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MASTER THESIS

Supersymmetric Quantum Mechanics, Index Theorems and Equivariant Cohomology

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Abstract

In this thesis, we investigate supersymmetric quantum mechanics (SUSYQM) and its relation to index theorems and equivariant cohomology. We define some basic constructions on super vector spaces in order to set the language for the rest of the thesis. The path integral in quantum mechanics is reviewed together with some related calculational methods and we give a path integral expression for the Witten index. Thereafter, we discuss the structure of SUSYQM in general. One shows that the Witten index can be taken to be the difference in dimension of the bosonic and fermionic zero energy eigenspaces. In the subsequent section, we derive index theorems. The models investigated are the supersymmetric non-linear sigma models with one or two supercharges. The former produces the index theorem for the spin-complex and the latter the Chern-Gauss-Bonnet Theorem. We then generalise to the case when a group action (by a compact connected Lie group) is included and want to consider the orbit space as the underlying space, in which case equivariant cohomology is introduced. In particular, the Weil and Cartan models are investigated and SUSYQM Lagrangians are derived using the obtained differentials. The goal was to relate this to gauge quantum mechanics, which was unfortunately not successful. However, what was shown was that the Euler characteristics of a closed oriented manifold and its homotopy quotient by $U(1)^n$ coincide.

Populärvetenskaplig sammanfattning

I naturen finns två sortes partiklar: bosoner och fermioner. *Supersymmetri* är en förmodad symmetri som relaterar dessa. En *symmetri* i en fysikalisk teori kan ses som en transformation som lämnar det fysikaliska innehållet i teorin oförändrat och supersymmetri är en symmetri som inbegriper ett utbyte av bosoner och fermioner. Det finns djupa relationer mellan supersymmetri och geometri. Ett av de centrala begreppen i denna uppsats är det så kallade *Witten-indexet* (även känt som det *supersymmetriska indexet*), som räknar skillnaden mellan antalet bosoniska tillstånd och fermioniska tillstånd. Samtidigt är det också en så kallad *topologisk invariant* av teorin, det vill säga att Witten-indexet är detsamma oberoende av hur vi kontinuerligt deformerar rummet som vår teori är definierad över, så länge vi kan kontinuerligt deformera tillbaka den. I detta arbete undersöker vi Witten-indexet genom supersymmetrisk kvantmekanik. Genom att starta med en klassisk teori kan man övergå till en kvantmekanisk teori via en process som kallas *kvantisering*. Här använder vi oss av två metoder: *kanonisk kvantisering* och *vägintegralkvantisering*.

I en supersymmetrisk kvantteori delas tillståndsrummet, *Hilbertrummet*, in i två delar: en bosonisk del och en fermionisk del. När man utför kanonisk kvantisering hittas Hilbertrummets struktur och de operatorer som agerar på tillstånden. Framförallt är vi intresserade av *superladdningarna*. Dessa operatorer omvandlar ett bosoniskt tillstånd till ett fermioniskt och vice versa. De uppstår från supersymmetrin i sig självt; enligt *Noethers sats* finns för varje (kontinuerlig) symmetri en kvantitet som är bevarad, vilket i detta fall motsvarar just superladdingen. Från dessa objekt kan vi sedan finna *Hamiltonianen*, operatorn som beräknar energin hos sina *egentillstånd* vilka är de tillstånd som lämnas oförändrade upp till en faktor av operatorn. Det visar sig att i alla energinivåer utom i just fallet när energin är noll, är bosonerna lika många som fermionerna. På så sätt beräknar Witten-indexet differensen mellan antalet bosoner och antalet fermioner med noll energi.

Vid vägintegralkvantisering används så kallade *vägintegraler*, vilka kan ses som ett sätt att beräkna sannolikheten för att en partikel i ett tillstånd ska hamna i ett annat. Det man i princip gör är att lägga ihop bidrag från alla möjliga vägar som en partikel skulle kunna ta och vikta varje väg med en faktor som talar om hur sannolik vägen i fråga är. Wittenindexet visar sig kunna skrivas som en vägintegral.

Det finns alltså två olika sätt att beräkna Witten-indexet på och genom att då relatera de två uttrycken som erhålls, får man en *indexsats*. Dessa matematiska satser relaterar, kort beskrivet, global topologisk data med lokal geometrisk information. De fysikaliska system som undersöks i detta fall är så kallade *supersymmetriska ickelinjära sigmamodeller* som ursprungligen består av en bosonisk partikel som rör sig i ett rum med krökning. Denna modell kan sedan göras supersymmetrisk. Vi undersöker fallen då vi har en och två superladdningar. Fallet med en superladdning resulterar i indexsatsen för det så kallade *spin-komplexet*. I fallet med två superladdningar får vi den välkända *Chern-Gauss-Bonnets sats*, en sats som grovt uttryckt relaterar antal hål i rummet (global data) med rummets kurvatur (lokal data).

Inspirerad av processen för härledningen av dessa indexsatser, vill vi generalisera fallet till när vi även har en slags väldefinierad omflyttning av punkterna i rummet (nyckelordet är *gruppverkan*). Till exempel kan rummet vara en sfär och omflyttningen av punkter skulle då kunna vara rotation kring axeln genom nord- och sydpol. Vi vill undersöka vad som händer om man definierar kvantmekanik över det rum som återstår när vi betraktar alla punkter som kan flyttas till varandra som en enda punkt. I fallet med sfären och rotation skulle alla punkter på samma breddgrad betraktas som samma punkt eftersom de kan roteras till varandra. Detta exempel är dock ett av många som medför vissa svårigheter - de resulterande objekten är inte alltid ”bra” rum. För att kringgå dessa effekter och därmed kunna använda de tekniker vi lärt oss, inför vi något som kallas *ekvivariant kohomologi*. Det finns olika modeller av ekvivariant kohomologi - här undersöks *Weil-modellen* och *Cartan-modellen*. Inom dessa definieras objekt som kan tolkas som superladdningar och via dem kan man skapa kvantmekaniska modeller. Målet var att relatera respektive modell till kvantmekanik med *gaugesymmetri*, vilket tyvärr inte lyckades. Vad som dock visades var att Witten-indexet i ett specialfall är detsamma som för den supersymmetriska ickelinjära sigmamodellen med två superladdningar.

Vi avslutar sammanfattningen med några ord om varför det är värdefullt att studera indexsatser. I naturen observerar vi inte bosoner och fermioner med samma massa. Detta innebär att supersymmetri ska vara en bruten symmetri. Detta är dock ofta svårt att bestämma, men i vissa modeller kan Witten-indexet tala om när symmetrin är obruten. Vi får således ett verktyg för att utesluta modeller som inte är möjliga kandidater till att beskriva verkligheten.

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Contents

1 Introduction

Supersymmetry, a symmetry which relates bosonic degrees of freedom with fermionic ones, has been shown to have deep connections with global questions in geometry [1, 2, 3, 4]. Given a supersymmetric quantum field theory, one may define a quantity $\mathrm{Tr}(-1)^{\mathrm{F}}$ which counts the difference in the number of bosonic and fermionic states of the Hilbert space. This number, known as the *supersymmetric index* or *Witten index*, is in fact a topological invariant of the theory. One can study this quantity using two different approaches - one in the canonical quantization picture and the other using path integral quantization. In the setting of supersymmetric quantum mechanics, by relating the two perspectives of the Witten index, one ends up with a mathematical expression recognized as the *Atiyah-Singer index theorem* for the complex that corresponds to the supersymmetric quantum mechanical system in question.

If supersymmetry turns out to be an actual property of nature, we need it to be spontaneously broken since we do not observe bosons or fermions of equal masses in nature [1, 2]. This happens if and only if the energy of the vacuum is non-vanishing. It is however difficult in many cases to determine whether supersymmetry is spontaneously broken or not. In certain classes of theories, namely non-linear sigma models [3], it turns out that the Witten index can tell if supersymmetry is unbroken (the converse is not true though). What makes the index an attractive quantity is due to the fact that it can be reliably calculated[1].

This thesis is structured as follows. We begin in Section 2 by reviewing some definitions of concepts we need to be able to discuss supersymmetry, mainly consisting of constructions on super vector spaces. In Section 3, we review both the bosonic and fermionic path integral. For both cases, we go through the path integral expression for the transition amplitude and how it is related to the partition function. In the part dealing with fermions, we furthermore introduce the *supertrace*, which will be one of the main objects of study in this thesis. The respective subsections end with a thorough calculation of a number of examples, some of which are used later in the thesis. Afterwards, the *saddle point method* is briefly considered in its simplest form. The next section, Section 4, deals with the general structure of supersymmetric quantum mechanics (SUSYQM abbreviated). We first give an axiomatic definition of SUSYQM, following up by displaying some basic features of this class of theories, for instance that all states of non-zero energy are paired up. We then define the ever so important *Witten index* which equals the supertrace. In the subsequent subsection, we consider some properties of $\mathcal{N}=2$ SUSYQM. In particular, when the $\mathbb{Z}_2$-grading refines to a $\mathbb{Z}$-grading, we find that the Witten index is actually the Euler characteristic of the complex defined by the supercharges acting on the graded Hilbert space. From the material covered in the preceding sections we get the necessary tools to derive the index theorems in Section 5. The first to be considered is the index of the spin complex. The corresponding quantum mechanical model is the SUSY $\mathcal{N}=1$ non-linear sigma model on a compact spin manifold of even dimension given by the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu D_t \psi^\nu. \tag{1.1}$$

This is done by first canonically quantizing the theory and finding the structure of the Hilbert space which in this setting will be the set of spinor fields. The supercharge takes the form of the Dirac operator. The Witten index can thus be computed. Afterwards, the same quantity is found by evaluating the path integral expression for it. Thus the index theorem for the spin complex is proven. By going through an analogous process using the $\mathcal{N}=2$ non-linear sigma model with Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu} (\bar{\psi}^\mu D_t \psi^\nu - D_t \bar{\psi}^\mu \psi^\nu) + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \bar{\psi}^\nu \psi^\rho \bar{\psi}^\sigma \tag{1.2}$$

we derive the index theorem for the de Rham complex, also known as the *Chern-Gauss-Bonnet theorem*. The last main section of the thesis, Section 6, deals with the case when there is a (compact and connected) Lie group acting on the manifold. In this case, we want to consider the resulting orbit space as the space the model is defined on. However the action is not always free which means the quotient space might not be a manifold. Instead one considers the *homotopy quotient* which is roughly speaking the orbit space of the manifold times a contractible space with respect to the diagonal action (which will always be free) of the group. For this we need the notion of *equivariant cohomology*, which we elaborate upon in the first part of the section. In particular, we will investigate two models of equivariant cohomology known as the *Weil model* and the *Cartan model*. Mimicking the process in section 5.2, i.e. for the $\mathcal{N} = 2$ non-linear sigma model, we use the underlying spaces of *equivariant differential forms* as Hilbert space. The corresponding obtained differentials and co-differentials are then the supercharges to construct the quantum mechanical models. The goal is to relate these to a gauging of the $\mathcal{N} = 2$ non-linear sigma model. By identifying the the group acting on the manifold (via smooth local isometries) as the gauge group, this seems plausible. Unfortunately, the attempt was not successful. We did however manage to show that the Euler characteristic of a closed oriented manifold and its homotopy quotient coincide in the abelian case.

Let us conclude the introduction by setting some conventions. We will use natural units $\hbar = c = 1$. The set of natural numbers $\mathbb{N}$ includes 0. We will use the summation convention, meaning we sum over repeated indices unless stated otherwise. If $A = \bigoplus_{i \in \mathbb{Z}} A^i$ is a $\mathbb{Z}$-graded vector space, we define

$$A^{\mathrm{odd}} := \bigoplus_{i \in \mathbb{Z}} A^{2i+1} \tag{1.3}$$

$$A^{\mathrm{even}} := \bigoplus_{i \in \mathbb{Z}} A^{2i}. \tag{1.4}$$

We will use $[,]$ do denote commutators and $\{,\}$ for anticommutators.

2 Super Vector Spaces and Superalgebras

In this section we will review some general mathematical structures governing supersymmetry by gathering some definitions in order to set the language for the rest of the thesis. We will unfortunately only scratch on the surface of this subject. For a more thorough coverage, check e.g. [5], which will be the one we will be following. A brief account is also given in the first parts of [6], which we also will take some elements from.

Let V be a vector space over a field $\mathbb{K}$. If V can be decomposed as a direct sum of subspaces V_i

$$V = \bigoplus_{i \in I} V_i \tag{2.1}$$

for some index set I, V is said to be a *I-graded* (or just *graded*) *vector space*.

Definition 2.1. A *super vector space* V is a $\mathbb{Z}_2$-graded vector space,

$$V = V_0 \oplus V_1. \tag{2.2}$$

The subspace V_0 is referred to as *even* and V_1 as *odd*.

We will mostly be concerned with super vector spaces. In the following, all vector spaces are super vector spaces unless stated otherwise.

Definition 2.2. Let $v \in V$ such that $v \in V_0$ or $v \in V_1$ (such elements are called *homogeneous*). The *parity* of $v \in V$, denoted by $|v|$ is defined by

$$|v| = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1. \end{cases} \tag{2.3}$$

As any element in V can be written as a linear combination of even and odd elements, it is sufficient to give the definitions in terms of homogeneous elements.

Definition 2.3. Let $\dim V_0 = p$ and $\dim V_1 = q$. The *superdimension* $\mathrm{sdim} V$ of V is the pair (p, q), denoted by $\mathrm{sdim} V = p|q$.

Definition 2.4. A *morphism* between super vector spaces is a linear map which preserves the $\mathbb{Z}_2$-grading. The vector space of morphisms from V to W is denoted by $\mathrm{Hom}(V, W)$.

The set of supervector spaces together with $\mathrm{Hom}(V,W)$ forms a category denoted by (smod). There is a specific functor naturally defined on this category known as the *parity reversing functor* $\Pi : (\mathrm{smod}) \to (\mathrm{smod})$ whose action on objects is specified by

$$(\Pi V)_0 = V_1, \quad (\Pi V)_1 = V_0. \tag{2.4}$$

The usual constructions, such as dual spaces and direct sums, carries over from regular linear algebra to this case. For instance, the tensor product of two super vector spaces V and W is again a super vector space by

$$(V\otimes W)_0 = (V_0\otimes W_0)\oplus(V_1\otimes W_1), \quad (V\otimes W)_1 = (V_0\otimes W_1)\oplus(V_1\otimes W_0). \tag{2.5}$$

Definition 2.5. A *superalgebra* is a super vector space A together with a bilinear multiplication $A\times A\to A$, $(a,b)\mapsto ab$ such that $|ab| = |a|+|b| \pmod 2$.

We say that A is *associative* if $a(bc)=(ab)c$, *unital* if there exists an element $1\in A$ such that $1a = a1 = a$ and (*super*)*commutative* if $de = (-1)^{|d||e|}ed$ for all $a,b,c\in A$ and all homogeneous $e,d\in A$.

An important example of a superalgebra arises when we consider Grassmann coordinates.

Example 2.1. Let

$$A = \mathbb{K}[t_1,\ldots,t_p,\theta_1,\ldots,\theta_q] \tag{2.6}$$

be the algebra of polynomials over $\mathbb{K}$ with even (commuting) indeterminates $t_1,\ldots,t_p$ and odd (anticommuting) indeterminates $\theta_1,\ldots,\theta_q$ known as *Grassmann numbers*, i.e.

$$\theta_i\theta_j = -\theta_j\theta_i \tag{2.7}$$

for all i,j. Hence, another way to describe A is

$$A = \mathbb{K}[t_1,\ldots,t_p]\otimes\Lambda(\theta_1,\ldots,\theta_q) \tag{2.8}$$

where $\Lambda(\theta_1,\ldots,\theta_q)$ is the exterior algebra generated by $\theta_1,\ldots,\theta_q$. The algebra A is actually an associative, supercommutative superalgebra with unit. Employing multiindex notation, $\theta_I = \theta_{i_1}\ldots\theta_{i_r}$, $|I| = r$ and $f_0, f_I \in \mathbb{K}[t_1,\ldots,t_p]$, the even and odd subspaces are

$$A_0 = \{f_0 + \sum_{|I|\text{ even}} f_I\theta_I | I = \{i_1<\cdots<i_r\}\} \tag{2.9}$$

$$A_1 = \{\sum_{|J|\text{ odd}} f_J\theta_J | J = \{j_1<\cdots<j_r\}\}. \tag{2.10}$$

We will need the notion of a derivative on superalgebras in the future. We give the algebraic definition.

Definition 2.6. Let A be a superalgebra and $D: A\to A$ be a $\mathbb{K}$-linear map. D is a (*super*)*derivation of degree* $|D|$ if it satisfies

$$D(ab) = D(a)b + (-1)^{|D||a|}aD(b) \tag{2.11}$$

with $a,b\in A$ and $D: A_i \to A_{i+|D| \pmod 2}$. If $|D|$ is odd, we say that $|D|$ is an *anti-derivation.*

Definition 2.7. A *Lie superalgebra* (or *super Lie algebra*) *of degree* ϵ is a super vector space L together with a bilinear map $[,]: L\times L\to L$ called the *super* (*Lie*) *bracket* which for homogeneous elements $x,y,z\in L$ satisfies

$$|[x,y]| = |x|+|y|+\epsilon \pmod 2 \tag{2.12}$$

together with

$$[x,y] + (-1)^{(|x|+\epsilon)(|y|+\epsilon)}[y,x] = 0 \tag{2.13}$$

which we refer to as *antisymmetry* and

$$(-1)^{(|x|+\epsilon)(|z|+\epsilon)}[x,[y,z]] + (-1)^{(|y|+\epsilon)(|x|+\epsilon)}[y,[z,x]] + (-1)^{(|z|+\epsilon)(|y|+\epsilon)}[z,[x,y]] = 0 \tag{2.14}$$

called the *Jacobi identity.* We say that L is *even* if $\epsilon = 0$ and *odd* if $\epsilon = 1$.

Note that any associative superalgebra A can be given an (even) Lie superalgebra structure by defining the bracket (called the *super commutator*)

$$[a, b] = ab - (-1)^{|a||b|} ba \tag{2.15}$$

with $a, b \in A$.

Definition 2.8. Let A be an associative and supercommutative algebra equipped with a super Lie bracket $[,] : A \times A \to A$ of degree ϵ such that $\mathrm{ad}_a = [a,] : A \to A$ is a derivation of degree $|a| + \epsilon$, i.e.

$$[a, bc] = [a, b]c + (-1)^{(|a|+\epsilon)|b|} b[a, c]. \tag{2.16}$$

For $\epsilon = 0$, A is called an *(even) Poisson (super)algebra* and the bracket $[,]$ is referred to as a *super Poisson bracket*. If $\epsilon = 1$, A is known as a *Gerstenhaber algebra* (or *odd Poisson algebra*) and $[,]$ an *antibracket*.

Further into the thesis, we will encounter the case when we also have a $\mathbb{Z}$-grading. Let us quickly extend some of the above definitions to include that case.

Definition 2.9. A *$\mathbb{Z}$-graded algebra* $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded vector space (over a field $\mathbb{K}$ of characteristic 0) together with a bilinear multiplication $A \times A \to A$, $(a, b) \to ab$ such that $|ab| = |a| + |b|$ where $|a| = i$ if $a \in A_i$ is the *degree* of a.

As before, A is *associative* if $a(bc) = (ab)c$ and unital if there is an element $1 \in A_0$ such that $1a = a1 = a$. It is *(graded-)commutative* if $ed = (-1)^{|e||d|} de$. The definition of a *graded derivation* extends to this case using degree in place of parity[1]. Let us now merge some structures together (the following definition can be found in e.g. [7]).

Definition 2.10. A unital associative graded-commutative algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that $A_i = 0$ for $i < 0$ is a *differential graded algebra* (abbreviated *DG*-algebra) if there is an anti-derivation $D : A_i \to A_j$ such that $D \circ D = 0$.

Example 2.2. The de Rham complex with the exterior derivative is an example of a *DG*-algebra.

Note that by gathering all odd summands as one vector space and all even as one, we retrieve the $\mathbb{Z}_2$-graded case. The extended definitions are compatible with this reconsideration, for instance the notion of being graded-commutative would be the same as being supercommutative in this context.

3 Path Integral Techniques

We will in this section cover the path integral formulation of quantum mechanics and compute a couple of useful examples. The treatment of this subject can be found in several places in the literature, for instance [8, 9, 10, 11, 12, 13]. We will also use some ideas from the lecture notes of Blau [14].

3.1 Bosonic Path Integration

Roughly speaking, one can compute the transition amplitude of going from a state $|q_i, t_i\rangle$ to a state $|q_f, t_f\rangle$ by integrating over all possible paths initiating at position q_i at a time t_i and ending up in q_f at a final time t_f, weighting each path with some quantity which describes how probable each of them are. One way to derive the expression for the transition amplitude is to divide the time interval into N time slices $\delta t := \dfrac{t_f - t_i}{N}$ and

[1] Of course, degrees are not added modulo 2.

then inserting complete set of position and momentum states. One finds that the amplitude, with arbitrary initial and final value of the momentum, is given by

$$\langle q_f, t_f | q_i, t_i \rangle = \lim_{\delta t \to 0} \int \left(\prod_{k=1}^{N-1} dq_k \right) \left(\prod_{j=1}^{N} \frac{dp_j}{2\pi} \right) e^{ip_j \frac{q_{j+1}-q_j}{\delta t} \delta t - iH(p_j, \bar{q}_j)\delta t} =: \int \tilde{\mathcal{D}}q \mathcal{D}p e^{i\int_{t_i}^{t_f} [p(t)\dot{q}(t) - H(p(t), q(t))]} \tag{3.1}$$

where $\bar{q}_j := \frac{1}{2}(q_j + q_{j+1})$ and $\dot{q}_j = \lim_{\delta t \to 0} \frac{q_{j+1} - q_j}{\delta t}$ and $H(p,q)$ is the Hamiltonian describing the system. Observe that the limit $\delta t \to 0$ is the same as taking the limit $N \to \infty$; making the time slices smaller is the same as dividing the interval into more and more pieces. As we already are aware of, the measures $\mathcal{D}q$ and $\mathcal{D}p$ are not well defined by themselves but the path integral in whole can be given meaning. If the Hamiltonian is no more than quadratic in the momenta, the integral over p is a product of Gaussian integrals with a linear term in the exponent and can be integrated out. For Hamiltonians of the form

$$H(p,q) = \frac{p^2}{2m} + V(q) \tag{3.2}$$

the expression for the transition amplitude is simplified to

$$\langle q_f, t_f | q_i, t_i \rangle = \lim_{\delta t \to 0} \left(\frac{m}{2\pi i \delta t} \right)^{\frac{N}{2}} \int \left(\prod_{k=1}^{N-1} dq_k \right) e^{i\sum_{j=1}^{N} \delta t \left[\frac{m}{2} \left(\frac{q_{j+1}-q_j}{\delta t} \right)^2 - V(\bar{q}_j) \right]} =: \int \hat{\mathcal{D}}q e^{iS} \tag{3.3}$$

where $S = \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t))$ is the action and L is the Lagrangian of the system, obtained from the Hamiltonian through a Legendre transform in usual order. Here, we have absorbed the constant from integrating out the momentum in the measure $\hat{\mathcal{D}}q$ (which means $\hat{\mathcal{D}}q$ differs from $\tilde{\mathcal{D}}q$ with a constant). Note that a general feature of path integrals is that they are invariant under redefinition of the integration variables since we are integrating over all possible paths.

Switching to Euclidean time, $\tau = it$, the transition amplitude takes the form

$$\langle q_f, t_f | q_i, t_i \rangle = \lim_{\delta t \to 0} \left(\frac{m}{2\pi \delta \tau} \right)^{\frac{N}{2}} \int \left(\prod_{k=1}^{N-1} dq_k \right) e^{-\sum_{j=1}^{N} \delta \tau \left[\frac{m}{2} \left(\frac{q_{j+1}-q_j}{\delta \tau} \right)^2 + V(\bar{q}_j) \right]} = \int \hat{\mathcal{D}}q e^{-S_E} \tag{3.4}$$

where

$$S_E = \int_{\tau_i}^{\tau_f} d\tau L_E = \int_{\tau_i}^{\tau_f} d\tau \left[\frac{1}{2} m \left(\frac{dq}{d\tau} \right)^2 + V(q) \right]. \tag{3.5}$$

Let us now define the *partition function* Z, for $\beta > 0$,

$$Z(\beta) = \mathrm{Tr}\ e^{-\beta H} \tag{3.6}$$

where the trace is taken over the Hilbert space on which H acts. Using the energy eigenbasis $\{E_n\}$, which satisfies $H|E_n\rangle = E_n|E_n\rangle$ and $\langle E_m|E_n\rangle = \delta_{mn}$,

$$Z(\beta) = \sum_n \langle E_n| e^{-\beta H} |E_n\rangle = \sum_n e^{-\beta E_n}. \tag{3.7}$$

The partition function can also be expressed using the position eigenstates $|q\rangle$,

$$Z(\beta) = \int dq \langle q| e^{-\beta H} |q\rangle. \tag{3.8}$$

The integrand of (3.8) can be written as

$$\langle q| e^{-\beta H} |q\rangle = \langle q| e^{-(\tau_f - \tau_i)H} |q\rangle = \langle q| e^{-\tau_f H} e^{\tau_i H} |q\rangle = \langle q, \tau_f | q, \tau_i \rangle. \tag{3.9}$$

By identifying β with the Euclidean time interval, $\beta \equiv \tau_f - \tau_i$, the partition function takes the form of a path integral with paths defined over the circle with circumference β or rephrased, periodic paths with period β. Note that integrating over periodic paths between τ_i and τ_f is the same as integrating between 0 and β. The partition function is therefore given by

$$Z(\beta) = \int dr \, \langle r, \tau_f | r, \tau_i \rangle = \int dr \int_{q(0)=q(\beta)=r} \hat{\mathcal{D}}q e^{-\int_0^\beta L_E} =: \int_{PBC} \mathcal{D}q e^{-\int_0^\beta L_E} \tag{3.10}$$

where PBC signifies the fact that we are integrating over paths with periodic boundary conditions (i.e. loops) and taking $dq_0 := dr$ (we chose 0 for notational convenience),

$$Z(\beta) = \lim_{N\to\infty} \left(\frac{m}{2\pi\delta\tau}\right)^{\frac{N}{2}} \int_{PBC} \left(\prod_{k=0}^{N-1} dq_k\right) e^{-\sum_{j=1}^{N} \delta\tau \left[\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\delta\tau}\right)^2 + V(\bar{q}_j)\right]}. \tag{3.11}$$

Observe that we now have equally many coordinate integrals as momentum integrals. It will turn out to be useful to compute the path integral as

$$Z(\beta) = \int_{PBC} \mathcal{D}q e^{-S_E} = \lim_{N\to\infty} \mathcal{C} \int_{PBC} \left(\prod_{k=0}^{N-1} dq_k\right) e^{-S_E} \tag{3.12}$$

where C is a normalisation constant we will determine in the following. We would like to again stress that this is not completely well-defined.

Example 3.1. In this thesis, we are mostly interested in partition functions. Let us compute the next simplest example: the one dimensional harmonic oscillator. The Lagrangian is given by

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2. \tag{3.13}$$

We will compute the partition function in two ways and then compare the results in order to determine $\mathcal{C}$. We can do this since $\mathcal{C}$ is universal; $\mathcal{C}$ does not depend on the potential. Recall that the eigenvalues of the Hamiltonian are given by

$$E_n = \left(n + \frac{1}{2}\right)\omega. \tag{3.14}$$

so in the canonical picture, the partition function (3.7) is (for $\omega > 0$)

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\omega} = e^{-\frac{1}{2}\beta\omega} \sum_{n=0}^{\infty} e^{-\beta\omega n} = e^{-\frac{1}{2}\beta\omega} \frac{1}{1-e^{-\beta\omega}} = \frac{1}{2} \frac{2}{e^{\frac{1}{2}\beta\omega} - e^{-\frac{1}{2}\beta\omega}} = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}. \tag{3.15}$$

Let us now do it using the path integral. The Euclidean action is

$$\begin{aligned} S_E &= \int_0^\beta d\tau \left[\frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 + \frac{1}{2}m\omega^2 q^2\right] = \int_0^\beta d\tau \left[-\frac{1}{2}mq\frac{d^2}{d\tau^2}q + \frac{1}{2}m\omega^2 q^2\right] \\ &= \frac{1}{2}m \int_0^\beta d\tau q \left(-\frac{d^2}{d\tau^2} + \omega^2\right) q \end{aligned} \tag{3.16}$$

where we integrated the first term by parts at the third equality while remembering that we are integrating over periodic paths. We next expand q in orthonormal eigenfunctions (which satisfy the boundary conditions) of the operator $-\dfrac{d^2}{d\tau^2} + \omega^2$,

$$q(\tau) = \sum_{n\in\mathbb{Z}} a_n f_n \tag{3.17}$$

where

$$f_{|n|} = \frac{1}{\sqrt{\beta}}\sin\left(\frac{2\pi n}{\beta}\tau\right) \tag{3.18}$$

$$f_{-|n|} = \frac{1}{\sqrt{\beta}}\cos\left(\frac{2\pi n}{\beta}\tau\right) \tag{3.19}$$

$$f_0 = \frac{1}{\sqrt{\beta}} \tag{3.20}$$

satisfying

$$\int_0^\beta d\tau\ f_m(\tau)f_n(\tau) = \delta_{mn} \tag{3.21}$$

with eigenvalues

$$\lambda_n = \left(\frac{2\pi n}{\beta}\right)^2 + \omega^2. \tag{3.22}$$

Plugging this into the action,

$$S_E = \frac{1}{2}m\int_0^\beta d\tau \sum_{l,n\in\mathbb{Z}} a_l f_l \left(-\frac{d^2}{d\tau^2}+\omega^2\right)(a_n f_n) = \frac{1}{2}m\int_0^\beta d\tau \sum_{l,n\in\mathbb{Z}} a_l f_l \lambda_n a_n f_n = \sum_{n\in\mathbb{Z}} \frac{1}{2}m\lambda_n a_n^2. \tag{3.23}$$

Observe now that in the path integral, integrating over all paths q is equivalent to integrating over every possible a_n. Note that the sum runs over all integers, while $\prod_{k=0}^{N-1} dq_k$ starts from zero[2]. By splitting the action as

$$S_E = \frac{1}{2}(\lambda_0 a_0^2 + \sum_{n=1}^{\infty}\lambda_n a_n^2 + \sum_{n=1}^{\infty}\lambda_{-n}a_{-n}^2) \tag{3.24}$$

we see that we can make the substitution[3] $q_0 \leftrightarrow a_0$, $q_{2n} \leftrightarrow a_{|n|}$ and $q_{2n+1} \leftrightarrow a_{-|n|}$. The Jacobian determinant from this change of variables is ± 1 since the transformation to the eigenbasis is orthogonal[4]. Under these considerations,

$$\begin{aligned} Z(\beta) &= \int_{PBC} \mathcal{D}q e^{-S_E} \\ &= \lim_{N\to\infty} \mathcal{C}\int_{-\infty}^{\infty} da_0 e^{-\frac{1}{2}\lambda_0 a_0^2}\int_{-\infty}^{\infty}\left(\prod_{k=1}^{N_+} da_k\right) e^{-\frac{1}{2}m\sum_{n=1}^{\infty}\lambda_n a_n^2}\int_{-\infty}^{\infty}\left(\prod_{k=1}^{N_-} da_{-k}\right) e^{-\frac{1}{2}m\sum_{n=1}^{\infty}\lambda_{-n}a_{-n}^2} \\ &= \lim_{N\to\infty}\mathcal{C}\prod_{n=-N_-}^{N_+}\int_{-\infty}^{\infty} da_n e^{-\frac{1}{2}m\lambda_n a_n^2} \end{aligned} \tag{3.25}$$

[2]This is fine since $|\mathbb{N}| = |\mathbb{Z}|$.

[3]In order for the change of variables to be well-defined, the number of a_k must equal the number of q_k. This is solved by setting $a_k = 0$ for $k > N-1$. This issue disappears in the $N\to\infty$ limit.

[4]If one attempts to compute the Jacobian in this setting, one would get something divergent. What one actually needs to do in order to show this [13] is to use the discrete version of the path integral (3.11) and expand the fields using the *discrete* Fourier transform $q_k = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} c_l e^{i2\pi kl/n}$. In the future, we will just assume this is what has been done and ignore computing the Jacobian for this particular change of variables.

where

$$N_+ = \frac{N-1-\sigma(N)}{2} \tag{3.26}$$

$$N_- = \frac{N-1-\sigma(N+1)}{2} \tag{3.27}$$

where we have defined $\sigma(n) = 0$ if n is even and $\sigma(n) = 1$ is n is odd. We have thus ended up with a product of Gaussian integrals

$$\begin{aligned} Z(\beta) &= \lim_{N\to\infty} \mathcal{C} \prod_{n=-N_-}^{N_+} \sqrt{\frac{2\pi}{m\lambda_n}} = \lim_{N\to\infty} \left(\mathcal{C} \prod_{n=-N_-}^{N_+} \sqrt{\frac{2\pi}{m}} \right) \prod_{n=-N_-}^{N_+} \frac{1}{\sqrt{\lambda_n}} \\ &= \left(\lim_{N\to\infty} \mathcal{C} \prod_{n=-N_-}^{N_+} \sqrt{\frac{2\pi}{m}} \right) \frac{1}{\sqrt{\mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}}. \end{aligned} \tag{3.28}$$

The functional determinant[5] consists of an infinite product of the eigenvalues and so needs to be regularized. We will make use of the so called *zeta-function regularization* following [11]. We define the *spectral zeta-function*

$$\zeta_O(s) = \sum_n \frac{1}{\lambda_n^s} \tag{3.29}$$

This function is convergent for large enough $\mathrm{Re}(s)$ and analytic with respect to s in that region. The function can be analytically continued to the whole s-plane (except for a finite number of points). Now, note that

$$\frac{d\zeta_O(s)}{ds}\Big|_{t=0} = \sum_n \frac{d}{ds} e^{-s\ln(\lambda_n)}\Big|_{t=0} = \sum_n -\ln(\lambda_n) \tag{3.30}$$

so

$$\prod_n \lambda_n = e^{-\frac{d\zeta_O(s)}{ds}\big|_{t=0}}. \tag{3.31}$$

Note that in our case, since $\lambda_n = \lambda_{-n}$, we may rewrite

$$\mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \omega^2\right) = \prod_{n=-\infty}^{\infty} \lambda_n = \lambda_0 \left(\prod_{n=1}^{\infty} \lambda_n\right)^2 \tag{3.32}$$

where

$$\prod_{n=1}^{\infty} \lambda_n = \prod_{n=1}^{\infty} \left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right] = \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^2 \left[1 + \left(\frac{\beta\omega}{2\pi n}\right)^2\right] = \frac{2\sinh\left(\frac{\beta\omega}{2}\right)}{\beta\omega} \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^2 \tag{3.33}$$

where we have used the well-known identity

$$\sinh(z) = z \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{\pi n}\right)^2\right) \tag{3.34}$$

so the product we need to regularize is

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^2. \tag{3.35}$$

[5] We will use capital D for functional determinants and lower case d for the finite dimensional case.

The spectral zeta-function is

$$\zeta_O(s) = \sum_{n\geq 1}\left(\frac{\beta}{2\pi n}\right)^{2s} = \left(\frac{\beta}{2\pi}\right)^{2s}\sum_{n\geq 1}\frac{1}{n^{2s}} = \left(\frac{\beta}{2\pi}\right)^{2s}\zeta(2s) = e^{2s\ln\left(\frac{\beta}{2\pi}\right)}\zeta(2s) \tag{3.36}$$

where $\zeta(s)$ is the Riemann-zeta function. The values at $s = 0$ are known and are given by

$$\zeta(0) = -\frac{1}{2}; \quad \zeta'(0) = -\frac{1}{2}\ln(2\pi). \tag{3.37}$$

Hence

$$\zeta_O'(0) = 2\ln\left(\frac{\beta}{2\pi}\right)\zeta(0) + 2\zeta'(0) = -\ln(\beta) \tag{3.38}$$

so by (3.31),

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^2 = \beta. \tag{3.39}$$

The functional determinant thus takes the form

$$\mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \omega^2\right) = 4\sinh^2\left(\frac{\beta\omega}{2}\right) \tag{3.40}$$

and the partition function becomes

$$Z(\beta) = \left(\lim_{N\to\infty}\mathcal{C}\prod_{n=-N_-}^{N_+}\sqrt{\frac{2\pi}{m}}\right)\frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}. \tag{3.41}$$

By comparing with (3.15), we finally deduce that

$$\left(\lim_{N\to\infty}\mathcal{C}\prod_{n=-N_-}^{N_+}\sqrt{\frac{2\pi}{m}}\right) = 1. \tag{3.42}$$

In the end, we have learnt from this example that we may compute the partition function as

$$Z(\beta) = \int \mathcal{D}q e^{-S_E} = \lim_{N\to\infty}\int_{PBC}\left(\prod_{k=0}^{N-1}\sqrt{\frac{m}{2\pi}}dq_k\right)e^{-S_E} \tag{3.43}$$

and in this particular example,

$$Z(\beta) = \int_{PBC}\mathcal{D}q e^{-S_E} = \frac{1}{\sqrt{\mathrm{Det}\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}}. \tag{3.44}$$

We can now compute the partition function for the free particle by following the same procedure as in the case of the harmonic oscillator but using (3.43), resulting in

$$Z(\beta) = \int_{PBC}\mathcal{D}q e^{-S_E} = \frac{1}{\sqrt{\mathrm{Det}'\left(-\frac{d^2}{d\tau^2}\right)}} = \frac{1}{\mathrm{Det}'\left(\frac{d}{d\tau}\right)} \tag{3.45}$$

with

$$L_E = \frac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 \tag{3.46}$$

where the $'$ in Det' symbolises the fact that we are taking the product of the non-zero eigenvalues. That the third equality in (3.45) is true can be seen by using the orthogonal eigenfunctions consisting of complex exponentials $e^{i\frac{2\pi n}{\beta}\tau}$ and investigating the functional determinant as a product of the corresponding eigenvalues. Note the independence of the mass parameter m in the final results. We may hence set it to 1 in the future when we compute partition functions.

