, the last term goes to zero as $(h_1, h_2) \to (0, 0)$. It is well know that the derivative of linear mappings are constant, see e.g. [60, Example 9.14], therefore the third order derivative of $\beta$ is zero everywhere, and we are done. \qed
In functional analysis, topological issues arise on Banach spaces. This has largely to do with the fact that infinite-dimensional Banach spaces are nowhere locally compact in the metric topology. In other words, the metric topology is too fine to be well-behaved. In order to remedy this, one can look at the weak topology of Banach spaces. This is the coarsest topology on a Banach space \( X \) so that all linear functionals in \( X^* \) still are continuous.

**Definition 2.8.** Let \( X \) be a Banach space. The **weak topology** on \( X \) is the topology generated by all sets of the form

\[
N(x_0, A, \varepsilon) = \{ x \in X : \| f(x) - f(x_0) \| < \varepsilon, \text{ for all } f \in A \},
\]

where \( x_0 \in X \), \( A \subseteq X^* \) a finite subset, and \( \varepsilon > 0 \).

Now, a subset of a Banach space is called **weakly open** (**weakly closed**, **weakly compact** etc.) if it is open (respectively closed, compact etc.) in the weak topology. A very suitable subclass of Banach spaces are weakly compactly generated Banach spaces. They shall be particularly useful as some smoothness properties of the norm hold for weakly compactly generated Banach spaces, see [40].

**Definition 2.9.** Let \( X \) be a Banach space. We say that \( X \) is **weakly compactly generated** if there is a weakly compact set \( K \subseteq X \) such that

\[
\text{cl}_X(\text{span}(K)) = X.
\]

Two important examples of weakly compactly generated Banach spaces are reflexive Banach spaces and separable Banach spaces.

2. **Topology**

Here, we shall remind the reader on some important topological results. The Urysohn’s metrization theorem is well-known but is stated here for reference. For a proof see [76, 77].

**Theorem 2.10 (Urysohn’s Metrization Theorem).** Every second countable regular topological space is metrizable.

There is also a related important metrization theorem, the Smirnov Metrization theorem. A proof can be found in [63, 64].

**Theorem 2.11 (Smirnov Metrization theorem).** A topological space is metrizable if, and only if, it is paracompact, Hausdorff and locally metrizable.

Remember that a topological space, \( X \), is locally metrizable, if for every point \( p \in X \) there is a neighbourhood around \( p \) that is homeomorphic to a metrizable space. Furthermore, Lemma 2.12 introduced in [50, Lemma 1] is rather useful.
Lemma 2.12. A Hausdorff topological space $X$ is paracompact if, and only if, every open cover of $X$ has a closed, locally finite refinement.

The following result is helpful when dealing with refinements of open covers of topological spaces. For a proof see Proposition 2.17 in [36].

Proposition 2.13. Let $Y$ be regular Hausdorff space, and let \( \{Y_i\}_{i \in I} \) be a locally finite family of closed subspaces of $Y$, such that $Y = \bigcup_{i \in I} Y_i$. If every subspace $Y_i$ is paracompact, then $Y$ is paracompact.

Following Definition I.3.2 in [6], we use the following notion of locally closed subset.

Definition 2.14. Let $X$ be a topological space, and let $A \subseteq X$ be a subset. If for some point $x \in N$ there is an open neighbourhood $V \subseteq X$ around $x$ such that $A \setminus V$ is closed in $V$, then we say that $A$ is locally closed at $x$. If $A$ is locally closed at all points $x \in A$, then we say that $A$ is locally closed on $X$.

Remark 2.15. Other equivalent definitions that are used in the literature includes that $A$ is the intersection of a closed subset and an open subset of $X$, or that $A$ is relatively open in the closure of $A$.

In differential geometry the following result is occasionally used, and we shall use it as well. For a proof see Theorem 41.7 in [56].

Proposition 2.16. Let $X$ be a paracompact Hausdorff topological space, and let \( \{U_a\}_{a \in A} \) be an open covering of $X$. Then, there exists a continuous partition of unity on $X$ subordinate to \( \{U_a\}_{a \in A} \).

The following claim can be implicitly found in many sources. But, as a proof of the statement seems to be hard to find, we shall also prove it.

Proposition 2.17. Let $X$ be a Hausdorff locally Banach topological space, then $X$ is completely regular, and hence in particular regular.

Proof. Let $p \in X$ be a point and let $A \subseteq X$ be a closed subset such that $p \notin A$. Since $X$ is locally Banach we can choose a neighbourhood $U$ around $p$ that is homeomorphic to some open subset of a Banach space. Let $\|\cdot\| : U \to \mathbb{R}$ be a norm for $U$ such that the metric topology induced by $\|\cdot\|$ is equivalent to the subspace topology of $U$. Since $X$ is Hausdorff, we can, for each $x \in \partial U$, choose a sequence $\{x_n\}$, $x_n \in U$, such that

$$\lim_{n \to \infty} x_n = x.$$ 

Now consider the number

$$r = \frac{1}{2} \min \left\{ \inf_{x \in \partial U} \left\{ \lim_{n \to \infty} \|p - x_n\| : x_n \to x \right\}, 1 \right\},$$

where $U$ is a neighbourhood of $p$ such that $\|p\| < r$. For each $x_n \in U$ there is a sequence $\{y_{n,k}\}$ such that $y_{n,k} \to x_{n+1}$ and $\|p - y_{n,k}\| < r$. By taking a subnet $\{y_{n_k}\}$ we can assume $y_{n_k} \to y$. Since $U$ is locally compact, there is a subnet $\{y_{n_{k_l}}\}$ such that $y_{n_{k_l}} \to z$. Then $y \leq z$ and $y_{n_{k_l}} \to z$. Since $\|p - y_{n_{k_l}}\| < r$, we have $\|p - z\| < r$. Hence $z = p$ and $\{y_{n_{k_l}}\}$ is a sequence converging to $p$.
and set $\tilde{A} = A \cap B(p, r)$.

Choose
\[ \varepsilon = \begin{cases} \alpha & \text{if } \tilde{A} = \emptyset, \\ \inf_{a \in \tilde{A}} \|p - a\| & \text{if } \tilde{A} \neq \emptyset. \end{cases} \]

The map
\[ f : X \to [0, 1] \]

defined by
\[ f(x) = \begin{cases} 1 - \frac{1}{\varepsilon}\|p - x\| & \text{if } x \in B(p, \varepsilon) \\ 0 & \text{otherwise,} \end{cases} \]
is continuous, $f(p) = 1$, and $f(a) = 0$ for all $a \in A$. Hence, $X$ is completely regular, and since it is Hausdorff, points are closed sets, and it holds that $X$ is regular.

□

Remark 2.18. The above result is not true if one would replace the assumption locally Banach with locally metrizable instead. A good counterexample is the $K$-topology on the real line, see [56, p. 197]. This shows a purely topological important distinction between metrizable spaces and normed spaces. One question coming to mind is whether Proposition 2.17 is true for Frechêt spaces as well, i.e. if locally convex Hausdorff spaces are completely regular. If it would be the case, one has incentive to ask whether M-polyfolds can be generalised to a theory built upon Frechêt spaces instead of Banach spaces. In other words, it would be convenient if the discussions related to the topology of M-polyfolds generalises to locally Frechêt spaces.

3. Multilinear Algebra

As we shall discuss a generalisation of differential geometry, multilinear algebra is indeed crucial. All the results in this section can be found in [7]. The reader that is interested in a broader and more in-depth discussion is referred to [58, 59]. Note, that we keep a rather general setting for this section. We do this since this general setting gives a very effective language.

Definition 2.19. Let $\mathcal{R}$ be an associative ring with unity. A (left) $\mathcal{R}$-module is an Abelian group $(V, +)$ together with a binary operator $\mathcal{R} \times V \to V$ denoted as $(a, v) \mapsto av$ for all $a \in \mathcal{R}$ and all $v \in V$, such that
\begin{enumerate}
  \item[(i)] $(a + b)v = av + bv$ for all $a, b \in \mathcal{R}$ and all $v \in V$;
  \item[(ii)] $a(v_1 + v_2) = av_1 + av_2$ for all $a \in \mathcal{R}$ and all $v_1, v_2 \in V$;
  \item[(iii)] $(ab)v = a(bv)$ for all $a, b \in \mathcal{R}$ and all $v \in V$;
  \item[(iv)] $1v = v$ for all $v \in V$ where $1 \in \mathcal{R}$ denotes the multiplicative unity.
\end{enumerate}
Remark 2.20. Observe that we could just as well defined so called right $\mathcal{R}$-modules by multiplication from the right instead of the left. Though, for our purposes these two distinctions does not matter.

Analogous to vector spaces, there are linear mappings between modules.

**Definition 2.21.** Let $X$ and $Y$ be $\mathcal{R}$-modules, and let $f : X \to Y$ be a mapping. If
\[
f(ax_1 + bx_2) = af(x_1) + bf(x_2),
\]
for all $a, b \in \mathcal{R}$ and for all $x_1, x_2 \in X$, then $f$ is called an $\mathcal{R}$-homomorphism. Furthermore, we shall denote the set of all $\mathcal{R}$-homomorphisms from $X$ to $Y$ by $\text{Hom}_{\mathcal{R}}(X, Y)$.

A very useful tool in most areas of mathematics is the language of category theory. A very important concept from category theory is the concept of commuting diagrams.

**Definition 2.22.** Let $A, B, C$ be objects, and let $f : A \to C, g : A \to B$, and $h : B \to C$ be morphisms in a category. Then, we say that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{f} & & \downarrow{h} \\
C
\end{array}
\]
commutes if, and only if, $f = h \circ g$.

Next, we shall introduce bilinear mappings. Probably, every reader has come into contact with the special case of bilinear mappings called matrices. Though, we need a way more abstract setting to work with.

**Definition 2.23.** Suppose that $\mathcal{R}$ is a commutative ring with unity, and let $M, N, P$ be $\mathcal{R}$-modules. A mapping $\phi : M \times N \to P$ is called an $\mathcal{R}$-bilinear mapping if
\[
\phi(am_1 + bm_2, n) = a\phi(m_1, n) + b\phi(m_2, n),
\]
and
\[
\phi(m, an_1 + bn_2) = a\phi(m, n_1) + b\phi(m, n_2),
\]
for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and all $a, b \in \mathcal{R}$.

We are now able to define the tensor product on modules. In principle, the tensor product is a linearisation of the product of two modules.

**Definition 2.24.** Let $\mathcal{R}$ be a commutative ring with unity, and let $M, N, T$ be $\mathcal{R}$-modules. Suppose that $\rho : M \times N \to T$ is an $\mathcal{R}$-bilinear mapping. The tuple $(T, \rho)$ is called the tensor product of $M$ and $N$ over $\mathcal{R}$ if for every
\( R \)-module, \( P \), and every \( R \)-bilinear mapping \( \varphi : M \times N \to P \), there exists an unique \( \mathcal{R} \)-homomorphism \( f : T \to P \) such that the diagram
\[
\begin{array}{ccc}
M \times N & \xrightarrow{\varphi} & T \\
\downarrow & & \downarrow f \\
& P & \\
\end{array}
\]
commutes.

The following definition is rather useful for the construction of the tensor product.

**Definition 2.25.** Let \( F \) be an \( \mathcal{R} \)-module. We say that \( F \) is a **free module** if there is a set, \( \{ e_i \}_{i \in J}, e_i \in F \), such that every element \( x \in F \) can be written as a unique linear combination of a finite number of \( e_i \) and with coefficients from \( \mathcal{R} \).

Now to the construction. Let \( \mathcal{R} \) be a commutative ring with unit, and let \( M, N \) be \( \mathcal{R} \)-modules. Define \( F \) as the free \( \mathcal{R} \)-module that has all pairs \((m,n)\in M \times N\) as its basis according to 2.25. Furthermore, suppose that \( F_0 \subseteq F \) is the submodule that is generated by the following elements in \( F \):
\[
\begin{align*}
(m_1 + m_2, n) - (m_1, n) - (m_2, n), & \quad (m, n_1 + n_2) - (m, n_1) - (m, n_2), \\
(rm, n) - r(m, n), & \quad (m, rn) - r(m, n),
\end{align*}
\]
where \( m_1, m_2, m \in M, n_1, n_2, n \in N \) and \( r \in \mathcal{R} \). We shall denote the quotient module \( F/F_0 \) by \( M \otimes_\mathcal{R} N \). Also, the image of the pair \((m,n)\in F\) onto \( F/F_0 \) is denoted as \( \rho((m,n)) := m \otimes n \).

Furthermore, we need to show that \((M \otimes_\mathcal{R} N, \rho)\) indeed is a tensor product of \( M \) and \( N \) over \( \mathcal{R} \). Observe, from the construction of \( F_0 \) and from that \( F_0 \) is the zero element of \( F/F_0 \), the following identities hold
\[
\begin{align*}
(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n, & m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2, \\
(rm) \otimes n &= r(m \otimes n), & m \otimes (rn) &= r(m \otimes n),
\end{align*}
\]
where \( m_1, m_2, m \in M, n_1, n_2, n \in N \) and \( r \in \mathcal{R} \). But then, it holds that \( \rho : M \times N \to M \otimes_\mathcal{R} N \) is \( \mathcal{R} \)-bilinear. Suppose that \( P \) is an arbitrary \( \mathcal{R} \)-module, and choose an \( \mathcal{R} \)-bilinear mapping \( \varphi : M \times N \to P \) arbitrarily.

Next, We shall show that there is exactly one \( \mathcal{R} \)-homomorphism
\[
f : M \otimes_\mathcal{R} N \to P
\]
such that the diagram
\[
\begin{array}{ccc}
M \times N & \xrightarrow{\varphi} & M \otimes_\mathcal{R} N \\
\downarrow & & \downarrow f \\
& P & \\
\end{array}
\]
commutes. To show that such a mapping $f$ exists, suppose that $f_0 : F \to P$ is an $\mathcal{R}$-homomorphism defined by $f_0(m, n) = \varphi(m, n)$. Using Definition 2.23, it holds that $f_0(F_0) = \{0\}$. To see this observe that

$$f_0((m_1 + m_2, n) - (m_1, n) - (m_2, n)) = \varphi(m_1 + m_2, n) - \varphi(m_1, n) - \varphi(m_2, n) = 0,$$

$$f_0((rm, n) - r(m, n)) = \varphi(rm, n) - r\varphi(m, n) = 0,$$

$$f_0((m, n_1 + n_2) - (m, n_1) - (m, n_2)) = \varphi(m, n_1 + n_2) - \varphi(m, n_1) - \varphi(m, n_2) = 0,$$

$$f_0((m, rn) - r(m, n)) = \varphi(m, rn) - r\varphi(m, n) = 0,$$

where $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $r \in \mathcal{R}$. Hence, $f_0$ induces an $\mathcal{R}$-homomorphism $f : F/F_0 \to P$ such that $f(m \otimes n) = \varphi(m, n)$. Uniqueness follows from that $f(m \otimes n) = \varphi(m, n)$ and from that the elements $m \otimes n$ generates $M \otimes \mathcal{R} N$. We have now shown that Definition 2.24 is well-defined.

Furthermore, it is not only the case that the tensor product is well-defined, but it is also unique up to $\mathcal{R}$-isomorphism.

**Proposition 2.26.** Let $M$ and $N$ be two $\mathcal{R}$-modules. Furthermore, suppose that $(T_1, \rho_1)$ and $(T_2, \rho_2)$ are two tensor products of $M$ and $N$ over $\mathcal{R}$. Then, there exists an $\mathcal{R}$-isomorphism $f : T_1 \to T_2$ such that the following diagram

\[
\begin{array}{ccc}
M \times N & \xrightarrow{f} & T_1 \\
\downarrow{\rho_1} & & \downarrow{\rho_1} \\
T_2 & \xrightarrow{f} & T_2
\end{array}
\]

commutes.

**Proof.** According to Definition 2.24, there exists exactly one $\mathcal{R}$-homomorphism $f : T_1 \to T_2$ and exactly one $\mathcal{R}$-homomorphism $f : T_1 \to T_2$ such that the diagrams

\[
\begin{array}{ccc}
M \times N & \xrightarrow{f} & T_1 \\
\downarrow{\rho_1} & & \downarrow{\rho_1} \\
T_2 & \xrightarrow{f} & T_2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M \times N & \xrightarrow{f} & T_1 \\
\downarrow{\rho_2} & & \downarrow{\rho_2} \\
T_2 & \xrightarrow{f} & T_2
\end{array}
\]

commute.
Then, it holds that the following diagrams commute. This is the case as \( g \circ (f \circ \rho_1) = g \circ \rho_2 = \rho_1 \) and \( f \circ (g \circ \rho_2) = f \circ \rho_1 = \rho_2 \). But these diagrams also commute if we replace \( g \circ f \) by the identity mapping \( \text{id}_1 : T_1 \to T_1 \). Similarly, these diagrams commute if we were to replace \( f \circ g \) by the identity mapping \( \text{id}_2 : T_2 \to T_2 \). According to Definition 2.24, it has to hold that \( g \circ f = \text{id}_1 \) och \( f \circ g = \text{id}_2 \), and we are done.

In physics, tensors are very important objects. As tensors are fibre-wise multilinear mappings, this calls for Definition 2.27.

**Definition 2.27.** Let \( \mathcal{R} \) be a commutative ring with a unit. Furthermore, suppose that \( M_i, i = 1, \ldots, k \), and \( P \) are \( \mathcal{R} \)-modules. A function

\[
\mu : M_1 \times \cdots \times M_k \to P
\]

is called an \( \mathcal{R} \)-**multilinear mapping** if for every \( i, i = 1, \ldots, k \) and every fixed

\[
(m_1, \ldots, \hat{m}_i, \ldots, m_k) \in M_1 \times \cdots \times \hat{M}_i \times \cdots \times M_k
\]

the mapping

\[
m \mapsto \mu(m_1, \ldots, m_{i-1}, m_{i+1}, m_{i+1}, \ldots, m_k),
\]

is an \( \mathcal{R} \)-module homomorphism. Notice that we denote a left-out element by \( \hat{\cdot} \).

Now, we are able to inductively define the \( k \)-fold tensor product

\[
\bigotimes_{i=1}^k M_i,
\]

as an \( \mathcal{R} \)-multilinear mapping \( \rho : M_1 \times \cdots \times M_k \to \bigotimes_{i=1}^k M_i \).

A very important subclass of multilinear mappings are alternating multilinear mappings. For the modules we are interested in these are the same as skew-symmetric multilinear mappings.

**Definition 2.28.** Let \( \mathcal{R} \) be a commutative ring with unity, and let \( M, N \) be \( \mathcal{R} \)-modules. A \( \mathcal{R} \)-multilinear mapping \( f : \prod^k M \to N \) is called **alternating** if

\[
f(m_1, \ldots, m_k) = 0
\]
whenever \( m_i = m_j \) for some \( i \neq j \). The set of all such alternating \( k \)-multilinear mapping is denoted as \( \text{Alt}^k(M, N) \).

Next we shall introduce the important alternation map (actually a functor), that maps multilinear mappings to alternating multilinear mappings. We denote the permutation of the letters \( 1, 2, \ldots, k \) by \( \sigma_k \), and the set of all such permutations as \( S_k \).

**Definition 2.29.** Let \( \mathcal{R} \) be a commutative ring with unity, and let \( M, N \) be \( \mathcal{R} \)-modules. For every \( k \)-multilinear mapping \( f : \times^k M \to N \), the alternation map \( \text{Alt} \), is defined as

\[
\text{Alt}(f)(m_1, \ldots, m_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma_k) f(m_{\sigma_k(1)}, m_{\sigma_k(2)}, \ldots, m_{\sigma_k(k)})
\]

for all \( m_1, \ldots, m_k \in M \).

4. Bochner Integrals

In short, Bochner integrals are generalised Lebesgue integrals. Instead of talking about real (or complex) valued measures, Bochner integrals deals with vector valued measures over Banach spaces. By merging Theorem 2.2 and Theorem 2.3 from Chapter XIII in [53] we summarise the following fundamental theorem of calculus for Bochner integrals.

**Theorem 2.30.** Let \( E \) be a Banach space, and let \( f : I \to E \), with \( I \subseteq \mathbb{R} \) a connected set, be a continuous function. Then, for \( x_0 \in I \), the function defined by

\[
F(x) = \int_{x_0}^{x} f(t)dt
\]

is a primitive function for \( f \) whenever \( x \in I \), i.e. \( F'(x) = f(x) \). Furthermore, if \( [a, b] \in I \), it holds that

\[
\int_{a}^{b} f(x)dx = F(b) - F(a).
\]
CHAPTER 3

Scale-Calculus

Here, the local models called scale-Banach spaces, in short sc-Banach spaces, shall be introduced. Put loosely, the goal of sc-Banach spaces are to generalise the notion of differentiability of maps on Banach spaces, to only require that the maps are differentiable on a dense Banach subspace of the original Banach space.

Chronologically, we shall first introduce the spaces that will be the fundamental building block on which everything else in this essay is built upon. Thereafter, we shall discuss some properties related to a new notion of differentiability. Then, a whole section is dedicated to the chain rule and sc-smooth retractions to emphasise their importance.

1. Introduction to Sc-Banach Spaces

We shall begin with the definition of sc-Banach spaces and sc-structures.

**Definition 3.1.** Let $E$ be a Banach space and let $\{E_m\}_{m \geq 0}$ be a sequence of Banach spaces such that

$$E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots.$$

We call $\{E_m\}_{m \geq 0}$ a sc-structure on $E$ if the following holds:

i) the inclusion operator $\iota : E_{m+1} \hookrightarrow E_m$ is compact;

ii) the set $E_\infty := \bigcap_{i=0}^\infty E_i$ is dense in $E_m$ for all $m \geq 0$.

A Banach space $E$ together with a sc-structure $\{E_m\}_{m \geq 0}$ is called a sc-Banach space.

**Remark 3.2.** The concept of a scaled Banach space is not entirely new, it has its place in interpolation theory, see e.g. [73, Section 1.19.4.]. Contrary to previous similar models, we need to assume that the inclusion operator is compact. It shall be seen later that this assumption is indeed crucial.

Henceforth, points in $E_\infty$ shall be referred to as smooth points and points in $E_m$ as points of regularity $m$. Note that if $E$ is finite-dimensional, i.e. it is isomorphic to some $\mathbb{R}^n$, the only subspace of $E$ that is dense in $E$ is $E$ itself. Also since $\mathbb{R}^n$ is locally compact in the Euclidean topology, the inclusion operator of $\mathbb{R}^n$ onto itself is compact. Therefore, all finite Banach
spaces $E$ only admit the constant sc-structure

$$E_0 = E_1 = \cdots = E_\infty.$$  

In contrast, infinite-dimensional Banach spaces are nowhere locally compact. Therefore, sc-structures on infinite-dimensional Banach spaces cannot be constant sc-structures. We give a quick non-trivial example of a sc-Banach space.

**Example 3.3.** Consider the space

$$E = \text{C}^1(S^1) = \left\{f : S^1 \to \mathbb{R} : \|f\|_\infty < \infty, \text{f is 1-times continuously differentiable}\right\},$$

i.e. the space of bounded one times continuously differentiable functions on the unit circle. It is a well known fact that $\text{C}^1(S^1)$ is a Banach space, see [10,61]. It is also known that the following subspaces also are Banach spaces

$$E_m = \text{C}^{m+1}(S^1)$$

where $\text{C}^{m+1}(S^1)$ denotes $m+1$-times everywhere continuously differentiable functions over $S^1$. Using the Arzelä-Ascoli Theorem, it can be shown that $E$ together with the collection $\{E_m\}_{m=0}^\infty$ is a sc-structure on $E$.

**Notation 3.4.** For a sc-Banach space $E$, a subset $A \subseteq E$ will inherit a filtration $\{A_m\}_{m=0}^\infty$, $A_m = A \cap E_m$. We shall by $A^k$ mean the set $A_k$ together with the induced filtration $\{A_{k+m}\}_{k=0}^\infty$.

It follows naturally that there are morphisms between sc-Banach spaces, i.e. linear operators that respect the filtration on the sc-Banach spaces. Hence, it is possible to consider the category of sc-Banach spaces.

**Definition 3.5.** Let $E, F$ be Banach spaces equipped with sc-structures $\{E_m\}, \{F_m\}$, respectively. A linear operator $L : E \to F$ is called a sc-operator if $L(E_m) \subseteq F_m$, and $L|_{E_m} : E_m \to F_m$ is continuous for all $m \geq 0$. Furthermore, an sc-operator $L : E \to F$ is said to be a sc-isomorphism if it is bijective and its inverse is also an sc-operator.

There is a useful special case of sc-operators that transfer the scale structure up one level, hence the following notation.

**Definition 3.6.** Let $E, F$ be Banach spaces equipped with sc-structures $\{E_m\}, \{F_m\}$, respectively. A sc-operator $L : E \to F$ is called a sc$^+$-operator, if $L(E_m) \subseteq F_{m+1}$ for all $m \geq 0$, and if $L : E \to F^1$ is a sc-operator.
Remark 3.7. It is important to note that by Definition 3.1, the inclusion \( \iota : E_m \hookrightarrow E_{m+1} \) is compact. Then, it follows that a sc\(^+-\)operator, \( L : E \rightarrow F \), is a compact operator, i.e. the map \[ L|_{E_m} : E_m \rightarrow F_m \] is a compact operator for all \( m \geq 0 \).

The definition of sc-subspaces of sc-Banach spaces follows in a natural way.

Definition 3.8. Let \( E \) be a Banach space equipped with a sc-structure \( \{E_m\}_{m \geq 0} \), and let \( F \subseteq E \) be a Banach subspace. We shall call \( F \) a sc-subspace if \( F \) is closed in \( E \) and if the sequence, \( \{E_m \cap F\} \), is the sc-structure on \( F \).

Notation 3.9. When sc-Banach spaces are concerned, we have need of the following convention. Henceforth, we shall assume that \( X \oplus Y \) denotes the direct sum of the Banach spaces \( X, Y \) as in Definition 2.4 together with the sc-structure \( \{X_m \oplus Y_m\}_{m=0}^\infty \).

Next, we define partial quadrants in sc-Banach spaces. These shall be used to model corners and edges in the same way that half spaces model edges on classical manifolds.

Definition 3.10. Let \( E \) and \( W \) be sc-Banach spaces, and let \( C \subseteq E \) be a closed (geometrically) convex subset. If there exists a sc-isomorphism \( L : E \rightarrow \mathbb{R}^n \oplus W \) with \( L(C) = [0, \infty)^n \oplus W \), then \( C \) is called a partial quadrant in \( E \).

To lessen the notational clutter, we shall follow the litterature and use open local M-Polyfold models.

Definition 3.11. Let \( E \) be a sc-Banach space, let \( C \subseteq E \) be a partial quadrant in \( E \), and let \( U \subseteq C \) be a relatively open set. Then, the tuple \( (U, C, E) \) shall be referred to as a open local M-polyfold model.

We need to extend our notion of scale-continuous operators in Definition 3.5 to open local M-polyfold models.

Definition 3.12. Let \( (U, C, E) \), \( (V, D, F) \) be open local M-polyfold models. A map \( f : U \rightarrow V \) is called sc\(^0\) (or class sc\(^0\), or sc-continuous), if \( f(U_m) \subseteq V_m \) and the maps \( f|_{U_m} : U_m \rightarrow V_m \) are continuous for all \( m \geq 0 \).

Further, sc-differentiability shall be necessary to introduce. Notice that although it might seem somewhat technical, observe that sc\(^0\), Fréchet differentiable maps are a special case of sc-differentiable maps.
Definition 3.13. Let \((U, C, E), (V, D, F)\) be open local M-polyfold models. A map \(f : U \to V\) is called \(\text{sc}^1\) (or of class \(\text{sc}^1\)) if the following conditions hold:

i) \(f\) is \(\text{sc}^0\); 

ii) for every \(x \subseteq U_1\) there exists a bounded linear operator \(A : E \to F\) such that for \(h \in E_1\) satisfying \(x + h \in U_1\), it holds that 
\[
\lim_{|h|_1 \to 0} \frac{|f(x + h) - f(x) - A(h)|_0}{|h|_1} = 0.
\]

We denote any such linear operator \(A\) satisfying the above by \(Df(x)\); 

iii) the mapping \(Df : U_1 \oplus E \to F\) defined by
\[
Df(x, h) := Df(x)h
\]
is \(\text{sc}^0\) for all \(x \in U_1\) and all \(h \in E\).

For \(k \geq 2\), we inductively define a function, \(f : U \to V\), to be \(\text{sc}^k\), if the mapping 
\[
D^{k-1}f : U^{k-1} \oplus \bigoplus_{j=1}^{k-1} E^{k-1-j} \to F.
\]
is \(\text{sc}^1\). Furthermore, \(f\) is called \(\text{sc}^\infty\) (or \(\text{sc-smooth}\)) if it is \(\text{sc}^k\) for all \(k \geq 0\).

Remark 3.14. Observe that for each fixed \(x \in U_k\), it is possible to view the mapping
\[
D^k f : U^k \oplus \bigoplus_{j=1}^k E^{k-j} \to F
\]
as a \(k\)-multilinear mapping
\[
D^k f(x) : \bigoplus_{j=1}^k E^{k-j} \to F.
\]
Depending on philosophy it is also possible to interpret \(D^k f(x)\) for each \(x \in U_k\) as
\[
D^k f(x) : x \mapsto L\left(E^{k-1}, L(E^{k-2}, L(E^{k-3}, \ldots, L(E^1, L(E, F)))\right).
\]

Though, for a fixed \(x \in U_1\) it does not in general hold that the map \(Df(x) : E \to F\) is sc-continuous. Nevertheless, it is indeed the case that \(Df(x) : E \to F\) is sc-continuous whenever \(x \in U_\infty\). For further discussion on this subject see [18].

