Periodic Structures with Higher Symmetries: Analysis and Applications

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To my dear Hadi
Abstract

In this thesis, periodic structures with higher symmetries are studied. Their wave propagation characteristics are investigated and their potential applications are discussed.

Higher-symmetric periodic structures are described with an additional geometrical operation beyond a translation operator. Two particular types of higher symmetry are glide and twist symmetries. Glide-symmetric periodic structures remain invariant under a translation of half a period followed by a reflection with respect to a glide plane. Twist-symmetric periodic structures remain invariant under a translation along followed by a rotation around a twist axis.

In a periodic structure with a higher symmetry, in which the higher order modes are excited, the frequency dispersion of the first mode is dramatically reduced. This feature overcomes the bandwidth limitations of conventional periodic structures. Therefore, higher-symmetric periodic structures can be employed for designing wideband metasurface-based antennas. For example, holey glide-symmetric metallic structures can be used to design low loss, wideband flat Luneburg lens antennas at millimeter waves, which find application in 5G communication systems. In addition, holey glide-symmetric structures can be exploited as low cost electromagnetic band gap (EBG) structures at millimeter waves, due to a wider stop-band achievable compared to non-glide-symmetric surfaces.

However, these attractive dispersive features can be obtained if holey surfaces are strongly coupled, so higher-order modes produce a considerable coupling between glide-symmetric holes. Hence, these structures cannot be analyzed using common homogenization methods based on the transverse resonance method. Thus, in this thesis, a mode matching formulation, taking the generalized Floquet theorem into account, is applied to analyze glide-symmetric holey periodic structures with arbitrary shape of the hole. Applying the generalized Floquet theorem, the computational domain is reduced to half of the unit cell. The method is faster and more efficient than the commercial software such as CST Microwave Studio. In addition, the proposed method provides a physical insight about the symmetry of Floquet modes propagating in these structures.

Moreover, in this thesis, the effect of twist symmetry and polar glide symmetry applied to a coaxial line loaded with holes is explained. A rigorous definition of polar glide symmetry, which is equivalent to glide symmetry in a cylindrical coordinate, is presented. It is demonstrated that the twist and polar glide symmetries provide an additional degree of freedom to engineer the dispersion characteristics of periodic structures. In addition, it is demonstrated that the combination of these two symmetries provides the possibility of designing reconfigurable filters. Finally, mimicking the twist symmetry effect in a flat structure possessing glide symmetry is investigated. The results demonstrate that the dispersion properties associated with twist symmetry can be mimicked in flat structures.
Sammanfattning

Denna avhandling behandlar periodiska strukturer med högre symmetrier. Deras vågutbredningsegenskaper undersöks och deras potentiella tillämpningar diskuteras.


I en högsymmetrisk periodisk struktur, innherillande flera högre ordningens moder, fås en dramatisk minskning av frekvensdispersionen för den första moden, varigenom den bandbreedsbegränsning som finns i konventionella periodiska strukturer kan övervinnas. Därigenom kan högsymmetriska strukturer användas vid utformandet av bredbandiga antenner baserade på metayer. Till exempel kan urkärnade glidsymmetriska metallstrukturer användas för att utforma bredbandiga Luneburg-linser med låga förluster, vilka för millimetravågor har tillämpningar inom femte generationens kommunikationssystem (5G). Dessutom kan dessa strukturer utnyttjas som kostnadseffektiva elektromagnetiska bandgap (EBG)-strukturer för millimetravågor.

På grund av förekomsten av högre ordningens moder kan emellertid inte högsymmetriska periodiska strukturer med starkt kopplade skikt analyseras med användning av den konventionella transversella resonansmetoden. Därför används i denna avhandling en modanpassningsmetod, vars formulering bygger på den generaliserade versionen av Floquets teorem, för att analysera glidsymmetriska urkärnade periodiska strukturer där hulen har godtyckligt tvärsnitt. Användningen av Floquets generaliserade teorem halverar storleken på beräkningsdomänen och metoden är både snabbare och effektivare än kommersiella programvaror som CST Microwave Studio. Dessutom bidrar den föreslagna metoden till fysikalisk förståelse genom symmetriegenskaperna hos de Floquet-moder som utbryter sig i de högsymmetriska strukturerna.

Preface

This thesis is in partial fulfillment for the Doctor of Philosophy degree at KTH Royal Institute of Technology, Stockholm, Sweden. The work presented in this thesis was performed at the Electromagnetic Engineering Department of the Electrical Engineering and Computer Science School of KTH in the second half of my PhD program (from September 2016 till December 2018). The work performed in the first half of my PhD program is presented in my Licentiate thesis entitled "Remote contact-free reconstruction of currents in two-dimensional parallel conductors" and published in 2016.

Professor Martin Norgren and Associate Professor Oscar Quevedo-Teruel from KTH and Associate Professor Guido Valerio from Sorbonne University have been supervised the work presented in this thesis. Parts of this thesis have been performed during my three months visit at the Laboratory of Electronics and Electromagnetism of Sorbonne University, supported by “Ericson E.C fond” foundation.
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Stockholm, November 2018
List of publications

This thesis is based on the following journal papers:


and the following conference paper:


Other journal papers related to but not included in this thesis:


**Parts of this thesis have been presented in the following peer-reviewed conference papers or workshops:**


**Other publications by the author (not related to the thesis):**


The author’s contribution to the journal papers included in this thesis:

Paper 1: O.Q.T. suggested the overall topic. I developed the mode matching formulation to analyze different type of doubled corrugated surfaces including glide-symmetric ones. I performed the coding and simulations, and prepared the figures and the manuscript. M.N. and O.Q.T. supervised the work. All authors reviewed and edited the manuscript.
Paper 2: O.Q.T. suggested the overall topic. Z.S. developed the initial concept. I and G.V. developed the mode matching formulation using Generalized Floquet theorem to analyze, respectively, 1D and 2D glide-symmetric holey surfaces. I performed the coding and simulations, and prepared the figures and the manuscript for the 1D part. O.Q.T. and Z.S. supervised the work. All authors reviewed and edited the manuscript.

Paper 3: O.Q.T. suggested the overall topic. I developed the mode matching formulation using a generalized Floquet theorem to analyze 2D glide-symmetric holey surfaces with circular holes. I performed the coding and simulations, and prepared the figures and the manuscript. G.V., M.N., and O.Q.T. supervised the work. All authors reviewed and edited the manuscript.

Paper 4: O.Q.T. and I developed the concept. I performed all the simulations and experiments and prepared the figures. I and O.Q.T. contributed equally to the main manuscript. O.Q.T. and M.N. supervised the work. All authors discussed the content, reviewed and edited the manuscript.

Paper 5: O.Q.T. suggested the overall topic. A.S. did some preliminary simulations under my supervision. I performed and verified the final simulations. I prepared the figures and the manuscript. O.Q.T. and M.N. supervised the work. All authors reviewed and edited the manuscript.
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Chapter 1

Introduction

In this chapter, I present a background about metasurfaces as periodic structures that enable controlling electromagnetics properties and wave propagation. In addition, potential applications of metasurfaces in antenna engineering and the significance of higher-symmetric metasurfaces are explained. The motivation of this study and the thesis outline are presented at the end.

1.1 Background

Metasurfaces for controlling electromagnetics properties

Since humankind learned how to have control over material properties, many significant technological breakthroughs have been achieved. By tinkering the raw material extracted from the Earth, early engineers produced artificial materials with desirable mechanical properties, such as steel and concrete, that revolutionized the architectural design. In the 20th century, engineers found how to control the electric properties of materials. This knowledge together with advances in semiconductor physics led to the transistor revolution in electronics.

In the last few decades, engineering the electromagnetic and optical properties of materials has attracted the attention of scientists. The possibility of controlling the electromagnetic wave or light propagation in desired ways has opened up a huge set of technological developments. For example, guiding light with fiber-optic cables has revolutionized the telecommunications industry.

Moreover, in recent years, complete control and manipulation of light propagation has become possible with periodic configurations of dielectrics, called photonic crystals [1]. They manipulate the light propagation in desired (possibly anomalous) ways. Analogous to photonic crystals, metamaterials and metasurfaces (two-dimensional metamaterial structures with subwavelength thickness) are periodic sub-wavelength metal/dielectric structures that provide the possibility of controlling and manipulating the electromagnetic wave propagation in desired
anomalous) ways [2]. These artificial materials have caused ground-breaking electromagnetic phenomena [3] such as

- realizing effective electric permittivity $\varepsilon$ and/or magnetic permeability $\mu$ that cannot be found in nature [4],
- electromagnetic invisibility [5],
- anomalous reflection and refraction properties for incident plane waves [6],
- prevent the propagation of electromagnetic waves in a desired direction [7],
- guiding surface waves to achieve desirable guided and radiating waves [8,9].

Due to these interesting characteristics, metasurfaces have found many applications in antenna and microwave engineering, especially at millimeter waves.

Metasurfaces for antenna applications

Recently, millimeter-wave frequencies have been considered for high data rate communications, point-to-point wireless communications, and high resolution imaging systems and radars [10]. For all of these applications, wide band, high gain, efficient, and low profile antennas are required. Integrated antenna arrays do not seem viable candidates at these frequencies since their feeding network would be very complicated and lossy. The other option proposed in literature is reflectarrays [11]. Reflectarrays are demonstrated as high efficient options at millimeter-wave frequencies [11]. However, they suffer from narrow bandwidth and radiation pattern degradation due to their feeding which is placed in front of the antenna. In addition, it is difficult to achieve reconfigurable radiation patterns by reflectarrays. Combination of a radiating element with a focusing lens has been also proposed for millimeter-wave frequencies [12]. Nevertheless, in case of using dielectric lenses, the antenna becomes bulky and suffers from the loss of dielectrics at these frequencies. Thus, metasurface-based lenses are proposed [13,14]. Transmitarrays that are capable of efficient phase and polarization control are another alternative for millimeter-wave antennas [6]. They are also based on periodic structures (metasurfaces) to control the electromagnetics wave propagation to obtain the desired radiation pattern.

Indeed, in transmitarrays and metasurface-based lenses, the usage of periodic structures provide degrees of freedom to control the wave propagation characteristics by creating spatial inhomogeneity over a subwavelength-thick surface. This is also called manipulation of the surface impedance [8]. This spatial inhomogeneity in metasurfaces resembles the spatially varying structural features in an array of antennas with subwavelength separation between adjacent elements. It means that the electromagnetic responses, such as scattering amplitude and phase, vary gradually in ways that lead to the desired wavefronts, far field radiation pattern and polarization [8,9]. This interesting feature along with the low manufacturing cost...
of metasurfaces, especially those realized through printed circuit board technology, make them appropriate structures in antenna engineering.

However, metasurfaces suffer from severe frequency dispersion, which leads to a narrow band of operation. This limitation hinders their use for practical applications. Using transformation optics and a full dielectric implementation of metasurfaces have been proposed as a solution to obtain ultra-wideband responses in antenna applications [15, 16]. Applying higher symmetries to metasurfaces is another solution proposed recently to reduce their intrinsic frequency dispersion. Thus, ultra-wideband antennas can be realized using higher-symmetric metasurfaces [17, 18].

Higher-symmetric metasurfaces

Higher-symmetric metasurfaces are periodic surfaces that are defined by means of an additional geometrical operation beyond their periodicity [19]. In recent years, these structures are proposed to overcome the intrinsic frequency dispersion of metasurfaces [20, 21]. This reduction in frequency dispersion is because the conventional stop-band between the first and second modes of periodic structures is removed by adding a higher symmetry [22–24]. Two particular types of higher symmetry are glide and twist (also called screw) symmetries [19]. A glide-symmetric structure coincides with itself after a translation and a reflection with respect to a so-called glide plane while a twist-symmetric structure coincides with itself after a translation and a rotation with respect to a twist axis.

Higher symmetries provide an additional degree of freedom to engineer the electromagnetic properties of periodic structures [25–31]. Additionally, higher values of equivalent refractive index can be realized with higher-symmetric structures [21, 27, 28, 32]. The potential of glide symmetry in creating non-dispersive (ultra-wideband) antennas has been also demonstrated [17, 33, 34]. These antennas can be applied in 5G communications systems [18, 35, 36]. Moreover, glide-symmetric ho-ley structures can be employed as low cost and broad-band electromagnetic band gap (EBG) surfaces at millimeter-wave frequencies [37, 38]. The applications of these EBG structures in designing waveguiding structures [39], flanges [40], and microwave components [41] have been demonstrated at millimeter-waves.

Polar glide symmetry is another type of higher symmetry that has been proposed recently [21]. Similar to glide symmetry, by applying twist and polar glide symmetries to periodic structures, the frequency dispersion is reduced dramatically [26, 42]. A promising application of periodic structures possessing these kinds of symmetry is low loss and wide-band leaky-wave antennas [21].

1.2 Motivation of the project

In all the above-mentioned applications for metasurfaces, one of the main steps is synthesizing the desired dispersion diagrams. For this purpose, analyzing and
modeling the different types of metasurfaces is of great importance. The extraordinary and promising characteristics of glide-symmetric metasurfaces motivated me to find a fast and efficient method to analyze these structures. I focused mainly on glide-symmetric corrugations and glide-symmetric holey structures, and I proposed an efficient mode matching formulation for analyzing them.

In addition, the dispersion-less behavior of twist-symmetric structures and their potential applications in designing wide-band leaky-wave antennas drove me to perform a comprehensive investigation on applying twist and polar glide symmetries to periodic structures. As a result, I came up with a rigorous definition of polar glide symmetry; and I demonstrated the possibility of designing reconfigurable filters by applying both twist and polar glide symmetries to a periodic structure.

Finally, since the twist symmetry is only applicable to cylindrical structures, which require a high cost for manufacturing and are not compatible with low-cost flat technologies, I was motivated to investigate the possibility of mimicking the twist symmetry effect in flat structures. I demonstrated that flat structures with mimicked twist symmetry show similar dispersion properties as structures with real twist symmetry.

1.3 Thesis outline

This thesis contains five chapters.

- Chapter 1 provides an overview about metasurfaces, higher-symmetric metasurfaces, their applications in antenna engineering, and the aim of this thesis.