Example 3.2. Let us add a source term to our considerations. Suppose we can write our Lagrangian (which at least contains the free Lagrangian) in the form

$$L_E = \frac{m}{2} qOq + Jq, \tag{3.47}$$

where O is a differential operator with orthonormal eigenfunctions f_n satisfying the periodic boundary conditions

$$\int_0^\beta dt f_m(t) f_n(t) = \delta_{mn} \tag{3.48}$$

with corresponding eigenvalues λ_n. For instance, O could be the operator $-\frac{d^2}{dt^2} + \omega^2$ as in the case of the harmonic oscillator in Example 3.1. To be able to compute the path integral, we make a change of variables $q(t) = r(t) - \frac{1}{m} O^{-1} J$ where O^{-1} is the inverse operator to O. In such case,

$$L_E = \frac{1}{2}(mrOr + Jr - O^{-1}JOr - \frac{1}{m} JO^{-1}J). \tag{3.49}$$

Note that this change in the path integral makes no difference - the measure in the path integral is translational invariant. Expanding r and J in terms of eigenfunctions $r = \sum_n a_n f_n$ and $J = \sum_n b_n f_n$ respectively,

$$\begin{aligned}
S_E &= \int_0^\beta dt \frac{1}{2} \sum_{l,n} (m a_l f_l O(a_n f_n) - b_l f_l a_n f_n - O^{-1}(b_l f_l) O(a_n f_n)) - \int_0^\beta dt \frac{1}{2m} JO^{-1}J \\
&= \sum_{l,n} \int_0^\beta dt \frac{1}{2} (m a_l f_l \lambda_n a_n f_n - b_l f_l a_n f_n - \lambda_l^{-1} b_l f_l \lambda_n a_n f_n) - \int_0^\beta dt \frac{1}{2m} JO^{-1}J \\
&= \frac{1}{2} \sum_n (m \lambda_n a_n^2 - b_n f_n - \lambda_n^{-1} b_n \lambda_n a_n) - \int_0^\beta dt \frac{1}{2m} JO^{-1}J \\
&= \frac{1}{2} m \sum_n \lambda_n a_n^2 - \int_0^\beta dt \frac{1}{2m} JO^{-1}J
\end{aligned} \tag{3.50}$$

where we used the orthonormality of f_n. Note that if J is zero, we would have gotten an integral in this form without the change of variables. The path integral for the partition function takes the form

$$\begin{aligned}
Z(\beta) = \int \mathcal{D}r e^{S_E} &= e^{\int_0^\beta dt \frac{1}{2m} JO^{-1}J} \lim_{N\to\infty} \int_{-\infty}^{\infty} \left(\prod_{k=0}^{N-1} \sqrt{\frac{m}{2\pi}} da_k \right) e^{-\frac{1}{2} m \sum_n \lambda_n a_n^2} \\
&= e^{\int_0^\beta dt \frac{1}{2m} JO^{-1}J} \frac{1}{\sqrt{\mathrm{Det}'(O)}}
\end{aligned} \tag{3.51}$$

Again, if $J = 0$, we would have lost the m-dependence. Note that "Det" is the same as "Det′" in the absence of zero eigenvalues. We will henceforth use "Det′" when it is not clear if we are in that situation or not[6].

Example 3.3. Let us also investigate the complex case. Like before, suppose we can rewrite our Lagrangian

$$L_E = \bar{q} O q \tag{3.52}$$

where O is a differential operator with orthonormal eigenfunctions f_n satisfying the periodic boundary conditions

$$\int_0^\beta dt \bar{f}_m(t) f_n(t) = \delta_{mn} \tag{3.53}$$

[6]However, one might have other quantities in the Lagrangian which may depend on the zero modes. In such cases, one needs to integrate over them as well.

with corresponding eigenvalues λ_n. Here, the $^-$ means taking complex conjugate. Expanding $q = \sum_n a_n f_n$ and $\bar{q} = \sum_n \bar{a}_n \bar{f}_n$, the action takes the form

$$S_E = \int_0^\beta dt \sum_{l,n} \bar{a}_l \bar{f}_l O(a_n f_n) = \sum_{l,n} \int_0^\beta dt \bar{a}_l \bar{f}_l \lambda_n a_n f_n = \sum_n \lambda_n |a_n|^2 \tag{3.54}$$

By defining $\mathrm{Re}(a_n) = \frac{1}{\sqrt{2}} c_n$ and $\mathrm{Im}(a_n) = \frac{1}{\sqrt{2}} d_n$ and changing integration variables from $\bar{q}$ and q to c and d. The absolute value of the Jacobian determinant for this change is just 1 (the matrix is built up by 2×2 blocks on the diagonal with determinant 1 each),

$$Z(\beta) = \int \mathcal{D}\bar{q}\mathcal{D}q e^{-S_E} = \lim_{N\to\infty} \int \left(\prod_{k=0}^{N} \frac{dc_k}{\sqrt{2\pi}} \frac{dd_k}{\sqrt{2\pi}} \right) e^{-\frac{1}{2}\sum_n \lambda_n c_n^2 - \frac{1}{2}\sum_n \lambda_n d_n^2} \tag{3.55}$$

Note that this is equivalent to having two independent real variables. This can also be seen by defining q_1 and q_2 satisfying $q = \frac{1}{\sqrt{2}}(q_1 + iq_2)$ and $\bar{q} = \frac{1}{\sqrt{2}}(q_1 - iq_2)$. The absolute value of the Jacobian determinant is again just 1. The Lagrangian takes the form

$$L_E = \frac{1}{2} q_1 O q_1 + \frac{1}{2} q_2 O q_2 \tag{3.56}$$

and the partition function is

$$\begin{aligned} Z(\beta) &= \int \mathcal{D}q_1 \mathcal{D}q_2 e^{-\int_0^\beta dt (\frac{1}{2} q_1 O q_1 + \frac{1}{2} q_2 O q_2)} = \int \mathcal{D}q_1 e^{-\int_0^\beta dt \frac{1}{2} q_1 O q_1} \int \mathcal{D}q_2 e^{-\int_0^\beta dt \frac{1}{2} q_2 O q_2} \\ &= \lim_{N\to\infty} \int \left(\prod_{k=0}^{N-1} \frac{dc_k}{\sqrt{2\pi}} \frac{dd_k}{\sqrt{2\pi}} \right) e^{-\frac{1}{2}\sum_n \lambda_n c_n^2 - \frac{1}{2}\sum_n \lambda_n d_n^2} = \frac{1}{\mathrm{Det}'(O)}. \end{aligned} \tag{3.57}$$

Example 3.4. Let us investigate a couple of simple cases in the setting of a d-dimensional manifold M. Let $\phi : \mathcal{T} \longrightarrow M$, with $\mathcal{T} = [0, \beta]$, be a curve on M with coordinates ϕ^μ. Let $O_{\mu\nu}$ be a differential operator with orthonormal eigenfunctions f_n satisfying the PBC,

$$\int_0^\beta dt f_m(t) f_n(t) = \delta_{mn} \tag{3.58}$$

and corresponding eigenvalues $\lambda_{\mu\nu,n}$. Suppose we can, yet again, write our Lagrangian (which at least contains the free Lagrangian) in the form

$$L_E = \frac{1}{2} \phi^\mu O_{\mu\nu} \phi^\nu. \tag{3.59}$$

An example of such an operator is $-\delta_{\mu\nu} \frac{d^2}{dt^2}$ in flat space (which would correspond to $L_E = \frac{1}{2}\dot{\phi}^\mu \dot{\phi}^\mu$). In the case when $O_{\mu\nu}$ is diagonal, the path integral splits as

$$Z(\beta) = \prod_{\mu=1}^{d} \int \mathcal{D}\phi^\mu e^{\frac{1}{2}\phi^\mu O_{\mu\mu} \phi^\mu} = \prod_{\mu=1}^{d} \frac{1}{\sqrt{\mathrm{Det}(O_{\mu\mu})}}. \tag{3.60}$$

We will later also encounter a case where $d = 2n$ and $O_{\mu\nu}$, viewed as a matrix $(O_{\mu\nu})$, is block diagonal with 2×2 blocks where each block is of the form

$$(O^i_{ab}) = \begin{pmatrix} O^i_{11} & O^i_{12} \\ -O^i_{12} & O^i_{11} \end{pmatrix} \tag{3.61}$$

where (O^i_{ab}) is the ith block of $(O_{\mu\nu})$. We will also assume that the operators in (O^i_{ab}) share the same set of orthonormal eigenbasis $\{f_n\}$. We will treat the computation of the path integral here as not to clog up the section in which this appears. In this setting, the path integral splits into (summing over a and b)

$$Z(\beta) = \prod_{i=1}^{n} \int \mathcal{D}\phi^{2i-1}\mathcal{D}\phi^{2i} e^{-\int_0^\beta dt \frac{1}{2}\phi^a O^i_{ab}\phi^b} \tag{3.62}$$

Consider one such block, e.g. the first one. Note that (O^1_{ab}) is normal, which is equivalent to it being unitarily diagonalisable. A unitary matrix doing the job is given by

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{3.63}$$

for which

$$(\tilde{O}^1_{ab}) = U(O^1_{ab})U^\dagger = \begin{pmatrix} O^1_{11} - iO^1_{12} & 0 \\ 0 & O^1_{11} + iO^1_{12} \end{pmatrix}. \tag{3.64}$$

In this basis, the new coordinates are

$$(\zeta^a) = \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi^1 + i\phi^2 \\ \phi^1 - i\phi^2 \end{pmatrix} \tag{3.65}$$

Since the transformation to this basis is unitary, the absolute value of the Jacobian determinant for changing to these coordinates is 1. We get

$$L_E = \frac{1}{2}(\zeta^a)^\dagger(\tilde{O}_{ab})(\zeta^a) = \frac{1}{2}\bar{\zeta}(O^1_{11} - iO^1_{12})\zeta + \frac{1}{2}\zeta(O^1_{11} + iO^1_{12})\bar{\zeta} \tag{3.66}$$

so expanding in the eigenbasis of O^1_{11} (O^1_{12}),

$$\begin{aligned} S_E &= \int_0^\beta dt \sum_{l,k} \frac{1}{2}[\bar{a}_l \bar{f}_l(\lambda_{11,k} - i\lambda_{12,k})a_k f_k + a_l f_l(\lambda_{11,k} + i\bar{\lambda}_{12,k})\bar{a}_k \bar{f}_k] \\ &= \sum_k [\lambda_{11,k} + \frac{i}{2}(\bar{\lambda}_{12,k} - \lambda_{12,k})]|a_k|^2 \end{aligned} \tag{3.67}$$

This is basically in the form as in the complex case. The partition function related to the first block is then

$$\begin{aligned} Z_1(\beta) &= \int \mathcal{D}\phi^1\mathcal{D}\phi^2 e^{-\int_0^\beta dt \frac{1}{2}\phi^a O^1_{ab}\phi^b} = \int \mathcal{D}\bar{\zeta}\mathcal{D}\zeta e^{-\int_0^\beta dt \frac{1}{2}\bar{\zeta}(O^1_{11} - iO^1_{12})\zeta + \frac{1}{2}\zeta(O^1_{11} + iO^1_{12})\bar{\zeta}} \\ &= \prod_{k=-\infty}^{\infty} [\lambda_{11,k} + \frac{i}{2}(\bar{\lambda}_{12,k} - \lambda_{12,k})]^{-1} = \frac{1}{\text{Det}'[O_{11} + \frac{i}{2}(\bar{O}_{12} - O_{12})]}. \end{aligned} \tag{3.68}$$

and as usual, skipping over the zero eigenvalue if it exists. The total partition function will just be the product of the partition functions related to each separate block,

$$Z(\beta) = \prod_{j=1}^{n} Z_j(\beta) = \prod_{j=1}^{n} \frac{1}{\text{Det}'[O_{2j-1,2j-1} + \frac{i}{2}(\bar{O}_{2j-1,2j} - O_{2j-1,2j})]}. \tag{3.69}$$

3.2 Fermionic Path Integration

In order to describe fermionic variables we need Grassmann numbers which we introduced in 2.1. We will begin this subsection by briefly going through some basic Grassmann calculus. Firstly, a function f of Grassmann numbers θ_i is given by the Taylor expansion of the said function. Note that the expansion will be finite since the product of two of the same Grassmann numbers is zero.

Definition 3.1. Let the Grassmann generators of the Grassmann algebra be $\{\theta_1, \ldots, \theta_q\}$. The *differentiation* with respect to θ_j, $\frac{\partial}{\partial \theta_j}$, is the anti-derivation satisfying

$$\frac{\partial \theta_i}{\partial \theta_j} = \delta_{ij} \tag{3.70}$$

The derivative satisfies

$$\left\{ \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\} = 0 \tag{3.71}$$

and is thus nilpotent. Furthermore

$$\left\{ \frac{\partial}{\partial \theta_i}, \theta_j \right\} = \delta^{ij} \tag{3.72}$$

which is proven by using that $\frac{\partial}{\partial \theta_i}$ is an anti-derivation. Let $p(\theta)$ be a polynomial of the Grassmann algebra. Then

$$\frac{\partial}{\partial \theta_i}\theta_j p(\theta) = \frac{\partial \theta_j}{\partial \theta_i} p(\theta) - \theta_j \frac{\partial}{\partial \theta_i} p(\theta) \iff \frac{\partial}{\partial \theta_i}\theta_j p(\theta) + \theta_j \frac{\partial}{\partial \theta_i} p(\theta) = \left\{ \frac{\partial}{\partial \theta_i}, \theta_j \right\} p(\theta) = \frac{\partial \theta_j}{\partial \theta_i} p(\theta) = \delta^{ij} p(\theta) \tag{3.73}$$

and the statement is thus proven.

Now that we have a notion of derivatives, we want to be able to integrate as well. As in Nakahara [11], let us temporarily denote derivation with respect to Grassmann variable by D and integration by I. Let A and B b two arbitrary functions of Grassmann variables. We would like the Grassmann integration to capture the following properties from ordinary integration:

1. The definite integral of a derivative yields zero: $ID = 0$.
2. The derivative of a definite integral is zero: $DI = 0$.
3. The integral is linear with respect to constants, i.e. numbers whose Grassmann derivative vanishes: $D(A) = 0 \implies I(BA) = I(B)A$.

It turns out that the correct result is reached if the integration is proportional to the differentiation. One chooses them to be equal.

Definition 3.2. The *integral* of a function $f(\theta_1, \ldots \theta_q)$ with respect to $\theta_{i_n}, \theta_{i_{n-1}}, \ldots, \theta_{i_1}$ is defined to be

$$\int d\theta_{i_n} d\theta_{i_{n-1}} \ldots d\theta_{i_1} f(\theta_1, \ldots \theta_q) = \frac{\partial}{\partial \theta_{i_n}} \frac{\partial}{\partial \theta_{i_{n-1}}} \cdots \frac{\partial}{\partial \theta_{i_1}} f(\theta_{i_1}, \ldots \theta_q) \tag{3.74}$$

Note that the order of the $d\theta_{i_k}$ and $\frac{\partial}{\partial \theta_{i_l}}$ is important. From the definition we find that

$$\int d\theta \theta = 1; \quad \int d\theta = 0. \tag{3.75}$$

A consequence of the equivalence of the integration and differentiation of Grassmann numbers is that the scaling from changing variables is different from the case of ordinary numbers. Consider the change of variables $\theta'_{i_k} = a_{i_k i_l} \theta_{i_l}$. Then after a direct calculation, one shows that the measure transforms as

$$d\theta_{i_n} d\theta_{i_{n-1}} \ldots d\theta_{i_1} = \det a \; d\theta'_{i_n} d\theta'_{i_{n-1}} \ldots d\theta'_{i_1} \tag{3.76}$$

which is the opposite of what one has for normal numbers.

Let us also define complex Grassmann numbers. Denote two sets of Grassmann generators by $\{\theta_1, \ldots, \theta_q\}$ and $\{\theta_1^*, \ldots, \theta_q^*\}$ such that taking the complex conjugate $(\theta_i)^* = \theta_i^*$ and $(\theta_i^*)^* = \theta_i$. We require

$$(\theta_i\theta_j)^* = \theta_j^*\theta_i^*. \tag{3.77}$$

We have thus given the necessary definitions for our purposes. In the following, we will start building the fermionic path integral.

In order to define the transition amplitude as a path integral between two fermionic states, it is useful to introduce what is known as *coherent states*. Assume we have fermionic creation operators c_i and annihilation operators $c_i^\dagger$ such that the anticommutation relation

$$\{c_i, c_j^\dagger\} = \delta_{ij} \tag{3.78}$$

is satisfied. The number operators is as usual defined by $c_i^\dagger c_i$ and has eigenstates $|n_1, \ldots, n_q\rangle$ with each n_i ranging between 0 and 1. For simplicity, let us consider the case when we only have one degree of freedom.

Definition 3.3. The fermonic *coherent states* are the eigenstates of the annihilation operator c such that

$$c\,|\psi\rangle = |\psi\rangle\,\psi \tag{3.79}$$

for some Grassmann valued ψ.

Explicitly, these states can be written as (no summation)

$$|\psi\rangle = |0\rangle + |1\rangle\,\psi \tag{3.80}$$

with the adjoint state given by (we will use both * and $^-$ to denote complex conjugate)

$$\langle\bar{\psi}| = \langle 0| + \bar{\psi}\,\langle 1| \tag{3.81}$$

so from this

$$\langle\bar{\psi}'|\psi\rangle = (\langle 0| + \bar{\psi}'\,\langle 1|)(|0\rangle + |1\rangle\,\psi) = 1 + \bar{\psi}'\psi = e^{\bar{\psi}'\psi}. \tag{3.82}$$

We also have the following completeness relation

$$\int d\bar{\psi}d\psi\,|\psi\rangle\,\langle\bar{\psi}|\,e^{-\bar{\psi}\psi} = I \tag{3.83}$$

which can be verified by a direct calculation.

By the coherent state construction, for a normal ordered Hamiltonian $H = H(c^\dagger, c)$,

$$\langle\bar{\psi}|\,H(c^\dagger, c)\,|\psi\rangle = \langle\bar{\psi}|\,H(\bar{\psi}, \psi)\,|\psi\rangle\,. \tag{3.84}$$

Let us now investigate the transition amplitude $\langle\bar{\psi}_f, t_f|\psi_i, t_i\rangle$ where $\psi_f = \psi(t_f)$ and $\psi_i = \psi(t_i)$. Going to the Schrödinger picture,

$$\langle\bar{\psi}_f, t_f|\psi_i, t_i\rangle = \langle\bar{\psi}_f|\,e^{-iH(c^\dagger,c)T}\,|\psi_i\rangle \tag{3.85}$$

where $T := t_f - t_i$. The procedure to find the path integral expression for the transition amplitude starts as the one for the bosonic case. Slicing the time in N equal sized pieces, $\delta t = \dfrac{T}{N}$ and inserting $N-1$ completeness relations,

$$\begin{aligned}\langle\bar{\psi}_f, t_f|\psi_i, t_i\rangle = \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k\right) \langle\bar{\psi}_f|\,e^{-iH\delta t}\,|\psi_{N-1}\rangle\,\langle\bar{\psi}_{N-1}|\,e^{-\bar{\psi}_{N-1}\psi_{N-1}}e^{-iH\delta t}\,|\psi_{N-2}\rangle \times \ldots \\ \cdots \times \langle\bar{\psi}_1|\,e^{-\bar{\psi}_1\psi_1}e^{-iH\delta t}\,|\psi_i\rangle\,.\end{aligned} \tag{3.86}$$

Note that $d\bar{\psi}_k d\psi_k$ always come in pairs. Note that $e^{-\bar{\psi}\psi}$ is Grassmann even. Hence, defining $\bar{\psi}_N := \bar{\psi}_f$ and $\psi_0 := \psi_i$ to ease the notation,

$$\begin{aligned}
\langle \bar{\psi}_f, t_f | \psi_i, t_i \rangle &= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{-\sum_{j=1}^{N-1} \bar{\psi}_j \psi_j} \prod_{l=1}^{N} \langle \bar{\psi}_l | \, e^{-iH(c^\dagger, c)\delta t} \, | \psi_{l-1} \rangle \\
&= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{-\sum_{j=1}^{N-1} \bar{\psi}_j \psi_j} \prod_{l=1}^{N} \langle \bar{\psi}_l | \psi_{l-1} \rangle \, e^{-iH(\bar{\psi}_l, \psi_{l-1})\delta t}
\end{aligned} \tag{3.87}$$

where we have used (3.84) together with the assumption that the Hamiltonian is Grassmann even. Now, from (3.82) and again using that our quantities in the exponent are Grassmann even, the transition amplitude takes the form

$$\begin{aligned}
\langle \bar{\psi}_f, t_f | \psi_i, t_i \rangle &= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{-\sum_{j=1}^{N-1} \bar{\psi}_j \psi_j} \prod_{l=1}^{N} e^{\bar{\psi}_l \psi_{l-1} - iH(\bar{\psi}_l, \psi_{l-1})\delta t} \\
&= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{\bar{\psi}_N \psi_N} e^{-\sum_{j=1}^{N} \bar{\psi}_j \psi_j} \prod_{l=1}^{N} e^{\bar{\psi}_l \psi_{l-1} - iH(\bar{\psi}_l, \psi_{l-1})\delta t} \\
&= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{\bar{\psi}_N \psi_N} e^{\sum_{j=1}^{N} \left(\bar{\psi}_j \frac{\psi_{j-1} - \psi_j}{\delta t} - iH(\bar{\psi}_j, \psi_{j-1}) \right) \delta t} \\
&= \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{\bar{\psi}_N \psi_N} e^{i \sum_{j=1}^{N} \left(i\bar{\psi}_j \frac{\psi_j - \psi_{j-1}}{\delta t} - H(\bar{\psi}_j, \psi_{j-1}) \right) \delta t}.
\end{aligned} \tag{3.88}$$

Finally, taking the $N \to \infty$ limit we see that the sum is just a Riemann sum and $\dfrac{\psi_j - \psi_{j-1}}{\delta t} \to \dot{\psi}$ so formally,

$$\begin{aligned}
\langle \bar{\psi}_f, t_f | \psi_i, t_i \rangle &= \lim_{N \to \infty} \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{\bar{\psi}_N \psi_N} e^{i \sum_{j=1}^{N} \left(i\bar{\psi}_j \frac{\psi_j - \psi_{j-1}}{\delta t} - H(\bar{\psi}_j, \psi_{j-1}) \right) \delta t} \\
&= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \; e^{\bar{\psi}_f \psi_f} e^{i \int_{t_i}^{t_f} dt (i\bar{\psi}\dot{\psi} - H(\bar{\psi}, \psi))} \\
&= \int \tilde{\mathcal{D}}\bar{\psi} \tilde{\mathcal{D}}\psi \; e^{\bar{\psi}_f \psi_f} e^{iS[\bar{\psi}, \psi]}
\end{aligned} \tag{3.89}$$

where S is the classical action. Note that unlike the bosonic expression (3.1), the indices in arguments in the Hamiltonian are slightly shifted, $H(\bar{\psi}_j, \psi_{j-1})$. Depending on the form of the Hamiltonian, one might need to take this into consideration. In Euclidean time, the transition amplitude becomes

$$\begin{aligned}
\langle \bar{\psi}_f, \tau_f | \psi_i, \tau_i \rangle &= \lim_{N \to \infty} \int \left(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{\bar{\psi}_N \psi_N} e^{-\sum_{j=1}^{N} \left(\bar{\psi}_j \frac{\psi_j - \psi_{j-1}}{\delta \tau} + H_E(\bar{\psi}_j, \psi_{j-1}) \right) \delta \tau} \\
&= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \; e^{\bar{\psi}_f \psi_f} e^{-\int_{t_i}^{t_f} d\tau (\bar{\psi} \frac{d}{d\tau} \psi + H_E(\bar{\psi}, \psi))} \\
&= \int \tilde{\mathcal{D}}\bar{\psi} \tilde{\mathcal{D}}\psi \; e^{\bar{\psi}_f \psi_f} e^{-S_E[\bar{\psi}, \psi]}
\end{aligned} \tag{3.90}$$

where as before, the subscript E denotes the corresponding Euclidean quantity. As before, the partition function is defined as

$$Z(\beta) := \mathrm{Tr}\; e^{-\beta H} = \sum_{n=0}^{1} \langle n | \, e^{-\beta H} \, | n \rangle \tag{3.91}$$

On the other hand, one may also compute the partition function via[7]

$$\text{Tr } e^{-\beta H} = \int d\bar{\psi}_0 d\psi_0 \, \langle -\bar{\psi}_0 | \, e^{-\beta H} \, |\psi_0\rangle \, e^{-\bar{\psi}_0 \psi_0}. \tag{3.92}$$

The minus sign in the bra-vector indicates that the ψ should satisfy antiperiodic boundary conditions. In other words, the following path integral expression for the partition function will be taken over antiperiodic paths over the circle with circumference β. The proof of this statement can be found in e.g. Nakahara [11]. Since $\langle \bar{\psi}_f, \tau_f | \psi_i, \tau_i \rangle = \langle \bar{\psi} | \, e^{-(\tau_f - \tau_i)H} \, |\psi_i\rangle$ (also, we may again shift the integration between 0 and β),

$$\begin{aligned}
\text{Tr } e^{-\beta H} &= \int d\bar{\psi}_0 d\psi_0 \int_{\psi(0) = -\psi(\beta) = \psi_0} \tilde{\mathcal{D}}\bar{\psi}\tilde{\mathcal{D}}\psi \; e^{\bar{\psi}_f \psi_f} e^{-S_E[\bar{\psi}, \psi]} e^{-\bar{\psi}_0 \psi_0} \\
&= \int d\bar{\psi}_0 d\psi_0 \int_{\psi(0) = -\psi(\beta) = \psi_0} \tilde{\mathcal{D}}\bar{\psi}\tilde{\mathcal{D}}\psi e^{-S_E[\bar{\psi}, \psi]} \\
&= \lim_{N \to \infty} \int_{ABC} \left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{-\sum_{j=1}^{N} \left(\bar{\psi}_j \frac{\psi_j - \psi_{j-1}}{\delta\tau} + H_E(\bar{\psi}_j, \psi_{j-1}) \right) \delta\tau} \\
&=: \int_{ABC} \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S_E[\bar{\psi}, \psi]}
\end{aligned} \tag{3.93}$$

where we on the second equality recalled that $\psi_f = \psi(\beta) = -\psi_0$ and $\bar{\psi}_f = \bar{\psi}(\beta) = -\bar{\psi}_0$. ABC stands for "antiperiodic boundary conditions". However in this thesis, the quantity we are interested in is the *supertrace* on fermions defined by

$$\begin{aligned}
\text{sTr } e^{-\beta H} &= \int d\bar{\psi}_0 d\psi_0 \, \langle \bar{\psi}_0 | \, e^{-\beta H} \, |\psi_0\rangle \, e^{-\bar{\psi}_0 \psi_0} \\
&= \int d\bar{\psi}_0 d\psi_0 \int_{\psi(0) = \psi(\beta) = \psi_0} \tilde{\mathcal{D}}\bar{\psi}\tilde{\mathcal{D}}\psi \; e^{\bar{\psi}_f \psi_f} e^{-S_E[\bar{\psi}, \psi]} e^{-\bar{\psi}_0 \psi_0} \\
&= \int d\bar{\psi}_0 d\psi_0 \int_{\psi(0) = \psi(\beta) = \psi_0} \tilde{\mathcal{D}}\bar{\psi}\tilde{\mathcal{D}}\psi e^{-S_E[\bar{\psi}, \psi]} \\
&= \lim_{N \to \infty} \int_{PBC} \left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k \right) e^{-\sum_{j=1}^{N} \left(\bar{\psi}_j \frac{\psi_j - \psi_{j-1}}{\delta\tau} + H_E(\bar{\psi}_j, \psi_{j-1}) \right) \delta\tau} \\
&=: \int_{PBC} \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S_E[\bar{\psi}, \psi]}
\end{aligned} \tag{3.94}$$

where now, the boundary conditions are periodic. On bosonic variables, the supertrace is defined to coincide with the trace. Define the operator $(-1)^F$ where $F = c^\dagger c$ is the fermion number operator, and note that $(-1)^F |\psi\rangle = (-1)^F |0\rangle + (-1)^F |1\rangle \psi = |0\rangle + |1\rangle (-\psi) = |-\psi\rangle$. We may thus in total write

$$\text{sTr} = \text{Tr } (-1)^F e^{-\beta H}. \tag{3.95}$$

This is a central piece in what will follow in the thesis. We will return to discuss more about this quantity in the next section.

Example 3.5. Let us as a check investigate the one dimensional fermionic harmonic oscillator described by the Hamiltonian

$$H = \left(c^\dagger c - \frac{1}{2} \right) \omega. \tag{3.96}$$

[7] The ψ_0 here is unrelated to ψ_i from the derivation of the path integral for the transition amplitude earlier. Instead, one should view them in analogy to the position eigenstates in (3.8). We have simply chosen this name to match with our notation later. The extra $e^{-\bar{\psi}_0 \psi_0}$ factor in the path integral expression for the fermionic partition function as compared to (3.8) comes from the fact that the completeness relation in this case look a bit different.

The energy eigenvalues are $\pm\frac{\omega}{2}$. The partition function is then given by

$$Z(\beta) = \text{Tr } e^{-\beta H} = \sum_{n=0}^{1} \langle n| e^{-\beta H} |n\rangle = e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}} = 2\cosh\left(\frac{\beta\omega}{2}\right). \tag{3.97}$$

Let us now compute the same quantity via the path integral. We commented before that we needed to take care when dealing with certain Hamiltoninans so starting form the the second row of (3.93),

$$\begin{aligned}
Z(\beta) &= \lim_{N\to\infty}\int_{ABC}\left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\right) e^{-\sum_{j=1}^{N}\left(\bar{\psi}_j\frac{\psi_j-\psi_{j-1}}{\delta\tau}+H_E(\bar{\psi}_j,\psi_{j-1})\right)\delta\tau} \\
&= \lim_{N\to\infty}\int_{ABC}\left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\right) e^{-\sum_{j=1}^{N}\left(\bar{\psi}_j\frac{\psi_j-\psi_{j-1}}{\delta\tau}+(\bar{\psi}_j\psi_{j-1}-\frac{1}{2})\omega\right)\delta\tau} \\
&= \lim_{N\to\infty}\int_{ABC}\left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\right) e^{-\sum_{j=1}^{N}\left((1-\delta\tau\omega)\bar{\psi}_j\frac{\psi_j-\psi_{j-1}}{\delta\tau}+\omega\bar{\psi}_j\psi_j-\frac{\omega}{2}\right)\delta\tau} \\
&= e^{\frac{\beta\omega}{2}}\lim_{N\to\infty}\int_{ABC}\left(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\right) e^{-\sum_{j=1}^{N}\left((1-\delta\tau\omega)\bar{\psi}_j\frac{\psi_j-\psi_{j-1}}{\delta\tau}+\omega\bar{\psi}_j\psi_j\right)\delta\tau} \\
&= e^{\frac{\beta\omega}{2}}\int_{ABC}\mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\int d\tau\bar{\psi}\left((1-\delta\tau\omega)\frac{d}{d\tau}+\omega\right)\psi}
\end{aligned} \tag{3.98}$$

As commented in Nakahara [11], it might seem strange to carry around the $\delta\tau$-term but it will turn out to give a finite contribution to the final result. Since we have antiperiodic boundary conditions, the orthonormal eigenfunctions to the $(1-\delta\tau\omega)\frac{d}{d\tau}+\omega$ operator takes the form

$$f_n = \frac{1}{\sqrt{\beta}}e^{\frac{i\pi(2n-1)}{\beta}\tau} \tag{3.99}$$

with corresponding eigenvalues

$$\lambda_n = (1-\delta\tau\omega)\frac{i\pi(2n-1)}{\beta}+\omega \tag{3.100}$$

Expanding the fermionic variables in these eigenfunctions

$$\psi = \sum_k a_k f_k \tag{3.101}$$

where a_k is Grassmann, the action then takes the form

$$S_E = \sum_l \lambda_l a_l^* a_l - \frac{\beta\omega}{2}. \tag{3.102}$$

Proceeding as in the bosonic case and noting that the Grassmann numbers all come in pairs,

$$\begin{aligned}
Z(\beta) &= e^{\frac{\beta\omega}{2}}\lim_{N\to\infty}\int\left(\prod_{k=-\frac{N}{4}}^{\frac{N}{4}} da_k^* da_k\right) e^{-\sum_l \lambda_l a_l^* a_l} = e^{\frac{\beta\omega}{2}}\lim_{N\to\infty}\left(\prod_{k=-\frac{N}{4}}^{\frac{N}{4}}\int da_k^* da_k e^{-\lambda_k a_k^* a_k}\right) \\
&= e^{\frac{\beta\omega}{2}}\lim_{N\to\infty}\left(\prod_{k=-\frac{N}{4}}^{\frac{N}{4}}\int da_k^* da_k(1-\lambda_k a_k^* a_k)\right) = e^{\frac{\beta\omega}{2}}\lim_{N\to\infty}\prod_{k=-\frac{N}{4}}^{\frac{N}{4}}\lambda_k \\
&= e^{\frac{\beta\omega}{2}}\text{Det}_{ABC}\left((1-\delta\tau\omega)\frac{d}{d\tau}+\omega\right).
\end{aligned} \tag{3.103}$$

Observe that since the variables were complex, each one contained two degrees of freedom. As a result, the product is truncated at $-\frac{N}{4} < k < \frac{N}{4}$. Continuing with the computation and recalling the formula for the exponential

$$e^x = \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N \tag{3.104}$$

and that $\delta\tau = \frac{\beta}{N}$ we get

$$\begin{aligned} Z(\beta) &= e^{\frac{\beta\omega}{2}} \lim_{N\to\infty} \prod_{k=-\frac{N}{4}}^{\frac{N}{4}} \left((1-\delta\tau\omega)\frac{i\pi(2k-1)}{\beta}+\omega\right) \\ &= e^{\frac{\beta\omega}{2}} \lim_{N\to\infty} \left(1-\frac{\beta\omega}{N}\right)^{\frac{N}{2}} \prod_{k=-\frac{N}{4}}^{\frac{N}{4}} \left(\frac{i\pi(2k-1)}{\beta}+\frac{\omega}{1-\delta\tau\omega}\right) \\ &= e^{\frac{\beta\omega}{2}} e^{-\frac{\beta\omega}{2}} \prod_{k=-\infty}^{\infty}\left(\frac{i\pi(2k-1)}{\beta}+\omega\right) = \prod_{k=-\infty}^{\infty}\left(\frac{i\pi(2k-1)}{\beta}+\omega\right) = \mathrm{Det}_{ABC}\left(\frac{d}{d\tau}+\omega\right) \end{aligned} \tag{3.105}$$

Rewriting the product and using the formula

$$\cosh\left(\frac{x}{2}\right) = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{\pi^2(2n-1)^2}\right), \tag{3.106}$$

we end up with

$$\begin{aligned} Z(\beta) &= \prod_{k=1}^{\infty}\left(\frac{i\pi(2k-1)}{\beta}+\omega\right)\left(\frac{i\pi(-2(k-1)-1)}{\beta}+\omega\right) \\ &= \prod_{k=1}^{\infty}\left(\left(\frac{\pi(2k-1)}{\beta}\right)^2+\omega^2\right) = \left[\prod_{k=1}^{\infty}\left(\frac{\pi(2k-1)}{\beta}\right)^2\right]\prod_{n=1}^{\infty}\left(1+\left(\frac{\omega\beta}{\pi(2n-1)}\right)^2\right) \\ &= \left[\prod_{k=1}^{\infty}\left(\frac{\pi(2k-1)}{\beta}\right)^2\right]\cosh\left(\frac{\beta\omega}{2}\right) \end{aligned} \tag{3.107}$$

The infinite product needs to be regularized. This can be done using the *Hurwitz ζ-function*

$$\zeta(s,a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^s} \tag{3.108}$$

in a similar way as previously done. We will not do it here though, the interested reader can e.g. check Nakahara [11]. One finds that

$$\prod_{k=1}^{\infty}\left(\frac{\pi(2k-1)}{\beta}\right)^2 = 2 \tag{3.109}$$

which finally gives us the desired result

$$Z(\beta) = 2\cosh\left(\frac{\beta\omega}{2}\right). \tag{3.110}$$

Let us now take a slightly different more heuristic approach by interpreting directly that in the path integral expression,

$$S_E = \int d\tau L_E \tag{3.111}$$

where L_E is the classical Euclidean Lagrangian, in this example given by[8]

$$L_E = \frac{1}{2}\bar{\psi}\frac{d}{d\tau}\psi + \frac{1}{2}\psi\frac{d}{d\tau}\bar{\psi} + \frac{\omega}{2}[\bar{\psi},\psi] \tag{3.112}$$

The classical (Euclidean) action, integrating the second term by parts and applying antiperiodic boundary conditions, is then given by

$$S_E = \int_0^\beta dt\left(\bar{\psi}\frac{d}{d\tau}\psi + \omega\bar{\psi}\psi\right) = \int_0^\beta dt\left(\bar{\psi}\left(\frac{d}{d\tau}+\omega\right)\psi\right). \tag{3.113}$$

As a matter of fact we again obtain

$$Z(\beta) = \int_{ABC}\mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\int_0^\beta dt\left(\bar{\psi}\left(\frac{d}{d\tau}+\omega\right)\psi\right)} = \mathrm{Det}_{ABC}\left(\frac{d}{d\tau}+\omega\right) = 2\cosh\left(\frac{\beta\omega}{2}\right). \tag{3.114}$$

Example 3.6. Let us from now on focus on the supertrace. As a first example, consider the free Dirac fermion

$$L_E = \frac{1}{2}\alpha\bar{\psi}\frac{d}{d\tau}\psi + \frac{1}{2}\alpha\psi\frac{d}{d\tau}\bar{\psi}. \tag{3.115}$$

for some constant α. The orthonormal eigenfunctions to $\alpha\frac{d}{d\tau}$ with periodic boundary conditions are

$$f_n(\tau) = \frac{1}{\sqrt{\beta}}e^{\frac{i2\pi n}{\beta}\tau} \tag{3.116}$$

with eigenvalues

$$\lambda_n = \alpha\frac{i2\pi n}{\beta}. \tag{3.117}$$

Expanding the variables as

$$\psi = \sum_n a_n f_n \tag{3.118}$$

the action, integrating the second term by parts and applying the periodic boundary conditions, can then be written as

$$S_E = \int_0^\beta d\tau\bar{\psi}\alpha\frac{d}{d\tau}\psi = \sum_n \lambda_n a_n^* a_n \tag{3.119}$$

Relabelling $\psi_j \to \psi_{j+1}$ and changing variables as $a_{|k|} \to \psi_{2k}$ and $a_{-|k|} \to \psi_{2k-1}$, $k \neq 0$ and defining

$$N_+ = \frac{N-1+\sigma(N)}{2} \tag{3.120}$$

$$N_- = \frac{N+\sigma(N+1)}{2} \tag{3.121}$$

where as before $\sigma(n) = 0$ for n even and $\sigma(n) = 1$ for odd n, we get

$$\begin{aligned}\mathrm{sTr}\ e^{-\beta H} &= \int_{PBC}\mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-S_E[\bar{\psi},\psi]} = \lim_{N\to\infty}\prod_{\substack{k=-N_-\\k\neq 0}}^{N_+}\int da_k^* da_k e^{-\lambda_k a_k^* a_k}\\ &= \lim_{N\to\infty}\prod_{\substack{k=-N_-\\k\neq 0}}^{N_+}\int da_k^* da_k(1-\lambda_k a_k^* a_k) = \lim_{N\to\infty}\prod_{\substack{k=-N_-\\k\neq 0}}^{N_+}\lambda_k = \mathrm{Det}'\left(\alpha\frac{d}{d\tau}\right).\end{aligned} \tag{3.122}$$

[8]What appears here is a different ordered system.

