Supplementary Material for “Bobrovsky-Zakai Bound for Filtering, Prediction and Smoothing of Nonlinear Dynamic Systems”

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1st June 2018

Report no.: LiTH-ISY-R-3105

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Abstract
This report contains supplementary material for the paper [1], and gives detailed proofs of all lemmas and theorems that could not be included into the paper due to space limitations. The notation is adapted from the paper.

Keywords: performance bounds, nonlinear dynamic systems, mean square error
Supplementary Material for “Bobrovsky-Zakai Bound for Filtering, Prediction and Smoothing of Discrete-Time Nonlinear Dynamic Systems”

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1 Recursive Computation of Filtering Bound

Lemma 1. The following identities hold ($0 < \ell < k$):

\[
L(Y_k; X_k + h_i, X_k) = M_0(x_1; x_0 + h_0^{(i)}, x_0),
\]
\[
L(Y_k; X_k + h_{n,\ell+i}, X_k) = K_\ell(x_{\ell+1}, y_\ell; x_\ell + h_\ell^{(i)}, x_\ell; x_{\ell-1}),
\]
\[
L(Y_k; X_k + h_{n,k+i}, X_k) = L_k(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}).
\]

Proof. We can simplify each ratio of joint densities as follows:

\[
L(Y_k; X_k + h_i, X_k) = \frac{p(Y_k, X_k + h_i)}{p(Y_k, X_k)} = \frac{p(y_1|x_1) \prod_{j=2}^{k} p(y_j|x_j)p(x_j|x_{j-1}) p(x_1|x_0 + h_0^{(i)})p(x_0 + h_0^{(i)})}{p(y_1|x_1) \prod_{j=2}^{k} p(y_j|x_j)p(x_j|x_{j-1}) p(x_1|x_0)p(x_0)} = \frac{p(x_1|x_0 + h_0^{(i)})p(x_0 + h_0^{(i)})}{p(x_1|x_0)p(x_0)} \triangleq M_0(x_1; x_0 + h_0^{(i)}, x_0),
\]

(2a)

\[
L(Y_k; X_k + h_{n,\ell+i}, X_k) = \frac{p(Y_k, X_k + h_{n,\ell+i})}{p(Y_k, X_k)} = \frac{p(y_{\ell+1}|x_{\ell+1}) \prod_{j=\ell+2}^{k} p(y_j|x_j)p(x_j|x_{j-1}) p(x_{\ell+1}|x_\ell + h_\ell^{(i)})p(x_\ell + h_\ell^{(i)})p(x_{\ell|\ell-1})}{p(y_{\ell+1}|x_{\ell+1}) \prod_{j=\ell+2}^{k} p(y_j|x_j)p(x_j|x_{j-1}) p(x_{\ell+1}|x_\ell)p(x_\ell|x_{\ell-1})} = \frac{p(x_{\ell+1}|x_\ell + h_\ell^{(i)})p(y_\ell|x_\ell + h_\ell^{(i)})p(x_\ell + h_\ell^{(i)})p(x_{\ell|\ell-1})}{p(x_{\ell+1}|x_\ell)p(y_\ell|x_\ell)p(x_\ell|x_{\ell-1})} \triangleq K_\ell(x_{\ell+1}, y_\ell; x_\ell + h_\ell^{(i)}, x_\ell; x_{\ell-1}),
\]

(2b)

\[
L(Y_k; X_k + h_{n,k+i}, X_k) = \frac{p(Y_k, X_k + h_{n,k+i})}{p(Y_k, X_k)} = \frac{p(y_k|x_k + h_k^{(i)})p(x_k + h_k^{(i)}|x_{k-1}) p(x_0) \prod_{j=1}^{k-1} p(y_j|x_j)p(x_j|x_{j-1}) p(x_0)}{p(y_k|x_k)p(x_k|x_{k-1}) p(x_0) \prod_{j=1}^{k-1} p(y_j|x_j)p(x_j|x_{j-1})} = \frac{p(y_k|x_k + h_k^{(i)})p(x_k + h_k^{(i)}|x_{k-1})}{p(y_k|x_k)p(x_k|x_{k-1})} \triangleq L_k(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}).
\]

(2c)

This concludes the proof of Lemma 1.
Lemma 2. For $\xi \leq k - 2$ it holds that
\[ J^{k|k}(\xi, k) = 0. \] (3)

Proof. We make use of the fact that due to the Markov property, the product terms inside the expectation in $J^{k|k}(\xi, k)$ become independent. In order to see this, let $\xi = k - 2$ and $\xi > 0$. Then, the $(i, j)$-th element of the matrix $J^{k|k}(k - 2, k)$ can be written as
\[
\left[J^{k|k}(k - 2, k) \right]_{i,j} = \mathbb{E} \left\{ L(Y_k; X_k + h_{n_i,(k-2)+j}, X_k) L(Y_k; X_k + h_{n_k,k+j}, X_k) \right\} - 1
\]
\[
= \mathbb{E} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h^{(i)}_{k-2}, x_{k-3}; x_{k-3}) L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\} - 1. \quad \text{(4)}
\]

With the tower property of conditional expectations, the above expectation may be rewritten as
\[
\mathbb{E} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h^{(i)}_{k-2}, x_{k-3}; x_{k-3}) L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\}
\]
\[
= \mathbb{E}_{x_{k-1}} \left\{ \mathbb{E}_{y_{k-2}, x_{k-2}, x_{k-3}|x_{k-1}} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h^{(i)}_{k-2}, x_{k-3}; x_{k-3}) \right\} \right\}
\]
\[
\times \mathbb{E}_{y_k, x_k|x_{k-1}} \left\{ L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\}. \quad \text{(5)}
\]

Since
\[
\mathbb{E}_{y_{k-2}, x_{k-2}, x_{k-3}|x_{k-1}} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h^{(i)}_{k-2}, x_{k-3}; x_{k-3}) \right\}
\]
\[
= \int \frac{p(x_{k-1}|x_{k-2} + h^{(i)}_{k-2}) p(y_{k-2}|x_{k-2} + h^{(i)}_{k-2}) p(x_{k-2} + h^{(i)}_{k-2})}{p(y_{k-2}|x_{k-2}) p(x_{k-2}|x_{k-3})} p(x_{k-2} + h^{(i)}_{k-2}) \, dy_{k-2} \, dx_{k-3}
\]
\[
= \int \frac{p(x_{k-1}|x_{k-2} + h^{(i)}_{k-2}) p(y_{k-2}|x_{k-2} + h^{(i)}_{k-2}) p(x_{k-2} + h^{(i)}_{k-2})}{p(y_{k-2}|x_{k-2}) p(x_{k-2}|x_{k-3})} \, dy_{k-2} \, dx_{k-3} \quad \times \frac{p(x_{k-1}|x_{k-2})}{p(x_{k-1})}
\]
\[
= \int \frac{p(x_{k-1}|x_{k-2} + h^{(i)}_{k-2}) p(y_{k-2}|x_{k-2} + h^{(i)}_{k-2}) p(x_{k-2} + h^{(i)}_{k-2})}{p(x_{k-1})} \, dy_{k-2} \, dx_{k-3}
\]
\[
= \int p(x_{k-1}|x_{k-2} + h^{(i)}_{k-2}) p(y_{k-2}|x_{k-2} + h^{(i)}_{k-2}) \, dy_{k-2} \int p(x_{k-2} + h^{(i)}_{k-2}) \, dx_{k-3}
\]
\[
= \frac{1}{p(x_{k-1})} \int p(x_{k-1}|x_{k-2} + h^{(i)}_{k-2}) \left[ \int p(y_{k-2}|x_{k-2} + h^{(i)}_{k-2}) \, dy_{k-2} \right] p(x_{k-2} + h^{(i)}_{k-2}, x_{k-3}) \, dx_{k-3}
\]
\[
= \frac{1}{p(x_{k-1})} \int p(x_{k-1}, x_{k-2} + h^{(i)}_{k-2}) \, dx_{k-2}
\]
\[
= 1 \quad \text{(6a)}
\]

and
\[
\mathbb{E}_{y_k, x_k|x_{k-1}} \left\{ L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\}
\]
\[
= \int \frac{p(y_k|x_k + h^{(j)}_k) p(x_k + h^{(j)}_k|x_{k-1})}{p(y_k|x_k) p(x_k|x_{k-1})} p(y_k|x_k) \, dy_k \, dx_k
\]
\[
= \int \frac{p(y_k|x_k + h^{(j)}_k) p(x_k + h^{(j)}_k|x_{k-1})}{p(y_k|x_k) p(x_k|x_{k-1})} \, dy_k \, dx_k
\]
\[
= \int \left[ \int p(y_k|x_k + h^{(j)}_k) \, dy_k \right] p(x_k + h^{(j)}_k|x_{k-1}) \, dx_k
\]
\[
= 1, \quad \text{(6b)}
\]

we obtain
\[
\left[J^{k|k}(k - 2, k) \right]_{i,j} = 0. \quad \text{(7)}
\]

This can be generalized for $0 < \xi < k - 2$ as follows:
\[
\left[J^{k|k}(\xi, k) \right]_{i,j} = \mathbb{E} \left\{ L(Y_k; X_k + h_{n_{\xi+1}}, x_{\xi+1}) L(Y_k; X_k + h_{n_{k+j}}, x_{k}) \right\} - 1
\]
\[
= \mathbb{E} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi+1} + h^{(i)}_{\xi}, x_{\xi+1}; x_{\xi+1}) L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\} - 1
\]
\[
= \mathbb{E}_{x_{\xi+1}} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi+1} + h^{(i)}_{\xi}, x_{\xi+1}; x_{\xi+1}) \right\} \mathbb{E}_{y_k, x_k, x_{k-1}} \left\{ L_k(y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\} - 1. \quad \text{(8)}
\]
Now, since

This concludes the proof of Lemma 2.

Finally, for \( \xi = 0 \) we have

holds, we arrive at

Finally, for \( \xi = 0 \) we have

This concludes the proof of Lemma 2.
Lemma 3. For $\xi \leq k - 3$ it holds that

$$J^{k|k}(\xi, k - 1) = 0.$$  \hfill (14)

Proof. Let $\xi = k - 3$ and $\xi > 0$. Then, we obtain

\[
\left[ J^{k|k}(k - 3, k - 1) \right]_{i,j} = \mathbb{E} \left\{ L(Y_k; X_k + h_{n_{k-3}i+1}, X_k)L(Y_k; X_k + h_{n_{k-3}+j}, X_k) \right\} - 1
\]

\[
= \mathbb{E} \left\{ K_{k-3}(x_{k-2}, y_{k-3} + h(i)_{k-4}, x_k)K_{k-1}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} - 1
\]

\[
= \mathbb{E}_{x_{k-2}} \left\{ \mathbb{E}_{x_k, y_k, x_{k-2}} \left\{ K_{k-3}(x_{k-2}; y_{k-3} + h(i)_{k-4}, x_k)K_{k-1}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} \right\} - 1
\]

\[
= 0,
\]

where the last equality follows from

\[
\mathbb{E}_{x_{k-3}, x_{k-2}, \ldots, x_{k-4}} \left\{ K_{k-3}(x_{k-2}; y_{k-3} + h(i)_{k-4}, x_k)K_{k-1}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} = 0.
\]

and

\[
\mathbb{E}_{x_k, y_k, x_{k-1}, x_{k-2}} \left\{ K_{k-3}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} = 0.
\]

This can be generalized for $0 < \xi < k - 3$ as follows:

\[
\left[ J^{k|k}(\xi, k - 1) \right]_{i,j} = \mathbb{E} \left\{ L(Y_k; X_k + h_{n_{\xi+i}+1}, X_k)L(Y_k; X_k + h_{n_{\xi+(k-1)+j}}, X_k) \right\} - 1
\]

\[
= \mathbb{E} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h(i)_{\xi}, x_{\xi-1})K_{k-1}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} - 1
\]

\[
= \mathbb{E}_{x_{\xi+1}, y_{\xi}, x_{\xi}-1} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h(i)_{\xi}, x_{\xi-1}) \right\}
\]

\[
\times \mathbb{E}_{x_k, y_k - 1, x_{k-1}, x_{k-2}} \left\{ K_{k-1}(x_k, y_k - 1; x_{k-1} + h_{k-1, k-1} x_{k-2}) \right\} - 1
\]

\[
= 0,
\]

(17)
where the last equality follows from (9a) and

\[
\begin{align*}
\mathbb{E}_{x_k, y_{k-1}, x_{k-1}, x_{k-2}} & \left\{ K_{k-1}(x_k, y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-2}) \right\} \\
= & \int \frac{p(x_k | x_{k-1}) + h_{k-1}^{(j)} \cdot p(y_{k-1} | x_{k-1} + h_{k-1}^{(j)})p(x_{k-1} + h_{j}^{(j)} | x_{k-2})}{p(y_{k-1} | x_{k-1})p(x_{k-1} | x_{k-2})} \\
& \times p(x_k, y_{k-1}, x_{k-1}, x_{k-2}) \, dx_k \, dy_{k-1} \, dx_{k-1} \, dx_{k-2} \\
= & \int \int \left( \frac{p(x_k | x_{k-1}) + h_{k-1}^{(j)} \cdot p(y_{k-1} | x_{k-1} + h_{k-1}^{(j)})}{p(y_{k-1} | x_{k-1})p(x_{k-1} | x_{k-2})} \right) p(x_{k-1} + h_{k-1}^{(j)} | x_{k-2})p(x_{k-1} + h_{j}^{(j)} | x_{k-2}) \\
& \quad \times p(x_k, y_{k-1}, x_{k-1}, x_{k-2}) \, dx_k \, dy_{k-1} \, dx_{k-1} \, dx_{k-2} \\
= & 1.
\end{align*}
\]  

