Nail H. Ibragimov

SELECTED WORKS

Volume V

Advanced topics to be added to my textbook

*A Practical Course in Differential Equations and Mathematical Modelling*

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Dedication

This volume is dedicated to the memory of my teacher and friend Lev Vasilyevich Ovsyannikov (22.04.1919–23.05.2014).
Figure 1: My last meeting with L.V. Ovsyannikov at his 95th birthday, Novosibirsk, 22 April 2014.
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Abstract. The Riccati equations reducible to first-order linear equations by an appropriate change of the dependent variable are singled out. All these equations are integrable by quadrature.

A wide class of linear ordinary differential equations reducible to algebraic equations is found. It depends on two arbitrary functions. The method for solving all these equations is given. The new class contains the constant coefficient equations and Euler’s equations as particular cases.

1 Introduction

The present paper is dedicated to quite old topics. Namely, it deals with a problem on integration by quadrature of Riccati equations investigated, in terms of elementary functions, by Francesco Riccati and Daniel Bernoulli some 280 years ago for the special Riccati equations (see, e.g. [75])

\[ y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.}, \]

and with an integration of higher-order linear equations by reducing them to algebraic equations. The later property was discovered by Leonard Euler in the 1740s for the constant coefficient equations

\[ y^{(n)} + A_1y^{(n-1)} + \cdots + A_{n-1}y' + A_ny = 0, \quad A_1, \ldots, A_n = \text{const.}, \]

as well as for the equations of the form

\[ x^ny^{(n)} + A_1x^{n-1}y^{(n-1)} + \cdots + A_{n-1}xy' + A_ny = 0, \quad A_1, \ldots, A_n = \text{const.}, \]
known as Euler’s equations.

It will be shown in what follows that these classical results can be extended to wide classes of equations.

# 2 Riccati equation

## 2.1 Preliminaries

Consider the special Riccati equation

\[ y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.} \]  

(2.1)

If \( \alpha = 0 \), Eq. (2.1) is integrable by the separation of variables:

\[
\frac{dy}{ay^2 + b} = dx.
\]

Another easily integrable case is \( \alpha = -2 \). Then the change of the dependent variable

\[ z = \frac{1}{y} \]

maps Eq. (2.1) to the homogeneous equation

\[
\frac{dz}{dx} = -\left[ a - b \left( \frac{z}{x} \right)^2 \right]
\]

which is integrable by quadrature.

F. Riccati and D. Bernoulli noted independently that Eq. (2.1) can be transformed to the case \( \alpha = 0 \), and hence integrable by quadrature in terms of elementary functions if \( \alpha \) takes the values from the following two series:

\[
\alpha = -\frac{4}{k}, \quad \frac{8}{3}, \quad \frac{12}{5}, \quad \frac{16}{7}, \ldots ;
\]

\[
\alpha = -\frac{4}{3}, \quad -\frac{8}{5}, \quad -\frac{12}{7}, \quad -\frac{16}{9}, \ldots .
\]

(2.2)

The series (2.2) are given by the formula

\[
\alpha = -\frac{4k}{2k - 1} \quad \text{with} \quad k = \pm 1, \pm 2, \ldots .
\]

(2.3)

It is manifest from (2.3) that both series in (2.2) have the limit \( \alpha = -2 \). For a derivation of the transformations mapping Eq. (2.1) with \( \alpha \) having the form (2.3) to an integrable form, see [75], Chapter 1, §6.
J. Liouville showed in 1841 that the solution to the special Riccati equation (2.1) is integrable by quadrature in terms of elementary functions only if $\alpha$ has the form (2.3).

It is well known that the general Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2$$

can be rewritten as a linear second-order equation. But this kind of linearization by raising the order does not solve the integration problem. Therefore, I will investigate a possibility of linearization of the Riccati equations without raising the order and will show that all Riccati equations of this type can be integrated by quadrature.

### 2.2 The linearizable Riccati equations

The following theorem is closely related to the theory of nonlinear superpositions discussed in [36], Section 6.7.

**Theorem 2.1.** The first-order ordinary differential equation

$$y' = f(x, y)$$

(2.4)

can be reduced to a linear first-order equation

$$\frac{dz}{dx} = p(x) + q(x)z$$

(2.5)

by a change of the dependent variable $y$,

$$z = z(y),$$

(2.6)

if and only if Eq. (2.4) can be written in the form

$$y' = T_1(x)x_1(y) + T_2(x)x_2(y)$$

(2.7)

such that the operators

$$X_1 = x_1(y)\frac{\partial}{\partial y}, \quad X_2 = x_2(y)\frac{\partial}{\partial y}$$

(2.8)

span a two-dimensional (or a one-dimensional if $X_1$ and $X_2$ are linearly dependent) Lie algebra called in [26] the VGL (Vessiot-Guldberg-Lie) algebra.
Proof. Let Eq. (2.4) be linearizable. Then we can assume that it is already reduced by a certain change of the dependent variable

\[ z = \zeta(y) \]  

(2.9)
to a linear equation (2.5),

\[ \frac{dz}{dx} = p(x) + q(x)z. \]

The VGL algebra associated with Eq. (2.5) is two-dimensional and is spanned by the operators

\[ X_1 = \frac{\partial}{\partial z}, \quad X_2 = z \frac{\partial}{\partial z}. \]  

(2.10)

The form of Eq. (2.7) and the algebra property \([X_1, X_2] = \alpha X_1 + \beta X_2\) remain unaltered under any change (2.6) of the dependent variable. Therefore, rewriting Eq. (2.5) in the original variable \(y = \zeta^{-1}(z)\) obtained by the inverse transformation to (2.9), we arrive at an equation of the form (2.7) for which the operators (2.8) span a two-dimensional Lie algebra. Since the equation obtained from Eq. (2.5) by the inverse transformation to (2.9) is the original equation (2.4) we have proved the “only if” part of the theorem.

Let us prove now the “if” part of the theorem. Namely, we have to demonstrate that any equation of the form (2.7) such that the operators (2.8) span a two-dimensional Lie algebra, is linearizable. If the operators (2.8) are linearly dependent, then \(\xi_2(x) = \gamma \xi_1(x), \gamma = \text{const.}\), and hence Eq. (2.7) has the form

\[ y' = [T_1(x) + \gamma T_2(x)] \xi_1(y). \]

It can be reduced to the linear equation

\[ z' = T_1(x) + \gamma T_2(x). \]

upon introducing a canonical variable \(z\) for

\[ X_1 = \xi_1(y) \frac{\partial}{\partial y} \]

by solving the equation \(X_2(z) = 1\).

Suppose now that the operators (2.8) are linearly independent. It is clear that Eq. (2.12) will be linearized if one transforms the operators (2.8) to the form (2.10). One can assume that the first operator (2.8) has been
already written, in a proper variable $z$, in the form of the first operator $X_1$ given in (2.10):

$$X_1 = \frac{\partial}{\partial z}.$$ 

Let the second operator (2.8) be written in the variable $z$ as follows:

$$X_2 = f(z) \frac{\partial}{\partial z}.$$ 

We have

$$[X_1, X_2] = f'(z) \frac{\partial}{\partial z}$$

and the requirement $[X_1, X_2] = \alpha X_1 + \beta X_2$ that $X_1, X_2$ span a Lie algebra $L_2$ yields the differential equation

$$f' = \alpha + \beta f,$$

where not both $\alpha$ and $\beta$ vanish because otherwise the operators $X_1$ and $X_2$ will be linearly dependent. hence $f'(x) \neq 0$. Solving the above differential equation, we obtain

$$f = \alpha z + C \quad \text{if} \quad \beta = 0,$$

$$f = C e^{\beta z} - \frac{\alpha}{\beta} \quad \text{if} \quad \beta \neq 0.$$

$$f = ax + C \quad \implies \quad X_2 = ax \frac{d}{dx} + CX_1, \quad \text{if} \quad b = 0,$$

$$f = Ce^{bx} - \frac{a}{b} \quad \implies \quad X_2 = Ce^{bx} \frac{d}{dx} - \frac{a}{b} X_1, \quad \text{if} \quad b \neq 0.$$

In the first case we have

$$X_2 = \alpha z \frac{\partial}{\partial z} + CX_1,$$

and hence a basis of $L_2$ is provided by (2.10). In the second case we have

$$X_2 = C e^{\beta z} \frac{\partial}{\partial z} - \frac{\alpha}{\beta} X_1,$$

and one can take basis operators, by assigning $\beta z$ as new $z$, in the form

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = e^{\beta z} \frac{\partial}{\partial z}.$$ 

Finally, substituting $\bar{z} = e^{-z}$ we arrive at the basis (2.10), thus completing the proof of the theorem.
The following theorem characterizes all Riccati equations that can be reduced to first-order linear equations by changing the dependent variable (see [22], Russian ed., Theorem 4.3; [26], Section 11.2.5 and Note [11.4]; see also Theorem 3.2.2 in [36]).

**Theorem 2.2.** The following two conditions 1° and 2° are equivalent and provide the necessary and sufficient conditions for the Riccati equation

\[ y' = P(x) + Q(x)y + R(x)y^2 \]  

(2.11)

to be linearizable by a change of the dependent variable (2.6), \( z = z(y) \).

1°. Eq. (2.11) has a constant solution \( y = c \) including \( c = \infty \).

2°. Eq. (2.11) has either the form

\[ y' = Q(x)y + R(x)y^2 \]  

(2.12)

with any functions \( Q(x) \) and \( R(x) \), or the form

\[ y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2 \]  

(2.13)

with any functions \( P(x), Q(x) \) and any constant \( k \).

**Proof.** The conditions 1° and 2° are equivalent. Indeed, let Eq. (2.11) have a constant solution \( y = c \). Then

\[ 0 = P(x) + Q(x)c + R(x)c^2. \]  

(2.14)

If \( c = 0 \), Eq. (2.14) yields \( P(x) = 0 \), and hence Eq. (2.11) has the form (2.12). If \( c \neq 0 \), Eq. (2.14) yields

\[ R(x) = -\frac{1}{c^2}[P(x) + cQ(x)] = -\frac{1}{c}[Q(x) + \frac{1}{c}P(x)]. \]

Denoting \( k = -1/c \) we get

\[ R(x) = k[Q(x) - kP(x)]. \]

Hence Eq. (2.11) has the form (2.13). Thus, we have proved that 1° \( \Rightarrow \) 2°.

Conversely, let Eq. (2.11) satisfy the condition 2°. It is manifest that Eq. (2.12) has the constant solution \( y = 0 \). Furthermore, one can verify that Eq. (2.13) has the constant solution \( y = -1/k \). This proves that 2° \( \Rightarrow \) 1°.

Let us turn to the necessary and sufficient conditions for linearization. The operators

\[ X_1 = y \frac{\partial}{\partial y}, \quad X_2 = y^2 \frac{\partial}{\partial y} \]
associated with Eq. (2.12) have the commutator $[X_1, X_2] = X_2$, and hence span a two-dimensional Lie algebra. Furthermore, the operators

$$X_1 = (1 - k^2 y^2) \frac{\partial}{\partial y}, \quad X_2 = (y + ky^2) \frac{\partial}{\partial y}$$

associated with Eq. (2.13) have the commutator $[X_1, X_2] = X_1 + 2kX_2$. Hence, they also span a two-dimensional Lie algebra. Therefore, according to Theorem 2.1, the condition 2° is sufficient for linearization.

Note that the equation $y' = P(x) + Q(x)y$ can be regarded as a particular case of Eq. (2.13) with $k = 0$. Since Eq. (2.13) has the constant solution $y = -1/k$, we conclude that the linear equation $y' = P(x) + Q(x)y$ has the constant solution $y = \infty$. We conclude that any linearizable equation has a constant solution because the change of the dependent variable $z = z(y)$ maps a constant solution into a constant solution of the transformed equation. Hence, the condition 1° is necessary for linearization. This completes the proof of the theorem due to the equivalence of the conditions 1° and 2°.

Now we will use Theorem 2.2 for integrating the linearizable Riccati equations. We will find linearizing transformations for the equations (2.12) and (2.13). We will assume that $k$ in Eq. (2.13) is a real number.

### 2.3 Linearization and integration of Equation (2.12)

Invoking Eqs. (6.7.5), (6.7.6) from [36], replacing there $t$ and $x^i$ by $x$ and $y$, respectively, and identifying $T_1(t)$ and $T_2(t)$ with $Q(x)$ and $R(x)$, respectively, we see that the VGL (Vessiot-Guldberg-Lie) algebra associated with Eq. (2.12) is a two-dimensional algebra $L_2$ spanned by the operators

$$X_1 = y \frac{\partial}{\partial y}, \quad X_2 = y^2 \frac{\partial}{\partial y}.$$  

Their commutator is $[X_1, X_2] = X_2$. Introducing the new basis

$$X_1' = X_2, \quad X_2' = -X_1$$

i.e. taking

$$X_1' = y^2 \frac{\partial}{\partial y}, \quad X_2' = -y \frac{\partial}{\partial y} \quad (2.15)$$

we have a basis in $L_2$ satisfying the commutator relation

$$[X_1', X_2'] = X_1'.$$  

(2.16)
Let us find a change of the dependent variable, \(z = z(y)\), such that Eq. (2.12) becomes a linear equation (2.5),

\[
\frac{dz}{dx} = p(x) + q(x)z.
\]

The VGL algebra of this equation is spanned by the operators (2.10),

\[
\overline{X}_1 = \frac{\partial}{\partial z}, \quad \overline{X}_2 = z \frac{\partial}{\partial z}
\]

whose commutator has the same form as Eq. (2.16), i.e. \([\overline{X}_1, \overline{X}_2] = \overline{X}_1\). Consequently, the linearizing transformation \(z = z(y)\) is determined by the equations \(X_1'(z) = 1\), \(X_2'(z) = z\), or

\[
y^2 \frac{dz}{dy} = 1, \quad -y \frac{dz}{dy} = z. \tag{2.17}
\]

Integrating the first equation (2.17) we obtain

\[
z = -\frac{1}{y} + A.
\]

Substituting in the second equation (2.17) we get \(A = 0\). Thus, the linearizing transformation is

\[
z = -\frac{1}{y}. \tag{2.18}
\]

In this variable, Eq. (2.12) becomes the linear equation

\[
z' = R(x) - Q(x)z, \tag{2.19}
\]

whence

\[
z = \left[ C + \int R(x)e^{\int Q(x)dx} \, dx \right] e^{-\int Q(x)dx}, \quad C = \text{const.}
\]

Substituting in Eq. (2.18) we finally arrive at the solution to Eq. (2.12):

\[
y = -\left[ C + \int R(x)e^{\int Q(x)dx} \, dx \right]^{-1} e^{\int Q(x)dx}. \tag{2.20}
\]

**Remark 2.1.** In Section 5.2 the solution (2.20) is obtained by an alternative method.
Example 2.1. Consider the equation
\[ y' = \frac{y}{x} + R(x)y^2.\]
By Eq. (2.20) yields its solution
\[ y = \frac{x}{C - \int xR(x)dx}.\]

Example 2.2. The equation
\[ y' = \frac{y}{x} + 3xy^2 \]
is a particular case of the previous equation. Its solution is
\[ y = \frac{x}{C - x^2}.\]

Example 2.3. The equation
\[ y' = y + \frac{1}{x}y^2 \]
has the solution
\[ y = \frac{e^x}{C + \text{Ei}(x)}, \quad \text{where} \quad \text{Ei}(x) = -\int_{-\infty}^{x} \frac{e^t}{t} dt.\]

2.4 Linearization and integration of Equation (2.13)
Now we identify the coefficients \( P(x) \) and \( Q(x) \) of Eq. (2.13),
\[ y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \quad (2.13) \]
with the coefficients \( T_1(t) \) and \( T_2(t) \) of Eqs. (6.7.5), (6.7.6) from [36] and associated with Eq. (2.13) the VGL algebra spanned by the operators
\[ X_1 = (1 - k^2y^2) \frac{\partial}{\partial y}, \quad X_2 = (y + ky^2) \frac{\partial}{\partial y}. \]
Their commutator is \([X_1, X_2] = X_1 + 2kX_2\). We assume that \( k \neq 0 \), since otherwise Eq. (2.13) is already linear. Therefore, setting
\[ X_1' = X_1 + 2kX_2, \quad X_2' = -X_1/(2k)\]
we obtain the new basis

\[ X'_{1} = (1 + ky)^2 \frac{\partial}{\partial y}, \quad X'_{2} = \frac{1}{2k} (k^2y^2 - 1) \frac{\partial}{\partial y}, \quad (2.21) \]

satisfying the commutator relation (2.16), \([X'_{1}, X'_{2}] = X'_{1}\).

The transformation \( z = z(y) \) of the operators (2.21) to the form (2.10) is determined by the equations

\[ X'_{1}(z) = 1, \quad X'_{2}(z) = z, \]

or

\[ (1 + ky)^2 \frac{dz}{dy} = 1, \quad \frac{1}{2k} (k^2y^2 - 1) \frac{dz}{dy} = z. \quad (2.22) \]

Integrating the first equation (2.22) we obtain

\[ z = A - \frac{1}{k(1 + ky)}. \]

Substituting in the second equation (2.22) we get \( A = 1/(2k) \). Thus, the linearizing transformation is

\[ z = \frac{ky - 1}{2k(ky + 1)}. \quad (2.23) \]

Eq. (2.23) yields:

\[ y = \frac{1 + 2kz}{k(1 - 2kz)}. \quad (2.24) \]

Substituting (2.24) in Eq. (2.13) we arrive at the linear equation

\[ z' = \frac{1}{2k} Q(x) + [Q(x) - 2kP(x)] z, \quad (2.25) \]

whence

\[ z = \frac{1}{2k} \left[ C + \int Q(x)e^{\int[2kP(x) - Q(x)]dx} dx \right] e^{\int(Q(x) - 2kP(x))dx}. \quad (2.26) \]

Substituting (2.26) in (2.24) we will obtain the solution to Eq. (2.13).

**Remark 2.2.** We have assumed that \( Q(x) \neq 0 \) because otherwise Eq. (2.13) is separable,

\[ y' = P(x)(1 - k^2y^2). \]
Remark 2.3. Eq. (2.13) has the constant solution
\[ y = -\frac{1}{k}. \] (2.27)
We excluded the singular solution (2.27) in the above calculations and assumed that \(1 + ky \neq 0\).

Example 2.4. Let us integrate the equation
\[ y' = x + 2xy + xy^2. \] (2.28)
Here
\[ P(x) = x, \quad Q(x) = 2x, \quad k = 1. \]
Therefore the linearized equation (2.25) is written \(z' = x\). Integrating it and denoting the constant of integration by \(C/2\), we obtain
\[ z = \frac{1}{2}(C + x^2) \]
in accordance with Eq. (2.26). Substituting the above \(z\) in Eq. (2.24) we obtain the following solution to Eq. (2.28):
\[ y = \frac{1 + C + x^2}{1 - C - x^2}. \] (2.29)
Remark 2.3 gives also the singular solution \(y = -1\).

Remark 2.4. Eq. (2.28) can also be integrated by separating the variables.

Example 2.5. The equation
\[ y' = x^2 + (x + x^2) y + \frac{1}{4}(2x + x^2) y^2 \] (2.30)
has the form (2.13) with
\[ P(x) = x^2, \quad Q(x) = x + x^2, \quad k = \frac{1}{2}. \]
The linearized equation (2.25) is written
\[ z' = x + x^2 + xz \]
and has the solution
\[ z = \left(C + \int (x + x^2)e^{-x^2/2} dx\right)e^{x^2/2}. \]
Substitution in Eq. (2.24) yields

\[ y = 2 \frac{1 + z}{1 - z}. \]

Hence, the solution to Eq. (2.30) is given by

\[ y = 2 \frac{1 + \left( C + \int (x + x^2)e^{-x^2/2} \, dx \right) e^{x^2/2}}{1 - \left( C + \int (x + x^2)e^{-x^2/2} \, dx \right) e^{x^2/2}}. \qquad (2.31) \]

3 Second-order linear equations reducible to algebraic equation

3.1 Preliminaries

The linear ordinary differential equations with constant coefficients

\[ y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.}, \quad (3.1) \]

and Euler’s equations

\[ x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} xy' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.}, \quad (3.2) \]

are discussed practically in all textbooks on differential equations. They are useful in applications. The most remarkable property is that both equations (3.1) and (3.2) are reducible to algebraic equations. Namely, their fundamental systems of solutions, and hence the general solutions can be obtained by solving algebraic equations. Then one can integrate the corresponding non-homogeneous linear equations by using the method of variation of parameters. It is significant to understand the nature of the reducibility and to extend the class of linear equations reducible to algebraic equations.

In this section, we investigate this problem for the second-order equations and find a wide class of linear ordinary differential equations that are reducible to algebraic equations. The new class depends on two arbitrary functions of \( x \) and contains the equations (3.1) and (3.2) as particular cases. The method for solving all these equations is given in Sections 3.5 and illustrated by examples in Section 3.8.

The similar results for the third-order equations are presented in Section 4.

The main statement for the second-order equations is as follows (Section 3.5).
Theorem 3.1. The linear second-order equations
\[ P(x)y'' + Q(x)y' + R(x)y = F(x) \]
whose general solution can be obtained by solving algebraic equations and by quadratures, have the form
\[ \phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = F(x), \quad (3.3) \]
where \(\phi = \phi(x), \sigma = \sigma(x)\) and \(F(x)\) are arbitrary (smooth) functions, and \(A, B = \text{const}\). The homogeneous equation (3.3),
\[ \phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0, \quad (3.4) \]
has the solutions of the form
\[ y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx}, \quad (3.5) \]
where \(\lambda\) satisfies the characteristic equation
\[ \lambda^2 + A\lambda + B = 0. \quad (3.6) \]
If the characteristic equation (3.6) has distinct real roots \(\lambda_1 \neq \lambda_2\), the general solution to Eq. (3.4) is given by
\[ y(x) = K_1 e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} \, dx} + K_2 e^{\int \frac{\sigma(x) + \lambda_2}{\phi(x)} \, dx}, \quad K_1, K_2 = \text{const.} \quad (3.7) \]
In the case of complex roots, \(\lambda_1 = \gamma + i\theta, \lambda_2 = \gamma - i\theta\), the general solution to Eq. (3.4) is given by
\[ y(x) = \left[ K_1 \cos \left( \theta \int \frac{dx}{\phi(x)} \right) + K_2 \sin \left( \theta \int \frac{dx}{\phi(x)} \right) \right] e^{\int \frac{\sigma(x) + \gamma}{\phi(x)} \, dx}. \quad (3.8) \]
If the characteristic equation (3.6) has equal roots \(\lambda_1 = \lambda_2\), the general solution to Eq. (3.4) is given by
\[ y = \left[ K_1 + K_2 \int \frac{dx}{\phi(x)} \right] e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} \, dx}, \quad K_1, K_2 = \text{const.} \quad (3.9) \]
The general solution of the non-homogeneous equation (3.3) can be obtained by the method of variation of parameters.

Remark 3.1. The equations with constant coefficients and Euler’s equation are the simplest representatives of Eqs. (3.4). Namely, setting \(\phi(x) = 1, \sigma(x) = 0\) we obtain the second-order equation with constant coefficients
\[ y'' + Ay' + By = 0, \]
and Eq. (3.5) yields the well-known formula
\[ y = e^{\lambda x}, \]
where \( \lambda \) is determined by the characteristic equation (3.5).

If we set \( \phi(x) = x, \sigma(x) = 0 \), we obtain the second-order Euler’s equation (3.2) written in the form
\[ x^2 y'' + (A + 1)xy' + By = 0. \]

Then Eq. (3.5) yields the particular solutions for Euler’s equations:
\[ y = x^\lambda, \]
where \( \lambda \) is determined again by the characteristic equation (3.5). For details, see Section 3.3. For other functions \( \phi(x) \) and \( \sigma(x) = 0 \), Eqs. (3.5) are new.

### 3.2 Equations with constant coefficients

Let us consider second-order equations with constant coefficients
\[ y'' + Ay' + By = 0, \quad A, B = \text{const}. \tag{3.10} \]

Eq. (3.10) is invariant under the one-parameter groups of translations in \( x \) and dilations in \( y \), since it does not involve the independent variable \( x \) explicitly (the coefficients \( A \) and \( B \) are constant) and is homogeneous in the dependent variable \( y \). In other words, Eq. (3.10) admits the generators
\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y} \tag{3.11} \]
of the translations in \( x \) and dilations in \( y \). We use them as follows [26].