The functional determinant is evaluated over periodic boundary conditions and as before, the $'$ means we are skipping the zero eigenvalue. Note that this procedure is quite general; $\alpha\frac{d}{d\tau}$ could be swapped with another operator O with orthonormal eigenfunctions f_n^O and corresponding eigenvalues λ_n^O to obtain

$$\text{sTr}e^{-\beta H} = \int_{PBC} \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-\int_0^\beta d\tau\bar{\psi}O\psi} = \text{Det}'(O) \tag{3.123}$$

and again skipping over the zero eigenvalue if it exists.

Example 3.7. Let us next consider the free Majorana fermion

$$L_E = \frac{1}{2}\alpha\psi\frac{d}{d\tau}\psi. \tag{3.124}$$

However, in this case there is a slight complication since we only have one variable. A trick is to compute the square of the supertrace

$$\left(\text{sTr}e^{-\beta H}\right)^2 = \int_{PBC} \mathcal{D}\psi_2\mathcal{D}\psi_1 e^{-\int_0^\beta d\tau\left(\frac{1}{2}\alpha\psi_2\frac{d}{d\tau}\psi_2+\frac{1}{2}\alpha\psi_1\frac{d}{d\tau}\psi_1\right)}. \tag{3.125}$$

By changing variables

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \tag{3.126}$$

$$\bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{3.127}$$

the action is written as

$$\begin{aligned} S_E &= \int_0^\beta d\tau\frac{1}{2}\alpha\left[-\frac{1}{2}(\psi-\bar{\psi})\frac{d}{d\tau}(\psi-\bar{\psi}) + \frac{1}{2}(\psi+\bar{\psi})\frac{d}{d\tau}(\psi+\bar{\psi})\right] = \int_0^\beta d\tau\frac{1}{2}\alpha(\bar{\psi}\frac{d}{d\tau}\psi + \psi\frac{d}{d\tau}\bar{\psi}) \\ &= \int_0^\beta d\tau\bar{\psi}\alpha\frac{d}{d\tau}\psi \end{aligned} \tag{3.128}$$

yet again remembering that we are integrating over periodic paths. Note now that for each j (referring to the time slicing),

$$\psi_j = \frac{1}{\sqrt{2}}(\psi_{1j} + i\psi_{2j}) \tag{3.129}$$

$$\bar{\psi}_j = \frac{1}{\sqrt{2}}(\psi_{1j} - i\psi_{2j}) \tag{3.130}$$

so

$$\prod_{k=0}^{N-1} d\psi_{2k}d\psi_{1k} = \prod_{k=0}^{N-1}(-i)d\bar{\psi}_k d\psi_k \tag{3.131}$$

the factors of $-i$ coming from the Jacobian

$$\frac{\partial(\psi_j,\bar{\psi}_j)}{\partial(\psi_{1j},\psi_{2j})} = \left[\det\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\right]^N = (-i)^N. \tag{3.132}$$

The eigenfunctions and corresponding eigenvalues are the same as in previous example so we get

$$(\text{sTr}\ e^{-\beta H})^2 = \lim_{N\to\infty}\prod_{\substack{k=-N_-\\k\neq 0}}^{N_+}(-i\lambda_k) = \lim_{N\to\infty}\left[\prod_{j=1}^{N_+}(-i\lambda_j)\right]\prod_{k=1}^{N_-} i\lambda_k \tag{3.133}$$

We need to regularize the infinite products. Let us do it for the slightly more general case

$$\prod_{k=1}^{\infty} i^a \lambda_k \tag{3.134}$$

where a is some integer. Following the same procedure as in Example 3.1, the spectral zeta function is

$$\zeta_O(s) = \sum_{k=1}^{\infty} \left(\frac{\beta}{i^{a+1}\alpha 2\pi k}\right)^s = \left(\frac{\beta}{i^{a+1}\alpha 2\pi}\right)^s \zeta(s) = e^{s\ln\left(\frac{\beta}{i^{a+1}\alpha 2\pi}\right)}\zeta(s) \tag{3.135}$$

so

$$\zeta_O'(0) = \ln\left(\frac{\beta}{i^{a+1}\alpha 2\pi}\right)\zeta(0) + \zeta'(0) = -\frac{1}{2}\ln\left(\frac{\beta}{i^{a+1}\alpha}\right) \tag{3.136}$$

Hence

$$\prod_{k=1}^{\infty} i^a \lambda_k = e^{\frac{1}{2}\ln\left(\frac{\beta}{i^{a+1}\alpha}\right)} = \sqrt{\frac{\beta}{i^{a+1}\alpha}} \tag{3.137}$$

which means

$$(\text{sTr } e^{-\beta H})^2 = \lim_{N\to\infty}\left[\prod_{j=1}^{N_+}(-i\lambda_j)\right]\prod_{k=1}^{N_-} i\lambda_k = \sqrt{\frac{\beta}{i^4\alpha}}\sqrt{\frac{\beta}{i^2\alpha}} = i\frac{\beta}{\alpha}. \tag{3.138}$$

Comparing with the regularized determinant

$$\text{Det}'\left(\alpha\frac{d}{d\tau}\right) = \lim_{N\to\infty}\prod_{\substack{k=-N_-\\ k\neq 0}}^{N_+}\lambda_k = \lim_{N\to\infty}\left[\prod_{j=1}^{N_+}(-\lambda_j)\right]\prod_{k=1}^{N_-}\lambda_k = \sqrt{\frac{\beta}{i^3\alpha}}\sqrt{\frac{\beta}{i^1\alpha}} = \frac{\beta}{\alpha} \tag{3.139}$$

we end up with the expression

$$\text{sTr } e^{-\beta H} = \int_{PBC}\mathcal{D}\psi\ e^{-\int_0^\beta d\tau\frac{1}{2}\alpha\psi\frac{d}{d\tau}\psi} = \sqrt{i\text{Det}'\left(\alpha\frac{d}{d\tau}\right)}. \tag{3.140}$$

3.3 A Short Note on The Saddle Point Method

Before we move on with the next part of the thesis, we will briefly illustrate the so called *saddle point method* (also known as the *method of steepest descent*) in its simplest setting. The discussion carries over to the path integral case, but for simplicity, we keep to this example. Consider an integral of the form

$$I = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-\frac{f(x)}{\hbar}}. \tag{3.141}$$

Assume that the function only has exactly one minimum at $x = x_0$ such that $f''(x_0) > 0$. Let us expand the integral asymptotically around its minimum in the limit $\hbar \to 0$. Let $x = x_0 + \sqrt{\hbar}y$. Then Taylor expanding f around x_0

$$f(x) = f(x_0) + \frac{1}{2}\hbar y^2 f''(x_0) + \frac{1}{3!}\hbar^{\frac{3}{2}}y^3 f^{(3)}(x_0) + \frac{1}{4!}\hbar^2 y^4 f^{(4)}(x_0) + \dots. \tag{3.142}$$

Plugging this back into I,

$$I = \sqrt{\hbar}e^{-\frac{f(x^0)}{\hbar}}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2 f''(x_0) - \left(\frac{1}{3!}\hbar^{\frac{1}{2}}y^3 f^{(3)}(x_0) + \frac{1}{4!}\hbar y^4 f^{(4)}(x_0) + \dots\right)}. \tag{3.143}$$

We now define the *moment* of y by

$$\langle y^n \rangle = \frac{\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} y^n e^{-y^2 \frac{f''(x_0)}{2}}}{\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2 \frac{f''(x_0)}{2}}}. \tag{3.144}$$

Note that for odd n, the moment is zero since y^n is odd and $e^{-y^2 \frac{f''(x_0)}{2}}$ is even. In the denominator we have an Gaussian integral

$$\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2 \frac{f''(x_0)}{2}} = \frac{1}{\sqrt{f''(x_0)}} \tag{3.145}$$

so we may rewrite I as

$$I = \frac{\sqrt{\hbar} e^{-\frac{f(x^0)}{\hbar}}}{\sqrt{f''(x_0)}} \left\langle \exp\left[-\frac{1}{3!} \hbar^{\frac{1}{2}} y^3 f^{(3)}(x_0) - \frac{1}{4!} \hbar y^4 f^{(4)}(x_0) - \dots \right] \right\rangle. \tag{3.146}$$

Expanding the exponential in the moment, we note that $\langle \dots \rangle = 1 + \mathcal{O}(\hbar)$ since $\langle y^3 \rangle = 0$. Later in the thesis, we will replace $\hbar$ with β and f with the action. The quantity we compute will in fact be independent of β and so the terms of $\mathcal{O}(\beta)$ will vanish which means we only need to care about the extrema of the action and the second order fluctuations.

4 Supersymmetric Quantum Mechanics

We begin this section by defining supersymmetric quantum mechanics (abbreviated SUSYQM) using the approach of Witten [2], following up with a discussion about some general properties of SUSYQM. The material covered here can be found in [1, 2, 11, 15, 16].

4.1 Definition and General Properties

Definition 4.1. *A quantum system with Hilbert space $\mathcal{H}$ and Hamiltonian H is $\mathcal{N} = n$ supersymmetric if the following is satisfied:*

- The Hilbert space is $\mathbb{Z}_2$-graded, $\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$. The states in $\mathcal{H}^B$ are called *bosonic* and the states in $\mathcal{H}^F$ are called *fermionic*. Define an operator $(-1)^F$ by $(-1)^F |\phi\rangle = |\phi\rangle$ for $|\phi\rangle \in \mathcal{H}^B$ and $(-1)^F |\psi\rangle = -|\psi\rangle$ for $|\psi\rangle \in \mathcal{H}^F$.
- There is a set of Hermitian operators $\{Q_1, \dots, Q_n\}$ called *supercharges*, each mapping $\mathcal{H}^B$ into $\mathcal{H}^F$ and vice versa, such that

$$\{(-1)^F, Q_i\} = 0 \tag{4.1}$$

$$\{Q_i, Q_j\} = 2\delta_{ij} H \tag{4.2}$$

for all i, j.

We would like to stress that SUSYQM is sometimes defined differently using non-Hermitian supercharges $\{\tilde{Q}_1, \tilde{Q}_1^\dagger, \dots \tilde{Q}_m, \tilde{Q}_m^\dagger\}$ such that $\{\tilde{Q}_i, \tilde{Q}_j^\dagger\} = 2\delta_{ij} H$ and $\{\tilde{Q}_i, \tilde{Q}_j\} = 0$. For instance, if n is even, one may take

$$\tilde{Q}_i = \frac{1}{\sqrt{2}}(Q_{2i} + iQ_{2i+1}); \quad \tilde{Q}_i^\dagger = \frac{1}{\sqrt{2}}(Q_{2i} - iQ_{2i+1}). \tag{4.3}$$

We will see this later in the thesis. In fact, given an operator A such that the adjoint $A^\dagger$ exists, one may always decompose it as a sum of a Hermitian operator A_H and a skew-Hermitian operator A_S: just take $A_H = \frac{1}{2}(A + A^\dagger)$ and $A_S = \frac{1}{2}(A - A^\dagger)$. This decomposition is also unique; assume there is another decomposition

$A = A'_H + A'_S$. Then $A_H - A'_H + A_S - A'_S = 0$ and $A_H^\dagger - A'^\dagger_H + A_S^\dagger - A'^\dagger_S = A_H - A'_H - (A_S - A'_S) = 0$. This implies that $A_H = A'_H$ and $A_S = A'_S$. Furthermore, a Hermitian operator can always be made skew-Hermitian and vice versa by multiplying it with the imaginary unit i. Hence the definition using non-Hermitian operators is contained in the one using Hermitian supercharges since any A described above can be written as in (4.3): $A = \frac{1}{2}[(A + A^\dagger) + i(-i)(A - A^\dagger)]$. The reason we give definition 4.1 is that it provides a nice way to include the instances with odd number of supercharges, in particular the $\mathcal{N} = 1$ case. Otherwise, one may speak of "$\mathcal{N} = \frac{1}{2}$ SUSYQM" as in e.g. [3].

The relation (4.2) immediately implies that $H = \frac{1}{2}Q_i^2 = \frac{1}{2n}\sum_{k=1}^{n} Q_k^2$. From this we may deduce two facts. Firstly, all supercharges commute with the Hamiltonian

$$[Q_i, H] = 0 \tag{4.4}$$

Secondly, taking into account the Hermiticity of the Q_i, we see that the eigenvalues of the Hamiltonian (which we call the *energy* of a given eigenstate) are non-negative. Using (4.1), we also have

$$\begin{aligned}
[(-1)^F, H] &= [(-1)^F, Q_i^2] = (-1)^F Q_i^2 - Q_i^2(-1)^F \\
&= \{(-1)^F, Q_i\}Q_i - Q_i(-1)^F Q_i - Q_i^2(-1)^F \\
&= \{(-1)^F, Q_i\}Q_i - Q_i\{(-1)^F, Q_i\} + Q_i^2(-1)^F - Q_i^2(-1)^F \\
&= 0
\end{aligned} \tag{4.5}$$

i.e. the operator $(-1)^F$ commutes with the Hamiltonian,

$$[(-1)^F, H] = 0. \tag{4.6}$$

We may decompose the Hilbert space in terms of eigenspaces of the Hamiltonian,

$$\mathcal{H} = \bigoplus_n \mathcal{H}_{(n)} \tag{4.7}$$

where the states in $\mathcal{H}_{(n)}$ has energy E_n and with $E_0 = 0$. States in $\mathcal{H}_0$ are known as *supersymmetric ground states* (a minor note, the non-existence of such states is equivalent to the statement $\mathcal{H}_0 \cong \{0\}$). We use the convention that E_n increases as n increases. The supercharges and $(-1)^F$ preserve the energy levels since they commute with the Hamiltonian so

$$Q_i, (-1)^F : \mathcal{H}_{(n)} \to \mathcal{H}_{(n)}. \tag{4.8}$$

In particular, this means that the $\mathbb{Z}_2$-grading of $\mathcal{H}$ is carried over to each subspace

$$\mathcal{H}_{(n)} = \mathcal{H}^B_{(n)} \oplus \mathcal{H}^F_{(n)} \tag{4.9}$$

with the supercharges mapping $\mathcal{H}^B_{(n)}$ into $\mathcal{H}^F_{(n)}$ and the other way around.

A particular feature of SUSYQM is that the states of non-zero energy are paired up.

Proposition 4.1. *The subspaces $\mathcal{H}^B_{(n)}$ and $\mathcal{H}^F_{(n)}$ are isomorphic for $n \neq 0$.*

Proof. Let $|b\rangle \in \mathcal{H}^B_{(n)}$ be any state with non-zero energy E_n. Choose one Q_i and consider the fermionic state defined by $|f\rangle := \frac{1}{\sqrt{E_n}} Q_i |b\rangle$ (this definition is just to ensure that $|f\rangle$ is normalised if $|b\rangle$ is). This gives us

$$Q_i |b\rangle = \sqrt{E_n} |f\rangle ; \qquad Q_i |f\rangle = \sqrt{E_n} |b\rangle . \tag{4.10}$$

Hence Q_i is invertible (with inverse $\frac{1}{E_n}Q_i$) and defines an isomorphism

$$\mathcal{H}^B_{(n)} \cong \mathcal{H}^F_{(n)} \tag{4.11}$$

as long as $E_n \neq 0$. □

What this means is that all states of non-zero energy are paired up with a unique state in the other subspace. This is however not generically true for the zero energy states (should there exist any). A state is annihilated by the Hamiltonian H if and only if it is annihilated by all Q_i,

$$0 = H|\alpha\rangle = \frac{1}{2}Q_i^2|\alpha\rangle \iff Q_i|\alpha\rangle = 0. \tag{4.12}$$

If $Q_i|\alpha\rangle = 0$, it follows immediately that $H|\alpha\rangle = 0$. Assume the converse, $H|\alpha\rangle = 0$. Taking the inner product

$$0 = \langle\alpha| H |\alpha\rangle = \frac{1}{2}\langle\alpha| Q_i Q_i |\alpha\rangle = \frac{1}{2}(Q_i|\alpha\rangle)^\dagger Q_i|\alpha\rangle \tag{4.13}$$

where we on the last equality used that Q_i is Hermitian. Positive-definiteness implies that $Q_i|\alpha\rangle = 0$ (Q_i is thus not generally invertible on $\mathcal{H}^B_{(0)}$ and $\mathcal{H}^F_{(0)}$).

We now want to consider a continuous deformation of our theory such that supersymmetry is preserved (i.e. our theory still has the properties of definition 4.1). By continuous deformation, we mean that the energy spectrum is continuously deformed. When this happens, the states may move around between energy levels. However, the isomorphism (4.11) constrains how the states can move around; the states can only move in pairs of one boson and one fermion. This implies that the difference between the number of bosonic states and fermionic states at each energy level is conserved under this deformation. From this, we define a quantity known as the *Witten index* or the *supersymmetric index*

$$\mathrm{Tr}(-1)^F e^{-\beta H} \tag{4.14}$$

which is a central object of study in this thesis. What the Witten index actually computes is the number of supersymmetric bosonic ground states minus the number of supersymmetric fermionic ground states. This can be seen as

$$\begin{aligned} \mathrm{Tr}(-1)^F e^{-\beta H} &= \sum_n \left(\mathrm{dim}\mathcal{H}^B_{(n)} e^{-\beta E_n} - \mathrm{dim}\mathcal{H}^F_{(n)} e^{-\beta E_n}\right) \\ &= \mathrm{dim}\mathcal{H}^B_{(0)} e^{-\beta E_0} - \mathrm{dim}\mathcal{H}^F_{(0)} e^{-\beta E_0} \\ &= \mathrm{dim}\mathcal{H}^B_{(0)} - \mathrm{dim}\mathcal{H}^F_{(0)} \end{aligned} \tag{4.15}$$

where we in the second equality have used the fact that $\mathrm{dim}\mathcal{H}^B_{(n)} = \mathrm{dim}\mathcal{H}^B_{(n)}$ for $n \neq 0$ (which is a direct consequence of (4.11)) and in the third equality that $E_0 = 0$. Note that the Witten index is actually independent of β. The factor $e^{-\beta H}$ is introduced as a way to regularize the otherwise ill-defined $\mathrm{Tr}(-1)^F$, obtained in the $\beta \to 0$ limit.

On the other hand, one can compute the the Witten index via path integration

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \left(\prod_i \mathcal{D}\phi_i\right)\left(\prod_j \mathcal{D}\psi_j\right) e^{-S_E[\phi_k,\psi_l]}. \tag{4.16}$$

where S_E is the (Euclidean) action corresponding to the Hamiltonian H. The Witten index is nothing but the supertrace defined in (3.95). This lies in the heart of the thesis; by equating the Witten index obtained from the canonical quantization picture (4.15) with the path integral formula in (4.16), one obtains a non-trivial relation. This relation is a so called *index theorem* and is a special case of the celebrated *Atiyah-Singer index theorem* [17, 18, 19].

4.2 Properties of $\mathcal{N} = 2$ Supersymmetric Quantum Mechanics

A case which we can deduce more useful properties is when we have two supercharges $\{Q_1, Q_2\}$. We will follow [15] closely in our discussions here. It is convenient to define

$$Q := \frac{1}{\sqrt{2}}(Q_1 + iQ_2); \qquad Q^\dagger := \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \tag{4.17}$$

We get the following anticommutation relations, proved by calculation and the usage of (4.1) and (4.2),

$$\{(-1)^F, Q_i\} = 0 \tag{4.18}$$

$$Q^2 = Q^{\dagger 2} = 0 \tag{4.19}$$

$$\{Q, Q^\dagger\} = 2H. \tag{4.20}$$

From this, we deduce that the newly defined supercharges are conserved,

$$[Q, H] = [Q^\dagger, H] = 0. \tag{4.21}$$

Since the Q and $Q^\dagger$ are just linear combinations of Q_1 and Q_2, they still preserve energy levels and map bosonic states to fermionic, and fermionic states to bosonic. Note also that the Hamiltonian H annihilates a state (i.e. the state has zero energy) if and only if Q and $Q^\dagger$ annihilates it

$$H|\alpha\rangle = 0 \iff Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0. \tag{4.22}$$

This is proved in a similar fashion to (4.13). The left implication is immediate. Assume hence that $H|\alpha\rangle = 0$. Taking the inner product, we get

$$0 = \langle\alpha| H |\alpha\rangle = \langle\alpha| (QQ^\dagger + Q^\dagger Q) |\alpha\rangle = \langle\alpha| QQ^\dagger |\alpha\rangle + \langle\alpha| Q^\dagger Q |\alpha\rangle \tag{4.23}$$

which implies that

$$\langle\alpha| QQ^\dagger |\alpha\rangle = -\langle\alpha| Q^\dagger Q |\alpha\rangle. \tag{4.24}$$

From the positive definiteness of inner products, we may conclude that indeed, $Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0$.

Since Q is nilpotent (4.19), we may consider the following complex[9] of vector spaces

$$\mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B \xrightarrow{Q} \mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B. \tag{4.25}$$

Since Q preserves each energy level, the complex above will decompose accordingly and we will get a collection of complexes

$$\mathcal{H}^F_{(n)} \xrightarrow{Q} \mathcal{H}^B_{(n)} \xrightarrow{Q} \mathcal{H}^F_{(n)} \xrightarrow{Q} \mathcal{H}^B_{(n)} \tag{4.26}$$

for each n. We now would like to investigate the cohomology of (4.25)

$$\begin{aligned} H^B(Q) &= \mathrm{Ker}(Q : \mathcal{H}^B \to \mathcal{H}^F)/\mathrm{Im}(Q : \mathcal{H}^F \to \mathcal{H}^B) \\ H^F(Q) &= \mathrm{Ker}(Q : \mathcal{H}^F \to \mathcal{H}^B)/\mathrm{Im}(Q : \mathcal{H}^B \to \mathcal{H}^F) \end{aligned} \tag{4.27}$$

by first considering the cohomology of the decomposed complex (4.26). The cohomologies (4.27) will be a direct sum of the corresponding cohomologies of (4.26).

Proposition 4.2. *For $n \neq 0$, (4.26) is an exact sequence.*

[9] Credit to Paul Taylor, the author to the package "diagrams" which was used to draw the (co)chain complex diagrams in this thesis.

Proof. Let $|\alpha\rangle$ be a Q-closed state, $Q|\alpha\rangle = 0$. Since $E_n \neq 0$, we my write the identity operator as

$$1 = \frac{2H}{2E_n} = \frac{QQ^\dagger + Q^\dagger Q}{2E_n}. \tag{4.28}$$

Hence

$$|\alpha\rangle = \frac{QQ^\dagger + Q^\dagger Q}{2E_n}|\alpha\rangle = \frac{QQ^\dagger}{E_n}|\alpha\rangle \tag{4.29}$$

which means $|\alpha\rangle$ is exact. Since $|\alpha\rangle$ was arbitrarily chosen, this means that all closed states are exact and we have trivial cohomology. This proves our claim. □

In the case when $n = 0$, (4.22) tells us that all states in $\mathcal{H}_{(0)}$ are closed and no exact states can exist[10]. We have thus determined the cohomology (4.27), and we get a canonical isomorphism

$$\begin{aligned} H^B(Q) &= \mathcal{H}^B_{(0)} \\ H^F(Q) &= \mathcal{H}^F_{(0)}. \end{aligned} \tag{4.30}$$

Sometimes, the Hilbert space may have a $\mathbb{Z}$-grading which reduces modulo 2 to a $\mathbb{Z}_2$-grading. One case when this might happen is if we have a Hermitian operator F with eigenvalues in $\mathbb{Z}$ and $e^{i\pi F} = (-1)^F$. We will encounter such an example so it is worthwhile to spend some time discussing this matter. If we are in this situation, the Hilbert space $\mathcal{H}$ will split into eigenspaces of F

$$\mathcal{H} = \bigoplus_{p\in\mathbb{Z}} \mathcal{H}^p \tag{4.31}$$

and the bosonic and fermionic spaces will just be

$$\mathcal{H}^B = \mathcal{H}^{\text{even}}; \qquad \mathcal{H}^F = \mathcal{H}^{\text{odd}} \tag{4.32}$$

since for $|\alpha_p\rangle \in \mathcal{H}^p$,

$$e^{i\pi F}|\alpha_p\rangle = \begin{cases} |\alpha_p\rangle & \text{if } p \text{ is even} \\ -|\alpha_p\rangle & \text{if } p \text{ is odd.} \end{cases} \tag{4.33}$$

A natural question to ask now is when Q will respect the $\mathbb{Z}$-grading due to F, i.e when Q will be a map $Q : \mathcal{H}^p \to \mathcal{H}^{p+1}$.

Proposition 4.3. *Q is a map $Q : \mathcal{H}^p \to \mathcal{H}^{p+1}$ if and only if $[F, Q] = Q$.*

Proof. Assume that $Q : \mathcal{H}^p \to \mathcal{H}^{p+1}$. Then for $|\alpha_p\rangle \in \mathcal{H}^p$,

$$[F,Q]|\alpha_p\rangle = FQ|\alpha_p\rangle - QF|\alpha_p\rangle = (p+1)Q|\alpha_p\rangle - pQ|\alpha_p\rangle = Q|\alpha_p\rangle \implies [F,Q] = Q. \tag{4.34}$$

Assume conversely that $[F, Q] = Q$. Then

$$\begin{aligned} [F,Q]|\alpha_p\rangle = Q|\alpha_p\rangle &\iff FQ|\alpha_p\rangle - QF|\alpha_p\rangle = Q|\alpha_p\rangle \iff FQ|\alpha_p\rangle - nQ|\alpha_p\rangle = Q|\alpha_p\rangle \\ &\iff FQ|\alpha_p\rangle = (n+1)Q|\alpha_p\rangle \end{aligned} \tag{4.35}$$

and our claim is thus proven. □

Remark. Similarly, one may prove that $Q^\dagger : \mathcal{H}^p \to \mathcal{H}^{p-1}$ if and only if $[F, Q^\dagger] = -Q^\dagger$.

[10]Since all states are closed, $\mathrm{Im}(Q : \mathcal{H}^B_{(0)} \to \mathcal{H}^F_{(0)}) = \mathrm{Im}(Q : \mathcal{H}^F_{(0)} \to \mathcal{H}^B_{(0)}) = 0$.

Hence, if we have that

$$[F, Q] = Q, \tag{4.36}$$

the complex (4.25) decomposes into a $\mathbb{Z}$-graded (cochain) complex

$$\dots \xrightarrow{Q} \mathcal{H}^{p-1} \xrightarrow{Q} \mathcal{H}^{p} \xrightarrow{Q} \mathcal{H}^{p+1} \xrightarrow{Q} \dots \tag{4.37}$$

with the nth cohomology group given by

$$H^n(Q) = \mathrm{Ker}(Q : \mathcal{H}^p \to \mathcal{H}^{p+1})/\mathrm{Im}(Q : \mathcal{H}^{p-1} \to \mathcal{H}^p). \tag{4.38}$$

From (4.30), we thus have

$$\begin{aligned} \mathcal{H}^B_{(0)} &= H^B(Q) = H^{\text{even}}(Q) \\ \mathcal{H}^F_{(0)} &= H^F(Q) = H^{\text{odd}}(Q). \end{aligned} \tag{4.39}$$

The Witten index then takes the form

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{p\in\mathbb{Z}} (-1)^p \mathrm{dim} H^p(Q) \tag{4.40}$$

which is recognized to be the Euler characteristic (denoted by χ) of the complex (4.37)[11].

5 Derivations of Index Theorems

We are now ready to demonstrate the proofs of a couple of index theorems using supersymmetric quantum mechanics. Let M be a compact manifold with metric g and $\phi : \mathcal{T} \to M$ be a path on M, locally represented by $\phi^\mu = x^\mu \circ \phi$. Our focus will be on so called *supersymmetric non-linear sigma models* which are models constructed from the Lagrangian $L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu$, i.e. a particle moving on M [3].

5.1 The $\mathcal{N} = 1$ Non-linear Sigma Model

In this part, we will follow the exposition of Nakahara [11] and Alvarez [20], together with the process in [15]. Let M be a $d = 2n$ even dimensional compact spin manifold (see Appendix A). The Lagrangian is given by

$$L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu D_t \psi^\nu \tag{5.1}$$

where $D_t \psi^\mu = D_{\dot{\phi}^\lambda \partial_\lambda \big|_t} \psi^\mu = \dot{\phi}^\lambda \partial_\lambda \psi^\mu + \dot{\phi}^\rho \Gamma^\mu_{\rho\sigma} \psi^\sigma = \dot{\psi}^\mu + \dot{\phi}^\rho \Gamma^\mu_{\rho\sigma} \psi^\sigma$ is just the covariant derivative along the path $\phi(t)$. The fermions ψ are locally represented by $\psi = \psi^\mu \frac{\partial}{\partial x^\mu}\big|_\phi$ where ψ^μ are Grassmann valued. The action is invariant under the supersymmetry transformation

$$\delta_\epsilon \phi^\mu = i\epsilon \psi^\mu \tag{5.2}$$

$$\delta_\epsilon \psi^\mu = -\epsilon \dot{\phi}^\mu \tag{5.3}$$

where ϵ is a Grassmann valued infinitesimal constant. Furthermore, the transformations are diffeomorphism covariant. Consider the coordinate transformation $x^\mu \to x'^\mu = x'^\mu(x)$, which means that $\phi^\mu \to \phi'^\mu = x'^\mu \circ \phi$ and $\psi^\mu \to \psi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \psi^\nu$. Then

$$\delta_\epsilon \phi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \delta_\epsilon \phi^\nu = \frac{\partial x'^\mu}{\partial x^\nu} i\epsilon \psi^\mu = i\epsilon \psi'^\mu \tag{5.4}$$

[11]An analogous analysis can be repeated for $Q^\dagger$ to yield the same result but with Q replaced with $Q^\dagger$.

and

$$\delta_\epsilon \psi'^\mu = \delta_\epsilon \frac{\partial x'^\mu}{\partial x^\nu}\psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\delta_\epsilon \psi^\nu = \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu}\delta_\epsilon \phi^\lambda \psi^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\delta_\epsilon \psi^\nu = \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu} i\epsilon\psi^\lambda \psi^\nu - \frac{\partial x'^\mu}{\partial x^\nu}\epsilon\dot{\phi}^\nu = -\epsilon\dot{\phi}'^\mu \quad (5.5)$$

where we on the fourth equality used that the antisymmetrisation of the fermions and the fact that partial derivatives commute to eliminate the first term. So the supersymmetry transformations are indeed covariant under coordinate transformations. By varying the Lagrangian (5.1) with δ_ϵ, one can show that

$$\delta_\epsilon L = \epsilon \frac{i}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu \dot{\phi}^\nu). \quad (5.6)$$

To find the supercharges, we go through Noether's procedure. By treating ϵ as a time-dependent object[12], $\epsilon \to \epsilon = \epsilon(t)$,

$$\begin{aligned}
\delta_\epsilon S &= \int_0^\beta dt \left[\frac{i}{2}g_{\mu\nu}\dot{\epsilon}(\psi^\mu\dot{\phi}^\nu + \dot{\phi}^\mu\psi^\nu) + \frac{i}{2}g_{\mu\nu}\psi^\mu\dot{\epsilon}(-\dot{\phi}^\nu + i\psi^\rho\Gamma^\nu_{\rho\sigma}\psi^\sigma) + \epsilon\frac{i}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\right] \\
&= \int_0^\beta dt \left[\frac{3i}{2}\dot{\epsilon}g_{\mu\nu}\psi^\mu\dot{\phi}^\nu + \epsilon\frac{i}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\right] \\
&= -\int_0^\beta dt i\epsilon\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)
\end{aligned} \quad (5.7)$$

where we in the first equality used that the terms without $\dot{\epsilon}$ simplfy to (5.6). On the second equality, we used the antisymmetric properties of the fermionic variables and on the third we integrated the first term by parts. Now, we remember that ϵ is actually *constant*. Then note that the requirement of the vanishing of an *arbitrary* variation of the action δS is equivalent to the fields satisfying the equations of motion. In particular, this means that

$$\delta_\epsilon S = 0 \quad (5.8)$$

whenever the equations of motions are satisfied[13]. In such case, we get from (5.7) that

$$\epsilon\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu) = 0 \quad (5.9)$$

for arbitrary ϵ. The conserved supercharge is hence given by

$$Q = g_{\mu\nu}\psi^\mu\dot{\phi}^\nu. \quad (5.10)$$

Let us now quantize the system. We will follow the prescription of [21, 22, 16]. We will only go through the process quickly, the interested reader is recommended to check the aforementioned references. Since the system is linear in the fermionic velocities, we need to use the *Dirac bracket*

$$\{F,G\}_{DB} = \{F,G\}_{PB} - \{F,\varphi_a\}_{PB}C^{ab}\{\varphi_b,G\}_{PB} \quad (5.11)$$

when determining the (anti-)commutators. Here, $\{,\}_{PB}$ is the super Poisson bracket (cf. (2.8))

$$\{F,G\}_{PB} = \left(\frac{\partial F}{\partial\phi^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial\phi^\mu}\right) + (-1)^{\varepsilon_F}\left(\frac{\partial^L F}{\partial\psi^\mu}\frac{\partial^L G}{\partial\pi_{\psi\mu}} + \frac{\partial^L F}{\partial\pi_{\psi\mu}}\frac{\partial^L G}{\partial\psi^\mu}\right) \quad (5.12)$$

with p_μ and $\pi_{\psi\mu}$ being the conjugate momenta to ϕ^μ and ψ^μ respectively and ε_F is the Grassman parity of F. The matrix C^{ab} is the inverse of $C_{ab} := \{\varphi_a, \varphi_b\}_{PB}$, where φ_a are the second class constraints of the system. The canonical conjugate momenta for ϕ^μ are

$$p_\mu = \frac{\partial L}{\partial\dot{\phi}^\mu} = g_{\mu\nu}\dot{\phi}^\nu + \frac{i}{2}\Gamma_{\rho\mu\sigma}\psi^\rho\psi^\sigma \quad (5.13)$$

[12]Constants are also time-dependent in the sense that they take the same value everywhere.
[13]This works since ϵ is actually constant.

and for ψ^μ

$$\pi_{\psi\mu} = \frac{\partial L}{\partial \dot{\psi}^\mu} = -\frac{i}{2} g_{\mu\nu}\psi^\nu. \tag{5.14}$$