Example 4.15 in [16] gives a very important non-trivial example of a sc-smooth map, the translation action on continuous functions on the unit circle.
Example 3.15. We shall construct a sc-smooth map onto the sc-Banach space similar to the one introduced in Example 3.3. Consider the translation action given by
\[ \tau : \mathbb{R} \oplus C^0(S^1), \quad (s, \gamma) \mapsto \gamma(\cdot + s). \]
As a matter of fact, \( \tau \) has directional derivatives only for points \((s_0, \gamma_0) \in \mathbb{R} \oplus C^1(S^1)\) and is nowhere classically differentiable. Despite this dire outlook, if \( C^0(S^1) \) is equipped with the sc-structure \( \{ C^k(S^1) \}_{k \geq 0} \) similarly to the one found in Example 3.3, \( \tau \) is sc\(^1\). For a point \((s_0, \gamma_0) \in C^1(S^1)\), the differential of \( \tau \) at \((s_0, \gamma_0)\) is the following bounded linear mapping
\[ D\tau(s_0, \gamma_0)(S, \Gamma) = S\gamma'_0(s_0 + \cdot) + \Gamma(s + \cdot) = S\tau(s_0, \gamma'_0) + \tau(s_0)(s_0, \Gamma), \]
where \((S, \Gamma) \in \mathbb{R} \oplus C^0(S^1)\).
\[ \square \]

For a motivation why one would be interested in sc-smooth maps given above the reader is referred to [79].

2. Properties of Scale-Differentiability

There are some eye-opening properties connected to scale-differentiability and classical continuous differentiability. We shall follow that of Proposition 2.1 in [35].

Proposition 3.16. Let \((U, C, E)\) be an open local \(M\)-polyfold model, and let \(F\) be a sc-Banach space. Furthermore, suppose that \(f : U \to F\) is a function of class sc\(^0\). Then, \(f\) is of class sc\(^1\) if, and only if, the following two conditions are true:

i) for every \(m \geq 1\), the induced map
\[ f|_{U_m} : U_m \to F_{m-1} \]
is of class \(C^1\), i.e. classically once continuously differentiable. In particular the map
\[ f' : U_m \to L(E_m, F_{m-1}) \]
is continuous;

ii) for every \(m \geq 1\) and every \(x \in U_m\), the bounded linear operator \(f'(x) : E_m \to F_{m-1}\) can be extended to a bounded linear operator \(Df(x) : E_{m-1} \to F_{m-1}\). Furthermore, the map
\[ U_m \oplus E_{m-1} \to F_{m-1}, \]
defined as \((x, h) \mapsto Df(x)h\) is continuous.
3.2. Properties of Scale-Differentiability

Proof. $\Leftarrow$: Condition $i)$ and $ii)$ of the Proposition above directly implies property $ii)$ and $iii)$ in Definition 3.13. Since property $i)$ is satisfied by assumption there is nothing left to prove.

$\Rightarrow$: Let $(U, C, E)$ be an open local $M$-polyfold model, let $F$ be a sc-Banach space, and let $f : U \to F$ be a function of class $\text{sc}^1$. By definition it follows directly that the induced map $f : U_1 \to F$ is (classically) differentiable at every point $x \in U_1$, and the derivative can be identified as $f'(x) = Df(x)|_{E_1} : L(E_1, F)$. Hence we can extend $f'(x)$ to the continuous linear map $Df(x) : E \to F$. We need to show that

$$f' : U_1 \to L(E_1, F), \quad x \mapsto f'(x)$$

is continuous. We shall argue by contradiction. Suppose there is a number $\epsilon > 0$, sequences $\{x_n\}$, $x_n \in U_1$, and $\{h_n\}$, $h_n \in E_1$, with $x_n \to x \in U_1$, $\|h_n\| = 1$ such that

$$\|f'(x_n)h_n - f'(x)h_n\| \geq \epsilon. \quad (3.1)$$

By choosing a subsequence $\{h_{n_j}\}$, of $\{h_n\}$, and using that the inclusion map $\iota : E_1 \hookrightarrow E_0$ is compact, we may assume that $h_{n_j} \to h$ as a sequence in $E_0$. By continuity it holds that,

$$\lim_{x_n \to x} \lim_{h_{n_j} \to h} f'(x_n)h_{n_j} = \lim_{x_n \to x} \lim_{h_{n_j} \to h} Df(x_n)h_{n_j} = Df(x)h$$

in $F_0$. Hence, it holds that

$$\lim_{x_n \to x} \lim_{h_{n_j} \to h} f'(x_n)h_{n_j} - f'(x)h_{n_j} = \lim_{x_n \to x} \lim_{h_{n_j} \to h} Df(x_n)h_{n_j} - Df(x)h_{n_j}$$

$$= Df(x)h - Df(x)h = 0$$

in $F_0$. But this contradicts (3.1), and therefore we can conclude that

$$f' : U_1 \to L(E_1, F), \quad x \mapsto f'(x)$$

is continuous.

Next, we shall prove that $f : U_{m+1} \to F_m$ is (classically) differentiable at $x \in U_{m+1}$ and has

$$f'(x) = Df(x)|_{E_{m+1}} \in L(E_{m+1}, F_m)$$

as derivative. Thus $Df(x) \in L(E_m, F_m)$ is a continuous extension to $f'(x)$. In the subsequent computations, observe that $f : U_1 \to F_0$ is of class $C^1$ and that $f'(x) = Df(x)|_{E_1}$ are true. Since the map

$$Df : U_{m+1} \oplus E_m \to F_m, \quad (x, h) \mapsto Df(x)h$$
is continuous, we can estimate for \( x \in U_{m+1} \) and \( h \in E_{m+1} \),
\[
\frac{1}{\|h\|_{m+1}} \|f(x + h) - f - Df(x)h\|_m
\]
\[
= \frac{1}{\|h\|_{m+1}} \left\| \int_0^1 [Df(x + th)h - Df(x)h]dt \right\|_m
\]
\[
\leq \int_0^1 \left\| Df(x + th) \frac{h}{\|h\|_{m+1}} - Df(x) \frac{h}{\|h\|_{m+1}} \right\|_m dt.
\]
Choose a sequence \( \{h_n\} \), \( h_n \in E_{m+1} \), such that \( \lim h_n \to 0 \). Since the inclusion operator \( i : E_{m+1} \to E_m \) is compact it holds that
\[
\lim_{n \to \infty} \frac{h}{\|h\|_{m+1}} = h_0
\]
for some \( h_0 \in E_m \). Using continuity one can conclude the following. The integrand above converges uniformly over \( t \), to \( \|Df(x)h_0 - Df(x)h_0\|_m = 0 \) as \( h \to 0 \), \( h \in E_{m+1} \). Thus, \( f : U_{m+1} \to F_m \) is differentiable at \( x \) with derivative
\[
f'(x) = Df(x)|E_{m+1} \in L(E_{m+1}, F_m),
\]
being a bounded linear operator. The continuity of the map
\[
x \mapsto f'(x) \in L(E_{m+1}, F_m)
\]
follows by the same argument as for \( f'(x) \in L(E_1, F_0) \). Hence, \( f : U_{m+1} \to F_m \) is of class \( C^1 \), and we are done.

The following proposition shows that \( k \)-times sc-differentiability is preserved under level raising, originally proved in [35] as Proposition 2.2.

**Proposition 3.17.** Let \((U, C, E), (V, D, F)\) be local M-polyfold models, and let \( f : U \to V \) be an sc\( k \) function. Then, the induced map \( f : U^1 \to V^1 \) is also sc\( k \).

**Proof.** Let \( f \) be given as stated in the Proposition above. We shall prove the statement inductively over \( k \). First let \( k = 1 \) and assume that \( f : U \to V \) is sc\(^1 \). This is equivalent to property \( i) \) and \( ii) \) in Proposition 3.16 for all \( m \geq 1 \). But property \( i) \) and \( ii) \) in Proposition 3.16 also holds if we replace \( E \) by \( E^1 \), \( F \) by \( F^1 \), \( U \) by \( U^1 \), and consider the restricted map \( f|_{U^1} : U^1 \to F^1 \). Hence by Proposition 3.16 \( f|_{U^1} : U^1 \to F^1 \) is also sc\(^1 \).

Now assume that for some \( k \geq 1 \) it is true that for all sc\(^k \) functions \( f : U \to V \) it holds that the restricted maps \( f|_{U^1} : U^1 \to V^1 \) are also sc\(^k \). Suppose that \( g : U \to V \) is a map of class sc\(^{k+1} \), by definition it holds that
\[
Dg : U^1 \oplus E \to F \text{ is sc}\(^k \).
\]
Using the induction hypothesis it then holds that the restricted map \( Dg|_{U^2 \oplus E^1} : U^2 \oplus E^1 \to F^1 \) is sc\(^k \). But if the derivative to the restricted function \( g|_{U^1} : U^1 \to V^1 \) is sc\(^k \), by definition it follows that \( g|_{U^1} : U^1 \to V^1 \) is sc\(^{k+1} \). \( \square \)
The following result can be found in [35] as Proposition 2.3. It shows that by raising the level of our domain of definition, we obtain classical continuous differentiability.

**Proposition 3.18.** Let \((U, C, E), (V, D, F)\) be open local M-polyfold models, and let \(f : U \to V\) be a \(sc^k\) function. Then, for all \(m \geq 0\), it holds that \(f|_{U_{m+k}} : U_{m+k} \to V_m\) is a \(C^k\) map (i.e. classically \(k\)-times continuously differentiable). Furthermore, \(f|_{U_{m+1}} : U_{m+1} \to V_m\) is of class \(C^l\) for all \(0 \leq l \leq k\) and all \(m \geq 0\).

**Proof.** First, note that the last sentence follows from the preceding one, since every \(sc^k\) function is \(sc^l\) whenever \(0 \leq l \leq k\). Therefore, it is sufficient to prove the first statement only.

Let \((U, C, E), (V, D, F)\) be open local M-polyfold models, and let \(f : U \to V\) be a \(sc^k\) function. Observe that we only need to check that \(f : U_k \to V_k\) is a \(C^k\) function. This is the case since by Proposition 3.17 it holds that \(f : U^m \to V^m\) is \(sc^k\), for all \(m \geq 0\). Thus the result follows by repeating the proof on \(f : U^m \to V^m\).

We shall prove the statement by induction over \(k\). For \(k = 0\) that \(f : U_0 \to F_0\) is a \(C^0\) function, is trivially true. Furthermore, note that the case \(k = 1\) is exactly condition i) in Proposition 3.16. Henceforth, assume that for some \(k \geq 1\), the statement that for all \(sc^k\) functions \(f : U \to V\), the induced map \(f : U_k \to F_0\) is \(C^k\), holds true. We now want to show that for all \(sc^{k+1}\) functions \(g : U \to V\) the induced map \(g : U_{k+1} \to F_0\) is \(C^{k+1}\).

In particular, it holds that \(g\) is \(sc^k\) and thus by the induction hypothesis, it is of class \(C^k\) on \(U_k\). Furthermore, the derivative \(Dg : U_1 \oplus E_0 \to F_0\) is also of class \(sc^k\) and hence \(\Phi = Dg|_{U_{k+1} \oplus E_k}\) is also of class \(C^k\). By taking the derivative of \(\Phi(x, h)\), with respect to \(x \in U_{k+1}\), \(k\)-times, the following continuous map is obtained,

\[
D^k\Phi : U_{k+1} \oplus E_k \to L^k(E_{k+1}, \ldots, E_{k+1}; F_0)
\]

defined by \((x, h) \mapsto (D^k\Phi(x))h\). Note that this map is linear over \(h \in E_k\) and continuous. Thus, we can consider the map

\[
\Gamma : U_{k+1} \to L^{k+1}(E_{k+1}, \ldots, E_{k+1}; F_0)
\]

defined by

\[
\Gamma(x) : E_{k+1} \oplus \cdots \oplus E_{k+1} \to F_0, \quad (h_1, \ldots, h_k, h) \mapsto (D^k\Phi(x))h(h_1, \ldots, h_k).
\]

We need to show that \(\Gamma\) is continuous. We shall argue by contradiction. Assume there is a point \(x \in U_{k+1}\), a number \(\epsilon > 0\), and \(k+2\) sequences

\[
\{x_n\}, \{h_{1n}\}, \ldots, \{h_{kn}\}, \text{ and } \{h_n\},
\]

with

\[
(x_n, h_{1n}, \ldots, h_{kn}, h_n) \in U_{k+1} \oplus E_{k+1} \oplus \cdots \oplus E_{k+1}.
\]
Choose these sequences such that \( x_n \to x \) in \( U_{k+1} \), \( \|h_{kn}\|_{k+1} = \|h_{kn}\|_{k+1} = 1 \) for all \( n, k \geq 0 \), and

\[
\| (\Gamma(x_n) - \Gamma(x))(h_{1,n}, \ldots, h_{k,n}, h_n) \|_0 \geq \epsilon > 0. \tag{3.2}
\]

Since the inclusion \( \iota : E_{k+1} \hookrightarrow E_k \) is compact, by taking a subsequence \( \{h_{n_j}\} \) of \( \{h_n\} \), we may assume that \( h_{n_j} \to h \) in \( E_k \). Thus, \( (x_n, h_{n_j}) \to (x, h) \) in \( U_{k+1} \oplus E_k \), and by continuity it holds that

\[
\| (D^k\Phi(x_n))h_{n_j} - (D^k\Phi(x))h \|_{L^{k+1}(E_{k+1}; F_0)} \to 0.
\]

But it holds that

\[
\| (\Gamma(x_n) - \Gamma(x))(h_{1,n}, \ldots, h_{k,n}, h_{n_j}) \|_0
\]

\[
= \| \left( (D^k\Phi(x_n))h_{n_j} - (D^k\Phi(x))h_{n_j} \right)(h_{1,n}, \ldots, h_{k,n}, h_{n_j}) \|_0
\]

\[
\leq \| (D^k\Phi(x_n))h_{n_j} - (D^k\Phi(x))h_{n_j} \|_{L^{k+1}(E_{k+1}; F_0)}.
\]

This is a contradiction to (3.2), and therefore \( \Gamma \) is continuous.

Finally, we shall show that \( g : U_{k+1} \to F_0 \) is of class \( C^{k+1} \). In other words we need to prove that

\[
\lim_{\|\delta x\|_{k+1} \to 0} \frac{\|D^kg(x + \delta x) - D^kg(x) - \Gamma(x)(\cdot, \delta x)\|_{L^k(E_{k+1}; F_0)}}{\|\delta x\|_{k+1}} = 0.
\]

Consider \( x \in U_{k+1} \), \( \delta x \in E_{k+1} \) such that \( x + \delta x \in U_{k+1} \), and \( t \in [0, 1] \).

By integrating the map

\[
(t, \delta x) \mapsto Df(x + t\delta x)\delta x
\]

with respect to \( t \) over \([0, 1]\), the following function of class \( C^k \) is obtained

\[
(x, \delta x) \mapsto f(x + \delta x) - f(x).
\]

For \( h_1, \ldots, h_k \in E_{k+1} \), as we differentiate this function \( k \)-times with respect to \( x \) we get that

\[
D^kf(x + \delta x)(h_1, \ldots, h_k) - D^kf(x)(h_1, \ldots, h_k)
\]

\[
= D^k(Df(x + \delta x) - f(x))(h_1, \ldots, h_k)
\]

\[
= D^k\left( \int_0^1 (Df(x + t\delta x)\delta x)dt \right)(h_1, \ldots, h_k)
\]

\[
= \int_0^1 D^k(Df(x + t\delta x)\delta x)dt(h_1, \ldots, h_k)
\]

\[
= \int_0^1 \Gamma(x + \delta x)dt(h_1, \ldots, h_k, \delta x).
\]
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Therefore,

$$\frac{1}{\|\delta x\|_{k+1}} \left[ (D^k g(x + \delta x) - D^k g(x))(h_1, \ldots, h_k) - \Gamma(x)(h_1, \ldots, h_k, \delta x) \right]$$

$$= \int_0^1 [\Gamma(x + \delta x) - \Gamma(x)] \left( h_1, \ldots, h_k, \frac{\delta x}{\|\delta x\|_{k+1}} \right) dt$$

By letting $\delta x \to 0$ in $E_{k+1}$ and by using that $\Gamma : U_{k+1} \to L^{k+1}(E_{k+1}, \ldots, E_{k+1}; F_0)$ is a continuous function, it follows that

$$\frac{1}{\|\delta x\|_{k+1}} \left[ (D^k g(x + \delta x) - D^k g(x))(h_1, \ldots, h_k) - \Gamma(x)(h_1, \ldots, h_k, \delta x) \right]$$

converges to 0 in the norm of $L^k(E_{k+1}, \ldots, E_{k+1}; F_0)$. We can conclude that $g|_{U_{k+1}}$ is of class $C^{k+1}$, and we are done. $\square$

3. The Chain Rule and sc-Smooth Retracts

Of course, in order for the notion sc-differentiability to be a version of a derivative we want it to satisfy the chain rule. Luckily, it does. We shall mostly follow the structure of the proof as it was originally done in [32, Theorem 2.16], though we shall modify the proof to follow a more general setting as seen in [36, Theorem 1.1].

Theorem 3.19. Let $E, F, G$ be sc-Banach spaces, let $C \subseteq E$, $D \subseteq F$, $Q \subseteq G$ be corresponding partial quadrants, and let $U \subseteq C$, $V \subseteq D$, $W \subseteq Q$ be relatively open sets. Furthermore, let $f : U \to V$ and $g : V \to W$ be sc$^1$-maps. Then, the function $g \circ f$ is a sc$^1$-map and

$$D(g \circ f) = D(g) \circ D(f).$$

Before we begin the proof we have need of a result on Bochner integrals. It is often used without much mention in functional analysis, see e.g. [37, Chapter 1.1].

Lemma 3.20. Let $E, F$ and $G$ be Banach spaces, and let $U \subseteq E$, and $V \subseteq F$ be open subsets. Furthermore, let $f : U \to E$, $g : V \to G$ be $C^1$ functions with $f(U) \subseteq V$. Fix a $x \in U$ and a $h \in E$ such that $f(x + h) \in B(f(x), \epsilon) \subseteq V$ for some $\epsilon > 0$. Then the following identity holds

$$g(f(x + h)) - g(f(x)) - D(f(x)) \circ Df(x)h$$

$$= \int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h]dt$$

$$+ \int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h dt.$$
**Proof of Lemma 3.20.** First we need to prove the existence of the above integral. Since $h$ is chosen small enough such that $f(x + h) \in B(f(x), \epsilon) \subseteq V$, and since open balls are convex by Proposition 2.6 it holds that $tf(x + h) + (1 - t)f(x) \in B(f(x), \epsilon)$ for all $t \in [0, 1]$, and in particular $tf(x + h) + (1 - t)f(x) \in V$ for all $t \in [0, 1]$. Hence the mappings

$$
t \mapsto Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h],
$$

$$
t \mapsto [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h,
$$

are continuous as maps $[0, 1] \to G$. By the fundamental theorem of calculus for Bochner integrals (Theorem 2.30) it holds that the integrals

$$
\int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h]dt,
$$

and

$$
\int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h dt
$$

exist. In particular, we can use Theorem 2.30 to compute the following,

$$
\int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h]dt
$$

$$
+ \int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h dt
$$

$$
= \int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x)]dt
$$

$$
- \int_0^1 Dg(f(x)) \circ Df(x)h dt
$$

$$
= \int_0^1 \frac{d}{dt} \left[ g(tf(x + h) + (1 - t)f(x)) \right] dt - \int_0^1 Dg(f(x)) \circ Df(x)h dt
$$

$$
= g(f(x + h)) - g(f(x)) - Dg(f(x)) \circ Df(x)h.
$$

\[ \square \]

**Proof of Theorem 3.19.** Assume, $f$ and $g$ are given as in the theorem above. We need to verify properties $(i)$ – $(iii)$ in Definition 3.13 for the function $g \circ f$. Directly we can conclude that $g \circ f$ is $sc^0$ since it is the composition of two $sc^0$ maps. By Proposition 3.18 it holds that $g : V_1 \to G$ and $f : U_1 \to F$ are of class $C^1$. For $x \in U_1$, we can view $Dg(f(x)) \circ Df(x)$ as an element in $L(E,G)$. Fix $x \in U_1$ and $h \in E_1$ such that $x + h \in U_1$
and \( f(x + h) \in V_1 \). From Lemma 3.20
\[
g(f(x + h)) - g(f(x)) - D(f(x)) \circ Df(x)h
= \int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h]dt
+ \int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h dt.
\]
If the first integral is divided by \( \|h\|_1 \), then
\[
\frac{1}{\|h\|_1} \int_0^1 Dg(tf(x + h) + (1 - t)f(x))[f(x + h) - f(x) - Df(x)h]dt
= \int_0^1 Dg(tf(x + h) + (1 - t)f(x)) \cdot \frac{1}{\|h\|_1} [f(x + h) - f(x) - Df(x)h] dt.
\]
(3.3)

For \( h \in E_1 \), each map \([0, 1] \to F_1\) defined by \( t \mapsto tf(x + h) + (1 - t)f(x)\) are continuous and converge in \( C^0([0, 1], F_1)\) to the map \( t \mapsto f(x)\) as \( \|h\|_1 \to 0 \). Furthermore, the identity
\[
a(h) := \frac{1}{\|h\|_1} [f(x + h) - f(x) - Df(x)h]
\]
converges to 0 since \( f \) is sc\(^1\). Therefore, because \( Dg(x)h \) is continuous, it holds that
\[
(t, h) \mapsto Dg(tf(x + h) + (1 - t)f(x)) \cdot [a(h)]
\]
as map \([0, 1] \times E_1 \to G_0\) converges, uniformly in \( t \), to 0 as \( h \to 0 \). Thus, one can conclude that integral (3.3) converges to 0 as \( \|h\| \to 0 \), for \( h \in E_1 \).

Next, we consider the second integral,
\[
\frac{1}{\|h\|_1} \int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x)h dt
= e \int_0^1 [Dg(tf(x + h) + (1 - t)f(x)) - Dg(f(x))] \circ Df(x) \frac{h}{\|h\|_1} dt.
\]
(3.4)

Because of the compactness requirement in Definition 3.1 it holds that the set of all points \( h/\|h\|_1 \in E_1 \) is relatively compact in \( E_0 \). Hence, using that \( Df(x) \in L(E_0, F_0) \) is continuous according to Definition 3.13, the set of all points
\[
Df(x) \frac{h}{\|h\|_1}
\]
is relatively compact in \( F_0 \). Consequently, every sequence \( \{h_n\}, h_n \in E_1 \), satisfying \( x + h_n \in U_1 \), and \( \|h\|_1 \to 0 \), has a subsequence such that the integrand in the integral (3.4) converges, uniformly in \( t \), to 0. Conclusively,
the integral in (3.4) also converges, uniformly in $t$, to $0 \in G_0$, when $\|h\|_1 \to 0$ and $x + h \in U_1$. We have shown that indeed,

$$\lim_{\|h\|_1 \to 0, x + h \in U_1} \|g(f(x + h)) - g(f(x)) - Dg(f(x)) \circ Df(x) h\|_0 = 0.$$ 

Hence, according to Definition 3.27 it holds that

$$D(g \circ f) = Dg(f(x)) \circ Df(x)L(E_0, G_0),$$

for $x \in U_1$. □

**Definition 3.21.** Let $(U, C, E), (V, D, F)$ be open local M-polyfold models and let $f : U \to V$ be a bijection. If $f$ and $f^{-1}$ both are sc-smooth, then we call $f$ a sc-smooth diffeomorphism. If $f$ and $f^{-1}$ both are sc-smooth, then we call $f$ a sc-smooth diffeomorphism.

In [32–34] something called *splicing cores* were used as the local models for M-polyfolds (for a definition see [32, Definition 3.2]), these can be viewed as images in a Banach space $E$ of some kind of projections on $\mathbb{R}^d \times E$. However, we shall follow [29, 35, 36] and use a more general (and also more modern) model for M-polyfolds, sc-smooth retracts. For a discussion on how these two different models go together see the discussion in [16, Chapter 5].

**Definition 3.22.** Let $(U, C, E)$ be an open local M-polyfold model. A sc-smooth map $r : U \to U$ is called a sc-smooth retraction if it holds that

$$r \circ r = r.$$

**Remark 3.23.** Notice that this is a somewhat direct generalisation of a differentiable retraction on Banach spaces as seen in [57]. Obviously, we generalise sc-smooth retractions from classical differentiable maps to sc-differentiable maps. Furthermore, sc-smooth retractions are also generalised in the sense that it is a map on subsets of partial quadrants of Banach spaces as opposed to the classical case which only considers maps of the total space.

The objects that we are interested in are not the sc-smooth retractions, but rather the image of the sc-smooth retractions.

**Definition 3.24.** Let $(U, C, E)$ be an open local M-polyfold model, and let $O \subseteq E$ be a subset. We shall call the tuple $(O, C, E)$ a sc-smooth retract, if there exists a sc-smooth retraction $r : U \to U$ such that $r(U) = O$.

Henceforth, we shall occasionally call sc-smooth retracts local M-polyfold models. This is to emphasise that sc-smooth retracts are the actual local models of the global spaces. Notice that in order to keep Definition 3.11 and Definition 3.24 separate the word open is omitted when we refer to
3.3. The Chain Rule and sc-Smooth Retracts

sc-smooth retracts. Quite naturally we can define the tangent space on a point in a sc-smooth retract.

**Definition 3.25.** Let \((O, C, E)\) be a sc-smooth retract, and let \(r : U \to U, \ U \subseteq C\) relatively open, be a sc-smooth retraction corresponding to \((O, C, E)\). We define the tangent space at a point \(x \in O_1\) as the subspace

\[T_xO = \text{im} \, Dr(x) \subseteq E_0.\]

**Remark 3.26.** Observe that in [36] the tangent space of sc-smooth retracts are defined as tuples \((T_xO, T_xC, T_xE)\). As \(T_xC = T_xE = E_0\) for all sc-smooth retracts, this notation is redundant for our purposes.

In particular, we are interested in sc-smooth maps between sc-smooth retracts.

**Definition 3.27.** Let \((U, C, E)\), \((V, D, F)\) be local M-polyfold models, and let \(O \subseteq U, \ O' \subseteq V\) be sets. Furthermore, suppose that there is a sc-smooth retraction \(r : U \to U\) for the sc-smooth retract \((O, C, E)\). A mapping \(f : O \to P\) between two sc-smooth retracts \((O, C, E)\) and \((P, D, F)\) is called sc-smooth, if

\[f \circ r : U \to F\]

is sc-smooth.

Of course we need to show that the definition above is independent on the choice of retraction \(r\). To show this, we shall first need a lemma. We modify the proof of Proposition 2.2 in [36] to follow the notation and conventions used in this essay.

**Lemma 3.28.** Let \((O, C, E)\) be a sc-smooth retract, and let \(r : U \to U, \ s : V \to V, \ with \ U, V \subseteq C\) relatively open, be corresponding sc-smooth retractions, i.e. \(r(V) = s(U) = O\). Then, for all \(x \in U_1\), and all \(y \in V_1\) such that \(r(x) = s(y)\), it holds that

\[\text{im} \, Dr(x) = \text{im} \, Ds(y).\]

**Proof.** Let \(x \in U\), then by surjectivity, there exists a \(y \in V\) such that \(r(x) = s(y)\). Therefore, it holds that \(s \circ r(x) = s \circ s(y) = s(y) = r(x)\), and similarly

\[r \circ s(y) = r \circ r(x) = r(x) = s(y).\]

Hence, we can conclude that \(s \circ r = r\), and \(r \circ s = s\).

For each \(x \in U_1\) consider \(h \in \text{im} \, Dr(x)\), then we want to show that \(h \in \text{im} \, Ds(y)\) for all \(y \in V_1\) such that \(r(x) = s(y)\). Furthermore, choose \(k \in E\)
such that $Dr(x)k = h$. Since $s \circ r = r$ and from the chain rule (Theorem 3.19), it holds that 
\[
Ds(y)h = Ds(r(x))h = Ds(r(x))[Dr(x)k] = D(s \circ r)(x)k = Dr(x)k = h.
\]
Therefore, it holds that $im Dr(x) \subseteq im Ds(y)$. Conversely, for each $y_1$ consider $u \in im Ds(y)$, now we want to show that $u \in im Dr(x)$ for all $x \in U_1$ such that $s(y) = r(x)$. Once again, choose $v \in E$ such that $Ds(y)v = u$. Since $r \circ s = s$ and by the chain rule (Theorem 3.19), it holds that 
\[
Dr(x)u = Dr(s(y))u = Dr(s(y))[Ds(y)v] = D(r \circ s)(y)v = Ds(y)v = u.
\]
We can conclude that $im Ds(y) \subseteq im Dr(x)$. Thus, $im Ds(y) = im Dr(x)$. □

From Lemma 3.28 we can now identify that indeed Definition 3.27 is independent of the choice of sc-smooth retraction, see also Proposition 2.3 in [36].

**Proposition 3.29.** Let $(O,C,E)$ and $(P,D,F)$ be sc-smooth retracts, and let $r : U \to U$, $s : V \to V$, with $U,V \subseteq C$ relatively open, be sc-smooth retractions corresponding to $(O,C,E)$. Furthermore, let $f : O \to P$ be a function, then the following holds:

i) $f \circ r : U \to F$ is sc-smooth if, and only if, $f \circ s : V \to F$ is sc-smooth;

ii) if $f \circ r$ is sc-smooth, then $D(f \circ r)(x) = D(f \circ s)(x)$ for every $x \in O_1$;

iii) let $t : W \to W$ be a sc-smooth retraction for $(P,D,F)$, if $f \circ r$ is sc-smooth, then $D(f \circ r)(x)h \in imDt(f(x))$ for all $x \in O_1$ and all $h \in E$.

**Proof.** i) : Assume that $f \circ r : U \to F$ is sc-smooth for some sc-smooth retraction $r : U \to U$. Since the map $s : V \to U \cap V$ is sc-smooth, and $O = r(U) \subseteq V$ it follows from the chain rule (Theorem 3.19) that $f \circ r \circ s : V \to F$ is sc-smooth. From the identity $r \circ s = s$ we can conclude that 
\[
f \circ r \circ s = f \circ s
\]
is sc-smooth. Since the choice of $r$ and $s$ was arbitrary statement i) holds.

ii) : Let $x \in O_1$ be a point, and let $h \in im Dr(x)$. By Lemma 3.28 it holds that $h \in im Ds(x)$ and $h = Ds(x)h$, using that $f \circ r \circ s = f \circ s$ is sc-smooth by step i), and the chain rule (Theorem 3.19), it holds that 
\[
D(f \circ r)(x)h = D(f \circ r)(x)[Ds(x)h] = D(f \circ r \circ s)(x)h = D(f \circ s)(x)h,
\]
and we have proved statement ii).
iii) Now, let \( t : W \to W \) be a sc-smooth retraction for \( (P, D, F) \) and suppose that \( f \circ r : V \to F \) is sc-smooth. It holds that \( t \circ f = f \) and therefore

\[
t \circ f \circ r = f \circ r
\]

holds true. Then, for all \( h \in \text{im } Dr(x) \) the chain rule (Theorem 3.19) implies that

\[
D(f \circ r)(x)h = D(t \circ f \circ r)(x)h = Dt(f(r(x))) \circ D(f \circ r)(x)h.
\]

Since \( D(f \circ r)(x)h \subseteq F \), it follows from the above that \( Df(f \circ r)(x)h \in \text{im } Dt(f(r(x))) \), and we are done. \( \square \)

For our purposes Lemma 3.30 is useful. In particular it shall be useful when discussing tangent spaces in Chapter 5.