Let us find the invariant solution for \( X = X_1 + \lambda X_2 \), i.e.
\[ X = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \quad \lambda = \text{const}. \tag{3.12} \]

The characteristic equation
\[ \frac{dy}{y} = \lambda \frac{dx}{x} \]
of the equation
\[ X(J) = \frac{\partial J}{\partial x} + \lambda y \frac{\partial J}{\partial y} = 0 \]
for the invariants $J(x, y)$ yields one functionally independent invariant

$$J = ye^{-\lambda x}.$$  

According to the general theory, the invariant solution is given by $J = C$ with an arbitrary constant $C$. Thus, the general form of the invariant solutions for the operator (3.12) is

$$y = Ce^{\lambda x}, \quad C = \text{const}.$$  

Since Eq. (3.10) is homogeneous one can set $C = 1$ and obtain Euler’s substitution:

$$y = e^{\lambda x}. \quad (3.13)$$  

As well known, the substitution (3.13) reduces Eq. (3.10) to the quadratic equation (characteristic equation)

$$\lambda^2 + A\lambda + B = 0. \quad (3.14)$$  

If Eq. (3.14) has two distinct roots, $\lambda_1 \neq \lambda_2$, then Eq. (3.13) provides two linearly independent solutions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x},$$  

and hence, a fundamental set of solutions. If the roots are real, the general solution to Eq. (3.10) is

$$y(x) = K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}, \quad K_1, K_2 = \text{const}. \quad (3.15)$$  

If the roots are complex, $\lambda_1 = \gamma + i\theta, \lambda_2 = \gamma - i\theta$, the general solution to Eq. (3.10) is given by

$$y(x) = [K_1 \cos(\theta x) + K_2 \sin(\theta x)] e^{\gamma x}, \quad K_1, K_2 = \text{const}. \quad (3.16)$$  

In the case of equal roots $\lambda_1 = \lambda_2$, standard texts in differential equations make a guess, without motivation, that the general solution has the form

$$y(x) = (K_1 + K_2 x) e^{\lambda_1 x}, \quad K_1, K_2 = \text{const}. \quad (3.17)$$  

The motivation is given in [26], Section 13.2.2, and states the following.

**Lemma 3.1.** Eq. (3.10) can be mapped to the equation $z'' = 0$ by a linear change of the dependent variable

$$y = \sigma(x) z, \quad \sigma(x) \neq 0, \quad (3.18)$$
if and only if the characteristic equation (3.14) has equal roots. Specifically, if \( \lambda_1 = \lambda_2 \), the substitution
\[
y = z e^{\lambda_1 x}
\]
(3.19)
carries Eq. (3.10) to the equation \( z'' = 0 \). Substituting in (3.19) the solution \( z = K_1 + K_2 x \) of the equation \( z'' = 0 \), we obtain the general solution (3.17) to Eq. (3.10) whose coefficients satisfy the condition of equal roots for Eq. (3.14):
\[
A^2 - 4B = 0.
\]
(3.20)

**Proof.** Calculation shows that after the substitution (3.18) the linear equation
\[
y'' + f(x)y' + g(x)y = 0.
\]
(3.21)
becomes (see, e.g. [36], Section 3.3.2)
\[
z'' + I(x)z = 0,
\]
where
\[
I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)
\]
and the function \( \sigma(x) \) in the transformation (3.18) has the form
\[
\sigma(x) = e^{-\frac{1}{2} \int f(x) dx}.
\]
(3.22)
Hence, Eq. (3.21) is carried into equation \( z'' = 0 \) if and only if \( I(x) = 0 \), i.e.
\[
f^2(x) + 2f'(x) - 4g(x) = 0.
\]
(3.23)
In the case of Eq. (3.10), the condition (3.23) is identical with Eq. (3.20), and the function \( \sigma(x) \) given by Eq. (3.22) becomes
\[
\sigma(x) = e^{\frac{-A}{2} x}.
\]
(3.24)
Furthermore, under the condition Eq. (3.20), the repeated root of Eq. (3.14) is \( \lambda_1 = -A/2 \). Therefore Eq. (3.24) can be written
\[
\sigma(x) = e^{\lambda_1 x}
\]
and we arrive at the substitution (3.19), and hence at the solution (3.17), thus proving the lemma.
3.3 Euler’s equation

Consider Euler’s equation

\[ x^2 y'' + A x y' + B y = 0, \quad A, B = \text{const.} \]  

(3.25)

It is double homogeneous (see [36], Section 6.6.1), i.e. admits the dilation groups in \( x \) and in \( y \) with the generators

\[ X_1 = x \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}. \]  

(3.26)

We proceed as in Section 3.2 and find the invariant solutions for the linear combination \( X = X_1 + \lambda X_2 \):

\[ X = x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const.} \]  

(3.27)

The characteristic equation

\[ \frac{dy}{y} = \lambda \frac{dx}{x} \]

of the equation \( X(J) = 0 \) for the invariants \( J(x, y) \) yields the invariant

\[ J = y x^{-\lambda} \]

for the operator (3.27). The invariant solutions are given by \( J = C \), whence

\[ y = C x^\lambda, \quad C = \text{const.} \]

Due to the homogeneity of Eq. (3.10) we can set \( C = 1 \) and obtain

\[ y = x^\lambda. \]  

(3.28)

Differentiating and multiplying by \( x \), we have:

\[ x y' = \lambda x^\lambda, \quad x^2 y'' = \lambda(\lambda - 1)x^\lambda. \]

Substituting in Eq. (3.27) and dividing by the common factor \( C x^\lambda \) we obtain the following characteristic equation for Euler’s equation (3.25):

\[ \lambda^2 + (A - 1) \lambda + B = 0. \]  

(3.29)

Remark 3.2. According to Eqs. (3.14), (3.29), the characteristic equation for Euler’s equation written in the form

\[ x^2 y'' + (A + 1) x y' + B y = 0 \]  

(3.30)

is identical with the characteristic equation (3.14) for Eq. (3.10) with constant coefficients.
3.4 New examples of reducible equations

Example 3.1. Let us find the linear second-order equations (3.21),

\[ y'' + f(x)y' + g(x)y = 0, \quad (3.21) \]

admitting the operator

\[ X_1 = x^\alpha \frac{\partial}{\partial x}, \quad (3.31) \]

where \( \alpha \) is any real-valued parameter. Taking the second prolongation of \( X_1 \),

\[ X_1 = x^\alpha \frac{\partial}{\partial x} - \alpha x^{\alpha-1} y' \frac{\partial}{\partial y'} - [\alpha(\alpha - 1)x^{\alpha-2} y' + 2x^{\alpha-1} y''] \frac{\partial}{\partial y''}, \]

we write the invariance condition of Eq. (3.21),

\[ X_1(y'' + f(x)y' + g(x)y) \bigg|_{(3.21)} = 0, \]

and obtain:

\[ x^{\alpha-2}[x^2 f' + \alpha x f - \alpha(\alpha - 1)]y' + x^{\alpha-1}[xg' + 2\alpha g]y = 0. \quad (3.32) \]

Since Eq. (3.32) should be satisfied identically in the variables \( x, y, y' \), it splits into two equations:

\[ x^2 f' + \alpha x f - \alpha(\alpha - 1) = 0, \quad xg' + 2\alpha g = 0. \quad (3.33) \]

Solving the first-order linear differential equations (3.33) for the unknown functions \( f(x) \) and \( g(x) \), we obtain:

\[ f(x) = \frac{\alpha}{x} + Ax^{-\alpha}, \quad g(x) = Bx^{-2\alpha}, \quad A, B = \text{const}. \]

Thus, we arrive at the following linear equation admitting the operator (3.31):

\[ x^{2\alpha} y'' + (Ax^\alpha + \alpha x^{2\alpha-1}) y' + By = 0, \quad A, B = \text{const}. \quad (3.34) \]

Since Eq. (3.34) is linear homogeneous, it admits, along with (3.31), the operator

\[ X_2 = y \frac{\partial}{\partial y}. \]

Now we proceed as in Section 3.3 and find the invariant solutions for the linear combination \( X = X_1 + \lambda X_2 \):

\[ X = x^\alpha \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const}. \]
Let $\alpha \neq 1$. The characteristic equation
\[
\frac{dy}{y} = \lambda \frac{dx}{x^\alpha}
\]
of the equation $X(J) = 0$ for the invariants $J(x, y)$ yields the invariant
\[
J = y e^{\frac{\lambda}{\alpha - 1} x^{1-\alpha}}.
\]
Hence, the invariant solutions are obtained by setting $J = C$, whence letting $C = 1$ we have
\[
y = e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}.
\]  
(3.35)

Differentiating we have:
\[
y' = \lambda x^{-\alpha} e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}, \quad y'' = \left[\lambda^2 x^{-2\alpha} - \lambda \alpha x^{-\alpha - 1}\right] e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}.
\]

Substituting in Eq. (3.34) and dividing by the non-vanishing factor $e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}$, we obtain the following characteristic equation for Eq. (3.34):
\[
\lambda^2 + A \lambda + B = 0.
\]  
(3.36)

Eq. (3.36) is identical with the characteristic equation (3.14) for the equation (3.10) with constant coefficients.

**Remark 3.3.** When $\alpha = 0$, Eq. (3.34) yields the equation (3.10) with constant coefficients. When $\alpha = 1$, Eq. (3.34), upon setting $A + 1$ as a new coefficient $A$, coincides with Euler’s equation (3.25).

**Remark 3.4.** Eq. (3.35) contains Euler’s substitution (3.13) for the equation (3.10) with constant coefficients as a particular case $\alpha = 0$.

**Remark 3.5.** Using the statement that Eq. (3.21) is mapped to the equation $z'' = 0$ if and only if the function
\[
I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)
\]
vanishes (see Lemma 3.1), one can verify that Eq. (3.35) is equivalent by function to the equation $z'' = 0$ if and only if
\[
A^2 - 4B = 0 \quad \text{and} \quad \alpha = 0 \text{ or } \alpha = 2.
\]  
(3.37)

The first equation in (3.37) means that the characteristic equation (3.36) has a repeated root, and hence there is only one solution of the form (3.35).
Then, using the reasoning of Lemma 3.1 we can show that the general
solution to Eq. (3.34) with \( A^2 - 4B = 0, \ \alpha = 2 \), i.e. of the equation

\[
y'' + \left( \frac{A}{x^2} + \frac{2}{x} \right) y' + \frac{B}{x^4} y = 0, \quad A^2 - 4B = 0,
\]

is given by

\[
y = \left( K_1 + \frac{K_2}{x} \right) e^{-\frac{2}{x}}, \quad (3.38)
\]

where \( K_1, K_2 \) are arbitrary constants and \( \lambda \) is the repeated root of the
characteristic equation (3.36). For a more general statement, see Section
3.8.

**Example 3.2.** Let us find the linear second-order equations (3.21) admit-
tting the projective group with the generator

\[
X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \quad (3.39)
\]

Taking the second prolongation of the operator (3.39),

\[
X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'} - 3xy'' \frac{\partial}{\partial y''},
\]

and writing the invariance condition of Eq. (3.21),

\[
X_1(y'' + f(x)y' + g(x)y) \bigg|_{(3.21)} = 0,
\]

we obtain:

\[
x(x f' + 2f)y' + (x^2 g' + 4xg + f)y = 0. \quad (3.40)
\]

Eqs. (3.40) yield:

\[
f(x) = \frac{A}{x^2}, \quad g(x) = \frac{B}{x^4} - \frac{A}{x^3}, \quad A, B = \text{const}.
\]

Thus, we arrive at the following linear equation admitting the operator
(3.39):

\[
x^4 y'' + Ax^2 y' + (B - Ax)y = 0, \quad A, B = \text{const}. \quad (3.41)
\]

Eq. (3.41) is linear homogeneous, and hence admits, along with (3.39),
the operator

\[
X_2 = y \frac{\partial}{\partial y}.
\]
Now we proceed as in Section 3.3 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x^2 \frac{\partial}{\partial x} + (x + \lambda)y \frac{\partial}{\partial y}, \; \lambda = \text{const.}$$

Rewriting the characteristic equation of the equation $X(J) = 0$ for the invariants $J(x, y)$ in the form

$$\frac{dy}{y} = \frac{x + \lambda}{x^2} dx$$

we obtain the invariant

$$J = \frac{y}{x} e^{\frac{\lambda}{x}}.$$

Setting $J = C$ and letting $C = 1$ we obtain the following form of the invariant solutions:

$$y = xe^{-\frac{i}{x}}. \quad (3.42)$$

Substituting (3.42) in Eq. (3.41) we reduce the differential equation (3.41) to the algebraic equation

$$\lambda^2 + A\lambda + B = 0$$

which is identical with the characteristic equation (3.14) for the equation (3.10) with constant coefficients.

**Example 3.3.** Solve the equation

$$y'' + \frac{\omega^2}{x^4} y = 0, \; \omega = \text{const.} \quad (3.43)$$

This is an equation of the form (3.41) with $A = 0$, $B = \omega^2$. The algebraic equation (3.14) yields $\lambda_1 = -i\omega$, $\lambda_2 = i\omega$, and hence we have two independent invariant solutions (3.42):

$$y_1 = x e^{\frac{i}{x}}, \quad y_2 = x e^{-i\frac{i}{x}}.$$

Taking their real and imaginary parts, just like in the case of constant coefficient equations, we obtain the following fundamental system of solutions:

$$y_1 = x \cos \left( \frac{\omega}{x} \right), \quad y_2 = x \sin \left( \frac{\omega}{x} \right). \quad (3.44)$$

Hence, the general solution to Eq. (3.42) is given by

$$y = x \left[ C_1 \cos \left( \frac{\omega}{x} \right) + C_2 \sin \left( \frac{\omega}{x} \right) \right].$$
We can also solve the non-homogeneous equation
\[ y'' + \frac{\omega^2}{x^4} y = F(x), \] (3.45)
e.g. by the method of variation of parameters, and obtain
\[ y = x \left[ C_1 \cos \left( \frac{\omega}{x} \right) + C_2 \sin \left( \frac{\omega}{x} \right) \right] \] (3.46)
\[ + \frac{x}{\omega} \left[ \cos \left( \frac{\omega}{x} \right) \int x F(x) \sin \left( \frac{\omega}{x} \right) \, dx \right. \left. - \sin \left( \frac{\omega}{x} \right) \int x F(x) \cos \left( \frac{\omega}{x} \right) \, dx \right]. \]

3.5 The general result for second-order equations

Note that the operators given in Eqs. (3.11), (3.26), (3.31), (3.39) are particular cases of the generator
\[ X_1 = \phi(x) \frac{\partial}{\partial x} + \sigma(x) y \frac{\partial}{\partial y} \] (3.47)
of the general equivalence group of all linear ordinary differential equations. We will find now all linear second-order equations (3.21) admitting the operator (3.47) with any fixed functions \( \phi(x) \) and \( \sigma(x) \).

Taking the second prolongation of the operator (3.47),
\[ X_1 = \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + \left[ \sigma' y + (\sigma - \phi') y' \right] \frac{\partial}{\partial y'} + \left[ \sigma'' y + (2 \sigma' - \phi'') y + (\sigma - 2 \phi') y'' \right] \frac{\partial}{\partial y''}, \]
and writing the invariance condition of Eq. (3.21),
\[ X_1 \left( y'' + f(x) y' + g(x) y \right) \bigg|_{(3.21)} = 0, \]
we obtain:
\[ (\phi f' + f \phi' + 2 \sigma' - \phi'') y' + (\phi g' + 2 \phi' g + f \sigma' + \sigma'') y = 0. \]

It follows:
\[ \phi f' + f \phi' + 2 \sigma' - \phi'' = 0, \]
\[ \phi g' + 2 \phi' g + f \sigma' + \sigma'' = 0. \] (3.48)
The first equation (3.48) is written
\[ (\phi f)' = (\phi' - 2 \sigma)' \]
and yields:
\[ f(x) = \frac{1}{\phi} \left[ A + \phi' - 2\sigma \right], \quad A = \text{const.} \quad (3.49) \]

Substituting this in the second equation (3.48), we obtain the following non-homogeneous linear first-order equation for determining \( g(x) \):
\[ \phi g' + 2\phi' g = -\sigma'' - \frac{\sigma'}{\phi} \left[ A + \phi' - 2\sigma \right]. \quad (3.50) \]

The homogeneous equation
\[ \phi(x) g' + 2\phi'(x) g = 0 \]
with a given function \( \phi(x) \) yields
\[ g = \frac{C}{\phi^2(x)}. \]

By variation of the parameter \( C \), we set
\[ g = \frac{u(x)}{\phi^2(x)}, \]

substitute it in Eq. (3.50) and obtain:
\[ u' = -A\sigma' + 2\sigma\sigma' - \phi'\sigma' - \phi\sigma'' \equiv -(A\sigma)' + (\sigma^2)' - (\phi\sigma)', \]
whence
\[ u = B - A\sigma + \sigma^2 - \phi\sigma', \quad B = \text{const.} \]

Therefore,
\[ g = \frac{1}{\phi^2(x)} \left[ B - A\sigma + \sigma^2 - \phi\sigma' \right]. \quad (3.51) \]

Thus, we have arrived at the following result.

**Theorem 3.2.** The homogeneous linear second-order equations (3.21) admitting the operator (3.47) with any given functions \( \phi = \phi(x) \) and \( \sigma = \sigma(x) \) have the form
\[ \phi^2 y'' + (A + \phi' - 2\sigma) \phi y' + (B - A\sigma + \sigma^2 - \phi\sigma') y = 0. \quad (3.52) \]

Now we use the homogeneity of Eq. (3.52) characterized by the generator
\[ X_2 = y \frac{\partial}{\partial y}. \]
Namely, we look for the invariant solutions with respect to the linear combination $X = X_1 + \lambda X_2$:

$$X = \phi(x) \frac{\partial}{\partial x} + (\sigma(x) + \lambda)y \frac{\partial}{\partial y}, \quad \lambda = \text{const.},$$

and arrive at the following statement reducing the problem of integration of the differential equation (3.52) to solution of the quadratic equation, namely, the characteristic equation as in the case of equations with constant coefficients.

**Theorem 3.3.** Eq. (3.52) has the invariant solutions of the form

$$y = e^\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx,$$

where $\lambda$ satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0.$$  \hspace{1cm} (3.54)

**Proof.** We solve the equation $X(J) = 0$ for the invariants $J(x, y)$, i.e. integrate the equation

$$\frac{dy}{y} = \frac{\sigma(x) + \lambda}{\phi(x)} \, dx$$

and obtain the invariant

$$J = ye^{-\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx}.$$  

Setting $J = C$ and letting $C = 1$ we obtain Eq. (3.53) for the invariant solutions. Thus, we have:

$$y = e^\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx, \quad y' = \frac{\sigma + \lambda}{\phi} \ e^\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx$$

$$y'' = \frac{1}{\phi^2} \left[ (\sigma + \lambda)^2 - (\sigma + \lambda)\phi' + \phi\sigma' \right] \ e^\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx.$$

Substituting (3.55) in Eq. (3.52) we obtain Eq. (3.54), thus completing the proof.

### 3.6 Distinct roots of the characteristic equation

It is manifest that if the characteristic equation (3.54) has distinct real roots $\lambda_1 \neq \lambda_2$, the general solution to Eq. (3.52) is given by

$$y(x) = K_1 e^\int \frac{\sigma(x) + \lambda_1}{\phi(x)} \, dx + K_2 e^\int \frac{\sigma(x) + \lambda_2}{\phi(x)} \, dx, \quad K_1, K_2 = \text{const.}$$

(3.56)
In the case of complex roots, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Eq. (3.52) is given by

$$y(x) = \left[K_1 \cos \left(\theta \int \frac{dx}{\phi(x)}\right) + K_2 \sin \left(\theta \int \frac{dx}{\phi(x)}\right)\right] e^{f \frac{\sigma(x) + \gamma}{\phi(x)} dx}. \quad (3.57)$$

### 3.7 The case of repeated roots

**Theorem 3.4.** If the characteristic equation (3.54) has equal roots $\lambda_1 = \lambda_2$, the general solution to Eq. (3.52) is given by

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)}\right] e^{f \frac{\sigma(x) + \lambda_1}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (3.58)$$

**Proof.** Let us new variables $t$ and $z$ defined by the linear first-order equations

$$X_1(t) \equiv \phi(x) \frac{\partial t}{\partial x} + \sigma(x) y \frac{\partial t}{\partial y} = 1, \quad X_2(t) \equiv y \frac{\partial t}{\partial y} = 0 \quad (3.59)$$

and

$$X_1(z) \equiv \phi(x) \frac{\partial z}{\partial x} + \sigma(x) y \frac{\partial z}{\partial y} = 0, \quad X_2(z) \equiv y \frac{\partial z}{\partial y} = z, \quad (3.60)$$

respectively. Eqs. (3.59) are easily solved and yield

$$t = \int \frac{dx}{\phi(x)}. \quad (3.61)$$

Integration of the second equation (3.60) with respect to $y$ gives

$$z = v(x)y.$$ 

Substituting this in the first equation (3.60) we obtain

$$\phi(x) \frac{dv}{dx} + \sigma(x) v = 0, \quad \text{whence} \quad v = e^{-f \frac{\sigma(x)}{\phi(x)} dx}.$$ 

Thus,

$$z = y e^{-f \frac{\sigma(x)}{\phi(x)} dx}. \quad (3.62)$$

The passage to the new variables (3.61), (3.62) converts the operator $X_1$ given by (3.47) to the translation generator without changing the form of the dilation generator $X_2$. In other words, upon introducing the new independent and dependent variables $t$ and $z$ given by (3.61) and (3.62),
respectively, we arrive at the operators (3.11). Hence, in the new variables, Eq. (3.62) becomes an equation with constant coefficients. Invoking that the equations (3.52) and (3.10) have Eq. (3.54) as their common characteristic equation, we use Lemma 3.1 and write

\[ z = (K_1 + K_2 t) e^{\lambda_1 t}. \]

Substituting this in Eq. (3.62) and replacing \( t \) and \( z \) by their expressions (3.61) and (3.62), respectively, and solving for \( y \), we finally arrive at Eq. (3.58).

**Remark 3.6.** We can easily solve the non-homogeneous equation Eq. (3.52):

\[ \phi^2 y'' + (A + \phi' - 2\sigma) \phi y' + (B - A\sigma + \sigma^2 - \phi' \sigma) y = F(x). \] (3.63)

Namely, we rewrite Eq. (3.52) in the form

\[ y'' + a(x) y' + b(x) y = P(x) \]

and employ the representation of the general solution (see, e.g. [36], Section 3.3.4)

\[ y = K_1 y_1(x) + K_2 y_2(x) - y_1(x) \int \frac{y_2(x) P(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x) P(x)}{W(x)} \, dx \] (3.64)

furnished by the method of variation of parameters. Here

\[ W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) \]

is the Wronskian of a fundamental system of solutions \( y_1(x), y_2(x) \) for the homogeneous equation

\[ y'' + a(x) y' + b(x) y = 0. \]

### 3.8 Illustrative examples

Euler’s substitution (3.13) as well as the solutions (3.28), (3.35) and (3.42) are encapsulated in Eq. (3.53). We will consider now these and several other examples.

**Example 3.4.** Let us take \( \phi(x) = 1, \sigma(x) = 0 \). Then Eqs. (3.52), (3.53) and (3.58) coincide with Eqs. (3.10), (3.13) and (3.17), respectively. Eq. (3.57) becomes (3.16).
Example 3.5. Let us take $\phi(x) = x$, $\sigma(x) = 0$. Then Eq. (3.52) becomes Euler’s equation written in the form (3.30), Eq. (3.53) yields Eq. (3.28) for invariant solutions, whereas Eq. (3.58) provides the general solution

$$y(x) = (K_1 + K_2 \ln x) x^{\lambda_1} \quad K_1, K_2 = \text{const.,}$$

(3.65)
to Euler’s equation (3.30) whose characteristic equation (3.14) has equal roots. Eq. (3.57) leads to the following solution for complex roots $\lambda_1 = \gamma + i\theta, \lambda_2 = \gamma - i\theta$:

$$y(x) = [K_1 \cos(\theta \ln x) + K_2 \sin(\theta \ln x)] x^{\gamma}.$$  

(3.66)

Example 3.6. Let us take $\phi(x) = x^\alpha$, $\sigma(x) = 0$. Then Eqs. (3.52) and (3.53) coincide with Eqs. (3.34) and (3.35), respectively, whereas Eq. (3.58) provides the general solution

$$y(x) = (K_1 + K_2 x^{1-\alpha}) e^{\lambda_1 x^{1-\alpha}} \quad K_1, K_2 = \text{const.,}$$

(3.67)
to Eq. (3.34) whose characteristic equation (3.36) has equal roots $\lambda_1 = \lambda_2$. Eq. (3.67) extends the solution (3.38) to all equations (3.34) with the coefficients $A, B, C$ satisfying the condition (3.20) of equal roots for the characteristic equation (3.36).