We see that the conjugate momenta $\pi_{\psi\mu}$ are proportional to ψ^μ which leads to the constraints

$$\varphi_\mu := \pi_{\psi\mu} + \frac{i}{2} g_{\mu\nu}\psi^\nu = 0. \tag{5.15}$$

They are second class; the (super) Poisson bracket of the constraints is given by

$$C_{\mu\nu} = \{\varphi_\mu, \varphi_\nu\}_{PB} = -ig_{\mu\nu} \implies C^{\mu\nu} = ig^{\mu\nu}. \tag{5.16}$$

From (5.11), after some calculation, we hence get

$$\{\phi^\mu, p_\nu\}_{DB} = \delta^\mu_\nu \tag{5.17}$$

$$\{p_\mu, p_\nu\}_{DB} = \frac{i}{2} g^{\rho\sigma}\partial_\mu g_{\rho\alpha}\partial_\nu g_{\sigma\beta}\psi^\alpha\psi^\beta \tag{5.18}$$

$$\{p_\mu, \psi^\nu\}_{DB} = \frac{1}{2} g^{\nu\rho}\partial_\mu g_{\rho\alpha}\psi^\alpha \tag{5.19}$$

$$\{p_\mu, \pi_{\psi^\nu}\}_{DB} = \frac{i}{4}\partial_\mu g_{\nu\alpha}\psi^\alpha \tag{5.20}$$

$$\{\psi^\mu, \psi^\nu\}_{DB} = -ig^{\mu\nu} \tag{5.21}$$

$$\{\psi^\mu, \pi_{\psi^\nu}\}_{DB} = -\frac{1}{2}\delta^\mu_\nu \tag{5.22}$$

$$\{\pi_{\psi^\mu}, \pi_{\psi^\nu}\}_{DB} = \frac{i}{4} g_{\mu\nu} \tag{5.23}$$

with the rest of the brackets vanishing. Not all of the brackets will be used but we have written all of them down for completeness sake. The (anti-)commutators are discovered to be

$$[\phi^\nu, p_\nu] = i\delta^\mu_\nu \tag{5.24}$$

$$[p_\mu, p_\nu] = -\frac{1}{2} g^{\rho\sigma}\partial_\mu g_{\rho\alpha}\partial_\nu g_{\sigma\beta}\psi^\alpha\psi^\beta \tag{5.25}$$

$$[p_\mu, \psi^\nu] = \frac{i}{2} g^{\nu\rho}\partial_\mu g_{\rho\alpha}\psi^\alpha \tag{5.26}$$

$$[p_\mu, \pi_{\psi^\nu}] = -\frac{1}{4}\partial_\mu g_{\nu\alpha}\psi^\alpha \tag{5.27}$$

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu} \tag{5.28}$$

$$\{\psi^\mu, \pi_\nu\} = -\frac{i}{2}\delta^\mu_\nu. \tag{5.29}$$

$$\{\pi_{\psi^\mu}, \pi_{\psi^\nu}\} = -\frac{1}{4} g_{\mu\nu} \tag{5.30}$$

We define the *kinetic momenta* $P_\mu = g_{\mu\nu}\dot{\phi}^\nu$, under which the the supercharge takes the form

$$Q = \psi^\mu P_\mu. \tag{5.31}$$

We would now like to find a representation of the Hilbert space. We will use the notion of Clifford algebras and spin structures in the following discussions. For a brief review, the reader is referred to Appendix A. We start by making a change of basis. Using orthonormal frame fields $\{e_\alpha\}$, i.e. a frame constituting a non-coordinate basis on each tangent space T_pM such that locally, ${e_a}^\mu {e_b}^\nu g_{\mu\nu} = \delta_{ab}$, (5.28) can be cast into the form

$$\{\tilde{\psi}^\alpha, \tilde{\psi}^\beta\} = \delta^{\alpha\beta} \tag{5.32}$$

with $\tilde{\psi}^\alpha := e^\alpha{}_\mu \psi^\mu$. Noting that this is nothing but the Clifford algebra (with a slight rescaling), we find a matrix representation for the $\tilde{\psi}^\alpha$ using the $2^n \times 2^n$ gamma matrices $\{\gamma^\alpha\}$ which satisfy

$$\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}, \tag{5.33}$$

meaning that $\tilde{\psi}^\alpha$ is represented by $\frac{1}{\sqrt{2}}\gamma^\alpha$. From the gamma matrices, we may define the *chirality operator*

$$\gamma_{d+1} := i^n \gamma^1 \dots \gamma^{2n}. \tag{5.34}$$

The chirality operator satisfies

$$\gamma_{d+1}^2 = 1 \tag{5.35}$$

which means it has eigenvalues ± 1. Furthermore, it anticommutes with the gamma matrices

$$\{\gamma_{d+1}, \gamma^\alpha\} = 0. \tag{5.36}$$

We find that a good candidate as our Hilbert space is the set of spinor fields $\Gamma(M, \mathfrak{S}_d)$, i.e. the set of sections on the spinor bundle $\mathfrak{S}_d$. It is equipped with an L^2-inner product (specified in Appendix A)

$$(\sigma, \sigma') = \int_M \langle \sigma, \sigma' \rangle \star (1) \tag{5.37}$$

where $\sigma, \sigma' \in \Gamma(M, \mathfrak{S}_d)$ are spinor fields and $\star(1)$ is the invariant volume form ($\star$ is the Hodge star), so it is indeed a Hilbert space. Furthermore, it splits under the eigenvalues of the chirality operator into

$$\Gamma(M, \mathfrak{S}_d) = \Gamma(M, \mathfrak{S}_d^+) \oplus \Gamma(M, \mathfrak{S}_d^-) \tag{5.38}$$

so it is $\mathbb{Z}_2$-graded as we need it to be. We hence identify $(-1)^F$ with γ_{d+1}. The bosonic and fermionic subspaces are therefore concluded to be $\Gamma(M, \mathfrak{S}_d^+)$ and $\Gamma(M, \mathfrak{S}_d^-)$ respectively. What is left is to see if the supercharge maps the bosonic part into the fermionic one and vice versa. Given a coordinate patch U on M (switching back to coordinate basis), the observables are represented by

$$\phi^\mu = x^\mu \times \tag{5.39}$$

$$P_\mu = -i\tilde{\nabla}_\mu \tag{5.40}$$

$$\psi^\mu = \frac{1}{\sqrt{2}}\gamma^\mu \tag{5.41}$$

where $x^\mu\times$ is multiplication by x^μ, $\tilde{\nabla}_\mu$ is the spin connection and $\gamma^\mu = e_\alpha{}^\mu \gamma^\alpha$. The supercharge then takes the form

$$Q = -\frac{i}{\sqrt{2}}\gamma^\mu \tilde{\nabla}_\mu = -\frac{i}{\sqrt{2}} \not{\nabla} \tag{5.42}$$

which is nothing but the Dirac operator - this operator maps $\Gamma(M, \mathfrak{S}_d^\pm)$ into $\Gamma(M, \mathfrak{S}_d^\mp)$. We hence get the spin complex

$$\Gamma(M, \mathfrak{S}_d^+) \underset{i\not{\nabla}}{\overset{i\not{\nabla}}{\rightleftarrows}} \Gamma(M, \mathfrak{S}_d^-). \tag{5.43}$$

If $Q : \Gamma(M, \mathfrak{S}_d^+) \to \Gamma(M, \mathfrak{S}_d^-)$, we denote it by Q_+. Similarly, we use the notation Q_- for $Q : \Gamma(M, \mathfrak{S}_d^-) \to \Gamma(M, \mathfrak{S}_d^+)$. The Hamiltonian now takes the form

$$H = \frac{1}{2}\{Q, Q\} = -\frac{1}{2}\not{\nabla}^2. \tag{5.44}$$

The fact that the Dirac operator is Hermitian implies that

$$Q\sigma = 0 \iff H\sigma = 0. \tag{5.45}$$

The Witten index is then given by

$$\mathrm{ind}(Q) = \mathrm{Tr}(-1)^F e^{-\beta H} = \dim(\ker Q_+) - \dim(\ker Q_-) \tag{5.46}$$

where $\ker Q_+$ and $\ker Q_-$ are the sets of harmonic spinor fields with eigenvalue $+1$ and -1 with respect to $(-1)^F = \gamma_{d+1}$, respectively. Note that numerical factors in Q are of no significance for the index.

5.1.1 Computing the path integral

We want to compute the path integral for the index of the Dirac operator $Q = -\frac{i}{\sqrt{2}}\nabla\!\!\!/$

$$\mathrm{ind}(Q) = \mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \mathcal{D}\phi\mathcal{D}\psi e^{-S_E} \tag{5.47}$$

where the action in Euclidean time is $S_E = \int_0^\beta d\tau L_E$ with L_E given by (with $\dot{}$ now denoting derivative with respect to τ)

$$L_E = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu D_\tau\psi^\nu \tag{5.48}$$

When computing this integral, we will utilize the fact that it is independent of β to make the calculations doable; in the $\beta \to 0$ limit, the integral localises around constant paths ϕ.

Rescaling the time parameter as $\tau = \beta s$, the action becomes

$$S = \int_0^1 ds \left(\frac{1}{2\beta}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu D_\tau\psi^\nu\right) \tag{5.49}$$

In the limit $\beta \to 0$ we see that for the non-constant paths $\dot{\phi}^\mu \neq 0$, the action blows up, giving an exponential suppression in the actual path integral. Thus the only paths having an actual contribution in this limit are the constant paths $\dot{\phi}^\mu = 0$ (which of course satisfy the periodic boundary conditions).

We would like to use the saddle point method around the constant paths to compute the path integral. In order to do that, we need to first verify that the constant paths are actually part of the set of extrema of the action. The extrema are well-known to be solutions to the classical equations of motions given by the Euler-Lagrange equations. For ψ^μ we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial\psi^\lambda} - \frac{d}{d\tau}\frac{\partial L}{\partial\dot{\psi}^\lambda} \\
&= \frac{1}{2}g_{\lambda\nu}D_t\psi^\nu - \frac{1}{2}g_{\mu\nu}\psi^\mu\dot{\phi}^\rho\Gamma^\nu_{\rho\lambda} + \frac{1}{2}\frac{d}{d\tau}(g_{\mu\lambda}\psi^\mu) \\
&= \frac{1}{2}\left[g_{\lambda\nu}D_\tau\psi^\nu - \psi^\mu\dot{\phi}^\rho\Gamma_{\mu\rho\lambda} + \partial_\rho g_{\mu\lambda}\dot{\phi}^\rho\psi^\mu + g_{\mu\lambda}\dot{\psi}^\mu\right] \\
&= \frac{1}{2}\left[g_{\lambda\mu}D_\tau\psi^\mu + g_{\lambda\mu}\dot{\psi}^\mu + \dot{\phi}^\rho\Gamma_{\lambda\rho\mu}\psi^\mu\right] \\
&= g_{\lambda\mu}D_\tau\psi^\mu
\end{aligned} \tag{5.50}$$

where we on the fourth equality used that $\partial_\rho g_{\mu\lambda} - \Gamma_{\mu\rho\lambda} = \partial_\rho g_{\mu\lambda} - \frac{1}{2}(\partial_\rho g_{\mu\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\mu g_{\rho\lambda}) = \Gamma_{\lambda\rho\mu}$. We hence obtain the equations

$$D_\tau\psi^\mu = \dot{\psi}^\mu + \dot{\phi}^\rho\Gamma^\mu_{\rho\sigma}\psi^\sigma = 0. \tag{5.51}$$

Next, let us tackle the ϕ^μ case

$$
\begin{aligned}
0 &= \frac{\partial L}{\partial \phi^\lambda} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}^\lambda} \\
&= \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}\partial_\lambda g_{\mu\nu}\psi^\mu D_\tau\psi^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu\dot{\phi}^\rho\partial_\lambda\Gamma^\nu_{\rho\sigma}\psi^\sigma - \frac{d}{d\tau}\left(g_{\lambda\nu}\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu\Gamma^\nu_{\lambda\sigma}\psi^\sigma\right) \\
&= -g_{\lambda\nu}\ddot{\phi}^\nu - \partial_\rho g_{\lambda\nu}\dot{\phi}^\rho\dot{\phi}^\nu + \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu \\
&\quad + \frac{1}{2}(\partial_\lambda g_{\mu\nu}\Gamma^\nu_{\rho\sigma} + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - \partial_\rho g_{\mu\nu}\Gamma^\nu_{\lambda\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma})\dot{\phi}^\rho\psi^\mu\psi^\sigma + \frac{1}{2}(\partial_\lambda g_{\mu\nu} + \Gamma_{\nu\lambda\mu} - \Gamma_{\mu\lambda\nu})\psi^\mu\dot{\psi}^\nu
\end{aligned}
\tag{5.52}
$$

Using the fact that the factor $\dot{\phi}^\rho\dot{\phi}^\nu$ is symmetric in its indices, we may rewrite $\partial_\rho g_{\lambda\nu}\dot{\phi}^\rho\dot{\phi}^\nu = \frac{1}{2}(\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho})\dot{\phi}^\rho\dot{\phi}^\nu$. Plugging this into our expression,

$$
\begin{aligned}
0 &= -g_{\lambda\rho}D_\tau\dot{\phi}^\rho + \frac{1}{2}(\partial_\lambda g_{\mu\nu}\Gamma^\nu_{\rho\sigma} + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - \partial_\rho g_{\mu\nu}\Gamma^\nu_{\lambda\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma})\dot{\phi}^\rho\psi^\mu\psi^\sigma + \Gamma_{\nu\lambda\mu}\psi^\mu\dot{\psi}^\nu \\
&= -g_{\lambda\rho}D_\tau\dot{\phi}^\rho + \frac{1}{2}(\partial_\lambda g_{\mu\nu}\Gamma^\nu_{\rho\sigma} + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - \partial_\rho g_{\mu\nu}\Gamma^\nu_{\lambda\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma})\dot{\phi}^\rho\psi^\mu\psi^\sigma - \Gamma_{\nu\lambda\mu}\psi^\mu\dot{\phi}^\rho\Gamma^\nu_{\rho\sigma}\psi^\sigma
\end{aligned}
\tag{5.53}
$$

where we on the second equality used the equations of motions (5.51) to get rid of the $\dot{\psi}^\nu$. Now, observe that $\Gamma_{\nu\lambda\mu}\Gamma^\nu_{\rho\sigma}\dot{\phi}^\rho\psi^\mu\psi^\sigma = \frac{1}{2}(\Gamma_{\nu\lambda\mu}\Gamma^\nu_{\rho\sigma} - \Gamma_{\nu\rho\mu}\Gamma^\nu_{\lambda\sigma})\dot{\phi}^\rho\psi^\mu\psi^\sigma$ due to the antisymmetrization of $\psi^\mu\psi^\sigma$. Finally,

$$
\begin{aligned}
0 &= -g_{\lambda\rho}D_\tau\dot{\phi}^\rho + \frac{1}{2}[g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma} + (\partial_\lambda g_{\mu\nu} - \Gamma_{\nu\lambda\mu})\Gamma^\nu_{\rho\sigma} - (\partial_\rho g_{\mu\nu} - \Gamma_{\nu\rho\mu})\Gamma^\nu_{\lambda\sigma}]\dot{\phi}^\rho\psi^\mu\psi^\sigma \\
&= -g_{\lambda\rho}D_\tau\dot{\phi}^\rho + \frac{1}{2}g_{\mu\nu}(\partial_\lambda\Gamma^\nu_{\rho\sigma} - \partial_\rho\Gamma^\nu_{\lambda\sigma} + \Gamma^\nu_{\lambda\alpha}\Gamma^\alpha_{\rho\sigma} - \Gamma^\nu_{\rho\alpha}\Gamma^\alpha_{\lambda\sigma})\dot{\phi}^\rho\psi^\mu\psi^\sigma \\
&= -g_{\lambda\rho}D_\tau\dot{\phi}^\rho + \frac{1}{2}R_{\mu\sigma\lambda\rho}\dot{\phi}^\rho\psi^\mu\psi^\sigma
\end{aligned}
\tag{5.54}
$$

The equations of motion can hence be written as

$$
D_\tau\dot{\phi}^\mu - \frac{1}{2}R^\mu{}_{\nu\rho\sigma}\dot{\phi}^\nu\psi^\rho\psi^\sigma = 0 \tag{5.55}
$$

We can now see that the constant paths ϕ_0 are indeed solutions to the equations of motion, provided $\psi = \psi_0$ are also constant. Note that the set of ϕ_0 is none other than M itself.

When we use the saddle point method (see subsection 3.3), recall that the path integral will be exact at second order expansion of the action; we need only to take care of the extrema of the action and the second order fluctuations. To simplify computations, we now choose to work in Riemann normal coordinates around $x = \phi_0$ in an orthonormal frame. With these conditions, the metric is subject to

$$
g_{\mu\nu}(\phi_0) = \delta_{\mu\nu} \tag{5.56}
$$

$$
\partial_\lambda g_{\mu\nu}(\phi_0) = 0 \tag{5.57}
$$

$$
\begin{aligned}
\partial_\alpha\partial_\beta g_{\mu\nu}(\phi_0) &= -\frac{1}{3}(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}) \\
&= -\frac{1}{3}(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu}) = \partial_\mu\partial_\nu g_{\alpha\beta}
\end{aligned}
\tag{5.58}
$$

Details can be found in e.g. [23]. In these coordinates, we see that $\det(g_{\mu\nu}) = 1$. Let us now perturb ϕ and ψ around ϕ_0 and ψ_0

$$
\phi^\mu(\tau) = \phi_0^\mu + \xi^\mu(\tau) \tag{5.59}
$$

$$
\psi^\mu(\tau) = \psi_0^\mu + \eta^\mu(\tau) \tag{5.60}
$$

where ξ^μ and η^μ are infinitesimal. We expand the Lagrangian (5.48)

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\psi^\mu(\dot{\psi}^\nu + \dot{\phi}^\rho\Gamma^\nu_{\rho\sigma}\psi^\sigma) \tag{5.48}$$

to second order in these (while remembering that the first order terms vanish since we expand round an extremum). Since every term comes with a derivative, there will be no zeroth order terms. The first term expand as

$$\frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu \sim \frac{1}{2}\dot{\xi}^\mu\dot{\xi}^\nu \tag{5.61}$$

and the second term

$$\frac{1}{2}g_{\mu\nu}\psi^\mu\dot{\psi}^\nu \sim \frac{1}{2}\eta^\mu\dot{\eta}^\mu. \tag{5.62}$$

The third term requires a bit of work to simplify,

$$\begin{aligned}
\frac{1}{2}\psi^\mu\dot{\phi}^\rho\Gamma_{\mu\rho\sigma}\psi^\sigma &\sim \frac{1}{2}(\psi_0^\mu + \eta^\mu)\dot{\xi}^\rho(\Gamma_{\mu\rho\sigma}\big|_{\phi_0} + \partial_\lambda\Gamma_{\mu\rho\sigma}\big|_{\phi_0}\xi^\lambda)(\psi_0^\sigma + \eta^\sigma) \\
&\sim \frac{1}{2}\partial_\mu\Gamma_{\rho\nu\sigma}\big|_{\phi_0}\psi_0^\rho\psi_0^\sigma\xi^\mu\dot{\xi}^\nu \\
&= \frac{1}{2}\frac{1}{2}(\partial_\mu\partial_\nu g_{\rho\sigma} + \partial_\mu\partial_\sigma g_{\rho\nu} - \partial_\mu\partial_\rho g_{\nu\sigma})\psi_0^\rho\psi_0^\sigma\xi^\mu\dot{\xi}^\nu \\
&= \frac{1}{2}\frac{1}{2}(\partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\nu\partial_\rho g_{\mu\sigma})\psi_0^\rho\psi_0^\sigma\xi^\mu\dot{\xi}^\nu
\end{aligned} \tag{5.63}$$

where we on the last equality used that $\psi_0^\rho\psi_0^\sigma$ is antisymmetric in conjunction to (5.58). However, again using the antisymmetry of $\psi_0^\rho\psi_0^\sigma$ and the fact that we are working in normal coordinates,

$$R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma = (\partial_\rho\Gamma_{\mu\sigma\nu} - \partial_\sigma\Gamma_{\mu\rho\nu})\psi_0^\rho\psi_0^\sigma = 2\partial_\rho\Gamma_{\mu\sigma\nu}\psi_0^\rho\psi_0^\sigma = (\partial_\rho\partial_\nu g_{\mu\sigma} + \partial_\sigma\partial_\mu g_{\rho\nu})\psi_0^\rho\psi_0^\sigma \tag{5.64}$$

so we end up with

$$\frac{1}{2}\psi^\mu\dot{\phi}^\rho\Gamma_{\mu\rho\sigma}\psi^\sigma \sim \frac{1}{2}\frac{1}{2}R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma\xi^\mu\dot{\xi}^\nu. \tag{5.65}$$

Defining $\tilde{R}_{\mu\nu} := \frac{1}{2}R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma$ and piecing everything together,

$$L \sim \frac{1}{2}\dot{\xi}^\mu\dot{\xi}^\nu + \frac{1}{2}\eta^\mu\dot{\eta}^\mu + \frac{1}{2}\tilde{R}_{\mu\nu}\xi^\mu\dot{\xi}^\nu. \tag{5.66}$$

The index can now be computed in the $\beta \to 0$ limit as

$$\operatorname{ind}(Q) = \int_{PBC}\mathcal{D}\xi\mathcal{D}\eta e^{-S_2} \tag{5.67}$$

where

$$S_2 := \int_0^\beta d\tau\left(\frac{1}{2}\dot{\xi}^\mu\dot{\xi}^\nu + \frac{1}{2}\eta^\mu\dot{\eta}^\mu + \frac{1}{2}\tilde{R}_{\mu\nu}\xi^\mu\dot{\xi}^\nu\right) \tag{5.68}$$

is the second order action. We have used the translational invariance of the measure $\mathcal{D}\phi\mathcal{D}\psi = \mathcal{D}\xi\mathcal{D}\eta$. We may integrate the first term in S_2 by parts and use the periodic boundary conditions to obtain

$$S_2 = \int_0^\beta d\tau\left[\frac{1}{2}\xi^\mu\left(-\delta_{\mu\nu}\frac{d^2}{d\tau^2} + \tilde{R}_{\mu\nu}\frac{d}{d\tau}\right)\xi^\nu + \frac{1}{2}\eta^\mu\delta_{\mu\nu}\frac{d}{d\tau}\eta^\nu\right]. \tag{5.69}$$

Since the paths we are integrating over are subject to periodic boundary conditions, we may Fourier expand the ξ and η as

$$\xi^\mu = \sum_{k=-\infty}^{\infty}\xi_k^\mu\frac{1}{\sqrt{\beta}}e^{2\pi ik\tau/\beta} \tag{5.70}$$

$$\eta^\mu = \sum_{k=-\infty}^{\infty}\eta_k^\mu\frac{1}{\sqrt{\beta}}e^{2\pi ik\tau/\beta}. \tag{5.71}$$

Note that the operators

$$-\delta_{\mu\nu}\frac{d^2}{d\tau^2}+\tilde{R}_{\mu\nu}\frac{d}{d\tau} \tag{5.72}$$

acting on ξ and

$$\delta_{\mu\nu}\frac{d}{d\tau} \tag{5.73}$$

acting on η kill the zero modes. This means we have to consider them separately since $\tilde{R}_{\mu\nu}$ depends on the zero modes. The integral splits into

$$\mathrm{ind}(Q)=\int\mathcal{D}\xi_0\mathcal{D}\eta_0\int_{PBC}\mathcal{D}\xi' e^{-\int_0^\beta d\tau\frac{1}{2}\xi^\mu\left(-\delta_{\mu\nu}\frac{d^2}{d\tau^2}+\tilde{R}_{\mu\nu}\frac{d}{d\tau}\right)\xi^\nu}\int_{PBC}\mathcal{D}\eta' e^{-\int_0^\beta d\tau\frac{1}{2}\eta^\mu\delta_{\mu\nu}\frac{d}{d\tau}\eta^\nu} \tag{5.74}$$

where $\mathcal{D}\xi'$ and $\mathcal{D}\eta'$ means integrating over the non-constant modes of (5.70) and (5.71). Now, since $\tilde{R}_{\mu\nu}=-\tilde{R}_{\nu\mu}$ due to the antisymmetry of the Riemann tensor together with the fact that M is even dimensional, we may block diagonalise $\tilde{R}_{\mu\nu}$ in the form (the change of basis is orthogonal so the corresponding Jacobi determinant will have modulus 1),

$$\tilde{R}_{\mu\nu}=\begin{pmatrix}0 & y_1 & & & \\ -y_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & y_n \\ & & & -y_n & 0\end{pmatrix}. \tag{5.75}$$

Taking one block at the time, we can now use (3.68)

$$Z_1(\beta)=\int\mathcal{D}\xi^1\mathcal{D}\xi^2 e^{-\int_0^\beta d\tau\frac{1}{2}\xi^a O^1_{ab}\xi^b}=\prod_{k\neq 0}[\lambda_{11,k}+\frac{i}{2}(\bar{\lambda}_{12,k}-\lambda_{12,k})]^{-1}=\frac{1}{\mathrm{Det}'[O_{11}+\frac{i}{2}(\bar{O}_{12}-O_{12})]} \tag{3.68}$$

to compute the middle part of the above path integral. The corresponding eigenvalues are

$$\lambda_{11,k}=\left(\frac{2\pi k}{\beta}\right)^2 \tag{5.76}$$

$$\lambda_{12,k}=\tilde{R}_{12}\frac{i2\pi k}{\beta} \tag{5.77}$$

so

$$\begin{aligned}\mathrm{Det}'[O_{11}+\frac{i}{2}(\bar{O}_{12}-O_{12})] &= \prod_{k\neq 0}\left[\left(\frac{2\pi k}{\beta}\right)^2+\tilde{R}_{12}\frac{2\pi k}{\beta}\right]\\ &= \prod_{k=1}^{\infty}\left[\left(\frac{2\pi k}{\beta}\right)^2+\tilde{R}_{12}\frac{2\pi k}{\beta}\right]\left[\left(\frac{2\pi k}{\beta}\right)^2-\tilde{R}_{12}\frac{2\pi k}{\beta}\right]\\ &= \prod_{k=1}^{\infty}\left[\left(\frac{i2\pi k}{\beta}\right)^4+\left(\tilde{R}_{12}\frac{i2\pi k}{\beta}\right)^2\right]\\ &= \prod_{k=1}^{\infty}\left(\frac{i2\pi k}{\beta}\right)^2\left[\left(\frac{i2\pi k}{\beta}\right)^2+\tilde{R}_{12}^2\right]\\ &= \prod_{k=1}^{\infty}\frac{i2\pi k}{\beta}\left(-\frac{i2\pi k}{\beta}\right)\left[\left(\frac{2\pi k}{\beta}\right)^2-\tilde{R}_{12}^2\right]\\ &= \mathrm{Det}'\left(\frac{d}{d\tau}\right)\left[\prod_{k=1}^{\infty}\left(\frac{2\pi k}{\beta}\right)^2\right]\prod_{l=1}^{\infty}\left[1-\left(\frac{\tilde{R}_{12}\beta}{2\pi k}\right)^2\right].\end{aligned} \tag{5.78}$$

The infinite product in the middle is regularized to β in (3.39) and

$$\prod_{l=1}^{\infty}\left[1-\left(\frac{\tilde{R}_{12}\beta}{2\pi k}\right)^2\right]=\frac{2\sin\left(\frac{\tilde{R}_{12}\beta}{2}\right)}{\tilde{R}_{12}\beta} \tag{5.79}$$

Hence

$$\mathrm{Det}'[O_{11}+\frac{i}{2}(\bar{O}_{12}-O_{12})]=\mathrm{Det}'\left(\frac{d}{d\tau}\right)\frac{2\sin\left(\frac{\tilde{R}_{12}\beta}{2}\right)}{\tilde{R}_{12}} \tag{5.80}$$

which means[14]

$$\begin{aligned}\int_{PBC}\mathcal{D}\xi' e^{-\int_0^\beta d\tau\frac{1}{2}\xi^\mu\left(-\delta_{\mu\nu}\frac{d^2}{d\tau^2}+\tilde{R}_{\mu\nu}\frac{d}{d\tau}\right)\xi^\nu}&=\prod_{j=1}^{n}\frac{1}{\mathrm{Det}'\left(\frac{d}{d\tau}\right)}\frac{\tilde{R}_{2j-1,2j}}{2\sin\left(\frac{\tilde{R}_{2j-1,2j}\beta}{2}\right)}\\&=\frac{1}{\left[\mathrm{Det}'\left(\frac{d}{d\tau}\right)\right]^n}\prod_{j=1}^{n}\frac{\tilde{R}_{2j-1,2j}}{2\sin\left(\frac{\tilde{R}_{2j-1,2j}\beta}{2}\right)}.\end{aligned} \tag{5.82}$$

Using (3.140), we may compute the other path integral in the expression for the index

$$\int_{PBC}\mathcal{D}\eta' e^{-\int_0^\beta d\tau\frac{1}{2}\eta^\mu\delta_{\mu\nu}\frac{d}{d\tau}\eta^\nu}=i^n\left[\mathrm{Det}'\left(\frac{d}{d\tau}\right)\right]^n. \tag{5.83}$$

The index now takes the intermediate form, recalling that we defined $y_j:=\tilde{R}_{2j-1,2j}$

$$\mathrm{ind}(Q)=i^n\int\mathcal{D}\xi_0\mathcal{D}\eta_0\prod_{j=1}^{n}\frac{\tilde{R}_{2j-1,2j}}{2\sin\left(\frac{\tilde{R}_{2j-1,2j}\beta}{2}\right)}=i^n\int\left(\prod_{\mu=1}^{2n}\frac{d\xi_0^\mu}{\sqrt{2\pi}}d\eta_0^\mu\right)\frac{1}{\beta^n}\prod_{j=1}^{n}\frac{\beta y_j/2}{\sin\left(\beta y_j/2\right)}. \tag{5.84}$$

By treating the factors in the j product as elements of a matrix, we note that they constitute a $2n\times 2n$ antisymmetric matrix with 2×2-blocks. Namely, we get a matrix

$$\frac{\beta\tilde{R}/2}{\sin\left(\beta\tilde{R}/2\right)} \tag{5.85}$$

with elements

$$\left(\frac{\beta\tilde{R}/2}{\sin\left(\beta\tilde{R}/2\right)}\right)_{2j-1,2j}=-\left(\frac{\beta\tilde{R}/2}{\sin\left(\beta\tilde{R}/2\right)}\right)_{2j,2j-1}=\frac{\beta y_j/2}{\sin\left(\beta y_j/2\right)} \tag{5.86}$$

and all else 0. We may hence write

$$\prod_{j=1}^{n}\frac{\beta y_j/2}{\sin\left(\beta y_j/2\right)}=\det\left(\frac{\beta\tilde{R}/2}{\sin\left(\beta\tilde{R}/2\right)}\right)^{\frac{1}{2}}. \tag{5.87}$$

Note that any Taylor expansion with respect to $\tilde{R}$ is finite since $\tilde{R}^p=0$ for $p>n$.

[14] Note that one may write

$$\mathrm{Det}'[O_{11}+\frac{i}{2}(\bar{O}_{12}-O_{12})]=\prod_{k=1}^{\infty}\left[\left(\frac{i2\pi k}{\beta}\right)^4+\left(\tilde{R}_{12}\frac{i2\pi k}{\beta}\right)^2\right]=\mathrm{Det}'\left(\det\begin{pmatrix}-\frac{d^2}{d\tau^2} & \tilde{R}_{12}\frac{d}{d\tau}\\ -\tilde{R}_{12}\frac{d}{d\tau} & -\frac{d^2}{d\tau^2}\end{pmatrix}\right)^{\frac{1}{2}}. \tag{5.81}$$

We have so far integrated the second order fluctuations around some fixed constant path (ϕ_0, ψ_0). We now need to treat the integration over these. Constant paths are just points so let us write $\phi_0^\mu = x^\mu$. We may to first order expand

$$x^\mu = x_0^\mu + \frac{1}{\sqrt{\beta}}\xi_0^\mu \tag{5.88}$$

where the factor $\dfrac{1}{\sqrt{\beta}}$ comes from the Fourier expansion of the fluctuation. Since we are integrating over all (constant) configurations the integral is translational invariant. From the expansion we may therefore conclude that $\dfrac{d\xi_0^\mu}{\sqrt{\beta}} = dx_0^\mu$. The same arguments apply for the Grassmann variables which means $d\psi_0 = \sqrt{\beta} d\eta_0^\mu$. Altogether, the index takes the form

$$\operatorname{ind}(Q) = i^n \int \left(\prod_{\mu=1}^{2n} \frac{dx_0^\mu}{\sqrt{2\pi}} d\psi_0^\mu \right) \frac{1}{\beta^n} \det\left(\frac{\beta\tilde{R}/2}{\sin\left(\beta\tilde{R}/2\right)} \right)^{\frac{1}{2}}. \tag{5.89}$$

Let us now scale away the apparent β-dependence by making the change

$$\psi_0^\mu = \frac{\chi_0^\mu}{\sqrt{2\pi\beta}} \implies d\psi_0^\mu = \sqrt{2\pi\beta}\, d\chi_0^\mu. \tag{5.90}$$

In the integrand we then get

$$\beta\tilde{R}_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma}\psi_0^\rho\psi_0^\sigma = \frac{1}{2\pi}\frac{1}{2} R_{\mu\nu\rho\sigma}\chi_0^\rho\chi_0^\sigma \tag{5.91}$$

so plugging everything into the integral we obtain

$$\operatorname{ind}(Q) = i^n \int \left(\prod_{\mu=1}^{2n} dx_0^\mu d\chi_0^\mu \right) \det\left(\frac{\frac{1}{2}\frac{1}{2\pi}\frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\chi_0^\rho\chi_0^\sigma}{\sin\left(\frac{1}{2}\frac{1}{2\pi}\frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\chi_0^\rho\chi_0^\sigma\right)} \right)^{\frac{1}{2}}. \tag{5.92}$$

Let us now switch to the language of differential forms. From the $\prod_{\mu=1}^{2n} d\chi_0^\mu$ integration, we see that only terms of order $2n$ in χ_0 will give a non-zero contribution. Also $\prod_{\mu=1}^{2n} dx_0^\mu$ is a volume element and the integration is over all points of the manifold. Let us therefore define the curvature two-form

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma. \tag{5.93}$$

Note now that since $\mathcal{R}/\sin(\mathcal{R})$ is even in $\mathcal{R}$ (which means only even number of pairs $dx^\rho \wedge dx^\sigma$ show up) n must be even i.e. d has to be a multiple of four in order for the integral to not vanish. As a consequence, i^n is ± 1. Let us expand

$$\frac{x}{\sin(x)} = \sum_{k=0}^{\infty} a_{2k} x^{2k} \tag{5.94}$$

and recall that $\sinh(x) = -i\sin(ix)$. We may hence conclude that

$$\frac{x}{\sinh(x)} = \frac{ix}{\sin(ix)} = \sum_{k=0}^{\infty} (-1)^k a_{2k} x^{2k}. \tag{5.95}$$

Looking at the product in (5.84), we now want to compare

$$i^n \prod_{j=1}^{n} \frac{\beta y_j/2}{\sin(\beta y_j/2)} = i^n \prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} a_{2k} y_j^{2k} \right) \tag{5.96}$$

and

$$\prod_{j=1}^{n} \frac{\beta y_j/2}{\sinh(\beta y_j/2)} = \prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} (-1)^k a_{2k} y_j^{2k} \right). \tag{5.97}$$

As we discussed earlier, only the terms of order n in y will contribute to the index. We have two cases to discuss: when $n/2$ is even and when it is odd. In the even case, $i^n = 1$ and the order n terms of (5.96) and (5.97) match. In the odd case, $i^n = -1$ but also the order n terms of (5.96) and (5.97) differ by a sign[15] In effect, this means we can make the replacement

$$i^n \det \left(\frac{\beta \tilde{R}/2}{\sin\left(\beta \tilde{R}/2\right)} \right)^{\frac{1}{2}} \to \det \left(\frac{\beta \tilde{R}/2}{\sinh\left(\beta \tilde{R}/2\right)} \right)^{\frac{1}{2}} \tag{5.98}$$

in the integral:

$$\operatorname{ind}(Q) = \int \left(\prod_{\mu=1}^{2n} dx_0^\mu d\chi_0^\mu \right) \det \left(\frac{\frac{1}{2}\frac{1}{2\pi}\frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\chi_0^\rho \chi_0^\sigma}{\sinh\left(\frac{1}{2}\frac{1}{2\pi}\frac{1}{2} R_{\mu\nu\rho\sigma}(x_0)\chi_0^\rho \chi_0^\sigma\right)} \right)^{\frac{1}{2}} \tag{5.99}$$

or in terms of differential forms

$$\operatorname{ind}(Q) = \int_M \det \left(\frac{\frac{1}{2}\frac{1}{2\pi}\mathcal{R}}{\sinh\left(\frac{1}{2}\frac{1}{2\pi}\mathcal{R}\right)} \right)^{\frac{1}{2}}. \tag{5.100}$$

Since $\mathcal{R}$ is skew-symmetric then just as before we may block diagonalise it

$$\frac{1}{2\pi}\mathcal{R}_{\mu\nu} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_n \\ & & & -x_n & 0 \end{pmatrix}. \tag{5.101}$$

We now define the *$\hat{A}$-genus* of M as the formal expansion

$$\hat{A}(M) = \prod_{j=1}^{n} \frac{x_j/2}{\sinh(x_j/2)} \tag{5.102}$$

and summarize our results in the following theorem:

Theorem 5.1. (Index theorem for the spin complex) *Let M be a $d = 2n$ dimensional compact spin manifold. Then the index of the Dirac operator Q defined on M is given by*

$$\operatorname{ind}(Q) = \int_M \hat{A}(M). \tag{5.103}$$

[15] Everything boils down to how we can split n into a sum of even numbers since we are comparing terms of (5.96) and (5.97) of order n; we examine terms whose sum of the exponents of the y^{2k}-factors equals n. Since n is even, we may write $n = 2(a_1 + \dots a_j + a_{j+1} + \dots + a_r)$ where we assume that a_i is odd for $i = 1, \dots, j$ and the rest even. Hence if j is even, $n/2$ will also be even and if j is odd, $n/2$ will be odd. By looking at (5.97), we see that j actually counts the number of signs appearing in a order n term (every y^{2k} with k odd comes with a sign). We may conclude from our discussion that if $n/2$ is even, the order n terms of (5.97) will have an even number of signs and if $n/2$ is odd, the order n terms will have an odd number of signs.