Finally, for $\xi = 0$ we have

\[
\left[ J_{k|k}(0, k-1) \right]_{i,j} = \mathbb{E} \left\{ L(Y_k; X_k + h_i, X_k)L(Y_k; X_k + h_{n_x(k-1)+j}, X_k) \right\} - 1
\]

\[
\begin{align*}
= & \mathbb{E} \left\{ M_0(x_1; x_0 + h_{0}^{(i)}, x_0)K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-1}; x_{k-2}) \right\} - 1 \\
= & \mathbb{E}_{x_{k-1}} \mathbb{E}_{x_{k-1}, x_{k-2}} \left\{ M_0(x_1; x_0 + h_{0}^{(i)}, x_0) \right\} \mathbb{E}_{x_{k-1}, y_{k-1}} \mathbb{E}_{x_{k-1}, x_{k-2}} \left\{ K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-1}; x_{k-2}) \right\} - 1, \\
& \quad k = 3, \quad k > 3.
\end{align*}
\]

Since

\[
\begin{align*}
\mathbb{E}_{x_0 | x_1} \left\{ M_0(x_1; x_0 + h_{0}^{(i)}, x_0) \right\} &= 1, \\
\mathbb{E}_{x_1, y_{2}, x_{2}} | x_1 \left\{ K_2(x_1; y_{2}; x_{2} + h_{2}^{(j)}, x_{2}; x_{1}) \right\} &= 1, \\
\mathbb{E}_{x_1, x_0} \left\{ M_0(x_1; x_0 + h_{0}^{(i)}, x_0) \right\} &= 1, \\
\mathbb{E}_{x_{k}, y_{k-1}, x_{k-1}, x_{k-2}} \left\{ K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-1}; x_{k-2}) \right\} &= 1
\end{align*}
\]

holds, we arrive at

\[
\left[ J_{k|k}(0, k-1) \right]_{i,j} = 0.
\]

This concludes the proof of Lemma 3.

Corollary 1. For $\xi \leq k-3$ it holds that

\[
J_{k-1|k-1}(\xi, k-1) = 0.
\]

Proof. Follows from Lemma 2.

Lemma 4. The following identity holds

\[
J_{k|k}(k-2, k-1) = J_{k-1|k-1}(k-2, k-1).
\]

Proof. For $k > 2$ it holds

\[
\begin{align*}
\left[ J_{k|k}(k-2, k-1) \right]_{i,j} &= \mathbb{E} \left\{ L(Y_k; X_k + h_{n_x(k-2)+1}, X_k)L(Y_k; X_k + h_{n_x(k-1)+j}, X_k) \right\} - 1 \\
&= \mathbb{E} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h_{k-2}^{(i)}, x_{k-2}; x_{k-3})K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-1}; x_{k-2}) \right\} - 1 \\
&= \mathbb{E} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h_{k-2}^{(i)}, x_{k-2}; x_{k-3})L_{k-1}(y_{k-1}; x_{k-1} + h_{k-1}^{(j)}, x_{k-1}; x_{k-2}) \right\} - 1.
\end{align*}
\]
where the last equality follows from

This concludes the proof of Lemma 4.

holds, we have proved the first part. In case $k = 2$, we have

Now, since

(25)

(26)

holds, we have proved the first part. In case $k = 2$, we have

(27)

But this is equal to

(28)

This concludes the proof of Lemma 4. ⊓⊔

Lemma 5. For $\xi, \eta \leq k - 2$ it holds that

(29)

Proof. For $0 < \xi, \eta \leq k - 2$ it follows

(30)

The expression inside the expectation depends neither on $y_k$ nor $x_k$. Hence, these variables can be integrated out yielding

(31)
For $\xi = \eta = 0$ (and $k \geq 2$) it follows
\[
\begin{align*}
\left[ J^{k|k}(0, 0) \right]_{i,j} &= \mathbb{E}_{Y_k, X_k} \{ L(Y_k; X_k + h_i, X_k) L(Y_k; X_k + h_j, X_k) \} - 1 \\
&= \mathbb{E}_{x_k, x_0} \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0) M_0(x_1; x_0 + h_0^{(j)}, x_0) \right\} - 1. 
\end{align*}
\]

This is equal to
\[
\begin{align*}
\left[ J^{k-1|k-1}(0, 0) \right]_{i,j} &= \mathbb{E}_{Y_{k-1}, x_{k-1}} \{ L(Y_{k-1}; X_{k-1} + h_i, X_{k-1}) L(Y_{k-1}; X_{k-1} + h_j, X_{k-1}) \} - 1 \\
&= \mathbb{E}_{x_{k-1}, x_0} \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0) M_0(x_1; x_0 + h_0^{(j)}, x_0) \right\} - 1. 
\end{align*}
\]

Similarly, for $\xi = 0$ and $0 < \eta \leq k - 2$ we obtain
\[
\begin{align*}
\left[ J^{k|k}(0, \eta) \right]_{i,j} &= \mathbb{E}_{Y_k, X_k} \{ L(Y_k; X_k + h_i, X_k) L(Y_k; X_k + h_{\eta}, X_k) \} - 1 \\
&= \mathbb{E}_{y_k, x_k} \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0) K_{y}(x_{\eta+1}, y_{\eta}; x_{\eta} + h_{\eta}^{(j)}, x_{\eta}; x_{\eta-1}) \right\} - 1. 
\end{align*}
\]

The expression inside the expectation depends neither on $y_k$ nor $x_k$. Hence, we can write
\[
\begin{align*}
\left[ J^{k|k}(0, \eta) \right]_{i,j} &= \mathbb{E}_{Y_{k-1}, x_{k-1}} \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0) K_{y}(x_{\eta+1}, y_{\eta}; x_{\eta} + h_{\eta}^{(j)}, x_{\eta}; x_{\eta-1}) \right\} - 1 \\
&= \left[ J^{k-1|k-1}(0, \eta) \right]_{i,j},
\end{align*}
\]

Essentially the same reasoning can be used to show that
\[
\left[ J^{k|k}(\xi, 0) \right]_{i,j} = \left[ J^{k-1|k-1}(\xi, 0) \right]_{i,j},
\]

where $0 < \xi \leq k - 2$. This concludes the proof of Lemma 5. \(\Box\)

**Theorem 1.** The MSE matrix for the filtering problem is bounded below by
\[
\mathbb{E}_{\mathbf{x}_k} \{ (\mathbf{x}_k - \mathbf{s}_k(Y_k)) (\mathbf{c})^T \} \succeq \mathbf{H}(k, k) \left[ J_{k|k} \right]^{-1} \mathbf{H}^T(k, k),
\]
where the information matrix $J_{k|k}$ can be computed recursively as
\[
J_{k|k} = J_{k|k}(k, k) - J_{k|k}(k, k - 1) \left[ J_{k|k}(k - 1, k - 1) - J_{k-1|k-1}(k - 1, k - 1) + J_{k-1|k}(k - 1, k - 1) \right]^{-1} J_{k|k}(k - 1, k),
\]

with initialization $J_{0|0} = J^{0|0}(0, 0)$. The $(i, j)$-th elements of the matrices $J^{k|k}(\xi, \eta)$ are given by
\[
\begin{align*}
\left[ J^{k|k}(k, k) \right]_{i,j} &= \mathbb{E} \{ L_k(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}) L_k(y_k; x_k + h_k^{(j)}, x_k; x_{k-1}) \}, \\
\left[ J^{k|k}(k, k - 1) \right]_{i,j} &= \left[ J^{k|k}(k - 1, k) \right]_{i,j}, \\
\left[ J^{k|k}(k - 1, k - 1) \right]_{i,j} &= \left\{ \begin{array}{ll}
\mathbb{E} \{ L_1(y_1; x_1 + h_1^{(i)}, x_1; x_0) M_0(x_1; x_0 + h_0^{(j)}, x_0) \} - 1, & k = 1, \\
\mathbb{E} \{ L_k(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}) M_0(x_1; x_0 + h_0^{(j)}, x_0) \} - 1, & k > 1,
\end{array} \right.
\end{align*}
\]

**Proof.** We partition the information matrix $J^{k-1|k-1}$ at time instance $k - 1$ as follows:
\[
J^{k-1|k-1} = \begin{bmatrix}
A_{k-1|k-1} & B_{k-1|k-1}
\end{bmatrix}
\begin{bmatrix}
A_{k-1|k-1} \quad B_{k-1|k-1}
\end{bmatrix}
\]

(40)
where

\[
A_{k-1|k-1} = \begin{bmatrix}
J^{k-1|k-1}(0, 0) & \cdots & J^{k-1|k-1}(0, k-2) \\
\vdots & \ddots & \vdots \\
J^{k-1|k-1}(k-2, 0) & \cdots & J^{k-1|k-1}(k-2, k-2)
\end{bmatrix}, \tag{41a}
\]

\[
B_{k-1|k-1} = \begin{bmatrix}
J^{k-1|k-1}(0, k-1) \\
\vdots \\
J^{k-1|k-1}(k-3, k-1) \\
J^{k-1|k-1}(k-2, k-1)
\end{bmatrix} \quad \text{Corollary 1} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix} \begin{bmatrix}
J^{k-1|k-1}(k-2, k-1)
\end{bmatrix}. \tag{41b}
\]

The information sub-matrix \(J_{k-1|k-1}\) for estimating \(x_{k-1}\) is the inverse of the \((n_x \times n_x)\) lower-right sub-block of \([J^{k-1|k-1}]^{-1}\) and can be obtained from block matrix inversion as follows:

\[
J_{k-1|k-1} = J^{k-1|k-1}(k-1, k-1) - B^T_{k-1|k-1} \left[ A_{k-1|k-1} \right]^{-1} B_{k-1|k-1}. \tag{42}
\]

Similarly, the information matrix \(J^{k|k}\) at time \(k\) can be partitioned according to

\[
J^{k|k} = \begin{bmatrix}
A_{k|k} & B_{k|k} & C_{k|k} \\
B_{k|k}^T & J^{k|k}(k-1, k-1) & J^{k|k}(k-1, k) \\
C_{k|k}^T & J^{k|k}(k, k-1) & J^{k|k}(k, k)
\end{bmatrix}, \tag{43}
\]

where

\[
A_{k|k} = \begin{bmatrix}
J^{k|k}(0, 0) & \cdots & J^{k|k}(0, k-2) \\
\vdots & \ddots & \vdots \\
J^{k|k}(k-2, 0) & \cdots & J^{k|k}(k-2, k-2)
\end{bmatrix} = A_{k-1|k-1}, \tag{44a}
\]

\[
B_{k|k} = \begin{bmatrix}
J^{k|k}(0, k-1) \\
\vdots \\
J^{k|k}(k-3, k-1) \\
J^{k|k}(k-2, k-1)
\end{bmatrix} = B_{k-1|k-1} \tag{44b}
\]

and

\[
C_{k|k} = \begin{bmatrix}
J^{k|k}(0, k) \\
\vdots \\
J^{k|k}(k-2, k)
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}. \tag{44c}
\]

Hence, the information matrix \(J^{k|k}\) can be reduced to

\[
J^{k|k} = \begin{bmatrix}
A_{k-1|k-1} & B_{k-1|k-1} & J^{k|k}(k-1, k-1) \\
B_{k-1|k-1}^T & J^{k|k}(k-1, k) & J^{k|k}(k, k)
\end{bmatrix} \begin{bmatrix}
0 \\
J^{k|k}(k-1, k-1)
\end{bmatrix}. \tag{45}
\]