- In Chapter 2, the definition of glide and twist symmetries and how to apply them to periodic structures are explained. In addition, by explaining the Floquet and generalized Floquet theorems, some general background about the wave propagation characteristics in conventional and higher-symmetric periodic structures is presented.

- In Chapter 3, the analysis of double-layer metallic corrugated surfaces, including glide-symmetric ones [43,44], and glide-symmetric metallic holey surfaces [45,46] is presented. A mode matching technique is used to analyze these structures.

- In Chapter 4, the effect of applying twist and polar glide symmetries to a coaxial line loaded with periodic holes and the possibility of designing reconfigurable filters by combing these symmetries are demonstrated [47]. The accurate definition of polar glide symmetry is also presented in this chapter. In addition, mimicking the twist symmetry effect in flat structures is investigated [48].

- Finally, in Chapter 5, the summary and conclusions of the presented work, future lines, and a brief discussion regarding the sustainability of higher-symmetric periodic structures are presented.
Chapter 2

Higher-symmetric periodic structures

In this chapter, first, the definition of higher symmetry in periodic structures is presented. Then, the Floquet theorem and some basic features of the wave propagation characteristics in periodic structures are explained. In addition, a generalized Floquet theorem is explained, and an overview on wave propagation in higher-symmetric periodic structures is discussed.

2.1 Higher symmetries in electromagnetics

In electromagnetics, two general types of higher symmetries are defined: higher symmetry with respect to the space operator, such as glide and twist [19], and higher symmetry with respect to the time operator such as parity time [49]. Applying these symmetries to periodic structures can cause a number of effects on their dispersion properties. Structures possessing higher symmetry with respect to the time operator are commonly expensive and lossy since they require a lattice alternating between lossy and gain scatterers [49, 50]. These structures are not in the scope of this thesis.

However, structures possessing spatial higher symmetries may be cost effective and show dispersive properties that cannot be found in conventional periodic structures. As explained in the previous chapter, these structures are an excellent candidate to realize ultra-wideband antennas and EBG structures at millimeter-wave frequencies. The two most-used higher symmetry types in electromagnetics, the glide and the twist, are defined and explained in this chapter. Additionally, the recently-discovered polar glide symmetry will be defined in Chapter 4.
CHAPTER 2. HIGHER-SYMMETRIC PERIODIC STRUCTURES

Definition of glide and twist symmetries

A glide reflection, illustrated in Fig. 2.1(a), is created by a translation along a line followed by a reflection with respect to that line. Twist or screw symmetry, illustrated in Fig. 2.1(b), is created by a translation along a twist axis followed by a rotation around this axis.

Applying glide and twist symmetries to periodic structures

As shown in Fig. 2.2, to apply glide symmetry to a periodic structure, it must be translated half a period along the periodicity direction and reflected with respect to a so-called glide plane. The glide plane must contain the periodicity axis, and could be either parallel or perpendicular to the surface of the periodic structure. Glide symmetry can be also applied to structures that are periodic in two directions. In that case, the translation in each direction must be equal to the half of the periodicity in that direction.

Twist symmetry is only applicable to structures that are periodic in one direction and match to a cylindrical coordinate system (see for example the periodic structure with periodicity $p$ depicted in Fig. 2.3(a)). As shown in Fig. 2.3(b), to apply $m$-
2.2 Wave propagation in periodic structures

Starting from Maxwell equations and assuming the time dependency of $e^{j\omega t}$, one can show that in a linear, isotropic, non-dispersive and loss-less material with relative permeability $\mu_r = 1$ and relative permittivity $\epsilon_r = \epsilon_r(r)$, the electric field mode profile $E(r)$ and magnetic field mode profile $H(r)$ satisfy the following equations [1]:

\[
\nabla \times [\nabla \times E(r)] = \left(\frac{\omega}{c}\right)^2 \epsilon_r(r) E(r),
\]

\[
\nabla \times \left[ \frac{1}{\epsilon_r(r)} \nabla \times H(r) \right] = \left(\frac{\omega}{c}\right)^2 H(r).
\]

Equation (2.2) can be written as

\[
\mathcal{L} H(r) = \left(\frac{\omega}{c}\right)^2 H(r),
\]

where

\[
\mathcal{L} \equiv \nabla \times \left[ \frac{1}{\epsilon_r(r)} \nabla \times \right]
\]

Figure 2.3: (a) Coaxial cable with periodic holes on its inner conductor with periodicity $p$. (b) Coaxial cable with four-fold twist-symmetric holes on its inner conductor with periodicity $p$. One unit cell consists of four subunit cells with a $p/4$ length.

fold twist symmetry to this periodic structure, where $m$ is called the degree of the twist symmetry, it must be translated for $p/m$ along and rotated for $2\pi/m$ around the twist axis, which is the periodicity axis of the structure.
is a linear differential operator. Taking the boundary conditions in electromagnetism into account, it has been proved in [1] that the operator $\mathcal{L}$ is symmetric [51]. It means, for any wave functions $F(r)$ and $G(r)$,
\begin{equation}
\langle F(r), L G(r) \rangle = \langle L F(r), G(r) \rangle, \tag{2.7}
\end{equation}
where $\langle . , . \rangle$ is the inner product operator. Considering the analogy between the wave functions and vector fields, in the physics literature such as in quantum mechanics [52] and photonic crystals [1], operator $\mathcal{L}$ is called a Hermitian operator.

Continuing the discussion using the same names and notation that are employed in [1], equation (2.5) is an eigenvalue equation for the Hermitian operator $\mathcal{L}$. Solving this equation, the eigenvectors $H(r)$, which is the spatial patterns of the modes, and their corresponding eigenvalues $(\omega/c)^2$ are obtained. Since the operator $\mathcal{L}$ is Hermitian, the eigenvalues are necessarily real [1]. In addition, two harmonic modes $H_1(r)$ and $H_2(r)$ are either orthogonal (if they have different corresponding frequencies) or degenerate (if they have the same corresponding frequencies). Degenerate modes are not necessarily orthogonal. Degeneracy occurs when more than one field pattern exist at one particular frequency. Usually, a symmetry in the structure is the reason behind having degenerate modes [1]. For example, in a structure that is invariant under a rotation, modes with spatial patterns that coincide with each other by the same angle of that rotation are expected to have the same frequency $\omega$ in their eigenvalues.

Now, let us assume that a structure has a special kind of geometrical symmetry $S$. This means that the structure is invariant under the operator $S$. Thus, operating on $H(r)$ with the operator $\mathcal{L}$ is equivalent to operating on it first with $S$, then with $\mathcal{L}$, and finally with the inverse of $S$:
\begin{equation}
\mathcal{L}H(r) = S^{-1}\mathcal{L}(SH(r)). \tag{2.8}
\end{equation}
This means the Hermitian operator $\mathcal{L}$ and the geometrical symmetry operator $S$ commute. Therefore,
\begin{equation}
S(\mathcal{L}H(r)) = \mathcal{L}(SH(r)) = (\frac{\omega}{c})^2(SH(r)), \tag{2.9}
\end{equation}
which tells us if $H(r)$ is an eigenfunction of the operator $\mathcal{L}$ with the frequency $\omega$, then $SH(r)$ is also an eigenfunction of $\mathcal{L}$ with the frequency $\omega$. Unless these two eigenfunctions (modes) are degenerate, they have to be a factor of each other since only one mode per frequency can exist. Thus, $SH(r) = \alpha H(r)$, which means $H(r)$ is also an eigenfunction of the operator $S$. It has been proved that even if these two modes are degenerate, a linear combination of them is an eigenfunction for the operator $S$ [52]. Therefore, it is concluded that the operator $\mathcal{L}$ and operator $S$ have some common eigenfunctions. This result is very helpful for finding $H(r)$ in (2.5) since usually the eigenfunctions of symmetry operators can be determined more easily than the eigenfunctions of the operator $\mathcal{L}$. Using this conclusion, in [1], the eigenfunctions of $\mathcal{L}$ for structures possessing inversion, translational, rotational, and mirror symmetries are constructed and cataloged based on the properties of these symmetries.
Floquet Theorem

To explain the Floquet theorem, wave propagation in a structure that is periodic only in one direction (x-direction), and invariant in the y-direction is considered (see Fig. 2.2(a)). The smallest section of the structure that makes the full structure by repetition is called the unit cell. The unit cell of the case under study is specified with two dashed lines in Fig. 2.2(a) and has a length of \( p \), which is called the lattice constant. The structure is invariant under translation operators with lattice vectors \( \mathbf{R} = \ell p \hat{x} \), where \( \ell \) is an integer. For \( \ell = 1 \), we have \( \mathbf{R} = p = p \hat{x} \), which is called the primitive lattice vector.

The solutions of (2.2) in \( x \) and \( y \) are separable. For the \( y \) dependency, since the structure is invariant under any translation vector along the \( y \)-direction, we have

\[
H(r) \propto e^{-jk_y y},
\]

(2.10)

where \( k_y \) is called the wave vector along \( y \)-direction. This kind of symmetry is called continuous translational symmetry in [1]. However, along \( x \)-direction, the structure has discrete translational symmetry. Thus, the solutions of (2.2) with respect to \( x \) is the eigenfunctions of the translation operator \( \mathcal{T}_R \). It is well-known that the eigenfunctions of translation operators are the modes with an exponential form:

\[
\mathcal{T}_R [e^{-jk_x x}] = e^{-jk_x (x-\ell p)} = (e^{jk_x \ell p}) e^{-jk_x x},
\]

(2.11)

where \( k_x \) is called the wave vector along \( x \)-direction. Paying careful attention to (2.11), it can be found out that the eigenfunction with \( k_x = k_{x, 0} \) and all the eigenfunctions with the \( k_x \) of the form \( k_{x, 0} + mq \), where \( m \) is an integer and \( q = 2\pi/p \), yield the same eigenvalue. Thus, these modes form a degenerate set. It should be mentioned that \( q \) is called primitive reciprocal lattice constant and \( \mathbf{q} = q \hat{x} \) is primitive reciprocal lattice vector.

Any linear combination of these degenerate modes is also an eigenfunction with the same eigenvalue. Therefore, using (2.10) and (2.11), the solution of (2.2) for the wave vector \( k_x \hat{x} + k_y \hat{y} \) can be expressed as

\[
H_{k_x, k_y}(r) = e^{-jk_y y} \sum_m c_{k_x, m}(z) e^{-j(k_x + mq)x} = e^{-jk_y y} e^{-jk_x x} \sum_m c_{k_x, m}(z) e^{-jmq x} = e^{-jk_y y} e^{-jk_x x} u_{k_x}(x, z),
\]

(2.12)

where \( c_{k_x, m}(z) \) is the expansion coefficients and \( u_{k_x}(x, z) \) is a periodic function in \( x \) with periodicity \( p \). Equation (2.12) tells us that the electromagnetic fields (mode profiles) in a periodic structure with periodicity \( p \) along the \( x \)-direction is a plane wave multiplied by a \( x \)-periodic function with the same periodicity:

\[
E(x, y, z) \propto e^{-jk_x x} u_x(x, y, z),
\]

(2.13)

\[
H(x, y, z) \propto e^{-jk_x x} u_x(x, y, z).
\]

(2.14)
Therefore, applying the translation operator \( T_p \) on \( E(x, y, z) \) and \( H(x, y, z) \) yields
\[
T_p [E(x, y, z)] = E(x + p, y, z) = e^{-j k_x p} E(x, y, z),
\]
\[
T_p [H(x, y, z)] = H(x + p, y, z) = e^{-j k_x p} H(x, y, z).
\]

This result is known as the Floquet theorem in mechanics and the Bloch theorem in solid-state physics. The mode with the form of (2.13) and (2.14) is called the Floquet mode or Bloch mode.

Dispersion relation and Brillouin zone

Substituting (2.14) in (2.2), a relation between \( \omega \) and \( k_x \), which is called the dispersion relation, is obtained. This relation provides the complete information about the wave propagating at the angular frequency \( \omega \) in the structure. As mentioned, changing \( k_x \) by integral multiples of \( q = 2\pi/p \) does not change the corresponding frequency. It means the dispersion relation \( \omega(k_x) \) is periodic with the periodicity \( q \).

Therefore, it is enough to find the dispersion relation for \(-\pi/p < k_x < \pi/p\). This region of \( k_x \) values is called the Brillouin zone.

It has also been proved in [1] that if a periodic structure has a rotation, mirror-reflection, or inversion symmetry, its dispersion diagram (\( \omega \) versus the wave vector \( \mathbf{k} \)) has the same symmetry. In these cases, it is not even necessary to determine \( \omega(k) \) for the whole Brillouin zone as there are some redundancies within it. Instead, it is enough to find \( \omega(k) \) only over the so-called the irreducible Brillouin zone which is the smallest region within the Brillouin zone where there is no relation between \( \omega \) and \( \mathbf{k} \) due to the symmetry.

The Brillouin zone for a general periodic structure is defined by the structure lattice vectors. A comprehensive explanation about this issue can be found in
Figure 2.5: (a) Lattice vectors of a two dimensional periodic structure with the periodicity \( p \) along both \( x \)- and \( y \)-direction. (b) The Brillouin zone (yellow square) and the irreducible Brillouin zone (orange triangle). (c) Irreducible Brillouin zone and the conventional name of its special points.

In case the structure has a square lattice \( (p_x = p_y = p) \), apart from the translation symmetry, it has also rotation symmetry since it is invariant under a 90° rotation (see Fig. 2.5(a)). In this case, the Brillouin zone is a square specified by \(-\pi/p < k_x < \pi/p\) and \(-\pi/p < k_y < \pi/p\) and the irreducible Brillouin zone is a triangle wedge whose area is 1/8 of the Brillouin zone area, both shown in Fig. 2.5(b). The dispersion diagram over the rest of the Brillouin zone is made with copies of the dispersion diagram over the irreducible Brillouin zone. Finally, it should be mentioned that for dispersion analysis of this periodic structure, it is common to obtain the dispersion diagrams over the edges of the irreducible Brillouin zone, which are the lines connecting point \( \Gamma \), \( X \), and \( M \) in Fig. 2.5(c). These names are used conventionally for the center, corner, and the face of the irreducible Brillouin zone.
2.3 Wave propagation in higher-symmetric periodic structures

As explained in the previous section, in a structure possessing the symmetry $S$, the electromagnetic fields profiles are the eigenfunctions of the operator $S$. The conventional periodic structures that have discrete translation symmetry are discussed. Floquet theorem states that the modes propagating in these structures, called Floquet modes, are the eigenfunctions of the translation operator $T_p$, assuming the periodicity is $p$. Similarly, the modes propagating in a glide-symmetric structure or an $m$-fold twist-symmetric structure are the eigenfunctions of the glide operator $G$ or the twist operator $S_m$. This result is called generalized Floquet theorem [19].