5.2 The $\mathcal{N}=2$ Non-linear Sigma Model

We will in this section utilize the results of section 4.2. The discussions in this subsection will largely be based on the treatment in [15]. Let M be a closed, oriented d-dimensional Riemannian manifold equipped with the metric g and let x be local coordinates on M. The Lagrangian under consideration is

$$L = \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{i}{2}g_{\mu\nu}(\bar{\psi}^\mu D_t\psi^\nu - D_t\bar{\psi}^\mu\psi^\nu) + \frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma \tag{5.104}$$

where $\bar{\psi}^\mu$ is the complex conjugate of ψ^μ. We may decompose $\psi^\mu = \psi_1^\mu + i\psi_2^\mu$ where ψ_1 and ψ_2 are Hermitian. Note that we recover the $\mathcal{N}=1$ Lagrangian (5.1) if $\psi_1^\mu = \psi_2^\mu = \frac{1}{\sqrt{2}}\psi^\mu$ [3]. The term with the Riemann tensor then vanishes due to the Bianchi identity. The action is invariant under the supersymmetry transformations

$$\delta_\epsilon\phi^\mu = \epsilon\bar{\psi}^\mu - \bar{\epsilon}\psi^\mu \tag{5.105}$$

$$\delta_\epsilon\psi^\mu = \epsilon(i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) \tag{5.106}$$

$$\delta_\epsilon\bar{\psi}^\mu = \bar{\epsilon}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) \tag{5.107}$$

where ϵ and $\bar{\epsilon}$ are infinitesimal Grassmann constants. We have used δ_ϵ to denote the (super)symmetry transformation, reserving δ for a general one. One can show that

$$\delta_\epsilon L = \epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu) \tag{5.108}$$

so the action is indeed invariant. We now would like to find the corresponding supercharges. Just like before we go through Noether's procedure. Promote $\epsilon \to \epsilon = \epsilon(t)$ and $\bar{\epsilon} \to \bar{\epsilon} = \bar{\epsilon}(t)$. Then

$$\begin{aligned}
\delta_\epsilon S = \int_0^\beta dt\, \Big\{ &\frac{1}{2}g_{\mu\nu}(\dot{\epsilon}\bar{\psi}^\mu - \dot{\bar{\epsilon}}\psi^\mu)\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu(\dot{\epsilon}\bar{\psi}^\nu - \dot{\bar{\epsilon}}\psi^\nu) \\
&+ \frac{i}{2}g_{\mu\nu}[\bar{\psi}^\mu\dot{\epsilon}(i\dot{\phi}^\nu - \Gamma^\nu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) + \bar{\psi}^\mu(\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\nu_{\rho\sigma}\psi^\sigma \\
&\qquad - \dot{\bar{\epsilon}}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma)\psi^\nu - (\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\mu_{\rho\sigma}\bar{\psi}^\sigma\psi^\nu] \\
&+ \epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\Big\} \\
= \int_0^\beta dt\, &\left(\dot{\epsilon}\frac{3}{2}g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu - \dot{\bar{\epsilon}}\frac{3}{2}g_{\mu\nu}\psi^\mu\dot{\phi}^\nu\right) + \int_0^\beta dt\left(\epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\right) \\
= \int_0^\beta dt\, &\left(-\epsilon\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) + \bar{\epsilon}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\right) \\
= \int_0^\beta dt\, &\left(i\epsilon\frac{d}{dt}(ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) + i\bar{\epsilon}\frac{d}{dt}(-ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\right)
\end{aligned} \tag{5.109}$$

where we on the first equality used that the terms without $\dot{\epsilon}$ or $\dot{\bar{\epsilon}}$ are just (5.108). To arrive to the second equality we used antisymmetry of the fermionic variables and for the third we simply integrated the first term by parts. Hence

$$\begin{aligned}
\frac{d}{dt}(ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) &= 0 \\
\frac{d}{dt}(-ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu) &= 0
\end{aligned} \tag{5.110}$$

for arbitrary ϵ and $\bar{\epsilon}$. We conclude that the conserved supercharges are given by

$$Q = ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu \tag{5.111}$$

$$Q^\dagger = -ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu. \tag{5.112}$$

There is another symmetry of this model; the Lagrangian is invariant under

$$\begin{aligned}\psi^\mu &\to e^{-i\gamma}\psi^\mu \\ \bar{\psi}^\mu &\to e^{i\gamma}\bar{\psi}^\mu.\end{aligned} \tag{5.113}$$

This symmetry is continuous so with infinitesimal γ,

$$\begin{aligned}\psi^\mu &\to (1-i\gamma)\psi^\mu \\ \bar{\psi}^\mu &\to (1+i\gamma)\bar{\psi}^\mu\end{aligned} \tag{5.114}$$

with the corresponding variation given by

$$\begin{aligned}\delta_\gamma\phi^\mu &= 0 \\ \delta_\gamma\psi^\mu &= -i\gamma\psi^\mu \\ \delta_\gamma\bar{\psi}^\mu &= i\gamma\bar{\psi}^\mu.\end{aligned} \tag{5.115}$$

Going through the Noether procedure yet again, we get the following expression for the conserved charge

$$F = g_{\mu\nu}\bar{\psi}^\mu\psi^\nu. \tag{5.116}$$

Note that F is Hermitian. The reason for naming the charge F will become evident as we proceed.

We now quantize the system through canonical quantization. The conjugate momenta for ϕ^μ are given by

$$p_\mu = \frac{\partial L}{\partial\dot{\phi}^\mu} = g_{\mu\nu}\dot{\phi}^\nu + \frac{i}{2}(\Gamma_{\rho\mu\sigma} - \Gamma_{\sigma\mu\rho})\bar{\psi}^\rho\psi^\sigma \tag{5.117}$$

and for ψ^μ and $\bar{\psi}^\mu$

$$\begin{aligned}\pi_{\psi\mu} &= \frac{\partial L}{\partial\dot{\psi}^\mu} = -\frac{i}{2}g_{\mu\nu}\bar{\psi}^\nu \\ \pi_{\bar{\psi}\mu} &= \frac{\partial L}{\partial\dot{\bar{\psi}}^\mu} = -\frac{i}{2}g_{\mu\nu}\psi^\nu\end{aligned} \tag{5.118}$$

which lead to the second class constraints

$$\begin{aligned}\varphi_\mu &:= \pi_{\psi^\mu} + \frac{i}{2}g_{\mu\nu}\bar{\psi}^\nu = 0 \\ \bar{\varphi}_\mu &:= \pi_{\bar{\psi}^\mu} + \frac{i}{2}g_{\mu\nu}\psi^\nu = 0.\end{aligned} \tag{5.119}$$

The Poisson brackets between the constraints are given by

$$\begin{aligned}\{\varphi_\mu, \varphi_\mu\}_{PB} &= \{\bar{\varphi}_\mu, \bar{\varphi}_\mu\}_{PB} = 0 \\ \{\bar{\varphi}_\mu, \varphi_\mu\}_{PB} &= \{\varphi_\mu, \bar{\varphi}_\mu\}_{PB} = -ig_{\mu\nu}.\end{aligned} \tag{5.120}$$

Using this, the Dirac brackets are computed,

$$\{\phi^\mu, p_\nu\}_{DB} = \delta^\mu{}_\nu \tag{5.121}$$

$$\{p_\mu, p_\nu\}_{DB} = \frac{i}{4} g^{\rho\sigma}(\partial_\mu g_{\rho\alpha}\partial_\nu g_{\sigma\beta} - \partial_\nu g_{\rho\alpha}\partial_\mu g_{\sigma\beta})\bar{\psi}^\alpha\psi^\beta \tag{5.122}$$

$$\{p_\mu, \psi^\nu\}_{DB} = \frac{1}{2} g^{\nu\alpha}\partial_\mu g_{\alpha\beta}\psi^\beta \tag{5.123}$$

$$\{p_\mu, \bar{\psi}^\nu\}_{DB} = \frac{1}{2} g^{\nu\alpha}\partial_\mu g_{\alpha\beta}\bar{\psi}^\beta \tag{5.124}$$

$$\{p_\mu, \pi_{\psi^\nu}\}_{DB} = \frac{i}{4}\partial_\mu g_{\nu\alpha}\bar{\psi}^\alpha \tag{5.125}$$

$$\{p_\mu, \pi_{\bar{\psi}^\nu}\}_{DB} = \frac{i}{4}\partial_\mu g_{\nu\alpha}\psi^\alpha \tag{5.126}$$

$$\{\psi^\mu, \bar{\psi}^\nu\}_{DB} = -ig^{\mu\nu} \tag{5.127}$$

$$\{\psi_\mu, \pi_{\psi^\nu}\}_{DB} = -\frac{1}{2}\delta^\mu{}_\nu \tag{5.128}$$

$$\{\bar{\psi}_\mu, \pi_{\bar{\psi}^\nu}\}_{DB} = -\frac{1}{2}\delta^\mu{}_\nu \tag{5.129}$$

$$\{\pi_{\psi^\mu}, \pi_{\bar{\psi}^\nu}\}_{DB} = \frac{i}{4} g_{\mu\nu} \tag{5.130}$$

with all other brackets vanishing. As before, we will not use all of the relations but still write them down for completeness. Following the procedure, impose the canonical (anti-)commutation relations

$$[\phi^\mu, p_\nu] = i\delta^\mu{}_\nu \tag{5.131}$$

$$[p_\mu, p_\nu] = -\frac{1}{4} g^{\rho\sigma}(\partial_\mu g_{\rho\alpha}\partial_\nu g_{\sigma\beta} - \partial_\nu g_{\rho\alpha}\partial_\mu g_{\sigma\beta})\bar{\psi}^\alpha\psi^\beta \tag{5.132}$$

$$[p_\mu, \psi^\nu] = \frac{i}{2} g^{\nu\alpha}\partial_\mu g_{\alpha\beta}\bar{\psi}^\beta \tag{5.133}$$

$$[p_\mu, \bar{\psi}^\nu] = \frac{i}{2} g^{\nu\alpha}\partial_\mu g_{\alpha\beta}\psi^\beta \tag{5.134}$$

$$[p_\mu, \pi_{\psi^\nu}] = -\frac{1}{4}\partial_\mu g_{\nu\alpha}\psi^\alpha \tag{5.135}$$

$$[p_\mu, \pi_{\bar{\psi}^\nu}] = -\frac{1}{4}\partial_\mu g_{\nu\alpha}\bar{\psi}^\alpha \tag{5.136}$$

$$\{\psi^\mu, \bar{\psi}^\nu\} = g^{\mu\nu} \tag{5.137}$$

$$\{\psi_\mu, \pi_{\psi^\nu}\} = -\frac{i}{2}\delta^\mu{}_\nu \tag{5.138}$$

$$\{\bar{\psi}_\mu, \pi_{\bar{\psi}^\nu}\} = -\frac{i}{2}\delta^\mu{}_\nu \tag{5.139}$$

$$\{\pi_{\psi^\mu}, \pi_{\bar{\psi}^\nu}\} = -\frac{1}{4} g_{\mu\nu} \tag{5.140}$$

with the remaining (anti-)commutators equal to zero. We may write the supercharges in terms of the kinetic momenta $P_\mu = g_{\mu\nu}\dot{\phi}^\nu$

$$Q = i\bar{\psi}^\mu P_\mu \tag{5.141}$$

$$Q^\dagger = -i\psi^\mu P_\mu \tag{5.142}$$

The quantum mechanical version of the Hamiltonian is computed via (4.20),

$$\{Q, Q^\dagger\} = 2H \tag{4.20}$$

which fixes the problem of operator ordering.

The $\psi, \bar{\psi}$ satisfy the harmonic oscillator algebra for fermions (5.137) with $\bar{\psi}$ as creation operator and ψ as annihilation operator. Let $|0\rangle$ be the state annihilated by the ψ^μ. A general state can be written as a linear combination of states consisting of a function times a number of ψ^μ and $\bar{\psi}^\mu$ times $|0\rangle$. However, using the anticommutation relation (5.137), we can always move all the ψ^μ next to $|0\rangle$ to annihilate it. In such a way, one may rewrite the state into a linear combination of states without ψ^μ. Consider thus a state

$$|r\rangle = \bar{\psi}^{\mu_1} \dots \bar{\psi}^{\mu_r} |0\rangle \tag{5.143}$$

We might already see that the Hilbert space has a $\mathbb{Z}$-grading, but for the sake of completeness, we wish to show that F indeed is the operator mentioned in section 4.2, bringing justice to its name. Acting with F on $|r\rangle$ and using the anticommutator (5.137),

$$F|r\rangle = g_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\psi}^{\mu_1} \dots \bar{\psi}^{\mu_r} |0\rangle = \bar{\psi}^{\mu_1} \dots \bar{\psi}^{\mu_r} |0\rangle + \bar{\psi}^{\mu_1} g_{\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\psi}^{\mu_2} \dots \bar{\psi}^{\mu_r} |0\rangle . \tag{5.144}$$

Doing this another $r-1$ times, we thus end up with

$$F|r\rangle = r|r\rangle \tag{5.145}$$

so F has integer eigenvalues, counting the number of $\bar{\psi}$ acting on $|0\rangle$. This implies that we have an operator $e^{\pi F} = (-1)^F$. We thus call F the *fermion number operator*. Furthermore, note that

$$\begin{aligned} [F, Q] &= Q \\ [F, Q^\dagger] &= -Q^\dagger \end{aligned} \tag{5.146}$$

which is nothing but (4.36), so by proposition 4.3, Q and $Q^\dagger$ respect the grading (which could also be detected directly from their structure). The above commutators furthermore imply that F is a conserved quantity also in the quantum theory,

$$[F, H] = 0. \tag{5.147}$$

Since M is compact, $L^2(M, \mathbb{C})$ are all possible complex valued functions on M. After all the discussion, a natural representation of the Hilbert space is the (complexified) algebra of differential forms

$$\mathcal{H} = \Omega(M)_{\mathbb{C}} := \Omega(M) \otimes \mathbb{C} \tag{5.148}$$

equipped with the inner product

$$(\omega, \eta) = \sum_k \int_M \bar{\omega}_k \wedge \star\eta_k \tag{5.149}$$

for $\omega = \sum_k \omega_k$ and $\eta = \sum_k \eta_k$, where ω_k and η_k are k-forms. Given a coordinate patch U on M with local coordinates x^μ, the observables are represented by

$$\phi^\mu = x^\mu \times \tag{5.150}$$

$$P_\mu = -i\nabla_\mu \tag{5.151}$$

$$\bar{\psi}^\mu = g^{\mu\nu}\iota_\nu^\dagger = dx^\mu \wedge := \iota^{\dagger\mu} \tag{5.152}$$

$$\psi^\mu = g^{\mu\nu}\iota_\nu =: \iota^\mu \tag{5.153}$$

where ∇_μ is the covariant derivative associated to the Levi-Civita connection on M and ι_V is the interior product[16] with the vector field V (when we write ι_μ, we mean the interior product with respect to the coordinate vector field ∂_μ). Note that for $\alpha = \frac{1}{r!}\alpha_{\rho_1 \dots \rho_r} dx^{\rho_1} \wedge \dots \wedge dx^{\rho_r} \in \Omega^r(M)$

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\alpha &= -\frac{1}{r!} g^{\sigma\lambda}(R_{\sigma\rho_1\mu\nu}\alpha_{\lambda\rho_2\dots\rho_r} + \dots + R_{\sigma\rho_r\mu\nu}\alpha_{\rho_1\dots\rho_{r-1}\lambda}) dx^{\rho_1} \wedge \dots \wedge dx^{\rho_r} \\ &= -R_{\sigma\rho\mu\nu} dx^\rho \wedge g^{\sigma\lambda}\iota_\lambda \alpha \end{aligned} \tag{5.154}$$

[16] One can show that the interior product ι_A is the adjoint to the wedge product $\alpha\wedge$ under the given inner product (5.149), where $A = \alpha^\mu\partial_\mu$ and $\alpha = \alpha_\mu dx^\mu$.

so

$$[\nabla_\mu, \nabla_\nu] = R_{\mu\nu\rho\sigma}\iota^{\dagger\rho}\iota^\sigma \tag{5.155}$$

or

$$[P_\mu, P_\nu] = -R_{\mu\nu\rho\sigma}\bar{\psi}^\rho\psi^\sigma. \tag{5.156}$$

Representing the state $|0\rangle$ by 1, the structure of the Hilbert space is manifest; a state $f_{\mu_1\dots\mu_r}(p)\bar{\psi}^{\mu_1}\dots\bar{\psi}^{\mu_r}|0\rangle$ corresponds to $f_{\mu_1\dots\mu_r}(p)dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$, where $f_{\mu_1\dots\mu_r}\in\Omega^0(M)_\mathbb{C}$. By (5.145), we see that the fermion number corresponding to an r-form is r. This means that the Hilbert space decomposes into a direct sum with respect to the grading by F,

$$\mathcal{H} = \bigoplus_{p=0}^{d}\Omega^p(M)_\mathbb{C} \tag{5.157}$$

as we would expect (of course if $p<0$ or $p>d$, $\Omega^p(M)_\mathbb{C}$ is trivial). The supercharges now take the form

$$Q = \iota^{\dagger\mu}\nabla_\mu = dx^\mu\wedge\nabla_\mu = d \tag{5.158}$$

$$Q^\dagger = -\iota^\mu\nabla_\mu = -g^{\mu\nu}\iota_\nu\nabla_\mu = d^\dagger \tag{5.159}$$

where d is the exterior derivative and $d^\dagger$ its adjoint. To see the second equality in (5.158), act with Q:

$$\begin{aligned} dx^\mu\wedge\nabla_\mu f &= dx^\mu\wedge(\partial_\mu f_{\mu_1\dots\mu_r} - \Gamma^\lambda_{\mu\mu_1}f_{\lambda\dots\mu_r} - \dots - \Gamma^\lambda_{\mu\mu_r}f_{\mu_1\dots\lambda})dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \\ &= \partial_\mu f_{\mu_1\dots\mu_r}dx^\mu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \\ &= d(f_{\mu_1\dots\mu_r}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r})) \end{aligned} \tag{5.160}$$

where we have used the antisymmetrisation of the dx to obliterate the terms with Christoffel symbols. To prove (5.159), recall the definition of the *Hodge star* operator[17] $\star:\Omega^r(M)\to\Omega^{d-r}(M)$. For an r-form $\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r}dx^{\mu_1}\wedge dx^{\mu_r}$,

$$\star\,\omega = \frac{\sqrt{\det(g_{\mu\nu})}}{(m-r)!r!}\omega_{\mu_1\dots\mu_r}\varepsilon^{\mu_1\dots\mu_r}{}_{\nu_{r+1}\dots\nu_m}dx^{\nu_{r+1}}\wedge\dots\wedge dx^{\nu_m}. \tag{5.161}$$

The Hodge star is simpler if we use an orthonormal basis. Let $\{e_a\}$ be a non-coordinate basis on the tangent space T_pM such that $e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \delta_{ab}$. We use greek letters to denote the curved indices and latin ones for the flat. Let $e^a{}_\mu$ be the inverse of $e_a{}^\mu$. The dual basis is $\{\theta^a\} = \{e^a{}_\mu dx^\mu\}$. The Hodge star in this setting takes the form

$$\star\,\omega = \frac{1}{(m-r)!r!}\omega_{a_1\dots a_r}\varepsilon^{a_1\dots a_r}{}_{b_{r+1}\dots b_m}\theta^{b_{r+1}}\wedge\dots\wedge\theta^{b_m} \tag{5.162}$$

Recall now that $d^\dagger$ is given by

$$d^\dagger = (-1)^{mr+r+1}\star d\,\star. \tag{5.163}$$

Acting with this on ω and using (5.158),

$$\begin{aligned} (-1)^{dr+r+1}\star d\star\omega &= \frac{(-1)^{dr+r+1}}{(r-1)!(d-r)!r!}\nabla_c\omega_{a_1\dots a_r}\varepsilon_{a_1\dots a_r b_{r+1}\dots b_d}\varepsilon_{cb_{r+1}\dots b_d d_1\dots d_{r-1}}\theta^{d_1}\wedge\dots\wedge\theta^{d_{r-1}} \\ &= \frac{(-1)^{dr+r+1}}{(r-1)!(d-r)!r!}\nabla_c\omega_{a_1\dots a_r}(-1)^{(r-1)(d-r)}\varepsilon_{a_1\dots a_r b_{r+1}\dots b_d}\varepsilon_{cd_1\dots d_{r-1}b_{r+1}\dots b_d}\theta^{d_1}\wedge\dots\wedge\theta^{d_{r-1}} \\ &= -\frac{1}{r!(d-r)!r!}\iota_c\nabla_c\omega_{a_1\dots a_r}\varepsilon_{a_1\dots a_r b_{r+1}\dots b_d}\varepsilon_{d_0 d_1\dots d_{r-1}b_{r+1}\dots b_d}\theta^{d_0}\wedge\theta^{d_1}\wedge\dots\wedge\theta^{d_{r-1}} \\ &= -\frac{1}{r!}\iota_c\nabla_c\omega_{a_1\dots a_r}\theta^{a_1}\wedge\dots\wedge\theta^{a_r}. \end{aligned} \tag{5.164}$$

[17] For compact Riemannian manifolds.

where we on the last equality used

$$\varepsilon_{a_1\ldots a_r b_{r+1}\ldots b_d}\varepsilon_{d_0 d_1\ldots d_{r-1} b_{r+1}\ldots b_d}\theta^{d_0}\wedge\theta^{d_1}\wedge\cdots\wedge\theta^{d_{r-1}} = r!(d-r)!\theta^{a_1}\wedge\cdots\wedge\theta^{a_r}. \tag{5.165}$$

Now, since all indices are contracted the expression is independent of choice of basis. Hence

$$d^\dagger = -\iota_c\nabla_c = -\delta^{ab}\iota_b\nabla_a = -g^{\mu\nu}\iota_\nu\nabla_\mu. \tag{5.166}$$

The quantum mechanical version of the Hamiltonian is, as stated before, given by (4.20),

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}(dd^\dagger + d^\dagger d) = \frac{1}{2}\Delta \tag{5.167}$$

which is none other than the Laplace operator on M. One can check that

$$H = \frac{1}{2}(-g^{\mu\nu}\nabla_\mu\nabla_\nu - R_{\mu\nu\rho\sigma}\iota^{\dagger\mu}\iota^\nu\iota^{\dagger\rho}\iota^\sigma - \iota^{\dagger\mu}[\nabla_\mu,\iota^\nu]\nabla_\nu - \iota^\nu[\nabla_\nu,\iota^{\dagger\mu}]\nabla_\mu) \tag{5.168}$$

using (5.155), (5.158) and (5.159). The corresponding classical Hamiltonian is given by

$$H = \frac{1}{2}g_{\mu\nu}P^\mu P^\nu - \frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma \tag{5.169}$$

which if Legendre transformed recovers the original Laplacian (5.104). Back to topic, the supersymmetric ground states are the (complexified) harmonic forms on (M, g)

$$\mathcal{H}_{(0)} = \mathrm{Harm}(M,g)_{\mathbb{C}} = \bigoplus_{p=0}^{d}\mathrm{Harm}^p(M,g)_{\mathbb{C}} \tag{5.170}$$

or with respect to the grading,

$$\mathcal{H}^p_{(0)} = \mathrm{Harm}^p(M,g)_{\mathbb{C}}. \tag{5.171}$$

However, recall also the arguments in section 4.2. With $Q = d$ and $\mathcal{H}^p = \Omega^p(M)_{\mathbb{C}}$, the complex (4.37) becomes the de Rham complex and

$$\mathcal{H}_{(0)} = H_{dR}(M)_{\mathbb{C}} = \bigoplus_{p=0}^{d} H^p_{dR}(M)_{\mathbb{C}} \tag{5.172}$$

where $H_{dR}(M)_{\mathbb{C}}$ is the (complexified) de Rham cohomology algebra of M. Taking into account the grading, i.e. the splitting in form degree of $\mathcal{H}_{(0)}$, we receive

$$\mathcal{H}^p_{(0)} = H^p_{dR}(M)_{\mathbb{C}}. \tag{5.173}$$

We thus end up with

$$\mathrm{Harm}^p(M,g)_{\mathbb{C}} = H^p_{dR}(M)_{\mathbb{C}} \tag{5.174}$$

which is basically the statement of Hodge's theorem. Note that the complexification does not change the dimension of the space. Finally, the Witten index is given by

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \sum_{p\in\mathbb{Z}}(-1)^p\mathrm{dim}(\mathrm{Harm}^p(M,g)) = \sum_{p\in\mathbb{Z}}(-1)^p\mathrm{dim}(H^p(M)) = \chi(M) \tag{5.175}$$

where $\chi(M)$ is the Euler characteristic of M. We again stress the fact that the Witten index is independent of β.

5.2.1 Computing the path integral

Let us now compute the other part of the index. The path integral to be evaluated is given by

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_E} \tag{5.176}$$

with the Euclidean action

$$S_E = \int_0^\beta d\tau \left[\frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}(\bar{\psi}^\mu D_\tau\psi^\nu - D_\tau\bar{\psi}^\mu\psi^\nu) - \frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma\right]. \tag{5.177}$$

The procedure of evaluating the path integral for this case can be done in a similar way as for the previous one. One can find computations of the Witten index in the works of DeWitt [24] and Alvarez-Gaumé [3]. Utilising the independence of β of the index, we will argue that the contributions to the integral will localise around constant paths in the $\beta \to 0$ limit. Then using Euler-Lagrange equations, we will verify that there are solutions, i.e. extrema of the action, with constant $\phi = \phi_0$. Again, due to the independence of β, the path integral will actually be exact at second order around the extrema, which will be useful when we evaluate the integration over non-constant configurations.

Just as before, to show that the integral localises around constant paths, change variables $\tau = \beta s$ in the action

$$S_E = \int_0^1 ds \left[\frac{1}{2\beta}g_{\mu\nu}\frac{d\phi^\mu}{ds}\frac{d\phi^\nu}{ds} + \frac{1}{2}g_{\mu\nu}(\bar{\psi}^\mu D_s\psi^\nu - D_s\bar{\psi}^\mu\psi^\nu) - \frac{\beta}{2}R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma\right]. \tag{5.178}$$

When β approaches zero, the non-constant paths ϕ will make the first term in the action grow towards infinity, giving an exponential suppression in the path integral. Thus only the constant paths $\dot{\phi} = 0$ will contribute to the final result in this limit.

We now would like to check that the constant paths are actually extrema of the action. The classical equations of motion are given by the Euler-Lagrange equations. For the fermionic fields,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial\psi^\lambda} - \frac{d}{d\tau}\frac{\partial L}{\partial\dot{\psi}^\lambda} \\
&= -\frac{1}{2}g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\rho\Gamma^\nu_{\rho\lambda} + \frac{1}{2}g_{\mu\lambda}D_\tau\bar{\psi}^\mu + \frac{1}{2}R_{\mu\lambda\rho\sigma}\bar{\psi}^\mu\bar{\psi}^\rho\psi^\sigma + \frac{1}{2}R_{\mu\nu\rho\lambda}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho + \frac{d}{d\tau}\left(\frac{1}{2}g_{\mu\lambda}\bar{\psi}^\mu\right)
\end{aligned} \tag{5.179}$$

Following the same steps as in the $\mathcal{N} = 1$ case, we arrive at

$$g_{\mu\nu}D_\tau\bar{\psi}^\nu - R_{\mu\nu\rho\sigma}\bar{\psi}^\nu\psi^\rho\bar{\psi}^\sigma = 0. \tag{5.180}$$

Similarly, with respect to $\bar{\psi}$, we get the equation of motion for the conjugate field

$$g_{\mu\nu}D_\tau\psi^\nu - R_{\mu\nu\rho\sigma}\psi^\nu\bar{\psi}^\rho\psi^\sigma = 0. \tag{5.181}$$

For the bosonic field, we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial\phi^\lambda} - \frac{d}{d\tau}\frac{\partial L}{\partial\dot{\phi}^\lambda} \\
&= \frac{1}{2}\partial_\lambda g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{1}{2}\partial_\lambda g_{\mu\nu}(\bar{\psi}^\mu D_\tau\psi^\nu - D_\tau\bar{\psi}^\mu\psi^\nu) + \frac{1}{2}g_{\mu\nu}(\bar{\psi}^\mu\dot{\phi}^\rho\partial_\lambda\Gamma^\nu_{\rho\sigma}\psi^\sigma - \dot{\phi}^\rho\partial_\lambda\Gamma^\mu_{\rho\sigma}\bar{\psi}^\sigma\psi^\nu) \\
&\quad - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma - \frac{d}{d\tau}(g_{\lambda\nu}\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\bar{\psi}^\mu\Gamma^\nu_{\lambda\sigma}\psi^\sigma - \frac{1}{2}g_{\mu\nu}\Gamma^\mu_{\lambda\sigma}\bar{\psi}^\sigma\psi^\nu) \\
&= -g_{\lambda\nu}\ddot{\phi}^\nu - \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})\dot{\phi}^\mu\dot{\phi}^\nu \\
&\quad + \frac{1}{2}(g_{\nu\sigma}\partial_\rho\Gamma^\nu_{\lambda\mu} - g_{\nu\sigma}\partial_\lambda\Gamma^\nu_{\rho\mu} + \partial_\rho g_{\nu\sigma}\Gamma^\nu_{\lambda\mu} - \partial_\lambda g_{\nu\sigma}\Gamma^\nu_{\rho\mu} \\
&\qquad + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma} + \partial_\lambda g_{\mu\nu}\Gamma^\nu_{\rho\sigma} - \partial_\rho g_{\mu\nu}\Gamma^\nu_{\lambda\sigma})\dot{\phi}^\rho\bar{\psi}^\mu\psi^\sigma \\
&\quad + \frac{1}{2}(\Gamma_{\sigma\lambda\mu} - \Gamma_{\mu\lambda\sigma} - \partial_\lambda g_{\mu\sigma})\dot{\bar{\psi}}^\mu\psi^\sigma + \frac{1}{2}(\Gamma_{\sigma\lambda\mu} - \Gamma_{\mu\lambda\sigma} + \partial_\lambda g_{\mu\sigma})\bar{\psi}^\mu\dot{\psi}^\sigma - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma
\end{aligned} \tag{5.182}$$

where we on the first line of the third equality have used that $\partial_\mu g_{\lambda\nu}\dot\phi^\mu\dot\phi^\nu = \left(\frac{1}{2}\partial_\mu g_{\lambda\nu} + \frac{1}{2}\partial_\nu g_{\mu\lambda}\right)\dot\phi^\mu\dot\phi^\nu$, utilising the symmetry of $\dot\phi^\mu\dot\phi^\nu$ and the metric tensor. Note also that the first row is just $-g_{\lambda\nu}D_\tau\dot\phi^\nu$. Using the equations of motion (5.180) and (5.181) for the fermions and the fact that $\Gamma_{\sigma\lambda\mu}-\Gamma_{\mu\lambda\sigma}-\partial_\lambda g_{\mu\sigma} = -2\Gamma_{\mu\lambda\sigma}$ and $\Gamma_{\sigma\lambda\mu}-\Gamma_{\mu\lambda\sigma}+\partial_\lambda g_{\mu\sigma} = 2\Gamma_{\sigma\lambda\mu}$,

$$\begin{aligned}
0 = &-g_{\lambda\nu}D_\tau\dot\phi^\nu - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma \\
&+\frac{1}{2}(g_{\nu\sigma}\partial_\rho\Gamma^\nu_{\lambda\mu} - g_{\nu\sigma}\partial_\lambda\Gamma^\nu_{\rho\mu} + \partial_\rho g_{\nu\sigma}\Gamma^\nu_{\lambda\mu} - \partial_\lambda g_{\nu\sigma}\Gamma^\nu_{\rho\mu} \\
&\quad + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma} + \partial_\lambda g_{\mu\nu}\Gamma^\nu_{\rho\sigma} - \partial_\rho g_{\mu\nu}\Gamma^\nu_{\lambda\sigma})\dot\phi^\rho\bar\psi^\mu\psi^\sigma \\
&+\Gamma_{\mu\lambda\sigma}(\dot\phi^\rho\Gamma^\mu_{\rho\nu}\bar\psi^\nu + R^\mu{}_{\nu\alpha\beta}\bar\psi^\nu\psi^\alpha\bar\psi^\beta)\psi^\sigma - \Gamma_{\sigma\lambda\mu}\bar\psi^\mu(\dot\phi^\rho\Gamma^\sigma_{\rho\nu}\psi^\nu - R^\sigma{}_{\nu\alpha\beta}\psi^\nu\bar\psi^\alpha\psi^\beta) \\
= &-g_{\lambda\nu}D_\tau\dot\phi^\nu - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma \\
&+\frac{1}{2}(g_{\nu\sigma}\partial_\rho\Gamma^\nu_{\lambda\mu} - g_{\nu\sigma}\partial_\lambda\Gamma^\nu_{\rho\mu} - (\Gamma_{\nu\rho\sigma}-\partial_\rho g_{\nu\sigma})\Gamma^\nu_{\lambda\mu} + (\Gamma_{\nu\lambda\sigma}-\partial_\lambda g_{\nu\sigma})\Gamma^\nu_{\rho\mu} \\
&\quad + g_{\mu\nu}\partial_\lambda\Gamma^\nu_{\rho\sigma} - g_{\mu\nu}\partial_\rho\Gamma^\nu_{\lambda\sigma} - (\Gamma_{\nu\mu\lambda}-\partial_\lambda g_{\mu\nu})\Gamma^\nu_{\rho\sigma} + (\Gamma_{\nu\rho\mu}-\partial_\rho g_{\mu\nu})\Gamma^\nu_{\lambda\sigma})\dot\phi^\rho\bar\psi^\mu\psi^\sigma \\
&+(\Gamma_{\alpha\lambda\sigma}R^\alpha{}_{\mu\nu\rho} + \Gamma_{\alpha\lambda\mu}R^\alpha{}_{\sigma\nu\rho})\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma \\
= &-g_{\lambda\nu}D_\tau\dot\phi^\nu - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma \\
&+\frac{1}{2}[g_{\nu\sigma}(\partial_\rho\Gamma^\nu_{\lambda\mu} - \partial_\lambda\Gamma^\nu_{\rho\mu} + \Gamma^\nu_{\rho\alpha}\Gamma^\alpha_{\lambda\mu} - \Gamma^\nu_{\lambda\alpha}\Gamma^\alpha_{\rho\mu}) + g_{\mu\nu}(\partial_\lambda\Gamma^\nu_{\rho\sigma} - \partial_\rho\Gamma^\nu_{\lambda\sigma} + \Gamma^\nu_{\lambda\alpha}\Gamma^\alpha_{\rho\sigma} - \Gamma^\nu_{\rho\alpha}\Gamma^\alpha_{\lambda\sigma})]\dot\phi^\rho\bar\psi^\mu\psi^\sigma \\
&+(\Gamma_{\alpha\lambda\sigma}R^\alpha{}_{\mu\nu\rho} + \Gamma_{\alpha\lambda\mu}R^\alpha{}_{\sigma\nu\rho})\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma
\end{aligned} \tag{5.183}$$

We thus end up with

$$-g_{\lambda\nu}D_\tau\dot\phi^\nu + R_{\mu\nu\lambda\rho}\bar\psi^\mu\psi^\nu\dot\phi^\rho + \left(\Gamma_{\alpha\lambda\sigma}R^\alpha{}_{\mu\nu\rho} + \Gamma_{\alpha\lambda\mu}R^\alpha{}_{\sigma\nu\rho} - \frac{1}{2}\partial_\lambda R_{\mu\nu\rho\sigma}\right)\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma = 0. \tag{5.184}$$

By inspection, we confirm that $(\phi^\mu, \psi^\mu, \bar\psi^\mu) = (\phi_0^\mu, 0, 0)$ with ϕ_0^μ constant is a solution to the three sets of equations and hence extremum to the action, which we want to expand around.