**Lemma 3.30.** Let \((O, C, E)\) be a sc-smooth retract together with a corresponding sc-smooth retraction \( r : U \to U, \) \( U \subseteq C \) relatively open, and let \( x \in U_1 \) be a point. Then, \( \text{im } Dr(x) \) for the linear mapping \( Dr(x) : E \to E \) is a Banach space.

**Proof.** Let \( \{a_n\}, a_n \in \text{im } Dr(x) \), be a Cauchy sequence in \( \text{im } Dr(x) \).

Since \( \text{im } Dr(x) \subseteq E \), \( \{a_n\} \) is a Cauchy sequence in \( E \) as well. Since \( E \) is a Banach space \( \{a_n\} \) converges to a point, say \( b \in E \). Assume that \( b \notin \text{im } Dr(x) \), we shall show that this leads to a contradiction. Let \( a = Dr(x)b \), thus by the assumption above \( b \neq a \). We have that for every \( \epsilon > 0 \) there is a \( N \in \mathbb{N} \) such that

\[
\|a_n - b\| < \frac{\epsilon}{\|Dr(x)\|}, \tag{3.5}
\]

for all \( n \geq N \). But notice that

\[
\|a_n - a\| = \|Dr(x)(a_n - b)\| \leq \|Dr(x)\|\|a_n - b\|. \tag{3.6}
\]

Taking Inequality (3.5) together with Equation (3.6), and using the fact that since \( r \) is sc-smooth \( Dr(x) \) is sc-smooth, and in particular continuous, we have that

\[
\|a_n - a\| < \epsilon
\]

for all \( n \geq N \). But this means that \( \{a_n\} \) converges to \( a \in \text{im } Dr(x) \), this is a contradiction to that the sequence converges to \( b \neq a \). Hence, \( \text{im } Dr(x) \) is a Banach space. \( \square \)

**Remark 3.31.** In Lemma 3.30 it is tempting to use the fact that linear bounded operators that are bounded below have closed image [17, Proposition 8.4], to then use that closed subsets of Banach spaces are Banach spaces of their own [14]*Chapter 1.5. But alas, \( Dr(x) : E \to E \) is not necessarily bounded below. This is easy to see considering that \( r : \mathbb{R}^2 \to \mathbb{R}^2 \), defined as

\[
r(x, y) = (x, 0)
\]
is a sc-smooth retraction.

The result following Proposition 2.4 in [36] shall be critical for generating the topology on M-polyfolds.

**Proposition 3.32.** Let \((O, C, E)\) be a sc-smooth retract. Then the following statements are true:

i) if \(O' \subseteq O\) is relatively open in \(C\), then \((O', C, E)\) is a sc-smooth retract;

ii) let \(W \subseteq O\) be a relatively open set in \(C\), and let \(s : W \rightarrow W\) be a sc-smooth map such that \(s \circ s = s\), then for \(O' = s(W)\), \((O', C, E)\) is a sc-smooth retract.

**Proof.** Throughout the proof, let \((O, C, E)\) be a sc-smooth retract.

i): Let \(O' \subseteq O\) be relatively open in \(C\). There exists a sc-smooth retraction \(r : V \rightarrow V\), with \(V \subseteq C\) relatively open, such that \(O = r(V)\). Since \(r\) is sc-smooth it implies that \(r\) is continuous. Therefore, it holds that \(U = r^{-1}(O')\) is an open subset in \(V\), hence \(U\) is also a relatively open subset of \(C\). Since \(r\) is a retraction \(O' \subseteq U\) and the map \(r|_U : U \rightarrow U\) is a sc-smooth retraction onto \(r(U) = O'\). Thus, \((O', C, E)\) is a sc-smooth retract.

ii): Let \(W \subseteq O\) be a relatively open set in \(C\), and let \(s : W \rightarrow W\) be a sc-smooth map such that \(s \circ s = s\) and \(O' = s(W)\). From part i) it follows that \((W, C, E)\) is a sc-smooth retract. Therefore, there exists a sc-smooth retraction \(r : U \rightarrow U\), with \(U \subseteq C\) open, such that \(r(U) = W\). Define the map \(\rho : U \rightarrow U\) by

\[
\rho := s \circ r.
\]

By the chain rule (Theorem 3.19) \(\rho\) is sc-smooth. Suppose that \(x \in U\), then it holds that \(r(x) \in W\) and therefore \(s(r(x)) \in W\). It follows that

\[
r\big(s(r(x))\big) = s(r(x)),
\]

and hence

\[
r \circ \rho = \rho.
\]

From that \(s \circ s = s\) it also follows that

\[
\rho \circ \rho = \rho.
\]

Conclusively, \(\rho\) is a sc-smooth retraction onto \(O'\) and \((O', C, E)\) is a sc-smooth retract. \(\square\)
CHAPTER 4

M-Polyfolds

In this chapter we shall first treat the main definitions related to M-polyfolds, our new version of differentiable manifolds. A loose description of M-polyfolds is that they are metrizable spaces that can change dimension drastically in a sc-smooth way. For example, in Chapter 7 we shall construct a M-polyfold which is the unit disk in $\mathbb{R}^2$ together with a 1-dimensional curve attached to it. For this to work, our main strategy is to replace Fréchet differentiability on classical Banach manifolds with sc-smoothness. Secondly, we shall discuss the boundary behaviour of M-polyfolds, and then finish this chapter by briefly discussing some results connected to sc-smooth partition of unity.

1. The Definition of M-polyfolds

Philosophically, M-polyfolds are constructed using building blocks similar to those used when constructing classical differentiable manifolds. Of course, the basic building block for M-polyfolds are charts. In contrast to classical manifolds these are bijections to sc-smooth retracts of Banach spaces as opposed to open subsets, this makes a significant difference for the behaviour of the dimensions.

**Definition 4.1.** Let $M$ be a set, let $V \subseteq M$ be a subset, let $(O, C, E)$ be a sc-smooth retract, and let $\phi : V \rightarrow O$ be a bijection. The tuple $(V, \phi, (O, C, E))$, is called a chart on $M$.

The definition for sc-smooth atlases generalises quite naturally from the definition of classical smooth atlases.

**Definition 4.2.** Let $M$ be a set and let $\{(V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in \mathcal{A}}$ be a collection of charts on $M$. The collection $\{(V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in \mathcal{A}}$ is called a sc-smooth atlas on $M$ if

1. $\bigcup_{\alpha \in \mathcal{A}} V_\alpha = M$;
2. for every two charts $(V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))$ and $(V_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta))$ such that $V_\alpha \cap V_\beta \neq \emptyset$, the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(V_\alpha \cap V_\beta)} : \phi_\alpha(V_\alpha \cap V_\beta) \rightarrow \phi_\beta(V_\alpha \cap V_\beta)$$

are sc-smooth diffeomorphisms.
4.1. The Definition of M-polyfolds

Obviously, we are inclined to discuss equivalent and maximal atlases. Their importance is evident.

**Definition 4.3.** Two sc-smooth atlases are called *equivalent* if the union of the two atlases also is a sc-smooth atlas. The *maximal* sc-smooth atlas for an sc-smooth atlas

\[ \{ (V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)) \}_{\alpha \in \mathcal{A}} \]

is the union of all equivalent sc-smooth atlases to \{ (V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)) \}_{\alpha \in \mathcal{A}}. Such a maximal sc-smooth atlas shall be referred to as a *M-polyfold structure*.

We need to show that the definition above indeed is an equivalence relation of atlases. We shall follow a similar strategy to that of [Chapter 5.2]. Reflexivity is obvious, and so is symmetry. Now suppose that \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \) are sc-smooth atlases for some M-polyfold \( M \). Furthermore, suppose that \( \mathcal{A}_1 \cup \mathcal{A}_2 \) and \( \mathcal{A}_2 \cup \mathcal{A}_3 \) both are sc-smooth atlases. The collection of charts \( \mathcal{A}_1 \cup \mathcal{A}_3 \) obviously satisfy property i) of Definition 4.2. Now let \((V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)) \in \mathcal{A}_1, (V_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta)) \in \mathcal{A}_2, \) and let \((V_\gamma, \phi_\gamma, (O_\gamma, C_\gamma, E_\gamma)) \in \mathcal{A}_3\) be charts such that \( V_\beta \cap V_\alpha \neq \emptyset \) and \( V_\beta \cap V_\gamma \neq \emptyset \). Then, the maps

\[ \phi_\beta \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\beta)} : \phi_\alpha(V_\alpha \cap V_\beta) \to \phi_\beta(V_\alpha \cap V_\beta), \]

and

\[ \phi_\gamma \circ \phi_\beta^{-1} |_{\phi_\beta(V_\beta \cap V_\gamma)} : \phi_\beta(V_\beta \cap V_\gamma) \to \phi_\gamma(V_\beta \cap V_\gamma). \]

are sc-smooth. If \( V_\alpha \cap V_\gamma \) is empty we are done. Therefore, suppose that \( V_\alpha \cap V_\gamma \) is nonempty. Then choose a point \( p \in V_\alpha \cap V_\gamma \) and choose \( \beta \) such that \( p \in V_\beta \), this is possible as an atlas covers the set \( M \). Then for the map

\[ \phi_\gamma \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)} : \phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta) \to \phi_\gamma(V_\alpha \cap V_\gamma \cap V_\beta), \]

it holds that

\[ \phi_\gamma \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)} = \phi_\gamma \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)} \]

\[ = (\phi_\gamma \circ \phi_\beta^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)}) \circ (\phi_\beta \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)}). \]

By the chain rule (Theorem 3.19) it holds that the above map is sc-smooth. Since \( p \) was chosen arbitrarily \( \phi_\gamma \circ \phi_\alpha^{-1} |_{\phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta)} \) is sc-smooth everywhere on \( V_\alpha \cap V_\gamma \), and transitivity follows.

There is one crucial detail that the above result depends on, i.e that \( \phi_\alpha(V_\alpha \cap V_\gamma \cap V_\beta) \subseteq O \) are sc-smooth retracts. Also when we generate a maximal atlas we need to use that sc-smooth retracts on open subsets of sc-smooth retracts also are sc-smooth retracts. All this follows from Proposition 3.32. Consequently, we define the *topology induced by an atlas*, \( \mathcal{A} \), on a set \( M \), as the topology that is generated by the collection of chart domains of the maximal sc-smooth atlas for \( \mathcal{A} \).
**Definition 4.4.** Let $M$ be a set together with a $M$-polyfold structure. If the topology induced by the $M$-polyfold structure is paracompact and Hausdorff, then $M$ is called a $M$-polyfold.

**Remark 4.5.** It is common in current research, see e.g. [16, 36], to use the definition above or to require that the topology is metrizable in the definition of $M$-polyfolds. By the Smirnov metrization theorem (Theorem 2.11) it holds that these two definitions are equivalent. In [32–34] paracompactness is replaced by second countability. For finite-dimensional manifolds the fact that locally Euclidean spaces are locally compact is used to prove that the second countable manifolds are paracompact, see [48, Theorem 4.77]. Alas, as infinite-dimensional Banach spaces are nowhere locally compact any such proof does not work here. By Proposition 2.17, the $M$-polyfold structure implies complete regularity, then by second countability and Uhrysohns metrization theorem (Theorem 2.10), it holds that all second countable Hausdorff spaces with a topology generated from a $M$-polyfold structure are metrizable. Paracompactness (and Hausdorff) or metrizability are mainly used in the definition for $M$-polyfolds, since some topological problems arise when second countability is used see Remark 5.18. More importantly assuming second countability is unnecessarily restrictive for the theory. Also, in some applications, zero sets for sc-Fredholm sections on $M$-polyfolds are compact which implies second countability of the zero set. Hence, in such cases all definitions are the same for the zero sets of sc-Fredholm sections. For further discussion on these issues see [16, Remark 5.2].

Observe that there is a completely new structure on $M$-polyfolds, there is a so called level structure on $M$-polyfolds induced by the sc-structure of the local $M$-polyfold models. We say that the point $p \in M$, $M$ a $M$-polyfold, is on the level $m$ if there is a compatible chart $(V, \phi, (O,C,E))$ around $p$ such that $\phi(p) \in O_m$. This definition is well-defined as it is independent of the choice of chart. To show this, suppose there is another chart around $p$, $(U, \psi, (P, D, F))$, belonging to the $M$-polyfold structure of $M$. Since these charts are compatible, the map

$$
\psi \circ \phi^{-1} |_{\phi(U \cap V)} : \phi(U \cap V) \to \psi(U \cap V)
$$

is a sc-differomorphism, in particular it is a $sc^0$ map. By Definition 3.12, it holds that $(\psi \circ \phi^{-1})(\phi(p)) \in P_m$, and by bijectivity $\psi(p) \in P_m$. Conversely, if $\psi(p) \in P_m$ a similar argument gives us that $\phi(p) \in O_m$. The set of all points in $M$ of level $m$ is denoted as $M_m$. The topology on $M_m$ is exactly that of the subspace topology. Consequently, charts on $M_m$ can be characterised as $(V \cap M_m, \phi |_{V \cap M_m}, (O_m, C_m, E_m))$. We shall denote $M^m$ to mean $M_m$ together with the level structure. This is referred to as raising the index by $m$ on $M$. 
From our previous discussion one can see that a M-polyfold, $M$, inherits a filtration of subspaces

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_\infty = \bigcap_{i=0}^{\infty} M_i$$

from the charts. Lemma 4.6 gives a useful result on the filtration of M-polyfolds. It substantiates the claim that $M_m$ indeed is a subspace of $M$.

Lemma 4.6. The inclusion map $\iota : M_{m+1} \hookrightarrow M_m$ is continuous for all $m \geq 0$.

Proof. The proof is straightforward. All we need to show is that every open subset in $M_m$ projects, i.e. the inverse of the inclusion map, to an open subset in $M_{m+1}$. Let $(V, \phi, (O, C, E))$ be chart belonging to the M-polyfold structure of $M$, and let $W \subseteq O_m$ be an open subset. We need to show that $\phi^{-1}(W) \cap M_{m+1}$ is open in $M_{m+1}$. Suppose that $p \in \phi^{-1}(W) \cap M_{m+1}$. From the induced filtration on $M$, it holds that $\phi(p) \in W \cap O_{m+1} \subseteq W \cap E_{m+1}$. Since $W$ is an open subset of $O_m$, there is an open subset $U \subseteq E_m$ such that $W = U \cap O$. Therefore, we have that

$$\phi(p) \in (U \cap E_{m+1}) \cap O = N \cap O,$$

where $N \subseteq E_{m+1}$ is a relatively open set such that $N = U \cap E_{m+1}$. Hence, $N \cap O$ is an open subset of $O_{m+1}$, and

$$\phi^{-1}(W) \cap M_{m+1} = \phi^{-1}(V \cap O).$$

The proof is finished since $\phi$ is a homeomorphism.

Furthermore, it is straightforward to translate smooth maps between manifolds to sc-smooth maps between M-polyfolds.

Definition 4.7. Let $M, N$ be M-polyfolds and let $f : M \to N$ be a mapping. Suppose that $p \in M$ is a point and that $(V, \phi, (O, C, E))$ is a chart around $p$ belonging to the M-polyfold structure on $M$. Furthermore, let $f(p) = q \in N$, and let $(U, \psi, (P, F, F))$ be chart around $q$ belonging to the M-polyfold structure of $N$ such that $f(V) \subseteq U$. The map $f$ is called \textit{sc-smooth} at $p$ if the map

$$\psi \circ f \circ \phi^{-1} : O \to P$$

is a sc-smooth map between the sc-smooth retracts as in Definition 3.27.

We can also talk about sub-M-polyfolds. Note that Definition 4.8 is a generalised notion of classical regular submanifolds.

Definition 4.8. Let $M$ be a M-polyfold and let $A \subseteq M$ be subset. If for every $a \in A$ there is an open neighbourhood $V \subseteq M$ and a sc-smooth retraction $r : V \to V$ with

$$r(V) = A \cap V,$$
then $A$ is called a \textit{sub-M-polyfold} of $M$.

\textbf{Remark 4.9.} Note that when we say sc-smooth retraction on an open subset $V \subseteq M$ of a M-polyfold $M$, it is not possible to refer to Definition 3.22. Instead what we mean is a sc-smooth map, in the sense of Definition 4.7, $r : V \to V$ such that $r \circ r = r$.

Before we go into more details on sub-M-polyfolds, we shall define the \textit{naturally induced M-polyfold structure}. We begin by constructing a sc-smooth atlas for $A$. Choose a point $a \in A$, and let $(V, \phi, (O, C, E))$ be a chart around $a$. By definition, there is an open neighbourhood $U$ of $a$, and a sc-smooth retraction $r : U \to U$ such that $r(U) = U \cap A$. Define a subset $W \subseteq M$, by

$$W := \left( r^{-1}(U \cap V) \right) \cap (U \cap V).$$

It holds that $W$ is open in $M$ and that $r(W) \subseteq W$, with $r(W) = W \cap A$. Thus, $\phi(W)$ is an open subset of $O$. Therefore, we can apply the first part of Proposition 3.32, and the tuple $(\phi(W), C, E)$ is a sc-smooth retract. Define the sc-smooth map $\rho : \phi(W) \to \phi(W)$ by

$$\rho = \phi \circ r \circ \phi^{-1}|_{\phi(W)}.$$

It follows from $r = r \circ r$ that $\rho = \rho \circ \rho$, and it holds that $\rho$ is a sc-smooth retraction onto $O' := \rho(\phi(W))$. Consequently, the second part of Proposition 3.32 yields that $(O', C, E)$ is a sc-smooth retract. Hence, there is a relatively open set $N \subseteq C$, and a sc-smooth retraction $s : N \to N$ such that $s(N) = O'$. To simplify notation, set $\psi := \phi|_{W \cap A}$. We have that

$$\psi(W \cap A) = \phi(W \cap A) = \phi \circ r(W) = \phi \circ r \circ \phi^{-1}(O) = \rho(O) = O'.$$

Therefore,

$$\psi : W \cap A \to O'$$

is a homeomorphism and the triple $(\psi, W \cap A, (O', C, E))$ is a chart on $A$.

The next step is to treat chart transformations between local models of $A$. Let $(\phi', V', (P', D, F))$ be another chart around $a \in A$ belonging to the M-polyfold structure of $M$. Equivalently to the earlier construction, define the corresponding chart $(\psi', W' \cap A, (P', D, F))$ of $A$. Consider the transition map

$$\psi' \circ \psi^{-1}|_{\psi((W \cap A) \cap (W' \cap A))} : \psi((W \cap A) \cap (W' \cap A)) \to \psi'((W \cap A) \cap (W' \cap A)).$$

From that $\psi((W \cap A) \cap (W' \cap A))$ is an open subset $O'$, and applying the second part of Proposition 3.32 we can argue as follows. There exists a relatively open subset $N' \subseteq C$, and a sc-smooth retraction $s' : N' \to N'$ such that

$$s'(N') = \psi((W \cap A) \cap (W' \cap A)).$$
By construction, it holds that
\[(\psi' \circ \psi^{-1}) \circ s' = (\phi' \circ \phi^{-1}) \circ s'.\]
The chart transformation \(\phi \circ \phi' : O \to P\) is sc-smooth and from the chain rule (Theorem 3.19), the right hand side is sc-smooth. Hence, the left hand side is also sc-smooth, and following Definition 3.27 \(\psi' \circ \psi^{-1}\) is sc-smooth.

We can conclude that the tuples \((W \cap A, \psi, (\psi(W \cap A), C, E))\) induce a sc-smooth atlas for \(A\). As per usual this defines a maximal atlas for \(A\) and hence, a M-polyfold structure on \(A\). We shall call this M-polyfold structure the natural M-polyfold structure for sub-M-polyfolds. With this in mind, we have two properties related to sub-M-polyfolds, see [36, Proposition 2.6].

**Proposition 4.10.** Let \(M\) be a M-polyfold, and let \(A \subseteq M\) be a sub-M-polyfold. Furthermore, suppose that the M-polyfold structure on \(A\) is the natural M-polyfold structure. Then, the following statements are true:

i) the inclusion map \(i : A \to M\) is sc-smooth and a homeomorphism onto its image;

ii) for every point \(a \in A\) and every sc-smooth retraction \(r : V \to V, V \subseteq M\) open, such that \(r(V) = A \cap V\) and \(a \in V\) the map \(i^{-1} \circ r : V \to A\) is sc-smooth.

**Proof.** i) : This follows directly from our previous discussion.

ii) : We shall consider all sets and charts as in the discussion previously. We prove that this statement holds for the local representation of \(i : A \to M\) which we denote as \(i : O' \to O\). Notice that \(i\) is sc-smooth since \(s : N \to N\) is a sc-smooth retraction. Conversely, the relations
\[\phi \circ r \circ \phi^{-1}(O) = O'\]
and \(U = \phi^{-1}(O)\) show that the map \(i^{-1} \circ r : U \to A\) is sc-smooth because the retraction \(r : U \to U\) is sc-smooth, and we are done. \(\square\)

We shall not follow the notation and conventions for germs found in Chapter 2.1 of [33]. We shall instead use conventions more in line with classical differential geometry.

**Definition 4.11.** Let \(M\) and \(N\) be M-polyfolds, and let \(p \in M\) be a point. Furthermore, let \(U, V \subseteq M\) be open neighbourhoods around \(p\), and suppose that \(f : U \to N\) and \(g : V \to N\) are sc-smooth (or sc\(^m\)) functions. Then, \(f\) and \(g\) are said to have the same germ at \(p\), if there is a open neighbourhood \(B \subseteq M\) around \(p\) such that \(f(x) = g(x)\) for all \(x \in B\). We denote the equivalence class \([f]_p\), as the set of all functions \(g : V \to N\) that have the same germ as \(f\) at \(p\). The set of all equivalence classes of functions \([f]_p\), is called the set of germs and is denoted as \(sc^\infty_p(M, N)\) (or \(sc^m_p(M, N)\)).
Let $K$ be a field, i.e. $\mathbb{R}$ or $\mathbb{C}$. We shall define scalar multiplication on germs, addition, and multiplication between germs $[f], [g] \in C^\infty_p(M, K)$ by

$$a[f] + b[g] := [af + bg],$$

$$[f][g] := [fg]$$

where $a, b \in K$. This is of course a $K$-algebra. Using that the evaluation of a germ at $p$, $ev_p([f]) = [f](p) := f(p)$ is well-defined, the proof is algebraically straightforward.

**Proposition 4.12.** Let $M$ be a $M$-polyfold and let $p \in M$ be a point. The set $sc^\infty_p(M, K)$ is a commutative $K$-algebra, where $K \in \{\mathbb{R}, \mathbb{C}\}$.

**Remark 4.13.** From [49], there are some good indications that almost complex $M$-polyfolds might be interesting objects as well. Though, they are considered to be beyond the scope of this essay.

## 2. Boundary Recognition and Degeneracy Index

To keep track on boundaries and corners of $M$-polyfolds, we shall construct the so called degeneracy index which will sense the boundary and the dimension of corners. First, this notion is defined locally. Later, we shall translate this local property to $M$-polyfolds.

**Definition 4.14.** Let $E$ be a sc-Banach space, and let $C \subseteq E$ be a partial quadrant. Furthermore, let $L : E \to \mathbb{R}^k \oplus W$ be a linear sc-isomorphism such that

$$L(C) = [0, \infty)^k \oplus W.$$  

The map $d_C : C \to \mathbb{N}$ defined as

$$d_C(x) = |\{i \in \{1, \ldots, k\} : T(x) = (a_1, \ldots, a_k, w) \text{ and } a_i = 0\}|$$

is called the local degeneracy index of the point $x \in C$.

Of course this definition is independent of the choice of linear sc-isomorphism, and is therefore well-defined.

**Proposition 4.15.** The local degeneracy index, $d_C : C \to \mathbb{N}$, is independent of the choice of linear sc-isomorphism.

**Proof.** Let $E$ be a sc-Banach space, and let $C \subseteq E$ be a partial quadrant. Furthermore, suppose that $L : E \to \mathbb{R}^k \oplus W$ and $L' : E \to \mathbb{R}^{k'} \oplus W'$ are linear sc-isomorphisms such that

$$L(C) = [0, \infty)^k \oplus W, \quad \text{and} \quad L'(C) = [0, \infty)^{k'} \oplus W'.$$

Define $S : \mathbb{R}^k \oplus W \to \mathbb{R}^{k'} \oplus W'$, by

$$S(x) = L' \circ L^{-1}(x).$$
Suppose that $S$ is a linear sc-isomorphism, and that $S([0, \infty)^k \oplus W) = [0, \infty)^{k'} \oplus W'$.

We shall show that $S(\{0\}^k \oplus W) = \{0\}^{k'} \oplus W'$. Suppose that $(a, w') = S(0, w)$ for some $(a, w') \in [0, \infty)^{k'} \oplus W'$. By linearity it holds that

$$S(0, tw) = tS(0, w) = t(a, w') = (ta, w') \in [0, \infty)^{k'} \oplus W'$$

for all $t \in \mathbb{R}$. Note that this only holds if $a = 0$. Hence, $S([0, \infty)^k \oplus W) \subseteq [0, \infty)^{k'} \oplus W'$, and since $S$ is an isomorphism, $S([0, \infty)^k \oplus W) = [0, \infty)^{k'} \oplus W'$. Observe that the set $\{0\}^k \oplus W \cong W$ has codimension $k$ in $\mathbb{R}^k \oplus W$. Furthermore, as $S$ is an isomorphism it holds that

$$S(\mathbb{R}^k \oplus W) = S(\mathbb{R} \oplus \{0\}) \oplus S(\{0\}^k \oplus W) = S(\mathbb{R}^k \oplus \{0\}) \oplus (\{0\}^{k'} \oplus W').$$

Therefore, $\{0\}^{k'} \oplus W'$ has codimension $k$ in $\mathbb{R}^{k'} \oplus W'$. Hence $k = k'$.

Next, we shall show that whenever $x = (a, w) \in [0, \infty)^k \oplus W$ and $y = S(x) = (a', w')$, it holds that

$$n = |\{i \in \{1, \ldots, k\} : a_i = 0\}| = |\{j \in \{1, \ldots, k\} : a'_j = 0\}| = m.$$

Define the subspaces $E_x \subseteq \mathbb{R}^k \oplus W$, and $E'_y \subseteq \mathbb{R}^{k'} \oplus W'$ by

$$E_x = \{(b, v) \in \mathbb{R}^k \oplus W : b_i = 0 \text{ for all } i \text{ such that } a_i = 0\},$$
$$E'_y = \{(b', v') \in \mathbb{R}^{k'} \oplus W' : b'_i = 0 \text{ for all } i \text{ such that } a'_i = 0\}.$$

Observe that $n$ is equal to the codimension of $E_x$ in $\mathbb{R}^k \oplus W$, and correspondingly $m$ is equal to the codimension of $E'_y$ in $\mathbb{R}^{k'} \oplus W'$. To show that $n = m$ it is enough to show that $S(E_x) \subseteq E'_y$. Let $u = (b, v) \in E_x$, and note that for some $|t| < \min\{|a_i| : a_i \neq 0\}$, it holds that

$$x + tu \in [0, \infty)^k \oplus W.$$

Then, we have

$$S(x + tu) = S(x) + tS(u) = y + tS(u).$$

Choose an $i \in \{1, \ldots, k\}$ such that $a'_i = 0$, then

$$S(x + tu)_i = y_i + tS(u)_i = tS(u)_i \in [0, \infty),$$

which is only true if $S(u)_i = 0$. Therefore, $S(E_x) \subseteq E'_y$. Since $S$ is an isomorphism it holds that $S(E_x) = E'_y$, and Proposition 4.15 follows.

Now, we are able to define the degeneracy index for $M$-polyfolds.

**Definition 4.16.** Let $M$ be a $M$-polyfold, and let $p \in M$ be a point. Suppose that

$$\{(V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in A}$$

is the collection of all compatible charts around $p$ belonging to the $M$-polyfold structure of $M$. The map $d_M : M \to \mathbb{N}$ defined by

$$d_M(p) = \min_{\alpha \in A} d_{C_\alpha}(\phi_\alpha(p))$$

is called the degeneracy index of $M$ at $p$. 

\[\square\]
is called the \textit{degeneracy index} for \( p \), where \( d_{C_n} \) is the local degeneracy index defined in Definition 4.14.

Next, Lemma 4.17, also found in [36, Lemma 2.2], gives some intuition on the boundary behaviour of M-polyfolds. That is, boundary and corner points always have a neighbourhood of points with the same or lower degeneracy index.

\textbf{Lemma 4.17.} Let \( M \) be a M-polyfold. Then, for every point \( p \in M \), there exists a neighbourhood \( N \) of \( p \) such that 
\[
d_{M}(p) \geq d_{M}(q)
\]
for all \( q \in N \).

\textbf{Proof.} Let \( M \) be a M-polyfold and let \( p \in M \) be a point. Arbitrarily choose a chart \((V, \phi, (O, C, E))\) around \( p \). We shall prove the statement by contradiction. Suppose there is a sequence of points \( \{ p_n \} \), \( p_n \in V \), such that \( \lim p_n \to p \) and such that \( d_{M}(p_n) > d_{M}(p) \) for all \( n > N \), for some number \( N \in \mathbb{N} \). Without loss of generality, suppose that the modelling space can be written as \( E = \mathbb{R}^k \oplus W \), and \( C = [0, \infty)^k \oplus W \). Since \( \phi \) is continuous, the sequence \( \{ \phi(p_n) \} \) converges to \( \phi(p) \) in \( O \subseteq C \). Denote
\[
\phi(p_n) = (a_1^n, \ldots, a_k^n, v^n) \in O \subseteq [0, \infty)^k \oplus W,
\]
and
\[
\phi(p) = (b_1, \ldots, b_k, w) \in O \subseteq [0, \infty)^k \oplus W.
\]
Suppose that \( a_j^n = 0 \) for all \( n > N \) for precisely the indices \( j \in J \subseteq \{ j \in \mathbb{N} : j \leq k \} \). But then for each \( j \in J \), \( a_j^n \to 0 = b_j \). This directly contradicts that \( d_{M}(p_n) > d_{M}(p) \) for all \( n > N \), and we are done. \( \square \)

Following Theorem 1.19 in [35], it holds that boundary behaviour is preserved under local sc-smooth diffeomorphisms.