Example 3.7. Let us take $\phi(x) = 1 + x^2$, $\sigma(x) = x$. Then Eq. (3.52) becomes

$$(1 + x^2)^2 y'' + (1 + x^2)Ay' + (B - Ax - 1)y = 0.$$  

(3.68)

Working out the integral in Eq. (3.53),

$$\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx = \int \frac{x}{1 + x^2} \, dx + \int \frac{\lambda}{1 + x^2} \, dx = \ln \sqrt{1 + x^2} + \lambda \arctan x,$$

we obtain the following expression for the invariant solutions:

$$y = \sqrt{1 + x^2} e^{\lambda \arctan x},$$

(3.69)

where $\lambda$ satisfies the characteristic equation (3.54):

$$\lambda^2 + A\lambda + B = 0.$$  

(3.70)

If the characteristic equation (3.70) has distinct real roots, $\lambda_1 \neq \lambda_2$, the general solution to Eq. (3.68) is given by

$$y(x) = \sqrt{1 + x^2} \left[ K_1 e^{\lambda_1 \arctan x} + K_2 e^{\lambda_2 \arctan x} \right].$$  

(3.71)
In the case of complex roots, $\lambda_1 = \gamma + i\theta$, $\lambda_2 = \gamma - i\theta$, the general solution to Eq. (3.68) is given by

$$y(x) = [K_1 \cos(\theta \arctan x) + K_2 \sin(\theta \arctan x)] \sqrt{1 + x^2} e^{\gamma \arctan x}. \quad (3.72)$$

Finally, if the characteristic equation (3.70) has equal roots $\lambda_1 = \lambda_2$, the general solution to Eq. (3.68) is given by

$$y(x) = (K_1 + K_2 \arctan x) \sqrt{1 + x^2} e^{\lambda_1 \arctan x}. \quad (3.73)$$

**Example 3.8.** Consider Eq. (3.68) with $A = 0$, $B = \omega^2$. Then, according to Example 3.7, the equation

$$(1 + x^2)^2 y'' + (\omega^2 - 1) y = 0 \quad (3.74)$$

has the following general solution:

$$y(x) = [K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x)] \sqrt{1 + x^2}. \quad (3.75)$$

**Example 3.9.** Let us solve the non-homogeneous equation

$$(1 + x^2)^2 y'' + (\omega^2 - 1) y = F(x). \quad (3.76)$$

Example 3.8 provides the fundamental system of solutions

$$y_1 = \sqrt{1 + x^2} \cos(\omega \arctan x), \quad y_2 = \sqrt{1 + x^2} \sin(\omega \arctan x)$$

for the homogeneous equation (3.74). We have:

$$y_1' = \frac{1}{\sqrt{1 + x^2}} [x \cos(\omega \arctan x) - \omega \sin(\omega \arctan x)],$$

$$y_2' = \frac{1}{\sqrt{1 + x^2}} [x \sin(\omega \arctan x) + \omega \cos(\omega \arctan x)].$$

Hence the Wronskian is $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \omega$. Now we rewrite Eq. (3.76), in accordance with Remark 3.6, in the form

$$y'' + \frac{\omega^2 - 1}{(1 + x^2)^2} y = \frac{F(x)}{(1 + x^2)^2}, \quad (3.77)$$

use Eq. (3.64) and obtain the following general solution to Eq. (3.76):

$$y(x) = \sqrt{1 + x^2} \left[ K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x) ight.$$

$$- \frac{1}{\omega} \cos(\omega \arctan x) \int \frac{F(x)}{(1 + x^2)^{3/2}} \sin(\omega \arctan x) \, dx \quad (3.78)$$

$$+ \frac{1}{\omega} \sin(\omega \arctan x) \int \frac{F(x)}{(1 + x^2)^{3/2}} \cos(\omega \arctan x) \, dx \right].$$
4 Third-order linear equations reducible to algebraic equation

The previous results can be extended to higher-order linear ordinary differential equations. I will discuss here the third-order equations

\[ y''' + f(x)y'' + g(x)y' + h(x)y = 0. \] (4.1)

**Theorem 4.1.** The homogeneous linear third-order equations (4.1) admitting the operator (3.47) with any given \( \phi = \phi(x) \) and \( \sigma = \sigma(x) \) have the form

\[
\phi^3 y''' + [A + 3(\phi' - \sigma)]\phi^2 y'' \\
+ [B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2] \phi y' \\
+ [C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'] y = 0.
\] (4.2)

**Proof.** We take the third prolongation of the operator (3.47):

\[
X_1 = \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + [\sigma' y + (\sigma - \phi')y'] \frac{\partial}{\partial y'} \\
+ [\sigma'' y + (2\sigma' - \phi'')y' + (\sigma - 2\phi')y''] \frac{\partial}{\partial y''} \\
+ [\sigma''' y + (3\sigma'' - \phi''')y' + 3(\sigma' - \phi'')y'' + (\sigma - 3\phi')y''' + (\sigma - 3\phi')y''''] \frac{\partial}{\partial y'''},
\]

and write the invariance condition of Eq. (4.1):

\[
X_1(y''' + f(x)y'' + g(x)y' + h(x)y) \bigg|_{(4.1)} = 0.
\]

We annul the coefficients for \( y'' \), \( y' \) and \( y \) of the left-hand side of the above equation and split it into the following three equations:

\[
\phi f' + \phi' f + 3(\sigma' - \phi'') = 0, \tag{4.3}
\]

\[
\phi g' + 2\phi' g + (2\sigma' - \phi'')f - \phi''' + 3\sigma'' = 0, \tag{4.4}
\]

\[
\phi h' + 3\phi' h + \sigma'' f + \sigma' g + \sigma''' = 0. \tag{4.5}
\]

Eq. (4.3) is written

\[
(\phi f)' = 3(\phi' - \sigma)'\]
and yields:

\[ f = \frac{1}{\phi} \left[ A + 3(\phi' - \sigma) \right], \quad A = \text{const.} \quad (4.6) \]

We substitute (4.6) in Eq. (4.4), integrate the resulting non-homogeneous linear first-order equation for \( g \) and obtain:

\[ g = \frac{1}{\phi^2} \left[ B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2 \right], \quad (4.7) \]

where \( B \) is an arbitrary constant. Now we substitute (4.6), (4.7) in Eq. (4.5), integrate the resulting first-order equation for \( h \) and obtain:

\[ h = \frac{1}{\phi^3} \left[ C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\sigma' + 3\phi\sigma\sigma' \right], \quad (4.8) \]

where \( C \) is an arbitrary constant. Finally, substituting (4.6), (4.7) and (4.8) in Eq. (4.1), we arrive at Eq. (4.2).

**Theorem 4.2.** Eq. (4.2) has the invariant solutions of the form (3.53),

\[ y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} \, dx}, \quad (3.53) \]

where \( \lambda \) satisfies the characteristic equation

\[ \lambda^3 + A\lambda^2 + B\lambda + C = 0. \quad (4.9) \]

**Proof.** Adding to Eqs. (3.55) the expression for the third derivative \( y''' \) and substituting in Eq. (4.2) we obtain Eq. (4.9).

### 5 Utilization of connection between Riccati and second-order linear equations

#### 5.1 Preliminaries

Recall that the Riccati equation

\[ y' = P(x) + Q(x)y + R(x)y^2, \quad R(x) \neq 0, \quad (5.1) \]

is mapped by the substitution

\[ y = -\frac{1}{R(x)} \frac{u'}{u} \quad (5.2) \]
to the linear second-order equation
\[ u'' + f(x)u' + g(x)u = 0 \] (5.3)
with the coefficients
\[ f(x) = -\left[ Q(x) + \frac{R'(x)}{R(x)} \right], \quad g(x) = P(x)R(x). \] (5.4)
Indeed, we have:
\[ y' = -\frac{1}{R} u'' + \frac{R'}{R^2} \frac{u'}{u} + \frac{1}{R} \frac{u^2}{u^2} \]
and
\[ P + Qy + Ry^2 = P - \frac{Q}{R} \frac{u'}{u} + \frac{1}{R} \frac{u^2}{u^2}. \]
Substituting these expressions in Eq. (5.1) and multiplying by \(-Ru\) we obtain the equation
\[ u'' - \frac{R'}{R} u' = Qu' - PRu, \]
i.e. Eq. (5.3) with the coefficients (5.4).

5.2 From Riccati to second-order equations
Applying Eqs. (5.3)-(5.4) to Eq. (2.12),
\[ y' = Q(x)y + R(x)y^2, \] (2.12)
we obtain the following second-order linear equation:
\[ u'' = \left( Q + \frac{R'}{R} \right) u'. \] (5.5)
The integration yields:
\[ \ln u' = \int \left( Q + \frac{R'}{R} \right) dx + \ln C_1 = \int Q dx + \ln R + \ln C_1. \]
Hence
\[ u' = C_1 R(x)e^{\int Q(x) dx} \] (5.6)
and
\[ u = C_1 \int R(x)e^{\int Q(x) dx} dx + C_2. \] (5.7)
Substituting (5.6) and (5.7) in Eq. (5.2) and denoting $C = C_2/C_1$, we arrive at the solution (2.20) to Eq. (2.12):

$$y = -\frac{e^{\int Q(x)dx}}{C + \int R(x)e^{\int Q(x)dx}dx}. \quad (2.20)$$

The following examples clarify how to use the linearizable equations (2.13),

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \quad k = \text{const.}, \quad (2.13)$$

for integrating the corresponding second-order linear equations (5.3).

**Example 5.1.** If we apply Eqs. (5.3)-(5.4) to Eq. (2.28) from Example 2.4,

$$y' = x + 2xy + xy^2,$$

we obtain the following second-order linear equation:

$$u'' - \left(2x + \frac{1}{x}\right)u' + x^2u = 0. \quad (5.8)$$

Let us integrate this equation. Writing Eq. (5.2) in the form

$$\frac{u'}{u} = -R(x)y,$$

substituting here $R(x) = x$ and the expression (2.29) for $y$, we get

$$\frac{u'}{u} = -x\frac{1 + C + x^2}{1 - C - x^2}.$$

Writing this equation in the form

$$\frac{d\ln u}{dx} = x + \frac{2x}{x^2 + C - 1}$$

and integrating we obtain the following general solution to Eq. (5.8):

$$u = K(x^2 + C - 1)^{1/2}, \quad C, K = \text{const.}. \quad (5.9)$$

**Example 5.2.** If we apply Eqs. (5.3)-(5.4) to Eq. (2.30) from Example 2.5,

$$y' = x^2 + (x + x^2)y + \frac{1}{4}(2x + x^2)y^2,$$
we obtain the following second-order linear equation:

\[ u'' - (1 + x) \left( x + \frac{2}{2x + x^2} \right) u' + \frac{1}{4} x^2 (2x + x^2) u = 0. \] (5.10)

Let us integrate this equation. Writing Eq. (5.2) in the form

\[ \frac{u'}{u} = -R(x)y, \]

substituting here

\[ R(x) = \frac{1}{4} (2x + x^2) \]

and the expression (2.31) for \( y \), we get

\[ \frac{u'}{u} = -\frac{1}{2} (2x + x^2) \frac{1 + \left( C + \int (x + x^2) e^{-x^2/2} \, dx \right) e^{x^2/2}}{1 - \left( C + \int (x + x^2) e^{-x^2/2} \, dx \right) e^{x^2/2}}. \]

The integration yields \( \ln |u| = \ln |K| + \phi(x) \), where

\[ \phi(x) = -\frac{1}{2} \int (2x + x^2) \frac{1 + \left( C + \int (x + x^2) e^{-x^2/2} \, dx \right) e^{x^2/2}}{1 - \left( C + \int (x + x^2) e^{-x^2/2} \, dx \right) e^{x^2/2}} \, dx. \]

Hence, the general solution to Eq. (5.10) has the form

\[ u = K e^{\phi(x)}, \] (5.11)

where \( \phi(x) \) is given above and \( K \) is an arbitrary constant.

Applying Eqs. (5.3)-(5.4) to Eq. (2.13) and using the solution procedure for Eq. (2.13) described in Section 2.4, we obtain the following general result.

**Theorem 5.1.** The general solution of the second-order linear equation

\[ u'' - \left[ Q(x)x + \frac{Q'(x) - kP'(x)}{Q(x) - kP(x)} \right] u' + k \left[ P(x)Q(x) - kP^2(x) \right] u = 0 \] (5.12)

with an arbitrary constant \( k \) and two arbitrary functions \( P(x) \) and \( Q(x) \) can be obtained by quadratures.
5.3 From second-order to Riccati equations

It is manifest from Eqs. (5.4) that two coefficients \( f(x) \) and \( g(x) \) of a given second-order equation (5.3) do not uniquely determine three coefficients \( P(x), Q(x), R(x) \) of the corresponding Riccati equation (5.1). Namely, if we know solutions of an equation (5.3), we can solve by using the formula (5.2) an infinite set of the Riccati equations

\[
y' = R(x)y^2 - \left[ f(x) + \frac{R'(x)}{R(x)} \right] y + \frac{g(x)}{R(x)}
\]

with an arbitrary function \( R(x) \neq 0 \).

**Example 5.3.** Consider the following equation with constant coefficients:

\[
u'' + u = 0.
\]

(5.14)

Here \( f = 0 \), \( g = 1 \). Hence, the corresponding Riccati equation (5.13) has the form

\[
y' = R(x)y^2 - \frac{R'(x)}{R(x)} y + \frac{1}{R(x)}.
\]

(5.15)

Substituting the general solution

\[u = C_1 \cos x + C_2 \sin x\]

of Eq. (5.14) in (5.2) we obtain the following solution to Eq. (5.15):

\[
y = \frac{1}{R(x)} \frac{C_1 \sin x - C_2 \cos x}{C_1 \cos x + C_2 \sin x}.
\]

If \( C_2 \neq 0 \) we denote \( K = C_2/C_1 \) and write the solution in the form

\[
y = \frac{1}{R(x)} \frac{\sin x - K \cos x}{\cos x + K \sin x}
\]

or, upon dividing the numerator and denominator by \( \cos x \),

\[
y = \frac{1}{R(x)} \frac{\tan x - K}{1 + K \tan x}, \quad K = \text{const.}
\]

(5.16)

If \( C_2 = 0 \) the solution becomes

\[
y = -\frac{\cot x}{R(x)}
\]

which can be obtained from (5.16) by letting \( K \to \infty \). Thus, the general solution to the Riccati equation (5.17) is given by (5.16) where \( -\infty \leq K \leq +\infty \).
In particular, taking in (5.17), (5.16) \( R(x) = e^x \) we conclude that the equation
\[
y' = e^x y^2 - y + e^{-x}
\] (5.17)
has the general solution
\[
y = \frac{\tan x - K}{1 + K \tan x} \ e^{-x}, \quad -\infty \leq K \leq +\infty.
\] (5.18)

Sometimes it is convenient to use a restricted correspondence between second-order and Riccati equations by writing Eqs. (5.3)-(5.4) corresponding to the Riccati equation (5.1) in the following form:
\[
R(x) u'' - \left[ R'(x) + Q(x) R(x) \right] u' + P(x) R^2(x) u = 0.
\] (5.19)

**Example 5.4.** Consider Euler’s equation written in the form (3.30):
\[
x^2 u'' + (A + 1) x u' + B u = 0, \quad A, B = \text{const}.
\] (5.20)

Comparing the equations (5.19) and (5.20) we take \( R(x) = x^2 \) and obtain
\[
Q(x) = -\frac{A + 3}{x} \quad P(x) = \frac{B}{x^4}.
\]
Thus, we have arrived at the following Riccati equation:
\[
y' = x^2 y^2 - \frac{A + 3}{x} y + \frac{B}{x^4}.
\] (5.21)

We know that the solutions of Eq. (5.20) have the form \( u = x^\lambda \). Substitution in (5.2) yields the following form of solutions to Eq. (5.21):
\[
y = -\frac{\lambda}{x^3}.
\] (5.22)

Substituting (5.22) in Eq. (5.21) we obtain again the characteristic equation (3.14):
\[
\lambda^2 + A\lambda + B = 0.
\] (5.23)

However, Eqs. (5.22), (5.23) provide only two particular solutions to Eq. (5.21). In order to find the general solution of the nonlinear equation (5.21), we have to construct the general solution \( u(x) \) of the linear equation (5.20) using the superposition principle and substitute \( u = u(x) \) in (5.2).

Using the results of Section 3.5 on integrability of Eq. (3.52),
\[
\phi^2 y'' + (A + \phi' - 2\sigma) \phi y' + (B - A\sigma + \sigma^2 - \phi\sigma') y = 0,
\]
we can formulate the following general result.
Theorem 5.2. The Riccati equation
\[
y' = R(x) y^2 - \left( \frac{A + \phi' - 2\sigma}{\phi} + \frac{R'}{R} \right) y + \frac{B - A\sigma + \sigma^2 - \phi\sigma'}{R\phi^2} \tag{5.24}\]
with three arbitrary functions \(R(x), \phi(x), \sigma(x)\) and two arbitrary constants \(A, B\) is integrable by quadratures.

5.4 Application to Ermakov’s equation
The above results on integration of linear equations
\[
u'' + a(x)v' + b(x)v = 0 \tag{5.25}\]
can be combined with Ermakov’s method for solving nonlinear equations of the following form (see [13], Editor’s preface):
\[
u'' + a(x)v' + b(x)v = \frac{\alpha}{u^3} e^{\int a(x)dx}, \quad \alpha = \text{const.} \tag{5.26}\]

Example 5.5. Using the solution (5.9) of Eq. (5.8) and applying Ermakov’s method one can solve the following nonlinear equation:
\[
u'' - \left( 2x + \frac{1}{x} \right) v' + x^2 v = \alpha x^2 e^{2x^2} u^{-3}. \tag{5.27}\]

Example 5.6. The nonlinear equation (5.26) associated with the integrable equation (3.52) has the form
\[
\phi^2 u'' + (A + \phi' - 2\sigma) \phi u' + (B - A\sigma + \sigma^2 - \phi\sigma') u = \frac{\alpha}{u^3} e^{\int (2\sigma - A)/\phi dx}, \tag{5.28}\]
where \(\phi\) and \(\sigma\) are arbitrary functions of \(x\), and \(A\) is an arbitrary constant. Eq. (5.28) is integrable by quadratures.
Paper 2

Two approaches to the method of variation of parameters

N.H. Ibragimov [35]

Abstract. An alternative approach to Lagrange’s method of variation of parameters is presented. Explicit formulas for solutions of arbitrary initial value problems for linear equations of the first, second and third order are provided. These formulas are simple and convenient for teachers and students.

1 Introduction

The simplest general method for integrating non-homogeneous linear ordinary differential equations of an arbitrary order is the method of variation of parameters. It was first discovered by Jean Bernoulli for first-order equations in 1697 and later extended by J.L. Lagrange to higher-order equations. Lagrange’s approach requires imposition of additional restrictions on varied parameters. I give here an alternative approach.

Theorem 4.1 provides an explicit formula for the solution of an arbitrary initial value problem for any linear first-order equation. Theorems 4.2 and 4.3 provide similar formulas for equations of the second and third order, respectively, provided that fundamental systems of solutions for the corresponding homogeneous equations are known. These formulas are as simple for practical using as the formula for roots of quadratic equations. Namely, the solutions to the Cauchy problem are obtained just by inserting the coefficients of the differential equations in question and the initial data.

Of course, the integrals involved in the solutions may be difficult or, in general, impossible to work out in terms of elementary functions. But this circumstance is unessential. Anyhow, equations whose solutions can be
expressed in terms of elementary functions are very rare and do not have a practical value. They appear mostly in textbooks as simple illustrations of various methods.

2 Traditional presentation

2.1 First-order equations

Jean Bernoulli noticed in 1697 that any non-homogeneous first-order linear ordinary differential equation

\[ y' + P(x)y = Q(x) \]  

(2.1)

can be easily integrated by varying the parameter \( C \) in the general solution

\[ y = Ce^{-\int Pdx} \]  

(2.2)

of the homogeneous equation

\[ y' + P(x)y = 0. \]  

(2.3)

Namely, we replace the constant \( C \) in (2.2) by an undetermined function \( u(x) \) and look for the solution of the non-homogeneous equation (2.1) in the form

\[ y = u(x)e^{-\int Pdx}. \]  

(2.4)

Substitution of (2.4) in Eq. (2.1) yields

\[ u'(x) = Q(x)e^{\int Pdx}, \]  

(2.5)

whence

\[ u(x) = \int Qe^{\int Pdx}dx + C, \quad C = \text{const}. \]  

(2.6)

Inserting the expression (2.6) for \( u(x) \) in (2.4) we obtain the general solution to the non-homogeneous linear equation (2.1) given by two quadratures:

\[ y = Ce^{-\int Pdx} + e^{-\int Pdx} \int Q(x)e^{\int Pdx}dx. \]  

(2.7)

Remark 2.1. Often the integrals in (2.7) cannot be worked out in terms of elementary functions. This fact does not mean, however, that the method of variation of parameter has disadvantages of its own. It only means that the general solution of the differential equation in question cannot be expressed in terms of elementary functions.
2.2 Second-order equations

Some 90 years later Lagrange showed that Bernoulli’s method of variation of parameters can be extended to higher-order equations. Lagrange’s method of variation of parameters allows one to integrate non-homogeneous linear ordinary differential equations of any order provided that one knows the fundamental set of solutions for the corresponding homogeneous equation. Recall the common way of presentation of this method, e.g., for second-order equations

\[ y'' + a(x)y' + b(x)y = f(x). \]  

(2.8)

Let us assume that we know two linearly independent solutions \( y_1 = y_1(x), \ y_2 = y_2(x) \) of the homogeneous equation

\[ y'' + a(x)y' + b(x)y = 0. \]  

(2.9)

Then the general solution to Eq. (2.9) is given by

\[ y = C_1 y_1 + C_2 y_2, \quad C_1, C_2 = \text{const}. \]  

(2.10)

The essence of the method of variation of parameters is the same as in the case of first-order equations. Namely, we replace the parameters \( C_1, C_2 \) by unknown functions \( u_1(x), u_2(x) \) and seek the solution of the non-homogeneous equation (2.8) in the form

\[ y = u_1(x) y_1 + u_2(x) y_2. \]  

(2.11)

Substituting (2.11) in Eq. (2.8) we will obtain only one equation for two unknown functions \( u_1(x) \) and \( u_2(x) \). Furthermore, computing the derivative of (2.11),

\[ y' = u_1 y'_1 + u_2 y'_2 + y_1 u'_1 + y_2 u'_2, \]  

(2.12)

we see that \( y'' \) will involve second derivatives of \( u_1, u_2 \). Therefore, the substitution (2.11) in Eq. (2.8) leads to a single differential equation of the second order for two unknowns \( u_1 \) and \( u_2 \). The common way to avoid this complication is to impose on \( u_1, u_2 \) the following restriction:

\[ y_1 u'_1 + y_2 u'_2 = 0. \]  

(2.13)

Then Eq. (2.12) reduces to

\[ y' = u_1 y'_1 + u_2 y'_2 \]  

(2.14)

and yields:

\[ y'' = u_1 y''_1 + u_2 y''_2 + y'_1 u'_1 + y'_2 u'_2. \]  

(2.15)
Substituting (2.15), (2.14) and (2.11) in Eq. (2.8) and invoking that the functions \( y_1(x) \) and \( y_2(x) \) solve the homogeneous equation (2.8), we obtain
\[
y' y_1' + y_2 y_2' = f(x).
\] (2.16)

Since \( y_1 = y_1(x) \), \( y_2 = y_2(x) \) are known functions, (2.13) and (2.16) provide two equations for determining two unknown functions \( u_1 \) and \( u_2 \):
\[
y y_1' + y_2 y_2' = 0,
\]
\[
y' y_1' + y_2 y_2' = f(x).
\] (2.17)
Solving the system (2.17) with respect to \( u_1', u_2' \):
\[
u_1' = -\frac{y_2 f(x)}{W[y_1, y_2]}, \quad u_2' = \frac{y_1 f(x)}{W[y_1, y_2]},
\] (2.18)
and integrating we obtain
\[
u_1 = -\int \frac{y_2 f(x)}{W[y_1, y_2]} \, dx + C_1, \quad u_2 = \int \frac{y_1 f(x)}{W[y_1, y_2]} \, dx + C_2,
\] (2.19)
where \( C_1, C_2 \) are arbitrary constants, and \( W[y_1, y_2] \) is the Wronskian:
\[
W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.
\] (2.20)
Inserting (2.20) in (2.11) we obtain the general solution to Eq. (2.8):
\[
y = C_1 y_1 + C_2 y_2 - y_1 \int \frac{y_2 f(x)}{W[y_1, y_2]} \, dx + y_2 \int \frac{y_1 f(x)}{W[y_1, y_2]} \, dx.
\] (2.21)

### 2.3 Remark

For an extension of the method to higher-order equations, it is useful to write the system (2.17) in the vector form
\[
MU = F,
\] (2.22)
where \( M \) is the \( 2 \times 2 \) matrix, and \( U, F \) are the column vectors defined as follows:
\[
M = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, \quad U = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}, \quad F = \begin{pmatrix} f(x) \end{pmatrix}.
\] (2.23)
The determinant of \( M \) is the Wronskian (2.20),
\[
\det M = W[y_1, y_2].
\]
Since the solutions \( y_1(x), \ y_2(x) \) are linearly independent, we have \( W[y_1, y_2] \neq 0 \). Hence, the matrix \( M \) is invertible and has the following inverse:

\[
M^{-1} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} y'_2 & -y_2 \\ -y'_1 & y_1 \end{pmatrix}.
\]  

(2.24)

Therefore the solution to Eq. (2.22) is given by

\[
U = M^{-1} F,
\]

or

\[
\begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} -y_2 f(x) \\ y_1 f(x) \end{pmatrix}.
\]  

(2.25)

In other words, we have arrived at Eqs. (2.18):

\[
u'_1 = -\frac{y_2 f(x)}{W[y_1, y_2]}, \quad u'_2 = \frac{y_1 f(x)}{[y_1, y_2]}.
\]

Note that for computing the solution (2.25) to Eq. (2.22) we need only the last column in the inverse matrix. Therefore we can write \( M^{-1} \) by keeping only the last column:

\[
M^{-1} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} \cdots & -y_2 \\ \cdots & y_1 \end{pmatrix}.
\]  

(2.26)

3 Alternative presentation

Imposition of the additional restriction (2.13) often becomes a stumbling block for students who consider it as an artificial trick that one has to remember. For higher-order equations the situation is more complicated. I give here an alternative presentation of the method of variation of parameters which is free from this disadvantage of the traditional presentation of the method.