The \((n_x \times n_x)\) information sub-matrix \(J_{k|k}\) is again obtained from block matrix inversion, yielding the recursion

\[
J_{k|k} = J^{k|k}(k, k) - \begin{bmatrix}
0 \\
J^{k|k}(k-1, k)
\end{bmatrix}^T \begin{bmatrix}
\star \\
\star
\end{bmatrix} + \begin{bmatrix}
0 \\
\star
\end{bmatrix} \begin{bmatrix}
J^{k|k}(k-1, k-1) - B^T_{k-1|k-1} \left[ A_{k-1|k-1} \right]^{-1} B_{k-1|k-1}
\end{bmatrix}^{-1} J^{k|k}(k-1, k). \tag{46}
\]
where the last equality follows by using (42). The matrix elements $J^{k|k}(\eta, \xi)$ constituting the above recursion can be obtained by making use of the results provided in Lemma 1. This yields

\[
\begin{bmatrix}
J^{k|k}(k, k)
\end{bmatrix}_{i,j} = \begin{bmatrix}
J^{k|k}
\end{bmatrix}_{n_{k}, i + 1, n_{k} + j},
\]

where the prediction information sub-matrix

\[L(y_{k}; x_{k} + h_{n_{k}+i}, x_{k})L(y_{k}; x_{k} + h_{n_{k}+j}, x_{k}^{-1}) - 1 = \]

\[
\begin{bmatrix}
p(y_{k}|x_{k} + h_{0}^{(i)})(p(y_{k}|x_{k} + h_{0}^{(j)}))p(x_{k} + h_{0}^{(i)}|x_{k}^{-1})p(x_{k} + h_{0}^{(j)}|x_{k}^{-1})
\end{bmatrix}_{i,j} - 1,
\] (47a)

\[
\begin{bmatrix}
J^{k|k}(k, k - 1)
\end{bmatrix}_{i,j} = \begin{bmatrix}
J^{k|k}
\end{bmatrix}_{n_{k}, i + 1, n_{k} - 1} + \begin{bmatrix}
J^{k|k}
\end{bmatrix}_{n_{k}, i + 1, n_{k} - 1 + j},
\]

where the recursive structure is given by

\[
\begin{bmatrix}
L(y_{k}; x_{k} + h_{n_{k}+i}, x_{k})L(y_{k}; x_{k} + h_{n_{k}+j}, x_{k}^{-1}) - 1 = \]

\[
\begin{bmatrix}
p(x_{k} + h_{0}^{(i)}|x_{k}^{-1})p(x_{k} + h_{0}^{(j)}|x_{k}^{-1})
\end{bmatrix}_{i,j} - 1,
\] (47b)

\[
\begin{bmatrix}
J^{k|k}(k, k - 1)
\end{bmatrix}_{i,j} = \begin{bmatrix}
J^{k}
\end{bmatrix}_{n_{k} - 1, i + 1, n_{k} - 1 + j},
\]

This concludes the proof of Theorem 1.

\[\square\]

2 Recursive Computation of Prediction Bound

**Theorem 2.** The MSE matrix for the one-step ahead prediction problem is bounded below by

\[
E_{x_{k+1}, Y_{k}}\{\{x_{k+1} - \tilde{x}_{k+1}(Y_{k})\}^{2}\} \geq H(k + 1, k + 1) \begin{bmatrix} J_{k+1|k} \end{bmatrix}^{-1} H^{T}(k + 1, k + 1),
\] (48)

where the prediction information sub-matrix $J_{k+1|k}$ can be computed from the recursion

\[
J_{k+1|k} = J^{k+1|k}(k + 1, k + 1) - J^{k+1|k}(k + 1, k) \begin{bmatrix} J^{k+1|k}(k, k) - J^{k|k}(k, k) + J_{k|k} \end{bmatrix}^{-1} J^{k+1|k}(k, k + 1),
\] (49)

with initialization $J_{0|0}$. The $(i, j)$-th element of the matrix $J^{k+1|k}(k + 1, k + 1)$ is given by

\[
\begin{bmatrix}
J^{k+1|k}(k + 1, k + 1)
\end{bmatrix}_{i,j} = \begin{bmatrix}
E\{p(x_{k+1} + h_{k+1}^{(i)}|x_{k})p(x_{k+1} + h_{k+1}^{(j)}|x_{k})
\end{bmatrix}_{i,j} - 1.
\] (50)

**Proof.** The one-step ahead prediction information matrix $J^{k+1|k}$ is partitioned into blocks $J^{k+1|k}(\xi, \eta), \xi, \eta = 0, \ldots, k + 1$ each of size $(n_{x} \times n_{x})$ as follows

\[
J^{k+1|k} = \begin{bmatrix}
J^{k+1|k}(0, 0) & J^{k+1|k}(0, 1) & \cdots & J^{k+1|k}(0, k) & J^{k+1|k}(0, k + 1) \\
J^{k+1|k}(1, 0) & J^{k+1|k}(1, 1) & \cdots & J^{k+1|k}(1, k) & J^{k+1|k}(1, k + 1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
J^{k+1|k}(k, 0) & J^{k+1|k}(k, 1) & \cdots & J^{k+1|k}(k, k) & J^{k+1|k}(k, k + 1) \\
J^{k+1|k}(k + 1, 0) & J^{k+1|k}(k + 1, 1) & \cdots & J^{k+1|k}(k + 1, k) & J^{k+1|k}(k + 1, k + 1)
\end{bmatrix},
\] (51)
where each block is given by
\[
\begin{bmatrix}
J^{k+1|k}(\xi, \eta)
\end{bmatrix}
= \begin{bmatrix}
J^{k+1|k}_{n_k \xi + n_k \eta + j}
\end{bmatrix}
= E_{Y_k, X_{k+1}} \{ L(Y_k, X_{k+1} + h_{n_k \xi + i}, X_{k+1}) L(Y_k, X_{k+1} + h_{n_k \eta + j}, X_{k+1}) \} - 1. \quad (52)
\]

The idea is now, similarly to the proof for the filtering bound, to identify blocks in the matrix \( J^{k+1|k} \) that are identical to the blocks in the matrix \( J^{k|k} \), i.e. they do not change over time, such that a recursive computation of the prediction information sub-matrix \( J^{k+1|k} \) in terms of the filtering information sub-matrix \( J_{k|k} \) is possible. In order to prove the above stated recursion, the following lemmas are required:

**Lemma 6.** The following identities hold \((0 < \ell < k + 1)\):

\[
\begin{align*}
L(Y_k; X_{k+1} + h_i, X_{k+1}) &= M_0(x_1; x_0 + h_0^{(i)}, x_0), \quad (53a) \\
L(Y_k; X_{k+1} + h_{n_k \xi + \ell} + \xi, X_{k+1}) &= K_\ell(x_{\ell+1}, Y_{\ell+1}; x_\ell + h_{\ell}^{(i)}, x_\ell; x_{\ell-1}), \quad (53b) \\
L(Y_k; X_{k+1} + h_{n_k (\ell+1)} + \xi, X_{k+1}) &= N_{k+1}(x_{k+1} + h_k^{(i)}, x_{k+1}; x_k). \quad (53c)
\end{align*}
\]

**Proof.** We can simplify each ratio of joint densities as follows:

\[
\begin{align*}
L(Y_k; X_{k+1} + h_i, X_{k+1}) &= \frac{p(Y_k, X_{k+1} + h_i)}{p(Y_k, X_{k+1})} \\
&= \frac{p(x_{k+1} | x_k) p(y_1 | x_1) \prod_{j=2}^{k} p(y_j | x_j) p(x_j | x_{j-1}) p(x_1 | x_0 + h_0^{(i)}) p(x_0 + h_0^{(i)})}{p(x_{k+1} | x_k) p(y_1 | x_1) \prod_{j=2}^{k} p(y_j | x_j) p(x_j | x_{j-1})} \\
&= \frac{p(x_1 | x_0 + h_0^{(i)}) p(x_0 + h_0^{(i)})}{p(x_1 | x_0) p(x_0)} \\
&= M_0(x_1; x_0 + h_0^{(i)}, x_0), \quad (54a)
\end{align*}
\]

\[
\begin{align*}
L(Y_k; X_{k+1} + h_{n_k \xi + \ell} + \xi, X_{k+1}) &= \frac{p(Y_{k+1}, X_{k+1} + h_{n_k \xi + \ell} + \xi)}{p(Y_k, X_{k+1})} \\
&= \frac{p(x_{k+1} | x_k) p(y_{\ell+1} | x_{\ell+1}) \prod_{j=\ell+2}^{k} p(y_j | x_j) p(x_j | x_{j-1})}{p(x_{k+1} | x_k) p(y_{\ell+1} | x_{\ell+1}) \prod_{j=\ell+2}^{k} p(y_j | x_j) p(x_j | x_{j-1})} \\
&\quad \times \frac{p(x_{\ell+1} | x_\ell + h_{\ell}^{(i)}) p(y_\ell | x_\ell + h_{\ell}^{(i)}) p(x_\ell + h_{\ell}^{(i)} | x_{\ell-1})}{p(x_{\ell+1} | x_\ell) p(y_\ell | x_\ell) p(x_\ell | x_{\ell-1})} \\
&\quad \times \frac{p(x_\ell | x_\ell)}{p(x_\ell) \prod_{j=1}^{\ell-1} p(y_j | x_j) p(x_j | x_{j-1})} \\
&= K_\ell(x_{\ell+1}, Y_{\ell+1}; x_\ell + h_{\ell}^{(i)}, x_\ell; x_{\ell-1}), \quad (54b)
\end{align*}
\]

\[
\begin{align*}
L(Y_k; X_{k+1} + h_{n_k (\ell+1)} + \xi, X_{k+1}) &= \frac{p(Y_k, X_{k+1} + h_{n_k (\ell+1)} + \xi)}{p(Y_k, X_{k+1})} \\
&= \frac{p(x_{k+1} + h_k^{(i)} | x_k) p(x_0) \prod_{j=1}^{k} p(y_j | x_j) p(x_j | x_{j-1})}{p(x_{k+1} + h_k^{(i)} | x_k) p(x_0) \prod_{j=1}^{k} p(y_j | x_j) p(x_j | x_{j-1})} \\
&= \frac{p(x_{k+1} + h_k^{(i)} | x_k)}{p(x_{k+1} | x_k)} \\
&\triangleq N_{k+1}(x_{k+1} + h_k^{(i)}, x_{k+1}; x_k). \quad (54c)
\end{align*}
\]

This concludes the proof of Lemma 6. \( \square \)

**Lemma 7.** For \( \xi \leq k - 1 \) it holds that

\[
J^{k+1|k}(\xi, k + 1) = 0. \quad (55)
\]
Proof. Let $\xi = k - 1$ and $\xi > 0$. Then, the $(i,j)$-th element of the matrix $J^{k+1|k}(k-1,k+1)$ can be written as

$$
\begin{align*}
\left[ J^{k+1|k}(k-1,k+1) \right]_{i,j} &= \mathbb{E} \left\{ L(Y_k; X_{k+1} + h_{n_1, n_2}(k-1, i, j) + X_{k+1}) \right\} - 1 \\
&= \mathbb{E} \left\{ K_{k-1}(x_k, y_{k-1}; x_k-1 + h_{n_1, n_2}(k-1, i, j))N_{k+1}(x_{k+1} + h_{n_1, n_2}(k-1, i, j)) \right\} - 1. 
\end{align*}
$$

(56)

The tower property of conditional expectations gives

$$
\begin{align*}
&\mathbb{E} \left\{ K_{k-1}(x_k, y_{k-1}; x_k-1 + h_{n_1, n_2}(k-1, i, j))N_{k+1}(x_{k+1} + h_{n_1, n_2}(k-1, i, j)) \right\} \\
&= \mathbb{E}_{x_k} \left\{ K_{k-1}(x_k, y_{k-1}; x_k-1 + h_{n_1, n_2}(k-1, i, j)) \right\} \\
&\quad \times \mathbb{E}_{x_{k+1}|x_k} \left\{ N_{k+1}(x_{k+1} + h_{n_1, n_2}(k-1, i, j)) \right\}. 
\end{align*}
$$

(57)

Since

$$
\begin{align*}
\mathbb{E}_{x_{k-1}, x_{k-2}|x_k} \left\{ K_{k-1}(x_k, y_{k-1}; x_k-1 + h_{n_1, n_2}(k-1, i, j)) \right\} = 1 
\end{align*}
$$

(58a)

and

$$
\begin{align*}
\mathbb{E}_{x_{k+1}|x_k} \left\{ N_{k+1}(x_{k+1} + h_{n_1, n_2}(k-1, i, j)) \right\} = \int \frac{p(x_{k+1} + h_{n_1, n_2}(k-1, i, j))}{p(x_{k+1}|x_k)} p(x_{k+1}|x_k) \, dx_{k+1} \\
&= 1, 
\end{align*}
$$

(58b)

we obtain

$$
\left[ J^{k+1|k}(k-1,k+1) \right]_{i,j} = 0. 
$$

(59)

This can be generalized for $0 < \xi < k - 1$ as follows:

$$
\begin{align*}
\left[ J^{k+1|k}(\xi,k+1) \right]_{i,j} &= \mathbb{E} \left\{ L(Y_k; X_{k+1} + h_{n_1, n_2}(\xi, i, j) + X_{k+1}) \right\} - 1 \\
&= \mathbb{E} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h_{n_1, n_2}(\xi, i, j))N_{k+1}(x_{k+1} + h_{n_1, n_2}(\xi, i, j)) \right\} - 1 \\
&= \mathbb{E}_{x_{\xi+1}, x_{\xi}, x_{\xi-1}} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h_{n_1, n_2}(\xi, i, j)) \right\} \mathbb{E}_{x_{k+1}|x_k} \left\{ N_{k+1}(x_{k+1} + h_{n_1, n_2}(\xi, i, j)) \right\} - 1. 
\end{align*}
$$

(60)

Now, since

$$
\begin{align*}
\mathbb{E}_{x_{\xi+1}, y_{\xi}, x_{\xi}, x_{\xi-1}} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h_{n_1, n_2}(\xi, i, j)) \right\} = 1 
\end{align*}
$$

(61a)

and

$$
\begin{align*}
\mathbb{E}_{x_{k+1}|x_k} \left\{ N_{k+1}(x_{k+1} + h_{n_1, n_2}(\xi, i, j)) \right\} = \int \left[ \int \frac{p(x_{k+1} + h_{n_1, n_2}(\xi, i, j))}{p(x_{k+1}|x_k)} p(x_{k+1}|x_k) \, dx_{k+1} \right] p(x_k) \, dx_k \\
&= 1, 
\end{align*}
$$