**Generalized Floquet Theorem**

Assume a periodic structure with the periodicity $p$ along the $x$-direction that is invariant under the higher symmetry operator $S_n$, also along the $x$-direction, such that $(S_n)^n = T_p$. Note that since all the operators act along the $x$-direction, it is omitted in the operator expression. Now, let us say $\psi = e^{-jk_xx}$ is a mode of this structure. Thus, we have

$$T_p[\psi] = e^{-jk_{x}(x-p)} = (e^{jk_{x}p})e^{-jk_xx} = t\psi,$$

(2.17)

where $t = e^{jk_{x}p}$ is the eigenvalue of the translation operator. Thus, one can write

$$(T_p - t)\psi = [(S_n)^n - t] \psi = \prod_{\nu=0}^{n-1} (S_n - \alpha_{\nu})\psi,$$

(2.18)

where $\alpha_{\nu} = t^{1/n}e^{-j(2\pi\nu/n)}$ for $\nu = 0, 1, \cdots, n - 1$. Therefore, if $\psi \neq 0$ is an eigenfunction of $T_p$, at least one of the $\alpha_{\nu}$, let us say $\alpha_{\nu} = s$, must satisfy

$$(S_n - s)\psi = 0.$$  

(2.19)

It means $\psi$ is also an eigenfunction of $S_n$ with the eigenvalue $s$. In other words, the modes propagating in a higher-symmetric structure are not only the eigenfunctions of the translation operator $T_p$, but also the eigenfunctions of the higher symmetry operator $S_n$. This result is known as the generalized Floquet theorem [19].

As explained in the previous section, inserting the eigenfunction $\psi$ in equation (2.2), the dispersion relation $\omega(k)$ can be obtained. It was also discussed that the dispersion diagram of a periodic structure with periodicity $p$ is periodic. This was because the eigenvalues of the operator $T_p$, which is $t(\omega) = e^{jk_{T_p}p}$, are periodic in $k_{T_p}$ with period $2\pi$. In case of a higher-symmetric structure, the modes are eigenfunctions of $S_n$. Thus, the eigenvalues of $S_n$, which is $s(\omega) = e^{jk_{S_n}p/n}$, specify the dispersion relation. Therefore, the dispersion diagram is periodic in
2.3. WAVE PROPAGATION IN HIGHER-SYMMETRIC PERIODIC STRUCTURES

$k_S p$ with period $2n\pi$. In addition, since $s^n(\omega) = t(\omega)$, there must be the following connection between $k_T p$ and $k_S p$:

$$k_T(\omega) p = k_S(\omega) p + 2\pi \nu \quad \nu = 0, 1, \cdots, n - 1.$$

This connection means that the Brillouin diagram for $k_T(\omega) p$ consists of $n$ subsets: the Brillouin diagram for $k_S(\omega) p$ and its $n - 1$ space harmonic branches (displacing along $k_S(\omega) p$ axis by integral multiples of $2\pi$). Note that there are only $n - 1$ independent translations since $k_S(\omega) p$ is periodic itself with period $2n\pi$. The existence of space harmonic branches of $k_S(\omega) p$ in the Brillouin diagram ($k_T(\omega) p$ diagram) of a structure possessing a higher symmetry eliminates some stop-bands that exist in the structure when it does not possess higher symmetry.

To shed more light on this discussion and demonstrate the effect of applying a higher symmetry to a periodic structure on its dispersion properties, I will present and compare the dispersion diagrams of a mirrored corrugated surface and a glide-symmetric corrugated surface (Fig. 2.6(a)). I will explain how the $k_T(\omega) p$ diagram can be obtained from the $k_G(\omega) p$ diagram, where $G \equiv S_2$ is the glide operator. In addition, in Chapter 4, I will come back to this discussion and explain how the $k_T(\omega) p$ diagram of an $m$-fold twist-symmetric structure can be obtained from the $k_{S_m}(\omega) p$ diagram, where $S_m$ is the $m$-fold twist operator.

The dispersion diagrams of the structures shown in Fig. 2.6(a) are obtained over the irreducible Brillouin zone using CST Microwave Studio and compared in Fig. 2.6(b) (red lines are for non-glide case and blue lines are for the glide case). These results demonstrate that the conventional stop-band between the first and second mode of the non-glide structure is absent in the glide structure. In addition, the frequency dispersion in the first mode of the glide structure has been dramatically reduced. These results can be justified by comparing the $k_T(\omega) p$ diagram for the non-glide structure (depicted in Fig. 2.6(c)) and the $k_G(\omega) p$ diagram for the glide structure (solid line in Fig. 2.6(d)) and its first space harmonic, obtained by $2\pi$ shift along $k_G p$ axis (dashed line in Fig. 2.6(d)). As explained, $k_T(\omega) p$ diagram for the glide structure is the composition of $k_G(\omega) p$ diagram and its first space harmonic (the composition of the solid line and dashed line in Fig. 2.6(d)). Therefore, the dispersion diagrams of these structures over the irreducible Brillouin zone (the zone between the black dashed lines in Fig. 2.6(c) and Fig. 2.6(d)) are those shown in Fig. 2.6(b). The plots in Fig. 2.6(d) clearly demonstrate that in glide-symmetric structures the first dominant mode is almost dispersion-free and there is no stop-band between the first and second modes of the structure.
Figure 2.6: (a) Simulated structures: a mirrored corrugated and a glide-symmetric corrugated surface. (b) Their dispersion diagrams over the irreducible Brillouin zone obtained using CST (red lines are for non-glide case and blue lines are for the glide case). (c) Plot of $k_T(\omega)p$ for the non-glide structure. (d) Plot of $k_G(\omega)p$ for the glide structure (solid line) and its first space harmonic branch (dashed line), obtained by $2\pi$ shift along $k_Gp$ axis. The results correspond to the following parameters: $w = 3$ mm, $p = 4$ mm, $h = 1.5$ mm, and a gap of 0.2 mm between the layers. The black dashed lines in (c) and (d) show the irreducible Brillouin zone.
Chapter 3

Analysis of glide-symmetric periodic structures

In the previous chapter, by presenting Floquet theorem and generalized Floquet theorem, the general properties of the dispersion relation in conventional and higher-symmetric periodic structures were explained. In this chapter, I focus on the dispersion analysis of glide-symmetric periodic structures. Thus, a discussion is first given about the previous studies related to the topic, together with a brief summary about the content of Papers 1-3. A brief explanation about the mode matching technique is also presented. Then, the mode matching technique is proposed to analyze the wave propagation in double-layered corrugated surfaces including glide-symmetric ones (related to Paper 1 and Paper 2). Finally, the mode matching method for the dispersion analysis of glide-symmetric holey surfaces using the generalized Floquet theorem is discussed (related to Paper 2 and Paper 3).

3.1 Previous studies and Papers 1-3

In 1960s and 1970s, the general characteristics of the waves propagating in one-dimensional glide-symmetric periodic structures were investigated for the first time [53–55]. These studies, performed in connection to the theory of periodic waveguides, led to proposing a generalized Floquet theorem by Oliner in 1973 [19]. However, the advent of metasurfaces and the recent demonstration of reducing frequency dispersion by applying glide symmetry to metasurfaces [17] have encouraged the development of new, fast and efficient methods to analyze glide-symmetric metasurfaces accurately.

Although several types of periodic structures have been modeled using the well-known homogenized impedance model [56, 57], strongly coupled glide-symmetric structures cannot be rigorously modeled by applying this technique [20]. The reason is that the dispersion reduction of glide-symmetric structures occurs when the two surfaces are strongly coupled to each other. This yields the excitation of higher-order modes.
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order modes which makes the homogenized impedance model non-applicable [20]. In addition, due to the strong coupling between the layers, glide-symmetric structures cannot be precisely modeled by the analysis of just one of the surfaces [58]. These features prevent driving a simple circuit model for glide-symmetric structures in general.

Nevertheless, a circuit model has been proposed for glide-symmetric corrugated surfaces having no overlap between the grooves in their upper and lower layers [20]. However, if the width of the grooves is smaller than half of the periodicity, these structures are equivalent to their non-glide counterpart with a period equal to half of the periodicity in the glide case [59]. These cases are considered as reducible glide-symmetric structures [59] and can be modeled using the available methods for conventional periodic structures. On the other hand, irreducible glide-symmetric structures, in which higher order modes have a significant effect on the dispersion properties [59], cannot be reduced to a non-glide case with a reduced period. Therefore, the available methods for analyzing conventional periodic structures cannot be applied to them.

Integral equation based formulations have been applied for dispersion analysis of different periodic structures with different geometries and materials [60–65]. However, for glide-symmetric structures, they cannot provide a dispersion equation highlighting the difference between glide and non-glide surfaces. The other powerful technique to analyze periodic structures is the mode matching method [66]. This technique has been successfully applied for dispersion analysis of holey surfaces with square holes [67–69]; and it has been demonstrated as a fast and efficient method for dispersion analysis of strongly interacting surfaces [69]. In addition, this method intrinsically provides a physical insight about the waves propagating in a structure. Therefore, in Paper 1, a mode matching technique has been applied to analyze the dispersion characteristics of strongly interacting corrugated surfaces including glide-symmetric ones. In Paper 2 by using a mode matching technique and the generalized Floquet theorem, glide-symmetric corrugated surfaces and holey surfaces with rectangular holes are analyzed. It is demonstrated that the generalized Floquet theorem leads to a reduction of the computational domain to one half of the unit cell, which makes the method more efficient than the mode-matching method presented in Paper 1. In Paper 3, the formulation presented in Paper 2 is extended to arbitrary shapes of the holes, like e.g. circular holes, since rectangular holes are not commonly used in practical applications.

3.2 Mode matching technique

Mode matching, also named modal analysis [66] or eigenmode expansion (EME) [70], is a well-known and powerful technique to analyze waveguide junctions and discontinuities. The technique is based on expressing the electromagnetic fields in the regions connected to the discontinuity as a summation of their local waveguide modes with unknown coefficients. The waveguide modes are obtained by solving
3.3 DOUBLE-LAYER CORRUGATED SURFACES

Maxwell’s equations in each region. Afterwards, by imposing boundary conditions at the discontinuity and projecting the boundary equations on the modes of one region, a set of linear equations are obtained. These equations relate the unknown coefficients at different regions with a so-called scattering matrix [70]. If there is an excitation, the right-hand side of the equations is non-zero and, therefore, the coefficients are obtained by solving the equations system. However, by assuming no excitation, the right-hand side is zero and the wave propagation characteristics of the structure are obtained by setting the determinant of the scattering matrix equal to zero.

3.3 Double-layer corrugated surfaces

In this section, a mode matching formulation has been derived to analyze the wave propagation in different types of two dimensional doubled corrugated metasurfaces including glide-symmetric ones. Figure 3.1 illustrates the general geometry of a two dimensional doubled corrugated metasurface. It is periodic along the $x$-direction, invariant along the $y$-direction, and bounded along the $z$-direction. Both layers have the same ridge width ($b$) and interspacing ($a$), resulting in the same periodicity $d = a + b$. However, the heights of the ridges are different ($h_1$ for the lower layer and $h_2$ for the upper one). There is a shift equal to $s$ between the layers and the plane $z = 0$ is located in the middle of the gap between the layers. There might be a dielectric inside the grooves, with the relative permittivity $\varepsilon_{h1}$ in the lower layer and $\varepsilon_{h2}$ in the upper layer, or between the layers, with the relative permittivity $\varepsilon_g$.

Note, glide-symmetric configuration occurs if $h_1 = h_2$, $\varepsilon_{h1} = \varepsilon_{h2}$, and $s = d/2$.

Fields expression

To analyze the structure using the mode matching technique, we should write the general expression of fields inside the lower and upper grooves and in the gap between them (see Figure 3.1). Afterwards, the boundary conditions (continuity of tangential electric and magnetic fields) need to be imposed at the surfaces $z = -g/2$ for $0 < x < d$, and $z = g/2$ for $s < x < d + s$. 

Figure 3.1: Cross section of the general geometry of a two dimensional doubled corrugated metasurface.
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As mentioned above, all the regions are filled with a homogeneous and lossless medium with a constant relative permittivity. Thus, first, we find the general field expression in such a medium, assuming and suppressing the time-dependency of $e^{j\omega t}$.

In a homogeneous and lossless medium with the intrinsic wave impedance $\eta = \eta_0 / \sqrt{\varepsilon_r}$ and the wave number $k = k_0 \sqrt{\varepsilon_r}$, where $\eta_0$ and $k_0$ are the intrinsic wave impedance and the wave number of free space, the fields satisfy the source free Maxwell's equations:

$$\nabla \times E = -j k_0 \eta_0 H, \quad \nabla \times H = j \frac{k}{\eta} E. \quad (3.1)$$

Since the structure is invariant along the $y$-direction, the $y$-dependency is written as $e^{-j k_y y} \Rightarrow \partial / \partial y = -j k_y y$ and omitted. In addition, the fields are decomposed with respect to the $y$-direction (TE$_y$ and TM$_y$). Thus, we have [71]

$$E_t = -j \frac{k}{k^2 - k_y^2} [k_y \nabla_t E_y - k_0 \eta_0 \hat{y} \times \nabla_t H_y], \quad (3.2)$$
$$H_t = -j \frac{k}{k^2 - k_y^2} \left[ k_y \nabla_t H_y + \frac{k}{\eta} \hat{y} \times \nabla_t E_y \right], \quad (3.3)$$

where $\nabla_t = \hat{x} \partial / \partial x + \hat{z} \partial / \partial z$, and

$$E_t = \hat{x} E_x + \hat{z} E_z, \quad (3.4)$$
$$H_t = \hat{x} H_x + \hat{z} H_z. \quad (3.5)$$

In this thesis, only the propagation characteristics of TE$_y$-modes (with $H_y \neq 0$ and $E_y = 0$) are considered as they are the first dominant modes that can propagate in the aforementioned structure. For these modes, the tangential field components $E_x$, $H_x$, and $H_y$ need to be matched between the regions. However, equations (3.2) and (3.3) yield

$$E_x = \frac{j k_0 \eta_0}{k^2 - k_y^2} \frac{\partial}{\partial z} H_y, \quad (3.6)$$
$$H_x = -\frac{j k_y}{k^2 - k_y^2} \frac{\partial}{\partial x} H_y, \quad (3.7)$$

which show continuity in $z$ of $E_x$ and $H_y$ implies the continuity in $z$ of $H_x$. Thus, we only need to know $E_x$ and $H_y$ in different regions. Finding $E_x$, $H_y$ can be readily expressed using (3.6).