3.1 Second-order equations

Let us rewrite the second-order equation (2.8) as the following non-homogeneous system of two first-order linear equations for two dependent variables \( y, z \):

\[
y' = z,
\]

\[
z' + a(x) z + b(x) y = f(x).
\]  

(3.1)
Two linearly independent solutions

\[ y_1 = y_1(x), \quad y_2 = y_2(x) \]

of the homogeneous equation (2.9), taken together with

\[ z_1 = y'_1(x), \quad z_2 = y'_2(x), \]

provide two linearly independent solutions

\[ Y_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \] (3.2)

of the homogeneous system

\[ y' = z, \]

\[ z' + a(x) z + b(x) y = 0. \] (3.3)

The linear combination of (3.2)

\[ Y = C_1 Y_1 + C_2 Y_2 = \begin{pmatrix} C_1 y_1 + C_2 y_2 \\ C_1 z_1 + C_2 z_2 \end{pmatrix} \]

furnishes the following general solution to the system (3.3):

\[ y = C_1 y_1 + C_2 y_2, \]

\[ z = C_1 z_1 + C_2 z_2. \] (3.4)

Now we replace the parameters \( C_1, C_2 \) in (3.4) by \( u_1(x), \ u_2(x) \) and seek the solution to the non-homogeneous system (3.1) in the form

\[ y = u_1(x) y_1 + u_2(x) y_2, \]

\[ z = u_1(x) z_1 + u_2(x) z_2. \] (3.5)

Substitution of (3.5) in Eqs. (3.1) yields:

\[ u_1 y'_1 + u_2 y'_2 + u'_1 y_1 + u'_2 y_2 = u_1 z_1 + u_2 z_2, \]

\[ u_1 z'_1 + u_2 z'_2 + u'_1 z_1 + u'_2 z_2 + a(x)(u_1 z_1 + u_2 z_2) + b(x)(u_1(x) y_1 + u_2(x) y_2) = f(x), \]

or upon rearranging the terms:

\[ u_1 y'_1 + u_2 y'_2 + u'_1 y_1 + u'_2 y_2 = u_1 z_1 + u_2 z_2, \] (3.6)

\[ [z'_1 + a(x) z_1 + b(x) y_1] u_1 + [z'_2 + a(x) z_2 + b(x) y_2] u_2 + u'_1 z_1 + u'_2 z_2 = f(x). \] (3.7)
Since \((y_1, z_1)\) and \((y_2, z_2)\) solve the homogeneous system (3.3), the terms in brackets in Eq. (3.7) vanish and Eqs. (3.6)-(3.7) are written
\[
\begin{align*}
u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2 &= u_1 y_1' + u_2 y_2', \\
u_1' z_1 + u_2' z_2 &= f(x),
\end{align*}
\]
whence
\[
\begin{align*}
y_1 u_1' + y_2 u_2' &= 0, \\
y_1' u_1' + y_2' u_2' &= f(x).
\end{align*}
\]
(3.8)
Thus, we have arrived at Eqs. (2.17) without imposing the restriction (2.13) \textit{a priori}.

### 3.2 Third-order equations

Let us rewrite the third-order equation
\[
y''' + a(x)y'' + b(x)y' + c(x)y = f(x)
\]
(3.9)
as the system of three first-order linear equations for three dependent variables \(y, z, v:\)
\[
\begin{align*}
y' &= z, \\
z' &= v, \\
v' + a(x)v + b(x)z + c(x)y &= f(x).
\end{align*}
\]
(3.10)
Three linearly independent solutions
\[
y_1 = y_1(x), \quad y_2 = y_2(x), \quad y_3 = y_3(x)
\]
of the homogeneous equation
\[
y''' + a(x)y'' + b(x)y' + c(x)y = 0
\]
(3.11)
taken together with
\[
\begin{align*}
z_1 &= y_1'(x), & v_1 &= y_1''(x), \\
z_2 &= y_2'(x), & v_2 &= y_2''(x), \\
z_3 &= y_3'(x), & v_3 &= y_3''(x),
\end{align*}
\]
provide three linearly independent solutions

\[ Y_1 = \begin{pmatrix} y_1 \\ z_1 \\ v_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_2 \\ z_2 \\ v_2 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} y_3 \\ z_3 \\ v_3 \end{pmatrix} \]  

(3.12)
of the homogeneous system

\[ y' = z, \]
\[ z' = v, \]
\[ v' + a(x)v + b(x)z + c(x)y = 0. \]  

(3.13)
The linear combination of (3.12)

\[ Y = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 = \begin{pmatrix} C_1 y_1 + C_2 y_2 + C_3 y_3 \\ C_1 z_1 + C_2 z_2 + C_3 z_3 \\ C_1 v_1 + C_2 v_2 + C_3 v_3 \end{pmatrix} \]

furnishes the following general solution to the system (3.13):

\[ y = C_1 y_1 + C_2 y_2 + C_3 y_3, \]
\[ z = C_1 z_1 + C_2 z_2 + C_3 z_3, \]
\[ v = C_1 v_1 + C_2 v_2 + C_3 v_3. \]  

(3.14)
Now we follow the procedure used in the case of second-order equations. Namely, we replace the parameters \( C_1, C_2, C_3 \) in (3.13) by \( u_1(x), u_2(x), u_3(x) \) and seek the solution to the non-homogeneous system (3.10) in the form

\[ y = u_1(x) y_1 + u_2(x) y_2 + u_3(x) y_3, \]
\[ z = u_1(x) z_1 + u_2(x) z_2 + u_3(x) z_3, \]
\[ v = u_1(x) v_1 + u_2(x) v_2 + u_3(x) v_3. \]  

(3.15)
Substituting (3.15) in Eqs. (3.10) and rearranging the terms we obtain:

\[ \sum_{i=1}^{3} u_i y_i' + \sum_{i=1}^{3} y_i u_i' = \sum_{i=1}^{3} u_i z_i, \]
\[ \sum_{i=1}^{3} u_i z_i' + \sum_{i=1}^{3} z_i u_i' = \sum_{i=1}^{3} u_i v_i, \]
\[ \sum_{i=1}^{3} \left( v'_i + a v_i + b z_i + c y_i \right) u_i + \sum_{i=1}^{3} v_i u_i' = f(x). \]  

(3.16)
Since \((y_i, z_i, v_i), \ i = 1, 2, 3\), solve the homogeneous system (3.13), we have
\[
y_i' = z_i, \quad z_i' = v_i, \quad v_i' + av_i + bz_i + cy_i = 0,
\]
and hence Eqs. (3.16) are written:
\[
\sum_{i=1}^{3} y_i u_i' = 0, \quad \sum_{i=1}^{3} z_i u_i' = 0, \quad \sum_{i=1}^{3} v_i u_i' = f(x).
\]
Thus, we have arrived at the equations
\[
y_1' u_1' + y_2' u_2' + y_3' u_3' = 0,
\]
\[
y_1'' u_1' + y_2'' u_2' + y_3'' u_3' = f(x).
\]
In the traditional approach the first two equations (3.17) are imposed \textit{a priori}.

One can readily solve the system of linear equations (3.17) for \(u_1', u_2', u_3'\) proceeding as in Section 2.3. Namely, we write the system (3.17) in the vector form
\[
MU = F,
\]
where \(M\) is the \(3 \times 3\) matrix, and \(U, F\) are column vectors defined as follows:
\[
M = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}, \quad U = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}.
\]
The determinant of the matrix \(M\) is the Wronskian of \(y_1, y_2, y_3\):
\[
\det M = W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.
\]
Since the solutions \(y_1(x), y_2(x), y_3(x)\) are linearly independent, we have
\[
W[y_1, y_2, y_3] \neq 0.
\]
Hence, the matrix \(M\) is invertible. For our purposes, it suffices to write the inverse matrix in the form (see, e.g., [32], Section 1.1.1)
\[
M^{-1} = \frac{1}{W[y_1, y_2, y_3]} \begin{pmatrix} W[y_2, y_3] \\ W[y_3, y_1] \\ W[y_1, y_2] \end{pmatrix},
\]
where
\[ W[y_i, y_k] = \begin{vmatrix} y_i & y_k \\ y_i' & y_k' \end{vmatrix} = y_i y_k' - y_k y_i', \quad i, k = 1, 2, 3. \quad (3.22) \]

Accordingly, the solution to Eq. (3.18) is given by
\[ \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \frac{1}{W[y_1, y_2, y_3]} \begin{pmatrix} W[y_2, y_3] f(x) \\ W[y_3, y_1] f(x) \\ W[y_1, y_2] f(x) \end{pmatrix}. \quad (3.23) \]

In other words,
\[ u'_1 = \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]}, \quad u'_2 = \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]}, \quad u'_3 = \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]}, \]

whence, upon integration:
\[ \begin{align*}
    u_1 &= \int \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]} \, dx + C_1, \\
    u_2 &= \int \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]} \, dx + C_2, \\
    u_3 &= \int \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]} \, dx + C_3.
\end{align*} \quad (3.24) \]

Substituting (3.24) in the first equation (3.15) we obtain the general solution of the non-homogeneous equation (3.9):
\[ y = \begin{cases} 
    C_1 y_1 + C_2 y_2 + C_3 y_3 + y_1 \int \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]} \, dx \\
    + y_2 \int \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]} \, dx + y_3 \int \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]} \, dx. 
\end{cases} \quad (3.25) \]

### 3.3 Higher-order equations

Consider an nth-order linear equation
\[ y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x) \quad (3.26) \]

with known linearly independent solutions \( y_1(x), \ldots, y_n(x) \) of the homogeneous equation. Proceeding as in Section 3.2, we obtain the equations
similar to (3.17):

\begin{align*}
y_1 u_1' + \cdots + y_n u_n' &= 0, \\
y_1' u_1' + \cdots + y_n' u_n' &= 0, \\
\cdots & \\
y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' &= f(x). \tag{3.27}
\end{align*}

Then we introduce the following matrix \( M \) and the vectors \( U, F \):

\begin{align*}
M &= \begin{pmatrix} y_1 & \cdots & y_n \\
y_1' & \cdots & y_n' \\
\cdots & \cdots & \cdots \\
y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}, \quad U = \begin{pmatrix} u_1' \\
u_2' \\
\cdots \\
u_n' \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\
0 \\
\cdots \\
f(x) \end{pmatrix} \tag{3.28}
\end{align*}

and write the system (3.27) in the vector form:

\[ MU = F. \tag{3.29} \]

In order to solve Eq. (3.29), we have to find the inverse matrix to \( M \) and write it in the form similar to (3.21).

Recall that the inverse to a matrix

\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{3.30} \]

with non-vanishing determinant \(|A| = \det A\) has the form

\[ A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}, \tag{3.31} \]

where \( A_{ij} \) is the cofactor to the element \( a_{ij} \) of the matrix (3.30).

Applying the formula (3.31) to the matrix \( M \) given in (3.27) and invoking that the determinant of the matrix \( M \) is the Wronskian

\[ W_n[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\cdots & \cdots & \cdots & \cdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \tag{3.32} \]
we obtain by keeping only the last column (cf. (2.26) and (3.21)):

\[
M^{-1} = \frac{1}{W_n[y_1, y_2, \ldots, y_n]} \begin{pmatrix}
\cdots 
& (-1)^{n-1}W_{n-1}[y_2, y_3, \ldots, y_n] \\
\cdots 
& (-1)^{n-2}W_{n-1}[y_1, y_3, \ldots, y_n] \\
\cdots 
& 
\cdots 
& W_{n-1}[y_1, y_2, \ldots, y_{n-1}]
\end{pmatrix}.
\] (3.33)

Accordingly, the solution to Eq. (3.29) is given by

\[
\begin{pmatrix}
u_1' \\
u_2' \\
\vdots \\
u_n'
\end{pmatrix} = \frac{1}{W_n[y_1, y_2, \ldots, y_n]} \begin{pmatrix}
\cdots 
& (-1)^{n-1}W_{n-1}[y_2, y_3, \ldots, y_n] f(x) \\
\cdots 
& (-1)^{n-2}W_{n-1}[y_1, y_3, \ldots, y_n] f(x) \\
\cdots 
& 
\cdots 
& W_{n-1}[y_1, y_2, \ldots, y_{n-1}] f(x)
\end{pmatrix}.
\] (3.34)

Integrating (3.34), we obtain \(u_i(x), \ i = 1, \ldots, n\), and hence the general solution

\[
y(x) = \sum_{i=1}^{n} u_i(x) y_i(x)
\] (3.35)

of the non-homogeneous equation (3.26).

**Example 3.1.** Let us solve the fourth-order equation

\[
\frac{d^4y}{dx^4} - y = f(x)
\] (3.36)

describing the phenomenon of “beating” of driving shafts due to the centrifugal force (see [36], Section 2.3.3) in presence of an external force \(f(x)\) such as friction, etc. The general solution of the homogeneous equation

\[
y'''' - y = 0
\]

is given by

\[
y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x.
\]

Hence, four linearly independent solutions of the homogeneous equation are

\[
y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = \cos x, \quad y_4(x) = \sin x.
\] (3.37)

In our case Eq. (3.34) is written

\[
\begin{pmatrix}
u_1' \\
u_2' \\
u_3' \\
u_4'
\end{pmatrix} = \frac{1}{W_4[y_1, y_2, y_3, y_4]} \begin{pmatrix}
-W_3[y_2, y_3, y_4] f(x) \\
W_3[y_1, y_3, y_4] f(x) \\
-W_3[y_1, y_2, y_4] f(x) \\
W_3[y_1, y_2, y_3] f(x)
\end{pmatrix}.
\] (3.38)
The Wronskian (3.32) of the functions (3.37) has the form

$$W_4[y_1, y_2, y_3, y_4] = \begin{vmatrix} e^x & e^{-x} & \cos x & \sin x \\ e^x & -e^{-x} & -\sin x & \cos x \\ e^x & -e^{-x} & -\cos x & -\sin x \\ e^x & e^{-x} & \sin x & -\cos x \end{vmatrix}. \quad (3.39)$$

Working out the determinant (3.39), one obtains

$$W_4[y_1, y_2, y_3, y_4] = -8. \quad (3.40)$$

Let us compute the Wronskians $W_3$ in (3.38). We have:

$$W_3[y_2, y_3, y_4] = \begin{vmatrix} e^{-x} & \cos x & \sin x \\ -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \end{vmatrix} = 2 e^{-x}. \quad (3.41)$$

$$W_3[y_1, y_3, y_4] = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = 2 e^x. \quad (3.42)$$

$$W_3[y_1, y_2, y_4] = \begin{vmatrix} e^x & e^{-x} & \sin x \\ e^x & -e^{-x} & \cos x \\ e^x & e^{-x} & -\sin x \end{vmatrix} = 4 \sin x. \quad (3.43)$$

$$W_3[y_1, y_2, y_3] = \begin{vmatrix} e^x & e^{-x} & \cos x \\ e^x & -e^{-x} & -\sin x \\ e^x & e^{-x} & -\cos x \end{vmatrix} = 4 \cos x. \quad (3.44)$$

Substituting (3.40)-(3.44) in (3.38) an integrating, we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} C_1 + \frac{1}{4} \int f(x) e^{-x} \, dx \\ C_2 - \frac{1}{4} \int f(x) e^x \, dx \\ C_3 + \frac{1}{2} \int f(x) \sin x \, dx \\ C_4 - \frac{1}{2} \int f(x) \cos x \, dx \end{pmatrix}.$$
Now the formula (3.35) gives the following general solution to Eq. (3.36):

\[ y = y_*(x) + C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x, \]

where \(C_1, \ldots, C_4\) are arbitrary parameters and \(y_*(x)\) is a particular solution of Eq. (3.36) defined by

\[
y_*(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx - \frac{1}{4} e^{-x} \int f(x) e^x \, dx + \frac{\cos x}{2} \int f(x) \sin x \, dx - \frac{\sin x}{2} \int f(x) \cos x \, dx.
\] (3.45)

Let us verify that the function \(y_*(x)\) solves Eq. (3.36). The differentiation yields:

\[
y_*(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx + \frac{1}{4} f(x) + \frac{1}{4} e^{-x} \int f(x) e^x \, dx - \frac{1}{4} f(x)
- \frac{\sin x}{2} \int f(x) \sin x \, dx + \frac{1}{2} f(x) \cos x \sin x
- \frac{\cos x}{2} \int f(x) \cos x \, dx - \frac{1}{2} f(x) \sin x \cos x.
\]

Hence,

\[
y_*(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx + \frac{1}{4} e^{-x} \int f(x) e^x \, dx
- \frac{\sin x}{2} \int f(x) \sin x \, dx - \frac{\cos x}{2} \int f(x) \cos x \, dx.
\]

Differentiating further, we obtain likewise:

\[
y_*'(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx - \frac{1}{4} e^{-x} \int f(x) e^x \, dx
- \frac{\cos x}{2} \int f(x) \sin x \, dx + \frac{\sin x}{2} \int f(x) \cos x \, dx,
\]

\[
y_*''(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx - \frac{1}{4} e^{-x} \int f(x) e^x \, dx
+ \frac{\sin x}{2} \int f(x) \sin x \, dx + \frac{\cos x}{2} \int f(x) \cos x \, dx,
\]

\[
y_*'''(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx + \frac{1}{4} e^{-x} \int f(x) e^x \, dx
- \frac{\cos x}{2} \int f(x) \sin x \, dx + \frac{\sin x}{2} \int f(x) \cos x \, dx.
\]
and finally

$$y^{(4)}_*(x) = \frac{1}{4} e^x \int f(x) e^{-x} \, dx - \frac{1}{4} e^{-x} \int f(x) e^x \, dx$$

$$+ \frac{\cos x}{2} \int f(x) \sin x \, dx - \frac{\sin x}{2} \int f(x) \cos x \, dx + f(x)$$

$$y_*(x) + f(x).$$

Hence, Eq. (3.36) is satisfied: $y^{(4)}_*(x) - y_*(x) = f(x)$.

### 4 Solution of initial value problems

The integral representations of the general solutions obtained by the method of variation of parameters are very convenient for solving arbitrary initial value problems (Cauchy’s problems). I will illustrate the statement using the solutions considered in the previous sections.

#### 4.1 First-order equations

Let us solve an arbitrary initial value problem for Eq. (2.1) at $x = x_0$:

$$y' + P(x)y = Q(x), \quad y(x_0) = Y_0,$$

where $x_0$ and $Y_0$ are arbitrary constants. We will use the integral representation (2.7) of the general solution:

$$y = C e^{-\int Pdx} + e^{-\int Pdx} \int Q(x) e^{\int Pdx} dx,$$

(2.7)

Recall that the function $e^{-\int Pdx}$ represents any solution of the homogeneous equations. We will chose for our convenience one of them, $y_1(x)$, which equals to 1 at $x = x_0$, namely

$$y_1(x) = e^{-\int_{x_0}^x P(\xi) d\xi}.$$  
(4.2)

Then Eq. (2.7) is written

$$y = C y_1(x) + y_1(x) \int \frac{Q(\xi)}{y_1(\xi)} \, d\xi,$$

(4.3)

where, according to (4.2),

$$\frac{1}{y_1(\xi)} = e^{\int_{x_0}^\xi P(\eta) d\eta}.$$  
(4.4)
The second term in (4.3) is an unspecified particular solution to the non-homogeneous equation. We can take any of them. We will chose the particular solution, \( y^* (x) \), which vanishes at \( x = x_0 \), namely

\[ y^* (x) = y_1 (x) \int_{x_0}^{x} \frac{Q(\xi)}{y_1 (\xi)} \, d\xi. \]  

(4.5)

Thus the general solution (2.7) of the non-homogeneous equation is written in the form

\[ y = C y_1 (x) + y^* (x), \]  

(4.6)

where the particular solution \( y_1 (x) \) of the homogeneous equation satisfies the initial condition \( y_1 (x_0) = 1 \) and the particular solution \( y^* (x) \) of the non-homogeneous equation satisfies the initial condition \( y^* (x_0) = 0 \). Now we substitute (4.6) in the initial condition of the problem (4.1) and obtain \( C = Y_0 \).

Finally, substituting in Eq. (4.6) \( C = Y_0 \) and invoking Eqs. (4.2), (4.4), (4.5), we obtain the following result.

**Theorem 4.1.** The solution to the initial value problem (4.1) with arbitrary \( x_0 \) and \( Y_0 \) has the form

\[ y(x) = Y_0 e^{- \int_{x_0}^{x} P(\xi) \, d\xi} + e^{- \int_{x_0}^{x} P(\eta) \, d\eta} \int_{x_0}^{x} Q(\xi) e^{\int_{x_0}^{\xi} P(\eta) \, d\eta} \, d\xi. \]  

(4.7)

### 4.2 Second-order equations

Let us solve an arbitrary initial value problem for Eq. (2.8) at \( x = x_0 \):

\[ y'' + a(x)y' + b(x)y = f(x), \quad y(x_0) = Y_0, \quad y'(x_0) = Y_1, \]  

(4.8)

where \( x_0 \), \( Y_0 \) and \( Y_1 \) are arbitrary constants.

Following the discussion of the initial value problem for the first-order equations, we will specify the integral representation (2.21) of the general solution as follows:

\[ y(x) = C_1 y_1 (x) + C_2 y_2 (x) + y^* (x). \]  

(4.9)

Here \( y^* (x) \) is a particular solution of the non-homogeneous equation (2.8) defined by

\[ y^* (x) = -y_1 (x) \int_{x_0}^{x} \frac{y_2 (\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y_2 (x) \int_{x_0}^{x} \frac{y_1 (\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi, \]  

(4.10)

where \( W[y_1, y_2](\xi) = y_1 (\xi) y_2' (\xi) - y_2 (\xi) y_1' (\xi) \).
Lemma 4.1. The particular solution (4.10) satisfies the following initial conditions:
\[ y_*(x_0) = 0, \quad y'_*(x_0) = 0. \] (4.11)

Proof. It is obvious from the definition (4.10) of \( y_*(x) \) that the first equation (4.11) is satisfied. Let us verify the second equation. We have by differentiation:
\[ y'_*(x) = -y'_1(x) \int_{x_0}^{x} \frac{y_2(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y'_2(x) \int_{x_0}^{x} \frac{y_1(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi \]
\[ - y_1(x) \frac{y_2(x) f(x)}{W[y_1, y_2](x)} + y_2(x) \frac{y_1(x) f(x)}{W[y_1, y_2](x)}. \]

Hence,
\[ y'_*(x) = -y'_1(x) \int_{x_0}^{x} \frac{y_2(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y'_2(x) \int_{x_0}^{x} \frac{y_1(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi. \] (4.12)

It is manifest from (4.12) that \( y'_*(x_0) = 0 \). This completes the proof.

Theorem 4.2. The solution to the initial value problem (4.8) with arbitrary \( x_0, \ Y_0 \) and \( Y_1 \) has the form
\[ y(x) = C_1 y_1(x) + C_2 y_2(x) \]
\[ - y_1(x) \int_{x_0}^{x} \frac{y_2(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y_2(x) \int_{x_0}^{x} \frac{y_1(\xi) f(\xi)}{W[y_1, y_2](\xi)} \, d\xi, \] (4.13)
where \( C_1, C_2 \) are determined by solving the system of linear algebraic equations
\[ C_1 y_1(x_0) + C_2 y_2(x_0) = Y_0, \]
\[ C_1 y'_1(x_0) + C_2 y'_2(x_0) = Y_1. \] (4.14)

Proof. We substitute the function \( y(x) \) defined by (4.9) and its derivative
\[ y'(x) = C_1 y'_1(x) + C_2 y'_2(x) + y'_*(x) \]
in the initial conditions (4.8), use Eqs. (4.11) and see that the initial conditions lead to Eqs. (4.14). This proves the theorem because the function (4.13) solves the differential equation in the problem (4.8).