\textbf{Theorem 4.18.} Let \((O, C, E)\) and \((P, D, F)\) be local M-polyfold models, and let \( f : O \to P \) be a sc-diffeomorphism. Then, it holds that
\[
d_{C}(x) = d_{D}(f(x))
\]
for all \( x \in O \).

\textbf{Proof.} Suppose that \((O, C, E)\) and \((P, D, F)\) are local M-polyfold models, and let \( f : O \to P \) be a sc-diffeomorphism. We shall first show that it is sufficient to prove the above equality for smooth points. Suppose that the statement holds for smooth points. Observe that for every point \( x \in O \) there exists a sequence \( \{ x_k \} \), \( x_k \in O_\infty \), such that \( \lim x_k \to x \). This is the case since \( O_\infty \) is dense in \( O_0 \). We can choose the sequence such that \( d_{C}(x_k) = d_{C}(x) \) for all \( k \in \mathbb{N} \). By assumption it holds that \( d_{D}(f(x_k)) = d_{C}(x_k) = d_{C}(x) \). By Lemma 4.17 and from that \( f(x_k) \to f(x) \),
it follows that \( d_D(f(x)) \geq d_C(x) \). Using the same argument applied to \( f^{-1} \) equality follows.

With the previous discussion in mind, assume that \( x \in O_{\infty} \). Without loss of generality assume that \( E = \mathbb{R}^n \oplus W \), \( C = [0, \infty)^n \oplus W \), and in a similar fashion \( F = \mathbb{R}^n \oplus W' \), \( D = [0, \infty)^n \oplus W' \). Let \( r : U \to U \), with \( U \subseteq C \) open, be the \( \text{sc} \)-smooth retraction corresponding to \( (O, C, E) \). Furthermore, for \( x = (a, w) \in U \), we denote

\[
N = \ker(id_E - Dr(x)).
\]

Then, it holds that \( N \) is a closed subspace of \( \mathbb{R}^n \oplus W \), then there is a \( \text{sc} \)-complement \( M \subseteq \{0\} \oplus W \) to \( N \). That is, such that \( N \oplus M \cong \mathbb{R}^n \oplus W \). Consequently, \( N \) is a direct sum of two closed subspaces as

\[
N = (N \cap \{0\} \oplus W) \oplus \{(q, p) \in N : (0, p) \in M\} =: N_1 \oplus N_2.
\]

We shall argue why this is the case. Suppose that \((p, q) \in N_1 \cap N_2\), then \( p = 0 \) and \((0, q) \in N \cap M\), thus \( q = 0 \) and \( N_1 \cap N_2 = \{(0, 0)\} \). It holds that we can uniquely decompose \((p, q) \in N\) by setting

\[
(p, 0) = (A, B) + (0, C) = (A, B + C)
\]

where \((A, B) \in N\) and \((0, C) \in M\). Inserting \( C = q - B \) and \( A = p \), we write

\[
(p, q) = (0, q - B) + (A, B)
\]

with

\[
(0, q - B) = (A, q) - (A, B) \in N_1
\]

and \((A, B) \in N_2\).

For \( x = (a, w) \in O \), let \( I \) denote the set of all indices \( 1 \leq i \leq n \) such that \( a_i = 0 \). Note that we have \( d_C(x) = |I| \). Let \( N_x \subseteq N \) denote the subspace of points \((p, q) \in N\) such that \( p_i = 0 \) whenever \( i \in I \). Using the decomposition \( N = N_1 \oplus N_2 \) it follows that \( N_x \) has codimension \( d_C(x) \) in \( N \). We shall make similar arguments for \((a', w') = f(x) \in P \). Let \( J \) denote the set of all indices \( 1 \leq j \leq m \) such that \( a_j' = 0 \). Furthermore denote

\[
N' = \ker(I - Ds(f(x)))
\]

where \( s : V \to V \), \( V \subseteq D \) open, is the \( \text{sc} \)-smooth retraction corresponding to \((P, D, F)\). Also, set \( N'_{f(x)} \subseteq N' \) as the subspace for which \((p, q) \in N'_{f(x)} \) whenever \( p_j = 0 \) for \( j \in J \). Notice that the codimension of \( N'_{f(x)} \) is \( d_D(f(x)) \). Let \((p, q) \in N_x \) be an arbitrary smooth point. It holds that \( x + \tau(p, q) \in U \cap C \) for \( 0 < \tau < \varepsilon \) and \( \varepsilon > 0 \) small enough, and therefore \( r(x + \tau(p, q)) \in O \). Hence,

\[
f \circ r(x + \tau(p, q)) \in P
\]

for all \( 0 < \tau < \varepsilon \). By the chain rule, \( f \circ r \) is \( \text{sc} \)-smooth, and then the map

\[
[0, \varepsilon) \to F_k, \quad \text{whenever } 0 < \tau \mapsto f \circ r(x + \tau(q, p))
\]
sc-smooth for all $k \geq 0$.

Next, for $1 \leq j \leq m$, we define the bounded linear functional $\lambda_j : \mathbb{R}^m \oplus W'$ by $\lambda_j(b, y) = b_j$. It follows that

$$0 \leq \lambda_j(f \circ r(x + \tau(p, q))), \quad 0 < \tau < \varepsilon.$$

Now, choose some $j \in J$. In the following discussion note that

$$f \circ r(x + \tau(p, q)) = f(x) + \tau Df(x)(p, q) + o_k(\tau),$$

where

$$\lim_{\tau \to 0^+} \frac{o_k(\tau)}{\tau} = 0.$$

Furthermore, since $\lambda_j(f(x)) = 0$, we have that

$$0 \leq \frac{1}{\tau} \lambda_j(f(x) + \tau Df(x)(p, q) + o_k(\tau)) = \lambda_j\left(Df(x)(p, q) + \frac{o_k(\tau)}{\tau}\right)$$

by letting $\tau \to 0^+$ on the right hand side we conclude that

$$0 \leq \lambda_j(Df(x)(p, q)).$$

In the case that $(p, q) \in N_x$, then $(-p, -q) \in N_x$. By inserting $(-p, -q)$ in the inequality above we end up with

$$\lambda_j(Df(x)(p, q)) = 0.$$

Therefore, $\lambda_j(Df(x)(p, q)) = 0$ for all $j \in J$ and all $(p, q) \in N_x$. Now we have shown that $Df(x)(N_x) \subseteq N'_f(x)$. Since $Df(x) : N \to N'$ is a sc-isomorphism, it holds that $Df(x)(N_x)$ has codimension $d_C(x)$ in $N'$. Consequently $d_C(x) \leq d_D(f(x))$. By using the same argument for the sc-smooth map $f^{-1} : P \to O$, it follows that $d_C(x) \geq d_D(f(x))$. We can conclude that $d_C(x) = d_D(f(x))$. □

As an immediate consequence we have Corollary 4.19, the degeneracy index is preserved under sc-differomorphic maps.

**Corollary 4.19.** Let $M, N$ be M-polyfolds, and let $f : M \to N$ be a sc-diffeomorphism. Then, it holds that

$$d_M(p) = d_N(f(p))$$

for all $p \in M$.

In line with classical differential geometry we can separate between M-polyfolds without boundary, M-polyfolds with boundary, and M-polyfolds with boundary and corners. Regardless, they are all simply referred to as M-polyfolds.
Definition 4.20. Let $M$ be a $M$-polyfold. If $d_M(p) = 0$ for all $p \in M$, then $M$ is called a $M$-polyfold without boundary. If $d_M(p) \leq 1$, then $M$ is called a $M$-polyfold with boundary otherwise, $M$ is called a $M$-polyfold with boundary and corners.

Analogously to classical differential geometry, we define points $p \in M$ to be interior points if $d_M(p) = 0$, boundary points if $d_M(p) = 1$, and corner points if $d_M(p) > 1$. We denote the boundary for a $M$-polyfold as

$$\partial M = \{ p \in M : d_M(p) \geq 1 \}.$$

There is a result connecting boundary behaviour to sub-$M$-polyfolds.

Proposition 4.21. Let $M$ be a $M$-polyfold, and let $A \subseteq M$ be a sub-$M$-polyfold. Then, it holds that

$$d_A(a) \leq d_M(a)$$

for all $a \in A$.

Proof. Let $a \in A$ be a point, choose a chart $(\phi, V, (O, C, E))$ around $a$ belonging to the $M$-polyfold structure of $M$ such that

$$d_M(a) = d_C(\phi(a)).$$

Consider the corresponding chart $(\psi, U, (O', C, E'))$, for $A$ defined as $U = V \cap A$, $\psi = \phi|_U$, and $O = \psi(V)$. Then, it holds that

$$d_C(\psi(a)) = d_C(\phi(a)) = d_M(a).$$

Taking the minimum of the left hand side over all compatible chart for $A$, it holds that

$$d_A(a) \leq d_M(a).$$

Remark 4.22. An important subclass of $M$-polyfolds are the tame $M$-polyfolds. In short, a tame $M$-polyfold, $M$, is a $M$-polyfold with an atlas $(V_a, \phi_a, (O_a, C_a, E_a))$ that satisfies the following:

i) there is a sc-smooth retraction $r : V \to V$, $V \subseteq C$ relatively open, corresponding to $(O, C, E)$ such that

$$d_C(r(x)) = d_C(x)$$

for all $x \in U$;

ii) for every point $x \in O_\infty$ there is a sc-subspace $A \subseteq E$ such that $E = T_x O \oplus A$ and

$$A \subseteq E_x \cong \{(a, v) \in \mathbb{R}^k \oplus W : a_i = 0 \text{ for all } 1 < i < k \text{ such that } x_i = 0\}.$$
Tame M-polyfolds are useful in some applications as one has to worry less about the boundary behaviour for a tame M-polyfold. Though, as it is sufficient for our purposes to keep the setting of general M-polyfolds, it should suffice to only mention them as a passing remark.

3. sc-Smooth Partition of Unity

Since M-polyfolds is an indirect generalisation of Banach manifolds, it also inherits some issues inherent to dimensional Banach spaces. Most famously, classically smooth bump functions on Banach spaces need not exist. For various examples and counterexamples see e.g. the extensive survey [23] on the subject. This problem persists to the theory of scale-Banach spaces as well. Though, it is still an open question for the theory of M-polyfolds whether the existence of sc-smooth bump functions is equivalent to the existence of sc-smooth partition of unity. In a nutshell, as there are some results in analysis on Banach spaces that also hold for sc-Banach spaces, we shall translate some of the results for smooth partition of unity on Banach spaces to sc-Banach spaces.

Definition 4.23. Let $M$ be a M-polyfold. A sc-smooth function $f : M \to \mathbb{R}$ is called a sc-smooth bump function if $\text{supp}(f)$ is non-empty. If, for every point $p \in M$ and every open neighbourhood $N \subseteq M$ of $p$, there exists a sc-smooth bump function with the support in $N$, then $M$ is said to admit sc-smooth bump functions.

Remark 4.24. In contrast to classical differential geometry, we can not assume that the support of sc-smooth bump functions are compact as done in [47, Chapter 1.5]. This is the case as infinite dimensional Banach spaces are nowhere locally compact.

In particular, every Hilbert space admits a (classically) smooth bump function, and hence every sc-Hilbert space admits a sc-smooth bump function. We need to make sure that this more generalised notion of smooth bump functions allow us to choose bump functions that satisfy our needs. That is so that they make good candidates for a sc-smooth partition of unity.

Proposition 4.25. Let $M$ be a M-polyfold that admits sc-smooth bump functions. Then for every point $p \in M$ and for every open neighbourhood $N \subseteq M$ of $p$, there exists a sc-smooth function

$$f : M \to [0, 1]$$

with $f(p) = 1$ and $\text{supp}(f) \subseteq N$. In particular, we can choose $f$ such that $f(y) = 1$ for all points $y \in V$, with $V \subseteq N$ an open neighbourhood of $p$. 
Proof. By assumption there exists a sc-smooth bump function $g : M \to \mathbb{R}$, such that $g(p) = 1$. Now, we shall construct a sc-smooth bump function $f : M \to [0, 1]$ with the required properties. Consider a (classically) smooth function $\sigma : \mathbb{R} \to [0, 1]$, such that $\sigma(s) = 0$ whenever $s \leq 0$ and $\sigma(s) = 1$ whenever $s \geq 1$. Now define

$$f = \sigma \circ g$$

and it holds that $f$ is sc-smooth. To get a sc-smooth bump function that is equivalently 1 on a neighbourhood of $p$, define the map $h : M \to [0, 1]$, by

$$h(y) = \sigma(\delta g(y))$$

where $\delta > 1$. □

It is rather interesting for us whether the local M-polyfold models for some M-polyfold admit sc-smooth bump functions. A definition is thus warranted.

Definition 4.26. Let $(O, C, E)$ be a local M-polyfold model. Then, $(O, C, E)$ is said to have the sc-smooth bump function property if, for every point $p \in O$ and every open neighbourhood $N \subseteq O$ of $p$ such that $\text{cl}_E(N) \subseteq O$, there exists a sc-smooth function $f : O \to \mathbb{R}$ such that $\text{supp}(f) \subseteq N$ is nonempty.

The following result, found in [36, Theorem 5.11], is almost immediate.

Theorem 4.27. Let $M$ be a M-polyfold. Then, $M$ admits sc-smooth bump functions if, and only if, $M$ has a sc-smooth atlas, such that the local M-polyfold models has the sc-smooth bump function property.

Proposition 4.28. Let $(O, C, E)$ be a local M-polyfold model such that the zero level $E_0$ is a Hilbert space. Then, $(O, C, E)$ has the sc-smooth bump function property.

Proof. Observe that the norm is a bilinear map restricted to the diagonal. Hence, it follows from Proposition 2.7 that the map $p \mapsto \|p\|^2$, where $p \in O$, is smooth in the sense of Fréchet. Since the codomain of this map has the constant sc-structure, it is sc-continuous and therefore sc-smooth. We can choose a smooth function $\beta : \mathbb{R} \to \mathbb{R}$ that has compact support. Then the map $p \mapsto \beta(\|p\|^2)$ is a sc-smooth map with bounded support, and we are done. □

Therefore, M-polyfolds modelled on sc-Hilbert spaces admits sc-smooth bump functions. Now to the definition of sc-smooth partition of unity.

Definition 4.29. Let $M$ be a M-polyfold, and let $\{\varphi_a\}_{a \in A}, \varphi_a : M \to \mathbb{R}$, be a collection of sc-smooth bump functions. Then, $\{\varphi_a\}_{a \in A}$ is called a sc-smooth partition of unity if:
i) \( \text{im} \varphi_a \subseteq [0,1] \) for all \( a \in A \);

ii) the collection of supports, \( \{ \text{supp}(\varphi_a) \}_{a \in A} \), is locally finite;

iii) \( \sum_{a \in A} \varphi_a(p) = 1 \) for all \( p \in M \).

If \( U = \{ U_a \}_{a \in A} \) is an open cover of \( M \) such that \( \text{supp}(\varphi_a) \subseteq U_a \) for all \( a \in A \), then we call \( \{ \varphi_a \}_{a \in A} \) a \( \text{sc-smooth} \) partition of unity \textit{subordinate} to \( U \). The M-polyfold \( M \) is said to \textit{admit} \( \text{sc-smooth} \) partitions of unity if, for every open cover of \( M \), there exists a \( \text{sc-smooth} \) partition of unity subordinate to the open cover.

We need some sufficient conditions for the existence of \( \text{sc-smooth} \) partition of unity on M-polyfolds.

\textbf{Definition 4.30.} Let \((O,C,E)\) be a local M-polyfold model. Then, \((O,C,E)\) is said to have the \textit{sc-smooth approximation property} if for every open set \( V \subseteq O \) such that \( \text{cl}_C(V) \subseteq O \), every continuous function \( f : O \to [0,1] \) with \( \text{supp}(f) \subseteq V \), and every number \( \varepsilon > 0 \), there exists a \( \text{sc-smooth} \) map \( g : O \to [0,1] \) such that \( \text{supp}(g) \subseteq V \) and \( |f(x) - g(x)| < \varepsilon \) for all \( x \in O \).

On their own right, local M-polyfold models with the \( \text{sc-smooth} \) approximation property are interesting. On such spaces we may approximate \( \text{sc-smooth} \) functions with \( \text{sc-continuous} \) maps. Particularly in some computational applications, such properties are central. Inspired by [36, Theorem 5.12], the \( \text{sc-smooth} \) approximation property of the local models is not only a sufficient condition, it is also a necessary condition for \( \text{sc-smooth} \) partition of unity.

\textbf{Theorem 4.31.} Let \( M \) be a M-polyfold. Then the following conditions are equivalent:

i) \( M \) admits \( \text{sc-smooth} \) partition of unity;

ii) there is an atlas belonging to the M-polyfold structure of \( M \) such that the local models have the \( \text{sc-smooth} \) approximation property.

\textbf{Proof.} i) \( \Rightarrow \) ii): Let \( \{ (U_a, \phi, (O_a, C_a, E_a)) \} \) be an atlas belonging to the M-polyfold structure of \( M \), and assume there is a \( \text{sc-smooth} \) partition of unity subordinate to \( \{ U_a \} \). We shall show that the local models \((O_a, C_a, E_a)\) has the \( \text{sc-smooth} \) approximation property. Let \( V \subseteq O_a \) be an open set, such that \( \text{cl}_{C_a}(V) \subseteq O_a \), let \( f : O_a \to [0,1] \) be a continuous function with \( \text{supp}(f) \subseteq V \), and let \( \varepsilon > 0 \) be a number. Note that the function

\[ f \circ \phi_a : U_a \to [0,1] \]

is continuous. Since \( M \) is paracompact and Hausdorff, Proposition 2.16 holds. Now, through a \( \text{continuous} \) partition of unity, extend \( f \circ \phi_a \) to a
4.3. sc-Smooth Partition of Unity

continuous function, \( g : M \to [0,1] \) that is zero outside \( U_\alpha \) and has its support inside \( W = \phi^{-1}(V) \). Define the set

\[
W = g^{-1}\left(\left(\frac{\varepsilon}{4}, 1\right]\right).
\]

Then, by continuity, it holds that \( \text{cl}_M(W) \subseteq W \). For every \( p \in \text{cl}_M(W) \) there exists an open neighbourhood \( N_p \subseteq M \) around \( p \) such that \( \text{cl}_M(N_p) \subseteq W \), and such that

\[
|g(p) - g(x)| < \frac{\varepsilon}{2}
\]

for all \( x \in N_p \). From the set

\[
N_0 = g^{-1}\left(\left[0, \frac{\varepsilon}{2}\right]\right),
\]

we have the open cover

\[
U = \{N_0\} \cup \{N_p\}_{p \in \text{cl}_M(W)}.
\]

By assumption there exists a sc-smooth partition of unity subordinate to \( U \), we shall denote the corresponding sc-smooth bump functions as \( \beta_0 : M \to [0,1] \) with \( \text{supp} \beta_0 \subseteq N_0 \), and \( \beta_p : M \to [0,1] \) with \( \text{supp} \beta_p \subseteq N_p \) for \( p \in \text{cl}_M(W) \). Then, we have that the map \( \hat{g} : M \to [0,1] \) defined as

\[
\hat{g}(x) = \sum_{p \in \text{cl}_M(W)} \beta_p(x)g(p)
\]

is sc-smooth since \( \{\text{supp} \beta_p\} \) is a locally finite collection of sc-smooth maps.

Next, we shall show that

\[
|g(x) - \hat{g}(x)| < \varepsilon
\]

for all \( x \in M \). The initial step is done by showing that \( \text{supp} \hat{g} \subseteq W \). Suppose that \( y \in \text{supp} \hat{g} \) and let \( \{y_k\}, y_k \in \hat{g}^{-1}((0,1]) \), be a sequence such that \( \lim_{k \to \infty} y_k = y \). Since it holds that \( \hat{g}(y_k) > 0 \), every open neighbourhood around \( y \) has a nonempty intersection with \( \text{supp} \beta_p \) for some \( p \in \text{cl}_M(W) \). Since the collection of supports, \( \{\text{supp} \beta_p\}_{p \in \text{cl}_M(W)} \) is locally finite, there exists an open neighbourhood \( Q \subseteq M \) around \( y \) such that \( Q \) intersects the sets \( \text{supp} \beta_p \) for only finitely many \( p_1, \ldots, p_n \in \text{cl}_M(W) \). Therefore, we have that

\[
0 < \hat{g}(y_k) = \sum_{i=1}^{n} \beta_{p_i}g(p_i).
\]

This implies that there is a number \( K \in \mathbb{N} \) such that

\[
y_k \in \bigcup_{i=1}^{n} \text{cl}_M(N_{p_i}) \subseteq W
\]
for all $k \geq K$. Hence, we have that $y \in W$, and we can conclude that $\text{supp}(\hat{g}) \subseteq W$.

Now we are able to finish the first part of the proof. Let $z \in U_\alpha$ be a point. There exists an open neighbourhood $Q \subseteq M$ around $z$ such that $Q$ intersects a finite collection of supports for the bump functions belonging to the sc-smooth partition of unity. In the case that $Q$ intersects $\text{supp}(\beta_0)$ only, then it holds that $Q \subseteq N_0$ and $\hat{g}(y) = 0$ for all $y \in Q$. It follows that

$$|g(y) - \hat{g}(y)| = |g(y)| < \frac{\varepsilon}{2} < \varepsilon$$

for all $y \in Q$. In the case that $Q$ intersects $\text{supp}(\beta_p)$ for some $p \in \text{cl}_M(N_p)$. Then, there are finitely many $p_1, \ldots, p_m$ such that $Q$ only intersects $\text{supp}(\beta_{p_i})$. Hence,

$$\hat{g}(y) = \sum_{i=1}^{m} \beta_{p_i}(y)g(p_i)$$

for all $y \in Q$. By definition we have that

$$|g(p) - g(y)| < \frac{\varepsilon}{2}$$

when $y \in N_p$, and

$$g(y) < \frac{\varepsilon}{2}$$

whenever $y \in N_0$. Conclusively, we have that

$$|g(y) - \hat{g}(y)| = \left| \beta_0(y)g(y) + \sum_{i=1}^{m} \beta_{p_i}(y)g(y) - \sum_{i=1}^{m} \beta_{p_i}(y)g(p_i) \right|$$

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{m} \beta_{p_i}(y)|g(y) - g(x_i)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{i=1}^{m} \beta_{p_i}(y) < \varepsilon,$$

holds for all $y \in Q$. Therefore, we can conclude that

$$|g(y) - \hat{g}(y)| < \varepsilon$$

for all $y \in U_\alpha$, and

$$|f(x) - \hat{g} \circ \phi^{-1}_\alpha(x)| < \varepsilon$$

for all $x \in O_\alpha$. The first part of the proof is done.

(ii) $\Rightarrow$ i): Let $\{(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in A}$ be an atlas belonging to the M-polyfold structure of $M$, and assume that the local M-polyfold models $(O_\alpha, C_\alpha, E_\alpha)$ all have the sc-smooth approximation property. Now, arbitrarily choose an open cover of $M$, $\{V_\beta\}_{\beta \in B}$, $V_\beta \subseteq M$ open. We are able to choose a locally finite refinement, $\{W_\beta\}_{\beta \in B}$, of $\{V_\beta\}_{\beta \in B}$ such that $W_\beta \subseteq V_\beta$ for all $\beta \in B$. Note that some of the $W_\beta$ may be empty and that we choose
the same index set. Also, this refinement is chosen with the property so that for every $\beta \in B$ we can choose an $\alpha(\beta) \in A$ such that
\[ \text{cl}_M(W_\beta) \subseteq U_{\alpha(\beta)}. \]
Furthermore, as $M$ is metrizable by the Smirnov metrization theorem (Theorem 2.11), and in particular normal, there are open sets $Q_\beta \subseteq M$ such that
\[ Q_\beta \subseteq \text{cl}_M(Q_\beta) \subseteq W_\beta. \]
and $\{Q_\beta\}_{\beta \in B}$ is a locally finite open cover of $M$. Also, $M$ is in particular completely regular, therefore there exist continuous functions $f_\beta : M \to [0,1]$ with the following properties
\[ f_\beta(Q_\beta) = \{1\} \quad \text{and} \quad \text{supp}(f_\beta) \subseteq W_\beta. \]
Succeedingly, we set $\varepsilon = \frac{1}{2}$, $V_\beta := \phi_{\alpha(\beta)}(W_\beta)$, and define $\tilde{f}_\beta : O_{\alpha(\beta)} \to [0,1]$ by $\tilde{f}_\beta = f_\beta \circ \phi_{\alpha(\beta)}^{-1}$. Since we assume that the local models have the sc-smooth approximation property, the following argument for the triple $(\tilde{f}_\beta, V_\beta, \frac{1}{2})$ holds by definition. There exists a sc-smooth function $g_\beta : O_{\alpha(\beta)} \to [0,1]$ such that $\text{supp}(g_\beta) \subseteq V_\beta$ and such that
\[ \left| \tilde{f}_\beta(x) - g_\beta(x) \right| < \frac{1}{2} \]
for all $x \in O_{\alpha(\beta)}$.
Left is to construct a sc-smooth partition of unity. Define the sc-smooth function $\hat{g}_\beta : M \to [0,1]$ by $f_\beta \circ \phi_{\alpha(\beta)}(x)$ whenever $x \in Q_\beta$ and zero otherwise. For $x \in M$, we are able to define the sc-smooth map $\gamma_\beta : M \to [0,1]$ by
\[ \gamma_\beta(x) = \frac{\hat{g}_\beta(x)}{\sum_{b \in B} \hat{g}_b(x)}. \]
The family of functions $\{\gamma_\beta\}_{\beta \in B}$ is exactly the sc-smooth partition of unity we are after. \hfill \Box

As was shown in [40], and further developed in [25], weakly compactly generated Banach spaces that admits smooth bump functions admits smooth partition of unity. For our purposes, it is enough to know that Hilbert spaces are weakly compactly generated, and therefore in the view of Proposition 4.28, sc-Hilbert spaces admit in particular sc-smooth partition of unity. This is stated as Corollary 5.2 in [36], and we shall restate it for convenience.

**Theorem 4.32.** Let $(O,C,E)$ be a local $M$-polyfold model such that the zero level $E_0$ is a Hilbert space. Then, $(O,C,E)$ has the sc-smooth approximation property.
Conclusively, M-polyfolds modelled on sc-Hilbert spaces are sufficiently nice to admit sc-smooth partition of unity.
CHAPTER 5

Tangent Space

In current research \([16, 32, 36]\), the tangent space at a point in a \(M\)-polyfold is defined using equivalence classes of charts. However, not until after(!) the definition of the tangent bundle over \(M\)-polyfolds. This could be argued to be significantly lacking, and thus we shall here define the tangent space for \(M\)-polyfolds in a more chronological order. We shall do so in the spirit of Jeffrey Lee \([47]\), i.e. we define the tangent space in three philosophically different ways. Then, we would like to show that these three definitions are equivalent, alas in contrast to classical differential geometry, this is not the case here. For a more comprehensive and in depth discussion on these different views on the tangent space, the reader is referred to \([67–71]\). After we have defined the different types of tangent spaces we define the tangent bundle on \(M\)-polyfolds. Then, we shall go into a version of the implicit function theorem on \(M\)-polyfolds and we shall finish the chapter with a discussion on finite-dimensional submanifolds of \(M\)-polyfolds.

1. Kinematic Tangent Space

In order to make the analogous definition of the Kinematic tangent space we need to introduce curves on \(M\)-polyfolds. Let \(M\) be a \(M\)-Polyfold and let \(p \in M_{\infty}\) be a point. If the point \(p\) is contained in the chart domain for \((V, \phi, (O, C, E))\), and if \(\phi(p)\) is relatively open in \(O\), then \(p\) can be viewed as a locally 0-dimensional Banach space and the tangent space at \(p\) will be identical to that of 0-dimensional manifolds, i.e. \(T_pM \cong \{0\}\). Thus, we shall henceforth assume that \(\phi(p)\) is not relatively open in \(O\). Let \(\varepsilon > 0\) be a real number. A \(sc^1\)-curve (respectively \(sc^k\)-curve, and \(sc\)-smooth-curve) on \(M\) going through \(p \in M\) is a \(sc^1\) map (respectively \(sc^k\) map, and \(sc\)-smooth map) \(c : (-\varepsilon, \varepsilon) \to M\), such that \(c(0) = p\).

**Definition 5.1.** Let \(M\) be a \(M\)-Polyfold, let \(p \in M_{\infty}\) be a point, and let \(\{(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in A}\) be an arbitrary collection of compatible charts on \(M\) such that \(p \in U_\alpha\). Let \(\varepsilon_1, \varepsilon_2 > 0\) be real numbers. Suppose there are two curves \(c_1 : (-\varepsilon_1, \varepsilon_1) \to M\) and
5.1. Kinematic Tangent Space

and \(c_2: (-\varepsilon_2, \varepsilon_2) \to M\) such that

\[c_1(0) = c_2(0) = p,\]

and \(c_1(-\varepsilon_1, \varepsilon_1), c_2(-\varepsilon_2, \varepsilon_2) \subseteq U_\alpha\) for all \(\alpha \in \mathcal{A}\), then \(c_1\) is said to be tangent to \(c_2\) if

\[D(\phi_\alpha \circ c_1) = D(\phi_\alpha \circ c_2),\]

holds for all \(\alpha \in \mathcal{A}\).

This is obviously an equivalence relation. We denote the equivalence class of a curve \(c: (-\varepsilon, \varepsilon) \to X\) as \([c]\). These equivalence classes are precisely the tangent vectors at \(p \in X\). The set of all equivalence classes of curves at \(p\) is denoted as \((T_pM)_{\text{kin}}\) and is called the kinematic tangent space.