(61b)

holds, we arrive at

$$
\left[ J^{k+1|k}(\xi,k+1) \right]_{i,j} = 0. 
$$

(62)

Finally, for $\xi = 0$ we have

$$
\begin{align*}
\left[ J^{k+1|k}(0,k+1) \right]_{i,j} &= \mathbb{E} \left\{ L(Y_k; X_{k+1} + h_i, X_{k+1}) \right\} - 1 \\
&= \mathbb{E} \left\{ M_0(x_i; x_0 + h_{n_1, n_2}(0, i, j) + x_0) \right\} - 1 \\
&= \mathbb{E}_{x_i, x_0} \left\{ M_0(x_i; x_0 + h_{n_1, n_2}(0, i, j)) \mathbb{E}_{x_2|x_1} \left\{ N_2(x_2 + h_{n_1, n_2}(0, i, j)) \right\} \right\} - 1, \quad k = 1, \\
&= \mathbb{E}_{x_i, x_0} \left\{ M_0(x_i; x_0 + h_{n_1, n_2}(0, i, j)) \mathbb{E}_{x_{k+1}|x_k} \left\{ N_{k+1}(x_{k+1} + h_{n_1, n_2}(0, i, j)) \right\} \right\} - 1, \quad k > 1. 
\end{align*}
$$

(63)
Since
\[ \mathbb{E}_{x_{i}|x_{1}} \left\{ M_{0}(x_{1}; x_{0} + h_{0}^{(i)}, x_{0}) \right\} = 1, \tag{64a} \]
\[ \mathbb{E}_{x_{i}|x_{1}} \left\{ N_{2}(x_{2} + h_{2}^{(j)}, x_{2}; x_{1}) \right\} = 1, \tag{64b} \]
\[ \mathbb{E}_{x_{1}, x_{0}} \left\{ M_{0}(x_{1}; x_{0} + h_{0}^{(i)}, x_{0}) \right\} = 1, \tag{64c} \]
\[ \mathbb{E}_{x_{k+1}, x_{k}} \left\{ N_{k+1}(x_{k+1} + h_{k+1}^{(j)}, x_{k+1}; x_{k}) \right\} = 1, \tag{64d} \]
holds, we arrive at
\[ \left[ J^{k+1|k}(0, k + 1) \right]_{i,j} = 0. \tag{65} \]
This concludes the proof of Lemma 7.

**Lemma 8.** For \( \xi \leq k - 2 \) it holds that
\[ J^{k+1|k}(\xi, k) = J^{k|k}(\xi, k) = 0. \tag{66} \]

**Proof.** Let \( \xi = k - 2 \) and \( \xi > 0 \). Then, we obtain
\[
\left[ J^{k+1|k}(k - 2, k) \right]_{i,j} = \mathbb{E} \left\{ L(Y_{k}; X_{k+1} + h_{n_{x}(k-2)+i}, X_{k+1})L(Y_{k}; X_{k+1} + h_{n_{x}+j}, X_{k+1}) \right\} - 1 \\
= \mathbb{E} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h_{k-2}^{(i)}, x_{k-2}; x_{k-3})K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1 \\
= \mathbb{E}_{x_{k-1}} \left\{ \mathbb{E}_{x_{k-2}, x_{k-3}|x_{k-1}} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h_{k-2}^{(i)}, x_{k-2}; x_{k-3}) \right\} \right\} \\
\times \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{k-1}} \left\{ K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1 \\
= 0, \tag{67} \]
where the last equality follows from
\[ \mathbb{E}_{x_{k-2}, x_{k-3}|x_{k-1}} \left\{ K_{k-2}(x_{k-1}, y_{k-2}; x_{k-2} + h_{k-2}^{(i)}, x_{k-2}; x_{k-3}) \right\} = 1, \tag{68a} \]
and
\[ \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{k-1}} \left\{ K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} = 1. \tag{68b} \]
This can be generalized for \( 0 < \xi < k - 2 \) as follows:
\[
\left[ J^{k+1|k}(\xi, k) \right]_{i,j} = \mathbb{E} \left\{ L(Y_{\xi}; X_{\xi+1} + h_{n_{x}\xi+i}, X_{\xi+1})L(Y_{k}; X_{k+1} + h_{n_{x}+j}, X_{k+1}) \right\} - 1 \\
= \mathbb{E} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h_{\xi}^{(i)}, x_{\xi}; x_{\xi-1})K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1 \\
= \mathbb{E}_{x_{\xi+1}, y_{\xi}, x_{\xi-1}} \left\{ K_{\xi}(x_{\xi+1}, y_{\xi}; x_{\xi} + h_{\xi}^{(i)}, x_{\xi}; x_{\xi-1}) \right\} \\
\times \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{\xi-1}} \left\{ K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1 \\
= 0, \tag{69} \]
where the last equality follows from (61a) and
\[ \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{\xi-1}} \left\{ K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} = 1. \tag{70} \]
Finally, for \( \xi = 0 \) we have
\[
\left[ J^{k+1|k}(0, k) \right]_{i,j} = \mathbb{E} \left\{ L(Y_{k}; X_{k+1} + h_{k}, X_{k+1})L(Y_{k}; X_{k+1} + h_{n_{x}+j}, X_{k+1}) \right\} - 1 \\
= \mathbb{E} \left\{ M_{0}(x_{1}; x_{0} + h_{0}^{(i)}, x_{0})K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1 \\
= \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{1}} \left\{ M_{0}(x_{1}; x_{0} + h_{0}^{(i)}, x_{0}) \right\} \mathbb{E}_{x_{k+1}, y_{k}, x_{k}|x_{1}} \left\{ K_{k}(x_{k+1}, y_{k}; x_{k} + h_{k}^{(j)}, x_{k}; x_{k-1}) \right\} - 1, \tag{71} \]
Since
\[
\mathbb{E}_{x_i | x_1} \left\{ M_0(x_1; x_0 + h^{(i)}_0, x_0) \right\} = 1, \tag{72a}
\]
\[
\mathbb{E}_{x_i, y, x_i | x_1} \left\{ K_2(x_1, y; x_2 + h^{(j)}_2, x_2; x_1) \right\} = 1, \tag{72b}
\]
\[
\mathbb{E}_{x_1, x_0} \left\{ M_0(x_1; x_0 + h^{(i)}_0, x_0) \right\} = 1, \tag{72c}
\]
\[
\mathbb{E}_{x_i, x_i, x_k, x_{k-1}} \left\{ K_k(x_{k+1}, y_k; x_k + h^{(j)}_k, x_k; x_{k-1}) \right\} = 1 \tag{72d}
\]
holds, we arrive at
\[
\left[ J^{k+1|k}(0, k) \right]_{i,j} = 0. \tag{73}
\]
Now comparing this with the result stated in Lemma 2 concludes the proof. □

**Lemma 9.** The following identity holds
\[
J^{k+1|k}(k - 1, k) = J^{k|k}(k - 1, k) \tag{74}
\]

**Proof.** For \( k > 1 \) it holds
\[
\left[ J^{k+1|k}(k - 1, k) \right]_{i,j} = E \left\{ L(Y_k; X_{k+1} + h_{n_k(k-1)+i}, X_{k+1})L(Y_k; X_{k+1} + h_{n_k(k-1)+j}, X_{k+1}) \right\} - 1
\]
\[
= E \left\{ K_{k-1}(x_{k-1}, y_{k-1}; x_{k-2} + h^{(i)}_{k-1}, x_{k-2}; x_{k-2})K_k(x_{k+1}, y_k; x_{k+1} + h^{(j)}_k, x_{k+1}; x_{k+1}) \right\} - 1
\]
\[
= E \left\{ \frac{p(x_k|x_{k-1} + h^{(i)}_{k-1})p(x_{k-1} + h^{(j)}_{k-1})}{p^2(x_k|x_{k-1})p(x_{k-1}|x_{k-1})} \right\} - 1, \tag{75}
\]
where the last equality follows from integrating out the variables \( x_{k+1}, y_k \) and \( y_{k-1} \). Similarly, integrating out the variables \( y_k \) and \( y_{k-1} \) in
\[
\left[ J^{k|k}(k - 1, k) \right]_{i,j} = E \left\{ L(Y_k; X_k + h_{n_k(k-1)+i}, X_k)L(Y_k; X_k + h_{n_k(k-1)+j}, X_k) \right\} - 1
\]
\[
= E \left\{ K_{k-1}(x_{k-1}, y_{k-1}; x_{k-2} + h^{(i)}_{k-1}, x_{k-2}; x_{k-2})L_k(y_k; x_k + h^{(j)}_k, x_k; x_k) \right\} - 1
\]
\[
= E \left\{ \frac{p(x_k|x_{k-1} + h^{(i)}_{k-1})p(x_{k-1} + h^{(j)}_{k-1})}{p^2(x_k|x_{k-1})p(x_{k-1}|x_{k-1})} \right\} - 1, \tag{76}
\]
yields the equality. For \( k = 1 \) we arrive at
\[
\left[ J^{2|1}(0, 1) \right]_{i,j} = E \left\{ L(Y_1; X_2 + h_0, X_2)L(Y_1; X_2 + h_{n_2} + j, X_2) \right\} - 1
\]
\[
= E \left\{ M_0(x_1; x_0 + h^{(i)}_0, x_0)K_1(x_2, y_1; x_1 + h^{(j)}_1, x_1; x_0) \right\} - 1
\]
\[
= E \left\{ \frac{p(x_1|x_0 + h^{(i)}_0)p(x_0 + h^{(j)}_1)}{p^2(x_1|x_0)p(x_0)} \right\} - 1, \tag{77}
\]
where the last equality follows from integrating out the variables \( X_2 \) and \( Y_1 \). Similarly, integrating out the variable \( Y_1 \) in
\[
\left[ J^{1|1}(0, 1) \right]_{i,j} = E \left\{ L(Y_1; X_1 + h_0, X_1)L(Y_1; X_1 + h_{n_2} + j, X_1) \right\} - 1
\]
\[
= E \left\{ M_0(x_1; x_0 + h^{(i)}_0, x_0)L_1(y_1; x_1 + h^{(j)}_1, x_1; x_0) \right\} - 1
\]
\[
= E \left\{ \frac{p(x_1|x_0 + h^{(i)}_0)p(x_0 + h^{(j)}_1)}{p^2(x_1|x_0)p(x_0)} \right\} - 1 \tag{78}
\]
yields the equality. This concludes the proof of Lemma 9. □

**Lemma 10.** For \( \xi, \eta \leq k - 1 \) it holds that
\[
J^{k+1|k}((\xi, \eta)) = J^{k|k}((\xi, \eta)). \tag{79}
\]
Proof. For $0 < \xi, \eta \leq k - 1$ it follows

$$
\begin{bmatrix}
J^{k+1|k}(\xi, \eta)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_{k+1}} \{ L(Y_k; X_{k+1} + h_{n,\xi+i,}, X_{k+1})L(Y_k; X_{k+1} + h_{n,\eta+j,} X_{k+1}) \} - 1
$$

Equivalently, we can write

$$
\begin{bmatrix}
J^{k|k}(\xi, \eta)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_k} \{ L(Y_k; X_k + h_{n,\xi+i,} X_k) L(Y_k; X_k + h_{n,\eta+j,} X_k) \} - 1
$$

For $\xi = \eta = 0$ (and $k \geq 1$) we have

$$
\begin{bmatrix}
J^{k+1|k}(0, 0)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_{k+1}} \{ L(Y_k; X_{k+1} + h_i, X_{k+1}) L(Y_k; X_{k+1} + h_j, X_{k+1}) \} - 1
$$

This is equal to

$$
\begin{bmatrix}
J^{k|k}(0, 0)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_k} \{ L(Y_k; X_k + h_i, X_k) L(Y_k; X_k + h_j, X_k) \} - 1
$$

Similarly, for $\xi = 0$ and $0 < \eta \leq k - 1$ we obtain

$$
\begin{bmatrix}
J^{k+1|k}(0, \eta)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_{k+1}} \{ L(Y_k; X_{k+1} + h_i, X_{k+1}) L(Y_k; X_{k+1} + h_{n,\eta+j,} X_{k+1}) \} - 1
$$

but this is equal to

$$
\begin{bmatrix}
J^{k|k}(0, \eta)
\end{bmatrix}_{i,j} = \mathcal{E}_{Y_k, X_k} \{ L(Y_k; X_k + h_i, X_k) L(Y_k; X_k + h_{n,\eta+j,} X_k) \} - 1
$$

Essentially the same reasoning applies to show that

$$
\begin{bmatrix}
J^{k+1|k}(\xi, 0)
\end{bmatrix}_{i,j} = \begin{bmatrix}
J^{k|k}(\xi, 0)
\end{bmatrix}_{i,j}
$$

holds, where $0 < \xi \leq k - 1$. This concludes the proof of Lemma 10.