Remark 4.1. The system (4.14) can be solved at once, giving
\[ C_1 = \frac{Y_0 y'_2(x_0) - Y_1 y_2(x_0)}{W[y_1, y_2](x_0)}, \]
\[ C_2 = \frac{Y_1 y_1(x_0) - Y_0 y'_1(x_0)}{W[y_1, y_2](x_0)}. \] (4.15)
4.3 Third-order equations

Let us solve an arbitrary initial value problem for Eq. (3.9) at \( x = x_0 \):

\[
y''' + a(x)y'' + b(x)y' + c(x)y = f(x),
\]

\[
y(x_0) = Y_0, \quad y'(x_0) = Y_1, \quad y''(x_0) = Y_2,
\]

where \( x_0, Y_0, Y_1 \) and \( Y_2 \) are arbitrary constants.

Proceeding as in Section 4.2, we rewrite the formula (3.25) for the general solution in the form

\[
y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) + y_s(x),
\]

where \( y_s(x) \) is a particular solution of the non-homogeneous equation (3.9) defined by

\[
y_s(x) = y_1(x) \int_{x_0}^x \frac{W[y_2, y_3](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi
\]

\[
+ y_2(x) \int_{x_0}^x \frac{W[y_3, y_1](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi + y_3(x) \int_{x_0}^x \frac{W[y_1, y_2](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi.
\]

**Lemma 4.2.** The particular solution (4.18) satisfies the following initial conditions:

\[
y_s(x_0) = 0, \quad y'_s(x_0) = 0, \quad y''_s(x_0) = 0.
\]

**Proof.** It is obvious from the definition (4.18) of \( y_s(x) \) that the first equation (4.19) is satisfied. In order to verify the second equation (4.19), we differentiate (4.18):

\[
y'_s(x) = y'_1(x) \int_{x_0}^x \frac{W[y_2, y_3](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi + y_1(x) \int_{x_0}^x \frac{W[y_2, y_3](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi
\]

\[
+ y_2(x) \int_{x_0}^x \frac{W[y_3, y_1](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi + y_2(x) \int_{x_0}^x \frac{W[y_3, y_1](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi
\]

\[
+ y_3(x) \int_{x_0}^x \frac{W[y_1, y_2](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi + y_3(x) \int_{x_0}^x \frac{W[y_1, y_2](\xi)}{W[y_1, y_2, y_3](\xi)} f(\xi) \, d\xi.
\]

The reckoning shows that

\[
y_1(x) W[y_2, y_3](x) + y_2(x) W[y_3, y_1](x) + y_3(x) W[y_1, y_2](x) = 0.
\]
Therefore
\[ y'_x(x) = y'_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y'_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y'_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi. \] (4.20)

Another differentiation yields:
\[ y''_x(x) = y''_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y''_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y''_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi. \] (4.21)

The reckoning shows that
\[ y'_1(x) W[y_2, y_3](x) + y'_2(x) W[y_3, y_1](x) + y'_3(x) W[y_1, y_2](x) = 0. \]

Therefore
\[ y''_x(x) = y''_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y''_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y''_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi. \] (4.21)

Eqs. (4.20), (4.21) yield that \( y'_x(x_0) = 0 \), \( y''_x(x_0) = 0 \). This completes the proof.

**Theorem 4.3.** The solution to the initial value problem (4.16) with arbitrary \( x_0 \), \( Y_0 \), \( Y_1 \) and \( Y_3 \) has the form
\[ y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) + y_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi \]
\[ + y_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi + y_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi) f(\xi)}{W[y_1, y_2, y_3](\xi)} \, d\xi, \] (4.22)
where $C_1, C_2$ are determined by solving the system of linear algebraic equations

\[
\begin{align*}
C_1 y_1(x_0) + C_2 y_2(x_0) + C_3 y_3(x_0) &= Y_0, \\
C_1 y'_1(x_0) + C_2 y'_2(x_0) + C_3 y'_3(x_0) &= Y_1, \\
C_1 y''_1(x_0) + C_2 y''_2(x_0) + C_3 y''_3(x_0) &= Y_2.
\end{align*}
\] (4.23)

The proof is similar to the proof of Theorem 4.2. Higher-order linear equations can be treated likewise.

5 Examples

Theorem 4.1 provides the simple formula (4.7) for the solution of an arbitrary initial value problem for any linear first-order equation. This formula is as simple for the practical use as the formula for roots of quadratic equations. The solution to the Cauchy problem (4.1) is obtained just by inserting the coefficients $P(x)$, $Q(x)$ of the differential equations in question and the initial data $x_0, Y_0$.

Theorems 4.2 and 4.3 play the same role for second-order and third-order equations, respectively, provided that fundamental systems of solutions for the corresponding homogeneous equations are known.

Of course, the integrals in (4.1), (4.13) and (4.22) may be difficult or impossible (this is the general case) to work out in terms of elementary functions. But this circumstance is unessential. Anyhow, equations whose solutions can be expressed in terms of elementary functions are very rare and do not have a practical value. They appear mostly in textbooks as simple illustrations of various methods.

5.1 First-order equations

Example 5.1. Let us solve the Cauchy problem

\[
y' - 2xy = x^3, \quad y(1) = Y_0.
\] (5.1)

Here

\[
P(x) = -2x, \quad Q(x) = x^3, \quad x_0 = 1,
\]

and the solution formula (4.7) is written:

\[
y(x) = Y_0 e^{\int_1^x 2\xi d\xi} + e^{\int_1^x 2\xi d\xi} \int_1^x \xi^3 e^{-\int_1^x (2\eta) d\eta} d\xi,
\]
or
\[ y(x) = Y_0 e^{x^2-1} + e^{x^2-1} \int_1^x \xi^3 e^{1-\xi^2} d\xi. \] (5.2)

One can leave the solution in the integral form (5.2). But in this particular case the integral can be easily worked out:
\[ \int_1^x \xi^3 e^{1-\xi^2} d\xi = \frac{e}{2} \int_1^x \xi^2 e^{-\xi^2} d(\xi^2) = -\frac{1}{2} (1 + \xi^2) e^{1-\xi^2} \bigg|_1^x \]
\[ = 1 - \frac{1}{2} (1 + x^2) e^{1-x^2}. \] (5.3)

Substituting (5.3) in (5.2) we obtain the solution to the problem (5.1) in elementary functions:
\[ y(x) = (1 + Y_0) e^{x^2-1} - \frac{1}{2} (1 + x^2). \] (5.4)

**Example 5.2.** Let us solve the Cauchy problem
\[ y' - y \cos x = x, \quad y(0) = Y_0. \] (5.5)

Here
\[ P(x) = - \cos x, \quad Q(x) = x, \quad x_0 = 0, \]
and the solution formula (4.7) is written:
\[ y(x) = Y_0 e^{\int_0^x \cos \xi d\xi} + e^{\int_0^x \cos \xi d\xi} \int_0^x \xi e^{-\int_0^\xi \cos \eta d\eta} d\xi. \]

Since
\[ e^{\int_0^x \cos \xi d\xi} = e^{\sin x} \]
we obtain the solution to the problem (5.5) containing one quadrature:
\[ y(x) = Y_0 e^{\sin x} + e^{\sin x} \int_0^x \xi e^{-\sin \xi} d\xi. \] (5.6)

### 5.2 Second-order equations

**Example 5.3.** Consider the Cauchy problem
\[ y'' + y = f(x), \]
\[ y(x_0) = Y_0, \quad y'(x_0) = Y_1. \] (5.7)

The functions
\[ y_1(x) = \cos x, \quad y_2(x) = \sin x \] (5.8)
provide a fundamental system of solutions for the homogeneous equation
\[ y'' + y = 0 \]
and have the Wronskian
\[ W[y_1, y_2] = 1. \]  
(5.9)
Furthermore, Eqs. (4.15) yield
\[
C_1 = Y_0 \cos x_0 - Y_1 \sin x_0, \\
C_2 = Y_1 \cos x_0 + Y_0 \sin x_0.
\]
Therefore
\[
C_1 y_1(x) + C_2 y_2(x) \\
= Y_0 \cos x_0 \cos x - Y_1 \sin x_0 \cos x + Y_1 \cos x_0 \sin x + Y_0 \sin x_0 \sin x \\
= Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0).
\]  
(5.10)
Substituting (5.8), (5.9) and (5.10) in the formula (4.13) we obtain the following solution to the problem (5.7):
\[
y(x) = Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0) \\
- \cos x \int_{x_0}^{x} f(\xi) \sin \xi \, d\xi + \sin x \int_{x_0}^{x} f(\xi) \cos \xi \, d\xi.
\]  
(5.11)
Exercise 5.1. Solve the Cauchy problem
\[
y'' + y = x^n, \quad (n = 1, 2, \ldots), \\
y(x_0) = Y_0, \quad y'(x_0) = Y_1.
\]  
(5.12)
Solution. Eq. (5.11) provides the following integral representation of solution to the problem (5.12):
\[
y(x) = Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0) \\
- \cos x \int_{x_0}^{x} \xi^n \sin \xi \, d\xi + \sin x \int_{x_0}^{x} \xi^n \cos \xi \, d\xi.
\]  
(5.13)
The solution (5.13) can be written in terms of elementary functions by using the well-known integrals
\[
\int \sin x \, dx = -\cos x, \quad \int \cos x \, dx = \sin x, \\
\int x \sin x \, dx = \sin x - x \cos x, \quad \int x \cos x \, dx = \cos x + x \sin x
\]
and the recursion formulae

\[
\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx, \\
\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx.
\]

**Exercise 5.2.** Solve the Cauchy problem

\[
y'' + y = \frac{1}{x+1} \quad (x \geq 0), \\
y(0) = Y_0, \quad y'(0) = Y_1. \tag{5.14}
\]

**Solution.** Substituting \( x_0 = 0 \) and

\[
f(x) = \frac{1}{x+1}
\]

in Eq. (5.11) we obtain the following solution to the problem (5.14):

\[
y(x) = Y_0 \cos x + Y_1 \sin x - \cos x \int_0^x \frac{\sin \xi}{\xi + 1} \, d\xi + \sin x \int_0^x \frac{\cos \xi}{\xi + 1} \, d\xi. \tag{5.15}
\]

The integrals in (5.15) cannot be worked out in terms of elementary functions. Nevertheless, one can readily verify that the function (5.15) satisfies the initial conditions and the differential equation of the problem (5.14). The condition \( y(0) = Y_0 \) is obviously satisfied. The differentiation of (5.15) yields:

\[
y'(x) = -Y_0 \sin x + Y_1 \cos x + \sin x \int_0^x \frac{\sin \xi}{\xi + 1} \, d\xi \\
\quad - \frac{\cos x \sin x}{x+1} + \cos x \int_0^x \frac{\cos \xi}{\xi + 1} \, d\xi + \frac{\sin x \cos x}{x+1} \\
= -Y_0 \sin x + Y_1 \cos x + \sin x \int_0^x \frac{\sin \xi}{\xi + 1} \, d\xi + \cos x \int_0^x \frac{\cos \xi}{\xi + 1} \, d\xi.
\]

It is manifest from the above expression for \( y'(x) \) that the condition \( y'(0) = \)}
Another differentiation yields:

\[
y''(x) = -Y_0 \cos x - Y_1 \sin x + \cos x \int_0^x \frac{\sin \xi}{\xi + 1} d\xi + \frac{\sin^2 x}{x + 1} - \sin x \int_0^x \frac{\cos \xi}{\xi + 1} d\xi + \frac{\cos^2 x}{x + 1}
\]

\[
= -Y_0 \cos x - Y_1 \sin x + \cos x \int_0^x \frac{\sin \xi}{\xi + 1} d\xi - \sin x \int_0^x \frac{\cos \xi}{\xi + 1} d\xi + \frac{1}{x + 1}
\]

\[
= -y + \frac{1}{x + 1}.
\]

Hence, the differential equation (5.14) is satisfied.

**Example 5.4.** Consider the Cauchy problem

\[
x^2 y'' + 3xy' + y = \frac{1}{x} \quad (x \geq 1),
\]

\[
y(1) = Y_0, \quad y'(1) = Y_1.
\]

Solving the homogeneous equation, i.e. the Euler equation

\[
x^2 y'' + 3xy' + y = 0,
\]

we obtain the following fundamental system of solutions:

\[
y_1(x) = \frac{1}{x}, \quad y_2(x) = \frac{\ln x}{x}.
\]

Their Wronskian is

\[
W[y_1, y_2](x) = \frac{1}{x^3}.
\]

Eqs. (4.14) yield

\[
C_1 = Y_0, \quad C_2 = Y_0 + Y_1.
\]

Now we write the differential equation of the problem (5.16) in the form (4.8),

\[
y'' + \frac{3}{x} y' + \frac{1}{x^2} y = \frac{1}{x^3},
\]

apply the formula (4.13) and, invoking Eqs. (5.17), (5.18), (5.19), obtain:

\[
y(x) = Y_0 \frac{1}{x} + (Y_0 + Y_1) \frac{\ln x}{x} - \frac{1}{x} \int_1^x \frac{\ln \xi}{\xi} d\xi + \frac{\ln x}{x} \int_1^x \frac{1}{\xi} d\xi.
\]
The integrals can be worked out at once, giving

\[
\int_1^x \frac{\ln \xi}{\xi} d\xi = \frac{1}{2} \left( \ln \xi \right)^2 \Big|_1^x = \left( \ln x \right)^2,
\]

\[
\int_1^x \frac{1}{\xi} d\xi = \ln \xi \Big|_1^x = \ln x.
\]

Substituting these expressions in (5.21) we obtain the solution to the problem (5.16) in elementary functions:

\[
y(x) = \frac{1}{x} \left[ Y_0 + (Y_0 + Y_1) \ln x + \frac{1}{2} \left( \ln x \right)^2 \right].
\]

(5.22)

It is useful to verify by direct substitution that the function (5.22) satisfies the differential equation and the initial conditions of the problem (5.16).

**Example 5.5.** Let us solve the following Cauchy problem:

\[
y'' - y' \cos x + y \sin x = f(x),
\]

\[
y(0) = Y_0, \quad y'(0) = Y_1.
\]

(5.23)

First, we will find two linearly independent solutions (a fundamental system of solutions) for the homogeneous equation

\[
y'' - y' \cos x + y \sin x = 0,
\]

(5.24)

noting that its order can be reduced. Indeed, it can be written in the form

\[
(y' - y \cos x)' = 0
\]

and integrated, giving

\[
y' - y \cos x = K_1.
\]

(5.25)

One can easily integrate the first-order equation (5.25) and find the general solution

\[
y = K_2 e^{\sin x} + K_1 e^{\sin x} \int e^{-\sin x} dx
\]

(5.26)

to Eq. (5.24) containing two arbitrary constants \( K_1, K_2 \). Hence, one can take for a fundamental system of solutions for Eq. (5.24) the following functions:

\[
y_1(x) = e^{\sin x}, \quad y_2(x) = e^{\sin x} \int_0^x e^{-\sin \xi} d\xi.
\]

(5.27)
Let us find the Wronskian of the functions (5.27). We have:

\[ y_1'(x) = \cos x e^{\sin x}, \quad y_2'(x) = 1 + \cos x e^{\sin x} \int_0^x e^{-\sin \xi} d\xi, \]

and hence

\[ W[y_1, y_2](x) = e^{\sin x}. \quad (5.28) \]

Eqs. (4.14) yield

\[ C_1 = Y_0, \quad C_2 = Y_1 - Y_0. \]

Substituting now (5.27) and (5.28) in (4.13) we obtain the following solution to the problem (5.23):

\[ y(x) = Y_1 e^{\sin x} + (Y_1 - Y_0) e^{\sin x} \int_0^x e^{-\sin \xi} d\xi \]

\[ - e^{\sin x} \int_0^x f(\xi) \left[ \int_0^\xi e^{-\sin \eta} d\eta \right] d\xi + e^{\sin x} \int_0^x e^{-\sin \xi} d\xi \int_0^x f(\xi) d\xi. \quad (5.29) \]

### 5.3 Third-order equations

**Example 5.6.** Let us solve the following Cauchy problem:

\[ y''' - y'' + y' - y = f(x), \]

\[ y(0) = Y_0, \quad y'(0) = Y_1, \quad y''(0) = Y_2. \quad (5.30) \]

The characteristic polynomial for the homogeneous equation

\[ y''' - y'' + y' - y = 0 \quad (5.31) \]

is written

\[ \lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1) \]

and has the roots \( \lambda_1 = 1, \quad \lambda_2 = i, \quad \lambda_3 = -i. \) Consequently, a fundamental system of solutions for Eq. (5.31) is provided by the functions

\[ y_1(x) = e^x, \quad y_2(x) = \cos x, \quad y_2(x) = \sin x. \quad (5.32) \]

Let us find the Wronskians (3.20) and (3.22). The reckoning yields

\[ W[y_1, y_2, y_3](x) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = 2 e^x \quad (5.33) \]
and
\[ W[y_2, y_3](x) = 1, \]
\[ W[y_3, y_1](x) = e^x (\sin x - \cos x), \]  \hspace{1cm} (5.34)
\[ W[y_1, y_2](x) = -e^x (\sin x + \cos x). \]

In the problem (5.30) we have \( x_0 \) and Eqs. (4.23), (5.32) yield:

\[ C_1 + C_2 = Y_0, \quad C_1 + C_3 = Y_1, \quad C_1 - C_2 = Y_2, \]

whence
\[ C_1 = \frac{1}{2} (Y_0 + Y_2), \quad C_2 = \frac{1}{2} (Y_0 - Y_2), \quad C_3 = \frac{1}{2} (2Y_1 - Y_0 - Y_2). \]  \hspace{1cm} (5.35)

Substituting (5.32)-(5.35) in (4.22) we obtain the following integral representation of the solution to the problem (5.30):

\[ y(x) = \frac{1}{2} \left[ (Y_0 + Y_2) e^x + (Y_0 - Y_2) \cos x + (2Y_1 - Y_0 - Y_2) \sin x \right. \]
\[ + e^x \int_0^x e^{-\xi} f(\xi) d\xi + \cos x \int_0^x (\sin \xi - \cos \xi) f(\xi) d\xi \]
\[ - \sin x \int_0^x (\sin \xi + \cos \xi) f(\xi) d\xi \left. \right] . \]  \hspace{1cm} (5.36)

Let us verify that (5.36) solves our problem. The first initial condition, \( y(0) = Y_0 \), is obviously satisfied. We differentiate (5.36):

\[ y'(x) = \frac{1}{2} \left[ (Y_0 + Y_2) e^x - (Y_0 - Y_2) \sin x + (2Y_1 - Y_0 - Y_2) \cos x \right. \]
\[ + e^x \int_0^x e^{-\xi} f(\xi) d\xi - \sin x \int_0^x (\sin \xi - \cos \xi) f(\xi) d\xi \]
\[ - \cos x \int_0^x (\sin \xi + \cos \xi) f(\xi) d\xi + f(x) \]
\[ + \cos x (\sin x - \cos x) f(x) - \sin x (\sin x + \cos x) f(x) \left. \right] , \]

whence
\[ y'(x) = \frac{1}{2} \left[ (Y_0 + Y_2) e^x - (Y_0 - Y_2) \sin x + (2Y_1 - Y_0 - Y_2) \cos x \right. \]
\[ + e^x \int_0^x e^{-\xi} f(\xi) d\xi - \sin x \int_0^x (\sin \xi - \cos \xi) f(\xi) d\xi \]
\[ - \cos x \int_0^x (\sin \xi + \cos \xi) f(\xi) d\xi \left. \right] . \]  \hspace{1cm} (5.37)
It is obvious now that \( y'(0) = Y_1 \). We differentiate (5.37) again and proceed as above to obtain:

\[
y''(x) = \frac{1}{2} \left[ (Y_0 + Y_2) e^x - (Y_0 - Y_2) \cos x - (2Y_1 - Y_0 - Y_2) \sin x 
+ e^x \int_0^x e^{-\xi} f(\xi) d\xi - \cos x \int_0^x (\sin \xi - \cos \xi) f(\xi) d\xi 
+ \sin x \int_0^x (\sin \xi + \cos \xi) f(\xi) d\xi \right].
\]

(5.38)

One can easily see that \( y''(0) = Y_2 \). We differentiate again and obtain:

\[
y'''(x) = \frac{1}{2} \left[ (Y_0 + Y_2) e^x + (Y_0 - Y_2) \sin x - (2Y_1 - Y_0 - Y_2) \cos x 
+ e^x \int_0^x e^{-\xi} f(\xi) d\xi + \sin x \int_0^x (\sin \xi - \cos \xi) f(\xi) d\xi 
+ \cos x \int_0^x (\sin \xi + \cos \xi) f(\xi) d\xi \right] + f(x).
\]

(5.39)

It follows from (5.36)-(5.39) that the differential equation (5.30) is satisfied.

For particular types of function \( f(x) \) the problem (5.30) may have the solution given by elementary functions. For instance, for the problem (5.30) with \( f(x) = x \) one can easily work out the integrals in (5.36) and obtain

\[
y(x) = -(1+x) + \frac{1}{2} \left[ (Y_0+Y_2+1) e^x + (Y_0-Y_2+1) \cos x + (2Y_1-Y_0-Y_2+1) \sin x \right].
\]
Abstract. A method of integration of nonlinear dynamical systems (systems of first-order nonlinear ordinary differential equations) admitting nonlinear superpositions is presented. The investigation is based on classification of Lie algebras. It is shown that the systems associated with one- and two-dimensional Lie algebras can be integrated by quadrature upon introducing Lie’s canonical variables. The knowledge of a symmetry group of a system in question is not needed in this approach. The systems associated with three-dimensional Lie algebras are classified into thirteen standard forms. Ten of them are integrable by quadrature. The remaining three standard forms lead to Riccati equations.

1 Introduction

Lie group analysis gives a systematic way for integrating scalar ordinary differential equations of the second and higher order. The efficiency of the group approach to these equations is guaranteed first of all by the fact that their symmetries can be calculated by solving over-determined systems of so called determining equations for infinitesimal symmetries. In the case of first-order ordinary differential equations the determining equations are not over-determined. Consequently, the Lie group methods are not sufficiently developed for integrating nonlinear systems of first-order ODEs. It was noticed in [24], Chapter 6 (see also [26], Section 11.2) that Lie algebras can
be useful for integration of systems admitting nonlinear superposition. I make here further steps in this direction.

The concept of a nonlinear superposition can be illustrated by the separable equation (see [26], Section 2.1.2)

\[ y' = p(x)q(y). \]  \hspace{1cm} (1.1)

The general solution to Eq. (1.1) is

\[ y = H^{-1}(\phi(x) + C), \]  \hspace{1cm} (1.2)

where

\[ H(y) = \int \frac{dy}{q(y)}, \quad \phi(x) = \int p(x)dx, \]

is the inverse to the function \( H(y) \) and \( C \) is an arbitrary constant. Let us take a particular solution \( y_0 \), e.g. by letting \( C = 0 \) in (1.2):

\[ y_0(x) = H^{-1}(\phi(x)). \]

Then the general solution (1.2) is written

\[ y = H^{-1}(H(y_0) + C). \]  \hspace{1cm} (1.3)

The presentation (1.3) of the general solution of the equation (1.1) is a nonlinear superposition (see also Example ?? in Section ??).

The nonlinear superposition formula (1.3) does not depend upon a choice of a particular solution \( y_0(x) \). Indeed, since (1.2) is the general solution, any particular solution corresponds to a particular choice of the arbitrary constant \( C \). Let

\[ y_1(x) = H^{-1}(\phi(x) + C_1) \]

be any particular solution. Then \( \phi(x) = H(y_1) - C_1 \), and the substitution into (1.2) yields:

\[ y = H^{-1}(H(y_1) + K), \quad K = C - C_1. \]

This is precisely the formula (1.3) with \( y_0 \) replaced by \( y_1 \) and with \( C \) replaced by a new arbitrary constant \( K \).

Consider the following example:

\[ y' = p(x)y^2. \]

Here

\[ H(y) = -\frac{1}{y} \]
and Eq. (1.2) is written

\[ y = \frac{1}{C - \phi(x)}. \]

Since \( H(y_0) = -1/y_0 \) and hence \( H(y_0) + C = (Cy_0 - 1)/y_0 \), the nonlinear superposition (1.3) has the form:

\[ y = \frac{y_0(x)}{1 - Cy_0(x)}. \]

It is natural to extend the nonlinear superposition to systems of ordinary differential equations

\[ \frac{dx^i}{dt} = f^i(t, x), \quad i = 1, \ldots, n. \]  

(1.4)

**Definition 1.1.** The system (1.4) is said to admit a nonlinear superposition of solutions if its general solution

\[ x = (x^1, \ldots, x^n) \]

can be written as a vector function

\[ x = \varphi(x_1, \ldots, x_m, C_1, \ldots, C_n) \]  

(1.5)

of a finite number of particular solutions

\[ x_1 = (x_1^1, \ldots, x_1^n), \ldots, x_m = (x_m^1, \ldots, x_m^n) \]  

(1.6)

and \( n \) arbitrary constants \( C_1, \ldots, C_n \). Here \( \varphi = (\varphi^1, \ldots, \varphi^m) \).