To simplify computations, again employ Riemann normal coordinates around $\phi = \phi_0$. Expanding around the extrema $(\phi^\mu, \psi^\mu, \bar\psi^\mu) = (\phi_0^\mu, 0, 0)$, we have

$$\phi^\mu(\tau) = \phi_0^\mu + \xi^\mu(\tau) \tag{5.185}$$
$$\psi^\mu(\tau) = \eta^\mu(\tau) \tag{5.186}$$
$$\bar\psi^\mu(\tau) = \bar\eta^\mu(\tau) \tag{5.187}$$

where ξ^μ, η^μ and $\bar\eta^\mu$ are infinitesimal. Expanding one part at the time in the Lagrangian to second order (as before, we ignore first order terms since we are expanding around an extremum),

$$\frac{1}{2}g_{\mu\nu}\dot\phi^\mu\dot\phi^\nu \approx \frac{1}{2}\delta_{\mu\nu}\dot\xi^\mu\dot\xi^\nu \tag{5.188}$$

$$\frac{1}{2}g_{\mu\nu}(\bar\psi^\mu D_\tau\psi^\nu - D_\tau\bar\psi^\mu\psi^\nu) \approx \frac{1}{2}\delta_{\mu\nu}(\bar\eta^\mu\dot\eta^\nu - \dot{\bar\eta}^\mu\eta^\nu) \tag{5.189}$$

$$\frac{1}{2}R_{\mu\nu\rho\sigma}\bar\psi^\mu\psi^\nu\bar\psi^\rho\psi^\sigma \approx 0. \tag{5.190}$$

Hence

$$L_E \approx \frac{1}{2}\dot\xi^\mu\dot\xi^\mu + \frac{1}{2}(\bar\eta^\mu\dot\eta^\mu - \dot{\bar\eta}^\mu\eta^\mu) \tag{5.191}$$

up to second order. Note that the last term of the Lagrangian (5.104) does not contribute as seen in (5.190). Also if the Lagrangian is evaluated for constant paths, it takes the form

$$L_E[\phi_0, \psi_0, \bar{\psi}_0] = -\frac{1}{2} R_{\mu\nu\rho\sigma}(\phi_0) \bar{\psi}_0^\mu \psi_0^\nu \bar{\psi}_0^\rho \psi_0^\sigma. \tag{5.192}$$

The result is that the integral splits into an integration over non-constant paths and an integration over constant paths. In the $\beta \to 0$ limit, when the contribution of non-constant paths are exponentially suppressed,

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H} = \int_{PBC} \mathcal{D}\phi \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{-\int_0^\beta d\tau [\frac{1}{2}\dot{\xi}^\mu \dot{\xi}^\mu + \frac{1}{2}(\bar{\eta}^\mu \dot{\eta}^\mu - \dot{\bar{\eta}}^\mu \eta^\mu)]} \ e^{\int_0^\beta dt \frac{1}{2} R_{\mu\nu\rho\sigma}(\phi_0) \bar{\psi}_0^\mu \psi_0^\nu \bar{\psi}_0^\rho \psi_0^\sigma} \tag{5.193}$$

As previously done, let us now Fourier expand the fields

$$\xi^\mu = \sum_{k=-\infty}^{\infty} \xi_k^\mu \frac{1}{\sqrt{\beta}} e^{2\pi i k\tau/\beta} \tag{5.194}$$

$$\eta^\mu = \sum_{k=-\infty}^{\infty} \eta_k^\mu \frac{1}{\sqrt{\beta}} e^{2\pi i k\tau/\beta} \tag{5.195}$$

$$\bar{\eta}^\mu = \sum_{k=-\infty}^{\infty} \bar{\eta}_k^\mu \frac{1}{\sqrt{\beta}} e^{2\pi i k\tau/\beta}. \tag{5.196}$$

By integrating out the non-constant fields, we note that their contribution is just 1 since from (3.45)

$$\int_{PBC} \mathcal{D}\xi' e^{-\int_0^\beta dt \frac{1}{2}\dot{\xi}^\mu \dot{\xi}^\mu} = \frac{1}{\sqrt{\mathrm{Det}'\left(-\frac{d^2}{d\tau^2}\right)}} = \frac{1}{\mathrm{Det}'\left(\frac{d}{d\tau}\right)} \tag{5.197}$$

and from (3.122)

$$\int_{PBC} \mathcal{D}\bar{\eta}' \mathcal{D}\eta' e^{-\int_0^\beta d\tau \frac{1}{2}(\bar{\eta}^\mu \dot{\eta}^\mu - \dot{\bar{\eta}}^\mu \eta^\mu)} = \int_{PBC} \mathcal{D}\bar{\eta}' \mathcal{D}\eta' e^{-\int_0^\beta d\tau \bar{\eta}^\mu \dot{\eta}^\mu} = \mathrm{Det}'\left(\frac{d}{d\tau}\right) \tag{5.198}$$

where the $'$ signifies that we are integrating over non-constant configurations. We are left with

$$\mathrm{Tr}(-1)^F e^{\beta H} = \int \mathcal{D}\xi_0 \mathcal{D}\bar{\eta}_0 \mathcal{D}\eta_0 e^{\frac{\beta}{2} R_{\mu\nu\rho\sigma} \bar{\psi}_0^\mu \psi_0^\nu \bar{\psi}_0^\rho \psi_0^\sigma} \tag{5.199}$$

Before, we integrated over fluctuations around $\phi = \phi_0$ and we could rely on the flatness of the metric under Riemann normal coordinates. Now however, we are integrating over ϕ_0, i.e. over the manifold itself. In order to find the form of the measure in our curved space setting, we introduce an orthonormal frame field in the spirit of general relativity. The idea is to use that we know what the measure looks like in flat space to get the expression for the curved case. Let $\{e_a\}$ be a basis on the tangent space T_pM such that ${e_a}^\mu {e_b}^\nu g_{\mu\nu} = \delta_{ab}$. As before, the greek letters denote the curved indices and the latin ones the flat. Let ${e^a}_\mu$ be the inverse of ${e_a}^\mu$. Going from the flat description to the curved one

$$\begin{aligned} \xi^a &= {e^a}_\mu \xi^\mu \\ \eta^a &= {e^a}_\mu \eta^\mu \\ \bar{\eta}^a &= {e^a}_\mu \bar{\eta}^\mu \end{aligned} \tag{5.200}$$

the measures change as follows:

$$\begin{aligned} \mathcal{D}\xi_0 &= \sqrt{\det(g_{\mu\nu})} \prod_{\mu=1}^d \frac{d\xi_0^\mu}{(2\pi)^{\frac{1}{2}}} = \sqrt{\det(g_{\mu\nu})} \frac{d^d \xi_0}{(2\pi)^{\frac{d}{2}}} \\ \mathcal{D}\bar{\eta}_0 \mathcal{D}\eta_0 &= \frac{1}{\det(g_{\mu\nu})} \prod_{\mu=1}^d d\bar{\eta}_0^\mu d\eta_0^\mu \end{aligned} \tag{5.201}$$

where we used that $\det(e^{\alpha}{}_{\mu}) = \sqrt{\det(g_{\mu\nu})}$. Henceforth, we will use the notation $g = \det(g_{\mu\nu})$. The above argumentation is quite heuristic, but it gives us the desired result. For a more complete treatment of path integrals in curved spaces, we suggest the interested reader to investigate e.g. [25]. Note that this step in the $\mathcal{N}=1$ case would just produce cancelling determinant factors, thus leaving the measure unchanged from the flat case.

Now, through the same argument as around (5.88), we observe that $d\xi_0^\mu = \sqrt{\beta}dx_0$, $d\eta_0^\mu = \frac{1}{\sqrt{\beta}}d\psi_0^\mu$ and $d\bar{\eta}_0^\mu = \frac{1}{\sqrt{\beta}}\bar{\psi}_0^\mu$. Therefore

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \frac{1}{\beta^{\frac{d}{2}}}\int_M \frac{d^dx}{(2\pi)^{\frac{d}{2}}}\frac{1}{\sqrt{g}}\int\left(\prod_{\alpha=1}^{d} d\bar{\psi}_0^\alpha d\psi_0^\alpha\right)e^{\frac{\beta}{2}R_{\mu\nu\rho\sigma}\bar{\psi}_0^\mu\psi_0^\nu\bar{\psi}_0^\rho\psi_0^\sigma} \tag{5.202}$$

Note that integrating over all constant paths ϕ_0 is the same as integrating over the manifold. Let us rescale away the apparent β dependence. Making the change $\psi_0'^\mu = \beta^{\frac{1}{4}}\psi_0^\mu$, which implies that $\prod_{\alpha=1}^{d} d\bar{\psi}_0^\alpha d\psi_0^\alpha = \beta^{\frac{d}{2}}\prod_{\alpha=1}^{d} d\bar{\psi}_0'^\alpha d\psi_0'^\alpha$, we get

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \frac{1}{\beta^{\frac{d}{2}}}\int_M \frac{d^dx}{(2\pi)^{\frac{d}{2}}}\frac{1}{\sqrt{g}}\int\beta^{\frac{d}{2}}\left(\prod_{\alpha=1}^{d} d\bar{\psi}_0'^\alpha d\psi_0'^\alpha\right)e^{\frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}_0'^\mu\psi_0'^\nu\bar{\psi}_0'^\rho\psi_0'^\sigma} \tag{5.203}$$

Removing the primes on the integration variables, we thus have

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int_M \frac{d^dx}{(2\pi)^{\frac{d}{2}}}\frac{1}{\sqrt{g}}\int\left(\prod_{\alpha=1}^{d} d\bar{\psi}_0^\alpha d\psi_0^\alpha\right)e^{\frac{1}{2}R_{\mu\nu\rho\sigma}(x)\bar{\psi}_0^\mu\psi_0^\nu\bar{\psi}_0^\rho\psi_0^\sigma}. \tag{5.204}$$

Observe that if the dimension d is odd, the integral will vanish since the expansion of the exponent yields terms with fields coming in multiples of four. This is expected as it is a well-known fact that the Euler number for odd-dimensional manifolds vanishes. Assume therefore that the dimension is $d = 2n$ for some positive integer n. To more directly get to the form we want, we make the change of variables

$$\begin{aligned}\psi_0^\mu &= \frac{1}{\sqrt{2}}(\psi_1^\mu + i\psi_2^\mu)\\ \bar{\psi}_0^\mu &= \frac{1}{\sqrt{2}}(\psi_1^\mu - i\psi_2^\mu),\end{aligned} \tag{5.205}$$

where $\frac{1}{\sqrt{2}}\psi_1$ and $\frac{1}{\sqrt{2}}\psi_2$ are the real and imaginary parts of ψ_0 respectively. The Jacobian for this change of variables is

$$\frac{\partial(\psi_0,\bar{\psi}_0)}{\partial(\psi_1,\psi_2)} = \left[\det\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\right]^d = (-i)^d = i^d. \tag{5.206}$$

Thus, the exponent in (5.204) takes the form

$$\frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}_0^\mu\psi_0^\nu\bar{\psi}_0^\rho\psi_0^\sigma = \frac{1}{4}R_{\mu\nu\rho\sigma}\psi_1^\mu\psi_1^\nu\psi_2^\rho\psi_2^\sigma \tag{5.207}$$

where we again have used the antisymmetric properties of the Riemann tensor in conjunction with the antisymmetrization due to the fermionic variables, together with the first Bianchi identity to simplify the expression. We end up with

$$
\begin{aligned}
\mathrm{Tr}(-1)^F e^{-\beta H} &= \frac{i^d}{(2\pi)^{\frac{d}{2}}} \int_M \frac{d^d x}{\sqrt{g}} \int \left(\prod_{\alpha=1}^{d} d\psi_2^{\alpha} d\psi_1^{\alpha} \right) e^{\frac{1}{4} R_{\mu\nu\rho\sigma} \psi_1^{\mu} \psi_1^{\nu} \psi_2^{\rho} \psi_2^{\sigma}} \\
&= \frac{i^d}{4^{\frac{d}{2}} \left(\frac{d}{2}\right)! (2\pi)^{\frac{d}{2}}} \int_M \frac{d^d x}{\sqrt{g}} \int \left(\prod_{\alpha=1}^{d} d\psi_2^{\alpha} d\psi_1^{\alpha} \right) \left(R_{\mu\nu\rho\sigma} \psi_1^{\mu} \psi_1^{\nu} \psi_2^{\rho} \psi_2^{\sigma} \right)^{\frac{d}{2}} \\
&= \frac{i^d (-1)^{\frac{d}{2}}}{4^{\frac{d}{2}} \left(\frac{d}{2}\right)! (2\pi)^{\frac{d}{2}}} \int_M \frac{d^d x}{\sqrt{g}} \int d\psi_2^d d\psi_1^d \dots d\psi_2^1 d\psi_1^1 \times \\
&\quad \times R_{\mu_1\nu_1\rho_1\sigma_1} \dots R_{\mu_n\nu_n\rho_n\sigma_n} \psi_1^{\mu_1} \psi_2^{\rho_1} \psi_1^{\nu_1} \psi_2^{\sigma_1} \dots \psi_1^{\mu_n} \psi_2^{\rho_n} \psi_1^{\nu_n} \psi_2^{\sigma_n} \\
&= \frac{1}{4^{\frac{d}{2}} \left(\frac{d}{2}\right)! (2\pi)^{\frac{d}{2}}} \int_M \frac{d^d x}{\sqrt{g}} \int d\psi_2^d \dots d\psi_2^1 d\psi_1^d \dots d\psi_1^1 \times \\
&\quad \times R_{\mu_1\nu_1\rho_1\sigma_1} \dots R_{\mu_n\nu_n\rho_n\sigma_n} \psi_1^{\mu_1} \psi_1^{\nu_1} \dots \psi_1^{\mu_n} \psi_1^{\nu_n} \psi_2^{\rho_1} \psi_2^{\sigma_1} \dots \psi_2^{\rho_n} \psi_2^{\sigma_n} \\
&= \frac{1}{4^{\frac{d}{2}} \left(\frac{d}{2}\right)! (2\pi)^{\frac{d}{2}}} \int_M \frac{d^d x}{\sqrt{g}} \int d\psi_2^d \dots d\psi_2^1 d\psi_1^d \dots d\psi_1^1 \times \\
&\quad \times \epsilon^{\mu_1\nu_1\dots\mu_n\nu_n} R_{\mu_1\nu_1\rho_1\sigma_1} \dots R_{\mu_n\nu_n\rho_n\sigma_n} \psi_1^1 \dots \psi_1^d \psi_2^{\rho_1} \psi_2^{\sigma_1} \dots \psi_2^{\rho_n} \psi_2^{\sigma_n} \\
&= \frac{1}{4^{\frac{d}{2}} \left(\frac{d}{2}\right)! (2\pi)^{\frac{d}{2}}} \int_M \left(\prod_{\mu=1}^{d} dx^{\mu} d\psi_2^{\mu} \right) \frac{1}{\sqrt{g}} \epsilon^{\mu_1\nu_1\dots\mu_n\nu_n} R_{\mu_1\nu_1\rho_1\sigma_1} \psi_2^{\rho_1} \psi_2^{\sigma_1} \dots R_{\mu_n\nu_n\rho_n\sigma_n} \psi_2^{\rho_n} \psi_2^{\sigma_n}
\end{aligned}
\tag{5.208}
$$

where we in equality three recalled the ordering of the measure (which came from how the coherent state path integral was derived)

$$
\prod_{\alpha=1}^{d} d\psi_2^{\alpha} d\psi_1^{\alpha} = d\psi_2^d d\psi_1^d \dots d\psi_2^1 d\psi_1^1. \tag{5.209}
$$

The Levi-Civita symbol arises since the ψ_1 antisymmetrise the μ and ν indices. Following the procedure of the $\mathcal{N} = 1$ case, we now switch to the language of differential forms,

$$
\begin{aligned}
\mathrm{Tr}(-1)^F e^{-\beta H} &= \frac{1}{2^n n! (2\pi)^n} \int_M \varepsilon^{\mu_1\nu_1\dots\mu_n\nu_n} \mathcal{R}_{\mu_1\nu_1} \wedge \dots \wedge \mathcal{R}_{\mu_n\nu_n} \\
&= \frac{1}{(2\pi)^n} \int_M \mathrm{Pf}((\mathcal{R})_{\mu\nu})
\end{aligned}
\tag{5.210}
$$

where $\mathcal{R}_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} dx^{\rho} \wedge dx^{\sigma}$ is the curvature 2-form and $\mathrm{Pf}((A)_{\mu\nu}) = \frac{1}{2^n n!} \varepsilon^{\mu_1\nu_1\dots\mu_n\nu_n} A_{\mu_1\nu_1} \dots A_{\mu_n\nu_n}$, with $(A)_{\mu\nu}$ being a $2n \times 2n$ skew-symmetric matrix, is the *Pfaffian* of $(A)_{\mu\nu}$. We have denoted the *Levi-Civita tensor* by $\varepsilon^{\mu_1\nu_1\dots\mu_n\nu_n} = \frac{1}{\sqrt{g}} \epsilon^{\mu_1\nu_1\dots\mu_n\nu_n}$. Our result is summarised in the following theorem:

Theorem 5.2. (Chern-Gauss-Bonnet) *Let M be a closed, oriented d-dimensional Riemannian manifold, equipped with the metric $g_{\mu\nu}$. Let $\mathcal{R}_{\mu\nu}$ be the curvature 2-form of the Levi-Civita connection associated to $g_{\mu\nu}$. Then*

$$
\chi(M) = \begin{cases} \dfrac{1}{(2\pi)^{\frac{d}{2}}} \displaystyle\int_M \mathrm{Pf}((\mathcal{R})_{\mu\nu}) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases} \tag{5.211}
$$

where $\chi(M)$ is the Euler characteristic of M.

Example 5.1. For $d = 2$, recall that the Riemann curvature tensor is given by

$$R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}) \tag{5.212}$$

where $K = \dfrac{R}{2}$ is the Gaussian curvature and R the Ricci scalar. We get

$$\begin{aligned}\chi(M) &= \frac{1}{8\pi}\int_M \frac{1}{\sqrt{g}}\epsilon^{\mu\nu}R_{\mu\nu\rho\sigma}dx^\rho \wedge dx^\sigma = \frac{1}{2\pi}\int_M \frac{1}{\sqrt{g}}R_{1212}dx^1 \wedge dx^2 \\ &= \frac{1}{2\pi}\int_M \frac{1}{\sqrt{g}}K(g_{11}g_{22} - g_{12}g_{21})dx^1 \wedge dx^2 = \frac{1}{2\pi}\int_M K\sqrt{g}dx^1 \wedge dx^2 \\ &= \frac{1}{2\pi}\int_M KdA\end{aligned} \tag{5.213}$$

which is the famous Gauss-Bonnet formula for closed and compact surfaces.

6 Equivariant Cohomomlogy and Supersymmetric Quantum Mechanics

We have so far investigated a few examples of index theorems on closed oriented Riemannian manifolds. However, in physics, there is often a group action on the manifold. A typical such case is when we are dealing with a gauge theory. Recall that given a d-dimensional closed oriented Riemannian manifold M and a nilpotent operator Q (represented by the exterior derivative d) acting on the associated Hilbert space (represented by the complexified space of differential forms $\Omega(M)_{\mathbb{C}} := \Omega(M) \otimes \mathbb{C}$), the Witten index for the $\mathcal{N} = 2$ non-linear sigma model is given by

$$\mathrm{Tr}(-1)^F e^{\beta H} = \sum_{r=1}^{d}(-1)^r \mathrm{dim}(\mathrm{Harm}^r(M, g)) = \sum_{r=1}^{d}(-1)^r \mathrm{dim}(H^r_{dR}(M)) \tag{6.1}$$

where $H = \dfrac{1}{2}\{Q, Q^\dagger\} = \dfrac{1}{2}(dd^\dagger + d^\dagger d) = \dfrac{1}{2}\Delta$ is the Hamiltonian of the system. We now would like to have a notion of cohomology for the orbit space. That is when the notion of *equivariant cohomology* comes into the picture.

6.1 Equivariant Cohomology

Let G be a topological group acting continuously on some topological space X (such a space X is referred to as a *G-space*). In the following, all group actions will be continuous. The main idea of equivariant cohomology is that if G acts freely on X (i.e. the stabilizer subgroups are all trivial, in other words, the action of a an element of G not being the identity has no fixed points), then the equivariant cohomology of X should be the regular singular cohomology for the orbit space X/G. The freeness of the action ensures that the quotient space will still be nice, for instance when X is a manifold, then X/G will also be a manifold. However, the action is not always free. Generically, quotients of non-free actions yield undesired results; the space could for instance no longer be Hausdorff or lose smoothness. Luckily, there is a way to construct a space X' homotopy equivalent to X such that the action on X' is free (recall that the cohomology is homotopy invariant). Such a space can be seen as the "correct" replacement for X/G.

Definition 6.1. Let G be a topological group and let $\pi : EG \to BG$ be a numerable[18] principal G-bundle. If for any G-bundle $\pi_P : P \to M$, there is a map $f : M \to BG$ unique up to homotopy (known as the *classifying map* of $P \to M$) such that P is isomorphic to the the pullback bundle $f^*(EG)$, $\pi : EG \to BG$ is called the *universal G-bundle* and BG is the *classifying space* of G.

[18]We will always assume that the principal bundles are numerable. For details and the definition of a *numerable* principal bundle, check [26].

The existence of universal bundles can be proved via the *Milnor construction* as was done in a paper by Milnor from 1956 [27]. We review the construction following Husemöller [28] without proving it has all the correct properties while also glancing over a few other details. We define the *join* $A = A_1 * \cdots * A_n$ of topological spaces A_i, $i = 1, \ldots n$ where a point in A is specified by

- non-negative real numbers $t_1, \ldots, t_n$ such that $t_1 + \cdots + t_n = 1$ and
- for each $t_i \neq 0$ a point a_i, where a point in A is written as $\langle x, t \rangle = (t_1 a_1, \ldots, t_n a_n)$. In the cases when $t_j = 0$, the a_j are arbitrarily chosen.

Next, we define an *infinite join* as the join of (countable) infinitely many topological spaces analogously, but with the extra requirement that all but finitely many t_i should vanish. Define now

$$EG = G * G * \ldots \tag{6.2}$$

as the infinite join of topological groups G, together with the right action of G

$$\langle g, t \rangle h = \langle gh, t \rangle = \langle t_1 g_1 h, t_2 g_2 h, \ldots \rangle \tag{6.3}$$

for $h \in G$. Note that this action is free; it is basically just the right action of the group on itself in each entry. This makes $\pi : EG \to BG$ into a principal G-bundle with $BG = EG/G$ and π the quotient map. This is the Milnor construction.

As an application of a theorem by Dold [26] we also have that a principal G-bundle is universal if and only if the total space EG is contractible. Define now the *homotopy quotient* X_G of a topological space X by G as the orbit space $(X \times EG)/G$ where the action of G is the diagonal action. Since the action of G on EG is free, it will automatically be free for $X \times EG$ regardless of the nature of the action on X. We have thus kept our promise about finding a space homotopic to the initial one with a free G-acion. It is time to define the concept of equivariant cohomology.

Definition 6.2. Let G be a topological group and $\pi : EG \to BG$ be a universal G-bundle. The *equivariant cohomology ring* of a topological space X with a continuous group action of G is

$$H^*_G(X) = H^*(X_G) \tag{6.4}$$

where X_G is the homotopy quotient of X by G and $H(X_G)$ is the singular cohomology ring of X_G.

Remark. In the case when G acts freely on X, the equivariant cohomology $H^*_G(X_G)$ reduces to $H^*(X/G)$, which agrees with the statement in the beginning of this section [29].

Example 6.1. In the case when X is a point,

$$H^*_G(X) = H^*((X \times EG)/G) = H^*(EG/G) = H^*(BG). \tag{6.5}$$

So far we have worked quite generally. In the following, we will explore a couple of models which construct equivariant cohomology in more specific settings. The models in consideration are the Cartan and Weil model. It will turn out that the cohomology computed by each model are in fact equal.

6.2 The Models

Both of the models in consideration aim at constructing equivariant cohomology in the case when $X = M$ is a smooth manifold and G is a compact connected Lie group acting smoothly on M (we refer to such M as a *G-manifold*). We start by recalling some terminology. Denoting the de Rham complex of M by $\Omega(M)$, a differential form $\omega \in \Omega(M)$ is *right G-invariant* if $(R_g)^*\omega = \omega$ for all $g \in G$, where $R_g : M \to M$ is the right action of g on M. Furthermore, if $\iota_X \omega = 0$ for all $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and ι_X is the interior product with X, we say that ω is *horizontal*. When we write ι_X we actually mean the interior product with respect to the fundamental vector field $X^\#$ generated by X, i.e. $\iota_X := \iota_{X^\#}$. The fundamental vector field is defined by $X^{\#}f(p) := \frac{d}{dt}[f(R_{\exp(tX)}(p))]\big|_{t=0}$. Note that $X^{\#}$ is just the directional derivative of the curve defined by $R_{\exp(tX)}(p)$, which is a curve in the direction of the group action. Hence, if $\pi : M \to M/G$ is a principal fibration with fibres $G_{[p]}$, $[p] \in M/G$, the space of $X^{\#}$ at a given point $p \in M$ is the tangent space of the fibre $G_{[p]}$ at p, also known as the *vertical subspace* at p.

Remark. We discuss some technical details. The pullback $(R_g)^*$ gives a representation on $\Omega(M)$; the map $\Phi : G \to \mathrm{End}(\Omega(M))$ defined by $\Phi(g) = (R_g)^*$ is a group homomorphism:

$$\Phi(g)\Phi(h) = (R_g)^*(R_h)^* = (R_h R_g)^* = (R_{gh})^* = \Phi(gh). \tag{6.6}$$

Let $\alpha \in \Omega(M)$. This induces a representation on the Lie algebra,

$$\mathcal{L}_X \alpha := \frac{d}{dt}[(R_{exp(tX)})^* \alpha]\big|_{t=0} = \frac{d}{dt}[(R_{exp(tX^{\#})})^* \alpha]\big|_{t=0} = \mathcal{L}_{X^{\#}} \alpha \tag{6.7}$$

which is nothing but the Lie derivative with respect to the fundamental vector field. If we would have chosen the left action l_g instead, the corresponding representation would have been given by $\Phi(g) = (l_g^{-1}) = l_{g^{-1}}$ instead since $l_g l_h = l_{gh}$. This in turn would have resulted in that the interior product would have been defined as $\iota_X = \iota_{-X^{\#}}$.

Let G act freely on a smooth manifold P. Then the projection $\pi : P \to P/G$ is a principal G-fibration. Define

$$\Omega(P)_{\text{bas}} := \pi^* \Omega(P) \subset \Omega(P) \tag{6.8}$$

as the *basic differential forms*. The pullback by the projection π^* is injective and as a consequence, $\pi^*\Omega(P)$ is isomorphic to $\Omega(P/G)$. Furthermore, the subalgebra $\pi^*\Omega(P)$ is closed under exterior derivation since the exterior derivative commutes with the pullback by a map. From here we see that we can use this observation as a starting point to the modelling of the equivariant cohomology of M; with the diagonal action, the projection $\pi : M \times EG \to M_G$ is a principal G-fibration. In other words,

$$\Omega(M \times EG)_{\text{bas}} \cong \Omega(M_G). \tag{6.9}$$

The de Rham theorem asserts that the de Rham cohomology of M_G is isomorphic to the singular cohomology of M_G, i.e.

$$H^*_{dR}(M_G) \cong H^*_G(M). \tag{6.10}$$

Proposition 6.1. *A differential form $\omega \in \Omega(P)$ is basic if and only if it is G-invariant and horizontal.*

Proof. We conduct the proof locally. If it is true for an arbitrary chart, it holds globally as well. Let $\omega = \frac{1}{r!}\omega_{\nu_1 \dots \nu_r} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r} \in \Omega^r(P)$. Assume first that ω is basic, i.e. $\omega = \pi^*\eta$ for some $\eta = \frac{1}{r!}\eta_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \in \Omega^r(P/G)$. Then we can write

$$\omega = \pi^* \eta = \frac{1}{r!} \eta_{\mu_1 \dots \mu_r} \circ \pi \; d(x^{\mu_1} \circ \pi) \wedge \dots \wedge d(x^{\mu_r} \circ \pi). \tag{6.11}$$

To check G-invariance, act with $(R_g)^*$,

$$\begin{aligned} (R_g)^* \omega &= \frac{1}{r!} \eta_{\mu_1 \dots \mu_r} \circ \pi \circ R_g \; d(x^{\mu_1} \circ \pi \circ R_g) \wedge \dots \wedge d(x^{\mu_r} \circ \pi \circ R_g) \\ &= \frac{1}{r!} \eta_{\mu_1 \dots \mu_r} \circ \pi \; d(x^{\mu_1} \circ \pi) \wedge \dots \wedge d(x^{\mu_r} \circ \pi) = \omega. \end{aligned} \tag{6.12}$$

The second equality is true since R_g just moves any $u \in P$ along the fibre which means if $\pi(u) = p$, then $\pi(R_g(u)) = p$. Horizontality is checked by acting with ι_X where $X \in \mathfrak{g}$

$$\iota_X \omega = \frac{1}{(r-1)!} \eta_{\mu_1 \dots \mu_r} \circ \pi \; (X^{\#})^{\mu_1} d(x^{\mu_2} \circ \pi) \wedge \dots \wedge d(x^{\mu_r} \circ \pi). \tag{6.13}$$

Note that $\eta_{\mu_1\ldots\mu_r}$ is zero in the fibre direction, which can be seen as follows. Let $\dim(P/G) = m$ and $\dim(P) = n$. Let U be a coordinate patch of P/G and V a coordinate patch of G. Then P can locally be written as $U \times V$. If the coordinates on U are $x^1 \ldots x^m$ and on V are $x^{m+1} \ldots x^n$, then $\omega_{\nu_1\ldots\nu_r}$ is zero for $\nu_i \geq m+1$ if $\omega = \pi^*\eta$ since the μ_i in $\eta_{\mu_1\ldots\mu_r}$ only run between $1 \leq \mu_i \leq m$. By construction, $(X^\#)^{\nu_1}$ can be non-zero only in the fibre direction, i.e. for $m+1 \leq \nu_i \leq n$. Hence $\iota_X\omega = 0$.

Assume conversely that ω is both G-invariant and horizontal. For ω to be a pullback projection, it needs to at each point live only in the cotangent space of P/G. The horizontality condition guarantees that $\omega_{\nu_1\ldots\nu_r}$ is zero for $m+1 \leq \nu_i \leq n$; just take the interior product with the fundamental coordinate basis vector fields of V. Furthermore, G-invariance means ω is constant in the fibre direction, hence ω can be viewed as a form on P/G. The pullback projection can now be constructed as $\omega = \pi^*\eta$ where $\eta_{\mu_1\ldots\mu_r} = \omega_{\mu_1\ldots\mu_r}$ where $1 \leq \mu_i \leq m$. $\square$

The connectedness of G implies that G-invariance is equivalent to the vanishing of the Lie derivative, $\mathcal{L}_X\omega = 0$ for all $X \in \mathfrak{g}$ [29]. Recall that the Lie derivative on differential forms satisfies

$$\mathcal{L}_X = \iota_X d + d\iota_X. \tag{6.14}$$

Given a connection form $\omega \in \mathfrak{g} \otimes \Omega^1(P)$ on P and its corresponding curvature form $\Omega \in \mathfrak{g} \otimes \Omega^2(P)$, we have the following relations:

$$\begin{gathered} \iota_X\omega = X \qquad \iota_X\Omega = 0 \\ \Omega = d\omega + \frac{1}{2}[\omega,\omega] \qquad d\Omega = [\Omega,\omega] \end{gathered} \tag{6.15}$$

where $[,]$ here denotes the Lie bracket of $\mathfrak{g}$. By modelling using these relations, we will get the desired result.

6.2.1 The Weil model

The Weil model is constructed over an object known as the *Weil algebra*

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \tag{6.16}$$

where $\Lambda(\mathfrak{g}^*)$ is the exterior algebra over the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. $S(\mathfrak{g}^*)$ is the symmetric algebra over $\mathfrak{g}^*$ which can be constructed via the tensor algebra by taking the quotient of the ideal generated by $u_1 \otimes u_2 - u_2 \otimes u_1$ for all u_1 and u_2 in $\mathfrak{g}^*$. In effect, this just means that all elements commute under the tensor product so the symmetric algebra can be identified with the polynomial algebra generated by a chosen basis $\{u^a\}$ of $\mathfrak{g}^*$,

$$S(\mathfrak{g}^*) \cong \mathbb{R}[u]. \tag{6.17}$$

Next, we give $W(\mathfrak{g})$ a $\mathbb{Z}$-grading by assigning to the generators $C^a \in \mathfrak{g}^*$ of the exterior algebra degree 1 and generators of the symmetric algebra $u^a \in \mathfrak{g}^*$ degree 2. In this way, $W(\mathfrak{g})$ is a commutative graded algebra, i.e. with $w^p w^q = (-1)^{pq} w^q w^p$ for $w^p \in W^p(\mathfrak{g})$ (we denote the subspace of elements of degree p with $W^p(\mathfrak{g})$) and $w^q \in W^q(\mathfrak{g})$, freely generated by the generators C^a of degree 1 and u^a of degree 2:

$$W(\mathfrak{g}) \cong \mathbb{R}[C; u]. \tag{6.18}$$

In this way, the Weil algebra takes on the structure of a supercommutative superalgebra (see Definition 2.5) with

$$W(\mathfrak{g}) = W^{\text{odd}}(\mathfrak{g}) \oplus W^{\text{even}}(\mathfrak{g}). \tag{6.19}$$

We would like to encode the relations (6.15) in the Weil algebra. Let $\{e_a\}$ be a basis for $\mathfrak{g}$ dual to $\{C^a\}$ and $\{u^a\}$ with the corresponding structure constants f^a_{bc}. We define a differential operator d_W on the generators of $W(\mathfrak{g})$ as

$$\begin{aligned} d_W C^a &= -\frac{1}{2} f^a_{bc} C^b C^c + u^a \\ d_W u^a &= -f^a_{bc} C^b u^c \end{aligned} \tag{6.20}$$

and extend it to all of $W(\mathfrak{g})$ as an anti-derivation

$$d_W(w^p w^q) = d_W w^p w^q + (-1)^p w^p d_W w^q. \tag{6.21}$$

The newly defined differential operator d_W is shown to be nilpotent via the Jacobi identity of the structure constants f^a_{bc}. We now define the action of the interior product ι_{e_a} and the Lie derivative $\mathcal{L}_X$ on $W(\mathfrak{g})$ as superderivations of degree -1 and 0 repectively, satisfying

$$\begin{aligned} \iota_{e_a} C^b &= \delta^b_a \\ \iota_{e_a} u^b &= 0 \\ \mathcal{L}_X &= \iota_X d_W + d_W \iota_X. \end{aligned} \tag{6.22}$$

In the future, we will use the notation $\iota_a := \iota_{e_a}$, and $\mathcal{L}_a := \mathcal{L}_{e_a}$. Just to to clarify, on $\Omega(M)$ the operator ι_a is $\iota_a = V^\mu \iota_\mu$ where $V^\mu \partial_\mu$ is the fundamental vector field corresponding to e^a (this carries over to $\mathcal{L}_a$ similarly). The cohomology of $W(\mathfrak{g})$ with respect to d_W yields [30, 31],

$$H^*_{d_W}(W(\mathfrak{g})) \cong \mathbb{R} \tag{6.23}$$

which is as we want since $W(\mathfrak{g})$ is expected to be a de Rham model for the contractible space EG (Poincaré Lemma).

What we wish to obtain is a de Rham model for $M_G = (M \times EG)/G$. It turns out that we obtain the correct results by using the basic subalgebra of $W(\mathfrak{g}) \otimes \Omega(M)$. This is where (6.22) comes in.

Definition 6.3. The complex

$$\begin{aligned} \Omega_G(M) &:= (W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}} \\ &= \left(\bigcap_{a=1}^{\dim(G)} \ker(\iota_a \otimes 1 + 1 \otimes \iota_a) \right) \cap \left(\bigcap_{b=1}^{\dim(G)} \ker(\mathcal{L}_b \otimes 1 + 1 \otimes \mathcal{L}_b) \right) \end{aligned} \tag{6.24}$$

is the differential graded algebra (see Definition 2.10) of *equivariant differential forms* on M under $d := d_W \otimes 1 + 1 \otimes d_M$, where d_M is the exterior derivative on $\Omega(M)$.

In order to make our notation easier, define $\iota_a := \iota_a \otimes 1 + 1 \otimes \iota_a$ and $\mathcal{L}_a := \mathcal{L}_a \otimes 1 + 1 \otimes \mathcal{L}_a$. Note that d, ι_a, and $\mathcal{L}_a$ act as graded derivations of degree 1, -1 and 0 respectively on $\Omega_G(M)$ in order to match with the algebraic structure of the space. Also, importantly, d squares to 0. One can check that a general equivariant differential form $\varphi \in \Omega_G(M)$ can be written in the form [32, 33]

$$\varphi = \prod_{a=1}^{n} (1 - C^a \iota_a)\alpha \tag{6.25}$$

where n is the number of generators in $\mathfrak{g}$ and $\alpha \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$. Here, we have denoted the infinitesimal G-invariant subalgebra of $S(\mathfrak{g}^*) \otimes \Omega(M)$ by an upper G.