We partition the prediction information matrix $J^{k+1|k}$ at time $k + 1$ as follows

$$
J^{k+1|k} = \begin{bmatrix}
A_{k+1|k} & B_{k+1|k} & C_{k+1|k} & D_{k+1|k} \\
B_{k+1|k}^T & J^{k+1|k}(k - 1, k - 1) & J^{k+1|k}(k - 1, k) & J^{k+1|k}(k - 1, k + 1) \\
C_{k+1|k}^T & J^{k+1|k}(k, k - 1) & J^{k+1|k}(k, k) & J^{k+1|k}(k, k + 1) \\
D_{k+1|k}^T & J^{k+1|k}(k + 1, k - 1) & J^{k+1|k}(k + 1, k) & J^{k+1|k}(k + 1, k + 1)
\end{bmatrix}.
$$

The elements in this matrix can be simplified as follows

$$
\begin{bmatrix}
D_{k+1|k} \\
J^{k+1|k}(k - 1, k + 1)
\end{bmatrix} \overset{\text{Lemma 7}}{=} \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix},
$$

$$
\begin{bmatrix}
C_{k+1|k} \\
J^{k+1|k}(k - 1, k)
\end{bmatrix} \overset{\text{Lemma 8,9}}{=} \begin{bmatrix}
0 \\
J^{k|k}(k - 1, k)
\end{bmatrix},
$$

and

$$
\begin{bmatrix}
A_{k+1|k} \\
B_{k+1|k}
\end{bmatrix} \overset{\text{Lemma 10}}{=} \begin{bmatrix}
A_{k-1|k-1} & B_{k-1|k-1} \\
B_{k-1|k-1} & J^{k|k}(k - 1, k - 1)
\end{bmatrix}.
$$
yielding

\[ J^{k+1|k} = \begin{bmatrix} A_{k-1|k-1} & B_{k-1|k-1} & 0 \\ J^k[k-1, k-1] & J^k[k-1, k] & J^{k+1}[k, k] \\ 0 & 0 & J^k[k+1, k] \end{bmatrix} \begin{bmatrix} 0 \\ J^k[k-1, k-1] \\ 0 \\ J^k[k+1, k] \end{bmatrix} \begin{bmatrix} 0 \\ J^{k+1}[k+1, k+1] \end{bmatrix}. \] (91)

Block-matrix inversion gives

\[ J^{k+1|k} = J^{k+1|k}(k+1, k+1) - \begin{bmatrix} 0 \\ 0 \\ J^{k+1|k}(k+1, k) \end{bmatrix}^T \begin{bmatrix} A_{k-1|k-1} & B_{k-1|k-1} & 0 \\ J^k[k-1, k-1] & J^k[k-1, k] & J^{k+1}[k, k] \\ 0 & 0 & J^k[k+1, k] \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ J^k[k-1, k-1] \\ 0 \\ J^k[k+1, k] \end{bmatrix}. \]

This concludes the proof of Theorem 2. □
3 Recursive Computation of Smoothing Bound

Theorem 3. The MSE matrix for the smoothing problem is bounded below by

\[
\mathbb{E}_{x_t, y_T} \{ (x_t - \hat{x}_t(y_T)) (\cdot)^T \} \succeq H(\ell, \ell) \left[ J_{\ell T} \right]^{-1} H^T(\ell, \ell),
\]

where \( 0 \leq \ell < T \) and the information matrix \( J_{\ell T} \) can be computed recursively as

\[
J_{\ell T} = \left[ J_{\ell T} + J_{\ell T}^T (\ell, \ell) - J_{\ell T}^T (\ell, \ell + 1) \right] \times \left[ J_{\ell T} + J_{\ell T}^T (\ell + 1, \ell) \left[ J_{\ell T} + J_{\ell T}^T (\ell, \ell) - J_{\ell T}^T (\ell, \ell + 1) \right]^{-1} J_{\ell T}^T (\ell, \ell + 1) \right]^{-1} J_{\ell T}^T (\ell + 1, \ell),
\]

with initialization \( J_{T T} \).

Proof. Let \( J_{T T} \) denote the sub-matrix of \( J_{T T}^T \) that contains the rows that correspond to time instance \( t_1 \) to \( t_2 \) and the columns that correspond to time instance \( t_3 \) to \( t_4 \). We decompose \( J_{T T}^T \) into blocks which correspond to time instants \( 0, 1, \ldots, \ell \) and \( \ell + 1, \ell + 2, \ldots, T \) with \( \ell < T \), yielding

\[
J_{T T} = \begin{bmatrix} J_{T T}^{0, \ell, 0, \ell} & J_{T T}^{\ell, \ell + 1, T} \\ J_{T T}^{\ell + 1, T, 0, \ell} & J_{T T}^{\ell + 1, T, \ell + 1, T} \end{bmatrix},
\]

where

\[
J_{T T}^{0, \ell, 0, \ell} = \begin{bmatrix} A & B \\ B^T & J_{T T}^T(\ell, \ell) \end{bmatrix},
\]

and

\[
J_{T T}^{\ell, \ell + 1, T} = \begin{bmatrix} J_{T T}(\ell + 1, \ell + 1) & J_{T T}(\ell + 1, \ell + 2) & 0 \\ J_{T T}(\ell + 2, \ell + 1) & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & J_{T T}(T - 1, T) \\ 0 & \ddots & \ddots & \ddots & \ddots & J_{T T}(T, T) \end{bmatrix}.
\]

Similarly, the inverse \( J_{T T}^{-1} \) can be decomposed according to

\[
J_{T T}^{-1} = \begin{bmatrix} J_{T T}^{-1}^{0, \ell, 0, \ell} & J_{T T}^{-1}^{\ell, \ell + 1, T} \\ J_{T T}^{-1}^{\ell + 1, T, 0, \ell} & J_{T T}^{-1}^{\ell + 1, T, \ell + 1, T} \end{bmatrix}.
\]

We further note that the information matrix \( J_{\ell T} \) for \( \ell < T \) can be partitioned as

\[
J_{\ell T} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B}^T & J_{\ell T}(\ell, \ell) \end{bmatrix},
\]

such that the information sub-matrix \( J_{\ell T} \) for estimating \( x_t \) is given by

\[
J_{\ell T} = J_{\ell T}(\ell, \ell) - \tilde{B}^T \tilde{A}^{-1} \tilde{B}.
\]

The matrices \( A = [ J_{T T} ]_{0, \ell - 1, 0, \ell - 1} \) and \( \tilde{A} = [ J_{\ell T} ]_{0, \ell - 1, 0, \ell - 1} \) are equivalent for \( \ell < T \), as can be shown by
repeated use of Lemma 5:

\[
\begin{align*}
\left[ J^{T} \right]_{0:T-2,0:T-2}^{T} &= \left[ J^{T-1} \right]^{T-1}_{0:T-2,0:T-2} \\
\left[ J^{T-1} \right]_{0:T-3,0:T-3}^{T-1} &= \left[ J^{T-2} \right]^{T-2}_{0:T-3,0:T-3} \\
\downarrow & \quad (101a) \\
\left[ J^{T} \right]_{0:T-3,0:T-3}^{T} &= \left[ J^{T-2} \right]^{T-2}_{0:T-3,0:T-3} \\
\left[ J^{T-2} \right]_{0:T-4,0:T-4}^{T-2} &= \left[ J^{T-3} \right]^{T-3}_{0:T-4,0:T-4} \\
\downarrow & \quad (101c) \\
\left[ J^{T} \right]_{0:T-4,0:T-4}^{T} &= \left[ J^{T-3} \right]^{T-3}_{0:T-4,0:T-4} \\
\quad \vdots & \quad (101e)
\end{align*}
\]

We introduce the following two lemmas:

**Lemma 11.** For \( \xi \leq k - 2 \) and \( k < T \) it holds that

\[
J^{T|T}(\xi, k) = 0.
\]  

(102)

**Proof.** Let \( \xi = k - 2 \) and \( \xi > 0 \). Then, we obtain

\[
\left[ J^{T}(k-2,k) \right]_{i,j} = \mathbb{E} \left\{ L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_{n_{i}(k-2)+1}, \mathbf{X}_T) L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_{n_{i}+j}, \mathbf{X}_T) \right\} - 1
\]

\[
= \mathbb{E} \left\{ K_{k-2}(\mathbf{x}_k-1, \mathbf{y}_k-2; \mathbf{x}_k-2, \mathbf{h}^{(i)}_{k-2}, \mathbf{x}_k-2, \mathbf{h}^{(i)}_{k-2}, \mathbf{x}_k-3) K_{k}(\mathbf{x}_{k+1}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_{k}, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1
\]

\[
= \mathbb{E}_{\mathbf{x}_{k-1}} \left\{ \mathbb{E}_{\mathbf{y}_{k-2}, \mathbf{x}_{k-2}, \mathbf{y}_{k-3}|x_{k-1}} \left\{ K_{k-2}(\mathbf{x}_k-1, \mathbf{y}_k-2; \mathbf{x}_k-2, \mathbf{h}^{(i)}_{k-2}, \mathbf{x}_k-2, \mathbf{h}^{(i)}_{k-2}, \mathbf{x}_k-3) \right\} \right\} - 1
\]

\[
\quad \times \mathbb{E}_{\mathbf{x}_{k+1}, \mathbf{y}_k, \mathbf{x}_k|x_{k-1}} \left\{ K_{k}(\mathbf{x}_{k+1}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_{k}, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1
\]

\[
= 0.
\]  

(103)

This can be generalized for \( 0 < \xi < k - 2 \) as follows:

\[
\left[ J^{T|T}(\xi, k) \right]_{i,j} = \mathbb{E} \left\{ L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_{n_{i}(\xi+i)}, \mathbf{X}_T) L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_{n_{i}+j}, \mathbf{X}_T) \right\} - 1
\]

\[
= \mathbb{E} \left\{ K_{\xi}(\mathbf{x}_{\xi+1}, \mathbf{y}_{\xi}; \mathbf{x}_\xi + \mathbf{h}^{(i)}_{\xi}, \mathbf{y}_{\xi}; \mathbf{x}_\xi-1) K_{k}(\mathbf{x}_{k+1}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_{k}, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1
\]

\[
= \mathbb{E}_{\mathbf{x}_{\xi+1}, \mathbf{y}_{\xi}, \mathbf{x}_{\xi-1}} \left\{ K_{\xi}(\mathbf{x}_{\xi+1}, \mathbf{y}_{\xi}; \mathbf{x}_\xi + \mathbf{h}^{(i)}_{\xi}, \mathbf{y}_{\xi}; \mathbf{x}_\xi-1) \right\}
\]

\[
\quad \times \mathbb{E}_{\mathbf{x}_{k+1}, \mathbf{y}_k, \mathbf{x}_k|x_{k-1}} \left\{ K_{k}(\mathbf{x}_{k+1}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_{k}, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1
\]

\[
= 0.
\]  

(104)

Finally, for \( \xi = 0 \) we have

\[
\left[ J^{T|T}(0, k) \right]_{i,j} = \mathbb{E} \left\{ L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_i, \mathbf{X}_T) L(\mathbf{Y}_T; \mathbf{X}_T + \mathbf{h}_{n_i+j}, \mathbf{X}_T) \right\} - 1
\]

\[
= \mathbb{E} \left\{ M_0(\mathbf{x}_i; \mathbf{x}_0 + \mathbf{h}^{(i)}_0, \mathbf{x}_0) K_{k}(\mathbf{x}_{k+1}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_k, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1
\]

\[
= \left\{ \begin{array}{ll}
\mathbb{E}_{\mathbf{x}_0|x_i} \left\{ M_0(\mathbf{x}_1; \mathbf{x}_0 + \mathbf{h}^{(i)}_0, \mathbf{x}_0) \right\} \mathbb{E}_{\mathbf{x}_{k+2}, \mathbf{y}_k|x_i} \left\{ K_{k}(\mathbf{x}_{k+2}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_k, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1, & k = 2,
\mathbb{E}_{\mathbf{x}_0|x_i} \left\{ M_0(\mathbf{x}_1; \mathbf{x}_0 + \mathbf{h}^{(i)}_0, \mathbf{x}_0) \right\} \mathbb{E}_{\mathbf{x}_{k+3}, \mathbf{y}_k,x_{k-1}|x_i} \left\{ K_{k}(\mathbf{x}_{k+3}, \mathbf{y}_k; \mathbf{x}_k + \mathbf{h}^{(i)}_k, \mathbf{x}_k; \mathbf{x}_{k-1}) \right\} - 1, & k > 2.
\end{array} \right.
\]

\[
= 0.
\]  

(105)

This concludes the proof of Lemma 11. \qed

**Lemma 12.** The following identity holds for \( k < T \)

\[
J^{T|T}(k-1, k) = J^{k|k}(k-1, k).
\]  

(106)
Proof. For \( k > 1 \) it holds
\[
\begin{align*}
\left[ J^T(k-1,k) \right]_{i,j} &= E \left\{ L(\mathbf{Y}_T; \mathbf{X}_T + h_{n_x(k-1)+i}, \mathbf{X}_T) L(\mathbf{Y}_T; \mathbf{X}_T + h_{n_x+k+j}, \mathbf{X}_T) \right\} - 1 \\
&= E \left\{ K_{k-1}(x_k, y_{k-1}; x_{k-1} + h^{(i)}_{k-1}, x_{k-1}; x_{k-2})K_{k}(x_{k+1}; y_k, x_{k} + h^{(j)}_{k}, x_{k-1}) \right\} - 1 \\
&= E \left\{ K_{k-1}(x_k, y_{k-1}; x_{k-1} + h^{(i)}_{k-1}, x_{k-1}; x_{k-2})L_k(y_k; x_{k} + h^{(j)}_{k}, x_{k-1}) \right\} - 1,
\end{align*}
\]