The problem on description of systems admitting nonlinear superposition was discussed independently by E. Vessiot [81] and A. Guldberg [17]. The final result is due to S. Lie [59], [60]. It is formulated in the following theorem (see also [36], Section 6.7).

**Theorem 1.1.** The system (1.4) admits a nonlinear superposition if and only if it has the form of generalized separation of variables:

\[ \frac{dx^i}{dt} = T_1(t)\xi_1^i(x) + \cdots + T_r(t)\xi_r^i(x), \quad i = 1, \ldots, n, \]  

(1.7)

where the coefficients \( \xi_i^\alpha(x) \) satisfy the condition that the operators

\[ X_\alpha = \xi_\alpha^i(x) \frac{\partial}{\partial x^i} \]  

(1.8)

span a finite-dimensional Lie algebra \( L_r \). The number \( m \) of necessary particular solutions (1.6) is estimated by

\[ nm \geq r. \]  

(1.9)

*We use here and further the usual summation convention in repeated indices.*
2 Semi-separable systems

Here we will consider the systems of the form

\[
\frac{dx^i}{dt} = T(t)\xi^i(x), \quad i = 1, \ldots, n. \tag{2.1}
\]

Eqs. (2.1) provide the simplest system possessing nonlinear superposition, namely the system (1.7) with the one-dimensional Vessiot-Guldberg-Lie algebra. In this case we have only one linearly independent operator (1.8):

\[
X = \xi^i(x) \frac{\partial}{\partial x^i}. \tag{2.2}
\]

The system (2.1) is an obvious generalization of the separable equation (1.1) with one dependent variable to the case of several dependent variables. I call (2.1) a semi-separable system because one can separate the variable \(t\) in all equations of the system (2.1), but this operation does not lead directly to the integration of the system in question. The integration requires one more operation, namely, a simplification of the vector \(\xi(x) = (\xi^1(x), \ldots, \xi^n(x))\).

2.1 Covariance of the system (1.7)

The following property of the general equations of the form (1.7) will be used in the integration procedure.

Proposition 2.1. The system

\[
\frac{dx^i}{dt} = T_1(t)\xi^i_1(x) + \cdots + T_r(t)\xi^i_r(x), \quad i = 1, \ldots, n, \tag{2.3}
\]

is covariant under invertible changes of variables

\[
\tilde{x}^i = \tilde{x}^i(x), \quad i = 1, \ldots, n. \tag{2.4}
\]

Specifically, Eqs. (2.3) are written in the new variables \(\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n)\) in the form

\[
\frac{d\tilde{x}^i}{dt} = T_1(t)\tilde{\xi}^i_1(\tilde{x}) + \cdots + T_r(t)\tilde{\xi}^i_r(\tilde{x}), \quad i = 1, \ldots, n, \tag{2.5}
\]

where the vectors \(\xi_\alpha = (\xi^1_\alpha, \ldots, \xi^n_\alpha)\) are transformed by the equations

\[
\tilde{\xi}^i_\alpha = \frac{\partial \tilde{x}^i(x)}{\partial x^j} \xi^j_\alpha, \quad i = 1, \ldots, n, \tag{2.6}
\]

and the coefficients \(T_\alpha(t)\) in Eqs. (2.5) are the same as in Eqs. (2.3).
Proof. The property stated in this proposition follows from the fact that the derivative $dx/dt$ of the vector $x = (x^1, \ldots, x^n)$ with respect to $t$ is a \textit{contravariant vector}. It means that $dx/dt$ obeys the same transformation law (2.6) as the vector $\xi_\alpha$:

$$ \frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i(x)}{\partial x^j} \frac{dx^j}{dt}, \quad i = 1, \ldots, n. $$  \hfill (2.7)

### 2.2 Integration of semi-separable systems

**Proposition 2.2.** The semi-separable system (2.1) can be rewritten in the separable form

$$ \frac{d\tilde{x}^1}{dt} = \cdots = \frac{d\tilde{x}^{n-1}}{dt} = 0, \quad \frac{d\tilde{x}^n}{dt} = T(t) $$  \hfill (2.8)

using the change of variables (2.4) defined by the following equations:

$$ X(\tilde{x}^1) = \cdots = X(\tilde{x}^{n-1}) = 0, \quad X(\tilde{x}^n) = 1. $$  \hfill (2.9)

**Proof.** Recall that the change of variables (2.4) maps the operator (2.2) into the form

$$ \tilde{X} = X(\tilde{x}^i) \frac{\partial}{\partial \tilde{x}^i}, $$  \hfill (2.10)

where

$$ X(\tilde{x}^i) = \xi^j(x) \frac{\partial \tilde{x}^i(x)}{\partial x^j}. $$

Eq. (2.10) is a direct consequence of the chain rule

$$ \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^k(x)}{\partial x^i} \frac{\partial}{\partial \tilde{x}^k}. $$

Eq. (2.10) shows that after introducing the variables $\tilde{x}^i$ (called \textit{canonical variables}) defined the equations (2.9) the operator (2.2) becomes

$$ \tilde{X} = \frac{\partial}{\partial \tilde{x}^n}. $$  \hfill (2.11)

Its coordinates are

$$ \tilde{\xi}^1 = \cdots = \tilde{\xi}^{n-1} = 0, \quad \tilde{\xi}^n = 1. $$

Therefore the statement of Proposition 2.2 follows from Proposition 2.1.
2.3 Example of integration of a semi-separable system

The system
\[ \frac{dx}{dt} = T(t) xy^2, \quad \frac{dy}{dt} = T(t) x^2 y \]  
(2.12)

has the form (2.1) with the following operator (2.2):
\[ X = xy^2 \frac{\partial}{\partial x} + x^2 y \frac{\partial}{\partial y}. \]  
(2.13)

Solution of the equations (2.9) gives the following canonical variables:
\[ \tilde{x} = x^2 - y^2, \quad \tilde{y} = \ln y - \ln x. \]  
(2.14)

In these variables, the operator (2.13) takes the form (2.11),
\[ X = \frac{\partial}{\partial \tilde{y}}, \]  
(2.15)

and the system is written in the separable form (2.8):
\[ \frac{d\tilde{x}}{dt} = 0, \quad \frac{d\tilde{y}}{dt} = T(t). \]  
(2.16)

Integrating the system (2.16), one obtains
\[ \tilde{x} = a, \quad \tilde{y} = b + \tau(t), \]  
(2.17)

where
\[ \tau(t) = \int T(t) dt \]

and \( a, b = \text{const.} \). We will assume that \( a \geq 0 \) for the sake of simplicity of further calculations.

\[ x = \sqrt{\frac{\tilde{x}}{1 - e^{2\tilde{y}}}}, \quad y = e^{\tilde{y}} \sqrt{\frac{\tilde{x}}{1 - e^{2\tilde{y}}}}. \]  
(2.18)

Eqs. (2.17), (2.18) provide the following general solution to the system (2.12):
\[ x = \sqrt{\frac{a}{1 - e^{2a[b+\tau(t)]}}}, \quad y = e^{a[b+\tau(t)]} \sqrt{\frac{a}{1 - e^{2a[b+\tau(t)]}}}. \]  
(2.19)
3 Systems associated with $L_2$

This section is devoted to integration of systems (1.7) associated with two-dimensional algebras. In other words, we will consider the systems of the form

$$\frac{dx^i}{dt} = T_1(t)\xi^i_1(x) + T_2(t)\xi^i_2(x), \quad i = 1, \ldots, n, \quad (3.1)$$

where $x = (x^1, \ldots, x^n)$, and the operators

$$X_1 = \xi^1_1(x)\frac{\partial}{\partial x^1}, \quad X_2 = \xi^1_2(x)\frac{\partial}{\partial x^1} \quad (3.2)$$

span a two-dimensional Lie algebra $L_2$. It means that the operators $X_1, X_2$ are linearly independent with constant coefficients and satisfied the condition

$$[X_1, X_2] = c_1X_1 + c_2X_2$$

with constant coefficients $c_1, c_2$, where

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

is the commutator.

Recall that Lie’s method of integration of second-order ordinary differential equations using their symmetries is based on existence of so-called \textit{canonical variables} in two-dimensional Lie algebras $L_2$ of operators

$$X = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

in the $(x, y)$ plane. Lie demonstrated that there exist precisely four distinctly different types of $L_2$ and showed that each type can be written in a simple form (often called a \textit{standard form}) by choosing appropriate canonical variables $t = t(x, y), \quad u = u(x, y)$ (for the proof, see[58], Chapter 18, §1; see also [26], Section 12.2.2).

I will show that Lie’s classification of two-dimensional Lie algebras $L_2$ of operators in the plane can be extended to $L_2$ spanned by the operators (3.2) in the $n$-dimensional space of the variables $x = (x^1, \ldots, x^n)$. Our calculations will give canonical variables for the Vessiot-Guldberg-Lie algebra. In these variables, the system (3.1) will take four different integrable forms.

3.1 Canonical variables in two-dimensional algebras

Let $L_2$ be the two-dimensional Lie algebra with the basis (3.2). The algebra $L_2$ can be either Abelian, $[X_1, X_2] = 0$, or non-Abelian, $[X_1, X_2] \neq 0$. In the
latter case the commutator can be taken to be \([X_1, X_2] = X_1\) by choosing a suitable basis in \(L_2\). Indeed, if \([X_1, X_2] = c_1X_1 + c_2X_2 \neq 0\), one obtains \([X'_1, X'_2] = X'_1\) in the new basis

\[
X'_1 = c_1X_1 + c_2X_2, \quad X'_2 = k_1X_1 + k_2X_2
\]

with \(k_1, k_2\) satisfying the equation \(c_1k_2 - c_2k_1 = 1\).

The operators (3.2) are said to be linearly connected if the equation

\[
\lambda_1(x)X_1 + \lambda_2(x)X_2 = 0
\]

holds identically in \(x\) with certain functions \(\lambda_1(x) \neq 0, \lambda_2(x) \neq 0\). The operators (3.2) are linearly connected if and only if the rank \(R\) of the \(2 \times n\) matrix of the coefficients of the operators \(X_1, X_2\):

\[
R = \text{rank} \begin{vmatrix}
\xi_1^1 & \cdots & \xi_1^n \\
\xi_2^1 & \cdots & \xi_2^n 
\end{vmatrix},
\]

is equal to 1.

Both Abelian and non-Abelian algebras have two non-similar realizations according to whether or not the basic operators (3.2) are linearly connected. The result is formulated as follows.

**Lemma 3.1.** Every two-dimensional Lie algebra \(L_2\) spanned by the operators (3.2) can be reduced, by choosing an appropriate basis, to one of the following four distinctly different types:

I. \([X_1, X_2] = 0, \quad R = 2\),

II. \([X_1, X_2] = 0, \quad R = 1\),

III. \([X_1, X_2] = X_1, \quad R = 2\),

IV. \([X_1, X_2] = X_1, \quad R = 1\).

The structure relations (3.4) are invariant with respect any change of variables \(x^i\). Using this invariance, we simplify the form of basis operators (3.2) of all types I–IV and obtain the following result.

**Theorem 3.1.** The two-dimensional Lie algebra spanned by the operators (3.2) can be transformed by choosing suitable canonical variables

\[
x, \ y, \ z^1, \ldots, z^{n-2}
\]

to one of Lie’s standard forms presented in the following table.
Table 3.1. Standard forms of $L_2$

<table>
<thead>
<tr>
<th>Type</th>
<th>Structure of $L_2$</th>
<th>Standard form of $L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$[X_1, X_2] = 0, \ R = 2$</td>
<td>$X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial y}$</td>
</tr>
<tr>
<td>II</td>
<td>$[X_1, X_2] = 0, \ R = 1$</td>
<td>$X_1 = \frac{\partial}{\partial u}, \ X_2 = x \frac{\partial}{\partial y}$</td>
</tr>
<tr>
<td>III</td>
<td>$[X_1, X_2] = X_1, \ R = 2$</td>
<td>$X_1 = \frac{\partial}{\partial y}, \ X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$</td>
</tr>
<tr>
<td>IV</td>
<td>$[X_1, X_2] = X_1, \ R = 1$</td>
<td>$X_1 = \frac{\partial}{\partial y}, \ X_2 = y \frac{\partial}{\partial y}$</td>
</tr>
</tbody>
</table>

**Proof.** To prove the statement, we will find the canonical variables (3.5),

\[
x = x(x^1, \ldots, x^n), \quad y = y(x^1, \ldots, x^n),
\]
\[
z^1 = z^1(x^1, \ldots, x^n), \ldots, \ z^{n-2} = z^{n-2}(x^1, \ldots, x^n),
\] (3.6)

for all types (3.4) of two-dimensional Lie algebras.

*Type I.* Using Proposition 2.2, we first choose the coordinates (3.6) so that the first operator $X_1$ (3.2) is written

\[X_1 = \frac{\partial}{\partial x}.\] (3.7)

Then the condition $[X_1, X_2] = 0$ for algebras of type I yields $(\xi_2)_x = 0$ which means that the coordinates of the second operator in (3.2) do not depend on the variable $x$. Let us write this operator in the form

\[X_2 = \xi(y, z) \frac{\partial}{\partial x} + \eta(y, z) \frac{\partial}{\partial y} + \zeta^{s}(y, z) \frac{\partial}{\partial z^{s}},\] (3.8)

where the summation in $s = 1, \ldots, n - 2$ is assumed, and $z$ is the vector

\[z = (z^1, \ldots, z^{n-2}).\]

According to the condition $R = 2$ for algebras of type I, the matrix (3.3) of the coefficients of the operators (3.7), (3.8),

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\xi & \eta & \cdots & \zeta^{n-2}_2
\end{bmatrix}
\]
has the rank two. We can assume, without loss of generality, that
\[
\det \begin{vmatrix} 1 & 0 \\ \xi & \eta \end{vmatrix} \neq 0.
\]
This condition implies that \( \eta(y, z) \neq 0 \). Thus, the first operator in (3.2) has the form (3.7), and the second operator in (3.2) has the form (3.8) with \( \eta(y, z) \neq 0 \).

We can simplify further the operator (3.8) without changing the operator (3.7). According to Eq. (2.10), the change of variables
\[
\tilde{x} = \hat{x}(x, y, z), \quad \tilde{y} = \hat{y}(x, y, z), \quad \tilde{z}^s(x, y, z), \quad s = 1, \ldots, n - 2,
\]
maps the operator (3.7) to the form
\[
\tilde{X}_1 = \frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial y} \frac{\partial}{\partial \tilde{y}} + \frac{\partial \tilde{z}^s}{\partial x} \frac{\partial}{\partial \tilde{z}^s},
\]
where the summation in the index \( s \) is assumed. We see that the transformation (3.10) does not change the form of the operator (3.7), i.e. \( \tilde{X}_1 = \partial / \partial \tilde{x} \) if \( \frac{\partial \tilde{x}}{\partial x} = 1, \frac{\partial \tilde{y}}{\partial x} = 0, \frac{\partial \tilde{z}^s}{\partial x} = 0, \quad s = 1, \ldots, n - 2 \).

These equations give the following transformation preserving the operator (3.7):
\[
\tilde{x} = x + f(y, z), \quad \tilde{y} = g(y, z), \quad \tilde{z}^s = h^s(y, z).
\] (3.11)

After the transformation (3.11), the operator (3.8) becomes
\[
\tilde{X}_2 = \left[ \xi(y, z) + \eta(y, z) \frac{\partial f}{\partial y} + \zeta^s(y, z) \frac{\partial f}{\partial z^s} \right] \frac{\partial}{\partial \tilde{x}} + \left[ \eta(y, z) \frac{\partial g}{\partial y} + \zeta^s(y, z) \frac{\partial g}{\partial z^s} \right] \frac{\partial}{\partial \tilde{y}} + \left[ \eta(y, z) \frac{\partial h^t}{\partial y} + \zeta^s(y, z) \frac{\partial h^t}{\partial z^s} \right] \frac{\partial}{\partial \tilde{z}^s}.
\]

It takes the required form \( \tilde{X}_2 = \partial / \partial \tilde{y} \) if the functions \( f(y, z), g(y, z) \) and \( h^s(y, z) \) in the transformation (3.11) solve the following first-order linear partial differential equations:
\[
\xi(y, z) + \eta(y, z) \frac{\partial f}{\partial y} + \zeta^s(y, z) \frac{\partial f}{\partial z^s} = 0, \quad \eta(y, z) \frac{\partial g}{\partial y} + \zeta^s(y, z) \frac{\partial g}{\partial z^s} = 1, \quad \eta(y, z) \frac{\partial h^t}{\partial y} + \zeta^s(y, z) \frac{\partial h^t}{\partial z^s} = 0, \quad t = 1, \ldots, n - 2.
\] (3.12)
The condition (3.9) guarantees the solvability of the second equation in the system (3.12). A simple analysis shows that the change of variables (3.11) determined by the system (3.12) is invertible. This completes the proof of the statement of Theorem 3.1 for algebras $L_2$ of type I.

**Type II.** In this case we start by choosing the coordinates (3.6) so that the first operator $X_1$ (3.2) is written

$$X_1 = \frac{\partial}{\partial y}.$$ 

Then we continue as in the previous case. The condition $R = 1$ means that $X_2 = \eta(x, y, z)X_1$, and the condition $[X_1, X_2] = 0$ yields $\eta_y = 0$. Hence, we have

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \eta(x, z) \frac{\partial}{\partial y}, \quad \eta(x, z) \neq 0. \quad (3.13)$$

The most general transformation preserving the form of the operator $X_1$ is obtained from (3.11) merely by interchanging $x$ and $y$:

$$\tilde{x} = f(x, z), \quad \tilde{y} = y + g(x, z), \quad \tilde{z}^s = h^s(x, z). \quad (3.14)$$

After the transformation (3.14), the operator $X_2$ from (3.13) becomes

$$\tilde{X}_2 = \eta(x, z) \frac{\partial}{\partial \tilde{y}}.$$ 

Therefore we set in (3.14) $f(x, z) = \eta(x, z), \ g(x, z) = 0$, choose $h^s(x, z)$ so that the set of functions $\eta(x, z), h^s(x, z)$ will be functionally independent, and obtain the change of variables

$$\tilde{x} = \eta(x, z), \quad \tilde{y} = y, \quad \tilde{z}^s = h^s(x, z)$$

transforming the operators (3.13) to the standard form of type II in Table 3.1:

$$\widetilde{X}_1 = \frac{\partial}{\partial \tilde{y}}, \quad \widetilde{X}_2 = \tilde{x} \frac{\partial}{\partial \tilde{y}}.$$ 

**Type III.** We take again $X_1$ as in (3.13) and employ the transformations (3.14). Using the conditions $[X_1, X_2] = X_1$, $R = 2$ and applying the reasoning similar to that used in the case of type I we obtain

$$X_1 = \frac{\partial}{\partial y}, \quad (3.15)$$

$$X_2 = \xi(x, z) \frac{\partial}{\partial x} + [y + \eta(x, z)] \frac{\partial}{\partial y} + \zeta^s(x, z) \frac{\partial}{\partial z^s}, \quad \xi(x, z) \neq 0.$$
After the transformation (3.14), the second operator (3.15) becomes

\[
\tilde{X}_2 = \left[ \xi(x, z) \frac{\partial f}{\partial x} + \zeta^s(x, z) \frac{\partial f}{\partial z^s} \right] \frac{\partial}{\partial \tilde{x}} 
+ \left[ y + \eta(x, z) + \xi(x, z) \frac{\partial g}{\partial x} + \zeta^s(x, z) \frac{\partial g}{\partial z^s} \right] \frac{\partial}{\partial \tilde{y}}
+ \left[ \xi(x, z) \frac{\partial h^t}{\partial x} + \zeta^s(x, z) \frac{\partial h^t}{\partial z^s} \right] \frac{\partial}{\partial \tilde{z}^t}.
\]

Therefore we solve the equations

\[
\begin{align*}
\xi(x, z) \frac{\partial f}{\partial x} + \zeta^s(x, z) \frac{\partial f}{\partial z^s} &= f(x, z), \\
\eta(x, z) + \xi(x, z) \frac{\partial g}{\partial x} + \zeta^s(x, z) \frac{\partial g}{\partial z^s} &= g(x, z), \\
\xi(x, z) \frac{\partial h^t}{\partial x} + \zeta^s(x, z) \frac{\partial h^t}{\partial z^s} &= 0,
\end{align*}
\]

and obtain the transformation (3.14) mapping the operators (3.15) to the form III in Table 3.1:

\[
\tilde{X}_1 = \frac{\partial}{\partial \tilde{y}}, \quad \tilde{X}_2 = \tilde{x} \frac{\partial}{\partial \tilde{x}} + \tilde{y} \frac{\partial}{\partial \tilde{y}}.
\]

Type IV. Using the conditions \([X_1, X_2] = X_1, R = 1\) and proceeding as in the previous case, we obtain

\[
X_1 = \frac{\partial}{\partial y}, \quad X_2 = [y + \eta(x, z)] \frac{\partial}{\partial y}.
\tag{3.16}
\]

It is transparent that the change of variables

\[
\tilde{x} = x, \quad \tilde{y} = y + \eta(x, z), \quad \tilde{z}^s = z^s, \quad s = 1, \ldots, n - 2,
\]

maps the operators (3.16) to the standard form IV:

\[
\tilde{X}_1 = \frac{\partial}{\partial \tilde{y}}, \quad \tilde{X}_2 = \tilde{y} \frac{\partial}{\partial \tilde{y}}.
\]

This completes the proof of Theorem 3.1.

The following corollary of Theorem 3.1 gives a convenient practical way for constructing the canonical variables in two-dimensional Lie algebras.
Corollary 3.1. The canonical variables (3.5) are found, for each type (3.4), by solving the following over-determined systems of first-order linear partial differential equations:

**Type I**: \( X_1(x) = 1, X_2(x) = 0; X_1(y) = 0, X_2(y) = 1; \)
\( X_1(z^s) = 0, X_2(z^s) = 0, \; s = 1, \ldots, n - 2. \)

**Type II**: \( X_1(x) = 0, X_2(x) = 0; X_1(y) = 1, X_2(y) = x; \)
\( X_1(z^s) = 0, X_2(z^s) = 0, \; s = 1, \ldots, n - 2. \) \( (3.17) \)

**Type III**: \( X_1(x) = 0, X_2(x) = x; X_1(y) = 1, X_2(y) = y; \)
\( X_1(z^s) = 0, X_2(z^s) = 0, \; s = 1, \ldots, n - 2. \)

**Type IV**: \( X_1(x) = 0, X_2(x) = 0; X_1(y) = 1, X_2(y) = y; \)
\( X_1(z^s) = 0, X_2(z^s) = 0, \; s = 1, \ldots, n - 2. \)

**Proof.** In the case of the change of variables (3.5), the transformation law (2.10) is written
\[ \tilde{X} = X(x) \frac{\partial}{\partial x} + X(y) \frac{\partial}{\partial y} + X(z^s) \frac{\partial}{\partial z^s}. \]

The requirement that this transformation maps the operators \( X_1 \) and \( X_2 \) to the standard forms given in Table 3.1 leads to the equations (3.17). Theorem 3.1 guarantees that the over-determined systems in (3.17) are solvable.

### 3.2 Integration of systems associated with \( L_2 \)

Proposition 2.1 and Theorem 3.1 provide the following method for integrating systems (3.1) associated with two-dimensional algebras.