In the gap between the layers, due to the periodicity, we have the Floquet modes:

$$E_x^{Gap} = \frac{1}{d} \sum_p e^{-jk_{x,p}x} \left[ A_p^x \sin(k_{z,p}z) + B_p^x \cos(k_{z,p}z) \right], \quad (3.8)$$
$$H_y^{Gap} = \frac{1}{d} \sum_p e^{-jk_{x,p}x} \left[ D_p^y \sin(k_{z,p}z) + F_p^y \cos(k_{z,p}z) \right], \quad (3.9)$$
where $k_{x,p} = k_{x,0} + 2\pi p/d$, $k_{z,p} = \sqrt{\varepsilon_g k_{0}^{2} - k_{y}^{2} - k_{z}^{2}}$ with $\Re\{k_{z,p}\} > 0$ and $\Im\{k_{z,p}\} < 0$. In addition, $A$ and $B$ coefficients are the amplitudes of the odd and even parts of each Floquet harmonic; and $D$ and $F$ coefficients are obtained using (3.6). Therefore,

$$
D_y^p = \frac{1}{j\eta_0 k_0} \frac{\varepsilon_g k_0^2 - k_y^2}{k_{z,p}} B_x^p, \quad \text{(3.10a)}
$$

$$
F_y^p = -\frac{1}{j\eta_0 k_0} \frac{\varepsilon_g k_0^2 - k_y^2}{k_{z,p}} A_x^p. \quad \text{(3.10b)}
$$

To express $E_x$ inside the grooves, they can be regarded as short-circuited parallel plate waveguides along the $z$ axis, with the length of $h_1$ for the lower layer and $h_2$ for the upper layer. Therefore, assuming $\varepsilon_{h1} = \varepsilon_{h2} = \varepsilon_h$, the $x$-component of the electric field in the lower layer ($E_{x,\text{low}}^l(x,z)$) and upper layer ($E_{x,\text{up}}^u(x,z)$) are expressed as

$$
E_{x,\text{low}}^l(x,z) = \sum_m C_{m}^{\text{low}}(x) [e^{iq_{z,m}(z+g/2)} - R_{m}^\text{low} e^{-iq_{z,m}(z+g/2)}], \quad \text{(3.11)}
$$

$$
E_{x,\text{up}}^u(x,z) = \sum_m C_{m}^{\text{up}} e^{-jk_{x,}\Phi_m(x - s)} [e^{-iq_{z,m}(z-g/2)} - R_{m}^\text{up} e^{iq_{z,m}(z-g/2)}], \quad \text{(3.12)}
$$

where $C_{m}^{\text{low}}$ and $C_{m}^{\text{up}}$ are the undetermined amplitudes of each mode, and

$$
\Phi_m(x) = \left\{ \begin{array}{ll} \sqrt{\frac{2}{a}} \cos(m\pi z/a) & m \neq 0 \\ \sqrt{\frac{2}{a}} & m = 0 \end{array} \right. \quad \text{(3.13)}
$$

are the normalized modal functions. In addition $q_{z,m} = \sqrt{\varepsilon_h k_0^2 - (m\pi/a)^2 - k_y^2}$. Note that $\Re\{q_{z,m}\} > 0$ and $\Im\{q_{z,m}\} < 0$. Additionally,

$$
R_{m}^\text{low} = \exp(-j2q_{z,m}h_1), \quad \text{(3.14)}
$$

$$
R_{m}^\text{up} = \exp(-j2q_{z,m}h_2), \quad \text{(3.15)}
$$

are the reflection coefficients at $z = -g/2 - h_1$ and $z = g/2 + h_2$.

Now, using (3.6), $H_{y,\text{low}}(x,z)$ and $H_{y,\text{up}}(x,z)$ are expressed as

$$
H_{y,\text{low}}(x,z) = -\frac{\varepsilon_h k_0^2 - k_y^2}{j\eta_0 k_0} \sum_m C_{m}^{\text{low}}(x) [e^{iq_{z,m}(z+\frac{g}{2})} + R_{m}^\text{low} e^{-iq_{z,m}(z+\frac{g}{2})}], \quad \text{(3.16)}
$$

$$
H_{y,\text{up}}(x,z) = \frac{\varepsilon_h k_0^2 - k_y^2}{j\eta_0 k_0} \times \\
\sum_m C_{m}^{\text{up}} e^{-jk_{x,}\Phi_m(x - s)} [e^{-iq_{z,m}(z-\frac{g}{2})} + R_{m}^\text{up} e^{iq_{z,m}(z-\frac{g}{2})}]. \quad \text{(3.17)}
$$
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Boundary conditions

First, the continuities of $E_x$ at $z = -g/2$ and $z = g/2$ are imposed:

$$E_{x}^{\text{Gap}}(x, z = -g/2) = \begin{cases} E_x^{\text{low}} (x, z = -g/2) : 0 < x < a \\ 0 : a < x < d \end{cases} \quad (3.18)$$

$$E_{x}^{\text{ Gap}}(x, z = g/2) = \begin{cases} E_x^{\text{ up}} (x, z = g/2) : s < x < a + s \\ 0 : a + s < x < d + s \end{cases} \quad (3.19)$$

Thus, we have

$$\frac{1}{d} \sum_p e^{-jk_{x,p}x} \left[ -A_p^z \sin(\theta_{z,p}) + B_p^z \cos(\theta_{z,p}) \right] = \begin{cases} \sum_m r_{m,\text{low}}^{-} C_m^{\text{low}} \Phi_m(x) : 0 < x < a \\ 0 : a < x < d \end{cases} \quad (3.20)$$

$$\frac{1}{d} \sum_p e^{-jk_{x,p}x} \left[ A_p^z \sin(\theta_{z,p}) + B_p^z \cos(\theta_{z,p}) \right] = \begin{cases} \sum_m r_{m,\text{up}}^{-} C_m^{\text{up}} e^{-jk_{x,s}x} \Phi_m(x - s) : s < x < a + s \\ 0 : a + s < x < d + s \end{cases} \quad (3.21)$$

where $\theta_{z,p} = k_{z,p}g/2$ and

$$r_{m,\text{low}}^{-} = 1 - R_{m}^{\text{low}}, \quad (3.22)$$

$$r_{m,\text{up}}^{-} = 1 - R_{m}^{\text{up}}. \quad (3.23)$$

Projecting (3.20) and (3.21) on $e^{jk_{x,p}x}$, respectively, over $0 < x < d$ and $s < x < d + s$, we obtain

$$\left[ -A_p^z \sin(\theta_{z,p}) + B_p^z \cos(\theta_{z,p}) \right] = \sum_m r_{m,\text{low}}^{-} C_m^{\text{low}} \tilde{\Phi}_m(k_{x,p}), \quad (3.24)$$

$$\left[ A_p^z \sin(\theta_{z,p}) + B_p^z \cos(\theta_{z,p}) \right] = e^{i(k_{x,s} - k_{x,0})} \sum_m r_{m,\text{up}}^{-} C_m^{\text{up}} \tilde{\Phi}_m(k_{x,p})$$

$$= e^{i(2\pi p/d)s} \sum_m r_{m,\text{up}}^{-} C_m^{\text{up}} \tilde{\Phi}_m(k_{x,p}), \quad (3.25)$$

where

$$\tilde{\Phi}_m(k_{x,p}) = \int_0^a \Phi_m(x) e^{jk_{x,p}x} dx \quad (3.26)$$
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is the Fourier transform of $\Phi_m(x)$ represented in (3.13). Solving (3.24) and (3.25), we obtain $A$ and $B$ coefficients based on $C$ coefficients:

$$A_p^x = \frac{-1}{2\sin(\theta_{z,p})} \sum_m \left[ r_{m,\text{low}}^+ C_m^{\text{low}} e^{i(2\pi p/d)s_r} - r_{m,\text{up}}^+ C_m^{\text{up}} \right] \tilde{\Phi}_m(k_{x,p}),$$  \hspace{1cm} (3.27)

$$B_p^x = \frac{1}{2\cos(\theta_{z,p})} \sum_m \left[ r_{m,\text{low}}^+ C_m^{\text{low}} + r_{m,\text{up}}^+ C_m^{\text{up}} \right] \tilde{\Phi}_m(k_{x,p}).$$  \hspace{1cm} (3.28)

The next step is imposing the continuity of $H_y$ at $z = -g/2$ over $0 < x < a$ and at $z = g/2$ over $s < x < a + s$:

$H_y^{\text{Gap}}(x, z = -g/2) = H_y^{\text{low}}(x, z = -g/2): 0 < x < a,$  \hspace{1cm} (3.29)

$H_y^{\text{Gap}}(x, z = g/2) = H_y^{\text{up}}(x, z = g/2): s < x < a + s.$  \hspace{1cm} (3.30)

Thus, we obtain

$$\frac{1}{d} \sum_p e^{-jk_{z,p}x} \left[ -D_p^x \sin(\theta_{z,p}) + F_p^y \cos(\theta_{z,p}) \right] \tilde{\Phi}_m'(-k_{z,p}) = -\delta_{m,m'} \frac{\varepsilon_h k_{0}^2 - k_y^2}{\eta_0 k_0} \sum_{m} C_m^{\text{low}} \phi_m(x) r_{m,\text{low}}^+,$$  \hspace{1cm} (3.31)

$$\frac{1}{d} \sum_p e^{-jk_{z,p}x} \left[ D_p^x \sin(\theta_{z,p}) + F_p^y \cos(\theta_{z,p}) \right] e^{-jk_{z,s}x} \tilde{\Phi}_m'(-k_{z,p}) = -\delta_{m,m'} \frac{\varepsilon_h k_{0}^2 - k_y^2}{\eta_0 k_0} \sum_{m} C_m^{\text{up}} \phi_m(x-s) r_{m,\text{up}}^+,$$  \hspace{1cm} (3.32)

where

$$r_{m,\text{low}}^+ = 1 + R_{m,\text{low}},$$ \hspace{1cm} (3.33)

$$r_{m,\text{up}}^+ = 1 + R_{m,\text{up}}.$$ \hspace{1cm} (3.34)

Now, projecting the boundary conditions (3.31) and (3.32), respectively, on the modal functions $\Phi_{m'}(x)$ (over $0 < x < a$) and $\Phi_{m'}(x-s)$ (over $s < x < a + s$), and using the orthogonality property of modal functions, we obtain

$$\frac{1}{d} \sum_p \left[ -D_p^x \sin(\theta_{z,p}) + F_p^y \cos(\theta_{z,p}) \right] \tilde{\Phi}_{m'}(-k_{z,p}) = -\delta_{m,m'} \frac{\varepsilon_h k_{0}^2 - k_y^2}{\eta_0 k_0} r_{m,\text{low}}^+ C_m^{\text{low}} \phi_m(x)$$  \hspace{1cm} (3.35)

$$\frac{1}{d} \sum_p \left[ D_p^x \sin(\theta_{z,p}) + F_p^y \cos(\theta_{z,p}) \right] e^{-jk_{z,s}x} \tilde{\Phi}_{m'}(-k_{z,p}) = -\delta_{m,m'} \frac{\varepsilon_h k_{0}^2 - k_y^2}{\eta_0 k_0} r_{m,\text{up}}^+ C_m^{\text{up}} \phi_m(x-s).$$  \hspace{1cm} (3.36)
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Substituting (3.27) and (3.28) into (3.10) and inserting the results into the above equations yield

\[
\sum_p \sum_m \frac{1}{2} \left[ g_{m,p} \cot(\theta_{z,p}) - h_{m,p} \tan(\theta_{z,p}) \right] \frac{\Phi_m(k_{x,p})\Phi_{m'}(-k_{x,p})}{k_{z,p}}
\]

\[
+ \delta_{m,m'} \frac{\text{id}}{q z_m \varepsilon g_{k_0^2} - q y_m \varepsilon g_{k_0^2}} C^\text{low}_m = 0,
\]

(3.37)

\[
\sum_p \sum_m \frac{1}{2} \left[ g_{m,p} \cot(\theta_{z,p}) + h_{m,p} \tan(\theta_{z,p}) \right] \frac{\Phi_m(k_{x,p})\Phi_{m'}(-k_{x,p})}{k_{z,p}} e^{-j(2\pi p/d)s}
\]

\[
- \delta_{m,m'} \frac{\text{id}}{q z_m \varepsilon g_{k_0^2} - q y_m \varepsilon g_{k_0^2}} C^\text{up}_m = 0,
\]

(3.38)

where

\[
g_{m,p} = \begin{cases} r^{-}_{m,\text{low}} C^\text{low}_m - e^{j(2\pi p/d)s} r^{-}_{m,\text{up}} C^\text{up}_m, & \text{if } m \text{ is even} \\ r^{+}_{m,\text{low}} C^\text{low}_m + e^{j(2\pi p/d)s} r^{+}_{m,\text{up}} C^\text{up}_m, & \text{if } m \text{ is odd} \end{cases}
\]

(3.39)

\[
h_{m,p} = \begin{cases} r^{-}_{m,\text{low}} C^\text{low}_m - e^{j(2\pi p/d)s} r^{-}_{m,\text{up}} C^\text{up}_m, & \text{if } m \text{ is even} \\ r^{+}_{m,\text{low}} C^\text{low}_m + e^{j(2\pi p/d)s} r^{+}_{m,\text{up}} C^\text{up}_m, & \text{if } m \text{ is odd} \end{cases}
\]

(3.40)

Finally, by truncating the number of modes to \(\max\{|m|\} = M\), \(\max\{|p|\} = P\), and rewriting (3.37) and (3.38) in a matrix form, we have

\[
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & C^\text{low}_m
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix}.
\]

(3.41)

Setting the determinant of the coefficient matrix in (3.41) equal to zero, the dispersion equation of the structure is obtained. Finding the set of \((k_0, k_x, k_y)\) that satisfies this dispersion equation gives the dispersion diagram of the structure.