**Remark 5.2.** Another common way to define tangent vectors, as done in e.g. [1], is to suppose that

\[D(\phi_\alpha \circ c_1)(0) = D(\phi_\alpha \circ c_2)(0),\]

is true for one chart \((U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\). Then, one shows that the choice of chart was arbitrary as it holds for all charts around that point.

We want to show that \((T_pM)_{\text{kin}}\) is a Banach space, to do so we shall first need Proposition 5.3 and Lemma 5.5. We begin with a standard method to obtain a vector space structure on a set.

**Proposition 5.3.** Let \(S\) be a set, let \(\{V_\alpha\}_{\alpha \in \mathcal{A}}\) be a collection of vector spaces over a field \(\mathbb{K}\), and for each \(\alpha \in \mathcal{A}\) suppose there is a bijection \(b_\alpha: V_\alpha \to S\). If for every \(\alpha, \beta \in \mathcal{A}\) the map \(b_\beta^{-1} \circ b_\alpha: V_\alpha \to V_\beta\) is a vector space isomorphism, then there is a unique vector space structure on \(S\) over \(\mathbb{K}\), such that each \(b_\alpha\) is a vector space isomorphism.

**Proof.** Let \(S, \{V_\alpha\}_{\alpha \in \mathcal{A}}\) and \(\{b_\alpha\}_{\alpha \in \mathcal{A}}\) be as stated in the proposition above. We start with defining addition over \(S\) by

\[s_1 + s_2 := b_\alpha(b_\alpha^{-1}(s_1) + b_\alpha^{-1}(s_2)),\]

for all \(s_1, s_2 \in S\). We need to show that this definition is independent of the choice of \(\alpha, \beta \in \mathcal{A}\). We have that

\[b_\alpha\left(b_\alpha^{-1}(s_1) + b_\alpha^{-1}(s_2)\right) = b_\alpha\left[b_\alpha^{-1} \circ b_\beta^{-1}(s_1) + b_\alpha^{-1} \circ b_\beta^{-1}(s_2)\right] = b_\alpha \circ b_\alpha^{-1} \circ b_\beta(b_\alpha^{-1}(s_1) + b_\alpha^{-1}(s_2)) = b_\beta(b_\beta^{-1}(s_1) + b_\beta^{-1}(s_2)),\]

where we used the fact that \(b_\alpha^{-1} \circ b_\beta\) is linear.

Similarly, we define scalar multiplication as

\[a \cdot s := b_\alpha\left(ab_\alpha^{-1}(s)\right),\]
for every \( a \in \mathbb{K} \) and every \( s \in S \). This is also independent of the choice of \( \alpha, \beta \in \mathcal{A} \) since,

\[
\begin{align*}
    b_{\alpha}(a^{-1}(s)) &= b_{\alpha}(a^{-1} \circ b_{\beta} \circ b_{\beta}^{-1}(s)) \\
    &= b_{\alpha} \circ a^{-1} \circ b_{\beta}(ab_{\beta}^{-1}(s)) \\
    &= b_{\beta}(ab_{\beta}^{-1}(s)),
\end{align*}
\]

where once again we used that \( b_{\alpha}^{-1} \circ b_{\beta} \) is linear. The vector space axioms on \((S, +, \cdot)\) follows directly from that the vector space axioms holds for each \( V_{\alpha} \). Hence, \((S, +, \cdot)\) is a vector space that is isomorphic to each \( V_{\alpha} \). \( \square \)

Suppose that \( M \) is a \( \mathcal{M} \)-polyfold, and \( p \in M_{\infty} \) is a point. Suppose that \((V, \phi, (O, C, E))\) is a chart around \( p \). From the proposition above, we are incentivised to construct a bijection from \((T_{p}M)_{\text{kin}}\) to \( T_{\phi_{\alpha}(p)}O \cap E_{\phi_{\alpha}(p)}\), where

\[
E_{\phi_{\alpha}(p)} = \{(a_{i}, v) \in \mathbb{R}^{k} \oplus W \cong E : a_{i} = 0 \text{ for all } i \text{ such that } \phi(p)_{i} = 0\}.
\]

**Remark 5.4.** In \([18, 36]\) this space is called the reduced tangent space. Furthermore observe that the point \( p \) is necessarily a smooth point as \( \text{sc-smooth curves} \ c : (-\varepsilon, \varepsilon) \to M \) are in particular \( \text{sc-continuous} \) and since the \( \text{sc-structure} \) on \( \mathbb{R} \) is the constant structure

\[
\mathbb{R}_{0} = \mathbb{R}_{1} = \cdots = \mathbb{R}_{\infty} = \mathbb{R},
\]

the image of \( c \) have to lie in \( M_{\infty} \).

**Lemma 5.5.** Let \( M \) be a \( \mathcal{M} \)-polyfold, and let \( p \in M_{\infty} \) be a point. For each chart

\[
(V_{\alpha}, \phi_{\alpha}, (O_{\alpha}, C_{\alpha}, E_{\alpha}))
\]

around \( p \) in the \( \mathcal{M} \)-polyfold structure of \( M \), there is a bijection

\[
b_{\alpha} : \text{im}(Dr(\phi_{\alpha}(p))) \cap E_{\phi_{\alpha}(p)} \to (T_{p}M)_{\text{kin}},
\]

where \( r : U \to U, \ U \subseteq C_{\alpha} \) relatively open, is a \( \text{sc-smooth retraction} \) corresponding to \((O_{\alpha}, C_{\alpha}, E_{\alpha})\). Furthermore, the bijections can be constructed such that

\[
b^{-1}_{\beta} \circ b_{\alpha} = D(\phi_{\alpha}^{-1} \circ \phi_{\alpha})(\phi_{\alpha}(p)).
\]

**Proof.** Suppose that \( M \) is a \( \mathcal{M} \)-polyfold and \( p \in M_{\infty} \) is a point. Suppose that

\[
\{(V_{\alpha}, \phi_{\alpha}, (O_{\alpha}, C_{\alpha}, E_{\alpha}))\}_{\alpha \in \mathcal{A}}
\]

is a collection of charts around \( p \) belonging to the \( \mathcal{M} \)-polyfold structure of \( M \). For each chart construct the map

\[
b_{\alpha} : \text{im}(Dr(\phi_{\alpha}(p))) \cap E_{\phi_{\alpha}(p)} \to (T_{p}M)_{\text{kin}},
\]
5.1. Kinematic Tangent Space

by for each \( v \in \text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi_\alpha(p)} \) mapping

\[
v \mapsto [\gamma_v], \text{ where } \gamma_v : t \mapsto \phi^{-1}_\alpha(\phi_\alpha(p) + tv),
\]

for \( t \) sufficiently small.

First we shall prove the injectivity of \( b_\alpha \). Notice that

\[
(\phi_\alpha \circ \gamma_v)'(0) = \left. \frac{d}{dt} \phi_\alpha \circ \phi^{-1}_\alpha(\phi_\alpha(p) + tv) \right|_{t=0}
= \left. \frac{d}{dt} (\phi_\alpha(p) + tv) \right|_{t=0}
= v.
\]

Suppose that for \( v, w \in \text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi_\alpha(p)} \) it holds that \([\gamma_v] = [\gamma_w]\).
Then, we have that

\[
v = (\phi_\alpha \circ \gamma_v)'(0) = (\phi_\alpha \circ \gamma_w)'(0) = w,
\]

thus \( b_\alpha \) is injective.

Next the surjectivity of \( b_\alpha \) is proved. Let \([c] \in (T_p M)_{\text{kin}}\) and let \( c : (-\varepsilon, \varepsilon) \to X \) be a representative curve centred at \( p \) for \([c]\).
Suppose that

\[
v = (\phi_\alpha \circ c)'(0) \in \text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi_\alpha(p)}.
\]

We have that \( b_\alpha(v) = [\gamma_v] \) for some

\[
\gamma_v : t \mapsto \phi^{-1}_\alpha(\phi_\alpha(p) + tv).
\]

Let \((f, U, (O, C, E))\) be an arbitrarily chosen chart, such that \( U \cap V_\alpha \neq 0 \) and such that the range of the curves are in \( U \), then we have that

\[
(f \circ \gamma_v)'(0) = \left. \frac{d}{dt} f \circ \phi^{-1}_\alpha(x_\alpha(p) + tv) \right|_{t=0}
= D(f \circ \phi^{-1}_\alpha)(\phi_\alpha(p))v
= D(f \circ \phi^{-1}_\alpha)(\phi_\alpha(p))(\phi_\alpha \circ c)'(0)
= (f \circ c)'(0),
\]

where we have used the chain rule (Theorem 3.19). Thus, we can conclude that \( b_\alpha \) is surjective. By definition of a tangent vector we know that the map \([c] \mapsto (\phi_\alpha \circ c)'(0)\) is well-defined, and from our previous work it is possible to identify this map as \( b_\alpha^{-1} \).
Therefore,

\[
b_{\beta}^{-1} \circ b_\alpha(v) = \left. \frac{d}{dt} \phi^{-1}_\alpha(\phi_\alpha(p) + tv) \right|_{t=0}
= D(\phi_{\beta} \circ \phi^{-1}_\alpha)(\phi_\alpha(p))v,
\]

and we are done.

\[\square\]

**Remark 5.6.** Notice in the proof above that \((\phi_\alpha \circ c)'(0) \in \text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi_\alpha(p)}\) is indeed true in the view of Definition 3.27, and it does not necessarily hold that \( \text{im}(Dr(\phi_\alpha(p))) = E \). This is one of the crucial details that makes polyfold theory stand out from classical differential geometry.
It is precisely this property that allows for M-polyfolds to locally change dimension.

We are ready to prove that \((T_pM)_{\text{kin}}\) indeed is a Banach space, and we shall do so using a construction.

**Theorem 5.7.** Let \(M\) be a M-polyfold, and let \(p \in M_\infty\) be a point. Suppose that there is a chart \((V, \phi, (O, C, E))\) around \(p\), and a sc-smooth retraction \(r : U \to U\) onto \(O\). Then \((T_pM)_{\text{kin}}\) can be equipped with a vector space structure, and a complete norm such that it is isomorphic to \(\text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi(p)}\), and therefore is a Banach space.

**Proof.** Let \(M\), \(p\) and \(r\) be given as stated in the proposition above. From Lemma 5.5 it holds that there exists bijections \(b_\alpha\) from \(\text{im}(Dr(\phi(p))) \cap E_{\phi(p)}\) to \((T_pM)_{\text{kin}}\). Then we can impose Proposition 5.3 so that \((T_pM)_{\text{kin}}\) has a vector space structure that is vector space isomorphic to \(\text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi(p)}\). What is left to check is that we can imbue a norm on \((T_pM)_{\text{kin}}\) and that it is indeed complete with respect to it. Let \(b_\alpha : \text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi_\alpha(p)} \to (T_pM)_{\text{kin}}\) be as given in the proof of Lemma 5.5.

Construct a norm on \((T_pM)_{\text{kin}}\) by defining

\[
\|\cdot\|_{\text{kin}} := v \mapsto \|b_\alpha^{-1}(v)\|
\]

for every \(v \in (T_pM)_{\text{kin}}\). That \(\|\cdot\|_{\text{kin}}\) is a norm is easy to check as it follows from that \(b_\alpha\) is linear. Next, we shall show that \(((T_pM)_{\text{kin}}, \|\cdot\|_{\text{kin}})\) is complete. Let \(\{a_n\}, a_n \in V\), be a Cauchy sequence. By definition, for every \(\varepsilon > 0\) there is a \(N \in \mathbb{N}\) such that

\[
\|a_n - a_m\|_{\text{kin}} < \varepsilon,
\]

for all \(n, m \geq N\). By construction and since \(b_\alpha\) is linear it holds that

\[
\|a_n - a_m\|_{\text{kin}} = \|b_\alpha(a_n - a_m)\| = \|b_\alpha(a_n) - b_\alpha(a_m)\| < \varepsilon.
\]

Hence, \(\{b_\alpha(a_n)\}\) is a Cauchy sequence in \(\text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi(p)}\). According to Lemma 3.30, \(\text{im}(Dr(\phi_\alpha(p)))\) is complete. Since \(E_{\phi(p)}\) is also complete, they are both closed subspaces of \(E\) and \(\text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi(p)}\) is a closed subspace of \(E\). Hence, \(\text{im}(Dr(\phi_\alpha(p))) \cap E_{\phi(p)}\) is complete. Thus the sequence converges to a point \(f \in \text{im}(Dr(\phi_\alpha(p)))\). That is, for every \(\varepsilon > 0\) there is a \(N \in \mathbb{N}\) such that

\[
\|b_\alpha(a_n) - f\| < \varepsilon
\]

for all \(n \geq N\). From the bijectivity of \(b_\alpha\) it follows that there is a point \(a \in (T_pM)_{\text{kin}}\) such that \(a = b_\alpha(f)\). Note that from the linearity of \(b_\alpha\) it holds that

\[
\|b_\alpha(a_n) - f\| = \|b_\alpha(a_n) - b_\alpha(a)\| = \|b_\alpha(a_n - a)\| = \|a_n - a\|_{\text{kin}}.
\]
Therefore, for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that

$$\|a_n - a\|_{\text{phys}} < \varepsilon$$

for all $n \geq N$. Hence, we can conclude that $(T_p M)_{\text{kin}}$ can be equipped with a vector space structure and norm so that it is a Banach space. □

2. Tangent Space via Charts

In contrast to the kinematic tangent space the viewpoint of the tangent space via charts warrant for less technical worry. This tangent space shall be referred to as the physical tangent space and is naturally extended from classical differential geometry to the more general setting of M-polyfolds.

**Definition 5.8.** Let $M$ be a M-polyfold, let $p \in M_1$ be a point, and let $\mathcal{M} = \{(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))\}_{\alpha \in A}$ be the maximal atlas that defines the M-polyfold structure for $M$. Define the following set of tuples

$$\Gamma_p := \{(p, v, (U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))) \in \{p\} \times T_p O \times \mathcal{M} : p \in U_\alpha\}.$$

Two elements $(p, v, (U, \phi, (O, C, E)))$, $(p, w, (V, \psi, (P, D, F))) \in \Gamma_p$ are said to be equivalent, if

$$v = D(\phi \circ \psi^{-1})(\psi(p))w.$$

It is trivial to check that this is indeed an equivalence relation, which we denote as $\sim$. We define the physical tangent space as

$$(T_p M)_{\text{phys}} := \Gamma_p / \sim.$$

Therefore, we identify each equivalence class of tuples

$$[(p, v, (\phi, U, (O, C, E)))]$$

as tangent vectors. Of course we can give $(T_p M)_{\text{phys}}$ a structure of a Banach space.

**Theorem 5.9.** Let $M$ be a M-polyfold, and let $p \in M_1$ be a point. Suppose that there is a chart $(V, \phi, (O, C, E))$ around $p$, and let $r : U \to U$, $U \subseteq C$ a relatively open subset, be a sc-smooth retraction corresponding to $(O, C, E)$. Then $(T_p M)_{\text{phys}}$ can be equipped with a vector space structure, and a norm such that it is isomorphic to $\text{im}(Dr(\phi_\alpha(p)))$, and is therefore a Banach space.

**Proof.** Let $M$ be a M-polyfold, and let $p \in M_1$ be a point. Suppose $(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))$ is a chart around $p$, and suppose that $r_\alpha : V_\alpha \to V_\alpha$, $V_\alpha \subseteq C_\alpha$ open, is a sc-smooth retraction corresponding to $(O_\alpha, C_\alpha, E_\alpha)$. For each chart $(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))$, we shall construct a bijection

$$b_\alpha : \text{im}(Dr(\phi_\alpha(p))) \to (T_p M)_{\text{phys}},$$
defined by
\[ b_\alpha : v \mapsto [(p, v, (U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))]]. \]

Surjectivity of \( b_\alpha \) is obvious. Now, suppose that \( b_\alpha(v) = b_\alpha(w) \), for \( w, v \in \text{im}(Dr(\phi_\alpha(p))) \). But then,
\[ (p, v, (\phi, U, (O, C, E))) \sim (p, w, (\phi, U, (O, C, E))) \]
and
\[ v = D(\phi \circ \phi^{-1})(\phi(p))w = w. \]

Hence, \( b_\alpha \) is bijective for all \( \alpha \in A \). By Proposition 5.3 it now follows that there is vector space structure on \((T_pM)_{\text{phys}}\) that is vector space isomorphic to \( \text{im}(Dr(\phi_\alpha(p))) \).

We construct a norm on \((T_pM)_{\text{phys}}\) by defining
\[ \|\cdot\|_{\text{phys}} : [(p, v, (U, \phi, (O, C, E)))] \mapsto \|b^{-1}_\alpha([(p, v, (U, \phi, (O, C, E)))]\|, \]
for all \([(p, v, (\phi, U, (O, C, E)))] \in (T_pM)_{\text{phys}}\). It is easy to see that \( \|\cdot\|_{\text{phys}} \) satisfies the required properties to be a norm. Left to check is that \((T_pM)_{\text{phys}}, \|\cdot\|_{\text{phys}}\) is complete. This is done using methods from standard real analysis. Let \( \{a_n\}, a_n \in V \), be a Cauchy sequence. By definition, for every \( \varepsilon > 0 \) there is a \( N \in \mathbb{N} \) such that
\[ \|a_n - a_m\|_{\text{phys}} < \varepsilon, \]
for all \( n, m \geq N \). By construction and since \( b_\alpha \) is linear it holds that
\[ \|a_n - a_m\|_{\text{phys}} = \|b_\alpha(a_n - a_m)\| = \|b_\alpha(a_n) - b_\alpha(a_m)\| < \varepsilon. \]

Hence, \( \{b_\alpha(a_n)\} \) is a Cauchy sequence in \( \text{im}(Dr(\phi_\alpha(p))) \), but according to Lemma 3.30, \( \text{im}(Dr(\phi_\alpha(p))) \) is complete, thus the sequence converges to a point \( f \in \text{im}(Dr(\phi_\alpha(p))) \). That is, for every \( \varepsilon > 0 \) there is a \( N \in \mathbb{N} \) such that
\[ \|b_\alpha(a_n) - f\| < \varepsilon \]
for all \( n \geq N \). From the bijectivity of \( b_\alpha \) it follows that there is a point \( a \in (T_pM)_{\text{kin}} \) such that \( a = b_\alpha(f) \). Note that from the linearity of \( b_\alpha \) it holds that
\[ \|b_\alpha(a_n) - f\| = \|b_\alpha(a_n) - b_\alpha(a)\| = \|b_\alpha(a_n - a)\| = \|a_n - a\|_{\text{phys}}. \]

Therefore, for every \( \varepsilon > 0 \) there is a \( N \in \mathbb{N} \) such that
\[ \|a_n - a\|_{\text{phys}} < \varepsilon \]
for all \( n \geq N \). Hence, we can conclude that \((T_pM)_{\text{phys}}\) can be equipped with a vector space structure and norm so that it is a Banach space. \( \square \)
3. Algebraic Tangent Space

The view of the algebraic tangent space, which we shall denote as 
\((T_p M)_{\text{alg}}\), is that of derivations. Philosophically, a tangent vector is viewed 
as a directional derivative for a germ at a point on a M-polyfold. Classically, 
this can be done in many ways. As the existence of a sc-smooth partition 
of unity is still an open problem for general M-polyfolds, we want avoid 
uncomfortable questions about existence of derivations of functions defined 
on \(M\) versus derivations of functions defined on open subsets \(U \subseteq X\). This 
problem is avoided by discussing derivations of germs instead of functions.

**Definition 5.10.** Let \(M\) be a M-polyfold, and let \(p \in M_1\) be a point. 
Let \(k \in \{\mathbb{R}, \mathbb{C}\}\), and let \(\mathcal{F}_p = sc^1_p(M, k)\) be the set of sc-smooth germs as in 
Definition 4.11. A **derivation** is a linear mapping \(D_p : \mathcal{F}_p \to k\) that satisfies 
Leibniz law, 
\[ D_p([f][g]) = ev_p([f])D_p([g]) + D_p([f])ev_p([g]). \]
The set of all such derivations around \(p\) is called the **algebraic tangent space**, 
\((T_p M)_{\text{alg}}\).

We need to show that derivations on sc-smooth germs are well-defined, 
i.e. the derivation is independent of the representative for the germ.

**Proposition 5.11.** Let \(M\) be a M-polyfold and let \(p \in M_1\) be a point. Let 
k \in \{\mathbb{R}, \mathbb{C}\} be a field, and let \([g], [h] \in sc^1_p(M, k)\) such that \([g] = [h]\), then 
\[ D_p([g]) = D_p([f]). \]
Furthermore, if \(h = c\), with \(c \in k\), is constant around an open neighbourhood 
of \(p\), then 
\[ D_p([h]) = 0. \]

**Proof.** First, suppose that \([g], [h] \in sc^1_p(M, k)\) such that \([g] = [h]\). We 
shall begin to show that \(D_p([0]) = 0\). Observe that \([0] = [f0] = [f][0]\), where 
\([f] \in sc^1_p(M, k)\) is arbitrary. Since \(D_p\) satisfies Leibniz law, it holds that 
\[ D_p([0]) = D_p([f][0]) = ev_p([f])D_p([0]) + D_p([f])ev_p([0]) = ev_p([f])D_p([0]). \]
Since this is true for all \([f] \in sc^1_p(M, k)\), this implies that \(D_p([0]) = 0\). Notice 
that for \([g] = [h]\), it holds that \([g - h] = [0]\). By linearity 
\[ 0 = D_p([0]) = D_p([g - h]) = D_p([g]) - D_p([f]), \]

hence 
\[ D_p([g]) = D_p([f]). \]
Secondly, let $c \in \mathbb{K}$ be a constant, and let $[c]$ be the equivalence class of functions being identically $c$ around an open neighbourhood of $p$. From linearity and Leibniz law we have that
\[
D_p([c]) = D_p([c \cdot 1])
= cD_p([1 \cdot 1])
= c(ev_p([1])D_p([1]) + D_p([1])ev_p([1]))
= 2cD_p([1])
= 2D_p([c]).
\]
Hence, $D_p([c]) = 0$. \qed

We impose the vector space structure on $(T_pM)_{\text{alg}}$ that is given as a subspace of the dual of $\mathcal{F}_p$. That $(T_pM)_{\text{alg}}$ is indeed closed under scalar multiplication is trivial, and closure under addition follows from that
\[
(D_p^1 + D_p^2)([f][g]) := D_p^1([f][g]) + D_p^2([f][g])
= ev_p([f])D_p^1([g]) + D_p^1([f])ev_p([g])
+ ev_p([f])D_p^2([g]) + D_p^2([f])ev_p([g])
= ev_p([f])\left(D_p^1([g]) + D_p^2([g])\right)
+ \left(D_p^2([f]) + D_p^1([f])\right)ev_p([g])
= ev_p([f])\left(D_p^1 + D_p^2\right)([g]) + \left(D_p^2 + D_p^1\right)([f])ev_p([g]),
\]
where $D_p^1, D_p^2 \in (T_pM)_{\text{alg}}$. The norm on $(T_pM)_{\text{alg}}$ is simply the operator norm. Sadly, for infinite dimensions the algebraic tangent space is not necessarily isomorphic to the other two, but the kinematic tangent space (respectively the physical tangent space) is always a closed vector space subspace of the algebraic tangent space.

**Theorem 5.12.** Let $M$ be a $M$-polyfold, and let $p \in M_\infty$, $q \in M_1$ be points. Then it holds that $(T_pM)_{\text{kin}}$ is vector space isomorphic to some subspace of $(T_pM)_{\text{phys}}$. Furthermore, it holds that $(T_qX)_{\text{phys}}$ is vector space isomorphic to a subspace of $(T_qX)_{\text{alg}}$.

**Proof.** That $(T_pM)_{\text{kin}}$ is isomorphic to a subspace of $(T_pM)_{\text{phys}}$ trivially follows from Theorem 5.7 and Theorem 5.9. Now, we shall construct an injective vector space homomorphism $\gamma : (T_pM)_{\text{phys}} \rightarrow (T_pM)_{\text{alg}}$. Let
\[
v_p = [p, v, (U_\alpha, \phi_{\alpha}, (O_\alpha, C_\alpha, E_\alpha))] \in (T_pM)_{\text{kin}}
\]
be given. Now for each $[f] \in \mathcal{F}_p$, we define
\[
\gamma(v_p)([f]) := \left. \frac{d}{dt}\right|_{t=0} f \circ \phi_{\alpha}^{-1}(\phi_{\alpha}(p) + tv).
\]
It is needed to show that this is a linear injection. Linearity follows directly by that composition of functions are distributive, and by that the differential operator is linear. Next, we show injectivity. Let

\[ v_p = [(p, v, (U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)))], \quad w_p = [(p, w, (U_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta)))] \]

be physical tangent vectors at \( p \). Then, for \([f] \in F_p\) suppose that

\[
\frac{d}{dt} \bigg|_{t=0} f \circ \phi_\alpha^{-1}(\phi_\alpha(p) + tv) = \frac{d}{dt} \bigg|_{t=0} f \circ \phi_\beta^{-1}(\phi_\beta(p) + tw).
\]

Since the transition map \( \phi_\alpha \circ \phi_\beta^{-1}|_{\phi_\beta(U_\alpha \cap U_\beta)} \) is sc-smooth we can use the chain rule on the left-hand side as follows

\[
\frac{d}{dt} \bigg|_{t=0} f \circ \phi_\alpha^{-1}(\phi_\alpha(p) + tv) = \frac{d}{dt} \bigg|_{t=0} f \circ \phi_\beta^{-1} \circ \phi_\beta \circ \phi_\alpha^{-1}(\phi_\alpha(p) + tv) = D(f \circ \phi_\beta^{-1})(\phi_\beta(p)) \circ D(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))v.
\]

Using the chain rule on the right-hand side one has that

\[
\frac{d}{dt} \bigg|_{t=0} f \circ \phi_\beta^{-1}(\phi_\beta(p) + tw) = D(f \circ \phi_\beta^{-1})(\phi_\beta(p))w.
\]

Therefore,

\[
D(f \circ \phi_\beta^{-1})(\phi_\beta(p)) \circ D(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))v = D(f \circ \phi_\beta^{-1})(\phi_\beta(p))w
\]

needs to hold true for all \([f] \in sc^1_p(M, K)\). Then, we necessarily have that

\[
D(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))v = w,
\]

and injectivity follows. \( \square \)

**Remark 5.13.** By Theorem 5.12 we can view the different definitions of tangent spaces as subspaces in the following way

\[
(T_pM)_{\text{kin}} \subseteq (T_pM)_{\text{phys}} \subseteq (T_pM)_{\text{alg}}.
\]

It is only in the finite-dimensional case and when \( p \notin \partial M \) that we can make sure that these three tangent spaces are isomorphic. For Banach manifolds, a sufficient condition that the kinematic and algebraic tangent spaces are isomorphic is that the modelling space is reflexive and has the (bornological) approximation property, see [44, Theorem 28.7]. It remains an open question whether the same is true for the physical and algebraic tangent spaces on M-polyfolds.
4. Tangent Bundle

Henceforth, we shall define the tangent space, $T_pM$, where $M$ is a $M$-polyfold and $p \in M_1$ is a point, as the physical tangent space. This choice is made to follow the conventions of the literature. The tangent bundle is defined as $TM := \bigsqcup_{p \in M_1} T_p M$. We want to impose charts on $TM$ such that the tangent bundle is a $M$-polyfold. Let $(V, \phi, (O, C, E))$ be a chart for $M$, then we define $TV := \bigsqcup_{p \in V_1} T_p M$. This induces a chart on $TM$ by defining $\bar{\phi}((p, v_p)) = (\phi(p), v)$ where $v_p = [(p, v, (O, \phi, (O, C, E)))] \in T_p M$. The set of all such chart domains, $TV$, induced by the $M$-polyfold structure of $M$ shall be denoted by $\mathcal{B}$. Furthermore we define the projection on $TM$ by $\pi : TM \to M^1$, $\pi(u) := p$ whenever $u \in T_p M$. Equivalent to our previous definition we have $TV = \pi^{-1}(V)$.

We define the tangent map as follows.

**Definition 5.14.** Let $M, N$ be $M$-polyfolds, and let $f : M \to N$ be a sc-smooth map. Furthermore, let $(V, \phi, (O, C, E))$ be a chart around a point $p \in M_1$, and let $(U, \psi, (P, D, F))$ be a chart around $f(p)$. We define the (pointed) tangent map, $T_p f : T_p M \to T_{f(p)} N$, at the point $p \in M_1$ by for each $v_p = [(p, v, (V, \phi, (O, C, E)))] \in T_p M$ mapping $v_p$ to the equivalence class of a representative in $T_{f(p)} N$ given by $(f(p), w, (U, \psi, (P, D, F)))$ where $w = D(\psi \circ f \circ \phi^{-1})(\phi(p))v$.

We define the tangent map $Tf : TM \to TN$ by for each $p \in M_1$ identifying $Tf|_{T_p M} : T_p M \to T_{f(p)} N$ as the pointed tangent map.

From the chain rule (Theorem 3.19) we directly obtain the chain rule of the tangent map.

**Theorem 5.15.** Let $M, N, K$ be $M$-polyfolds, and let $f : M \to N$, $g : N \to K$ be sc-smooth maps. Then, it holds that $T(f \circ g) = Tf \circ Tg$. 

5.4. Tangent Bundle

Following the proof of Proposition 2.5 in [36], we can prove that the tangent bundle is a M-polyfold. For convenience, we split the proof into Proposition 5.16 and Theorem 5.17.

**Proposition 5.16.** Let $M$ be a M-polyfold. Then it holds that:

i) $\mathcal{B}$ is a basis for a Hausdorff topology on $TM$;

ii) the projection map $\pi : TM \to M^1$ is continuous and open.