**Theorem 3.2.** Any system of first-order equations (3.1),
\[ \frac{dx^i}{dt} = T_1(t)\xi^i_1(x^1, \ldots, x^n) + T_2(t)\xi^i_2(x^1, \ldots, x^n), \quad i = 1, \ldots, n, \] (3.18)
where the operators
\[ X_1 = \xi^i_1 \frac{\partial}{\partial x^i}, \quad X_2 = \xi^i_2 \frac{\partial}{\partial x^i} \] (3.19)
span a two-dimensional Lie algebra \( L_2 \), can be reduced to an integrable form by introducing the canonical variables (3.6),
\[ x = x(x^1, \ldots, x^n), \quad y = y(x^1, \ldots, x^n), \]
\[ z^1 = z^1(x^1, \ldots, x^n), \quad \ldots, \quad z^{n-2} = z^{n-2}(x^1, \ldots, x^n). \] (3.6)
The canonical variables (3.6) are obtained by solving the corresponding system from (3.17) in accordance with the type of the algebra $L_2$. In these variables, the system (3.18) with the algebra $L_2$ takes the following forms:

**Type I**: 
\[
\frac{dx}{dt} = T_1(t), \quad \frac{dy}{dt} = T_2(t), \quad \frac{dz_1}{dt} = \ldots = \frac{dz_{n-2}}{dt} = 0, \quad (3.20)
\]

**Type II**: 
\[
\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = T_1(t) + T_2(t)x, \quad \frac{dz_1}{dt} = \ldots = \frac{dz_{n-2}}{dt} = 0, \quad (3.21)
\]

**Type III**: 
\[
\frac{dx}{dt} = T_2(t)x, \quad \frac{dy}{dt} = T_1(t) + T_2(t)y, \quad \frac{dz_1}{dt} = \ldots = \frac{dz_{n-2}}{dt} = 0, \quad (3.22)
\]

**Type IV**: 
\[
\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = T_1(t) + T_2(t)y, \quad \frac{dz_1}{dt} = \ldots = \frac{dz_{n-2}}{dt} = 0. \quad (3.23)
\]

**Example 3.1.** In investigating invariant solutions of the following system of partial differential equations from nonlinear optics:

\[
\left( \frac{\partial}{\partial z} - i\Delta \right) E_1 = |E_2|^2 E_1, \quad \left( \frac{\partial}{\partial z} + i\Delta \right) E_2 = |E_1|^2 E_2,
\]

the complex vLUWS amplitudes of incident and phase conjugated (amplified) light waves, respectively, I have arrived at the problem on integration of the nonlinear system*

\[
\frac{dx}{dt} = xy^2 - \frac{x}{2t}, \quad \frac{dy}{dt} = x^2y - \frac{y}{2t}. \quad (3.24)
\]

Let us apply Theorem 3.2 to the system (3.24).

The equations (3.24) have the form (3.1) with two dependent variables $x^1 = x, \ x^2 = y$. Here

\[
T_1(t) = 1, \quad \xi_1^1(x, y) = xy^2, \quad \xi_1^2(x, y) = x^2y,
\]

\[
T_2(t) = \frac{1}{t}, \quad \xi_2^1(x, y) = -\frac{x}{2}, \quad \xi_2^2(x, y) = -\frac{y}{2}, \quad (3.25)
\]

and the operators (3.2) are written:

\[
X_1 = xy^2 \frac{\partial}{\partial x} + x^2y \frac{\partial}{\partial y}, \quad X_2 = -\frac{1}{2} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right). \quad (3.26)
\]

*See [26], Section 11.2.7, and [36], Section 7.2.6.
We have:
\[
[X_1, X_2] = X_1, \quad \xi_1^2 - \xi_2^2 = \frac{xy}{2} (x^2 - y^2) \neq 0.
\]

Hence, the operators (3.26) span a two-dimensional Lie algebra satisfying the conditions of type III in the classification (3.4). Therefore we can reduce the system (3.24) to the form (3.22).

According to Eqs. (3.17) for Type III, canonical variables \(\tilde{x}, \tilde{y}\) for the first operator (3.26) are determined by the equations
\[
X_1(\tilde{x}) \equiv xy^2 \frac{\partial \tilde{x}}{\partial x} + x^2 y \frac{\partial \tilde{x}}{\partial y} = 0, \quad X_1(\tilde{y}) \equiv xy^2 \frac{\partial \tilde{y}}{\partial x} + x^2 y \frac{\partial \tilde{y}}{\partial y} = 1. \quad (3.27)
\]

The characteristic equation
\[
\frac{dx}{y} - \frac{dy}{x} = 0
\]
for the equation \(X_1(\tilde{x}) = 0\) has the first integral \(x^2 - y^2 = \text{const}\). Hence, the general solution of the first equation of the system (3.27) is given by
\[
\tilde{x} = F(x^2 - y^2) \quad (3.28)
\]
with an arbitrary function \(F\).

Let us solve the equation \(X_1(\tilde{y}) = 1\). Consider its characteristic system
\[
\frac{dx}{xy^2} = \frac{dy}{x^2y} = d\tilde{y}.
\]

Using the first integral \(x^2 - y^2 = a^2\) of the equation \(X_1(\tilde{x}) = 1\), we write the second equation \(dx/(xy^2) = d\tilde{y}\) of the characteristic system in the form
\[
\frac{dx}{x(x^2 - a^2)} = d\tilde{y}
\]
or
\[
d\tilde{y} = \frac{1}{a^2} \left[ \frac{1}{2(x - a)} + \frac{1}{2(x + a)} - \frac{1}{x} \right] dx.
\]

Hence, we have the following first integral:
\[
\tilde{y} = \frac{1}{a^2} \left[ \ln \sqrt{x^2 - a^2} - \ln |x| \right] = C.
\]

Since \(a^2 = x^2 - y^2\), we obtain the general solution to the equation \(X_1(\tilde{y}) = 1\):
\[
\tilde{y} = \frac{\ln |y| - \ln |x|}{x^2 - y^2} + H(x^2 - y^2).
\]
The second term in the above expression is a function of \( \tilde{x} \) due to Eq. (3.28). Therefore we let \( H = 0 \) and arrive at the following solution of Eqs. (3.27):

\[
\tilde{x} = F(x^2 - y^2), \quad \tilde{y} = \frac{\ln |y| - \ln |x|}{x^2 - y^2}.
\]  

(3.29)

We turn again to Eqs. (3.17) for Type III. According to these equations, the variables \( \tilde{x}, \tilde{y} \) should satisfy the equations

\[
X_2(\tilde{x}) \equiv -\frac{1}{2} \left( x \frac{\partial \tilde{x}}{\partial x} + y \frac{\partial \tilde{x}}{\partial y} \right) = \tilde{x}, \quad X_2(\tilde{y}) \equiv -\frac{1}{2} \left( x \frac{\partial \tilde{y}}{\partial x} + y \frac{\partial \tilde{y}}{\partial y} \right) = \tilde{y}.
\]  

(3.30)

One can verify that upon replacing \( \tilde{x}, \tilde{y} \) with their expressions (3.29), the second equation of the system (3.30) is satisfied identically, and the first equation of the system (3.30) gives the following first-order differential equation for \( F \):

\[ -(x^2 - y^2) F'(x^2 - y^2) = F(x^2 - y^2). \]

Solving this equation, we obtain

\[ F(x^2 - y^2) = \frac{A}{x^2 - y^2}, \quad A = \text{const}. \]

We set \( A = 1 \) and obtain the following canonical variables:

\[
\tilde{x} = \frac{1}{x^2 - y^2}, \quad \tilde{y} = \frac{\ln |y| - \ln |x|}{x^2 - y^2}.
\]  

(3.31)

Solving Eqs. (3.29) with respect to \( |x| \) and \( |y| \) we have:

\[
|x| = \frac{1}{\sqrt{\tilde{x}} (1 - e^{2\tilde{y}/\tilde{x}})}, \quad |y| = \frac{e^{\tilde{y}/\tilde{x}}}{\sqrt{\tilde{x}} (1 - e^{2\tilde{y}/\tilde{x}})}.
\]  

(3.32)

Upon introducing the canonical variables, the operators (3.26) become

\[
\tilde{X}_1 = \frac{\partial}{\partial \tilde{y}}, \quad \tilde{X}_2 = \tilde{x} \frac{\partial}{\partial \tilde{x}} + \tilde{y} \frac{\partial}{\partial \tilde{y}}
\]

and the system (3.24) is written in the form (3.22):

\[
\frac{d\tilde{x}}{dt} = \frac{1}{t} \tilde{x}, \quad \frac{d\tilde{y}}{dt} = 1 + \frac{1}{t} \tilde{y}.
\]  

(3.33)

The solution of Eqs. (3.33) is given by

\[
\tilde{x} = C_1 t, \quad \tilde{y} = C_2 t + t \ln t
\]  

(3.34)
with a positive constant $C_1$ and an arbitrary constant $C_2$. Now we substitute (3.34) in (3.32) and obtain the following general solution to Eqs. (3.24):

$$|x| = \sqrt{\frac{k}{t(1 - \zeta^2)}}, \quad |y| = \zeta \sqrt{\frac{k}{t(1 - \zeta^2)}}. \quad (3.35)$$

Here $\zeta = Ct^k$ with $C = e^{C_2/C_1}, \ k = 1/C_1$.

### 4 Systems associated with $L_3$

The next step is to consider the systems of the form

$$\frac{dx}{dt} = T_1(t)\xi^i_1(x) + T_2(t)\xi^i_2(x) + T_3(t)\xi^i_3(x), \quad i = 1, \ldots, n, \quad (4.1)$$

i.e. to systems (1.7) with the three-dimensional Vessiot-Guldberg-Lie algebra $L_3$ spanned by the operators

$$X_1 = \xi^i_1(x)\frac{\partial}{\partial x^i}, \quad X_2 = \xi^i_2(x)\frac{\partial}{\partial x^i}, \quad X_3 = \xi^i_3(x)\frac{\partial}{\partial x^i}. \quad (4.2)$$

I will restrict the consideration to $n = 2$, denote $x^1 = x, \ x^2 = y$ and write the system (4.1) in the form

$$\frac{dx}{dt} = T_1(t)\xi^i_1(x, y) + T_2(t)\xi^i_2(x, y) + T_3(t)\xi^i_3(x, y), \quad (4.3)$$

$$\frac{dy}{dt} = T_1(t)\xi^i_1(x, y) + T_2(t)\xi^i_2(x, y) + T_3(t)\xi^i_3(x, y).$$

In this notation, the operators (4.2) are written as follows:

$$X_1 = \xi^i_1(x, y)\frac{\partial}{\partial x} + \xi^i_2(x, y)\frac{\partial}{\partial y},$$

$$X_2 = \xi^i_2(x, y)\frac{\partial}{\partial x} + \xi^i_3(x, y)\frac{\partial}{\partial y}, \quad (4.4)$$

$$X_3 = \xi^i_3(x, y)\frac{\partial}{\partial x} + \xi^i_3(x, y)\frac{\partial}{\partial y}.$$

### 4.1 Lie’s classification of $L_3$

Lie showed that the basis $X_1, X_2, X_3$ of any three-dimensional algebra $L_3$ of operators in two variables can be mapped, by a complex change of variables, to one of the following 13 standard forms (see [58], Chapter 22; see also [26], Section 7.3.8).
Table 4.1. Standard forms of three-dimensional Lie algebras

A. The first derived algebra has the dimension three:

1) \( X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, \)

2) \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \)

3) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = y \frac{\partial}{\partial y}, \quad X_3 = y^2 \frac{\partial}{\partial y}. \)

B. The first derived algebra has the dimension two:

4) \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \)

5) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = (1 - c)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} (c \neq 0, c \neq 1), \)

6) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \)

7) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \)

8) \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (x + y) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \)

9) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \)

C. The first derived algebra has the dimension one:

10) \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x}, \)

11) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \)

12) \( X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial y}. \)

D. The first derived algebra has the dimension zero:

13) \( X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial y}, \quad X_3 = p(x) \frac{\partial}{\partial y}, \)

where \( p(x) \) is any given function.

Remark 4.1. Lie uses the algebras 1) and 2) also in the following alternative forms:

1') \( X_1 = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \quad X_3 = (x^2 - y) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \)

2') \( X_1 = x \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial x}. \)
Recall that the derived algebra $L'_3$ of the Lie algebra $L_3$ with a basis $X_1, X_2, X_3$ is the algebra spanned by the commutators

$$[X_1, X_2], \quad [X_1, X_3], \quad [X_2, X_3].$$

The higher derivatives are defined by induction, $L''_3 = (L'_3)'$, etc. A Lie algebra is *solvable* if its derivative of a certain order vanishes. It is obvious that $L_3$ is solvable if $\dim L'_3 \leq 2$ and not solvable if $\dim L'_3 = 3$.

**Remark 4.2.** The classification in the real domain gives the following four additional standard forms of $L_3$ (see [62] and [25], Section 8.2.2):

14) $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y}$, $X_3 = (bx + y)\frac{\partial}{\partial x} + (by - x)\frac{\partial}{\partial y}$,

15) $X_1 = \frac{\partial}{\partial y}$, $X_2 = x\frac{\partial}{\partial y}$, $X_3 = (1 + x^2)\frac{\partial}{\partial x} + (x + b)y\frac{\partial}{\partial y}$,

16) $X_1 = \frac{\partial}{\partial y}$, $X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $X_3 = 2xy\frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial}{\partial y}$,

17) $X_1 = (1 + x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$, $X_2 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$,

$$X_3 = xy\frac{\partial}{\partial x} + (1 + y^2)\frac{\partial}{\partial y}.$$ A discussion of complex changes of variables bringing the forms 14)–17) to Lie’s standard form in Table 4.1 can be found in [26], Section 7.3.8.

### 4.2 Integration of systems associated with $L_3$

Using Lie’s classification of three-dimensional algebras and Proposition 2.1, we arrive at the following result (compare with Theorem 3.2 in Section 3.2).

**Theorem 4.1.** The system (4.3) can be solved by spanned by the operators (4.4) is solvable and reduced to integration of Riccati equations if $L_3$ is not solvable.

**Proof.** We transform the basis (4.4) of the algebra $L_3$ to an appropriate standard form given in Table 4.1 and map Eqs. (4.3) to the forms given in the following Table 4.2.
The alternative forms in Remark 4.1 provide the following alternative standard forms of Eqs. (4.3):

Remark 4.3. completes the proof of the theorem.

It is manifest that the systems of forms B, C and D can be solved by quadratures. This is also obvious that the systems in A require integration of Riccati equations and, in general, cannot be solved by quadratures. This completes the proof of the theorem.

**Remark 4.3.** The alternative forms in Remark 4.1 provide the following alternative standard forms of Eqs. (4.3):

1') \( \dot{x} = T_1(t) + T_2(t)x + T_3(t)(x^2 - 1), \quad \dot{y} = T_1(t)x + 2T_2(t)y + T_3(t)xy; \)

2') \( \dot{x} = T_2(t)x + T_3(t)y, \quad \dot{y} = T_1(t)x - T_2(t)y. \)
In certain particular cases the systems in A can be integrated either by quadratures or in terms of special functions. The simplest case is $T_3(t) = 0$. Then the equations 1) - 3) become easily solvable linear systems. Furthermore, if $T_1(t) = 0$ the Riccati equations in systems 1) - 3) can be linearized by a change of the dependent variables (see [26], Section 11.2.5, or [33], Chapter 1). Moreover, it is demonstrated in [26] that the Riccati equations in systems 1), 3) can be linearized by a change of the dependent variables if

$$T_3(t) = k[T_2(t) - kT_1(t)], \quad k = \text{const}.$$  

In the case of the system 2) this condition is replaced by

$$T_3(t) = k[2T_2(t) - kT_1(t)].$$

It is well known that if $T_3(t) \neq 0$, one can transform the Riccati equations in question to the equivalent form with $T_3(t) = -1$, $T_2(t) = 0$. Assuming that this transformation has been done, let us consider, e.g., the system 2),

$$\dot{x} + x^2 = T_1(t), \quad \dot{y} + xy = 0.$$  \hspace{1cm} (4.5)

We set $x = (\ln |u|)'$ and rewrite the first equation of this system in the form of a linear second-order equation

$$u'' = T_1(t)u,$$

where $u'$ is the derivative of $u$ with respect to $t$. The above equation can be solved in terms of special functions if $T_1(t)$ is a linear function. Indeed, let $T_1(t) = \alpha t + \beta$, $\alpha \neq 0$. Then our equation

$$u'' = (\alpha t + \beta)u$$

becomes the Airy equation

$$\frac{d^2u}{d\tau^2} - \tau u = 0$$

upon introducing the new independent variable

$$\tau = \alpha^{-2/3}[\alpha t + \beta].$$

The general solution to the Airy equation is given by the linear combination

$$u = C_1 \text{Ai}(\tau) + C_2 \text{Bi}(\tau)$$
of the *Airy functions*

\[
\text{Ai}(\tau) = \frac{1}{\pi} \int_0^\infty \cos \left( s\tau + \frac{1}{3} s^3 \right) ds, \\
\text{Bi}(\tau) = \frac{1}{\pi} \int_0^\infty \left[ \exp \left( s\tau - \frac{1}{3} s^3 \right) + \sin \left( s\tau + \frac{1}{3} s^3 \right) \right] ds.
\]

Assuming that \( C_1 \neq 0 \) and introducing the new constant \( K_1 = C_2/C_1 \) we obtain

\[
x(t) = \frac{d}{dt} \ln \left| \text{Ai} \left( \alpha^{-2/3}[\alpha t + \beta] \right) + K_1 \text{Bi} \left( \alpha^{-2/3}[\alpha t + \beta] \right) \right|. \tag{4.6}
\]

Now we substitute (4.6) in the second equation of the system (4.5) and obtain by integration:

\[
y(t) = K_2 \left\{ \text{Ai} \left( \alpha^{-2/3}[\alpha t + \beta] \right) + K_1 \text{Bi} \left( \alpha^{-2/3}[\alpha t + \beta] \right) \right\}^{-1}. \tag{4.7}
\]

Thus, the solution of the system (4.5) with \( T_1(t) = \alpha t + \beta \) is given by the special functions (4.6), (4.7).

**Example 4.1.** Consider the nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= -T_1(t) y e^{\arctan(y/x)} + T_2(t) x + T_3(t)y, \\
\frac{dy}{dt} &= T_1(t) x e^{\arctan(y/x)} + T_2(t) y - T_3(t)x
\end{align*} \tag{4.8}
\]

with arbitrary coefficients \( T_1(t), T_2(t), T_3(t) \).

The operators (4.4) associated with Eqs. (4.8) have the form

\[
X_1 = e^{\arctan(y/x)} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right), \\
X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \\
X_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}
\]

and span a three-dimensional Lie algebra \( L_3 \) with the following commutator relations:

\[
[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = 0. \tag{4.10}
\]
It follows that the derived algebra $L'_3$ has the dimension one, and hence our algebra $L_3$ belongs to the category $C$ of Table 4.1. Specifically, comparison of the commutator relations (4.10) with the commutators of the standard operators 10), 11) or 12) from Table 4.1 shows that the operators (4.9) can be mapped by a change of variables (2.4) either to 10) or to 11). However, it is easy to show that they cannot be mapped to the form 11). Indeed, the change of variables (2.4) converts (4.9) to the form 11),

$$X_1 = \frac{\partial}{\partial \tilde{y}}, \quad X_2 = \tilde{x} \frac{\partial}{\partial \tilde{y}}, \quad X_3 = \tilde{x} \frac{\partial}{\partial \tilde{x}} + \tilde{y} \frac{\partial}{\partial \tilde{y}},$$

if $\tilde{x}$ and $\tilde{y}$ solve the following over-determined systems:

\begin{align*}
X_1(\tilde{x}) &= 0, & X_1(\tilde{y}) &= 1, \\
X_2(\tilde{x}) &= 0, & X_2(\tilde{y}) &= \tilde{x}, \\
X_3(\tilde{x}) &= \tilde{x}, & X_3(\tilde{y}) &= \tilde{y},
\end{align*}

where $X_1, X_2, X_3$ are the operators (4.9). These equations are not compatible. For example, the equations $X_1(\tilde{x}) = 0$ and $X_3(\tilde{x}) = \tilde{x}$ contradict each other because $X_1$ differs from $X_3$ by a non-vanishing factor only. For another reasoning, see the general construction of similarity transformations given in [26], Section 7.3.7.

Let us find the change of variables (2.4) mapping (4.9) to the form 10),

$$X_1 = \frac{\partial}{\partial \tilde{y}}, \quad X_2 = \frac{\partial}{\partial \tilde{x}}, \quad X_3 = \tilde{x} \frac{\partial}{\partial \tilde{x}}. \quad (4.11)$$

Now $\tilde{x}$ and $\tilde{y}$ should solve the following over-determined systems:

\begin{align*}
X_1(\tilde{x}) &= 1, & X_1(\tilde{y}) &= 0, \\
X_2(\tilde{x}) &= 0, & X_2(\tilde{y}) &= 1, \\
X_3(\tilde{x}) &= \tilde{x}, & X_3(\tilde{y}) &= 0.
\end{align*}

Substituting the expressions (4.9) for $X_1, X_2, X_3$ we write these equations in the form

\begin{align*}
x \frac{\partial \tilde{x}}{\partial y} - y \frac{\partial \tilde{x}}{\partial x} &= e^{-\arctan(y/x)}, & x \frac{\partial \tilde{y}}{\partial y} - y \frac{\partial \tilde{y}}{\partial x} &= 0, \\
x \frac{\partial \tilde{x}}{\partial x} + y \frac{\partial \tilde{x}}{\partial y} &= 0, & x \frac{\partial \tilde{y}}{\partial x} + y \frac{\partial \tilde{y}}{\partial y} &= 1, \quad (4.12) \\
x \frac{\partial \tilde{x}}{\partial y} - y \frac{\partial \tilde{x}}{\partial x} &= \tilde{x}, & x \frac{\partial \tilde{y}}{\partial y} - y \frac{\partial \tilde{y}}{\partial x} &= 0.
\end{align*}
Comparison of the first and third equations for $\tilde{x}$ yields $\tilde{x} = e^{-\arctan(y/x)}$. One can readily verify that this function solves all three equations (4.12) for $\tilde{x}$. The equations for $\tilde{y}$ are easy to solve and yield $\tilde{y} = \ln \sqrt{x^2 + y^2}$. Thus, the canonical variables mapping the operators (4.9) to the standard form (4.11) are given by

$$\tilde{x} = e^{-\arctan(y/x)}, \quad \tilde{y} = \ln \sqrt{x^2 + y^2}.$$  \hspace{1cm} (4.13)

In the variables (4.13) Eqs. (4.8) are written in the integrable form (10) of Table 4.2:

$$\frac{d\tilde{x}}{dt} = T_1(t) + T_3(t) \tilde{x}, \quad \frac{d\tilde{y}}{dt} = T_2(t).$$ \hspace{1cm} (4.14)
Nonlinear self-adjointness in constructing conservation laws

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Abstract. The general concept of nonlinear self-adjointness of differential equations is introduced. It includes the linear self-adjointness as a particular case. Moreover, it embraces the previous notions of self-adjoint [28] and quasi self-adjoint [30] nonlinear equations. The class of nonlinearly self-adjoint equations includes, in particular, all linear equations. Conservation laws associated with symmetries can be constructed for all nonlinearly self-adjoint differential equations and systems. The number of equations in systems can be different from the number of dependent variables.

Part 1
Nonlinear self-adjointness

1 Preliminaries

The concept of self-adjointness of nonlinear equations was introduced [28, 29] for constructing conservation laws associated with symmetries of differential equations. To extend the possibilities of the new method for constructing conservation laws the notion of quasi self-adjointness was suggested in [30]. I introduce here the general concept of nonlinear self-adjointness. It embraces the previous notions of self-adjoint and quasi self-adjoint equations and includes the linear self-adjointness as a particular case. But the set of nonlinearly self-adjoint equations is essentially wider and includes, in particular, all linear equations. The construction
of conservation laws demonstrates a practical significance of the nonlinear self-adjointness. Namely, conservation laws can be associated with symmetries for all nonlinearly self-adjoint differential equations and systems. In particular, this is possible for all linear equations and systems.

1.1 Notation
We will use the following notation. The independent variables are denoted by
\[ x = (x^1, \ldots, x^n). \]
The dependent variables are
\[ u = (u^1, \ldots, u^m). \]
They are used together with their first-order partial derivatives \( u^{(1)} \):
\[ u^{(1)} = \{u^\alpha_i\}, \quad u^\alpha_i = D_i(u^\alpha), \]
and higher-order derivatives \( u^{(2)}, \ldots, u^{(s)}, \ldots \), where
\[ u^{(2)} = \{u^\alpha_{ij}\}, \quad u^\alpha_{ij} = D_iD_j(u^\alpha), \ldots, \]
\[ u^{(s)} = \{u^\alpha_{i_1 \ldots i_s}\}, \quad u^\alpha_{i_1 \ldots i_s} = D_{i_1} \cdots D_{i_s}(u^\alpha). \]
Here \( D_i \) is the total differentiation with respect to \( x^i \):
\[ D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots. \quad (1.1) \]

A locally analytic function \( f(x, u, u^{(1)}, \ldots, u^{(k)}) \) of any finite number of the variables \( x, u, u^{(1)}, u^{(2)}, \ldots \) is called a differential function. The set of all differential functions is denoted by \( \mathcal{A} \). For more details see [26], Chapter 8.

1.2 Linear self-adjointness
Recall that the adjoint operator \( F^* \) to a linear operator \( F \) in a Hilbert space \( H \) with a scalar product \( (u, v) \) is defined by
\[ (Fu, v) = (u, F^*v), \quad u, v \in H. \quad (1.2) \]

Let us consider, for the sake of simplicity, the case of one dependent variable \( u \) and denote by \( H \) the Hilbert space of real valued functions \( u(x) \) such that \( u^2(x) \) is integrable. The scalar product is given by
\[ (u, v) = \int_{\mathbb{R}^n} u(x)v(x)dx. \]
Let $F$ be a linear differential operator in $H$. Its action on the dependent variable $u$ is denoted by $F[u]$. The definition (1.2) of the adjoint operator $F^*$ to $F$,

$$(F[u], v) = (u, F^*[v]),$$

can be written, using the divergence theorem, in the simple form

$$vF[u] - uF^*[v] = D_i(p^i), \quad (1.3)$$

where $v$ is a new dependent variable, and $p^i$ are any functions of $x, u, v, u(1), v(1), \ldots$.