Glide-symmetric Corrugations

Now, let us consider what will happen in case of having glide symmetry in the structure. In that case, \(h_1 = h_2\) and \(s = d/2\) which yield \(e^{j(2\pi p/d)s} = (-1)^p\) and

\[
R^\text{low}_m = R^\text{up}_m = R_m,
\]

\[
r_{m,\text{low}}^\pm = r_{m,\text{up}}^\pm = r^\pm_m.
\]

(3.42)

(3.43)

Thus,

\[
g_{m,p} = r^{-}_m \left[ C^\text{low}_m - (-1)^p C^\text{up}_m \right],
\]

(3.44)

\[
h_{m,p} = r^{-}_m \left[ C^\text{low}_m + (-1)^p C^\text{up}_m \right],
\]

(3.45)
which change (3.37) and (3.38) to

\[
\sum_p \sum_m \frac{1}{2} \left\{ \frac{r_m^-}{C_m^\text{low}} \cot(\theta_{z,p}) - \left[ C_m^\text{low} + (-1)^p C_m^\text{up} \right] \tan(\theta_{z,p}) \right\} \frac{\tilde{f}_m(k_{x,p}) \Phi_m'( -k_{x,p} )}{k_{x,p}} - \frac{j \eta_k d}{\varepsilon_y k_0^2 - k_y^2} \frac{\varepsilon_h k_0^2 - k_y^2}{\eta_k} r_m^+ C_m^\text{low} \frac{q_{z,m}}{m} = 0, \tag{3.46}
\]

\[
\sum_p \sum_m \frac{1}{2} \left\{ \frac{r_m^-}{C_m^\text{up}} \cot(\theta_{z,p}) - \left[ C_m^\text{up} + (-1)^p C_m^\text{low} \right] \tan(\theta_{z,p}) \right\} \frac{\tilde{f}_m(k_{x,p}) \Phi_m'( -k_{x,p} )}{k_{x,p}} - \frac{j \eta_k d}{\varepsilon_y k_0^2 - k_y^2} \frac{\varepsilon_h k_0^2 - k_y^2}{\eta_k} r_m^+ C_m^\text{up} \frac{q_{z,m}}{m} = 0. \tag{3.47}
\]

Note that these two equations are unchanged if we exchange \( C_m^\text{low} \leftrightarrow C_m^\text{up} \). This means that in (3.41), the coefficient matrix is symmetric and also \( \alpha_{11} = \alpha_{22} \), which yield \( C_m^\text{low} = C_m^\text{up} = C_m \). Using the generalized Floquet theorem, this result could have been known from the beginning. Therefore, in case of glide symmetry, it is enough to enforce the boundary conditions at only one of the surfaces, and solve the simplified dispersion equation:

\[
\sum_m \alpha_{m',m} C_m = 0, \tag{3.48}
\]

where

\[
\alpha_{m',m} = \sum_p \tilde{f}_p \frac{\tilde{f}_m(k_{x,p}) \Phi_m'( -k_{x,p} )}{k_{x,p}} - \delta_{m,m'} \frac{j d}{q_{z,m}} \frac{\varepsilon_h k_0^2 - k_y^2}{\varepsilon_y k_0^2 - k_y^2} \frac{r_m^+}{r_m^+} \tag{3.49}
\]

and

\[
\tilde{f}_p = \frac{1}{2} \frac{(-1)^p}{2} \cot(\theta_{z,p}) - \frac{1 + (-1)^p}{2} \tan(\theta_{z,p}) = \begin{cases} -\tan(\theta_{z,p}) & \text{p even} \\ \cot(\theta_{z,p}) & \text{p odd} \end{cases} \tag{3.50}
\]

is called the vertical spectral function [45]. Thus, the dispersion diagram is obtained by finding the set of \((k_0, k_{x,0}, k_y)\) that makes the determinant of a matrix with the elements \(\alpha_{m',m}\) zero.

It is also interesting to note that in case of glide symmetry, (3.27) and (3.28) are simplified to

\[
A_p^z = \frac{-1}{2 \sin(\theta_{z,p})} \sum_m r_m^- C_m \left[ 1 - (-1)^p \right] \Phi_m(k_{x,p}), \tag{3.51}
\]

\[
B_p^z = \frac{1}{2 \cos(\theta_{z,p})} \sum_m r_m^- C_m \left[ 1 + (-1)^p \right] \Phi_m(k_{x,p}). \tag{3.52}
\]

which means that for even/odd Floquet harmonics (when \(p\) is even/odd), \(A_p^z/B_p^z\) becomes zero and the transverse electric field along the \(z\)-direction, \(E_x^\text{Gap}(x = \text{constant}, z)\), is even/odd with respect to the plane \(z = 0\).
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Corrugated surface with a metallic top cover

In this part, we consider a corrugated surface with a metallic plate at distance g/2 above it. It means that the plate is located at z = 0 in Fig. 3.1. For this case, we can use the above formulation only with some modifications. First, since $E_{gap}^2(x, z = 0) = 0$, $B_{p}^{x} = 0$ in (3.8), which causes $D_{p}^{y} = 0$ in (3.9). In addition, in this case, $h_2 = 0$ and $s$ can be set to zero since the shift between the layers becomes undefined. Finally, since there is no upper corrugation, $C_{up}^m = 0$ and only the boundary conditions at $z = -g/2$ must be imposed. Thus, considering $r_{m, low} = r_m$ and $C_{low}^m = C_m$, the boundary equation (3.35) changes to

$$\frac{1}{d} \sum_p \int y \cos(\theta_{z,p}) \Phi_{m'}(-k_{x,p}) + \delta_{m,m'} \frac{\varepsilon_k k_0^2 - k_y^2}{\eta_0 k_0} r_m q_{z,m} = 0,$$

which yields (3.37) changes to

$$\sum_p \sum_m C_m \cot(\theta_{z,p}) \Phi_m(k_{x,p}) \Phi_{m'}(-k_{x,p})$$

$$- \delta_{m,m'} \frac{j d}{q_{z,m}} \varepsilon_k k_0^2 - k_y^2 r_m q_{z,m} C_m = 0.$$  

Note that the above equation can be expressed in the same form as (3.48) with the same $\alpha_{m', m}$ as (3.49), except that, in this case, $f_p = \cot(\theta_{z,p})$.

Results

The dispersion properties of three different double-layer corrugated surfaces (conventional corrugated surface with a metallic top cover, mirrored corrugated and glide-symmetric corrugated surface) are obtained using the mode matching method presented above. For all the structures, the dispersion relation is obtained for propagation in all directions. The results are presented in Paper 1 and compared with the results of the commercial software CST Microwave Studio. In addition, the dispersion diagrams of glide-symmetric corrugated surfaces with different values of geometrical parameters are obtained using the mode matching technique and the generalized Floquet theorem. The results are presented in section III.A of Paper 2 and compared with the CST results. By all of these case studies, we demonstrate that for analyzing strongly-coupled surfaces, including glide-symmetric ones, the proposed mode matching method is much faster and more efficient than the finite-method-based commercial software such as CST Microwave Studio.

3.4 Glide-symmetric holey metasurfaces

The general mode matching formulation to analyze glide-symmetric holey metasurfaces with arbitrary shape of the hole has been proposed in Paper 4. The
3.4. GLIDE-SYMMETRIC HOLEY METASURFACES

The unit cell of (a) a glide-symmetric holey structure and (b) its corresponding non-glide case with a metallic plane above it.

![Figure 3.2: The unit cell of (a) a glide-symmetric holey structure and (b) its corresponding non-glide case with a metallic plane above it.](image)

The generalized Floquet theorem has been used in the formulation to reduce the computational cost of the problem. Additionally, in Paper 4, the formulation has been verified for the case with circular holes as this type of hole is the most-used type in practical applications. Here, we only present a summary of the mode matching formulation, obtained in detail in the paper. However, the case with circular holes are explained in more detail here as it was explained briefly in the paper.

### Summary of mode matching formulation

In this section, we present a summary of the mode matching formulation for dispersion analysis of glide-symmetric holey metasurfaces (Fig. 3.2(a)) and its corresponding non-glide case with a metallic plane above it (Fig. 3.2(b)). The glide-symmetric structure is periodic along the $x$- and $y$-direction with a periodicity of $d$ and bounded along the $z$-direction. The gap between the two surfaces is denoted by $g$, the plane $z = 0$ is located in the middle of the gap, and the cross section of the hole can have any shape, for example square, circle, etc. The corresponding non-glide case contains the lower surface of the glide structure and a metallic plate at the distance of $g/2$ above it, where the plane $z = 0$ is located on the metallic plane. For both structures, the gap and the waveguides are filled with air.

As explained in the previous section and demonstrated in Paper 2 and Paper 3, using the generalized Floquet theorem in analyzing glide-symmetric structures simplifies the mode matching formulation such that it is enough to impose the boundary conditions only on one surface of the structure. It yields the same dispersion equation for the glide structure and its corresponding non-glide case with a metallic plane above it, taking into account the appropriate vertical spectral
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function \( \tilde{f} \) for each case.

Fields expression

Assuming and suppressing the time-dependency of \( e^{j\omega t} \), for both structures depicted in Fig. 3.2, the expression of tangential fields inside the waveguide can be written as

\[
E_{t}^{WG}(\rho, z = -g/2) = \sum_{m=1}^{M} r_{m}^{-} C_m \Phi_m(\rho),
\]

\[
H_{t}^{WG}(\rho, z = -g/2) = \sum_{m=1}^{M} r_{m}^{+} Y_m C_m [\hat{z} \times \Phi_m(\rho)],
\]

where \( M \) is the number of modal functions taken into account, \( C_m \) is the unknown coefficient of the \( m \)-th mode and \( \Phi_m \) and \( Y_m \) are the corresponding cross section modal function and wave admittance of the \( m \)-th mode. Note that, to keep the formulation simple, we do not distinguish between TE and TM modes and assume that the value of \( m \) specifies if the mode is TE or TM. In addition,

\[
r_{m}^{\pm} = 1 \pm \exp (-j2k_{zm}h) \tag{3.57}
\]

are the magnetic-field and electric-field reflection coefficients due to the short circuit at the end of the waveguide, where \( k_{zm} = \sqrt{k_0^2 - k^2_{tm}} \) is the longitudinal wave number and \( k_{zm} \) is the transversal wave number.

Due to the periodicity, the tangential electric field in the gap region can be expressed as a summation of Floquet harmonics for both structures:

\[
E_{t}^{Gap}(z) = \frac{1}{d^2} \sum_{pq} e^{-j(k_{x,p}x + k_{y,q}y)} \tilde{e}_{t,pq}^{Gap}(z),
\]

where \( k_{x,p} = k_{x,0} + 2\pi p/d, k_{y,q} = k_{y,0} + 2\pi p/d, \) and the amplitude of each Floquet harmonic can be written as

\[
\tilde{e}_{t,pq}^{Gap}(z) = \begin{pmatrix} A_{x,pq}^r \cr A_{y,pq}^r \end{pmatrix} \sin(k_{z,pq}z) + \begin{pmatrix} B_{x,pq}^r \cr B_{y,pq}^r \end{pmatrix} \cos(k_{z,pq}z),
\]

where \( k_{z,pq} = \sqrt{k_0^2 - k_{x,p}^2 - k_{y,q}^2} \) is the vertical wavenumber of the \((p,q)\)th harmonic. The tangential magnetic field in the gap region can be obtained from (3.58), using Maxwell equations [45].

Dispersion equation

Imposing the boundary equations, as explained in detail in Paper 4, a system of linear equations is obtained:

\[
\sum_{m=1}^{M} \alpha_{n,m} C_m = 0 \quad n = 1, ..., M
\]

(3.60)
where

\[
\alpha_{n,m} = \bar{\mu}_0 \eta_0 r^2 Y_m I_{nm} + \sum_{pq} \bar{f}_{pq}(k_{z,pq}) \beta_{n,m}(k_{pq}). \tag{3.61}
\]

Setting the determinant of the coefficient matrix in (3.60) to zero, the dispersion equation is obtained. In the following, we will explain about \(\beta_{n,m}(k_{pq})\), \(I_{nm}\), and \(\bar{f}_{pq}(k_{z,pq})\).