**Proof.** i) : We begin by showing that $\mathcal{B}$ is indeed a topological basis for $TM$. Let $W_\alpha, W_\beta \in \mathcal{B}$ such that $W_\alpha \cap W_\beta \neq \emptyset$. We want to show that for

$$v_p = [(p, h, (V, \phi, (O, C, E)))] \in W_\alpha \cap W_\beta$$

there exists a $W \in \mathcal{B}$ such that $v_p \in W \subseteq W_\alpha \cap W_\beta$. By definition

$$W_\alpha = (T\phi_\alpha)^{-1}(U_\alpha), \text{ and } W_\beta = (T\phi_\beta)^{-1}(U_\beta)$$

where $(V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)), (V_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta))$ are charts belonging to the M-polyfold structure of $M$ and where $U_\alpha, U_\beta$ are open subsets of $TO_\alpha$ and $TO_\beta$ respectively. Observe that $W_\alpha$ contains

$$v_{p_\alpha} = [(p_\alpha, h_\alpha, (V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)))]$$

and that $W_\beta$ contains

$$v_{p_\beta} = [(p_\beta, h_\beta, (V_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta)))]$$

This means that if

$$p_\alpha = p_\beta = p,$$

then

$$h_\alpha = D(\phi_\alpha \circ \phi^{-1})(\phi(p))h \quad \text{and} \quad h_\beta = D(\phi_\beta \circ \phi^{-1})(\phi(p))h.$$ 

Also, we have that

$$(\phi_\alpha(p_\alpha), h_\alpha) = (\phi_\alpha(p), h_\alpha) \in U_\alpha, \quad \text{and} \quad (\phi_\beta(p_\beta), h_\beta) = (\phi_\beta(p), h_\beta) \in U_\beta.$$ 

We define the sets

$$W_\alpha' = \left[T(\phi_\alpha \circ \phi^{-1})(\phi(x))\right]^{-1}(U_\alpha),$$

and

$$W_\beta' = \left[T(\phi_\beta \circ \phi^{-1})(\phi(x))\right]^{-1}(U_\beta).$$

Following from the above results we have that $(\phi(p), h) \in W_\alpha' \cap W_\beta'$, and if $U = W_\alpha' \cap W_\beta'$ with $W = (T\phi)^{-1}(U)$, then

$$v_p = [(p, h, (V, \phi, (O, C, E)))] \in W \subseteq W_\alpha \cap W_\beta.$$ 

We can conclude that $\mathcal{B}$ indeed is a topological basis.

Next, we shall prove that the topology generated by $\mathcal{B}$ is Hausdorff. Let

$$v_{p_\alpha} = [(p_\alpha, h_\alpha, (V_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha))]$$
and
\[ v_{p_\alpha} = [(p_\alpha, h_\alpha, (V_\alpha, \phi_\alpha, (O_\alpha, C, E_\alpha)))] \]
be two points on \( TM \) such that \( v_{p_\alpha} \neq v_{p_\beta} \). Then either \( p_\alpha \neq p_\beta \) or if \( p_\alpha = p_\beta \), then
\[ h_\beta \neq T(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p_\alpha))h_\alpha. \]
In the case that \( p_\alpha \neq p_\beta \) we simply choose smaller open neighbourhoods \( N_\alpha, N_\beta \subseteq X \) of \( p_\alpha \) and \( p_\beta \) respectively such that \( N_\alpha \cap N_\beta = \emptyset \). Then, the restricted charts give two disjoint open subsets
\[ v_{p_\alpha} = [(p_\alpha, h_\alpha, (N_\alpha, \phi_\alpha|_{N_\alpha}, (\phi_\alpha(N_\alpha), C_\alpha, E_\alpha)))] \]
and
\[ v_{p_\beta} = [(p_\beta, h_\beta, (N_\beta, \phi_\beta|_{N_\beta}, (\phi_\beta(N_\beta), C_\beta, E_\beta)))] \].
In the case that \( p_\alpha = p_\beta = p \), choose open neighbourhoods \( W_\alpha \subseteq TO_\alpha \), \( W_\beta \subseteq TO_\beta \) of \( (p, h_\alpha) \) and \( (p, h_\beta) \) respectively, such that
\[ T(\phi_\alpha \circ \phi_\beta^{-1})(W_\beta) \cap W_\alpha \neq \emptyset. \]
Then it holds that \( v_{p_\alpha} \in (T\phi_\alpha)^{-1}(W_\alpha) = \overline{W}_\alpha \) and \( v_{p_\beta} \in (T\phi_\beta^{-1})(W_\beta) = \overline{W}_\beta \).
Both \( \overline{W}_\alpha \) and \( \overline{W}_\beta \) are open sets. Since they are disjoint, we can conclude that \( TM \) is Hausdorff.

\( ii) \): First, we shall prove that the map \( \pi : TM \to M^1 \) is open. This is done by showing that, for every \( W \in B \), \( \pi(W) \) is open in \( M_1 \). Let \( (V, \phi, (O, C, E)) \) be a chart for \( M \), we have that \( T\phi : TV \to TO \) be the associated chart map for \( TM \). Furthermore, let \( W \subseteq TO \) be an open subset, and define \( \overline{W} = (T\phi)^{-1}(W) \). Denote \( \sigma : TO \to O^1 \) as the projection of the local tangent bundle \( TO \). We have the following
\[ \pi(W) = \phi^{-1} \circ \sigma \circ (T\phi)(W). \]
It holds that \( \sigma \) is an open map, \( T\phi \) is open by construction, and since \( \phi \) is a homeomorphism, it follows that \( \pi(W) \) is an open subset of \( M_1 \).

Let \( (V, \phi, (O, C, E)) \) be a chart for \( M \), and choose an open set \( U \subseteq M_1 \) such that \( U \subseteq V_1 \). We have that
\[ \pi^{-1}(U) = (T\phi)^{-1}((\phi(U) \times E) \cap TO). \]
Since \( (\phi(U) \times E) \cap TO \) is an open subset of \( TO \), it holds that \( \pi^{-1}(U) \in B \). Since the chart domains openly covers \( M \), and in particular \( M_1 \), it follows that \( \pi : TM \to M^1 \) is continuous.

This gives us sufficient groundwork to prove that the tangent bundle is a \( M \)-polyfold.

**Theorem 5.17.** Let \( M \) be a \( M \)-polyfold, then \( TM \) is a \( M \)-polyfold.
5.4. Tangent Bundle

Proof. We shall first show that $TM$ is paracompact. Consider an atlas

$$\mathcal{V} = \{(V^j, \phi^j, (O^j, C^j, E^j))\}_{j \in J}$$

that belongs to the M-polyfold structure of $M$ and such that $\{V^j\}_{j \in J}$ is an open locally finite cover of $M$. By Lemma 2.12 there exists a closed locally finite refinement, $\mathcal{G} = \{D^j\}_{j \in J}$, of $\mathcal{V}$, then it holds that $\mathcal{G}_1 = \{D_1^j\}_{j \in J}$ is a closed, locally finite refinement of $\mathcal{V}_1 = \{V_1^j\}_{j \in J}$. The sets

$$K^j := \bigsqcup_{x \in \phi(D^j_1)} T_x O^j$$

are closed in $TO^j$ for each $j \in J$. Therefore, the sets $\tilde{K}^j := (T\phi^j)^{-1}(K^j)$ are closed subsets of $TV^j$. In particular, it holds that each $\tilde{K}^j$ is paracompact as they are closed subsets of metrizable spaces. Furthermore, the collection $\{\tilde{K}^j\}_{j \in J}$ is locally finite. We shall argue why. Choose an element $v_p = [(h, (V, \phi, (O, C, E)))] \in TM$.

Then there is a neighbourhood $U_1 \subset M_1$, of $M$ such that only finitely many $D^j_1$ intersect $U_1$, say for all $j \in I \subseteq J$. Since, $M_1$ is in particular regular we can choose $U_1$ sufficiently small such that $M$ belongs to all $D^j_1$ with $j \in I$. Now define

$$W := \bigsqcup_{q \in \phi(U_1)} T_q O,$$

which is an open subset of $TO$. It holds that

$$\tilde{W} := (T\phi)^{-1}(W)$$

intersects only those $\tilde{K}^j$ such that $j \in I$. We are now able to use Proposition 2.13 and we can conclude that $TM$ is paracompact.

Next to show is that the bundle charts defined previously defines a M-polyfold structure. Consider two charts

$$(V, \phi, (O, C, E)),$$

and

$$(U, \psi, (P, D, F))$$

belonging to the M-polyfold structure of $M$. In the case that $U \cap V = \emptyset$ we are done. Suppose that $U \cap V \neq \emptyset$, then the bundle charts

$$T\phi : TV \to TO,$$

and

$$T\psi : TU \to TP$$

can be composed to a map

$$T\phi \circ T\psi^{-1}|_{T\psi(TU \cap TV)} : T\psi(TU \cap TV) \to T\phi(TU \cap TV)$$

which is explicitly

$$T\phi \circ T\psi^{-1}|_{T\psi(TU \cap TV)}(a, h) = (\phi \circ \psi^{-1}(a), D(\phi \circ \psi^{-1})(a)h),$$

where $a \in \psi(U_1 \cap V_1)$ and $h \in F$. Since the map $\phi \circ \psi^{-1}$ is sc-smooth, we have that the above map is also sc-smooth. Furthermore, as $(TO, TC, TE)$
is a sc-smooth retract, all tuples of the form \((TV, T\phi, (TO, TC, TE))\) defines a sc-smooth atlas on \(TM\). This together with Proposition 5.16 shows that \(TM\) is a M-polyfold. □

**Remark 5.18.** Here we can observe that if we would require that M-polyfolds are second countable we can only guarantee that \(TM\) would be a M-polyfold if the modelling spaces are separable. This follows from that infinite-dimensional Banach spaces are separable if and only if they are second countable. This is one of the many reasons that the assumption of paracompactness is a preferable axiom for M-polyfolds instead of second countability. Requiring that the modelling spaces are separable is an unnecessarily strict assumption.

Next, we shall define what is meant by the dimension of a M-polyfold. Note that we can not define it as the dimension of the underlying Banach space, a contradiction would be the M-polyfolds defined in Chapter 7.

**Definition 5.19.** Let \(M\) be a M-polyfold. A point \(p \in M_1\) is of dimension \(n \in [0, \infty]\) if \(T_pM\) is a \(n\)-dimensional vector space. Furthermore, \(M\) is called finite-dimensional if

\[
\sup_{p \in M_1} (\dim T_pM) < \infty.
\]

There are two pleasant properties connecting the sub-M-polyfold and tangent spaces.

**Proposition 5.20.** Let \(M\) be a M-polyfold, and let \(A \subseteq M\) be a sub-M-polyfold together with the natural M-polyfold structure for \(A\). Then, the following statements are true:

i) the tangent space \(T_aA\) at the point \(a \in A_\infty\) has a sc-complement in \(T_aM\);

ii) let \(a \in A_\infty\) be a point, and let \(s : W \to W\), with \(a \in W \subseteq M\) open, be a sc-smooth retraction such that \(s(W) = W \cap A\). Then, the induced map \(W \to A\) is sc-smooth and \(T_a s(T_aM) = T_aA\).

**Proof.** Assume all is given as stated as in the first part of the Proposition above. We shall prove that the statement holds locally. Let \((\phi, V, (O, C, E))\) be a chart for \(M\), with \(a \in V\), and let \(r : U \to U\), with \(U \subseteq C\) open, a sc-smooth retraction corresponding to \((O, C, E)\). Then, \(\phi(A \cap U)\) is a subset of \(O\) and since \(A\) is a sub-M-polyfold, for every \(a \in A \cap V\), there is an open neighbourhood \(N \subseteq O\) of \(\phi(a)\), and a sc-smooth retraction \(s : N \to N\) such that \(s(N) = \phi(A \cap V) \cap N\). In the case that \(a \in A\) is a smooth point, define the map \(t := s \circ r : U \to U\). It holds that \(t = t \circ t\) and that \(t\) is...
5. An Implicit Function Theorem

The goal of this section is to introduce a version of the implicit function theorem on M-polyfolds. It shall not be central for most of our applications, but since the implicit function theorem is important in its own right, any essay would be incomplete without it.

Let \((U, C, E)\) be a local M-polyfold model, and let \(F\) be a sc-Banach space. The strong filtration product, denoted \(\preceq\), is defined by \(U \preceq F\) being the set \(U \times F\) together with the double filtration for \(0 \leq k \leq m + 1\)
\[
(U \preceq F)_{m,k} := U_m \oplus F_k.
\]
In particular, for \(i = 0, 1\), we are interested in the following sc-Banach spaces denoted by
\[
((U \preceq F)[i])_m = U_m \oplus F_{m+i}.
\]
Locally, we shall build what is called strong bundles on sc-Banach spaces. In principle these bundles have strong filtration products as a total space.

**Definition 5.21.** Let \((U, C, E)\), \((V, D, F)\) be local M-polyfold models, and let \(G, H\) be sc-Banach spaces. A function \(f : U \preceq G \to V \preceq H\) is called a strong map if the following holds:

i) \(f((U \preceq G)_{m,k}) \subseteq (V \preceq H)_{m,k}\) for all \(m \geq 0\) and all \(0 \leq k \leq m + 1\);

ii) \(f\) is of the form
\[
f(u, h) = (g(u), \Gamma(u, h))
\]
where \(\Gamma(u, h)\) is linear over \(h \in G\), and \(g : U \to V\) is a map;

iii) for \(i = 0, 1\), the maps
\[
f[i] : (U \preceq G)[i] \to (V \preceq H)[i]
\]
are sc-smooth.
In other words strong maps are sc-smooth maps that preserve the strong filtration.

**Remark 5.22.** In [36] these strong maps are called strong bundle maps. We shall only call them strong bundle maps when we talk about strong bundles on M-polyfolds.

From the strong maps we can define strong retracts, that is retracts on strongly filtrated products.

**Definition 5.23.** Let \((U, C, E)\) be a local M-polyfold model, and let \(F\) be a sc-Banach space. A strong map \(R : U \triangleleft F \rightarrow U \triangleleft F\) is called a **strong retraction** if \(R = R \circ R\).

Note that strong retractions can be written of the form

\[ R(u, h) = (r(u), \Gamma(u, h)) \]

where \(r : U \rightarrow U\) is a sc-smooth retraction. Also note that if \(r(u) = u\) it holds that

\[ \Gamma(u, \Gamma(u, h)) = \Gamma(u, h) \]

and therefore

\[ h \mapsto \Gamma(u, h) : F \rightarrow F \]

is a projection. Furthermore, in the case that \(u \in U_\infty\) it holds that this map is a sc-operator. From this we can naturally define strong retracts.

**Definition 5.24.** Let \((U, C, E)\) be a local M-polyfold model, and let \(F\) be a sc-Banach space. The tuple

\( (K, C \triangleleft F, E \triangleleft F) \)

is called a **local strong retract** if there exists a strong retraction

\( \rho : U \triangleleft F \rightarrow U \triangleleft F \)

such that \(\rho(U \triangleleft F) = K\).

In a similar fashion to previous discussions, for \(m \geq 0\) and \(0 \leq k \leq m + 1\), there is a double filtration on \(K\), defined by

\[ K_{m,k} = K \cap (U_m \oplus F_k) \]

Consequently, we have the associated spaces \(K[i]\), for \(i = 0, 1\) being the set

\[ K[i] = K_{0,i} \]

together with the filtration

\( (K[i])_m = K_{m,m+i} \)

where \(m \geq 0\). Now to the global constructions.
Definition 5.25. Let $M$ and $N$ be $M$-polyfolds. The tuple $(N, \pi, M, E_x)$, where $\pi : N \to M$ is a surjective sc-smooth map called the projection map, is called a strong bundle over the $M$-polyfold $M$ if the following holds:

(i) the typical fiber $\pi^{-1}(x)$ is isomorphic to some Banach space $E_x$ at every point $x \in M$;

(ii) for every point $p \in M_i$ there is a compatible chart $(U, \phi, (O, C, E))$ around $p$ for $M$ and a sc-diffeomorphism $\psi : \pi^{-1}(U) \to K$, where $(K, C \circ F, E \circ F)$ is a strong retract, such that the following diagram commutes.

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi} & K \\
\downarrow{\pi} & & \downarrow{pr} \\
U & \xrightarrow{\phi} & O
\end{array}
$$

The maps $\psi : \pi^{-1}(N) \to K$ are called strong bundle charts.

Remark 5.26. In the literature [32, 36] strong bundles are defined as topological spaces that have $M$-polyfold charts generated by the strong bundle charts. In favour of consistency, we use a more differential geometric flavoured notion of the strong bundle.

Note that the strong bundle charts induce a double filtration $N_{m,k}$ on $N$, where $m \geq 0$ and $0 \leq k \leq k + 1$. This double filtration induces two important $M$-polyfolds $Y[i] = Y_{0,i}$, where $i = 0, 1$, together with the filtrations

$$(Y[i])_m = Y_{m,m+i}.$$ 

Note that the projections $\pi[i] : N[i] \to M$ are sc-smooth for $i = 0, 1$. Now we are able to generalise the concept of sections to sections on strong $M$-polyfold bundles. Observe that we have two different kinds of sc-smooth sections.

Definition 5.27. Let $(N, \pi, M, E_x)$ be strong bundle, and let $f : M \to N$ be a function. We call $f$ a sc-smooth section if $\pi \circ f = \mathbb{1}_X$ and if the map $f[0] : M \to N[0]$ is sc-smooth. Similarly, we call $f$ a sc$^+$-section if $\pi \circ f = \mathbb{1}_X$ and if the map $f[1] : M \to N[1]$ is sc-smooth.

Quite naturally we can define the pullback bundle.
Definition 5.28. Let \((N, \pi, M, E_x)\) be a strong bundle, let \(K\) be a \(M\)-polyfold, and let \(f : K \to M\) be an injective sc-smooth map. Then, the pullback bundle is defined as the tuple \((f^*N, \tau, K, G_x)\) satisfying the following:

i) \(f^*N = \{(k, n) \in K \times N : \pi(n) = f(k)\}\);

ii) \(\tau : f^*N \to K\) is the projection defined by \(\tau(k, n) = k\);

iii) for the projection map to the second factor, \(\text{pr}_2 : f^*N \to N\), the diagram

\[
\begin{array}{ccc}
    f^*N & \xrightarrow{\text{pr}_2} & N \\
    \downarrow\tau & & \downarrow\pi \\
    K & \xrightarrow{f} & M
\end{array}
\]

commutes;

iv) the typical fiber \(\tau^{-1}(x)\) is isomorphic to some Banach space \(G_x\) at every point \(x \in K\).

Of course we want the pullback bundle to be a strong bundle, and this is indeed the case as is shown in Proposition 4.11 in [32].

Proposition 5.29. The pullback bundle \((f^*N, \tau, K, G_x)\) carries naturally a strong bundle structure over the \(M\)-polyfold \(K\).

The following implicit function theorem for interior points holds, we shall not prove it but simply state it as it follows from another technical result, see [36, Theorem 3.13].

Theorem 5.30 (Implicit Function Theorem). Let \((N, \pi, M, E_X)\) be a strong bundle with \(d_M(x) = 0\) for all \(x \in M\), and let \(f : M \to N\) be a sc-Fredholm section (for the definition see [36, Definition 3.8]). Suppose that \(p_0 \in M_\infty\) such that \(f(p_0) = 0\). Then the map \(f'(p_0) : T_{p_0}M \to \pi^{-1}(p_0)\) is a sc-Fredholm operator. If \(f'(p_0)\) is surjective, then there exists an open neighbourhood \(U \subseteq M\) of \(p_0\) such that the set

\[S := \{p \in U : f(p) = 0\}\]

naturally inherits the structure of a finite dimensional manifold, with the dimension equal to the Fredholm index of \(f\). Furthermore, \(U\) can be chosen such that

\[f'(p) : T_pX \to \pi^{-1}(p), \quad \text{for } p \in S,\]

is surjective and \(\ker(f'(p)) = T_pS\).

Remark 5.31. The assumption of sc-Fredholm is not only necessary for the proof of Theorem 5.30, but as is shown in [20] there are non-sc-Fredholm sections for which the implicit function theorem does not hold. This is one of the more radical differences between sc-calculus and classical calculus.
Remark 5.32. There is also a version of the Implicit Function Theorem for strong bundles at boundary points, see Theorem 3.5 in [36]. But this requires that we limit the behaviour of the boundary of the base M-polyfold, i.e. we need to consider tame M-polyfolds. Furthermore, one needs to restrict the behaviour of the Kernel of $f'(p) : T_pM \to \pi^{-1}(p)$, for one, it has to have nonempty interior in the partial quadrant on which $M$ is modelled upon.

6. Submanifolds of M-polyfolds

In applications to Fredholm theory, it is very important that M-polyfolds can contain finite-dimensional smooth manifolds. We shall give a brief overview on some important results connected to finite-dimensional manifolds.

Definition 5.33. Let $M$ be a M-polyfold, and let $A \subseteq M$ be a subset. If for every point $a \in A$ there is an open neighbourhood $U \subseteq M$ around $a$ and a sc$^+$-retraction $r : U \to U$, such that $r(U) = A \cap U$, then $A$ is called a smooth finite-dimensional submanifold of $M$.

Remark 5.34. Note that smooth finite dimensional submanifolds are not defined as a special case of sub-M-polyfolds.

That $A$ indeed is a smooth manifold follows from a few observations. Since $r : N \to N$ is a retraction it holds that $r = r \circ r$, and as it is sc$^+$ we have that $r(U_m) \subseteq U_{m+1}$ for all $m \geq 0$. These two observations yield that

$$r(U_m) = r \circ r(U_m) \subseteq r(U_{m+1}) = r \circ r(U_{m+1}) \subseteq r(U_{m+2}) = \ldots$$

for all $m \geq 0$, and hence $r(U) \subseteq M_\infty$, i.e. all points on $A$ are smooth. Furthermore, the tangent space at $a \in A$ is

$$T_aA = T_a r(T_aM),$$

and since $r$ is sc$^+$-smooth, $T_a r$ is also sc$^+$-smooth. In view of Remark 3.7 $T_a r$ is a compact operator onto $T_a(A \cap N) \subseteq T_a(M_\infty)$. But as infinite-dimensional spaces are nowhere locally compact and since $A \cap U$ is open in the subspace topology, $T_a A$ must be finite-dimensional. Note also that these submanifolds here are more general than classical manifolds as these really are manifolds with boundary and corners, see Definition 2.4 in [51]. We sum up some of properties for these submanifolds.

Proposition 5.35. Let $M$ be a M-polyfold, and let $A \subseteq M$ be a finite dimensional submanifold of $M$. Then, $A$ has the following properties:

i) $A \subseteq M_\infty$, and $A$ inherits the M-polyfold structure from $M$. In particular, $A$ has a tangent space at every point in $A$ and the degeneracy index $d_A$ is defined on $A$;
ii) $A$ is locally closed in $M$;

iii) $\partial A = \{a \in A : d_A(a) \geq 1\} \subseteq \partial M$.

**Proof.** i) : This follows directly from our previous discussion.

ii) Let $a \in A$ be a point, and let $V \subseteq M$ be an open neighbourhood of $a$ such that there is a sc$^+$-smooth retraction, $s : V \to V$ with $s(V) = A \cap V$. Suppose that $b \in \text{cl}_V(A \cap V)$, then there is a sequence $\{a_k\}$, $a_k \in A$, such that $\lim a_k \to b$. Since $V$ is open and $b \in V$ there is a $N \in \mathbb{N}$ such that $a_k \in A \cap V$ for all $k \geq N$. Therefore $s(a_k) = a_k$ for all $k \geq N$, and we can conclude that

$$b = \lim_{k \to \infty} a_k = \lim_{k \to \infty} s(a_k) = s(b).$$

Hence, $b \in V \cap A$ and since $a$ and $b$ were arbitrary it holds that $A \cap V$ is closed in $V$ for all points $a \in A$.

iii) By definition we have that $\partial A = \{a \in A : d_A(a) \geq 1\}$. We shall show that if $d_A(a) \geq 1$ for some $a \in A$, then $d_M(a) \geq 1$. Suppose the contrary, that $d_A(a) \geq 1$ and $d_M(a) = 0$ for some $a \in A$. In this case we can find a neighbourhood $V$ of $a$ that is sc-diffeomorphic to a sc-smooth retract $(O, E, E)$. This implies that there is an open neighbourhood $U \subseteq A$ of $M$ such that $U$ is sc-diffeomorphic to some sc-smooth retract $(P, E, E)$. Therefore, $d_A(a) = 0$ which is a contradiction to our assumption. □

**Remark 5.36.** In the current state of research, equivalency between the manifold structure of submanifolds and the M-polyfold structure corresponding to the same set considered as a sub-M-polyfold is still an open question. Though, it is true for smooth finite-dimensional submanifolds of tame M-polyfolds, see Remark 4.22, if the submanifold is in *good position*, see [36, Theorem 4.2].
CHAPTER 6

sc-Differential Forms and Tensors

Here, we shall introduce differential forms and tensors in the context of M-polyfolds. We shall begin this chapter by entering the world of sc-differential forms. Because M-polyfolds allow for drastic changes of the topology, the whole subject of sc-differential forms demands some update. Therefore, many of the results and ideas here are new. Though, there are some open problems left, see Conjecture 6.4. We shall finish the chapter by constructing some fundamental tensor calculus on M-polyfolds, which has not been done before.

1. sc-Differential Forms

Already for classical Banach manifolds we need to be really careful about how we choose to define differential forms. As seen in Table 33.21 in [44] there are at least 12 different notions of differential forms on Banach manifolds that are equivalent for finite dimensional manifolds but are not in general equivalent for infinite dimensions. Also, take note that here we define differential forms for real M-polyfolds, i.e. where the modelling spaces are Banach spaces on $\mathbb{R}$. Furthermore, as we do not require our modelling spaces to be reflexive, care is heeded when we try to interpret the notions of sc-smooth differential forms on M-polyfolds. As non-reflexivity especially becomes problematic for the interpretation of sc-smooth tensors on M-polyfolds, we shall keep our generalised setting for sc-differential forms only, and restrict our definition of tensors to spaces modelled on reflexive spaces. For this reason we shall first discuss sc-differential forms and later tensors.

**Definition 6.1.** Let $M$ be a M-polyfold, and let $\kappa : M_1 \to \mathbb{N}$ be a mapping such that

$$\bigcup_{p \in M_1} \left( \bigoplus_{\kappa(p)} T_p M \right)$$

is a M-polyfold. A $sc^G$-differential form $\omega$ is a sc-smooth map

$$\omega : \bigcup_{p \in M_1} \left( \bigoplus_{\kappa(p)} T_p M \right) \to \mathbb{R},$$
such that $\omega$ is $\kappa(p)$-multilinear and an anti-symmetric mapping for each fixed $p \in M_1$. The map $\kappa : M_1 \to \mathbb{N}$ shall be referred to as the order map for $\omega$. If $\kappa(p) = k \in \mathbb{N}$ for all $p \in M_1$, then $\omega$ is called a sc-differential $k$-form. The set of all sc-differential $k$-forms is denoted as $\Omega^k(M)$.

**Notation 6.2.** To avoid confusion we distinguish between sc-differential $k$-forms and $\text{sc}^G$-differential forms, short for generalised sc-differential forms, by adding a $G$ as a superscript. When we talk about the set of all $\text{sc}^G$-differential forms on a M-polyfold $M$ we shall denote it as $\Omega^G(M)$, and notice that we shall leave the notation of classical $k$-forms so that it is consistent with the literature.

**Remark 6.3.** Note that this definition of $\text{sc}^G$-differential forms is a completely new notion. In [30, 36] only the notion of $k$-forms is defined. The main motivation for $\text{sc}^G$-differential forms is that we are interested in integration over volume-forms on M-polyfolds. In the finite-dimensional case, we want to end up with an integral over the Hausdorff measure where the dimension of the Hausdorff measure depends on the local dimension of the point. Furthermore, note that we have not yet said anything about how we should choose $\kappa(p)$ so that $\bigoplus_{p \in M_1} (\bigoplus_{\kappa(p)} T_pM)$ is a M-polyfold, i.e. such that sc-smoothness is well-defined. Observe that if we do not make this requirement some troublesome choices of $\kappa : M_1 \to \mathbb{N}$ are eliminated.

Conjecture 6.4 seems like a plausible requirement on how we can choose the order map $\kappa : M_1 \to \mathbb{N}$. Though, no proof for it has been successfully furnished as of yet.

**Conjecture 6.4.** Let $M$ be a M-polyfold, and let $\{N_\alpha\}$ be an open cover for $M$. For each $N_\alpha$, let $r_\alpha : N_\alpha \to N_\alpha$ be sc-smooth retractions such that, whenever $N_\alpha \cap N_\beta \neq \emptyset$ and $p \in (N_\alpha \cap N_\beta)_1$, 

$$\dim(T_{ra(p)r_\alpha r_\alpha(p)}M) = \dim(T_{r_\alpha(p)r_\alpha r_\alpha(p)}M) < \infty.$$ 

Then, by defining $\kappa : M_1 \to \mathbb{N}$, for $p \in (N_\alpha)_1$, $\kappa(p) = \dim(T_{ra(p)r_\alpha r_\alpha(p)}M)$, it holds that 

$$\bigcup_{p \in M_1} \left( \bigoplus_{\kappa(p)} T_pM \right)$$ 

is a M-polyfold.

In particular, we are interested in differential forms when 

$$\kappa(p) = \dim(T_pM),$$ 

for all $p \in M_1$. Since the identity map on a M-polyfold is a sc-smooth mapping on a M-polyfold, the following result follows from Conjecture 6.4.
Corollary 6.5. Let $M$ be a finite-dimensional $M$-polyfold, and let $\kappa : M_1 \to \mathbb{N}$ be defined by $\kappa(p) = \dim(T_pM)$. Then,

$$\bigcup_{p \in M_1} \left( \bigoplus_{\kappa(p)} T_pM \right)$$

is a $M$-polyfold.

In the following discussion we shall treat the Lie bracket of sc-smooth vector fields, we shall do so inspired by [30, Chapter 4.2]. We shall begin by a local construction and then show that the local definition on $M$-polyfolds are chart independent.

Let $M$ be a $M$-polyfold, let $(N, \phi, (O, C, E))$ be a chart around $p \in M_1$ belonging to $M$, and let $\mu, \nu : M_1 \to TM$ be two sc-smooth vector fields. Suppose that $r : U \to U$, with $U \subseteq C$ open, be a sc-smooth retraction corresponding to $(O, C, E)$. Then, following Theorem 5.9, for each $p \in N_1$, it holds that there is a linear isomorphism $\psi : T_p(X) \to \text{im}(Dr(\phi(p)))$.