Now, since
\[
\begin{align*}
\left[ J^{k|k-1}(k-1,k) \right]_{i,j} &= E \left\{ L(\mathbf{Y}_k; \mathbf{X}_k + h_{n_x(k-1)+i}, \mathbf{X}_k) L(\mathbf{Y}_k; \mathbf{X}_k + h_{n_x+k+j}, \mathbf{X}_k) \right\} - 1 \\
&= E \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0)K_1(x_2, y_1; x_1 + h_1^{(j)}, x_1; x_0) \right\} - 1 \\
&= E \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0)L_1(y_1; x_1 + h_1^{(j)}, x_1; x_0) \right\} - 1.
\end{align*}
\]

holds, we have proved the first part. In case \( k = 1 \), we have
\[
\begin{align*}
\left[ J^{1|1}(0,1) \right]_{i,j} &= E \left\{ L(\mathbf{Y}_1; \mathbf{X}_1 + h_1, \mathbf{X}_1) L(\mathbf{Y}_1; \mathbf{X}_1 + h_{n_x+j}, \mathbf{X}_1) \right\} - 1 \\
&= E \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0)L_1(y_1; x_1 + h_1^{(i)}, x_1; x_0) \right\} - 1.
\end{align*}
\]

This concludes the proof of Lemma 12. \(\square\)

We are now in the position to prove that the matrices \( \mathbf{B} = [J^T]_{0:T-1,\ell} \) and \( \tilde{\mathbf{B}} = [J^{\ell|T}]_{0:T-1,\ell} \) are equivalent for \( \ell < T \):

Lemma 11, \( k = T - 1 \):
\[
\left[ J^{T|T} \right]_{0:T-3,T-1:T-1} = 0
\]
(111a)

Lemma 2, \( k = T - 1 \):
\[
\left[ J^{-1|T-1} \right]_{0:T-3,T-1:T-1} = 0
\]
(111b)

Lemma 12, \( k = T - 1 \):
\[
\left[ J^{T|T} \right]_{0:T-2,T-1:T-1} = \left[ J^{T-1|T-1} \right]_{0:T-2,T-1:T-1}
\]
(111c)

Lemma 11, \( k = T - 2 \):
\[
\left[ J^{T|T} \right]_{0:T-4,T-2:T-2} = 0
\]
(111e)

Lemma 2, \( k = T - 2 \):
\[
\left[ J^{T-2|T-2} \right]_{0:T-4,T-2:T-2} = 0
\]
(111f)

Lemma 12, \( k = T - 2 \):
\[
\left[ J^{T|T} \right]_{0:T-3,T-2:T-2} = \left[ J^{T-2|T-2} \right]_{0:T-3,T-2:T-2}
\]
(111g)

\[\vdots\]

Hence, the expression for \( J_{\ell|T} \) can be written as
\[
J_{\ell|T} = J^{\ell|T}(\ell, \ell) - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}.
\]

The information matrix inverse \( [J^{T|T}]^{-1} \) corresponding to the bound for the smoothing problem is given by the \((n_x \times n_x)\) lower-right sub-block of \( [J^{T|T}]^{-1} \) \( \ell, \ell \), or equivalently
\[
[J^{\ell|T}]^{-1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T [J^{T|T}]^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix},
\]
(113)
where \( \mathbf{0} \) is a matrix of zeros of appropriate size. Accordingly, the information matrix inverse \( [\mathbf{J}_{\ell+1|T}]^{-1} \) is given by the \((n_x \times n_x)\) upper-left sub-block of \( \left[ [\mathbf{J}^{T|T}]^{-1} \right]_{\ell+1:T,\ell+1:T} \), i.e.

\[
[\mathbf{J}_{\ell+1|T}]^{-1} = \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \left[ [\mathbf{J}^{T|T}]^{-1} \right]_{\ell+1:T,\ell+1:T} \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{0} \end{bmatrix}. \tag{114}
\]

The inverse \( \left[ [\mathbf{J}^{T|T}]^{-1} \right]_{0:\ell,0:\ell} \) can be expressed using block matrix inversion as follows

\[
\left[ [\mathbf{J}^{T|T}]^{-1} \right]_{0:\ell,0:\ell} = \left[ [\mathbf{J}^{T|T}]_{0:\ell,0:\ell} \right]^{-1} + \left[ [\mathbf{J}^{T|T}]_{0:\ell,\ell+1:T} \left[ [\mathbf{J}^{T|T}]_{\ell+1:T,\ell+1:T} \right]^{-1} [\mathbf{J}^{T|T}]_{\ell+1:T,0:\ell} \right] \times \left[ [\mathbf{J}^{T|T}]_{0:\ell,0:\ell} \right]^{-1}. \tag{115}
\]

Inserting this expression into (113) yields

\[
[\mathbf{J}_{\ell|T}]^{-1} = \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1} + \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1} \mathbf{J}^{T|T}(\ell, \ell + 1) \left[ \mathbf{J}_{\ell+1|T} \right]^{-1} \times \mathbf{J}^{T|T}(\ell + 1, \ell) \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1}, \tag{116}
\]

where we have used

\[
\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T \left[ [\mathbf{J}^{T|T}]_{0:\ell,0:\ell} \right]^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T \begin{bmatrix} * & * \\ * & \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix} = \left[ \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}^{-1} \right]^{-1} = \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1}. \tag{117a}
\]

\[
\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T \left[ \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T \right]^{-1} \left[ \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix} \right] = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix}^T \begin{bmatrix} * & * \\ * & \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_x} \end{bmatrix} = \left[ \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}^{-1} \right]^{-1} = \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1}. \tag{117b}
\]

The Woodbury formula gives

\[
[\mathbf{A} - \mathbf{U} \mathbf{C}^{-1} \mathbf{V}]^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{U} \left[ \mathbf{C} - \mathbf{V} \mathbf{A}^{-1} \mathbf{U} \right]^{-1} \mathbf{V} \mathbf{A}^{-1}. \tag{118}
\]

Now let \( \mathbf{U} \triangleq \mathbf{J}^{T|T}(\ell, \ell + 1) \), \( \mathbf{A}^{-1} \triangleq \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1} \) and \( \mathbf{V} \triangleq \mathbf{J}^{T|T}(\ell + 1, \ell) \). Then, we obtain

\[
\mathbf{C} = \mathbf{J}_{\ell+1|T} + \mathbf{J}^{T|T}(\ell + 1, \ell) \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1} \mathbf{J}^{T|T}(\ell, \ell + 1). \tag{119}
\]

Thus, we finally obtain the recursion

\[
\mathbf{J}_{\ell|T} = \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right] - \mathbf{J}^{T|T}(\ell, \ell + 1)
\times \left[ \mathbf{J}_{\ell+1|T} + \mathbf{J}^{T|T}(\ell + 1, \ell) \left[ \mathbf{J}_{\ell|\ell} + \mathbf{J}^{T|T}(\ell, \ell) - \mathbf{J}^{T|T}(\ell, \ell) \right]^{-1} \mathbf{J}^{T|T}(\ell, \ell + 1) \right]^{-1} \mathbf{J}^{T|T}(\ell, \ell + 1). \tag{120}
\]

This concludes the proof of Theorem 3.
4 Relation to Bayesian Cramér-Rao Lower Bound

Lemma 13 (Lemma 6 in the paper). For the particular choice of the matrix $H^{(T)} = h \cdot I_{n_x (T+1) \times n_x (T+1)}$ with $h \to 0$, a Taylor series expansion of the $J$-terms ($0 \leq \ell < T$) yields

\begin{align}
J^{k|k}(k, k) &= h^2 D_{x_{k-1}} \mathcal{D} + o(h^2), \quad (121a) \\
J^{k|k}(k-1, k) &= h^2 D_{x_{k-1}} \mathcal{D} + o(h^2), \quad (121b) \\
J^{k|k}(k-1, k-1) - J^{k-1|k-1}(k-1, k-1) &= h^2 D_{x_{k-1}} \mathcal{D} + o(h^2), \quad (121c) \\
J_{\mathcal{T}T}(\ell, \ell) - J^{\ell|\ell}(\ell, \ell) &= h^2 D_{x_{\ell+1}} \mathcal{D} + o(h^2), \quad (121d) \\
J_{\mathcal{T}T}(\ell, \ell+1) &= h^2 D_{x_{\ell+2}} \mathcal{D} + o(h^2). \quad (121f)
\end{align}

Proof. A multi-dimensional Taylor series expansion of the numerator of $L_\ell(y_k; x_k + h_k^{(i)}, x_k; x_{k-1})$ around $(x_k - h_k^{(i)})$ with $h_k^{(i)} = [0_{1 \times n_x - 1}, h, 0_{1 \times n_x + 1}]^T$ gives

\begin{equation}
p(y_k|x_k + h_k^{(i)})p(x_k + h_k^{(i)}|x_{k-1}) = p(y_k|x_k)p(x_k|x_{k-1}) + \frac{\partial [p(y_k|x_k)p(x_k|x_{k-1})]}{\partial x_{k,i}} h + o(h), \quad (122)
\end{equation}

where the Little-o notation $o(h)$ is used to express that contributions from higher-order terms of the Taylor series expansion can be neglected as $h \to 0$. Hence, $L_\ell$ can be written as

\begin{equation}
L_\ell(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}) = \frac{p(y_k|x_k + h_k^{(i)})p(x_k + h_k^{(i)}|x_{k-1})}{p(y_k|x_k)p(x_k|x_{k-1})} = 1 + h \left( \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} + \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right) + o(h). \quad (123)
\end{equation}

Similarly, a Taylor series expansion of the numerator of $K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_k^{(i)}; x_{k-1}; x_{k-2})$ around $(x_{k-1} - h_{k-1}^{(i)})$ gives

\begin{align}
p(x_k|x_{k-1} + h_{k-1}^{(i)})p(y_{k-1}|x_{k-1} + h_{k-1}^{(i)})p(x_{k-1} + h_{k-1}^{(i)}|x_{k-2}) &= \\
p(x_k|x_{k-1})p(y_{k-1}|x_{k-1})p(x_{k-1}|x_{k-2}) + \frac{\partial [p(x_k|x_{k-1})p(y_{k-1}|x_{k-1})p(x_{k-1}|x_{k-2})]}{\partial x_{k-1,i}} h + o(h). \quad (124)
\end{align}

Thus, $K_{k-1}$ can be written as

\begin{equation}
K_{k-1}(x_k; y_{k-1}; x_{k-1} + h_{k-1}^{(i)}; x_{k-1}; x_{k-2}) = \frac{p(x_k|x_{k-1} + h_{k-1}^{(i)})p(y_{k-1}|x_{k-1} + h_{k-1}^{(i)})p(x_{k-1} + h_{k-1}^{(i)}|x_{k-2})}{p(x_k|x_{k-1})p(y_{k-1}|x_{k-1})p(x_{k-1}|x_{k-2})} = 1 + h \left( \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(y_{k-1}|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k-1,i}} \right) + o(h). \quad (125)
\end{equation}

We further introduce the following intermediate relations

\begin{align}
\mathbb{E}_{y_k|x_k} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \right\} &= \frac{\partial}{\partial x_{k,i}} \int p(y_k|x_k) dy_k = \frac{\partial}{\partial x_{k,i}} \left( \int p(y_k|x_k) dy_k \right) = 0, \quad (126a) \\
\mathbb{E}_{x_{k-1}|x_k} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} \right\} &= \frac{\partial}{\partial x_{k-1,i}} \left( \int p(x_k|x_{k-1}) dx_k \right) = 0, \quad (126b) \\
\mathbb{E}_{x_{k-1}|x_k} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right\} &= \frac{\partial}{\partial x_{k,i}} \left( \int p(x_k|x_{k-1}) dx_k \right) = 0. \quad (126c)
\end{align}
The \((i,j)\)-th element of the matrix \(J^{k|k}(k, k)\) can be thus rewritten as

\[
J^{k|k}(k, k)_{i,j} = \mathbb{E} \left\{ L_k(y_k; x_k + h_k^{(i)}, x_k; x_{k-1}) L_k(y_k; x_k + h_k^{(j)}, x_k; x_{k-1}) - 1 \right\} = \mathbb{E} \left\{ \left( 1 + h \left( \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} + \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right) + o(h) \right) \times \left( 1 + h \left( \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,j}} + \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right) + o(h) \right) - 1 \right\} = 1 + h \cdot \mathbb{E} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \right\} + h \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right\} + h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,j}} \right\} + h \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} + h^2 \mathbb{E} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right\} + h^2 \mathbb{E} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,j}} \right\} + o(h^2) - 1 \right\} = h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \right\} + h^2 \mathbb{E} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \right\} + o(h^2),
\]

where we have used (126) and the tower property of conditional expectations

\[
\mathbb{E}_{y_k|x_k,x_{k-1}} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} = \mathbb{E}_{x_k} \left\{ \mathbb{E}_{y_k|x_k}|x_{k-1} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} \right\} = \mathbb{E}_{x_k} \left\{ \mathbb{E}_{y_k|x_k} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \right\} \cdot \mathbb{E}_{x_{k-1}|x_k} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} \right\} = 0.
\]