It is on the space $\Omega_G(M)$ which the Weil model is defined. We will now sketch a motivation for why this is sensible, more or less taken straight from [32]. From the connection and curvature forms of P and the C^a and u^a, we get an induced homomorphism of differential algebras

$$W(\mathfrak{g}) \to \Omega(P). \tag{6.26}$$

From the way we have defined the exterior derivatives and interior products, this is indeed a homomorphism of differential algebras. Together with the lifting of differential forms on M to forms on $P \times M$, we receive the homomorphism

$$w : W(\mathfrak{g}) \otimes \Omega(M) \to \Omega(P \times M) \tag{6.27}$$

which in turn induces a homomorphism of subalgebras

$$\bar{w} : \Omega_G(M) \to \Omega((P \times M)/G). \tag{6.28}$$

In the literature, $\bar{w}$ is known as the *Chern-Weil homomorphism* [32]. This in turn induces a homomorphism on the level of cohomology

$$H_d^*(\Omega_G(M)) \to H_{dR}^*((P \times M)/G) \tag{6.29}$$

which, if G is compact, can be shown to be an isomorphism. With $P = EG$, we have thus found that

$$H_d^*(\Omega_G(M)) \cong H_{dR}^*(M_G) \cong H_G^*(M). \tag{6.30}$$

Details can be found in [30, 32].

6.2.2 The Cartan model

The Cartan model is built on another subalgebra of $W(\mathfrak{g}) \otimes \Omega(M)$, namely

$$(S(\mathfrak{g}^*) \otimes \Omega(M))^G \cong (W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}} = \Omega_G(M). \tag{6.31}$$

The algebra isomorphism is induced by the map $\varepsilon : W(\mathfrak{g}) \otimes \Omega(M) \to S(\mathfrak{g}^*) \otimes \Omega(M)$, defined by $\varepsilon(C^a) = 0$. In fact, as stated in [33], the map $(W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}} \to (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ given by

$$\prod_{a=1}^{n} (1 - C^a \iota_a)\alpha \mapsto \alpha \tag{6.32}$$

is an algebra isomorphism, known as the *Mathai-Quillen isomorphism.* We define the derivation D on $S(\mathfrak{g}^*) \otimes \Omega(M)$, for $\alpha \in \Omega(M)$, by

$$\begin{aligned} Du^a &= 0 \\ D\alpha &= d\alpha - u^a \iota_a \alpha = (1 \otimes d_M - u^a \otimes \iota_a)(1 \otimes \alpha) \end{aligned} \tag{6.33}$$

which is nilpotent on the subalgebra $(S(\mathfrak{g}^*) \otimes \Omega(M))^G$. In usual order, we extend it to the whole algebra as an anti-derivation. This makes the isomorphism into an isomorphism of differential algebras, proven in two different ways by Mathai and Quillen [32] and by Kalkman [31]. By defining d_M as an anti-derivation on $(S(\mathfrak{g}^*) \otimes \Omega(M))^G$ satisfying $d_M u^a := 0$, we are allowed to write

$$D = d_M - u^a \iota_a. \tag{6.34}$$

In this form, we check that indeed

$$D^2 = d_M^2 + u^a u^b \iota_a \iota_b - u^a \{d_M, \iota_a\} = -u^a \mathcal{L}_a = 0. \tag{6.35}$$

6.2.3 An example: the abelian case

We investigate the case $G = U(1)$, closely following the procedure of Atiyah and Bott [30] in the first part of the example. In the second part, we add some structures to the models in the form of inner products. Let us begin with the case when the Lie algebra is $\mathfrak{g} = \mathfrak{u}(1) \cong \mathbb{R}$ so we only have one generator which we will call e. All structure constants vanish, which means that

$$d_W C = u; \quad d_W u = 0. \tag{6.36}$$

Let C be the dual generator of $\mathfrak{u}(1)^*$, $\iota_e C = 1$. As before, let u be the generator of $S(\mathfrak{g}^*)$. Then $W(\mathfrak{g}) \cong \mathbb{R}[C, u]$ so we have that $\varphi \in W(\mathfrak{g}) \otimes \Omega(M)$ decomposes as the finite sum

$$\varphi = a_k u^k + C b_l u^l \tag{6.37}$$

where $a_k, b_l \in \Omega(M)$. Furthermore, if φ is to be basic, we require that

$$\begin{aligned} \iota_e\varphi &= (\iota_e a_k)u^k + b_l u^l - C\iota_e b_l u^l = 0 \\ \mathcal{L}_e\varphi &= (\mathcal{L}_e a_k)u^k + C(\mathcal{L}_e(b_l))u^l = 0 \end{aligned} \tag{6.38}$$

where we have used that $L_eC = L_eu = 0$ which is implied by the vanishing of the structure constants. We get the equivalent conditions for φ to be basic

$$\begin{aligned} \mathcal{L}_e a_k &= 0 \\ b_k &= -\iota_e a_k. \end{aligned} \tag{6.39}$$

In other words, the elements in $(W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}}$ will be of the form

$$\varphi = a_k u^k - C\iota_e \alpha_l u^l = (1 - C\iota_e)a_k u^k \tag{6.40}$$

with the condition $\mathcal{L}_e a_k = 0$, i.e. $a_k \in \Omega(M)^G$.

Consider the algebra of $U(1)$-invariant differential forms $\Omega(M)^G = \ker(\mathcal{L}_e)$. Then let $\Omega(M)^G[u]$ be the polynomial ring generated by u with coefficients in $\Omega(M)^G$. Note that the fact that $L_eu = 0$ means that $S(\mathfrak{g}^*) = S(\mathfrak{g}^*)^G$. In other words, $L_e\alpha = 0$ for all $\alpha \in \Omega(M)^G[u]$. Hence $\Omega(M)^G[u] \cong (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ which is nothing but the space for the Cartan model. Inspired by (6.40), define the algebra homomorphism

$$j : \Omega(M)^G[u] \to W(\mathfrak{g}) \otimes \Omega(M) \tag{6.41}$$

given by

$$j = 1 - C\iota_e \tag{6.42}$$

or equivalently, for $a \in \Omega(M)^G$,

$$\begin{aligned} j(a) &= a - C\iota_e a \\ j(u) &= u. \end{aligned} \tag{6.43}$$

By the conditions (6.39) for an element in $W(\mathfrak{g}) \otimes \Omega(M)$ to be basic, the homomorphism

$$j : \Omega(M)^G[u] \xrightarrow{\sim} (W(\mathfrak{g}) \otimes \Omega(M))_{\mathrm{bas}} = \Omega_G(M) \tag{6.44}$$

is an algebra isomorphism. To make this into an isomorphism of differential algebras, we require j to be a chain map, i.e.

$$jD = dj \tag{6.45}$$

where d is the differential operator on $\Omega_G(M)$ (given in definition 6.3 with d_W satisfying (6.36)) and D is the differential operator on $\Omega(M)^G[u]$, to be found. We now have

$$\begin{aligned} dj(a) &= d(a - C\iota_e a) \\ &= d_M a - d_W C\iota_e a + C d_M \iota_e a \\ &= d_M a - u\iota_e a - C\iota_e d_M a \\ &= j(d_M a) - j(u\iota_e a) \\ &= j(d_M a - u\iota_e a) = j(d_M a - u\iota_e a) \end{aligned} \tag{6.46}$$

where we on the third equality used (6.14) and (6.39), and on the fourth the fact that the interior product squares to zero. Hence we obtain

$$Da = d_M a - u\iota_e a. \tag{6.47}$$

Since $du = d_W u = 0$ (i.e. u is closed) we see that

$$Du = 0. \tag{6.48}$$

We have thus obtained the differential D on $\Omega(M)^G[u]$, which is nothing but the differential defined by (6.33). This example is just a special case of the proof provided by Mathai and Quillen [32] for the equivalence of the Weil model and the Cartan model for a general compact and connected Lie group.

The space $\Omega(M)^G[u]_{\mathbb{C}} \cong \Omega(M)^G_{\mathbb{C}}[u]$ can be endowed an L^2-inner product in the case when M is compact and oriented. Fix a $X = se \in \mathfrak{u}(1)$ with $s \in \mathbb{R} \setminus \{0\}$. Let $\beta, \beta' \in \Omega(M)^G[u]_{\mathbb{C}}$ with $\beta = \beta_k u^k$ and $\beta' = \beta'_k u^k$ where $\beta_k, \beta'_k \in \Omega(M)_{\mathbb{C}}$. Note that we may define an inner product on $\mathbb{R}[u]$ which on homogeneous elements is given by

$$\langle u^k, u^l \rangle = l!\delta^{kl} \tag{6.49}$$

and then extending it linearly to the whole algebra. This bilinear map is symmetric since δ^{kl} vanishes unless $k = l$. Under this inner product, the adjoint operation $u^\dagger$ to multiplying with u, is found through

$$\langle uu^k, u^l \rangle = l!\delta^{k+1,l} = l(l-1)!\delta^{k,l-1} = \langle u^k, lu^{l-1} \rangle \tag{6.50}$$

thence

$$u^\dagger = \frac{\partial}{\partial u} \tag{6.51}$$

where

$$\frac{\partial}{\partial u} u^k = ku^{k-1}. \tag{6.52}$$

Note that this works for the case when $l = 0$ as well since $k + 1 > 0$ for which $\delta^{k+1,l} = 0$. We may extend this to an Hermitian inner product on $\Omega(M)^G_{\mathbb{C}}[u]$ as

$$\langle \beta, \beta' \rangle = \langle \beta_k u^k, \beta'_l u^l \rangle := (\beta_k, \beta'_l)\langle u^k, u^l \rangle \tag{6.53}$$

where $(,)$ is the regular inner product (5.149) on $\Omega(M)_{\mathbb{C}}$

$$(\omega, \eta) = \sum_k \int_M \bar{\omega}_k \wedge \star\eta_k. \tag{5.149}$$

By furthermore assuming that M has no boundary (M is hence assumed to be closed and oriented), the usual adjoint $d^\dagger$ to the exterior derivative d is defined. In this setting we may find the adjoint to D, which we denote by $D^\dagger$, using (6.53). Differentiating β,

$$D\beta = d_M \beta_k u^k - \iota_e \beta_k u^{k+1} \tag{6.54}$$

we get

$$\begin{aligned} \langle D\beta, \beta' \rangle &= (d_M\beta, \beta_l)\langle u^k, u^l \rangle - (\iota_e\beta_k, \beta_l)\langle u^{k+1}, u^l \rangle \\ &= (\beta_k, d_M^\dagger \beta_l)\langle u^k, u^l \rangle - (\beta_k, \iota_e^\dagger \beta_l)\langle u^k, \frac{\partial}{\partial u} u^l \rangle \\ &= \langle \beta, d_M^\dagger \beta'_l u^l - \iota_e^\dagger \beta'_l \frac{\partial}{\partial u} u^l \rangle. \end{aligned} \tag{6.55}$$

The operator $d_M^\dagger$ is just the co-differential from Hodge theory and $\iota_e^\dagger = \varepsilon^\# \wedge$ where, locally, $\varepsilon^\# = V_\mu dx^\mu$ such that $e^\# = V^\mu \partial_\mu$ is the fundamental vector field generated by e. By setting the relations

$$\frac{\partial}{\partial u} \alpha = 0 \tag{6.56}$$

$$d_M^\dagger u = 0 \tag{6.57}$$

$$\iota_e^\dagger u = 0 \tag{6.58}$$

with $\alpha \in \Omega(M)^G_{\mathbb{C}}$, the adjoint of D

$$D^\dagger = d_M^\dagger - \frac{\partial}{\partial u} \iota_e^\dagger \tag{6.59}$$

is thus found. As we have hoped for, the adjoint $D^\dagger$ preserves the $\mathbb{Z}$-grading of $\Omega(M)^G_\mathbb{C}$ and maps the subalgebra to itself. Also, $D^\dagger$ is nilpotent,

$$(D^\dagger)^2 = (d^\dagger)^2 + (\frac{\partial}{\partial u})^2(\iota^\dagger)^2 - \frac{\partial}{\partial u}\{d^\dagger, \iota^\dagger\} = -\frac{\partial}{\partial u}\mathcal{L}_e^\dagger = 0 \tag{6.60}$$

on the invariant subalgebra, which is precisely what we want.

The case $G = U(1)$ extends naturally to the case when $G = T^n = U(1)^n$ is a rank n torus. What needs to be done then is, for a chosen basis $\{e_a\}$ of its Lie algebra and dual basis $\{u^a\}$, to extend $\Omega(M)^G_\mathbb{C}[u] \cong (S(\mathfrak{g}^*) \otimes \Omega(M))^G_\mathbb{C}$ so that it has indeterminates $\{u^a\}$ ($L_a u^b = 0$ holds for this case as well since the structure constants remain zero). The corresponding derivative D on this space is given by

$$D = d_M - u^a \iota_a. \tag{6.61}$$

The inner product on $\mathbb{C}[u^a]$ is similarly constructed by defining on homogeneous elements

$$\left\langle \prod_{i=1}^n (u^{a_i})^{k_i}, \prod_{j=1}^n (u^{a_j})^{l_j} \right\rangle_u := \prod_{i=1}^n \langle (u^{a_i})^{k_i}, (u^{a_i})^{l_i}\rangle_u := \prod_{i=1}^n l_i! \delta^{k_i l_i}. \tag{6.62}$$

and then extended linearly to the entirety of $\mathbb{C}[u^a]$. To find the adjoint to u^c, assume that $c = a_r$ for some $r \in \{1, \dots, n\}$. It is sufficient to derive it for homogeneous elements as they constitute a basis for $\mathbb{C}[u^a]$. Then

$$\begin{aligned}
\left\langle u^c \prod_{i=1}^n (u^{a_i})^{k_i}, \prod_{j=1}^n (u^{a_j})^{l_j} \right\rangle_u &= \left\langle (u^{a_r})^{k_r+1} \prod_{\substack{i=1\\ i\neq r}}^n (u^{a_i})^{k_i}, \prod_{j=1}^n (u^{a_j})^{l_j} \right\rangle_u \\
&= \langle (u^{a_r})^{k_r+1}, (u^{a_r})^{l_r}\rangle_u \prod_{\substack{i=1\\ i\neq r}}^n \langle (u^{a_i})^{k_i}, (u^{a_i})^{l_i}\rangle_u \\
&= \langle (u^{a_r})^{k_r}, l_r (u^{a_r})^{l_r-1}\rangle_u \prod_{\substack{i=1\\ i\neq r}}^n \langle (u^{a_i})^{k_i}, (u^{a_i})^{l_i}\rangle_u \\
&= \left\langle \prod_{i=1}^n (u^{a_i})^{k_i}, l_r (u^{a_r})^{l_r-1} \prod_{\substack{j=1\\ j\neq r}}^n (u^{a_j})^{l_j} \right\rangle_u
\end{aligned} \tag{6.63}$$

where we on the third equality used the same computation as in (6.50). We find that

$$(u^a)^\dagger = \frac{\partial}{\partial u^a} \tag{6.64}$$

where

$$\frac{\partial}{\partial u^a}(u^b)^k = \delta^{ab} k (u^b)^{k-1}. \tag{6.65}$$

From this derivation, observe that $\frac{\partial}{\partial u^c}$ picks out only the factor with $a_r = c$. Since all the factors commute, we may conclude that $\frac{\partial}{\partial u^c}$ commutes with everything except for the case $a_r = c$. Moreover, due to the fact that u^c acts linearly, $\frac{\partial}{\partial u^c}$ is a linear operator and thus a (degree -2) derivation. Following the previous procedure, we define the inner product on $\Omega(M)^G_\mathbb{C}[u^a]$ via, with $\beta, \beta' \in \Omega(M)^G_\mathbb{C}[u^a]$,

$$\begin{aligned}
\langle \beta, \beta' \rangle &= \langle \beta_{a_1 \dots a_n k_1 \dots k_n} (u^{a_n})^{k_n} \dots (u^{a_n})^{k_n}, \beta'_{a_1 \dots a_n l_1 \dots l_n} (u^{a_n})^{k_n} \dots (u^{a_n})^{l_n} \rangle \\
&:= (\beta_{a_1 \dots a_n k_1 \dots k_n}, \beta'_{a_1 \dots a_n l_1 \dots l_n}) \langle (u^{a_n})^{k_n} \dots (u^{a_n})^{k_n}, (u^{a_n})^{k_n} \dots (u^{a_n})^{l_n} \rangle_u
\end{aligned} \tag{6.66}$$

From there, the adjoint to D can be found as

$$D^\dagger = d_M^\dagger - \frac{\partial}{\partial u^a}\iota_a^\dagger \tag{6.67}$$

where we define $\iota_a^\dagger = \iota_{e^a}^\dagger$ analogously to $\iota_e^\dagger$. We have thus the case when G is a compact connected abelian Lie group covered.

Let us now repeat the analysis, but in the framework of the Weil model. Inspired by [34] and looking at (6.20), we see that we can write the differential in the toroidal case $G = U(1)^n$ as

$$d = d_M + u^a \frac{\partial}{\partial C^a}. \tag{6.68}$$

We may construct an inner product in $\Lambda(\mathfrak{g}^*)$ as in (6.62) by defining on homogeneous elements

$$\left\langle \prod_{i=1}^n (C^{a_i})^{k_i}, \prod_{j=1}^n (C^{a_j})^{l_j} \right\rangle_C := \prod_{i=1}^n \langle (C^{a_i})^{k_i}, (C^{a_i})^{l_i} \rangle_C := \prod_{i=1}^n \delta^{k_i l_i} \tag{6.69}$$

where k_i and l_i are either 0 or 1, and then extending it by linearity to the entire algebra. By a similar computation as before, one shows that the adjoint to C^a is the degree -1 differential

$$C^{a\dagger} := \frac{\partial}{\partial C^a} \tag{6.70}$$

which satisfies

$$\frac{\partial}{\partial C^a}(C^b)^k = k\delta^{ab}. \tag{6.71}$$

An L_2-product on $\Omega_G(M)$ can now be constructed as

$$\langle , \rangle := (,)\langle , \rangle_u \langle , \rangle_C. \tag{6.72}$$

Using this construction, the adjoint to d will be given by

$$d^\dagger = d_M^\dagger + C^a \frac{\partial}{\partial u^a}. \tag{6.73}$$

This operator preserves the $\mathbb{Z}$-grading of $\Omega_G(M)$ and is nilpotent.

Let us conclude this part by having a brief look at the (co)differential in the non-abelian case as well for the Weil model, inspired by above discussions. By investigating (6.20) and (6.68), we see that

$$d = d_M + u^a \frac{\partial}{\partial C^a} - \frac{1}{2} f_{bc}^d C^b C^c \frac{\partial}{\partial C^d} + f_{bc}^d u^b C^c \frac{\partial}{\partial u^d}. \tag{6.74}$$

From what we learnt before, the adjoint is then given by

$$d^\dagger = d_M^\dagger + C^a \frac{\partial}{\partial u^a} + \frac{1}{2} f_{bc}^d C^d \frac{\partial}{\partial C^b}\frac{\partial}{\partial C^c} + f_{bc}^d u^d \frac{\partial}{\partial C^c}\frac{\partial}{\partial u^b}. \tag{6.75}$$

6.3 Supersymmetric Quantum Mechanics from Equivariant Cohomology

We would now like to construct quantum mechanics using both the Weil model and the Cartan model of the equivariant theory. The goal is to see the connection between what arises from the respective models and relate them to gauge theory. As the equivariant cohomology of these models computes the de Rham cohomology for the orbit space of a group action of a (compact and connected) Lie group, we expect the obtained quantum mechanics to be gauged with the given Lie group as gauge group.

6.3.1 Gauged $\mathcal{N}=2$ non-linear sigma model

Let us first investigate the simplest case without supersymmetry

$$L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu \tag{6.76}$$

together with the transformation

$$\delta_a \phi^\mu = a V^\mu(\phi) \tag{6.77}$$

where a is infinitesimal and V is a vector field. Varying the action with respect to this transformation, one gets

$$\delta_a S = \int dt \frac{1}{2} a (\partial_\rho g_{\mu\nu} V^\rho + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho) \dot{\phi}^\mu \dot{\phi}^\nu. \tag{6.78}$$

The action is invariant if and only if

$$\partial_\rho g_{\mu\nu} V^\rho + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho = \mathcal{L}_V g = 0 \tag{6.79}$$

which we recognize as the Killing equation. Rephrased, the action with Lagrangian (6.76) is invariant under (6.77) if and only if V is a Killing vector field, i.e. δ_a is an (infinitesimal) isometry. From now on, we assume that V is such a vector field. Let us as a next step make the variation local by letting $a = a(t)$. In such case, the original action will no longer be invariant under the transformation (6.77). For things to work out, we need to introduce a *gauge field* $A = A(t)$ and modify the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} (\dot{\phi}^\mu + A V^\mu)(\dot{\phi}^\nu + A V^\nu) \tag{6.80}$$

for which

$$\delta_a S = \int dt g_{\mu\nu} (\dot{a} + \delta_a A) V^\mu (\dot{\phi}^\nu + A V^\nu) \tag{6.81}$$

so the action is invariant provided

$$\delta_a A = -\dot{a}. \tag{6.82}$$

Let us now return to the case which is interesting for us, namely the $\mathcal{N}=2$ non-linear sigma model[19]. Recall that the Lagrangian is given by

$$L = \frac{1}{2} g_{\mu\nu} \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu} (\bar{\psi}^\mu D_t \psi^\nu - D_t \bar{\psi}^\mu \psi^\nu) + \frac{1}{2} R_{\mu\nu\rho\sigma} \bar{\psi}^\mu \psi^\nu \bar{\psi}^\rho \psi^\sigma \tag{5.104}$$

where $D_t \psi^\mu = \dot{\psi}^\mu + \dot{\phi}^\rho \Gamma^\mu_{\rho\sigma} \psi^\sigma$ is the covariant derivative along the path $\phi(t)$. The action is invariant under the transformation

$$\delta_a \phi^\mu = a V^\mu \tag{6.83}$$
$$\delta_a \psi^\mu = a \partial_\lambda V^\mu \psi^\lambda \tag{6.84}$$
$$\delta_a \bar{\psi}^\mu = a \partial_\lambda V^\mu \bar{\psi}^\lambda. \tag{6.85}$$

This is verified after some calculation, using that V is a Killing field. This also means that the Lie derivative of the Riemann tensor with respect to this vector field vanishes, $\mathcal{L}_V R = 0$. Following the same procedure as above, make the parameter a local by adding a t-dependence and introduce a gauge field $A(t)$. The Lagrangian needs to be modified by replacing

$$\dot{\phi}^\mu \to \tilde{\dot{\phi}}^\mu = \dot{\phi}^\mu + A V^\mu \tag{6.86}$$
$$\dot{\psi}^\mu \to \tilde{\dot{\psi}}^\mu = \dot{\psi}^\mu + A \partial_\lambda V^\mu \psi^\lambda \tag{6.87}$$
$$\dot{\bar{\psi}}^\mu \to \tilde{\dot{\bar{\psi}}}^\mu = \dot{\bar{\psi}}^\mu + A \partial_\lambda V^\mu \bar{\psi}^\lambda \tag{6.88}$$

[19]The following could be modified to work for the $\mathcal{N}=1$ case as well. We will not do it here since it is not really relevant for what we are considering later.

so

$$L = \frac{1}{2} g_{\mu\nu} \tilde{\phi}^\mu \tilde{\phi}^\nu + \frac{i}{2} g_{\mu\nu} \left[\bar{\psi}^\mu (\tilde{\psi}^\nu + \tilde{\phi}^\rho \Gamma^\nu_{\rho\sigma} \psi^\sigma) - (\tilde{\bar{\psi}}^\mu + \tilde{\phi}^\rho \Gamma^\mu_{\rho\sigma} \bar{\psi}^\sigma) \psi^\nu \right] + \frac{1}{2} R_{\mu\nu\rho\sigma} \bar{\psi}^\mu \psi^\nu \bar{\psi}^\rho \psi^\sigma. \tag{6.89}$$

Observe that

$$\delta_a \tilde{\phi}^\mu = a \partial_\lambda V^\mu \tilde{\phi}^\lambda + (\dot{a} + \delta_a A) V^\mu \tag{6.90}$$

$$\delta_a \tilde{\psi}^\mu = a \partial_\lambda V^\mu \tilde{\psi}^\lambda + a \partial_\lambda \partial_\rho V^\mu \tilde{\phi}^\lambda \psi^\rho + (\dot{a} + \delta_a A) \partial_\lambda V^\mu \psi^\lambda \tag{6.91}$$

$$\delta_a \tilde{\bar{\psi}}^\mu = a \partial_\lambda V^\mu \tilde{\bar{\psi}}^\lambda + a \partial_\lambda \partial_\rho V^\mu \tilde{\phi}^\lambda \bar{\psi}^\rho + (\dot{a} + \delta_a A) \partial_\lambda V^\mu \bar{\psi}^\lambda \tag{6.92}$$

so by comparing with what we would have in the case when a is a global parameter,

$$\delta_a \dot{\phi}^\mu = a \partial_\lambda V^\mu \dot{\phi}^\lambda \tag{6.93}$$

$$\delta_a \dot{\psi}^\mu = a \partial_\lambda V^\mu \dot{\psi}^\lambda + a \partial_\lambda \partial_\rho V^\mu \dot{\phi}^\lambda \psi^\rho \tag{6.94}$$

$$\delta_a \dot{\bar{\psi}}^\mu = a \partial_\lambda V^\mu \dot{\bar{\psi}}^\lambda + a \partial_\lambda \partial_\rho V^\mu \dot{\phi}^\lambda \bar{\psi}^\rho \tag{6.95}$$

we see that the variation of the action is

$$\begin{aligned} \delta_a S = \int dt \Big\{ & g_{\mu\nu} (\dot{a} + \delta_a A) V^\mu \tilde{\phi}^\nu + \frac{i}{2} g_{\mu\nu} [\bar{\psi}^\mu (\dot{a} + \delta_a A) \partial_\lambda V^\nu \psi^\lambda + \bar{\psi}^\mu (\dot{a} + \delta_a A) V^\rho \Gamma^\nu_{\rho\sigma} \psi^\sigma \\ & - (\dot{a} + \delta_a A) \partial_\lambda V^\mu \bar{\psi}^\lambda \psi^\nu - (\dot{a} + \delta_a A) V^\rho \Gamma^\mu_{\rho\sigma} \bar{\psi}^\sigma \psi^\nu] \Big\}. \end{aligned} \tag{6.96}$$

Again, the variation vanishes provided

$$\delta_a A = -\dot{a}. \tag{6.97}$$

Given this result, the model is indeed invariant under local gauge transformation, corresponding to local isometry transformations which means the isometry group is identified with the gauge group.

6.3.2 Supersymmetric quantum mechanics in the Cartan model - the abelian case

We start with the simplest example to illustrate the process of deriving the quantum mechanics corresponding to the different models. We will begin by investigating the case of the Cartan model. Recall that in section 5.2, the supercharges where given by the de Rham differential and the co-differential. Mimicking that fact, define supercharges

$$Q := D = d_M - u^a \iota_a \tag{6.98}$$

$$Q^\dagger := D^\dagger = d_M^\dagger - \frac{\partial}{\partial u^a} \iota_a^\dagger. \tag{6.99}$$

Recall that in our notation $\iota_a = V^\mu \iota_\mu$ and $\iota_a^\dagger = V_\mu dx^\mu \wedge$. In order to determine the structure of the system we make use of the (anti-)commutators

$$[d_M, f(x)] = (\nabla_\mu f) \iota^{\dagger\mu} \tag{6.100}$$

$$[d_M^\dagger, f(x)] = -(\nabla_\mu f) \iota^\mu \tag{6.101}$$

$$\{d_M, \iota_X^\dagger\} = (\nabla_\mu X_\rho) \iota^{\dagger\mu} \iota^{\dagger\rho} \tag{6.102}$$

$$\{d_M^\dagger, \iota_X\} = -(\nabla_\mu X_\rho) \iota^\mu \iota^\rho \tag{6.103}$$

$$\{\iota_X, \iota_Y^\dagger\} = g_{\mu\nu} X^\mu Y^\nu \tag{6.104}$$

$$\left[\frac{\partial}{\partial u^a}, u^b \right] = \delta^{ab} \tag{6.105}$$

for some function f and vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$. Relations (6.100), (6.102) and (6.104) can be derived by using the fact that d_M and ι_X are anti-derivations. For instance, with $\alpha \in \Omega(M)_{\mathbb{C}}$ and recalling (5.158),

$$\begin{gathered} d_M(\iota_X^\dagger \alpha) = d_M(X_\mu dx^\mu \wedge \alpha) = d_M(X_\mu dx^\mu) \wedge \alpha - X_\mu dx^\mu \wedge d_M \alpha \\ \Longleftrightarrow \\ \{d_M, \iota_X^\dagger\}\alpha = d_M(\iota_X^\dagger \alpha) + X_\mu dx^\mu \wedge d_M \alpha = d_M(X_\mu dx^\mu) \wedge \alpha = (\nabla_\nu X_\mu) dx^\nu \wedge dx^\mu \wedge \alpha \end{gathered} \tag{6.106}$$

which precisely gives (6.102). Equations (6.101) and (6.103) are just adjoints to equations (6.100) and (6.102) respectively. As an example,

$$[d_M^\dagger, f(x)] = (-[d_M, f(x)])^\dagger = -(\nabla_\mu f)\iota^\mu \tag{6.107}$$

which is (6.101). The commutator (6.105) is also derived by using that $\dfrac{\partial}{\partial u^a}$ is a derivation. Take an element $\beta \in \Omega(M)_{\mathbb{C}}^G[u](M)$. Then

$$\frac{\partial}{\partial u^a}(u^b \beta) = \frac{\partial u^b}{\partial u^a}\beta + u^b \frac{\partial}{\partial u^a}\beta \iff \left[\frac{\partial}{\partial u^a}, u^b\right]\beta = \frac{\partial u^b}{\partial u^a}\beta = \delta^{ab}\beta \tag{6.108}$$

from which the commutator can be read off.

Using the commutation relations, we may derive the Hamiltonian

$$H = \frac{1}{2}\{D, D^\dagger\} = \frac{1}{2}\left[\Delta + (\nabla_\mu V_{a\nu})(u^a \iota^\mu \iota^\nu - \frac{\partial}{\partial u^a}\iota^{\dagger\mu}\iota^{\dagger\nu}) + V_a^\mu V_a^\nu \iota_\mu^\dagger \iota_\nu + g_{\mu\nu} V_a^\mu V_b^\nu u^a \frac{\partial}{\partial u^b}\right] \tag{6.109}$$

with classical counterpart (using (5.169))

$$\begin{aligned} H = \frac{1}{2}\Big[& g_{\mu\nu} P^\mu P^\nu - R_{\mu\nu\rho\sigma} \bar{\psi}^\mu \psi^\nu \bar{\psi}^\rho \psi^\sigma \\ & + \nabla_\mu V_{a\nu}(\bar{b}^a \psi^\mu \psi^\nu - b^a \bar{\psi}^\mu \bar{\psi}^\nu) + V_{a\mu} V_{a\nu} \bar{\psi}^\mu \psi^\nu + g_{\mu\nu} V_a^\mu V_b^\nu \bar{b}^a b^b \Big] \end{aligned} \tag{6.110}$$

where we have defined fields corresponding to u^a and $\dfrac{\partial}{\partial u^a}$

$$\bar{b}^a \longleftrightarrow u^a \tag{6.111}$$

$$b^a \longleftrightarrow \frac{\partial}{\partial u^a}. \tag{6.112}$$

Continuing to utilize the commutators, we can compute the infinitesimal transformations of the fields by

$$\delta O = [\epsilon Q + \bar{\epsilon} Q^\dagger, O] = [\epsilon d_M + \bar{\epsilon} d_M^\dagger, O] - [\epsilon u^a \iota_a + \bar{\epsilon}\frac{\partial}{\partial u^a}\iota_a^\dagger, O] \tag{6.113}$$

where the brackets above stand for the super commutator (see (2.15)) and O is some operator. The infinitesimal transformations of the fields take the form

$$\delta\phi^\mu = \epsilon\bar{\psi}^\mu - \bar{\epsilon}\psi^\mu \tag{6.114}$$

$$\delta\psi^\mu = \epsilon(i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) - \bar{\epsilon} V_a^\mu b^a \tag{6.115}$$

$$\delta\bar{\psi}^\mu = \bar{\epsilon}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) - \epsilon V_a^\mu \bar{b}^a \tag{6.116}$$

$$\delta\bar{b}^a = -\bar{\epsilon} V_{a\mu}\bar{\psi}^\mu \tag{6.117}$$

$$\delta b^a = \epsilon V_{a\mu}\psi^\mu \tag{6.118}$$

after translating back to the classical case. Note that the first term in (6.113) gives rise to the infinitesimal supersymmetry transformations in Section 5.2 for ϕ^μ, ψ^μ and $\bar{\psi}^\mu$.

Recall that in the Cartan model, we are working with invariant forms. In other words, we require that the Lie derivative along the direction of the group action to vanish,

$$\mathcal{L}_a = \{d_M, \iota_a\} = 0. \tag{6.119}$$

Note that in particular $\mathcal{L}_a g = 0$, which means the fundamental vector fields V_a are necessarily Killing vector fields. Observe that $\{d_M, \iota^\dagger_\mu\} = 0 = \{d^\dagger_M, \iota_\mu\}$. Hence

$$\{d_M, \iota_a\} = V_a^\mu \{d_M, \iota_\mu\} + (\nabla_\mu V_{a\nu})\iota^{\dagger\mu}\iota^\nu = \frac{1}{\epsilon} V_{a\mu}\delta_{dR}(\iota^\mu) + (\nabla_\mu V_{a\nu})\iota^{\dagger\mu}\iota^\nu \tag{6.120}$$

where δ_{dR} is the SUSY variation in Section 5.2; we have that $\delta_{dR}(g^{\mu\nu}\iota_\nu) = \{\epsilon d_M + \bar{\epsilon} d^\dagger_M, g^{\mu\nu}\iota_\nu\}$. Classically, this corresponds to $\delta_{dR}\psi^\mu$. Note that division by Grassmann valued numbers are generally not well defined, in this case this just means that we remove ϵ from the term. Hence

$$V_a^\lambda \nabla_\lambda - V_{a\lambda}\Gamma^\lambda_{\rho\sigma}\iota^{\dagger\rho}\iota^\sigma + (\nabla_\mu V_{a\nu})\iota^{\dagger\mu}\iota^\nu = 0 \tag{6.121}$$

or classically,

$$V_{a\lambda}(i\dot{\phi}^\lambda - \Gamma^\lambda_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) + (\nabla_\mu V_{a\nu})\bar{\psi}^\mu\psi^\nu = 0. \tag{6.122}$$

This could also have been done using the commutation relations. We may derive more constraints from the vanishing of the Lie derivative. If $\mathcal{L}_a = 0$, then $\delta\mathcal{L}_a$ must also vanish. We have that

$$\delta\mathcal{L}_a = [\epsilon D + \bar{\epsilon} D^\dagger, \mathcal{L}_a] = \epsilon[d_M, \mathcal{L}_a] + \bar{\epsilon}[d^\dagger_M, \mathcal{L}_a] - \epsilon u^b[\iota_b, \mathcal{L}_a] + \bar{\epsilon}\frac{\partial}{\partial u^b}[\iota^\dagger_b, \mathcal{L}_a]. \tag{6.123}$$

The first vanishes automatically. The second term vanishes thanks to V_a being Killing vector fields. Let us calculate the third and fourth commutator. The third commutator is proportional to

$$[\mathcal{L}_a, \iota_b] = [d_M\iota_a, \iota_b] + [\iota_a d_M, \iota_b]. \tag{6.124}$$

Taking the first term,

$$[d_M\iota_a, \iota_b] = d_M\{\iota_a, \iota_b\} - \{d_M, \iota_b\}\iota_a = 0. \tag{6.125}$$

The first anticommutator vanishes directly and the second one vanishes by the constraint (6.119). Similarly, $[\iota_a d_M, \iota_b] = 0$ and hence

$$[\mathcal{L}_a, \iota_b] = 0. \tag{6.126}$$

For the fourth term, let $\omega \in \Omega(M)$. Since $\mathcal{L}_a$ is a derivation (also, recall the notation $\iota^\dagger_a = V_{a\mu}\iota^{\dagger\mu} = V_{a\mu}dx^\mu\wedge$),

$$\mathcal{L}_a\iota^\dagger_d\omega = \mathcal{L}_a V_{b\mu}dx^\mu \wedge \omega = \mathcal{L}_a(V_{b\mu}dx^\mu)\wedge\omega + V_{b\mu}dx^\mu\wedge\mathcal{L}_a\omega \iff [\mathcal{L}_a, \iota^\dagger_b] = \mathcal{L}_a(V_{b\mu}dx^\mu)\wedge. \tag{6.127}$$

In other words

$$\begin{aligned}\mathcal{L}_a(V_{b\mu}dx^\mu) &= (d_M\iota_a + \iota_a d_M)(V_{b\mu}dx^\mu) = \nabla_\nu(V_a^\mu V_{b\mu})dx^\nu + V_a^\nu\nabla_\nu V_{b\mu}dx^\mu - V_a^\mu\nabla_\mu V_{b\nu}dx^\mu \\ &= V_b^\nu\nabla_\mu V_{a\nu}dx^\mu + V_a^\nu\nabla_\nu V_{b\mu}dx^\mu = (V_b^\nu\nabla_\mu V_{a\nu} - V_a^\nu\nabla_\mu V_{b\nu})dx^\mu\end{aligned} \tag{6.128}$$

where we on the last equation used the Killing equation $\nabla_\mu V_{b\nu} + \nabla_\nu V_{b\mu} = 0$. Hence

$$0 = \delta\mathcal{L}_a = \bar{\epsilon}(V_b^\nu\nabla_\mu V_{a\nu} - V_a^\nu\nabla_\mu V_{b\nu})\frac{\partial}{\partial u^b}\iota^{\dagger\mu} \tag{6.129}$$

which classically corresponds to

$$(V_b^\nu\nabla_\mu V_{a\nu} - V_a^\nu\nabla_\mu V_{b\nu})b^b\psi^\mu = 0. \tag{6.130}$$

Note that we need to enforce these constraints by hand since $Q = D$ is not automatically nilpotent (recall (4.19) which requires this), but only on the invariant subalgebra.