The parameter \(\beta_{n,m}(k_{pq})\), in (3.61) is expressed as

\[
\beta_{n,m}(k_{pq}) = \beta_{n,m}(k_{x,p}, k_{y,q}, k_{z,pq}) = k_0^2 - k_{y,q}^2 \tilde{\varphi}_m(k_{x,p}, k_{y,q}) \tilde{\varphi}_n(-k_{x,p}, -k_{y,q}) + k_{x,p} k_{y,q} \tilde{\varphi}_m(k_{x,p}, k_{y,q}) \tilde{\varphi}_n(-k_{x,p}, -k_{y,q}) + k_0^2 - k_{x,p}^2 \tilde{\varphi}_m(k_{x,p}, k_{y,q}) \tilde{\varphi}_n(-k_{x,p}, -k_{y,q}), \tag{3.62}
\]

where \(\tilde{\varphi}_m^x\) and \(\tilde{\varphi}_m^y\) are respectively the \(x\) and \(y\) component of the Fourier transform of the modal function \(\Phi_m\), which means

\[
\tilde{\Phi}_m(k_{x,p}, k_{y,q}) = \int_{S_{\text{hole}}} \Phi_m(\rho) e^{j(k_{x,p} \rho + k_{y,q} \rho)} d\rho = \hat{x} \tilde{\varphi}_m^x(k_{x,p}, k_{y,q}) + \hat{y} \tilde{\varphi}_m^y(k_{x,p}, k_{y,q}). \tag{3.63}
\]

Note that, to find \(\beta_{n,m}\) in (3.62), it is enough to calculate \(\tilde{\Phi}_m(k_{x,p}, k_{y,q})\) since \(\Phi_m(-k_{x,p}, -k_{y,q})\) is its complex conjugate. In addition, \(I_{nm}\) in (3.61) is the inner product of the \(n\)-th and \(m\)-th modal functions

\[
I_{nm} = \int_{S_{\text{hole}}} \Phi_n(\rho) \cdot \Phi_m(\rho) d\rho, \tag{3.64}
\]

which is non-zero only if \(m = n\), since the modes in lossless metallic waveguides are always orthogonal regardless of the shape of the cross section [72]. Finally, \(\bar{f}_{pq}(k_{z,pq})\) in (3.61) is the vertical spectral function and defined as

\[
\bar{f}_{pq} = \begin{cases} 
\frac{-\tan (k_{z,pq} g/2)}{\cot (k_{z,pq} g/2)} & p + q \text{ even} \\
\frac{\cot (k_{z,pq} g/2)}{\tan (k_{z,pq} g/2)} & p + q \text{ odd}
\end{cases} \tag{3.65}
\]

for the glide structure,

\[
\bar{f}_{pq}(k_{z,pq}) = \cot (k_{z,pq} g/2) \tag{3.66}
\]

for the non-glide structure, and \(\bar{f}_{pq}(k_{z,pq}) \to j\) for the case of a holey periodic surface in the air since in that case \(g \to +\infty\) and all \(k_{z,pq}\) are imaginary for bound
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(none-radiating) modes. Therefore, the presented formulation can be used for all holey periodic structures regardless of the shape of the hole, whether the structure possesses glide symmetry or is bounded by a metallic plate with a gap in between or is unbounded along the z-direction (a holey periodic structure in the air). Different shapes of the hole cross section can be handled by identifying the appropriate modal functions $\Phi_m$ and substituting them in the formulation. Glide-symmetric, non-glide with a metallic plate, or unbounded structure can be dealt with choosing the proper vertical spectral function $\tilde{f}_{pq}(k_z, \rho_0)$.

Numerical implementation for circular holes

In this section, the presented mode matching formulation is applied to a case with circular holes with the radius $a$. The equations are derived in detail for this case and the numerical considerations are explained.

Since we do not distinguish between the TE and TM modes in the formulation, to implement it numerically, we first arrange the propagating modes inside the holes based on their cut-off frequency in a look up table. In case of a circular waveguide, the modes are arranged as $\text{TE}_{11}$, $\text{TM}_{01}$, $\text{TE}_{21}$, $\text{TE}_{01}$, $\text{TM}_{11}$, and so on. Considering $N$ modes inside the holes means selecting the first $N$ modes of this table, such that $n = 1$ corresponds to $\text{TE}_{11}$, $n = 2$ corresponds to $\text{TM}_{01}$, and so on.

In addition, since in circular waveguides, there are two degenerate modes for the TE and TM modes that do not possess azimuthal symmetry, we assume two terms in the series in (3.55) for each mode. It means if we consider $N$ mode inside the hole, there will be $M = 2N$ terms in (3.55) such that $m = 2n - 1$ and $m = 2n$ correspond to the $n$-th mode. By this assumption, there will be all-zero rows and columns in the final coefficient matrix of the equations system (3.60) since there is only one mode associated with the modes with azimuthal symmetry such as $\text{TM}_{01}$ or $\text{TE}_{01}$. These all-zero rows and columns must be excluded from the coefficient matrix before calculating the determinant.

Now, for each $m$, we need to express the modal function $\Phi_m$ in (3.55) and its corresponding $Y_m$ in (3.56). The general expression for $\Phi_m$ in circular waveguides is $\Phi_m(\rho) = \hat{\rho}E_{\rho m} + \hat{\varphi}E_{\varphi m}$. For $m = 2n - 1$ and $m = 2n$ where $n$ corresponds to a TE mode (assume $\text{TE}_{ns}$), $Y_m = k_{zm}/(\eta_0 k_0)$ and we have [72]

\[
\begin{align*}
E_{\rho m} &= \frac{\rho}{\rho_0} J_r(k_{c,m}\rho) \sin(r\varphi) & \text{if } m = 2n - 1, \\
E_{\varphi m} &= k_{c,m} J'_r(k_{c,m}\rho) \cos(r\varphi)
\end{align*}
\]

\[
\begin{align*}
E_{\rho m} &= -\frac{\rho}{\rho_0} J_r(k_{c,m}\rho) \cos(r\varphi) & \text{if } m = 2n, \\
E_{\varphi m} &= k_{c,m} J'_r(k_{c,m}\rho) \sin(r\varphi)
\end{align*}
\]

where $k_{c,m} = x_{r,s}'/a$. However, for $m = 2n - 1$ and $m = 2n$ where $n$ corresponds
to a TM mode (assume TM\(_n\)), \(Y_m = k_0/\left(\eta_0 k_{zm}\right)\) and we have [72]

\[
\begin{cases}
E_{pm} = k_{c,m}J'_r(k_{c,m}\rho) \sin(r\varphi) \\
E_{\varphi m} = + \frac{\mu}{\nu} J_r(k_{c,m}\rho) \cos(r\varphi)
\end{cases}
\text{if } m = 2n - 1, \quad (3.69)
\]

\[
\begin{cases}
E_{pm} = k_{c,m}J'_r(k_{c,m}\rho) \cos(r\varphi) \\
E_{\varphi m} = - \frac{\mu}{\nu} J_r(k_{c,m}\rho) \sin(r\varphi)
\end{cases}
\text{if } m = 2n, \quad (3.70)
\]

where \(k_{c,m} = x_{r,s}/a\). In these expressions, \(J_r\) and \(J'_r\) are the Bessel function of the first kind and its first derivative; \(x_{r,s}\) and \(x'_{r,s}\) are the s-th roots of \(J_r\) and \(J'_r\).

Having the modal functions \(\Phi_m(\rho)\), the integrals in (3.63) and (3.64) must be calculated to obtain \(\Phi_m(x_{r,p},y_{q,q})\) and \(I_{nm}\). The integral in (3.63) becomes

\[
\Phi_m(x_{r,p},y_{q,q}) = \tilde{\phi}_m^x(x_{r,p},y_{q,q}) + \tilde{\phi}_m^y(x_{r,p},y_{q,q}) = \int_0^{2\pi} \int_0^{\rho_m} \left[ \hat{\rho} E_{\rho m} + \hat{\varphi} E_{\varphi m} e^{[k_{r,p}x + k_{y,q}y]} \rho \rho \varphi \right] d\rho d\varphi. \quad (3.71)
\]

Note that in the above integral, \(x\) and \(y\) should be replaced by \(\rho \cos(\varphi)\) and \(\rho \sin(\varphi)\), respectively. To simplify the integral calculation, we define \(k_{p,q} = \sqrt{k_{x,p}^2 + k_{y,q}^2}\) and \(\alpha = \tan^{-1}(k_{y,q}/k_{x,p})\) which changes the exponential part \(e^{[k_{x,p}x + k_{y,q}y]}\) into \(e^{jk_{p,q}p\cos(\varphi - \alpha)}\). Thus, we have

\[
\tilde{\phi}_m^x = \int_0^{2\pi} \int_0^{\rho_m} \left[ \cos(\varphi) E_{\rho m} - \sin(\varphi) E_{\varphi m} \right] e^{jk_{p,q}p\cos(\varphi - \alpha)} \rho d\varphi, \quad (3.72)
\]

\[
\tilde{\phi}_m^y = \int_0^{2\pi} \int_0^{\rho_m} \left[ \sin(\varphi) E_{\rho m} + \cos(\varphi) E_{\varphi m} \right] e^{jk_{p,q}p\cos(\varphi - \alpha)} \rho d\varphi. \quad (3.73)
\]

We will show that both \(\tilde{\phi}_m^x\) and \(\tilde{\phi}_m^y\) can be expressed in closed forms for four different cases: \(m\) is odd and corresponds to a TE mode, \(m\) is even and corresponds to a TM mode, \(m\) is odd and corresponds to a TE mode, \(m\) is even and corresponds to a TM mode.

Considering \(E_{\rho m}(\rho, \varphi) = f_{\rho m}(\rho)g_{\rho m}(\varphi)\) and \(E_{\varphi m}(\rho, \varphi) = f_{\varphi m}(\rho)g_{\varphi m}(\varphi)\), for an odd \(m\), \(g_{\rho m}(\varphi) = \sin(r\varphi)\) and \(g_{\varphi m}(\varphi) = \cos(r\varphi)\). Thus, we obtain

\[
\tilde{\phi}_m^x = \int_0^{\rho_m} \rho d\rho \int_0^{2\pi} \left[ \cos(\varphi) \sin(r\varphi)f_{\rho m}(\rho) - \sin(\varphi) \cos(r\varphi)f_{\varphi m}(\rho) \right] e^{jk_{p,q}p\cos(\varphi - \alpha)} d\varphi, \quad (3.74)
\]

\[
\tilde{\phi}_m^y = \int_0^{\rho_m} \rho d\rho \int_0^{2\pi} \left[ \sin(\varphi) \sin(r\varphi)f_{\rho m}(\rho) + \cos(\varphi) \cos(r\varphi)f_{\varphi m}(\rho) \right] e^{jk_{p,q}p\cos(\varphi - \alpha)} d\varphi. \quad (3.75)
\]
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However, if \( m \) is even, \( g_{\rho m}(\varphi) = \cos(r\varphi) \) and \( g_{\varphi m}(\varphi) = \sin(r\varphi) \), which yield

\[
\tilde{\phi}_m^x = \int_0^a \rho \, d\rho \int_0^{2\pi} [\cos(\varphi) \cos(r\varphi) f_{\rho m}(\rho) - \sin(\varphi) \sin(r\varphi) f_{\varphi m}(\rho)] e^{j k_{\rho,p,q} \rho \cos(\varphi - \alpha)} \, d\varphi, 
\]

(3.76)

\[
\tilde{\phi}_m^y = \int_0^a \rho \, d\rho \int_0^{2\pi} [\sin(\varphi) \cos(r\varphi) f_{\rho m}(\rho) + \cos(\varphi) \sin(r\varphi) f_{\varphi m}(\rho)] e^{j k_{\rho,p,q} \rho \cos(\varphi - \alpha)} \, d\varphi. 
\]

(3.77)

Since the product of two trigonometric functions can be expressed as the sum of two other trigonometric functions, to calculate the above integrals with respect to \( \varphi \), we need to find

\[
\int_0^{2\pi} \sin(\gamma \varphi) e^{j k_{\rho,p,q} \rho \cos(\varphi - \alpha)} \, d\varphi, 
\]

(3.78)

\[
\int_0^{2\pi} \cos(\gamma \varphi) e^{j k_{\rho,p,q} \rho \cos(\varphi - \alpha)} \, d\varphi, 
\]

(3.79)

where \( \gamma \) is an integer. In other words, we need to calculate

\[
I = \int_0^{2\pi} e^{\pm j(\gamma \varphi)} e^{j k_{\rho,p,q} \rho \cos(\varphi - \alpha)} \, d\varphi. 
\]

(3.80)

Since the integrand in the above integral is periodic with the period of \( 2\pi \), we can rewrite it by changing the integral variable as \( \varphi' = \varphi - \alpha \), but keeping the same integral limits. Finally, using the integral expression of the Bessel function of the first kind [73], we obtain

\[
I = e^{\pm j(\gamma \alpha)} \int_0^{2\pi} e^{\pm j(\gamma \varphi')} e^{j k_{\rho,p,q} \rho \cos(\varphi')} \, d\varphi' = e^{\pm j(\gamma \alpha)} (2\pi)^1 J_\gamma(k_{\rho,p,q} \rho). 
\]

(3.81)

With the above discussion and having the solution of the integral \( I \), the result of the integrals in (3.72) and (3.73) with respect to \( \varphi \) can be written after some simple calculations for both odd and even values of \( m \). The final results for an odd \( m \) are

\[
\tilde{\phi}_m^x = \pi (j)^{r-1} \sin((r-1)\alpha) \int_0^a [f_{\rho m}(\rho) + f_{\varphi m}(\rho)] J_{r-1}(k_{\rho,p,q} \rho) \, d\rho \\
+ \pi (j)^{r+1} \sin((r+1)\alpha) \int_0^a [f_{\rho m}(\rho) - f_{\varphi m}(\rho)] J_{r+1}(k_{\rho,p,q} \rho) \, d\rho, 
\]

(3.82)

\[
\tilde{\phi}_m^y = \pi (j)^{r-1} \cos((r-1)\alpha) \int_0^a [f_{\rho m}(\rho) + f_{\varphi m}(\rho)] J_{r-1}(k_{\rho,p,q} \rho) \, d\rho \\
+ \pi (j)^{r+1} \cos((r+1)\alpha) \int_0^a [-f_{\rho m}(\rho) + f_{\varphi m}(\rho)] J_{r+1}(k_{\rho,p,q} \rho) \, d\rho; 
\]

(3.83)
and for an even \( m \) are

\[
\tilde{\phi}_m^x = \pi (j)^{r-1} \cos((r - 1)\alpha) \int_0^\alpha [f_{\rho m}(\rho) - f_{\rho m}(\rho)] J_{r-1}(k_{p, pq \rho}) d\rho \\
+ \pi (j)^{r+1} \cos((r + 1)\alpha) \int_0^\alpha [f_{\rho m}(\rho) + f_{\rho m}(\rho)] J_{r+1}(k_{p, pq \rho}) d\rho, \tag{3.84}
\]

\[
\tilde{\phi}_m^y = \pi (j)^{r-1} \sin((r - 1)\alpha) \int_0^\alpha [-f_{\rho m}(\rho) + f_{\rho m}(\rho)] J_{r-1}(k_{p, pq \rho}) d\rho \\
+ \pi (j)^{r+1} \sin((r + 1)\alpha) \int_0^\alpha [f_{\rho m}(\rho) + f_{\rho m}(\rho)] J_{r+1}(k_{p, pq \rho}) d\rho. \tag{3.85}
\]