Note, further, that the following identities hold

\[
\left[D^{22,a}_{k-1}\right]_{i,j} = \mathbb{E} \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,i}} \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} = -\mathbb{E} \left\{ \frac{\partial^2 \ln p(x_k|x_{k-1})}{\partial x_{k,i} \partial x_{k,j}} \right\}, \quad (129a)
\]

\[
\left[D^{22,b}_{k-1}\right]_{i,j} = \mathbb{E} \left\{ \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,i}} \frac{\partial \ln p(y_k|x_k)}{\partial x_{k,j}} \right\} = -\mathbb{E} \left\{ \frac{\partial^2 \ln p(y_k|x_k)}{\partial x_{k,i} \partial x_{k,j}} \right\}. \quad (129b)
\]

Hence, \(J^{k|k}(k, k)\) can be written as

\[
J^{k|k}(k, k) = h^2 D^{22}_{k-1} + o(h^2),
\]

where

\[
D^{22}_{k-1} = D^{22,a}_{k-1} + D^{22,b}_{k-1} = \mathbb{E} \{-\Delta x_k \ln p(x_k|x_{k-1})\} + \mathbb{E} \{-\Delta x_k \ln p(y_k|x_k)\}.
\]

(131)
The \((i, j)\)-th element of the matrix \(J^{k}(k-1,k-1)\) for \(k > 1\) can be written as

\[
J^{k}(k-1,k-1)_{i,j} = E \left\{ K_{k-1}(x_k, y_{k-1}; x_{k-1} + h^{(i)}_{k-1}, x_{k-1}; x_{k-2}) - 1 \right\} - 1
\]

\[
\quad = E \left\{ \left( 1 + h \left( \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(y_{k-1}|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k-1,i}} + o(h) \right) \right) \times \left( 1 + h \left( \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,j}} + \frac{\partial \ln p(y_{k-1}|x_{k-1})}{\partial x_{k-1,j}} + \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k-1,j}} + o(h) \right) \right) - 1 \right\}
\]

\[
\quad = 1 + h \cdot E \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} + h^2 \cdot E \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,j}} \right\} - 1 \right\} - 1
\]

\[
\quad = h^2 \cdot E \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,j}} \right\}
\]

where we have used (126) and

\[
E \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} \right\} = 0,
\]

\[
E \left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,j}} \right\} = 0,
\]

We further note that

\[
J^{k-1,k-1}(k-1,k-1)_{i,j} = h^2 \cdot E \left\{ \frac{\partial \ln p(y_{k-1}|x_{k-1})}{\partial x_{k-1,i}} \right\}
\]

\[
\quad = h^2 \cdot E \left\{ \frac{\partial \ln p(y_{k-1}|x_{k-1})}{\partial x_{k-1,j}} \right\}
\]

Thus, after rearranging we arrive at

\[
J^{k,k-1}(k-1,k-1)_{i,j} = h^2 \cdot D^{11}_{k-1} + o(h^2),
\]
where

\[
D_{k-1}^{11} = \mathbb{E}\left\{ -\Delta x_{k-1} \ln p(x_k|x_{k-1}) \right\}.
\] (137)

For \( k = 1 \), we use a Taylor series expansion of \( p(x_1|x_0 + h_0^{(i)})p(x_0 + h_0^{(i)}) \) around \( (x_0 - h_0^{(i)}) \)

\[
p(x_1|x_0 + h_0^{(i)})p(x_0 + h_0^{(i)}) = p(x_1|x_0)p(x_0) + \frac{\partial p(x_1|x_0)p(x_0)}{\partial x_{0,i}} h + o(h).
\] (138)

Thus, we arrive at

\[
\left[ J^{11}(0,0) \right]_{i,j} = \mathbb{E}\left\{ M_0(x_1;x_0 + h_0^{(i)},x_0)M_0(x_1;x_0 + h_0^{(j)},x_0) \right\} - 1 = \left[ \mathbb{E}\left\{ \left( 1 + h \left( \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,i}} + \frac{\partial \ln p(x_0)}{\partial x_{0,j}} \right) + o(h) \right) \left( 1 + h \left( \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,j}} + \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \right) + o(h) \right) \right\} - 1 = h^2 \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,i}} + \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,j}} \right\} + o(h^2),
\] (139)

where we have used (126) and

\[
\mathbb{E}_{x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \right\} = \int \frac{\partial p(x_0)}{\partial x_{0,i}} dx_0 = \left[ p(x_0) \right]_{-\infty}^\infty = 0,
\] (140a)

\[
\mathbb{E}_{x_1,x_0} \left\{ \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,i}} \frac{\partial \ln p(x_0)}{\partial x_{0,j}} \right\} = \mathbb{E}_{x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \cdot \mathbb{E}_{x_1|x_0} \left\{ \frac{\partial \ln p(x_1|x_0)}{\partial x_{0,j}} \right\} \right\} = 0.
\] (140b)

We further note that

\[
\left[ J^{00}(0,0) \right]_{i,j} = h^2 \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \frac{\partial \ln p(x_0)}{\partial x_{0,j}} \right\} + o(h^2).
\] (141)

Hence, after rearranging we arrive at the following expression

\[
J^{11}(0,0) - J^{00}(0,0) = h^2 D_0^{11} + o(h^2).
\] (142)

A Taylor series expansion of \( p(x_k|x_{k-1} + h_{k-1}^{(i)})p(x_{k-1} + h_{k-1}^{(i)}) \) around \( (x_{k-1}, x_{k-2}) \) and of \( p(x_k + h_k^{(j)})x_{k-1} \) around \( (x_k - h_k^{(j)}) \) gives

\[
p(x_k|x_{k-1} + h_{k-1}^{(i)})p(x_{k-1} + h_{k-1}^{(i)}) = p(x_k|x_{k-1})p(x_{k-1}|x_{k-2}) + \frac{\partial [p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})]}{\partial x_{k-1,i}} h + o(h),
\] (143)

\[
p(x_k + h_k^{(j)}|x_{k-1}) = p(x_k|x_{k-1}) + \frac{\partial p(x_k|x_{k-1})}{\partial x_{k,j}} h + o(h).
\] (144)

Thus, the \((i,j)\)-th element of the matrix \( J^{i|k}(k-1,k) \) for \( k > 1 \) can be written as

\[
\left[ J^{i|k}(k-1,k) \right]_{i,j} = \mathbb{E}\left\{ K_{k-1}(x_k,y_k-1|x_{k-1} + h_{k-1}^{(i)}),x_{k-1} + h_{k-1}^{(i)},x_{k-2}) L_k(y_k;x_k + h_k^{(j)},x_k) \right\} - 1
\]

\[
= \mathbb{E}\left\{ \frac{p(x_k|x_{k-1} + h_{k-1}^{(i)})p(x_{k-1} + h_{k-1}^{(i)}|x_{k-2})}{p^2(x_k|x_{k-1})p(x_{k-1}|x_{k-2})} \right\} - 1
\]

\[
= \mathbb{E}\left\{ \left( 1 + h \left( \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} + \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k-1,i}} \right) + o(h) \right) \left( 1 + h \left( \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} + \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k,j}} \right) + o(h) \right) \right\} - 1
\]

\[
= 1 + h \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} + h \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} \right\} + h \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k-1,i}} \right\} + h \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k,j}} \right\} + h^2 \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_{k-1}|x_{k-2})}{\partial x_{k,j}} \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} \right\} + o(h^2) - 1
\]

\[
= h^2 \cdot \mathbb{E}\left\{ \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k-1,i}} \frac{\partial \ln p(x_k|x_{k-1})}{\partial x_{k,j}} \right\} + o(h^2)
\] (145)
where we have used (126) and the tower property of conditional expectations

\[
\mathbb{E}_{x_k; x_{k-1}, x_{k-2}} \left\{ \frac{\partial \ln p(x_{k-1} | x_{k-2})}{\partial x_{k-1,i}} \cdot \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\}
\]

\[
= \mathbb{E}_{x_{k-1}} \left\{ \mathbb{E}_{x_k; x_{k-2} | x_{k-1}} \left\{ \frac{\partial \ln p(x_{k-1} | x_{k-2})}{\partial x_{k-1,i}} \cdot \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\} \right\}
\]

\[
= \mathbb{E}_{x_{k-1}} \left\{ \mathbb{E}_{x_k | x_{k-1}} \left\{ \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,i}} \right\} \cdot \mathbb{E}_{x_{k-2} | x_{k-1}} \left\{ \frac{\partial \ln p(x_{k-1} | x_{k-2})}{\partial x_{k,j}} \right\} \right\}
\]

\[
= 0.
\]

Now, since the following identity holds

\[
[D_{k-1}^{12}]_{i,j} \triangleq \mathbb{E} \left\{ \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k-1,i}} \cdot \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\} = -\mathbb{E} \left\{ \frac{\partial^2 \ln p(x_k | x_{k-1})}{\partial x_{k-1,i} \partial x_{k,j}} \right\},
\]

the expression \( J^{k|k}(k-1, k) \) can be written as

\[
J^{k|k}(k-1, k) = h^2 D_{k-1}^{12} + o(h^2),
\]

where

\[
D_{k-1}^{12} = \mathbb{E} \left\{ -\Delta_{x_{k-1}} \ln p(x_k | x_{k-1}) \right\}.
\]

For \( k = 1 \) we arrive at

\[
\left[ J^{1|1}(0, 1) \right]_{i,j} = \mathbb{E} \left\{ M_0(x_1; x_0 + h_0^{(i)}, x_0)L_1(y_1; x_1 + h_1^{(j)}, x_1; x_0) \right\} - 1
\]

\[
= \mathbb{E} \left\{ \frac{p(x_1 | x_0 + h_0^{(i)})p(x_0 + h_0^{(i)} | x_0)}{p^2(x_1 | x_0)p(x_0)} \right\} - 1
\]

\[
= \mathbb{E} \left\{ \left( \frac{1}{h} \left( \frac{\partial \ln p(x_1 | x_0)}{\partial x_{0,i}} + \frac{\partial \ln p(x_0)}{\partial x_{0,j}} \right) + o(h) \right) \left( 1 + h \left( \frac{\partial \ln p(x_1 | x_0)}{\partial x_{1,j}} \right) + o(h) \right) \right\} - 1
\]

\[
= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_1 | x_0)}{\partial x_{0,i}} \cdot \frac{\partial \ln p(x_1 | x_0)}{\partial x_{1,j}} \right\} + o(h^2)
\]

\[
= h^2 D_0^{12} + o(h^2),
\]

where we have used (126) and

\[
\mathbb{E}_{x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \right\} = 0,
\]

\[
\mathbb{E}_{x_1, x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \cdot \frac{\partial \ln p(x_1 | x_0)}{\partial x_{1,j}} \right\} = \mathbb{E}_{x_1, x_0} \left\{ \frac{\partial \ln p(x_1 | x_0)}{\partial x_{1,j}} \cdot \mathbb{E}_{x_0} \left\{ \frac{\partial \ln p(x_1 | x_0)}{\partial x_{0,i}} \right\} \right\} = 0.
\]

The \((i,j)\)-th element of the matrix \( J^{k|k-1}(k, k) \) can be written as

\[
\left[ J^{k|k-1}(k, k) \right]_{i,j} = \mathbb{E} \left\{ \frac{p(x_k + h_k^{(i)} | x_{k-1})p(x_k + h_k^{(j)} | x_{k-1})}{p^2(x_k | x_{k-1})} \right\} - 1
\]

\[
= \mathbb{E} \left\{ \left( 1 + h \left( \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,i}} \right) + o(h) \right) \left( 1 + h \left( \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right) \right) \right\} - 1
\]

\[
= 1 + h \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,i}} + \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\}
\]

\[
+ h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,i}} \cdot \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\} + o(h^2)
\]

\[
= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,i}} \cdot \frac{\partial \ln p(x_k | x_{k-1})}{\partial x_{k,j}} \right\} + o(h^2).
\]

Hence,

\[
J^{k|k-1}(k, k) = h^2 D_{k-1}^{22} + o(h^2).
\]
The \((i,j)\)-th element of the matrix \(J^{T|T}(\ell, \ell)\) for \((0 < \ell < T)\) can be written as
\[
\begin{align*}
\left[ J^{T|T}(\ell, \ell) \right]_{i,j} &= \mathbb{E} \left\{ K_\ell(x_{\ell+1}, y; x_{\ell} + h^{(i)}_\ell, x_{\ell}; x_{\ell-1}) K_\ell(x_{\ell+1}, y; x_{\ell} + h^{(j)}_\ell, x_{\ell}; x_{\ell-1}) \right\} - 1 \\
&= \mathbb{E} \left\{ \left( 1 + h \left( \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell,i}} + \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,i}} + \frac{\partial \ln p(x_{\ell}\mid x_{\ell-1})}{\partial x_{\ell,i}} \right) + o(h) \right) \right. \\
&\quad \times \left. \left( 1 + h \left( \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell,j}} + \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,j}} + \frac{\partial \ln p(x_{\ell}\mid x_{\ell-1})}{\partial x_{\ell,j}} \right) + o(h) \right) \right\} - 1 \\
&= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell,i}} \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell,j}} \right\} + h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,i}} \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,j}} \right\} + o(h^2).
\end{align*}
\]
(154)