Since $r$ is a sc-smooth retraction, the following holds

$$Dr(\phi(p))\psi(\mu(p)) = \psi(\mu(p)) \quad \text{and} \quad Dr(\phi(p))\psi(\nu(p)) = \psi(\nu(p)). \quad (6.1)$$

At a point $p \in N_2$ we can take the directional derivative of the first identity of (6.1) in the direction of $\psi(\nu(p))$

$$D^2r(\phi(p))(\psi(\mu(p)), \psi(\nu(p))) + Dr(\phi(p))D\psi(\mu)(\phi(p))\psi(\nu(p)) = D\psi(\mu)(\phi(p))\psi(\nu(p)).$$

In a similar fashion we take the directional derivative at a point $p \in N_2$ of the second identity in (6.1)

$$D^2r(\phi(p))(\psi(\nu(p)), \psi(\mu(p))) + Dr(\phi(p))D\psi(\nu)(\phi(p))\psi(\mu(p)) = D\psi(\nu)(\phi(p))\psi(\mu(p)).$$

Since the second derivative is symmetric, i.e. it holds that

$$D^2r(\phi(p))(\psi(\mu(p)), \psi(\nu(p))) = D^2r(\phi(p))(\psi(\nu(p)), \psi(\mu(p))),$$

one finds that the following holds

$$Dr(\phi(p))[D\psi(\nu)(\phi(p))\psi(\mu(p)) - D\psi(\mu)(\phi(p))\psi(\nu(p))] = D\psi(\nu)(\phi(p))\psi(\mu(p)) - D\psi(\mu)(\phi(p))\psi(\nu(p)).$$

Hence, one can conclude that the identity

$$\psi^{-1}[D\psi(\nu)(\phi(p))\psi(\mu(p)) - D\psi(\mu)(\phi(p))\psi(\nu(p))]$$

belongs to $(T_pN)_1$ for all $p \in N_2$.

Therefore, we are able to define the Lie bracket of two vector fields.
Definition 6.6. Let $M$ be a $M$-polyfold, and let $\mu, \nu : M_1 \to TM$ be two sc-smooth vector fields. Then we define the map $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M^1)$ by

$$[\mu, \nu](p) = D\mu(p)\nu(p) - D\nu(p)\mu(p),$$

where $p \in M_2$.

It is important to note that $[\mu, \nu]$ is a sc-smooth map $M^2 \to (TM)^1$. In other words it is a sc-smooth vector field of level 1. That our definition of the Lie bracket is chart-independent follows from the following lemma, see Lemma 4.3 in [30].

Lemma 6.7. Let $M$ be a $M$-polyfold, let $\varphi : M \to M$ be a local sc-diffeomorphism, and let $\mu, \nu : M_1 \to TM$ be two sc-smooth vector fields. Suppose that

$$T\varphi(\mu(p)) = \mu(\varphi(p)) \quad \text{and} \quad T\varphi(\nu(p)) = \nu(\varphi(p))$$

holds for all $p \in M_2$, then

$$T\varphi([\mu, \nu](p)) = [\mu, \nu](\varphi(p))$$

is true for all $p \in M_2$.

Proof. We shall prove this locally. Let

$$(N_1, \phi, (O, C, E)), \quad \text{and} \quad (N_2, \psi, (P, D, F))$$

be charts belonging to $M$ such that $\varphi|_{N_1} : N_1 \to N_2$ is a sc-diffeomorphism. Then $O$ and $P$ are sc-diffeomorphic by the map $\chi := \psi \circ \varphi \circ \phi^{-1}$. To further ease notational clutter set $\gamma := T\phi \circ \mu \circ \phi^{-1}$ and $\delta := T\phi \circ \nu \circ \phi^{-1}$. Observe that from the principal part of our assumptions on the vector fields we have that

$$D(\chi)\gamma = \gamma \circ \chi \quad \text{and similarly} \quad D(\chi)\delta = \delta \circ \chi. \quad (6.2)$$

Now, take the directional derivative of the first identity in (6.2) along $\nu$ at points $p \in (N_1)_2$ and get

$$D(\gamma \circ \chi)\delta = D^2(\chi)(\gamma, \delta) + D(\chi)[D(\gamma)\delta].$$

In a similar fashion we take the directional derivative of the second identity in (6.2) along $\mu$ at points $p \in (N_1)_2$ and we get

$$D(\delta \circ \chi)\gamma = D^2(\chi)(\delta, \gamma) + D(\chi)[D(\delta)\gamma].$$
Since the second order derivative is symmetric and using the chain rule, we obtain by taking the difference of the above
\[ D(\chi)[\gamma, \delta] = D(\chi)[D(\gamma)\delta - D(\delta)\gamma] = D(\gamma \circ \chi)\delta - D(\delta \circ \chi)\gamma = D(\gamma) \circ \chi[D(\chi)\delta] - D(\delta) \circ \chi[D(\chi)\gamma] = D(\gamma) \circ \chi[\delta \circ \chi] - D(\delta) \circ \chi[\gamma \circ \chi] = [\gamma, \delta] \circ \chi. \]

This is the local representation of what is stated in Lemma 6.7 and since our choice of chart was arbitrary the proof follows. □

As we have levels on M-polyfolds we can also talk about the set of sc-differential forms on each level. From the inclusions \( M^i \hookrightarrow M \), \( M^i \) a M-polyfold and \( i \geq 0 \), we have that differential forms on \( M^i \) pulls back to differential forms on \( M \). We get the following directed system for \( k \)-forms
\[ \Omega^k(M) \to \Omega^k(M^1) \to \cdots \to \Omega^k(M^i) \to \Omega^k(M^{i+1}) \to \cdots \]
where we shall denote the direct limit as \( \Omega^k_\infty(M) \). As per usual we end up with the following direct sum of sc-smooth differential forms on \( M_\infty \),
\[ \Omega^\ast_\infty(M) = \bigoplus_{k=0}^{\infty} \Omega^k_\infty(M). \]

Since the sc-smooth Lie bracket has been established, we can define the exterior derivative of \( k \)-forms.

**Definition 6.8.** Let \( M \) be a M-polyfold. We define the **exterior derivative**
\[ d : \Omega^k(M^i) \to \Omega^{k+1}(M^{i+1}), \]
where \( i \geq 0 \), by for each \( \omega \in \Omega^k(M^i) \) defining
\[ \omega(A_0, A_1, \ldots, A_k) = \sum_{j=0}^{k} (-1)^j D(\omega(A_0, \ldots, \hat{A}_j, \ldots, A_k))A_j + \sum_{j<l} (-1)^{j+l} \omega([A_j, A_l], A_0, \ldots, \hat{A}_j, \ldots, \hat{A}_l, \ldots, A_k), \]
where \( A_0, A_1, \ldots, A_k \in \mathfrak{X}(M) \) and where \( \hat{A}_i \) denotes that the \( i \)th element is omitted.

**Remark 6.9.** One might be interested to know whether there is an exterior derivative for sc\( ^G \)-differential forms. If such a definition would be well-defined, the order map for the exterior derivative of the sc\( ^G \)-differential should be the order map plus one. Though, for such a definition to be well-defined the definition above has to be rewritten quite significantly.
6.2. sc-Differential Forms

Since the diagram
\[
\begin{array}{ccc}
\Omega^k(M^i) & \xrightarrow{d} & \Omega^{k+1}(M^i) \\
\downarrow & & \downarrow \\
\Omega^k(M^{i+1}) & \xrightarrow{d} & \Omega^{k+1}(M^{i+1})
\end{array}
\]
commutes, we induce the differential form on the limit of the system
\[d : \Omega^*_\infty(M) \to \Omega^*_\infty(M)\).

We have as per usual that \(d^2 = 0\) and thus we can talk about de Rahm cohomology on M-polyfolds, see also [36, Definition 4.10].

**Definition 6.10.** Let \(M\) be a M-polyfold and let
\[d : \Omega^*_\infty(M) \to \Omega^*_\infty(M)\]
be the exterior derivative. The quotient group
\[H^*_\text{sc}(X) := \frac{\ker(d)}{\text{im}(d)}\]
is called the sc-de Rahm cohomology group.

As per usual, we want to equip the set of differential forms with the exterior product (also called wedge product). The definition of the exterior product between sc-\(k\)-forms is completely analogous to that of classical \(k\)-forms.

**Definition 6.11.** Let \(M\) be a M-polyfold and let \(\omega, \xi \in \Omega^G(M)\) be sc\(^G\)-differential forms with the respective order maps \(\kappa_1, \kappa_2 : M_1 \to \mathbb{N}\). The exterior product, \(\wedge : \Omega^G(M) \times \Omega^G(M) \to \Omega^G(M)\) is defined, for every \(p \in M_1\), as the map
\[\wedge(\omega, \xi)(p) = \frac{(\kappa_1 + \kappa_2)!}{\kappa_1! \kappa_2!} \text{Alt}^{\kappa_1(p)+\kappa_2(p)}(\omega(p) \otimes \xi(p)).\]
We shall denote the exterior product between \(\omega\) and \(\xi\) as \(\omega \wedge \xi\).

The easy part is to state the definition of the exterior product. Next to ask is whether this map is well defined. At least \(\omega \wedge \xi\) is a section on the the Whitney bundle
\[\pi : \bigcup_{p \in M_1} \left( \bigoplus_{\kappa_1(p)} T_pM \right) \oplus \bigcup_{p \in M_1} \left( \bigoplus_{\kappa_2(p)} T_pM \right) \to M_1.\]
We need to check whether \(\omega \wedge \xi\) is sc-smooth in general. This is still an open question.
2. Tensors on M-polyfolds

Now, we shall introduce tensors on M-polyfolds. However, in [1, Chapter 5], the cotangent space is not defined as the algebraic dual to the tangent space (which is what is used here). Instead, a restricted set of functionals are used so that \( E^{**} \cong E \), \( E \) a Banach space, even though \( E \) might not be reflexive. As we do not follow this convention we shall instead restrict our discussion of tensors on M-polyfolds modelled on reflexive spaces, for convenience we shall simply call such M-polyfolds reflexive M-polyfolds. Furthermore, observe that all the concepts used here have been done for Banach manifolds but not for M-polyfolds. Therefore, all following concepts are new in the context of M-polyfolds.

**Definition 6.12.** Let \( M \) be a reflexive M-polyfold. The set \( T^s_r(M) \) denotes the set of sc-smooth maps

\[
\tau : \bigoplus_{n=0}^r T^* M \oplus \bigoplus_{m=0}^s TM \to \mathbb{R}
\]

such that for each \( p \in M_1 \), the map

\[
\tau : \bigoplus_{n=0}^r T^*_p X \oplus \bigoplus_{m=0}^s T_p M \to \mathbb{R}
\]

is \((r + s)\)-multilinear.

As we want to apply physics to M-polyfolds, we need to define semi-Riemannian metrics on M-polyfolds. Naturally it shall be a special case of a sc-smooth symmetric tensor field. In Section 5.2 in [1] two versions of semi-Riemannian metrics are used, a weak and a strong notion of a semi-Riemannian metric.

**Definition 6.13.** Let \( M \) be a reflexive M-polyfold. A weak semi-Riemannian metric on \( M \) is a symmetric tensor field \( g \in \mathcal{T}^0_2(M) \) such that for each \( p \in M_1 \),

\[
g(p)(v_p, w_p) = 0
\]

for all \( w_p \in T_p M \) implies that \( v_p = 0 \). The tensor field \( g \in \mathcal{T}^0_2(M) \) is called a weak Riemannian metric on \( M \) if in addition,

\[
g(p)(v_p, v_p) > 0
\]

for all \( p \in M_1 \) and all \( v_p \in T_p M_1 \) with \( v_p \neq 0 \).

See e.g [52] for some applications of weak Riemannian metrics. In [42, 46] strong semi-Riemannian metrics are used.
Definition 6.14. Let $M$ be a reflexive M-polyfold. A strong semi-Riemannian metric on $M$ is a symmetric tensor field $g \in \mathcal{T}_2^0(M)$ such that for each $p \in M_1$, the map

$$v_p \mapsto g(p)(v_p, \cdot)$$

is a vector space isomorphism from $T_p M$ to $T^*_p M$. The tensor field $g \in \mathcal{T}_2^0(M)$ is called a strong Riemannian metric on $M$ if in addition,

$$g(p)(v_p, v_p) > 0$$

for all $p \in M_1$ and all $v_p \in T_p M$ with $v_p \neq 0$.

As the name implies a strong semi-Riemannian metric is a special case of a weak semi-Riemannian metric and is often more convenient to use. Though, they are not always applicable, see e.g. [9, Chapter 2] where the use of the weak metric on the manifold of metrics is necessary but causes some technical difficulties. These technicalities arise mostly because some general results does not hold for weak metrics, even though they are true for strong metrics.

Remark 6.15. As in the case of classical Banach manifolds, a necessary condition for a M-polyfold to admit a strong semi-Riemannian metric is that the M-polyfold is modelled on a sc-Hilbert space. Though, it does not hold that every weak semi-Riemannian metric is a strong semi-Riemannian metric on M-polyfolds modelled on sc-Hilbert spaces. A counterexample is the Hilbert space $H$ of real sequences $a = \{a_n\}$, $a_n \in \mathbb{R}$, such that

$$\|a\|^2 := \sum_{n=1}^{\infty} n^2 a_n^2 < \infty.$$  

Then, the map $g : H \times H \to \mathbb{R}$

$$g(a, b) = \sum_{n=1}^{\infty} a_n b_n,$$

where $a, b \in H$, is a weak Riemannian metric on $H$ but is not a strong Riemannian metric on $H$.

Furthermore, some issues arise when dealing with the existence of strong (or weak) semi-Riemannian metrics on M-polyfolds. This is the case, as we would want sc-smooth partition of unity to furnish a simple proof. To avoid this we restrict the proof of existence of semi-Riemannian metrics to that of M-polyfolds modelled on sc-Hilbert spaces. This is done in order to guarantee the sc-smooth bump function property, see Definition 4.26 and Theorem 4.31, and hence Theorem 4.32 holds. Also, since Hilbert spaces are reflexive, such M-polyfolds are good candidates for admitting (strong and weak) semi-Riemannian metrics.
Proposition 6.16. Let $M$ be a $M$-polyfold modelled on sc-Hilbert spaces. Then there exists a strong Riemannian metric on $M$.

Proof. For each $p \in M_1$, let $(U, \phi, (O, C, E))$ be a chart around $p$. Define the map $g_\alpha : T_pM \oplus T_pM \to \mathbb{R}$ by

$$g_\alpha(u, v)(p) = \langle T_p\phi(u_p), T_p\phi(v_p) \rangle$$

where $u_p, v_p \in T_pM$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $E_0$. We have that $g_\alpha$ is a smooth map by Proposition 2.7, and in particular sc-smooth. It holds that $g : TU \oplus TU \to \mathbb{R}$ is a strong metric on $U$. By $\mathcal{O} = \{U_\alpha\}$ we denote an open cover on $M$ constructed from charts of the form $(U, \phi, (O, C, E))$. By Theorem 4.31 and Theorem 4.32 there is a sc-smooth partition of unity subordinate to $\mathcal{O}$, which we denote as $\{\varphi_\alpha\}$. Define $g : TM \times TM \to \mathbb{R}$ by

$$g(u, v) = \sum_\alpha \varphi_\alpha g_\alpha(u, v).$$

where $u, v \in TM$. This yields the strong Riemannian metric we want. □

In order to define the musical isomorphisms, $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$, we need to restrict the discussion to strong semi-Riemannian metrics. The reason why we need to use strong metrics is obvious from the following definition.

Definition 6.17. Let $M$ be a reflexive $M$-polyfold, and let $g : TM \times TM \to \mathbb{R}$ be a strong semi-Riemannian metric. The $M$-polyfold bundle isomorphism $\flat : TM \to T^*M$ defined by

$$\flat(v_p) := g(p)(v_p, \cdot)$$

for every $v_p \in T_pM$, is called the flattening operator. The $M$-polyfold bundle isomorphism $\sharp : T^*M \to TM$ is defined as the inverse map of $\flat$ and is called the sharpening operator.

These operators are used (usually without mention) in physics as index raising and index lowering. Note that in general the metric $g$ is not constant and therefore index lowering might be computationally non-trivial. In physics, specifically electrodynamics and general relativity, we want nothing other than the Lorentz metric, which as we know is a special case of a semi-Riemannian metric. All this is analogous to classical differential geometry. Let $E$ be a sc-Hilbert space. Henceforth, we shall, without loss of generality, decompose the local spaces of $M$-polyfolds as

$$E = \mathbb{R} \oplus W,$$

where $W \subset E$ is a sc-Banach space. Locally, we define the Lorentz-metric on $E$, for $\eta_1, \eta_2 \in E$, by

$$g(\eta_1, \eta_2) := \langle \eta_1, (1, 0) \rangle \cdot \langle \eta_2, (1, 0) \rangle - \langle \eta_1 - \langle \eta_1, (1, 0) \rangle, \eta_2 - \langle \eta_2, (1, 0) \rangle \rangle.$$
That is, we take the inner product between the two vectors \( \eta_1, \eta_2 \) projected to the first factor, minus the inner product of the parts of \( \eta_1 \) and \( \eta_2 \) projected onto \( W \). This is our version of a generalised Minkowski space. Similar to Proposition 6.16 we can always induce a Lorentz-metric on M-polyfolds modelled on \( E \). This is the construction we shall use in later applications.

**Definition 6.18.** Let \((M, g)\) be a finite dimensional, reflexive semi-Riemannian M-polyfold. For each \( p \in M_1 \), we denote the *signature of \( g \) at \( p \)* as the set \((n, \dim(T_p M) - n)_p\), if for each chart \((U, \phi, (O, C, E))\) around \( p \in M_1 \), we can decompose the modelling space as \( E = \mathbb{R}^n \oplus W \), \( W \) a sc-Banach space, such that \( g \) is negative definite on \( \phi^{-1}(O \cap \mathbb{R}^n) \) and positive definite on \( \phi^{-1}(O \cap W) \).

**Remark 6.19.** Observe that we can not require that semi-Riemannian metrics should have constant signature as the dimension of \( M \) may change locally.

Naturally, a Lorentz metric on a M-polyfold, \( M \), is a semi-Riemannian metric with signature \((1, \dim(T_p M) - 1)_p\) for all \( p \in M_1 \). Notice that we choose the signature to follow the convention of \( [39, 45] \).

The last thing we need, is the Hodge-operator on M-polyfolds. Though, as it requires a volume form on the M-polyfold, we are interested in some notion of orientable M-polyfolds. It is not intuitive whether a direct generalisation of some of the definitions of orientability in classical geometry, really works. What is an volume form on a space that changes dimension locally, what is an orientable boundary of said space? One could also use a purely topological formulation using a two-fold covering space, see Proposition 3.25 in \([27]\), then a sufficient, but not necessary, condition for orientability is to require that the fundamental group for the covering space does not have a subgroup of index 2. The strong suit of this formulation is that it holds for all topological spaces, but as it is not a necessary condition we shall not use it here. For convenience, we shall go with the first version of the definition of orientability, that is using volume forms. Therefore, we need to define volume-forms, observe that this notion is completely new as it requires generalised sc-differential forms.

**Definition 6.20.** Let \( M \) be a finite-dimensional M-polyfold. A *volume form* on \( M \) is a nowhere vanishing sc\(^G\)-differential form on \( M \) that has the order map, \( \kappa : M_1 \to \mathbb{N} \), given as \( \kappa(p) = \dim(T_p M) \).

Obviously this definition is consistent with volume forms on classically differentiable manifolds. Also, as we know, some M-polyfolds may not admit such nowhere vanishing sc\(^G\)-differential forms. The classical example is the Möbius strip, which is a finite-dimensional differentiable manifold
and thus a M-polyfold. Therefore, we distinguish M-polyfolds that admit volume forms from M-polyfolds that do not.

**Definition 6.21.** Let $M$ be a finite-dimensional M-polyfold. We say that $M$ is **orientable** if there exists a volume form $\varpi \in \Omega^G(M)$.

We say that two volume forms, $\eta, \nu \in \Omega^G(M)$ on a M-polyfold $M$, are orientation equivalent if there exists a sc-smooth map $f : X^1 \to \mathbb{R}$, with $f(p) > 0$ for all $p \in M_1$, such that $\eta = f\nu$. We shall denote this equivalence class of volume forms by $[\varpi]$. An **orientation** on an orientable M-polyfold $M$, is defined as the equivalence class $[\varpi]$, with $\varpi \in \Omega^G(M)$ a volume form. But, we can also talk about an orientable atlas on a M-polyfold.

**Definition 6.22.** Let $M$ be a M-polyfold, and let $\mathcal{A}$ be an atlas with charts belonging to the M-polyfold structure of $M$. Then, $\mathcal{A}$ is called an **orientable atlas** if for each $p \in M_1$, and for every two charts $(U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)), (U_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta)) \in \mathcal{A}$ around $p$, it holds that

$$\det\left\{ D(\phi_\alpha \circ \phi_\beta^{-1})|_{U_\alpha \cap U_\beta}(\phi_\beta(p)) \right\} > 0.$$ 

Here, note that we see the differential as a linear mapping

$$D(\phi_\alpha \circ \phi_\beta^{-1})|_{U_\alpha \cap U_\beta}(\phi_\beta(p)) : T_{\phi_\beta(p)}O_\beta \to T_{\phi_\alpha(p)}O_\alpha.$$ 

We want these two definitions to be equivalent. Theorem 6.23 shows that this is indeed the case.

**Theorem 6.23.** Let $M$ be a finite-dimensional M-polyfold that admits sc-smooth partition of unity such that

$$\{ p \in M_1 : \dim(T_pM) = 1 \text{ and } p \in \partial M \} = \emptyset.$$ 

Then, $M$ is orientable if, and only if, $M$ admits an orientable atlas.

**Proof.** $\Rightarrow$: Suppose that $M$ is an orientable M-polyfold and that $\varpi \in \Omega^G(M)$ is a volume form. Let $\mathcal{A}$ be an atlas belonging to the M-polyfold structure of $M$. Fix a point $p \in M_1$. Choose a chart $(U, \phi, (O, C, E))$ around $p$. Set $k = \max_{q \in U_1}(\dim(T_qM))$, then choose a basis $\{e_i\}_{i=1}^k$ for a $k$-dimensional vector space $V \supseteq O$. Without loss of generality suppose that $\mathbb{R}^k = V$ and that $\{e_i\}_{i=1}^k$ is the standard basis for $\mathbb{R}^k$. Then, we denote $\phi^1, \ldots, \phi^k$ as each component of the chart map. Set $n = \dim(T_pM)$, then choose $n$ indices, $I_p \subseteq \{i \in \mathbb{N} : 1 \leq i \leq k\}$, for the elements $\phi^i$ so that $\wedge_{i \in I_p} d\phi^i$ is non-zero. Make this choice for all $p \in U_1$. Then, there is a sc-smooth function $f : U \to \mathbb{R}$ with $f \neq 0$ such that

$$\varpi = f d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_n}.$$
It follows that
\[ \varpi(\partial_I^{I_1}, \ldots, \partial_I^{I_n}) = f. \]
In the case that \( f > 0 \) we keep the chart as it is. In the case that \( f < 0 \) we change the chart by replacing \( \phi^1 \) by \(-\phi^1\). Note that in such a case we also change \( O \), through a reflection. Furthermore this step is only possible as long as we guarantee that
\[ \{ p \in M_1 : \dim(T_pM) = 1 \text{ and } p \in \partial M \} = \emptyset. \]
We do this for all charts in \( \mathcal{A} \), call this new atlas \( \mathcal{B} \). Suppose that \((U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)) \), \((U_\beta, \phi_\beta, (O_\beta, C_\beta, E_\beta)) \) \( \in \mathcal{B} \) and suppose that
\[ \varpi = f d\phi_\alpha^{I_1} \wedge \cdots \wedge d\phi_\alpha^{I_n} = g d\phi_\beta^{I_1} \wedge \cdots \wedge d\phi_\beta^{I_n} \]
for \( f, g : M \to \mathbb{R} \) sc-smooth. Then,
\[ 0 < \varpi(\partial_\alpha^{I_1}, \ldots, \partial_\alpha^{I_n}) = \det \left\{ D(\phi_\alpha \circ \phi_\beta^{-1})|_{U_\alpha \cap U_\beta} \right\} \varpi(\partial_\beta^{I_1}, \ldots, \partial_\beta^{I_n}) \]
\[ = \det \left\{ D(\phi_\alpha \circ \phi_\beta^{-1})|_{U_\alpha \cap U_\beta} \right\} f. \]
Since \( f > 0 \) this implies that \( \det \left\{ D(\phi_\alpha \circ \phi_\beta^{-1})|_{U_\alpha \cap U_\beta}(\phi_\beta(p)) \right\} > 0 \). Hence, we have constructed an oriented atlas \( \mathcal{B} \).
\[ \Leftarrow : \text{Assume } M \text{ admits an orientable atlas } \mathcal{A}. \] For each chart \((U_\alpha, \phi_\alpha, (O_\alpha, C_\alpha, E_\alpha)) \) \( \in \mathcal{A} \), we define a local expression for the volume form in a similar way as in first part of the proof
\[ \varpi_\alpha = d\phi_\alpha^{I_1} \wedge \cdots \wedge d\phi_\alpha^{I_n}. \]
Now, choose a sc-smooth partition of unity \( \{ \rho_\alpha \} \) subordinate to the chart domains in \( \mathcal{A} \). We directly get a volume form on \( M \) by setting
\[ \varpi = \sum_\alpha \rho_\alpha \varpi_\alpha. \]
Hence, \( M \) is orientable. \( \square \)

**Remark 6.24.** By the proof above, one can see that sc-smooth partition of unity is necessary for the proof to work. A proof where admittance of sc-smooth partition of unity is not assumed might still be possible.

Now that we have sufficiently established orientability, we are ready for the notion of the Hodge operator on generalised differential forms.

**Definition 6.25.** Let \((M, g)\) be an orientable, finite-dimensional, semi-Riemannian M-polyfold, and let \( \varpi \in \Omega^G(M) \) be a volume form generated by \( g \). Next, suppose that the sc\(^G\)-differential forms \( \alpha, \beta \in \Omega^G(M) \) has the
same order map \( \kappa : M_1 \to \mathbb{N} \) with \( \kappa(p) \leq \dim(T_pM) \) for all \( p \in M_1 \). For each \( p \in M_1 \), Define the fibre-wise Hodge-star operator

\[ *_p : \text{Alt}^{\kappa(p)}(T_pM, \mathbb{R}) \to \text{Alt}^{(\dim(T_pM) - \kappa(p))}(T_pM, \mathbb{R}) \]

by the linear map defined by

\[ \alpha(p) \wedge *_p \beta(p) = \frac{1}{\kappa(p)!} \alpha(p)(\sharp^{\kappa(p)} \beta(p)) \varpi(p). \]

Observe that in the last part we have interpreted \( \beta(p) \) as an element of \( \bigoplus_{\kappa(p)} T^*_pM \), and that \( \sharp^{\kappa(p)} \beta(p) \) is interpreted as sharpening each element of \( T^*_pM \) to \( T_pM \). Then, the Hodge operator is the map \( * : \Omega^G(M) \to \Omega^G(M) \) defined as \( *_p \) on each fibre over each \( p \in M_1 \).

**Remark 6.26.** Note the factor \( \frac{1}{\kappa(p)!} \). It is needed to ensure the smoothness (and even continuity) of the factor in front of the volume form. As an example, consider \( \mathbb{R}^4 \) together with standard Euclidean coordinates \((x, y, z, w)\) and the Euclidean metric \( \delta_{ij} \). Then, for \( dx \wedge dy \wedge dz \in \Omega^3(X) \)

\[
(dx \wedge dy \wedge dz) \wedge * (dx \wedge dy \wedge dz) = \frac{2}{6} \left( dx \left( \frac{\partial}{\partial x} \right) + dy \left( \frac{\partial}{\partial y} \right) + dz \left( \frac{\partial}{\partial z} \right) \right) \cdot dx \wedge dy \wedge dz \wedge dw = dx \wedge dy \wedge dz \wedge dw.
\]

This follows the convention used by Morita in [55, p. 150].

We need to show the fibre-wise uniqueness of this version of the Hodge operator, fibre-wise. Proposition 6.27 shows this uniqueness, as the proof is identical to that of classical differential geometry we shall leave it out, for the proof see e.g. [47, Theorem 9.22].

**Proposition 6.27.** Let \((M, g)\) be a finite-dimensional, orientable, semi-Riemannian \( M \)-polyfold, and let \( \varpi \in \Omega^G(M) \) be a volume form. The Hodge operator as defined in Definition 6.25 is unique.
CHAPTER 7

Constructions

In this chapter we shall construct some non-trivial examples of M-polyfolds. Inspired by Example 1.22 in [35], we shall construct a class of local models for M-polyfolds. Though, we shall do so in a more general context in order to be able to generate a much larger class of M-polyfolds on the get go. Then, we shall go into some explicit computations on these M-polyfolds to validate our previous work.

1. Construction of the Local Models

Let \( \{ \delta_m \}_{m \in \mathbb{N}_0} \) be a strictly increasing sequence with \( \delta_0 = 0 \). Define \( E = L^2(\mathbb{R}) \), it is a standard fact that \( L^2(\mathbb{R}) \) is a Hilbert space, see Chapter 3 and Example 4.5 in [61]. For \( m \in \mathbb{N}_0 \), we shall equip a scale structure on \( E \), by defining

\[
E_m := W^{m, \delta_m}(\mathbb{R})
\]

where

\[
W^{m, \delta_m}(\mathbb{R}) := \left\{ [f] \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left( |f(s)|^2 + |D^1 f(s)|^2 + \cdots + |D^m f(s)|^2 \right) e^{\delta_m \beta(s)} ds < \infty \right\}
\]

is the weighted Sobolev space, here \( \beta : \mathbb{R} \to [0, 1] \) is some smooth symmetric function such that \( \beta(s) \geq 0 \) for all \( s \geq 0 \) and \( \beta(s) = 1 \) for all \( s \geq 1 \). We need to show that this is indeed a sc-Banach space, at very least [26] tells us that they are Banach spaces. Furthermore, from Lemma 4.10 in [16], we have the following result.