Taking the conjugate momenta to b^a and $\bar{b}^a$ to be

$$P_{b^a} = \frac{i}{2}\bar{b}^a \tag{6.131}$$

$$P_{\bar{b}^a} = -\frac{i}{2}b^a \tag{6.132}$$

the Lagrangian takes the form

$$L = L_{dR} + L_C \tag{6.133}$$

where

$$L_{dR} := \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu\dot{\phi}^\nu + \frac{i}{2}g_{\mu\nu}(\bar{\psi}^\mu D_t\psi^\nu - D_t\bar{\psi}^\mu\psi^\nu) + \frac{1}{2}R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma \tag{6.134}$$

is just the $\mathcal{N} = 2$ non-linear sigma model Lagrangian given in (5.104) and

$$L_C := \frac{i}{2}\bar{b}^a\dot{b}^a - \frac{i}{2}\dot{\bar{b}}^a b^a - \nabla_\mu V_{a\nu}(\bar{b}^a\psi^\mu\psi^\nu - b^a\bar{\psi}^\mu\bar{\psi}^\nu) - V_{a\mu}V_{a\nu}\bar{\psi}^\mu\psi^\nu - g_{\mu\nu}V_a^\mu V_b^\nu\bar{b}^a b^b. \tag{6.135}$$

This Lagrangian is proven to be a total derivative under the infinitesimal SUSY transformation δ

$$\delta L = \epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu), \tag{6.136}$$

given the constraints (6.122) and (6.130). As a verification, let us go through the Noether procedure to obtain the supercharges

$$\begin{aligned}
\delta_\epsilon S = \int_0^\beta dt \Bigg\{ &\frac{1}{2}g_{\mu\nu}(\dot{\epsilon}\bar{\psi}^\mu - \dot{\bar{\epsilon}}\psi^\mu)\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu(\dot{\epsilon}\bar{\psi}^\nu - \dot{\bar{\epsilon}}\psi^\nu) \\
&+ \frac{i}{2}g_{\mu\nu}[\bar{\psi}^\mu(\dot{\epsilon}(i\dot{\phi}^\nu - \Gamma^\nu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) - \dot{\bar{\epsilon}}V_a^\nu b^a) + \bar{\psi}^\mu(\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\nu_{\rho\sigma}\psi^\sigma \\
&\qquad - (\dot{\bar{\epsilon}}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) - \dot{\epsilon}V_a^\mu\bar{b}^a)\psi^\nu - (\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\mu_{\rho\sigma}\bar{\psi}^\sigma\psi^\nu] \\
&+ \frac{i}{2}\bar{b}^a\dot{\epsilon}V_{a\mu}\psi^\mu + \frac{i}{2}\dot{\bar{\epsilon}}V_{a\mu}\bar{\psi}^\mu b^a \\
&+ \epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu)\Bigg\} \\
= \int_0^\beta dt \Bigg\{ &\frac{1}{2}g_{\mu\nu}(\dot{\epsilon}\bar{\psi}^\mu - \dot{\bar{\epsilon}}\psi^\mu)\dot{\phi}^\nu + \frac{1}{2}g_{\mu\nu}\dot{\phi}^\mu(\dot{\epsilon}\bar{\psi}^\nu - \dot{\bar{\epsilon}}\psi^\nu) \\
&+ \frac{i}{2}g_{\mu\nu}[\bar{\psi}^\mu\dot{\epsilon}(i\dot{\phi}^\nu - \Gamma^\nu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) + \bar{\psi}^\mu(\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\nu_{\rho\sigma}\psi^\sigma \\
&\qquad - \dot{\bar{\epsilon}}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma)\psi^\nu - (\dot{\epsilon}\bar{\psi}^\rho - \dot{\bar{\epsilon}}\psi^\rho)\Gamma^\mu_{\rho\sigma}\bar{\psi}^\sigma\psi^\nu] \\
&+ \epsilon\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu) - \bar{\epsilon}\frac{1}{2}\frac{d}{dt}(g_{\mu\nu}\psi^\mu\dot{\phi}^\nu) \\
&+ i\dot{\epsilon}\bar{b}^a V_{a\mu}\psi^\mu + i\dot{\bar{\epsilon}}b^a V_{a\mu}\bar{\psi}^\mu\Bigg\} \\
= \int_0^\beta dt &\left(i\epsilon\frac{d}{dt}(ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu - \bar{b}^a V_{a\mu}\psi^\mu) + i\bar{\epsilon}\frac{d}{dt}(-ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu - b^a V_{a\mu}\bar{\psi}^\mu)\right)
\end{aligned} \tag{6.137}$$

Note that this is up to a few terms the same computation as in (5.109). The supercharges can thus be read off,

$$Q = ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu - \bar{b}^a V_{a\mu}\psi^\mu \tag{6.138}$$

$$Q^\dagger = -ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu - b^a V_{a\mu}\bar{\psi}^\mu \tag{6.139}$$

which after quantization exactly correspond to D and $D^\dagger$. However, it is at this point a bit unclear how to relate the obtained Lagrangian to the Lagrangian (6.89).

6.3.3 Supersymmetric quantum mechanics in the Weil model - the abelian case

In this case, following what we did before, the supercharges will be given by

$$Q := d = d_M + u^a \frac{\partial}{\partial C^a} \tag{6.140}$$

$$Q^\dagger := d^\dagger = d_M^\dagger + C^a \frac{\partial}{\partial u^a} \tag{6.141}$$

The Hilbert space is $\Omega_G(M) = (W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}}$ with inner product (6.72). The following commutation relations will be needed:

$$\left[\frac{\partial}{\partial u^a}, u^b\right] = \delta^{ab} \tag{6.142}$$

$$\left\{\frac{\partial}{\partial C^a}, C^b\right\} = \delta^{ab} \tag{6.143}$$

The commutation relation (6.142) is proved in the previous section and the proof for (6.143) is given in (3.73).

Using the commutation relations, the Hamiltonian corresponding to the supercharges can be computed,

$$H = \frac{1}{2}\{d, d^\dagger\} = \frac{1}{2}\left[\Delta + u^a \frac{\partial}{\partial u^a} + C^a \frac{\partial}{\partial C^a}\right]. \tag{6.144}$$

Note that the de Rham part is decoupled from u^a, $\frac{\partial}{\partial u^a}$, C^a and $\frac{\partial}{\partial C^a}$. We denote the classical counterparts by

$$\phi^\mu \longleftrightarrow x^\mu \times \tag{6.145}$$

$$\psi^\mu \longleftrightarrow g^{\mu\nu}\iota_{M\nu} \tag{6.146}$$

$$\bar{\psi}^\mu \longleftrightarrow g^{\mu\nu}\iota_{M\nu}^\dagger \tag{6.147}$$

$$\bar{b}^a \longleftrightarrow u^a \tag{6.148}$$

$$b^a \longleftrightarrow \frac{\partial}{\partial u^a} \tag{6.149}$$

$$\bar{c}^a \longleftrightarrow C^a \tag{6.150}$$

$$c^a \longleftrightarrow \frac{\partial}{\partial C^a}, \tag{6.151}$$

where $\iota_{M\mu}$ and $\iota_{M\mu}^\dagger$ are just the operators ι_μ and $\iota_\mu^\dagger$ acting only on the $\Omega(M)$ part of $(W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}}$ (and supercommuting with the rest of the elements). The corresponding classical Hamiltonian takes the form

$$H = \frac{1}{2}\left[g_{\mu\nu}P^\mu P^\nu - R_{\mu\nu\rho\sigma}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma + \bar{b}b + \bar{c}c\right]. \tag{6.152}$$

Continuing on as in the previous section, the infinitesimal SUSY transformation of the fields are given by

$$\delta\phi^\mu = \epsilon\bar{\psi}^\mu - \bar{\epsilon}\psi^\mu \tag{6.153}$$
$$\delta\psi^\mu = \epsilon(i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) \tag{6.154}$$
$$\delta\bar{\psi}^\mu = \bar{\epsilon}(-i\dot{\phi}^\mu - \Gamma^\mu_{\rho\sigma}\bar{\psi}^\rho\psi^\sigma) \tag{6.155}$$
$$\delta b^a = -\epsilon c^a \tag{6.156}$$
$$\delta\bar{b}^a = \bar{\epsilon}\bar{c}^a \tag{6.157}$$
$$\delta c^a = \bar{\epsilon} b^a \tag{6.158}$$
$$\delta\bar{c}^a = \epsilon\bar{b}^a \tag{6.159}$$

Note that the transformation of ϕ, ψ and $\bar{\psi}$ are as in 5.2, this reflects the fact that they are decoupled from the new fields. This also means that the conjugate momenta for the ϕ, ψ and $\bar{\psi}$ are the same as before. Taking the conjugate momenta to the new fields to be

$$P_{b^a} = \frac{i}{2}\bar{b}^a \tag{6.160}$$
$$P_{\bar{b}^a} = -\frac{i}{2}b^a \tag{6.161}$$
$$P_{c^a} = -\frac{i}{2}\bar{c}^a \tag{6.162}$$
$$P_{\bar{c}^a} = -\frac{i}{2}c^a, \tag{6.163}$$

the (classical) Lagrangian takes the form

$$L = L_{dR} + L_W \tag{6.164}$$

where

$$L_W = -\frac{1}{2}\bar{b}^a b^a - \frac{1}{2}\bar{c}^a c^a + \frac{i}{2}\bar{b}^a\dot{b}^a - \frac{i}{2}\dot{\bar{b}}^a b^a + \frac{i}{2}\bar{c}^a\dot{c}^a - \frac{i}{2}\dot{\bar{c}}^a c^a \tag{6.165}$$

and L_{dR} is given in (6.134). One can show that L_W is invariant under δ,

$$\delta L_W = 0. \tag{6.166}$$

Going through the Noether procedure, with $S_g := \int_0^\beta dt L_W$,

$$\begin{aligned}\delta S_g &= \int_0^\beta dt\left[-\frac{i}{2}\bar{b}^a\dot{\epsilon}c^a - \frac{i}{2}\dot{\bar{\epsilon}}\bar{c}^a b^a - \frac{i}{2}\dot{\epsilon}\bar{b}^a c^a + \frac{i}{2}\bar{c}^a\dot{\bar{\epsilon}}b^a\right]\\ &= \int_0^\beta dt[-i\dot{\epsilon}\bar{b}^a c^a - i\dot{\bar{\epsilon}}b^a\bar{c}^a]\\ &= \int_0^\beta dt\left[i\epsilon\frac{d}{dt}(\bar{b}^a c^a) + i\bar{\epsilon}\frac{d}{dt}(b^a\bar{c}^a)\right]\end{aligned} \tag{6.167}$$

where we integrated by parts in the third equality. This together with the variation of $S_{dR} := \int_0^\beta dt L_{dR}$, given in (5.109), gives us

$$\delta S = \int_0^\beta dt\left(i\epsilon\frac{d}{dt}(ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu + \bar{b}^a c^a) + i\bar{\epsilon}\frac{d}{dt}(-ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu + b^a\bar{c}^a)\right) \tag{6.168}$$

which gives us the charges

$$Q = ig_{\mu\nu}\bar{\psi}^\mu\dot{\phi}^\nu + \bar{b}^a c^a \tag{6.169}$$
$$Q^\dagger = -ig_{\mu\nu}\psi^\mu\dot{\phi}^\nu + b^a\bar{c}^a \tag{6.170}$$

which precisely correspond to d and $d^\dagger$. However, as in the case of the Cartan model, it is not entirely obvious how to relate the Lagrangian here to the Lagrangian (6.89).

Let us conclude this section by computing the index. The path integral splits into three parts

$$\begin{aligned}\chi(M_G) &= \int_{PBC} \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{b}\mathcal{D}b\mathcal{D}\bar{c}\mathcal{D}c e^{-S_E} \\ &= \int_{PBC} \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_{dR}} \int_{PBC} \mathcal{D}\bar{b}\mathcal{D}b e^{-S_b} \int_{PBC} \mathcal{D}\bar{c}\mathcal{D}c e^{-S_c}.\end{aligned} \tag{6.171}$$

where now S_{dR} is given in Euclidean time and

$$S_b := \int_0^\beta d\tau \left(\frac{1}{2}\bar{b}^a\dot{b}^a - \frac{1}{2}\dot{\bar{b}}^a b^a + \frac{1}{2}\bar{b}^a b^a\right) = \int_0^\beta d\tau \left(\bar{b}^a\dot{b}^a + \frac{1}{2}\bar{b}^a b^a\right) = \int_0^\beta d\tau\ \bar{b}^a\left(\frac{d}{d\tau} + \frac{1}{2}\right) b^a \tag{6.172}$$

and

$$S_c := \int_0^\beta d\tau \left(\frac{1}{2}\bar{c}^a\dot{c}^a - \frac{1}{2}\dot{\bar{c}}^a c^a + \frac{1}{2}\bar{c}^a c^a\right) = \int_0^\beta d\tau \left(\bar{c}^a\dot{c}^a + \frac{1}{2}\bar{c}^a c^a\right) = \int_0^\beta d\tau\ \bar{c}^a\left(\frac{d}{d\tau} + \frac{1}{2}\right) c^a \tag{6.173}$$

where $\dot{}$ now means derivative with respect to τ. Using (3.57), we obtain

$$\int \mathcal{D}\bar{b}\mathcal{D}b e^{-S_b} = \mathrm{Det}\left(\frac{d}{d\tau} + \frac{1}{2}\right)^{-1} \tag{6.174}$$

and from (3.123) we get

$$\int \mathcal{D}\bar{c}\mathcal{D}c e^{-S_c} = \mathrm{Det}\left(\frac{d}{d\tau} + \frac{1}{2}\right). \tag{6.175}$$

This results in a relation between Euler characteristics of the homotopy quotient $M_G = (M \times EG)/G$ and with M itself,

$$\chi(M_G) = \int_{PBC} \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_{dR}} = \chi(M). \tag{6.176}$$

In other words, the Euler characteristic is invariant under taking homotopy quotients.

7 Concluding Remarks

Let us conclude by giving a brief summary of the results and some possible continuations of the project.

When deriving the index theorems, the central quantity of interest is the Witten index $\mathrm{Tr}(-1)^F e^{\beta H}$. In this thesis, we investigated the supersymmetric non-linear sigma models, in the case when we have $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supercharges. In the initial setting, we computed the Witten index in both the canonical picture and through the use of the path integral. For the $\mathcal{N} = 1$ case, we ended up with the index theorem for the spin complex and for $\mathcal{N} = 2$ we received the index theorem for the de Rham complex, also known as the Chern-Gauss-Bonnet Theorem. Thereafter we considered a (compact and connected) Lie group acting on the manifold and instead of considering the whole manifold, we wanted to investigate the orbit space. The key tool in this analysis was equivariant cohomology, which is a cohomology theory dealing with homotopy quotients of manifolds. The goal of this section was to, with the $\mathcal{N} = 2$ non-linear sigma model in mind, through the Weil and Cartan model try to derive corresponding Lagrangians and relate them to gauge quantum mechanics. Unfortunately, the attempt was not successful. However, what we did find out through computing the path integral expression for the Witten index in the Weil model setting was that the Euler characteristic remains invariant under taking the homotopy quotient, i.e. $\chi(M) = \chi((M \times EG)/G)$, at least if $G = U(1)^n$.

In the future, one should keep investigating the relation between equivariant cohomology and gauge quantum mechanics. After all, by interpreting the group acting on the manifold as the gauge group (acting on the manifold through local isometries) one should be able to relate this, in some way, to gauge quantum mechanics. Furthermore, fixing the gauge means making a choice of a smooth section on the principal fibration $\pi : M \times EG \to (M \times EG)/G$ (i.e. the particle moves horizontally, or in other words, not in the direction of the fibres), which correspond to making a smooth choice of representatives for the elements of the orbit space $(M \times EG)/G$. A quick first guess is that the different Lagrangians obtained from the different models should correspond to choice of different gauges, based on the fact that equivariant cohomology is built on a quotient with the would be gauge group. What might be an obstacle is the fact that one is considering $M \times EG$ in place of the usual case with only M as the total pace of some principal bundle. One might also want to consider different models for equivariant cohomology than covered in this thesis, for instance the BRST-model. This might lead to deeper insight to the problem of gauging. It would indeed be interesting to see how the different Lagrangians of the different models relate to each other.

As a generalization of the considerations in the last section, one can repeat the computations for the non-abelian case.

A Clifford Algebras and Spin Structures

In this section, we will review some general facts about Clifford algebras, spin structures and related topics, which can be found in [35] (we will however use a different convention). No proofs will be presented. Let V be a d-dimensional vector space over $\mathbb{R}$ and let the inner product $\langle , \rangle$ be defined. From the inner product, we may as usual obtain the norm $||v|| := \sqrt{\langle v, v\rangle}$.

Definition A.1. The *Clifford algebra* $\mathrm{Cl}(V)$ (also written as $\mathrm{Cl}(d)$) over V is the quotient of the tensor algebra $\bigoplus_{k\geq 0} V^{\otimes k}$ with the ideal generated by $v \otimes v - ||v||^2$ for $v \in V$.

We hence get the multiplication rule, from here on omitting the tensor product sign,

$$\{v, w\} = vw + wv = 2\langle v, w\rangle \tag{A.1}$$

If we choose an ON-basis $\{e_1, \dots, e_d\}$, we get the relation

$$e_i e_j = \begin{cases} -eje_i & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{A.2}$$

for fixed i, j. The Clifford algebra carries a natural grading and we may write

$$\mathrm{Cl}(V) = \bigoplus_{k\geq 0} \mathrm{Cl}^k(V) \tag{A.3}$$

where $\mathrm{Cl}^0(V) = \mathbb{R}$ and $\mathrm{Cl}^1(V) = V$. Each $\mathrm{Cl}^k(V)$ is generated by $e_\alpha := e_{\alpha_1} \dots e_{\alpha_k}$, where $\alpha_1 < \cdots < \alpha_k$ and with $e_0 := 1$ generating $\mathrm{Cl}^0(V)$. From this, we see that the (vector space) dimension of $\mathrm{Cl}(V)$ is $\sum_{k=0}^{d} \binom{d}{k} = 2^d$. Note that the Clifford algebra $\mathrm{Cl}(V)$ is, as a vector space, isomorphic to the exterior algebra ΛV.

The subspace $\mathrm{Cl}^2(V)$ forms a Lie algebra with Lie bracket given by

$$[a, b] = ab - ba. \tag{A.4}$$

We denote this by $\mathfrak{spin}(V)$ and it turns out to be the Lie algebra for the Lie group $\mathrm{Spin}(V)$, defined below.

From the Clifford algebra, we can define the $\mathrm{Spin}(d)$-group.

Definition A.2. Let $a_i \in V$ such that $||a_i|| = 1$. $\mathrm{Pin}(V)$ (or $\mathrm{Pin}(d)$) is the group of elements of $\mathrm{Cl}(V)$ of the form $a = a_1 \dots a_k$. $\mathrm{Spin}(V)$ (or $\mathrm{Spin}(d)$) is the group of elements in $\mathrm{Cl}(V)$ of the form $a = a_1 \dots a_{2m}$, i.e. $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^{\mathrm{even}}(V)$.

Using the anti-automorphism of $\mathrm{Cl}(V)$

$$a \mapsto a^t \tag{A.5}$$

defined on $e_\alpha = e_{\alpha_1} \dots e_{\alpha_k}$ as reversing the order of the e_{α_i}

$$(e_{\alpha_1} \dots e_{\alpha_k})^t = e_{\alpha_k} \dots e_{\alpha_1}, \tag{A.6}$$

we may define the homomorphism $\varphi : \mathrm{Spin}(d) \to \mathrm{SO}(d)$ defined by the action

$$\varphi(a)v := ava^t \tag{A.7}$$

on $v \in V$. In fact, the homomorphism φ turns out to be a double covering map.

Consider now the complexification of the Clifford algebra $\mathrm{Cl}(V)_{\mathbb{C}} := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$. We have the following property:

Theorem A.1. *There is an algebra isomorphism*

$$\mathrm{Cl}(V)_{\mathbb{C}} \cong \begin{cases} \mathbb{C}^{2^n \times 2^n} \oplus \mathbb{C}^{2^n \times 2^n} & \text{if } d = 2n+1 \\ \mathbb{C}^{2^n \times 2^n} & \text{if } d = 2n. \end{cases} \tag{A.8}$$

where $\mathbb{C}^{k \times l}$ *denotes the set of complex* $k \times l$*-matrices.*

We recognize the contents in Theorem A.1 from physics via the *gamma matrices*, which is a matrix representation of the Clifford algebra. Let ρ be that representation. Then the basis vector e_i is associated to the gamma matrix $\gamma_i = \rho(e_i)$ such that

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} I. \tag{A.9}$$

which is exactly of the form (A.1) for basis vectors and hence respects (A.2).

We now choose an orientation of V; we define $\{e_1, \ldots, e_d\}$ to be a positive ON-basis.

Definition A.3. The *chirality operator* is defined by

$$\Gamma := i^n e_1 \ldots e_d \tag{A.10}$$

where $n = d/2$ if d is even and $n = (d+1)/2$ if d is odd.

The chirality operator is independent of choice of positive ON-basis. Also, $\Gamma^2 = 1$, which means it has eigenvalues ± 1 and we receive a decomposition into eigenspaces[20] of Γ, $\mathrm{Cl}(V)_{\mathbb{C}} = \mathrm{Cl}^+(V)_{\mathbb{C}} \oplus \mathrm{Cl}^-(V)_{\mathbb{C}}$, where $\mathrm{Cl}^{\pm}(V)_{\mathbb{C}}$ has eigenvalue ± 1. Furthermore, for $v \in V$

$$\Gamma v = \begin{cases} v\Gamma & \text{if } d \text{ is odd} \\ -v\Gamma & \text{if } d \text{ is even.} \end{cases} \tag{A.11}$$

In the case when d is even, we have the fact that multiplication by v maps $\mathrm{Cl}^{\pm}(V)_{\mathbb{C}}$ to $\mathrm{Cl}^{\mp}(V)_{\mathbb{C}}$ since for $a_\pm \in \mathrm{Cl}^{\pm}(V)_{\mathbb{C}}$

$$\Gamma v a_\pm = -v\Gamma a_\pm = \mp v a_\pm. \tag{A.12}$$

In terms of gamma matrices, the chirality operator takes the form

$$\gamma_{d+1} := \rho(\Gamma) = i^n \gamma_1 \ldots \gamma_d. \tag{A.13}$$

From here on, we only consider the case when $d = 2n$ is even. In $V_{\mathbb{C}}$, consider the subspace W spanned by

$$\eta_j := \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j}) \tag{A.14}$$

with $j = 1, \ldots, n$. By extending the inner product $\langle , \rangle$ to $V_{\mathbb{C}}$ via complex linearity, we obtain

$$\langle \eta_i, \eta_j \rangle_{\mathbb{C}} = 0 \text{ for all } j \tag{A.15}$$

which in particular implies that

$$\langle w, w \rangle_{\mathbb{C}} = 0 \text{ for all } w \in W. \tag{A.16}$$

Defining $\bar{W}$ as the subspace spanned by $\bar{\eta}_j = \dfrac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, we obtain the decomposition $V_{\mathbb{C}} = W \oplus \bar{W}$. $\bar{W}$ can be identified with the dual space of W with the bilinear form $\langle , \rangle_{\mathbb{C}}$.

Definition A.4. The exterior algebra of W, $S := \Lambda W$, is called the *spinor space*. One may also write S_d to emphasize the dimension of V.

[20] From linear algebra, we know that involutory operators are diagonalisable if the ground field is not of characteristic 2.

Any vector $v \in V$ can be decomposed as $v = w + w'$ for some $w \in W$ and $w' \in \bar{W}$. Let $s \in S$. One may define the algebra homomorphism $\rho : \mathrm{Cl}(V)_{\mathbb{C}} \to \mathrm{End}(S)_{\mathbb{C}}$ (i.e. a representation of $\mathrm{Cl}(V)_{\mathbb{C}}$) by

$$\begin{aligned} \rho(w)s &:= \sqrt{2}\epsilon(w)s \\ \rho(w') &:= -\sqrt{2}\iota(w')s \end{aligned} \tag{A.17}$$

where ϵ denotes the exterior product and ι the interior product. The map ρ is extended to the entirety of $\mathrm{Cl}(V)_{\mathbb{C}}$ by the rule $\rho(vv') = \rho(v)\rho(v')$ for $v, v' \in V$. In fact more can be said:

Theorem A.2. *For $d = 2n$, the map ρ defined above is an algebra isomorphism*

$$\mathrm{Cl}(V)_{\mathbb{C}} \cong \mathrm{End}(S)_{\mathbb{C}}. \tag{A.18}$$

The famous fact that endomorphisms on (finite dimensional) vector spaces can be represented by square matrices can now be observed; the the gamma matrices can be seen to act on the spinor space S as linear operators.

Observe that $\eta_j\bar{\eta}_j - \bar{\eta}_j\eta_j = 2ie_{2j-1}e_{2j}$ so we way rewrite the chirality operator as

$$\Gamma = \frac{1}{2^n}(\eta_1\bar{\eta}_1 - \bar{\eta}_1\eta_1)\dots(\eta_n\bar{\eta}_n - \bar{\eta}_n\eta_n). \tag{A.19}$$

The operator acts on S via the representation ρ as

$$\rho(\Gamma) = (-1)^n(\epsilon(\eta_1)\iota(\bar{\eta}_1) - \iota(\bar{\eta}_1)\epsilon(\eta_1))\dots(\epsilon(\eta_n)\iota(\bar{\eta}_n) - \iota(\bar{\eta}_1)\epsilon(\eta_1)) \tag{A.20}$$

Take an element $\eta_{m_1} \wedge \dots \wedge \eta_{m_k}$ from $S = \Lambda^k W$ and act on it with $(\epsilon(\eta_m)\iota(\bar{\eta}_m) - \iota(\bar{\eta_m})\epsilon(\eta_m))$. There are two cases: (i) $m \neq m_i$ for all i and (ii) $m = m_i$ for some i. In case (i) only the second term contributes

$$(\epsilon(\eta_m)\iota(\bar{\eta}_m) - \iota(\bar{\eta_m})\epsilon(\eta_m))\eta_{m_1} \wedge \dots \wedge \eta_{m_k} = -\eta_{m_1} \wedge \dots \wedge \eta_{m_k} \tag{A.21}$$

and in case (ii) the second term vanishes

$$(\epsilon(\eta_m)\iota(\bar{\eta}_m) - \iota(\bar{\eta_m})\epsilon(\eta_m))\eta_{m_1} \wedge \dots \wedge \eta_{m_k} = \eta_{m_1} \wedge \dots \wedge \eta_{m_k}. \tag{A.22}$$

When acting with $\rho(\Gamma)$ on $\eta_{m_1} \wedge \dots \wedge \eta_{m_k}$ case (i) happens $n - k$ times and case (ii) occurs k times. Hence

$$\rho(\Gamma) = (-1)^n(-1)^{n-k}\eta_{m_1} \wedge \dots \wedge \eta_{m_k} = (-1)^k\eta_{m_1} \wedge \dots \wedge \eta_{m_k} \tag{A.23}$$

and we can identify $\rho(\Gamma)$ with the operator $(-1)^k$ which results in the splitting of S into eigenspaces $S = S^+ \oplus S^-$, where $S^+ = \Lambda^{\mathrm{even}}W$ and $S^- = \Lambda^{\mathrm{odd}}W$ are the subspaces of even and odd exterior powers respectively.

Since $\mathrm{Spin}(V) \subset \mathrm{Cl}(V) \subset \mathrm{Cl}(V)_{\mathbb{C}}$, any representation of $\mathrm{Cl}(V)_{\mathbb{C}}$ restricts to a (group) representation of $\mathrm{Spin}(V)$ which means

$$\rho : \mathrm{Spin}(V) \to \mathrm{End}(S)_{\mathbb{C}} \tag{A.24}$$

is a representation of the spin group. Also, note that for $a \in \mathrm{Spin}(V)$

$$\rho(\Gamma)\rho(a) = \rho(\Gamma a) = \rho(a\Gamma) = \rho(a)\rho(\Gamma) \tag{A.25}$$

in virtue of (A.11) and the fact that $\mathrm{Spin}(V) \subset \mathrm{Cl}^{\mathrm{even}}(V)$. This implies that $S^{\pm}$ is left invariant under the action of $\rho(\mathrm{Spin}(V))$ and so the representation is reducible. The representation on $S^{\pm}$ is however irreducible.

Definition A.5. The representation $\rho : \mathrm{Spin}(V) \to \mathrm{End}(S)_{\mathbb{C}}$ is called the *spinor representation*. The representations on $\rho : \mathrm{Spin}(V) \to \mathrm{End}(S^{\pm})_{\mathbb{C}}$ are called the *half spinor representation.*

We now would like to extend the inner product $\langle , \rangle$ on V as a Hermitian inner product on $V_{\mathbb{C}}$ defined by

$$\langle \alpha^i e_i, \beta^j e_j \rangle = \delta_{ij} \alpha^i \bar{\beta}^j. \tag{A.26}$$

By letting elements in ΛV of the form $e_{i_1} \wedge \cdots \wedge e_{i_k}$ constitute an ON-basis, this inner product naturally extends to ΛV (which we mentioned before is isomorphic to $\mathrm{Cl}(V)$). One can show that the spinor representation is unitary with respect the Hermitian inner product, for $a \in \mathrm{Spin}(V)$

$$\langle \rho(a)s, \rho(a)s' \rangle = \langle s, s' \rangle \tag{A.27}$$

for all $s, s' \in S$.

Let M be a oriented Riemannian manifold with metric g. We would now like to define spin structures. We will use the tangent space T_pM in each point as the vector space V for the construction of Clifford algebras. The fact that M is equipped with a Riemannian metric makes it possible to reduce the structure group of the tangent bundle TM to $\mathrm{SO}(d)$. We may therefore construct the *oriented orthonormal frame bundle* $\pi : F_{\mathrm{SO}}M \to M$ of M, which is an associated principal $\mathrm{SO}(d)$-bundle.

Definition A.6. A *spin structure* on M is a a principal $\mathrm{Spin}(d)$-bundle $\tilde{\pi} : P \to M$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[dr, "\tilde{\pi}"'] & F_{\mathrm{SO}}M \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes, where $\varphi : \mathrm{Spin}(d) \to \mathrm{SO}(d)$ is the double covering map given by in (A.7), defined to act on each fibre. If M has a fixed spin structure, it is called a *spin manifold*.

Let U_a be a local trivialization patch for F_{SO} and $t_{ab} : U_a \cap U_b \to \mathrm{SO}(d)$ be the transition function taking us from U_a to U_b. On the level of transition functions we require that all transition functions $\tilde{t}_{ab}$ on P must satisfy $\rho(\tilde{t}_{ab}) = t_{ab}$ for all a, b. Note that the spin structure might not always exist. In fact, the topological nature of M determines this; the existence of a spin structure is equivalent to the vanishing of the so called *second Stiefel-Whitney class*. We will not discuss this matter in this thesis; we will simply assume the existence here.

The definition of spin structure works in any dimension, but let us return to the assumption that $d = 2n$ is even. We will now put the preceding discussions to use. The fibre of P, $\mathrm{Spin}(d)$ (note the switch of notation), acts on the spinor space S_d via the spinor representation and on the half spinor spaces $S_d^{\pm}$ via the half spinor representations. Arising from this are hence associated vector bundles[21] with $\mathrm{Spin}(d)$:

Definition A.7. The associated vector bundle $\mathfrak{S}_d := P \times_{\mathrm{Spin}(d)} S_d$ is called the *spinor bundle associated with the spin structure* P. The associated vector bundles $\mathfrak{S}_d^{\pm} := P \times_{\mathrm{Spin}(d)} S_d^{\pm}$ are called the *half spinor bundles associated with the spin structure* P. The sections of these bundles are called *(half) spinor fields.* We denote the set of spinor fields by $\Gamma(M, \mathfrak{S}_d)$ and half spinor fields by $\Gamma(M, \mathfrak{S}_d^{\pm})$.

Observe that the spinor bundle decomposes into a sum of the half spinor bundles $\mathfrak{S}_d := \mathfrak{S}_d^+ \oplus \mathfrak{S}_d^-$ so the space[22] of (half) spinor fields, also splits up

$$\Gamma(M, \mathfrak{S}_d) = \Gamma(M, \mathfrak{S}_d^+) \oplus \Gamma(M, \mathfrak{S}_d^-). \tag{A.28}$$

[21]Let $\pi : P \to M$ be a principal G-bundle and V a k-dimensional vector space. Let ρ be a k-dimensional representation of G. The vector bundle associated with P, denoted by $P \times_G V$, is the quotient $(P \times V)/G$ given by identifying $(u, v) \sim (ug, \rho(g)^{-1}v)$ with $(u, v) \in P \times V$ and $g \in G$. Denoting the equivalence classes by $[u, v]$, the projection $\tilde{\pi} : P \times_G V \to M$ is given by $\tilde{\pi}([u, v]) = \pi(u)$. The local trivializations are $\psi_i : U_i \times V$ with transition functions $\rho(t_{ij}(u))$, where $t_{ij}(u) \in G$ are the transition functions of P.

[22]The set of sections of a vector bundle always carry the structure of a vector space. The vector addition and scalar multiplication is defined pointwise. The additive neutral element is the null section, which always exists.

Note that these bundles are pointwise naturally equipped with the Hermitian inner product (A.26) which is invariant under the action of Spin(d) on each fibre (A.27). In the case when M is compact, this induces a L^2-product on the spinor fields; for $\sigma, \sigma' \in \Gamma(M, \mathfrak{S}_d)$

$$(\sigma, \sigma') := \int_M \langle \sigma(p), \sigma'(p) \rangle \star (1) \tag{A.29}$$

where $\star(1) = \sqrt{|\det(g_{\mu\nu})|}dx^1 \wedge \cdots \wedge dx^d$ is the invariant volume form ($\star$ is the Hodge star).

We end this section with a brief discussion of the (very important) *Dirac operator*. Roughly speaking, the Levi-Civita connection for the tangent bundle of M induces a connection on the oriented orthonormal frame bundle which then via the double covering map φ can be pulled back to the spinor bundle. The resulting connection is known as the *spin connection*, which we denote by ω. Let us introduce the orthonormal frame fields ${e_\alpha}^\mu$. Using the gamma matrix representation of the Spin(d)-group[23], the spin connection is locally given by [11]

$$\omega_\mu = \frac{1}{2} i {\omega_\mu}^{\alpha\beta} \Sigma_{\alpha\beta} \tag{A.30}$$

with $\Sigma_{\alpha\beta} := \dfrac{i}{4}[\gamma_\alpha, \gamma_\beta]$ and ${\omega_\mu}^{\alpha\beta} = {e^\alpha}_\nu \nabla_\mu e^{\beta\nu}$, where ∇_μ is the covariant derivative associated to the Levi-Civita connection on TM. The covariant derivative induced by this connection, which we call $\tilde{\nabla}_\mu$, is given by

$$\tilde{\nabla}_\mu = \partial_\mu + \omega_\mu. \tag{A.31}$$

Definition A.8. The *Dirac operator* is an operator $i\nabla\!\!\!\!/ : \Gamma(M, \mathfrak{S}_d) \to \Gamma(M, \mathfrak{S}_d)$ locally given by

$$i\nabla\!\!\!\!/\,\sigma(p) := i\gamma^\mu \tilde{\nabla}_\mu \sigma(p) \tag{A.32}$$

where $\sigma \in \Gamma(M, \mathfrak{S}_d)$.

It can be shown in the case when M is compact that the Dirac operator is Hermitian with respect to the L^2-product (A.29). We also have the following result:

Theorem A.3. *If M is even dimensional, the Dirac operator maps $\Gamma(M, \mathfrak{S}_d^\pm)$ to $\Gamma(M, \mathfrak{S}_d^\mp)$.*

We thus get a two termed complex, the so called *spin complex*

$$\Gamma(M, \mathfrak{S}_d^+) \underset{i\nabla\!\!\!\!/}{\overset{i\nabla\!\!\!\!/}{\rightleftarrows}} \Gamma(M, \mathfrak{S}_d^-). \tag{A.33}$$

Furthermore, keeping the compactness condition, observe that the fact that the Dirac operator is Hermitian means

$$(i\nabla\!\!\!\!/^2 \sigma, \sigma) = (i\nabla\!\!\!\!/\,\sigma, i\nabla\!\!\!\!/\,\sigma) \tag{A.34}$$

which implies that

$$i\nabla\!\!\!\!/\,\sigma = 0 \iff (i\nabla\!\!\!\!/)^2\sigma. \tag{A.35}$$

A spinor field satisfying $(i\nabla\!\!\!\!/)\sigma = 0$ is referred to as *harmonic*.

[23] Which we by now can identify with the spinor representation setting $\rho(e_\alpha) = \gamma_\alpha$, where the e_α constitute an ON-basis for T_pM.

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