We further note that
\[
\begin{align*}
\left[ J^{T}(\ell, \ell) \right]_{i,j} &= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,i}} \frac{\partial \ln p(y_{\ell}\mid x_{\ell})}{\partial x_{\ell,j}} \right\} \\
&\quad + h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_{\ell}\mid x_{\ell-1})}{\partial x_{\ell,i}} \frac{\partial \ln p(x_{\ell}\mid x_{\ell-1})}{\partial x_{\ell,j}} \right\} + o(h^2).
\end{align*}
\]
(155)

Hence, after rearranging we arrive at the following expression
\[
J^{T|T}(\ell, \ell) - J^{T}(\ell, \ell) = h^2 D^{11}_\ell + o(h^2).
\]
(156)

For the case \(\ell = 0\), we arrive at
\[
\begin{align*}
\left[ J^{T|T}(0, 0) \right]_{i,j} &= \mathbb{E} \left\{ M_0(x_1; x_0 + h^{(i)}_0, x_0) M_0(x_1; x_0 + h^{(j)}_0, x_0) \right\} - 1 \\
&= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_1\mid x_0)}{\partial x_{0,i}} \frac{\partial \ln p(x_1\mid x_0)}{\partial x_{0,j}} \right\} + h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_0)}{\partial x_{0,i}} \frac{\partial \ln p(x_0)}{\partial x_{0,j}} \right\} + o(h^2).
\end{align*}
\]
(157)

This can be rearranged, yielding
\[
J^{T|T}(0, 0) - J^{00}(0, 0) = h^2 D^{11}_0 + o(h^2).
\]
(158)

The \((i,j)\)-th element of the matrix \(J^{T|T}(\ell, \ell + 1)\) for \(\ell = T - 1\) and \(\ell > 0\) can be written as
\[
\begin{align*}
\left[ J^{T|T}(T - 1, T) \right]_{i,j} &= \mathbb{E} \left\{ K_{T-1}(x_T, y_{T-1}; x_{T-1} + h^{(i)}_{T-1}, x_{T-1}; x_{T-2}) L_{T}(y_T; x_T + h^{(j)}_{T}, x_T; x_{T-1}) \right\} - 1 \\
&= \mathbb{E} \left\{ \frac{p(x_T\mid x_{T-1} + h^{(i)}_{T-1}) p(x_{T-1} + h^{(i)}_{T-1}) p(x_{T-2}) p(x_T + h^{(j)}_{T} \mid x_{T-1})}{p(x_T\mid x_{T-1}) p(x_{T-1} \mid x_{T-2})} \right\} - 1 \\
&= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_T\mid x_{T-1})}{\partial x_{T-1,i}} \frac{\partial \ln p(x_T\mid x_{T-1})}{\partial x_{T-1,j}} \right\} + o(h^2).
\end{align*}
\]
(159)

The \((i,j)\)-th element of the matrix \(J^{T|T}(\ell, \ell + 1)\) for \(0 < \ell < T - 1\) is given by
\[
\begin{align*}
\left[ J^{T|T}(\ell, \ell + 1) \right]_{i,j} &= \mathbb{E} \left\{ K_\ell(x_{\ell+1}, y; x_{\ell} + h^{(i)}_\ell, x_{\ell}; x_{\ell-1}) K_{\ell+1}(x_{\ell+2}, y_{\ell+1}; x_{\ell+1} + h^{(j)}_{\ell+1}, x_{\ell+1}; x_{\ell}) \right\} - 1 \\
&= \mathbb{E} \left\{ \frac{p(x_{\ell+1}\mid x_{\ell} + h^{(i)}_\ell) p(x_{\ell} + h^{(i)}_\ell \mid x_{\ell-1}) p(x_{\ell+1} + h^{(j)}_{\ell+1} \mid x_{\ell})}{p^2(x_{\ell+1}\mid x_{\ell}) p(x_{\ell}\mid x_{\ell-1})} \right\} - 1 \\
&= h^2 \cdot \mathbb{E} \left\{ \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell,i}} \frac{\partial \ln p(x_{\ell+1}\mid x_{\ell})}{\partial x_{\ell+1,j}} \right\} + o(h^2).
\end{align*}
\]
(160)

Hence, \(J^{T|T}(\ell, \ell + 1)\) can be written as
\[
J^{T|T}(\ell, \ell + 1) = h^2 D^{12}_\ell + o(h^2).
\]
(161)
For the case $\ell = 0$ we obtain
\[
\left[ J_{T}^{T}(0,1) \right]_{i,j} = E \left\{ M_0(x_1; x_0 + h_0^{(i)} , x_0) K_1(x_2; y_1; x_1 + h_1^{(j)} , x_1; x_0) \right\} - 1
\]
\[
= E \left\{ \frac{p(x_1| x_0 + h_0^{(i)}) p(x_0 + h_0^{(i)}| x_0) p(x_1 + h_1^{(j)}| x_1)}{p^2(x_1| x_0) p(x_0)} \right\} - 1
\]
\[
= E \left\{ \left( 1 + h \left( \frac{\partial \ln p(x_1| x_0)}{\partial x_0,i} + \frac{\partial \ln p(x_0)}{\partial x_0,i} \right) + o(h) \right) \left( 1 + h \left( \frac{\partial \ln p(x_1| x_0)}{\partial x_1,j} + o(h) \right) \right) \right\} - 1
\]
\[
= h^2 E \left\{ \frac{\partial \ln p(x_1| x_0)}{\partial x_0,i} \frac{\partial \ln p(x_1| x_0)}{\partial x_1,j} \right\} + o(h^2)
\]
where we have used (126) and the fact that
\[
E_{x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_0,i} \right\} = 0,
\]
\[
E_{x_1; x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_0,i} \frac{\partial \ln p(x_1| x_0)}{\partial x_1,j} \right\} = E_{x_0} \left\{ \frac{\partial \ln p(x_0)}{\partial x_0,i} \cdot E_{x_1| x_0} \left\{ \frac{\partial \ln p(x_1| x_0)}{\partial x_1,j} \right\} \right\} = 0
\]
holds. This concludes the proof of Lemma 13.

\[ \square \]

**Theorem 4** (Theorem 5 in the paper). For the particular choice of the matrix $H^{(T)} = h I_{n_x(T+1) \times n_x(T+1)}$ with $h \to 0$, and given the fact that the joint distribution $p(Y_k, X_k)$ satisfies the BCRLB regularity conditions, the BZLB recursion for filtering, prediction and smoothing presented in Theorem 1, 2 and 3, reduces to the BCRLB recursion given in Theorem 4 of the paper.

**Proof.** The filtering recursion (38) can be stated using (121) as follows
\[
J_{k|k} = h^2 D_{k-1}^{22} + o(h^2) - \left[ h^2 D_{k-1}^{21} + o(h^2) \right] \left[ J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2) \right]^{-1} \left[ h^2 D_{k-1}^{12} + o(h^2) \right]
\]
\[
= h^2 \left[ D_{k-1}^{22} - D_{k-1}^{21} \left[ J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2) \right]^{-1} D_{k-1}^{12} \right] - o(h^2) \left[ J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2) \right]^{-1} D_{k-1}^{12}
\]
\[
+ \left[ D_{k-1}^{21} J_{k-1|k-1} + h^2 D_{k-1}^{11} \right]^{-1} o(h^2) - \left[ h^2 J_{k-1|k-1} + h^4 D_{k-1}^{11} + o(h^2) \right]^{-1} o(h^2)
\]
\[
+ \frac{o(h^2)}{h^2}.
\]

Let $\tilde{J}_{k|k}$ denote the inverse of the filtering BZLB in the case of $h \to 0$, i.e.
\[
\tilde{J}_{k|k} = \lim_{h \to 0} \left[ H(k,k) J_{k|k}^{-1} H^T(k,k) \right]^{-1} = \lim_{h \to 0} \frac{J_{k|k}}{h^2}.
\]

Then, the recursion can be written as
\[
\tilde{J}_{k|k} = \lim_{h \to 0} \left[ D_{k-1}^{22} - D_{k-1}^{21} \left[ J_{k-1|k-1} + h^2 D_{k-1}^{11} \right]^{-1} D_{k-1}^{12} \right] = D_{k-1}^{22} - D_{k-1}^{21} \left[ \lim_{h \to 0} \frac{J_{k-1|k-1}}{h^2} + D_{k-1}^{11} \right]^{-1} D_{k-1}^{12}
\]
\[
= D_{k-1}^{22} - D_{k-1}^{21} \left[ \tilde{J}_{k-1|k-1} + D_{k-1}^{11} \right]^{-1} D_{k-1}^{12}
\]
with initialization
\[
J_{0|0} = h^2 E \{- \Delta x_0^0 \ln p(x_0) \} + o(h^2)
\]
\[
\lim_{h \to 0} \frac{J_{0|0}}{h^2} = E \{- \Delta x_0^0 \ln p(x_0) \} + \lim_{h \to 0} \frac{o(h^2)}{h^2}
\]
\[
\tilde{J}_{0|0} = E \{- \Delta x_0^0 \ln p(x_0) \}.
\]
The prediction recursion (49) can be stated using (121) as follows

\[
J_{k|k-1} = h^2 D_{k-1}^{22,a} + o(h^2) - [h^2 D_{k-1}^{21} + o(h^2)] [J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2)]^{-1} [h^2 D_{k-1}^{12} + o(h^2)]
\]

\[
= h^2 \left[ D_{k-1}^{22,a} - D_{k-1}^{21} \left( \frac{J_{k-1|k-1}}{h^2} + D_{k-1}^{11} + o(h^2) \right)^{-1} D_{k-1}^{12} - D_{k-1}^{21} [J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2)]^{-1} D_{k-1}^{12} \right. \\
\left. - D_{k-1}^{21} [J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2)]^{-1} o(h^2) - o(h^2) [h^2 J_{k-1|k-1} + h^2 D_{k-1}^{11} + o(h^2)]^{-1} o(h^2) + o(h^2) \right]
\]

(168)

Let \( \tilde{J}_{k|k-1} \) denote the inverse of the prediction BZLB in the case of \( h \to 0 \), i.e.

\[
\tilde{J}_{k|k-1} = \lim_{h \to 0} \left[ H(k,k) [J_{k|k-1}]^{-1} H^T(k,k) \right]^{-1} = \lim_{h \to 0} \frac{J_{k|k-1}}{h^2}.
\]

(169)

Then, the recursion can be written as

\[
\tilde{J}_{k|k-1} = \lim_{h \to 0} \left[ D_{k-1}^{22,a} - D_{k-1}^{21} \left[ J_{k-1|k-1} + D_{k-1}^{11} \right]^{-1} D_{k-1}^{12} \right]
\]

\[
= D_{k-1}^{22,a} - D_{k-1}^{21} \left[ J_{k-1|k-1} + D_{k-1}^{11} \right]^{-1} D_{k-1}^{12}
\]

(170)

with initialization \( \tilde{J}_{0|0} \). The smoothing recursion (95) can be stated using (121) as follows

\[
J_{\ell|T} = [J_{\ell+1|T} + h^2 D_{\ell}^{11} + o(h^2)] - [h^2 D_{\ell}^{12} + o(h^2)] \times [J_{\ell+1|T} + h^2 D_{\ell}^{21} + o(h^2)] [J_{\ell|T} + h^2 D_{\ell}^{11} + o(h^2)]^{-1} [h^2 D_{\ell}^{12} + o(h^2)]^{-1} [h^2 D_{\ell}^{21} + o(h^2)]
\]

\[
= h^2 \left[ \frac{J_{\ell|T}}{h^2} + D_{\ell}^{11} + o(h^2) \right] - D_{\ell}^{12} \times \left[ \frac{J_{\ell+1|T}}{h^2} + D_{\ell}^{21} + o(h^2) \right]^{-1} [D_{\ell}^{12} + o(h^2)]^{-1} \left[ D_{\ell}^{21} + o(h^2) \right]
\]

(171)

Let \( \tilde{J}_{\ell|T} \) denote the inverse of the smoothing BZLB in the case of \( h \to 0 \), i.e.

\[
\tilde{J}_{\ell|T} = \lim_{h \to 0} \left[ H(\ell,\ell) [J_{\ell|T}]^{-1} H^T(\ell,\ell) \right]^{-1} = \lim_{h \to 0} \frac{J_{\ell|T}}{h^2}.
\]

(172)

Then, the recursion can be written as

\[
\tilde{J}_{\ell|T} = \lim_{h \to 0} \left( \frac{J_{\ell|T}}{h^2} + D_{\ell}^{11} \right) - D_{\ell}^{12} \left[ \frac{J_{\ell+1|T}}{h^2} + D_{\ell}^{21} [J_{\ell|T} + D_{\ell}^{11}]^{-1} D_{\ell}^{12} \right]^{-1} D_{\ell}^{21}
\]

(173)

This concludes the proof of Theorem 4.

\[\square\]

References

Supplementary Material for “Bobrovsky-Zakai Bound for Filtering, Prediction and Smoothing of Nonlinear Dynamic Systems”

This report contains supplementary material for the paper [1], and gives detailed proofs of all lemmas and theorems that could not be included into the paper due to space limitations. The notation is adapted from the paper.

Keywords: performance bounds, nonlinear dynamic systems, mean square error