Considering the field expressions in (3.67)-(3.70), we find that, to calculate the above integrals, it is enough to find the results of

\[
\int_0^\alpha \rho k_{c, m} J_r'(k_{c, m} \rho) J_{r-1}(k_{p, pq \rho}) d\rho =
\]

\[
\int_0^\alpha [\rho k_{c, m} J_{r-1}(k_{c, m} \rho) - r J_r(k_{c, m} \rho)] J_{r-1}(k_{p, pq \rho}) d\rho, \tag{3.86}
\]

\[
\int_0^\alpha \rho k_{c, m} J_r'(k_{c, m} \rho) J_{r+1}(k_{p, pq \rho}) d\rho =
\]

\[
\int_0^\alpha [r J_r(k_{c, m} \rho) - \rho k_{c, m} J_{r+1}(k_{c, m} \rho)] J_{r+1}(k_{p, pq \rho}) d\rho, \tag{3.87}
\]

and

\[
\int_0^\alpha \frac{r}{\rho} J_r(k_{c, m} \rho) J_{r+1}(k_{p, pq \rho}) d\rho = r \int_0^\alpha J_r(k_{c, m} \rho) J_{r+1}(k_{p, pq \rho}) d\rho. \tag{3.88}
\]

However, substituting the proper \( f_{\rho m}(\rho) \) and \( f_{\rho m}(\rho) \) in \( \tilde{\phi}_m^x \) and \( \tilde{\phi}_m^y \) expressions and using (3.86)-(3.88), we see that all the integrals in the form of (3.88) cancel each other and we obtain

\[
\tilde{\phi}_m^x = \pi (j)^{r-1} \sin((r - 1)\alpha) I_{r-1} + \pi (j)^{r+1} \sin((r + 1)\alpha) I_{r+1}, \tag{3.89}
\]

\[
\tilde{\phi}_m^y = \pi (j)^{r-1} \cos((r - 1)\alpha) I_{r-1} - \pi (j)^{r+1} \cos((r + 1)\alpha) I_{r+1}. \tag{3.90}
\]

if \( m \) is odd and corresponds to a TE mode;

\[
\tilde{\phi}_m^x = \pi (j)^{r-1} \sin((r - 1)\alpha) I_{r-1} - \pi (j)^{r+1} \sin((r + 1)\alpha) I_{r+1}, \tag{3.91}
\]

\[
\tilde{\phi}_m^y = \pi (j)^{r-1} \cos((r - 1)\alpha) I_{r-1} + \pi (j)^{r+1} \cos((r + 1)\alpha) I_{r+1}. \tag{3.92}
\]

if \( m \) is odd and corresponds to a TM mode;

\[
\tilde{\phi}_m^x = -\pi (j)^{r-1} \cos((r - 1)\alpha) I_{r-1} - \pi (j)^{r+1} \cos((r + 1)\alpha) I_{r+1}, \tag{3.93}
\]

\[
\tilde{\phi}_m^y = \pi (j)^{r-1} \sin((r - 1)\alpha) I_{r-1} - \pi (j)^{r+1} \sin((r + 1)\alpha) I_{r+1}. \tag{3.94}
\]
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if \( m \) is even and corresponds to a TE mode;

\[
\tilde{\phi}_x^m = \pi (j)^{r-1} \cos((r-1)\alpha)I_{r-1} - \pi (j)^{r+1} \cos((r+1)\alpha)I_{r+1},
\]

(3.95)

\[
\tilde{\phi}_y^m = \pi (j)^{r-1} \sin((r-1)\alpha)I_{r-1} - \pi (j)^{r+1} \sin((r+1)\alpha)I_{r+1},
\]

(3.96)

if \( m \) is even and corresponds to a TM mode. In these equations,

\[
I_{r-1} = \int_0^a \rho k_{c,m} J_{r-1}(k_{c,m}\rho) J_{r-1}(k_{p,pq}\rho) d\rho,
\]

(3.97)

\[
I_{r+1} = \int_0^a \rho k_{c,m} J_{r+1}(k_{c,m}\rho) J_{r+1}(k_{p,pq}\rho) d\rho.
\]

(3.98)

The solution of these integrals can be expressed in a closed form [74]:

\[
I_{r-1} = k_{c,m} \frac{k_{c,m} J_{r-1}(k_{c,m}\rho) - k_{p,pq} J_{r-1}(k_{c,m}\rho)}{k_{c,m}^2 - k_{p,pq}^2},
\]

(3.99)

\[
I_{r+1} = k_{c,m} \frac{k_{p,pq} J_{r+1}(k_{c,m}\rho) - k_{c,m} J_{r+1}(k_{c,m}\rho)}{k_{c,m}^2 - k_{p,pq}^2}.
\]

(3.100)

Thus, as mentioned in the beginning of this discussion, both \( \tilde{\phi}_x^m \) and \( \tilde{\phi}_y^m \) are obtained in closed forms to be substituted in (3.62).

Now, to find \( I_{mm} \) for the circular hole with the radius \( a \), (3.64) is rewritten as

\[
I_{mm} = \int_0^{2\pi} \int_0^a [E_{\rho m}(\rho, \varphi) + E_{\varphi m}(\rho, \varphi)] \rho d\rho d\varphi
\]

\[
= \int_0^{2\pi} \int_0^a [f_{\rho m}(\rho) g_{\varphi m}(\varphi) + f_{\varphi m}(\rho) g_{\rho m}(\varphi)] \rho d\rho d\varphi
\]

\[
= \int_0^{2\pi} g_{\rho m}(\varphi) d\varphi \int_0^a f_{\rho m}(\rho) d\rho + \int_0^{2\pi} g_{\varphi m}(\varphi) d\varphi \int_0^a f_{\varphi m}(\rho) d\rho.
\]

(3.101)

Taking look at the field expressions in (3.67)-(3.70), it is clear that \( I_{mm} \) is the same for \( m = 2n - 1 \) and \( m = 2n \), where \( n \) corresponds to a mode without azimuthal symmetry (a mode with \( r \neq 0 \)). In addition, the results of integrals with respect to \( \varphi \) in (3.101) are well-known:

\[
\int_0^{2\pi} \sin^2(r\varphi) d\varphi = \begin{cases} 0 & r = 0 \\ \pi & r \neq 0 \end{cases},
\]

(3.102)

\[
\int_0^{2\pi} \cos^2(r\varphi) d\varphi = \begin{cases} 2\pi & r = 0 \\ \pi & r \neq 0 \end{cases}.
\]

(3.103)
Therefore, we find $I_{mm}$ first for those $m$ that correspond to TE$_{0s}$ or TM$_{0s}$ modes; and then for those $m$ that correspond to TE$_{rs}$ or TM$_{rs}$ modes with $r \neq 0$.

For an $m$ that corresponds to a TE$_{0s}$ mode, $I_{mm} = 0$ when $m$ is even; and when $m$ is odd,

$$I_{mm} = 2\pi \int_0^a (k_{c,m} J'_0(k_{c,m}\rho))^2 \rho d\rho.$$  \hspace{1cm} (3.104)

However for an $m$ that corresponds to a TM$_{0s}$ mode, $I_{mm} = 0$ when $m$ is odd; and when $m$ is even,

$$I_{mm} = 2\pi \int_0^a (k_{c,m} J'_0(k_{c,m}\rho))^2 \rho d\rho,$$  \hspace{1cm} (3.105)

which is exactly same as (3.104). Defining $u = k_{c,m}\rho$ and using the recurrence formulas for Bessel functions yield

$$I_{mm} = 2\pi \int_0^{k_{c,m}a} u(J'_0(u))^2 du = 2\pi \int_0^{k_{c,m}a} uJ_1^2(u) du.$$  \hspace{1cm} (3.106)

From the integral tables for Bessel functions [74], one finds

$$\int uJ_1^2(u) du = u^2 \left[ J_0^2(u) + J_1^2(u) \right] - ruJ_{r-1}(u)J_r(u)$$  \hspace{1cm} (3.107)

$$= u^2 \left[ J_{r+1}(u) + J_0^2(u) \right] - ruJ_{r+1}(u)J_r(u).$$  \hspace{1cm} (3.108)

Thus,

$$I_{mm} = 2\pi \left[ \frac{u^2}{2} (J_0^2(u) + J_1^2(u)) - uJ_0(u)J_1(u) \right]^{k_{c,m}a}_0$$

$$= \pi k_{c,m}^2 a^2 \left[ J_0^2(k_{c,m}a) + J_1^2(k_{c,m}a) \right] - 2\pi k_{c,m}a J_0(k_{c,m}a)J_1(k_{c,m}a)$$

$$= \begin{cases} 
\pi k_{c,m}^2 a^2 J_0^2(k_{c,m}a) & \text{m is odd and corresponds to TE}_{0s} \\
\pi k_{c,m}^2 a^2 J_1^2(k_{c,m}a) & \text{m is even and corresponds to TM}_{0s} 
\end{cases},$$  \hspace{1cm} (3.109)

since $J_1(k_{c,m}a) = 0$ for TE$_{0s}$ modes and $J_0(k_{c,m}a) = 0$ for TM$_{0s}$ modes.

Now, let find $I_{mm}$ for those $m$ (even or odd) that correspond to a TE$_{rs}$ or TM$_{rs}$ mode with $r \neq 0$. For these cases,

$$I_{mm} = \pi \int_0^a \left[ f_{pm}^2(\rho) + f_{rm}^2(\rho) \right] \rho d\rho$$

$$= \pi \int_0^a \left[ \left( \frac{r}{\rho} J_r(k_{c,m}\rho) \right)^2 + (k_{c,m}J'_r(k_{c,m}\rho))^2 \right] \rho d\rho.$$  \hspace{1cm} (3.110)
CHAPTER 3. ANALYSIS OF GLIDE-SYMMETRIC PERIODIC STRUCTURES

Again, defining \( u = k_{c,m} \rho \) and using the recurrence formulas for Bessel functions yield

\[
I_{mm} = \pi \int_0^{k_{c,m}a} \left[ r^2 J_r^2(u) + (uJ_r'(u))^2 \right] \frac{1}{u} \, du
\]

\[
= \pi \int_0^{k_{c,m}a} \left[ r^2 J_r^2(u) + (uJ_{r-1}(u) - rJ_r(u))^2 \right] \frac{1}{u} \, du
\]

\[
= 2\pi \int_0^{k_{c,m}a} J_r(u) \left[ \frac{r}{u} J_r(u) - J_{r-1}(u) \right] \, du + \pi \int_0^{k_{c,m}a} u J_{r-1}^2(u) \, du
\]

\[
= 2\pi \int_0^{k_{c,m}a} J_r(u) \left[ -J_r'(u) \right] \, du + \pi \int_0^{k_{c,m}a} u J_{r-1}^2(u) \, du.
\]  \hspace{1cm} (3.111)

Using (3.108) to express the result of \( \int u J_{r-1}^2(u) \, du \) in (3.111), a closed form solution for \( I_{mm} \) in (3.111) is obtained:

\[
I_{mm} = \left[ -r \pi J_r^2(u) + \frac{\pi u^2}{2} \left[ J_r^2(u) + J_{r-1}^2(u) \right] - \pi (r - 1) u J_r(u) J_{r-1}(u) \right]_{u=0}^{k_{c,m}a}
\]

\[
= \frac{\pi k_{c,m}^2 a^2}{2} \left[ J_r^2(k_{c,m}a) + J_{r-1}^2(k_{c,m}a) \right]
\]

\[
- r \pi J_r^2(k_{c,m}a) - \pi (r - 1) k_{c,m} a J_r(k_{c,m}a) J_{r-1}(k_{c,m}a).
\]  \hspace{1cm} (3.112)

Note that in case of TM\(_{rs}\) modes, since \( J_r(k_{c,m}a) = 0 \), \( I_{mm} \) is simplified to

\[
I_{mm} = \frac{\pi k_{c,m}^2 a^2}{2} J_{r-1}^2(k_{c,m}a).
\]  \hspace{1cm} (3.113)

As demonstrated, for the case with circular holes, the results of these integrals are found in closed forms. However, for an arbitrary cross section, numerical integration methods might be used to calculate the integrals in (3.63) and (3.64) [75].

Results

The dispersion diagrams of glide-symmetric holey surfaces with rectangular and circular holes are obtained for different values of geometrical and physical parameters using the mode matching technique and the generalized Floquet theorem. The results for the cases with rectangular holes are presented in section III.B of Paper 2 and the results for the cases with circular holes are presented in Paper 3. For all cases, the obtained dispersion diagrams are in a decent agreement with the CST results. However, the proposed method is more efficient and provides the result in a much shorter time. This is due to using the mode matching method and the generalized Floquet theorem which reduces the computational domain to half of the unit cell. In addition, the proposed mode matching method provides a physical insight into the special symmetry of the waves propagating in glide-symmetric structures, which is explained in both Paper 2 and Paper 3.
Chapter 4

Twist and polar glide symmetries

The definition of twist symmetry and the generalized Floquet theorem have been presented in Chapter 2. In this chapter, the generalized Floquet theorem is used to explain the effect on the dispersion properties when adding twist symmetry to a periodic structure. In addition, the definition of polar glide symmetry and its influence on the propagation characteristics of periodic structures are explained. The potential of designing reconfigurable filters by the combination of twist and polar glide symmetry, demonstrated in Paper 4, is briefly discussed. Finally, mimicking the twist symmetry effect in flat structures, discussed in detail in Paper 5, is presented.

4.1 Twist symmetry effect

In Chapter 2, by applying the generalized Floquet theorem, the effect of glide symmetry on the dispersion characteristics of periodic structures is discussed. Here, in the similar manner, I explain the impact of twist symmetry by comparing the dispersion diagrams of two coaxial lines loaded with periodic holes on their inner conductors: one with a single hole in the unit cell and the other one with 3-fold twist-symmetric holes (see Fig. 4.1(a)). For both structures, the same periodicity ($p = 12$ mm), and the same length ($\ell = 2.4$ mm) and opening angle ($180^\circ$) for the holes are assumed. In addition, the radius of their inner conductors, denoted by $d$, are the same and equal to 1.5 mm. The gap between the inner and outer conductors is 0.1 mm in both cases.