**Lemma 7.1.** Let \( l \in \mathbb{N}_0 \) and \( \delta_0 \in \mathbb{R} \) be numbers. Then, the weighted Sobolev space \( W^{l, \delta_0}(\mathbb{R}) \) can be equipped with sc-structures

\[
E_k = W^{l+k, \delta_k}(\mathbb{R}),
\]

where \( k \in \mathbb{N}_0 \) for some sequence \( \{ \delta_k \}_{k=0}^{\infty}, \delta_k \in \mathbb{R} \), with \( \delta_k > \delta_j \) whenever \( k > j \).
7.1. Construction of the Local Models

**Proof.** Using Sobolev’s embedding theorem (see e.g. [2, 65, 66], and see Chapter 2.6 in [75]), the inclusions $E_k = W^{l+k, \delta_k}(\mathbb{R}) \hookrightarrow W^{l+j, \delta_j}(\mathbb{R}) = E_j$ for $k > j$ exist since $\epsilon^{\delta_k s\beta(s)} \geq \epsilon^{\delta_j s\beta(s)}$ for all $s \in \mathbb{R}$. It holds that for some $R \geq 1$ the restrictions

$$W^{l+k, \delta_k}(\mathbb{R}) \rightarrow W^{l+k, \delta_k}([-R, R])$$

are compactly embedded. This is the case as the Sobolev compact embedding theorem applies here, see [26]. Hence, the inclusion operator is compact.

The smooth points $u \in E_\infty$ consists of equivalence classes of smooth maps whose weak derivatives decay exponentially,

$$\sup_{s \in \mathbb{R}} \epsilon^{\delta_s \beta(s)} |D^N u(s)| < \infty$$

for all $N \in \mathbb{N}_0$ and for every submaximal weight $\delta < \sup_{k \in \mathbb{N}_0} \delta_k$. In the case that the sequence $\{\delta_k\}$ is unbounded, this means that the maps decay faster than any linear exponential function. In particular, the compactly supported smooth functions are a subset $C_0^\infty(\mathbb{R}) \subseteq E_\infty$, and since $C_0^\infty$ is dense in $L^2$, we can conclude in particular that $E_\infty$ is dense in $E_0$. □

From Lemma 7.1 it holds that our constructed sequence of Banach spaces $\{E_m\}$ is indeed a sc-structure. Following Example 1.22 in [35] and Example 5.8 in [16] we shall construct sc-smooth retracts on this sc-Banach space. We shall begin by choosing $m$ compactly supported, mutually orthogonal functions $\{\gamma_i : \mathbb{R} \rightarrow [0, \infty)\}_{i=1}^m$ such that,

$$\int_\mathbb{R} \gamma_i(s)^2 ds = 1,$$

for all $m \leq i \geq 1$. We define a family of sc-operators $\pi_i^m : E \rightarrow E$ in the following way. For $t \leq 0$, set $\pi_t^m = 0$, and for $t > 0$ define

$$\pi_t^m([f]) = \langle f, \gamma_1(\cdot + e^{\frac{t}{m}}) \rangle_{L^2} \cdot \gamma_1(\cdot + e^{\frac{1}{m}}) + \cdots + \langle f, \gamma_m(\cdot + e^{\frac{1}{m}}) \rangle_{L^2} \cdot \gamma_m(\cdot + e^{\frac{1}{m}}),$$

where $[f] \in E$. For $t \geq 0$, this is a $L^2$-orthogonal projection from $L^2(\mathbb{R})$ onto the $m$-dimensional subspace that is spanned by the equivalence classes of the functions $\gamma_m$ with the arguments shifted by $e^{\frac{1}{m}}$. Furthermore, we consider the mapping $r : \mathbb{R}^n \oplus \mathbb{R} \oplus E \rightarrow \mathbb{R}^n \oplus \mathbb{R} \oplus E$, defined as

$$r(q, t, [f]) = (q, t, \pi_t^m([f])),$$

where $q \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $[f] \in E$. Since the $L_2$-norm of $\gamma_m(s)$ is 1 and because the maps $f_i$ are mutually orthogonal, it holds that

$$r = r \circ r.$$ 

Now we claim that this map is sc-smooth, and hence that $r$ is a sc-smooth retraction. We do this inspired by Lemma 1.23 in [35].
Lemma 7.2. The map $\Phi : \mathbb{R} \oplus E \to E$ defined by

$$
\Phi(t, [f]) = \begin{cases} 
(f, \gamma(\cdot + e^{\frac{t}{2}}))_{L^2} \cdot \gamma(\cdot + e^{\frac{t}{2}}) & \text{if } t > 0, \\
0 & \text{if } t \leq 0,
\end{cases}
$$

is sc-smooth.

**Proof.** It is clear that the restricted map $\Phi|_{\mathbb{R} \backslash \{0\}} \oplus E_m \to E_m$ is sc-smooth. For ease of notation, denote $F(t,s) = \gamma(s + e^{\frac{t}{2}})$, where $s \in \mathbb{R}$ and $t > 0$. The $k$-th order derivative of $F(t,s)$ with respect to $t$ can be estimated as

$$
|F^{(k)}(ts)| \leq p(t),
$$

where $p(t) = P(e^{\frac{t}{2}}, \frac{1}{t})$ is a polynomial of degree $k$ in $e^{\frac{1}{2}}$ and of degree $2k$ in $\frac{1}{t}$. Observe that this polynomial is dependent on the choice of $\gamma$ and that it has nonnegative coefficients. Furthermore, suppose that $[-A, A]$, with $A > 0$, is the compact support for $\gamma$, then the map $s \mapsto F(s,t)$ has compact support $I_t := [-A - e^{\frac{t}{2}}, A - e^{\frac{t}{2}}]$, for each fixed $t > 0$. In particular, observe that there exists $t > 0$ small enough such that $I_t \subseteq (-\infty, -1)$.

For $[u] \in E_m$ and $t > 0$, we have the following estimate,

$$
\langle u, F(t, \cdot) \rangle^2_{L^2} = \int_{\mathbb{R}} |u(s)|^2 F(t, s)^2 ds
= \int_{I_t} |u(s)|^2 F(t, s)^2 e^{\delta_m s} e^{-\delta_m s} ds
\leq C e^{-\delta_m e^{\frac{t}{2}}} \int_{I_t} |u(s)| F(t, s)^2 e^{-\delta_m s} ds
$$

(7.1)

where the inequality

$$
\max_{s \in I_t} |F(s, t)| e^{\delta_m s} \leq C e^{-\delta_m e^{\frac{t}{2}}}
$$

has been used. In a similar fashion we estimate

$$
\langle u, F^{(j)}(t, \cdot) \rangle^2_{L^2} = \int_{\mathbb{R}} |u(s)|^2 F^{(j)}(t, s)^2 ds
= \int_{I_t} |u(s)|^2 F^{(j)}(t, s)^2 e^{\delta_m s} e^{-\delta_m s} ds
\leq C p(t)^2 e^{-\delta_m e^{\frac{t}{2}}} \int_{I_t} |u(s)| F(t, s)^2 e^{-\delta_m s} ds.
$$

(7.2)

From the first estimate (7.1) we shall show that $\Phi$ is sc$^0$. We shall do this by proving continuity of each restricted map, $\Phi|_{\mathbb{R} \oplus E_m}$, at all points $(0, u_0)$. Let $\epsilon > 0$ be a number, choose $h \in E_m$ such that $\|h\|_m = 0$ and set
7.1. Construction of the Local Models

$u := u_0 + h$. Furthermore choose $t$ small enough such that $e^{\frac{t}{4}} > A + 1$. By construction, it holds that $\Phi(0, u_0) = 0$, therefore we have

$$||\Phi(t, u) - \Phi(0, u_0)||_m^2 = \langle u, F(t, \cdot) \rangle_{L^2} \sum_{i=0}^{m} \int_{I_t} |D^i_s F(t, s)|^2 e^{-\delta_m s} ds$$

$$\leq Ce^{-\delta_m e^t} \left( \int_{I_t} |u(s)|^2 F(t, s)^2 e^{-2\delta_m s} ds \right) \left( \int e^{-\delta_m s} ds \right)$$

$$\leq C \int_{I_t} |u(s)|^2 e^{-\delta_m s} ds$$

$$\leq C \left( \int_{I_t} |u_0(s)|^2 e^{-\delta_m s} ds + \int |h|^2 e^{-\delta_m s} ds \right)$$

$$\leq C \int_{I_t} |u_0(s)|^2 e^{-\delta_m s} ds + Ce.$$ 

Since $\int_{I_t} |u_0(s)|^2 e^{-\delta_m s} ds \to 0$ as $t \to 0^+$, we can conclude that $\Phi(t, u) \to (0, u_0)$ as $(t, u) \to (0, u_0)$ in $\mathbb{R} \oplus E_m$. Hence, it holds that $\Phi : \mathbb{R} \oplus E \to E$ is $sc^0$. Next, we shall prove that $\Phi$ is $sc^0$-smooth, we shall do so by induction. The inductive statement reads as follows.

$(S_k)$. The map $\Phi : \mathbb{R} \oplus E \to E$ is $sc^k$ and the map $T^k \Phi(t_1, u_1, \ldots, t_{2^k}, u_{2^k}) = 0$, if $t_1 \leq 0$.

Furthermore, for every $m \geq 0$, let $\pi : T^k E \to E_m$ be the projection of $T^k E$ onto $E_m$. Then, the mapping $\pi \circ T^k \Phi$ at the point $(t_1, u_2, \ldots, t_{2^k}, u_{2^k})$ with $t_1 > 0$, is a linear combination of functions, $\Gamma : \mathbb{R}^{k+1} \oplus E_m \to E_m$, defined as

$$(t_1, t_2, \ldots, t_{k+1}, u) \mapsto \langle u, F^{(i)}(t_1, \cdot) \rangle_{L^2} F^{(j)}(t_1, \cdot) t_2 \ldots t_{i+j},$$

where $n \geq m$ and $i + j \leq k$.

First, we shall prove that $(S_1)$ holds true. Start by considering the map $D\Phi(t_1, u_1) : \mathbb{R} \oplus E_m \to E_m$ at the point $(t_1, u_1) \in \mathbb{R} \oplus E_m$ defined as

$$D\Phi(t_1, u_1) = 0$$

if $t_1 \leq 0$, and whenever $t_1 > 0$

$$D\Phi(t_1, u_1)(t_2, u_2) = \langle u_2, F(t_1, \cdot) \rangle_{L^2} F(t_1, \cdot) + \langle u_1, F(t_1, \cdot) \rangle_{L^2} F(t_1, \cdot) t_2$$

$$\quad + \langle u_1, F(t_1, \cdot) \rangle_{L^2} F(t_1, \cdot) t_2.$$ (7.3)

It is clear that $D\Phi(t_1, u_1) : \mathbb{R} \oplus E_m \to E_m$ is a bounded linear map. Moreover, when $t_1 \neq 0$, the above map is the derivative of $\Phi : \mathbb{R} \oplus E_{m+1} \to E_m$ at the point $(t_1, u_1)$. We shall show that the derivative at the point $(0, u_1)$ is the zero map. In order to show this, we need to estimate $||\Phi(t, u)||_m$ for $u \in E_n$ and $n > m$. Let $t > 0$ be small enough such that $e^{\frac{t}{4}} > A + 1$. From
(7.1) we have that
\[ \|\Phi(t, u)\|_m^2 = \langle u, F(t, u) \rangle_{L^2} \sum_{i \leq m} \int_{I_t} |D^i_s F(t, s)|^2 e^{-\delta_m s} ds \]
\[ \leq C e^{\delta_n e^t} \left( \int_{I_t} |u|^2 e^{\delta_n s} ds \right) \left( \int_{I_t} e^{\delta_m s} ds \right) \]
\[ \leq C e^{-(\delta_n - \delta_m)e^t} \int_{I_t} |u|^2 \]
\[ \leq C e^{-(\delta_n - \delta_m)e^t} \|u\|_n^2. \]

In the case that \( n = m + 1 \), since \( \delta_{m+1} > \delta_m \), we can conclude that
\[ \frac{1}{|\delta t| + \|\delta u_1\|_{m+1}} \|\Phi(\delta t, u_1 + \delta u_1)\|_m \leq C \frac{e^{-(\delta_{m+1} - \delta_m)e^t}}{|\delta t|} \|u_1 + \delta u_1\|_{m+1} \to 0 \]
as \( |\delta t| + \|\delta u_1\| \to 0 \). Hence, \( D\Phi(0, u) = 0 \). Furthermore, from (7.2) as \( t_1 \to 0 \) it holds that
\[ D\Phi(t_1, u_1)(t_2, u_2) \to 0 \]
in \( E_m \) where \( (t_1, u_1) \in \mathbb{R} \oplus E_{m+1} \), and \( (t_2, u_2) \in \mathbb{R} \oplus E_m \). Therefore, the tangent map
\[ \Gamma : \mathbb{R} \to TE \]
is \( sc^0 \), and hence the map \( \Phi : \mathbb{R} \oplus E \to E \) is \( sc^1 \). In view of (7.3), for \( t_1 > 0 \), it holds that the map \( \pi \circ T\Phi \) is of the form as stated in \( S_1 \). It has been shown that \( S_1 \) is true.

Next, assume that the statement \( S_k \) holds true, we shall show that \( S_{k+1} \) is true. It is enough to consider the map \( \Gamma : \mathbb{R}^{k+1} \oplus E_n \to E_m \) defined by
\[ \Gamma(t_1, \ldots, t_{i+j}, u) = 0 \]
for \( t_1 \leq 0 \), and
\[ \Gamma(t_1, \ldots, t_{i+j}, u) = \langle u, F^{(i)}(t_1, \cdot) \rangle_{L^2} F^{(j)}(t_1, \cdot) t_2 \cdots t_{i+j} \]
whenever \( t_1 > 0 \), where \( n \geq m \) and \( i+j \leq k \). We shall show that \( \Gamma \) is of class \( sc^1 \). This map is obviously \( sc \)-smooth when \( t_1 \neq 0 \). Let \( (t_1, \ldots, t_{k+1}, u) \in \mathbb{R}^{k+1} \oplus E_{n+1} \). When \( t_1 < 0 \) it holds that \( D\Gamma(t_1, \ldots, t_{k+1}, u) \), we shall show that this also holds for \( t_1 = 0 \). Consider the expression
\[ \frac{1}{|\delta t_1| + \cdots + |\delta t_{i+j}| + \|u\|_{n+1}} \|\Gamma(\delta t_1, t_2 + \delta t_2, \ldots, t_{i+j} + \delta t_{i+j}, u + \delta u)\|_m. \]
If $\delta t_1 \leq 0$, then $\Gamma$ is zero and for $\delta_1 > 0$, using a very similar argument as before, one has the following estimate
\[
\left\| \Gamma(t_1, t_2, \ldots, t_{i+j}, u + \delta u) \right\|_{m} \leq C \frac{e^{-(\delta_{n+1}-\delta_m)e^{1}}}{|\delta t_1|} \left\| u + \delta u \right\|_{n+1}
\]
goesto zero as $\delta t_1 \to 0^+$. Thus, it holds that $D\Gamma(0, t_2, \ldots, t_{i+j}, u) = 0$. If $t_1 > 0$, then the map
\[
D\Gamma(t_1, t_2, \ldots, t_{i+j}, u) : \mathbb{R}^{k+1} \oplus E_n \to E_m
\]
is equal to
\[
D\Gamma(t_1, \ldots, t_{i+j}, u)(\delta t_1, \ldots, \delta t_{i+j}, \delta u) = \langle \delta u, F(i)(t_1, \cdot) \rangle_{L^2} F(j)(t_1, \cdot) t_2 \cdots t_{i+j}
\]
\[+ \langle u, F^{(i+1)}(t_1, \cdot) \rangle_{L^2} F(j)(t_1, \cdot) \delta t_1 \cdot t_2 \cdots t_{i+j}
\]
\[+ \langle u, F(i)(t_1, \cdot) \rangle_{L^2} F^{(j+1)}(t_1, \cdot) \delta t_1 \cdot t_2 \cdots t_{i+j}
\]
\[+ \sum_{l=2}^{i+j} \langle u, F^{(j)}(t_1, \cdot) \rangle_{L^2} F^{(j)}(t_1, \cdot) t_2 \cdots \delta t_l \cdot t_{i+j}.
\]
It is easy to check that the rest of the statements of $S_{k+1}$ are satisfied. Hence, we can conclude that the map $\Phi$ is $sc^k$ for all $k \in \mathbb{N}$, and thus $\Phi$ is $sc^{\infty}$. $\square$

This choice of local model for our constructions is convenient also as the weighted Sobolev spaces admit smooth bump functions, following Theorem 1 in [23] (see also [5]). Hence by Theorem 4.27 and by Proposition 4.28, the M-polyfolds modelled on these $sc$-smooth retracts admits $sc$-smooth bump functions. Consequently, in view of Theorem 4.32 and Theorem 4.32 all such M-polyfolds admits $sc$-smooth partition of unity.

Now to the geometry of the M-polyfolds. The geometric interpretation of the $sc$-smooth retraction $r$, is that $r(\mathbb{R}^n \oplus \mathbb{R} \oplus E)$ is the set
\[
\left\{ (x_1, \ldots, x_n, t, 0, \ldots, 0) : t \leq 0, x_1, \ldots, x_n \in \mathbb{R} \right\}
\]
\[\cup \left\{ (x_1, \ldots, x_n, s, s_1 t \gamma_1 \left( \cdot + e^{1/2} \right), \ldots, s_m t \gamma_m \left( \cdot + e^{1/2} \right)) : t > 0,
\]
\[x_1, \ldots, x_n, s_1, \ldots, s_m \in \mathbb{R} \right\}
\]
which is homeomorphic to the set
\[
\mathbb{R}^n \times \left[ \left( \{0\} \cup \mathbb{R}^- \times \prod_{i=1}^{m} \{0\} \right) \cup \left( \mathbb{R}^+ \times \mathbb{R}^n \right) \right].
\]
We shall denote $r(\mathbb{R}^n \oplus \mathbb{R} \oplus E)$ as $\Gamma^m_\alpha$ and we interpret this geometrically as the open half plane in $\mathbb{R}^{n+m+1}$ stitched together with a $n + 1$-dimensional
subspace in $\mathbb{R}^{n+m+1}$. In the special case that $n = 0$ and $m = 1$ we have the open half plane together with a horizontal line, as seen in Figure 7.1.

![Figure 7.1](image)

**Figure 7.1** – A sketch of the set $\mathbb{R}^n \times \left[\left(\{0\} \cup \mathbb{R}^-\right) \times \prod_{i=1}^{m} \{0\}\right] \cup \left(\mathbb{R}^+ \times \mathbb{R}^m\right)$ when $n = 0$ and $m = 1$.

All these geometric objects generate a class of local models for finite dimensional M-polyfolds. They allow for the construction of geometric objects that radically change dimension.

### 2. Examples of M-polyfolds

We are now able to construct the following geometrical object, and imbue it with a M-polyfold structure.

![Figure 7.2](image)

**Figure 7.2** – A finite-dimensional M-polyfold embedded in $\mathbb{R}^3$ that is locally 3-dimensional, the ball to the left, locally 1-dimensional, the black curves, and locally 2-dimensional, the two disks to the right one with normal in the $z$-direction and one with the normal in the $x$-direction.
We shall consider the M-polyfold $M$ as the geometrical object seen in Figure 7.2, together with an obvious atlas modelled on $(\Gamma^n_m, E, E)$ as sc-smooth retracts. Using set-theoretic notation to precisely describe the set depicted in Figure 7.2 would be very tedious and not really interesting. Therefore, to simplify our discussion we shall instead refer to the ball as $M_B$. We shall further denote the disk with the normal in the $z$-direction as $M_{D_1}$, and the disk with the normal in the $x$-direction as $M_{D_2}$. Finally, we denote the smooth curve connecting $M_B$ and $M_{D_1}$ as $\gamma_1$, the smooth curve connecting $M_B$ and $M_{D_2}$ as $\gamma_2$, and the smooth curve connecting $M_{D_1}$ and $M_{D_2}$ as $\gamma_3$.

As disjoint sets the dimension of each part is obvious. Though, as $M$ does not constitute of an disjoint union of these sets, we can not arrive to an answer that quickly. What is the dimension of the tangent space at the point where $\gamma_1$ is attached to $M_{D_1}$? In contrast to classical differential geometry, this point is not viewed as a boundary point, it is in fact an interior point! This can be argued from Definition 4.20, and since we have modelled $M$ on sc-smooth retracts on the total space $E$, $M$ has no boundary points at all. From the proof of Lemma 7.2 we can conclude that the tangent space is of dimension 1 at the point where the curve attaches to the ball.

Next to ask is whether $M$ is orientable, and if so what is a good choice for the volume form? First, we shall define the volume form locally on $\Gamma^n_m$. Let $(x_1, \ldots, x_{n+m+1})$ denote the standard basis of $\mathbb{R}^{n+m+1}$ and we shall view $\Gamma^n_m$ as a subset of $\mathbb{R}^{n+m+1}$, without loss of generality. Define the volume form on $\Gamma^n_m$ by

$$\varpi = \begin{cases} dx_1 \wedge \cdots \wedge dx_{n+1} & \text{if } x_{n+1} \leq 0, \\ dx_1 \wedge \cdots \wedge dx_{n+1} \wedge \cdots \wedge dx_{n+m+1} & \text{if } x_{n+1} > 0. \end{cases}$$

Furthermore, we need the map

$$\varpi : \bigcup_{p \in \Gamma^n_m} \left( \bigoplus_{\dim(T_p \Gamma^n_m)} T_p \Gamma^n_m \right) \rightarrow \mathbb{R}$$

to be sc-smooth. The first step is to verify that the domain of definition for $\varpi$ is also of the form $\Gamma^N_M$. First, observe that since $\Gamma^n_m$ is the finite-dimensional image of a sc-smooth retract, it inherits the trivial sc-structure. Hence, we can sloppily write down the domain of definition in that way, pedantically we want the disjoint union to be over points of level one. Fix a $p \in \Gamma^n_m$ and denote $N = \dim(T_p \Gamma^n_m)$, we then have that

$$\bigoplus_{\dim(T_p \Gamma^n_m)} T_p \Gamma^n_m \cong \mathbb{R}^N.$$
Therefore, we can identify the domain of definition for $\varpi$ as

$$\Gamma_{m^m}^{((n+1)(n+3)-1)}.$$  

For example the volume form on $\Gamma_0^4$, depicted in Figure 7.1, has $\Gamma_0^4$ as domain of definition. But then, the domain of definition for the volume forms on our local models are also sc-smooth retracts.

We have that $\varpi$ is at least classically smooth on all points except at 0. Following Definition 3.27, the volume form $\varpi$ is indeed sc-smooth everywhere. This is the case as with Proposition 3.29 $iii$) in mind, it holds that $\varpi$ is sc-smooth at 0. Moreover, a similar argument tells us that the exterior derivative on sc$^G$-differential forms that have M-polyfolds modelled using $\Gamma_m^n$ as domain of definition is well-defined.

Indeed $M$ is oriented. To see this, we shall prove orientability for a simpler M-polyfold. Then, the discussion for the M-polyfold $M$ shall follow analogously. By $N$ we denote the set

$$N = B((0,0),1) \cup \{(x,y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1 \}$$

seen in Figure 7.3.

![Figure 7.3](image_url)

**Figure 7.3** – The set $N = B((0,0),1) \cup \{(x,y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1 \}$ as a subset of $\mathbb{R}^2$.

We shall construct charts on $N$ in the following way. As local models, the sc-smooth retracts $\Gamma_0^4$ shall be used. Observe that this sc-smooth retract is sc-diffeomorphic to the subspace seen in Figure 7.1. Define the subsets

$$V_1 = B((0,0),1) \cup \{(x,y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1, x < 1, y > 0 \},$$

$$V_2 = B((0,0),1) \cup \{(x,y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1, x < 1, y < 0 \},$$
and

\[ V_3 = \left\{ (x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1, x > \frac{1}{2} \right\}. \]

It is easy to see that \( N = \bigcup_{i=1}^{3} V_i \). These shall be our chart domains. Now to the corresponding chart maps. Define the map \( \phi_1 : V_1 \to \Gamma_0^1 \) by

\[
\phi_1(x, y) := \begin{cases} 
\frac{(1-x^2-y^2, \sqrt{3}x-y)}{(x^2+y^2+1)^{1/2}} & \text{if } (x, y) \in B((0,0), 1) \cap V_1, \\
\frac{(1-x^2-y^2, 0)}{(x^2+y^2+1)^{1/2}} & \text{if } (x, y) \in V_1 \setminus B((0,0), 1).
\end{cases}
\]

In a similar fashion, we have the map \( \phi_2 : V_2 \to \Gamma_0^1 \) defined as

\[
\phi_2(x, y) := \begin{cases} 
\frac{(1-x^2-y^2, y+\sqrt{3}x)}{(x^2+y^2+1)^{1/2}} & \text{if } (x, y) \in B((0,0), 1) \cap V_2, \\
\frac{(1-x^2-y^2, 0)}{(x^2+y^2+1)^{1/2}} & \text{if } (x, y) \in V_2 \setminus B((0,0), 1).
\end{cases}
\]

Lastly, we define the map \( \phi_3 : V_3 \to \mathbb{R} \) by setting

\[ \phi_3(x, y) := \frac{2y}{x^2 + y^2}. \]

The injectivity of these maps follows directly from observing that they are modified Möbius transformations, that is Möbius transformations composed with rotations, reflections and the natural homeomorphism from \( \mathbb{C} \) to \( \mathbb{R}^2 \). We shall obtain surjectivity by restricting the codomains to the image of the maps. Observe that on \( \phi_1(B((0,0), 1)) \), the inverse map

\[ \phi_1^{-1}|_{\phi_1(B((0,0), 1))} : \phi_1(B((0,0), 1)) \to B((0,0), 1) \]

is

\[ \phi_1^{-1}|_{\phi_1(B((0,0), 1))}(x, y) = \frac{1}{y^2 + (x+1)^2} \cdot \left( -\sqrt{3}y - \frac{1}{2}(x^2 + y^2 - 1), y - \frac{\sqrt{3}}{2}(x^2 + y^2 - 1) \right). \]

Furthermore, observe that \( V_1 \cap V_2 = B((0,0), 1) \). It holds that the transition map \( \phi_2 \circ \phi_1^{-1}|_{\phi_1(B((0,0), 1))} \) is smooth and thus sc-smooth. In a similar fashion, we have the transition maps

\[
(\phi_3 \circ \phi_1^{-1}|_{\phi_1(V_1 \cap V_3)} : x \mapsto \frac{\sqrt{3}(x+1)^2}{1-x^2}
\]

and

\[
(\phi_3 \circ \phi_2^{-1}|_{\phi_2(V_2 \cap V_3)} : x \mapsto \frac{\sqrt{3}(x+1)^2}{x^2 - 1}.
\]

One can check that the determinant of the Jacobian of all these transition maps is positive on all the restricted domains. Therefore, the atlas we have chosen here is actually orientable. Furthermore, since \( N \) is modelled on sc-Hilbert spaces it admits a sc-smooth partition of unity according to
Theorem 4.32 and Theorem 4.31, and because $\partial N = \emptyset$, we can apply Theorem 6.23. Hence, $N$ is orientable in the sense of $sc^G$-differential forms. A similar, but tedious, construction of orientable charts for $M$ is possible and one can argue implicitly that $M$ is also orientable.
CHAPTER 8

Electrodynamics on M-polyfolds

We shall first discuss electrodynamics on the Minkowski space, on which most classical electrodynamics are computed in practice. We denote \( \mathbb{R}^4 \) as Minkowski space and we denote the natural coordinates as \((t, x, y, z)\). That is, Minkowski space is the spacetime where time is one-dimensional and space is three-dimensional together with the Lorentz-metric. As per usual let \( \mathbf{E}, \mathbf{B} \in \mathfrak{X}(\mathbb{R}^3) \) stand for the Electric field and the magnetic field on the spatial part of \( \mathbb{R}^4 \). Now we shall make the following convenient index lowering of the vector fields

\[
\mathcal{E} = b(\mathbf{E}) \quad \text{and} \quad \mathcal{B} = \ast(b(\mathbf{B})).
\]

Next we embed \( \mathcal{E} \) and \( \mathcal{B} \) as differential forms on \( \Omega(\mathbb{R}^3) \) to differential forms on \( \Omega(\mathbb{R}_1^4) \). Then, one can define the electromagnetic field tensor as

\[
\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt. \quad (8.1)
\]

Consequently, the Maxwell’s equations of the electromagnetic field tensor is defined as

\[
d\mathcal{F} = 0 \quad \text{and} \quad \ast d \ast \mathcal{F} = \mathcal{J}, \quad (8.2)
\]

where \( \mathcal{J} = b(J) \) and \( J \) is the 4-current.

Next, we shall translate this to M-polyfolds, see Definition 4.4. We shall suppose that the finite-dimensional Lorentz M-polyfold \((M, g)\) is orientable, see Definition 6.21. We shall further assume that the local spaces \((O, C, E)\), Definition 3.24, can be decomposed as \((U \oplus P, \mathbb{R} \oplus D, \mathbb{R} \oplus F)\), \( U \subseteq \mathbb{R} \) open. Therefore, \( g \) is locally positive definite restricted to the first factor of the sc-smooth retract. A construction of the electromagnetic field tensor as done on Minkowski space would not only be mathematically impossible, but also not physical since on the global space, we have not assumed a reference frame. Instead, we shall construct the electromagnetic field tensor using the electromagnetic potential. Let \( A \in \mathfrak{X}(M) \) and then set \( \mathcal{A} = b(A) \). As most physicists well-versed in relativistic electrodynamics know, the next step is to define the electromagnetic field tensor as

\[
\mathcal{F} = d\mathcal{A}.
\]

Since \( d^2 = 0 \), the first equation in Maxwell’s equations (8.2) is satisfied automatically. From all our previous work, we can define Maxwell’s equations
on M-polyfolds as
\[ d\mathcal{F} = 0 \quad \text{and} \quad *d*\mathcal{F} = \mathcal{J}, \]
where \( \mathcal{J} = b(J) \) and where \( J \in \mathfrak{X}(M) \) is the contravariant current (on Minkowski space also referred to as 4-current). All this is made possible by \( \text{sc}^G \)-differential forms, see Definition 6.1, which allows for the Hodge operator \( * \) see Definition 6.25.

We shall conclude the essay by speculating on a plausible real world physical application of this theory. Suppose that we attach a ball together with two thin disks using a thin wire to couple them. Also, suppose that its configuration corresponds to the configuration seen in Figure 7.2. Furthermore, assume that this object is lowered into a highly conductive medium, i.e. so that we can approximate all the electromagnetic fields to be zero in the medium. Now, we should be able to model this object as a M-polyfold, and use the Maxwell’s equations to compute various electrodynamic quantities. Take care observing that an endeavour using classical electromagnetism would straight out fail trying to describe this accurately.
CHAPTER 9

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CHAPTER 10

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