Based on the generalized Floquet theorem, the $k_T(\omega)p$ diagram of the 3-fold twist-symmetric structure can be obtained from the $k_{S_3}(\omega)p$ diagram, where $S_3$ is the 3-fold twist operator. Indeed, the $k_T(\omega)p$ diagram is the composition of the $k_{S_3}(\omega)p$ diagram and its first and second space harmonic branches, obtained by $2\pi$ and $4\pi$ shift along $k_{S_3}(\omega)p$ axis. Note that the $k_{S_3}(\omega)p$ diagram is periodic with periodicity $3 \times 2\pi = 6\pi$.

The dispersion diagrams of the structures shown in Fig. 4.1(a) are obtained
Figure 4.1: (a) Simulated structures: a coaxial line with single hole and 3-fold twist-symmetric holes on their inner conductors. (b) Their dispersion diagrams over the irreducible Brillouin zone obtained using CST (red lines are for the non-twist case and blue lines are for the twist case). These results demonstrate that the stop-bands between the first and second
modes and the second and third modes are absent in the 3-fold twist-symmetric structure. In addition, the first and second modes of the twist structure are almost dispersion-free. These results can be justified by comparing the $k_T(\omega)p$ diagram for the non-twist structure (depicted in Fig. 4.1(c)) and the $k_{S_3}(\omega)p$ diagram for the twist case and its first and second space harmonics, obtained by $2\pi$ and $4\pi$ shift along $k_{S_3}p$ axis (all shown in Fig. 4.1(d)). As we explained, $k_T(\omega)p$ diagram for the twist-symmetric structure is the composition of $k_{S_3}(\omega)p$ diagram and its first and second space harmonics (the composition of the solid line, dashed line, and dot line in Fig. 4.1(d)). Therefore, the dispersion diagrams of these structures over the irreducible Brillouin zone (the zone between the black dashed lines in Fig. 4.1(c) and Fig. 4.1(d)) are those shown in Fig. 4.1(b). The plots in Fig. 4.1(d) clearly demonstrate that in 3-fold twist-symmetric structures the first and second dominant modes are almost dispersion-free and there is no stop-band between the first and second modes and second and third modes of the structure.

In Paper 4, the dispersion diagrams of coaxial lines with two-fold and four-fold twist-symmetric holes on their inner conductors are presented and compared with the non-twist structure shown in Fig. 4.1(a). It has been demonstrated that a higher value of equivalent refractive index can be realized by applying a twist symmetry and increasing the degree of the twist symmetry. This is because, by increasing the degree of the twist symmetry, the structure becomes denser. In addition, the comparison results demonstrate that the location and the width of stop-bands can be controlled by the degree of twist symmetry. This phenomenon, as explained above, is justified by using the generalized Floquet theorem.

Moreover, the dispersion diagrams of two coaxial lines with four aligned and four-fold twist-symmetric holes on their inner conductors are compared in Paper 4. In both cases, the holes are located with a distance of $p/4$ from each other along the coaxial axis. The results demonstrate that an $m$-fold twist-symmetric structure with the periodicity $p$ shows different dispersion properties from a corresponding non-twist case with the periodicity $p/m$. Indeed, an $m$-fold twist-symmetric structure has a higher value of equivalent refractive index.

To experimentally verify the twist symmetry effect, a prototype of a coaxial transmission line with four-fold twist-symmetric holes on its inner conductor has been manufactured and its dispersion diagram has been obtained. The prototype consists of three unit cells and the matching sections. To find the dispersion diagram, the scattering parameter $S_{21}$ is measured for the whole prototype and for the case that the matching sections are connected to each other. The difference between the phase of $S_{21}$ in these two cases is equal to the phase shift along the structure, which is equal to the propagation constant times the length of the structure. The final result presented in Paper 4 shows an excellent agreement with the simulation result.
4.2 Polar glide symmetry

Polar glide symmetry, as its name reveals, is the glide symmetry in a polar (or cylindrical) coordinate system. As we explained in Chapter 2, glide symmetry consists of a translation and a reflection with respect to a plane, which is a flat surface. More precisely, we can call it Cartesian glide symmetry (Fig. 4.2(a)). On the other hand, if the reflection is performed in a cylindrical surface with a circular cross section, a polar glide symmetry can be achieved (Fig. 4.2(b)). Similar to twist symmetry, polar glide symmetry can be only applied to the structures that match with a cylindrical coordinate system such as coaxial lines.

Polar glide symmetry was first introduced in [21] and applied to a coaxial line loaded with pins. In this work, however, instead of applying a true reflection with respect to a cylinder, a structure was considered as a polar glide-symmetric structure if its flat approximation possessed Cartesian glide symmetry [21]. It means an identical length is considered for the pins protruding from the inner and outer conductors of the coaxial line. Therefore, the characteristics of glide symmetry was not seen in the dispersion diagrams of the polar glide-symmetric structure considered in [21].

However, in Paper 4, we apply a true polar glide symmetry to a coaxial line loaded with holes. We demonstrate that the polar glide symmetry shows the same properties as Cartesian-glide symmetry provided the gap between the inner and outer conductors is small enough such that the electric field does not have a tangential component in the gap between the conductors. A similar approach has been followed in transformation optics [76]. Same as Cartesian glide-symmetric structures, polar glide-symmetric structures do not have a stop-band between the first and second modes and the first mode shows a low frequency dispersion.
4.3 FLAT STRUCTURES WITH MIMICKED TWIST SYMMETRY

Combining twist and polar glide symmetries
In addition, in Paper 4, we demonstrate that applying both twist and polar-glide symmetries to a coaxial line loaded with holes provides the possibility of designing reconfigurable filters. For this purpose, we first prove that a case with 2-fold twist symmetric holes on its inner conductor and 2-fold twist symmetric holes on its outer conductor such that it has a polar glide symmetry in its sub-unit cell mimics the behavior of a case with 4-fold twist-symmetric holes on its inner conductor. This means that there is no stop-band between the first and second modes, the second and third modes, and the third and fourth modes. Now, if the inner conductor rotates such that the polar glide symmetry within the sub-unit cell breaks, the mimicked four-fold twist symmetry breaks and the structure will only have the 2-fold twist symmetry. As a result, there will be a stop-band between its second and third modes. Therefore, by only rotating the inner conductor, one can create or remove the stop-band between the second and third modes. Thus, reconfigurable filters can be designed by combining twist and polar glide symmetries. A prototype of a reconfigurable filter based on this idea was designed, manufactured and measured. A more detailed discussion and the results are presented in Paper 4.

4.3 Flat structures with mimicked twist symmetry
Twist symmetry is only applicable to structures compatible with a cylindrical coordinate system. However, the manufacturing of these structures may be difficult and expensive and they cannot be integrated with components in flat technologies. Therefore, the possibility of mimicking the twist symmetry effect in flat structures is investigated and presented in Paper 5. For this purpose, a holey metallic layer with a metallic plate above is assumed as a flat realization of a coaxial line with holes in its inner conductor. To mimic a situation similar to the real m-fold twist-symmetric holes, the holes in the flat structure are placed in a zigzag pattern, having m holes with a separation p/m from each other in one unit cell.

Assuming that m is even, these holey flat structures with mimicked m-fold twist-symmetric holes possess glide symmetry. Thus, they have a linear dispersion relation for their first modes and there is no stop-band between their first and second modes. But, what is the effect of the degree m on their dispersion properties? It is demonstrated in Paper 5 that, same as coaxial lines with twist-symmetric holes on their inner conductors, an optically denser material can be realized by increasing the degree of twist symmetry applied to a holey flat structure. In addition, a comparison between two of these structures with the period p, having four aligned and mimicked four-fold twist symmetric holes, separated p/4 in both case, demonstrates that the twist-symmetric case is not equivalent to the non-twist case with four aligned holes. This result can be generalized to m-fold twist-symmetric structures. This means that an m-fold twist-symmetric structure with the periodicity p is not equivalent to its non-twist counterpart with the reduced periodicity of p/m.
Chapter 5

Conclusion, future lines, and discussion on sustainability

5.1 Conclusion

In this thesis, the definition of glide and twist symmetry, as particular cases of higher symmetry, is presented. In addition, the rigorous definition of polar glide symmetry, which corresponds to the glide symmetry in a cylindrical coordinate, is proposed. The characteristics of the waves propagating in periodic structures possessing these higher symmetries are discussed. It is demonstrated that applying a higher symmetry to a periodic structure provides an additional degree of freedom to control its dispersion properties. Moreover, it is explained that due to the special dispersion characteristics of higher-symmetric periodic structures, they are considered as great candidates for realizing fully-metallic wideband flat lens antennas, leaky wave antennas, and low-loss guiding structures at millimeter waves. They can be also used in designing reconfigurable filters.

The main characteristics of higher-symmetric periodic structures are the linear dispersion relation of their first mode and the absence of the stop-band between their first and second modes. Indeed, their first and second modes are degenerate with non-zero group velocity at the edge of the Brillouin zone. The generalized Floquet theorem, explained in Chapter 2, is used to explain these dispersion behaviors of higher-symmetric periodic structures. In addition, it is demonstrated that the width and the position of stop-bands in the dispersion diagram of a periodic structure can be engineered by applying a higher symmetry or changing the degree of the twist symmetry.

In higher-symmetric periodic structures that are composed of two layers, a strong coupling between the layers that excites the higher order modes is needed to obtain the aforementioned desirable dispersion characteristics (linear dispersion relation of the first mode). This strong coupling can be achieved with a very small gap between the layers. In addition, in the case of holey glide-symmetric structures,
overlapping between the holes in different layers intensifies the coupling between the layers and has a significant contribution in exciting higher order modes. As explained in Chapter 3, such glide-symmetric periodic structures with strong coupling between the layers cannot be reduced to the conventional periodic structures with a reduced periodicity. In addition, due to the existence of the higher order modes, they cannot be modeled with the well-known transverse resonance method and it is not possible to derive a simple circuit model for them. On the other hand, the small gap between the layers causes a long simulation time for analyzing these structures with commercial software such as CST Microwave Studio.

Therefore, in Chapter 3, a mode matching formulation was derived for analyzing glide-symmetric corrugated and holey surfaces with an arbitrary shape of the holes. The generalized Floquet theorem was applied in the formulation to reduce the computational domain to half of the unit cell. Compared to the commercial software using finite-method algorithms such as CST Microwave Studio, the proposed method is faster and more efficient. Thus, it can be used to design the unit cell of all-metal glide-symmetric holey metasurfaces that can be employed in low-cost and low-loss graded-index planar lenses and gap waveguide technology. In addition, this method provides a physical insight toward the symmetry of the Floquet modes propagating in these glide-symmetric structures.

The twist symmetry effect was explained in Chapter 4 using the generalized Floquet theorem. It was demonstrated that a precise control over the propagation characteristics of periodic structures compatible with a cylindrical coordinate system can be achieved by applying a twist symmetry and changing its degree. In addition, the possibility of designing reconfigurable filters by combining twist symmetry and polar glide symmetry was demonstrated in Chapter 4.

Furthermore, in Chapter 4, the possibility of mimicking the twist symmetry effect in flat holey structures was discussed. Note that the real twist symmetry cannot be applied to flat structures and technologies. However, it was demonstrated that a holey flat structure possessing a mimicked twist symmetry shows a similar dispersion behavior as structures possessing a real twist symmetry. In other words, the number and the configuration of holes have a significant effect on the dispersion properties and equivalent refractive index of these flat structures. Same as structures with a real $m$-fold twist symmetry, in a flat structure with a mimicked $m$-fold twist symmetry, an optically denser material is achieved by increasing $m$. In addition, these flat structures are not equivalent to their non-twist counterpart with $m$ aligned holes with a distance of $p/m$.

5.2 Future lines

There are several options for the continuation of this work:

- As explained in Chapter 3, to find the dispersion diagrams, a set of $(k_0, k_x, k_y)$ that makes the determinant of the coefficient matrix zero must be found. The dispersion diagrams presented in Papers 1-3 are obtained simply by
5.3 Discussion on the sustainability of higher-symmetric periodic structures

As explained in this thesis, higher-symmetric structures are great candidates for designing ultra-wideband antennas and low-loss microwave components at millimeter waves, which is the frequency band for the future telecommunications systems. These components are the heart of telecommunications systems, and they have a significant impact on the overall efficiency and sustainability of the system. In this section, the effect of employing higher-symmetric structures in antennas and microwave components on the efficiency and sustainability of telecommunications systems is discussed.

By increasing the demand from the telecommunications industry, its power consumption is continuously increasing [81]. This increase is mainly due to the high data traffic and the high requirement for the network hardware capabilities. Therefore, to have a sustainable world, it is of great importance to take into account the
efficiency and sustainability of telecommunications systems. For this purpose, the sustainability of these systems can be investigated in three levels: the sources that provide the power for the system, the efficiency in the system level, and the efficiency in the component level. Among these factors, higher-symmetric structures have an impact on the efficiency of the system in the component level.

**Sustainability of power sources**—Fortunately, many efforts have been taken to incorporate different sustainable sources for electrical energy production. In addition, solar cells are explored and exploited for direct integration on the base station sites. Other sustainable sources can be also investigated to be applied to the system in the same way.

**Sustainability in the system level**—The main source of power consumption in the system level is the system resource planning. Therefore, the efficiency can be increased by implementing smart algorithms to control the power consumption of the system. Currently, many researchers in the telecommunication area work on these algorithms.

**Sustainability in the component level**—It has been discussed in [82] that power amplifiers and antennas including their microwave and radio frequency (RF) interconnecting parts consume the largest portion of the power in the component level. Therefore, one critical factor to ensure sustainability in this level is designing efficient antennas and microwave/RF components. Higher-symmetric metallic structures can be used as artificial dielectrics at millimeter waves where dielectric materials are very lossy. Therefore fully-metallic antennas based on these structures are low loss. Indeed, the only source of losses are the metals which is almost negligible [83]. Compared to the phased array antennas, higher-symmetric based lens antennas cost less and have much lower loss at millimeter waves. Thus, they are an efficient alternative for phased array antennas for these frequencies. Moreover, higher-symmetric holey metallic structures are demonstrated to be low cost and low loss EBG structures that can be used in microwave components and flanges at millimeter waves [39–41].
Bibliography