It is manifest from Eq. (1.3) that the operators $F$ and $F^*$ are mutually adjoint,

$$(F^*)^* = F. \quad (1.4)$$

In other words, the adjointness of linear operators is a symmetric relation.

The linear operator $F$ is said to be self-adjoint if $F^* = F$. In this case we say that the equation $F[u] = 0$ is self-adjoint. Thus, the self-adjointness of a linear equation $F[u] = 0$ can be expressed by the equation

$$F^*[v]_{v = u} = F[u]. \quad (1.5)$$

### 1.3 Adjoint equations to nonlinear differential equations

Let us consider a system of $m$ differential equations (linear or nonlinear)

$$F_{\alpha}(x, u, u(1), \ldots, u(s)) = 0, \quad \alpha = 1, \ldots, m, \quad (1.6)$$

with $m$ dependent variables $u = (u^1, \ldots, u^m)$. Eqs. (1.6) involve the partial derivatives $u(1), \ldots, u(s)$ up to order $s$.

**Definition 1.1.** The adjoint equations to Eqs. (1.6) are given by

$$\left. F^*_\alpha(x, u, v, u(1), v(1), \ldots, u(s), v(s)) \right|_{v = u} = 0, \quad \alpha = 1, \ldots, m, \quad (1.7)$$

with

$$F_{\alpha}(x, u, v, u(1), v(1), \ldots, u(s), v(s)) = \frac{\delta \mathcal{L}}{\delta u^\alpha}, \quad (1.8)$$

where $\mathcal{L}$ is the formal Lagrangian for Eqs. (1.6) defined by *

$$\mathcal{L} = \sum_{\beta=1}^m v^\beta F_{\beta}. \quad (1.9)$$

*See [28]. An approach in terms of variational principles is developed in [5].
Here \( v = (v^1, \ldots, v^m) \) are new dependent variables, \( v_{(1)}, \ldots, v_{(s)} \) are their derivatives, e.g. \( v_{(1)} = \{v^α_i\} \), \( v^α_i = D_i(v^α) \). We use \( \delta/\delta u^α \) for the Euler-Lagrange operator

\[
\frac{\delta}{\delta u^α} = \frac{\partial}{\partial u^α} + \sum_{s=1}^{\infty} (-1)^s D_i \cdots D_s \frac{\partial}{\partial u^α_i \cdots i_s}, \quad \alpha = 1, \ldots, m,
\]

so that

\[
\frac{\delta (v^β F^β)}{\delta u^α} = \frac{\partial (v^β F^β)}{\partial u^α} - D_i \left( \frac{\partial (v^β F^β)}{\partial u^α_i} \right) + D_i D_k \left( \frac{\partial (v^β F^β)}{\partial u^α_{ik}} \right) - \cdots.
\]

The total differentiation (1.1) is extended to the new dependent variables:

\[
D_i = \frac{\partial}{\partial x^i} + u^α_i \frac{\partial}{\partial u^α} + v^α_i \frac{\partial}{\partial v^α} + u^α_{ij} \frac{\partial}{\partial u^α_j} + v^α_{ij} \frac{\partial}{\partial v^α_j} + \cdots
\]  

(1.10)

The adjointness of nonlinear equations is not a symmetric relation. In other words, nonlinear equations, unlike the linear ones, do not obey the condition (1.4) of mutual adjointness. Instead, the following equation holds:

\[
(F^*)^* = \hat{F}
\]  

(1.11)

where \( \hat{F} \) is the linear approximation to \( F \) defined as follows. We use the temporary notation \( F[u] \) for the left-hand side of Eq. (1.6) and consider \( F[u + w] \) by letting \( w \ll 1 \). Then neglecting the nonlinear terms in \( w \) we define \( \hat{F} \) by the equation

\[
F[u + w] \approx F[u] + \hat{F}[w]
\]  

(1.12)

For linear equations we have \( \hat{F} = F \), and hence Eq. (1.11) is identical with Eq. (1.4).

Let us illustrate Eq. (1.11) by the equation

\[
F \equiv u_{xy} - \sin u = 0.
\]  

(1.13)

Eq. (1.8) yields

\[
F^* \equiv \frac{\delta}{\delta u}[v(u_{xy} - \sin u)] = v_{xy} - v \cos u
\]  

(1.14)

and

\[
(F^*)^* \equiv \frac{\delta}{\delta v}[w(v_{xy} - v \cos u)] = w_{xy} - w \cos u.
\]  

(1.15)
Let us find $\hat{F}$ by using Eq. (1.12). Since $\sin w \approx w$, $\cos w \approx 1$ when $w \ll 1$, we have

$$F[u + w] \equiv (u + w)_{xy} - \sin(u + w)$$

$$= u_{xy} + w_{xy} - \sin u \cos w - \sin w \cos u,$$

$$\approx u_{xy} - \sin u + w_{xy} - w \cos u,$$

$$= F[u] + w_{xy} - w \cos u.$$

Hence, by (1.12) and (1.15), we have

$$\hat{F}[w] = w_{xy} - w \cos u = (F^*)^*$$

(1.16)

in accordance with Eq. (1.11).

### 1.4 The case of one dependent variable

Let us consider the differential equation

$$F(x, u, u^{(1)}, \ldots, u^{(s)}) = 0$$

(1.17)

with one dependent variable $u$ and any number of independent variables. In this case Definition 1.1 of the adjoint equation is written

$$F^*(x, u, v, u^{(1)}, v^{(1)}, \ldots, u^{(s)}, v^{(s)}) = 0,$$

(1.18)

where

$$F^*(x, u, v, u^{(1)}, v^{(1)}, \ldots, u^{(s)}, v^{(s)}) = \frac{\delta(vF)}{\delta u}.$$  

(1.19)

### 1.5 Construction of adjoint equations to linear equations

The following statement has been formulated in [28, 29].

**Proposition 1.1.** In the case of linear differential equations and systems, the adjoint equations determined by Eq. (1.8) and by Eq. (1.3) coincide.

**Proof.** The proof is based on the statement (see Proposition 7.1 in Section 7.2) that a function $Q(u, v)$ is a divergence, i.e. $Q = D_i(h^i)$, if and only if

$$\frac{\delta Q}{\delta u^\alpha} = 0, \quad \frac{\delta Q}{\delta v^\alpha} = 0, \quad \alpha = 1, \ldots, m.$$  

(1.20)
Let the adjoint operator $F^*$ be constructed according to Eq. (1.3). Let us consider the case of many dependent variables and write Eq. (1.3) as follows:

\[ v^\beta F_\beta[u] = u^\beta F^*_\beta[v] + D_i(p^i). \]  

(1.21)

Applying to (1.21) the variational differentiations and using Eqs. (1.20) we obtain

\[ \frac{\delta(v^\beta F_\beta[u])}{\delta u^\alpha} = \delta^\beta_\alpha F^*_\beta[v] \equiv F^*_\alpha[v]. \]

Hence, (1.8) coincides with $F^*_\alpha[v]$ given by (1.3).

Conversely, let $F^*_\alpha[v]$ be given by (1.8),

\[ F^*_\beta[v] = \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta}. \]

Consider the expression $Q$ defined by

\[ Q = v^\beta F_\beta[u] - u^\beta F^*_\beta[v] \equiv v^\beta F_\beta[u] - u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta}. \]

Applying to the first expression for $Q$ the variational differentiations $\delta/\delta u^\alpha$ we obtain

\[ \frac{\delta Q}{\delta u^\alpha} = \frac{\delta(v^\beta F_\beta[u])}{\delta u^\alpha} - \delta^\beta_\alpha F^*_\beta[v] = F^*_\alpha[v] - \delta^\beta_\alpha F^*_\beta[v] = 0. \]

Applying $\delta/\delta v^\alpha$ to the second expression for $Q$ we obtain

\[ \frac{\delta Q}{\delta v^\alpha} = \delta^\beta_\alpha F_\beta[u] - \frac{\delta}{\delta v^\alpha} \left[ u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right] \equiv F^*_\alpha[u] - \frac{\delta}{\delta v^\alpha} \left[ u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right]. \]

The reckoning shows that

\[ \frac{\delta}{\delta v^\alpha} \left[ u^\beta \frac{\delta(v^\gamma F_\gamma[u])}{\delta u^\beta} \right] = F^*_\alpha[u]. \]  

(1.22)

Thus $Q$ solves Eq. (1.20) and hence Eq. (1.21) is satisfied. This completes the proof.

**Remark 1.1.** Let us discuss the proof of Eq. (1.22) in the case of a second-order linear operator for one dependent variable:

\[ F[u] = a^{ij}(x)u_{ij} + b'(x)u_i + c(x)u. \]

Then we have:

\[ u \frac{\delta(v F[u])}{\delta u} = u \left[ cv - vD_i(b') + vD_iD_j(a^{ij}) - b'v_i + 2v_iD_j(a^{ij}) + a^{ij}v_{ij} \right]. \]
Whence, after simple calculations we obtain
\[
\frac{\delta}{\delta v} \left[ u \frac{\delta(vF[u])}{\delta u} \right] = [cu + b^i u_i + a^{ij} u_{ij}]
\]
\[+ \{D_i D_j (a^{ij} u) - D_i (a^{ij} u_j) - D_i [u D_j (a^{ij})]\}
\]
and, noting that the expression in the braces vanishes, arrive at Eq. (1.22).

Let us illustrate Proposition 1.1 by the following simple example.

**Example 1.1.** Consider the heat equation
\[
F[u]] \equiv u_t - u_{xx} = 0
\]
and construct the adjoint operator to the linear operator
\[
F = D_t - D_x^2
\]
by using Eq. (1.3). Noting that
\[
v u_t = D_t(uv) - uv_t,
\]
\[
v u_{xx} = D_x(vu_x) - v_x u_x = D_x(vu_x - uv_x) + uv_{xx}
\]
we have:
\[
v F[u] \equiv v(u_t - u_{xx}) = u(-v_t - v_{xx}) + D_t(uv) + D_x(uv_x - uv_x).
\]
Hence,
\[
v F[u] - u(-v_t - v_{xx}) = D_t(uv) + D_x(uv_x - uv_x).
\]
Therefore, denoting \( t = x^1, \ x = x^2, \) we obtain Eq. (1.3) with
\[
F^*[v] = -v_t - v_{xx}
\]
and
\[
p^1 = uv, \quad p^2 = uv_x - vu_x.
\]
Thus, the adjoint operator to the linear operator (1.24) is
\[
F^* = -D_t - D_x^2
\]
and the adjoint equation to the heat equation (1.23) is written \(-v_t - v_{xx} = 0,\)
or
\[
v_t + v_{xx} = 0.
\]

The derivation of the adjoint equation (1.26) and the adjoint operator (1.25) by the definition (1.19) is much simpler. Indeed, we have:
\[
F^* = \frac{\delta(v u_t - vu_{xx})}{\delta u} = -D_t(v) - D_x^2(v) = -(v_t + v_{xx}).
\]
1.6 Self-adjointness and quasi self-adjointness

Recall that a linear differential operator $F$ is called a self-adjoint operator if it is identical with its adjoint operator, $F = F^*$. Then the equation $F[u] = 0$ is also said to be self-adjoint. Thus, the self-adjointness of a linear differential equation $F[u] = 0$ means that the adjoint equation $F^*[v] = 0$ coincides with $F[u] = 0$ upon the substitution $v = u$. This property has been extended to nonlinear equations in [28]. It will be called here the strict self-adjointness and defined as follows.

**Definition 1.2.** We say that the differential equation (1.17) is strictly self-adjoint if the adjoint equation (1.18) becomes equivalent to the original equation (1.17) upon the substitution $v = u$.

It means that the equation

$$F^*(x, u, u, \ldots, u(s), u) = \lambda F(x, u, \ldots, u(s))$$

holds with a certain (in general, variable) coefficient $\lambda$.

**Example 1.2.** The Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x$$

is strictly self-adjoint [29]. Indeed, its adjoint equation (1.18) has the form

$$v_t = v_{xxx} + uv_x$$

and coincides with the KdV equation upon the substitution (1.27).

In the case of linear equations the strict self-adjointness is identical with the usual self-adjointness of linear equations.

**Example 1.3.** Consider the linear equation

$$u_{tt} + a(x)u_{xx} + b(x)u_x + c(x)u = 0.$$  \hspace{1cm} (1.29)

According to Eqs. (1.18)-(1.19), the adjoint equation to Eq. (1.29) is written

$$\frac{\delta}{\delta u}\{v[u_{tt} + a(x)u_{xx} + b(x)u_x + c(x)u]\} \equiv D_t^2(v) + D_x^2(av) - D_x(bv) + cv = 0.$$  \hspace{1cm} (1.30)

Upon substituting $v = u$ and performing the differentiations it becomes

$$u_{tt} + au_{xx} + (2a' - b)u_x + (a'' - b' + c)u = 0.$$  \hspace{1cm} (1.31)

According to Definition 1.2, Eq. (1.29) is strictly self-adjoint if Eq. (1.30) coincides with Eq. (1.29). This is possible if

$$b(x) = a'(x).$$  \hspace{1cm} (1.31)
Definition 1.2 is too restrictive. Moreover, it is inconvenient in the case of systems with several dependent variables $u = (u^1, \ldots, u^m)$ because in this case Eq. (1.27) is not uniquely determined as it is clear from the following example.

Example 1.4. Let us consider the system of two equations

$$
\begin{align*}
  u^1_y + u^2 u_x^2 - u^2_t &= 0, \\
  u^2_y - u^1_x &= 0
\end{align*}
$$

(1.32)

with two dependent variables, $u = (u^1, u^2)$, and three independent variables $t, x, y$. Using the formal Lagrangian (1.9)

$$
\mathcal{L} = v^1 (u^1_y + u^2 u_x^2 - u^2_t) + v^2 (u^2_y - u^1_x)
$$

and Eqs. (1.8) we write the adjoint equations (1.7), changing their sign, in the form

$$
\begin{align*}
  v^2_y + u^2 v^1_x - u^1_t &= 0, \\
  v^1_y - v^2_x &= 0.
\end{align*}
$$

(1.33)

If we use here the substitution (1.27), $v = u$ with $v = (v^1, v^2)$, i.e. let

$$
\begin{align*}
  v^1 &= u^1, \\
  v^2 &= u^2,
\end{align*}
$$

then the adjoint system (1.33) becomes

$$
\begin{align*}
  u^2_y + u^2 u^1_x - u^1_t &= 0, \\
  u^1_y - u^2_x &= 0,
\end{align*}
$$

which is not connected with the system (1.32) by the equivalence relation (1.28). But if we set

$$
\begin{align*}
  v^1 &= u^2, \\
  v^2 &= u^1,
\end{align*}
$$

the adjoint system (1.33) coincides with the original system (1.32).

The concept of quasi self-adjointness generalizes Definition 1.2 and is more convenient for dealing with systems (1.6). This concept was formulated in [30] as follows.

The system (1.6) is \textit{quasi self-adjoint} if the adjoint system (1.7) becomes equivalent to the original system (1.6) upon a substitution

$$
\begin{align*}
  v &= \varphi(u)
\end{align*}
$$

(1.34)

such that its derivative does not vanish in a certain domain of $u$,

$$
\varphi'(u) \neq 0, \quad \text{where} \quad \varphi'(u) = \left\| \frac{\partial \varphi^\alpha(u)}{\partial u^\beta} \right\|. 
$$

(1.35)
Remark 1.2. The substitution (1.34) defines a mapping
\[ v^\alpha = \varphi^\alpha(u), \quad \alpha = 1, \ldots, m, \]
from the \( m \)-dimensional space of variables \( u = (u^1, \ldots, u^m) \) into the \( m \)-dimensional space of variables \( v = (v^1, \ldots, v^m) \). It is assumed that this mapping is continuously differentiable. The condition (1.35) guarantees that it is invertible, and hence Eqs. (1.7) and (1.6) are equivalent. The equivalence means that the following equations hold with certain coefficients \( \lambda^\beta_\alpha \):\[ F^*_\alpha(x, u, \varphi, \ldots, u(s), \varphi(s)) = \lambda^\beta_\alpha F_\beta(x, u, \ldots, u(s)), \quad \alpha = 1, \ldots, m, \quad (1.36) \]
where \[ \varphi = \{ \varphi^\alpha(u) \}, \quad \varphi(\sigma) = \{ D_{i_1} \cdots D_{i_s}(\varphi^\alpha(u)) \}, \quad \sigma = 1, \ldots, s. \quad (1.37) \]
It can be shown that the matrix \( \| \lambda^\beta_\alpha \| \) is invertible due to the condition (1.35).

Example 1.5. The quasi self-adjointness of nonlinear wave equations of the form
\[ u_{tt} - u_{xx} = f(t, x, u, u_t, u_x) \]
is investigated in [51]. The results of the paper [51] show that, e.g. the equation
\[ u_{tt} - u_{xx} + u^2_t - u^2_x = 0 \quad (1.38) \]
is quasi self-adjoint and that in this case the substitution (1.34) has the form\[ v = e^u. \quad (1.39) \]
Indeed, the adjoint equation to Eq. (1.38) is written
\[ v_{tt} - v_{xx} - 2v u_{tt} - 2u_t v_t + 2v u_{xx} + 2u_x v_x = 0. \quad (1.40) \]
After the substitution (1.39) the left-hand side of Eq. (1.40) takes the form (1.36):
\[ v_{tt} - v_{xx} - 2v u_{tt} - 2u_t v_t + 2v u_{xx} + 2u_x v_x = -e^u [u_{tt} - u_{xx} + u^2_t - u^2_x]. \quad (1.41) \]
It is manifest from Eq. (1.41) that \( v \) given by (1.39) solves the adjoint equation (1.40) if one replaces \( u \) by any solution of Eq. (1.38).

In constructing conservation laws one can relax the condition (1.35). Therefore I generalize the previous definition of quasi self-adjointness as follows.
Definition 1.3. The system (1.6) is said to be quasi self-adjoint if the adjoint equations (1.7) are satisfied for all solutions \( u \) of the original system (1.6) upon a substitution

\[ v^\alpha = \varphi^\alpha(u), \quad \alpha = 1, \ldots, m, \]  

such that

\[ \varphi(u) \neq 0. \]  

In other words, the equations (1.36) hold after the substitution (1.42), where not all \( \varphi^\alpha(u) \) vanish simultaneously.

Remark 1.3. The condition (1.43), unlike (1.35), does not guarantee the equivalence of Eqs. (1.7) and (1.6) because the matrix \( |\lambda_\alpha^\beta| \) may be singular.

Example 1.6. It is well known that the linear heat equation (1.23) is not self-adjoint (not strictly self-adjoint in the sense of Definition 1.2). It is clear from Eqs. (1.23) and (1.26). Let us test Eq. (1.23) for quasi self-adjointness. Letting \( v = \varphi(u) \), we obtain

\[ v_t = \varphi' u_t, \quad v_x = \varphi' u_x, \quad v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2, \]

and the condition (1.36) is written:

\[ \varphi'(u)[u_t + u_{xx}] + \varphi''(u)u_x^2 = \lambda[u_t - u_{xx}]. \]

Whence, comparing the coefficients of \( u_t \) in both sides, we obtain \( \lambda = \varphi'(u) \). Then the above equation becomes

\[ \varphi'(u)[u_t + u_{xx}] + \varphi''(u)u_x^2 = \varphi'(u)[u_t - u_{xx}]. \]

This equation yields that \( \varphi'(u) = 0 \). Hence, Eq. (1.23) is quasi self-adjoint with the substitution \( v = C \), where \( C \) is any non-vanishing constant. This substitution does not satisfy the condition (1.35).

Example 1.7. Let us consider the Fornberg-Whitham equation [14]

\[ u_t - u_{txx} - uu_{xxx} - 3u_x u_{xx} + uu_x + u_x = 0. \]  

Eqs. (1.18)-(1.19) give the following adjoint equation:

\[ F^* \equiv -v_t + v_{txx} + uv_{xxx} - uv_x - v_x = 0. \]

It is manifest from the equations (1.44) and (1.45) that the Fornberg-Whitham equation is not strictly self-adjoint. Let us test it for quasi self-adjointness. Inserting in (1.45) the substitution \( v = \varphi(u) \) and its derivatives

\[ v_t = \varphi' u_t, \quad u_x = \varphi' u_x, \quad v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2, \quad v_{tx} = \varphi' u_{tx} + \varphi'' u_t u_x, \ldots, \]

then writing the condition (1.36) and comparing the coefficients for \( u_t, \ u_{tx}, \ u_{xx}, \ldots \) one can verify that \( \varphi'(u) = 0 \). Hence, Eq. (1.44) is quasi self-adjoint but does not satisfy the condition (1.35).
2 Strict self-adjointness via multipliers

It is commonly known that numerous linear equations used in practice, e.g. linear evolution equations, are not self-adjoint in the classical meaning of the self-adjointness. Likewise, useful nonlinear equations such as the nonlinear heat equation, the Burgers equation, etc. are not strictly self-adjoint. We will see here that these and many other equations can be rewritten in a strictly self-adjoint equivalent form by using multipliers. The general discussion of this approach will be given in Section 3.7.

2.1 Motivating examples

Example 2.1. Let us consider the following second-order nonlinear equation

$$u_{xx} + f(u)u_x - u_t = 0. \tag{2.1}$$

Its adjoint equation (1.18) is written

$$v_{xx} - f(u)v_x + v_t = 0. \tag{2.2}$$

It is manifest that the substitution $v = u$ does not map Eq. (2.2) into Eq. (2.1). Hence Eq. (2.1) is not strictly self-adjoint.

Let us clarify if Eq. (2.1) can be written in an equivalent form

$$\mu(u) [u_{xx} + f(u)u_x - u_t] = 0 \tag{2.3}$$

with a certain multiplier $\mu(u) \neq 0$ so that Eq. (2.3) is strictly self-adjoint. The formal Lagrangian for Eq. (2.3) is

$$\mathcal{L} = v\mu(u)[u_{xx} + f(u)u_x - u_t].$$

We have:

$$\frac{\delta \mathcal{L}}{\delta u} = D_x^2[\mu(u)v] - D_x[\mu(u)f(u)v] + D_t[\mu(u)v]$$

$$+ \mu'(u)v[u_{xx} + f(u)u_x - u_t] + \mu(u)f'(u)v u_x,$$

whence, upon performing the differentiations,

$$\frac{\delta \mathcal{L}}{\delta u} = \mu v_{xx} + 2\mu' v u_{xx} + 2\mu' u_x v_x + \mu'' v u_x^2 - \mu f v_x + \mu v_t.$$

The strict self-adjointness requires that

$$\frac{\delta \mathcal{L}}{\delta u} \bigg|_{v=u} = \lambda [u_{xx} + f(u)u_x - u_t].$$
This provides the following equation for the unknown multiplier $\mu(u)$:

$$(\mu + 2u\mu')u_{xx} + (2\mu' + u\mu'')u_x^2 - \mu f u_x + \mu u_t = \lambda[u_{xx} + f(u)u_x - u_t].$$  \hspace{1cm} (2.4)

Since the right side of Eq. (2.4) does not contain $u_x^2$ we should have $2\mu' + u\mu'' = 0$, whence $\mu = C_1u^{-1} + C_2$. Furthermore, comparing the coefficients of $u_t$ in both sides of Eq. (2.4) we obtain $\lambda = -\mu$. Now Eq. (2.4) takes the form

$$(C_2 - C_1u^{-1})u_{xx} - (C_1u^{-1} + C_2)f u_x = -(C_1u^{-1} + C_2)[u_{xx} + f(u)u_x]$$

and yields $C_2 = 0$. Thus, $\mu = C_1u^{-1}$. We can let $C_1 = -1$ and formulate the result.

**Proposition 2.1.** Eq. (2.1) becomes strictly self-adjoint if we rewrite it in the form

$$\frac{1}{u}[u_t - u_{xx} - f(u)u_x] = 0.$$ \hspace{1cm} (2.5)

**Example 2.2.** One can verify that the $n$th-order nonlinear evolution equation

$$\frac{\partial u}{\partial t} - f(u)\frac{\partial^n u}{\partial x^n} = 0, \quad f(u) \neq 0,$$ \hspace{1cm} (2.6)

with one spatial variable $x$ is not strictly self-adjoint. The following statement shows that it becomes strictly self-adjoint after using an appropriate multiplier.

**Proposition 2.2.** Eq. (2.6) becomes strictly self-adjoint upon rewriting it in the following equivalent form:

$$\frac{1}{uf(u)} \left[ \frac{\partial u}{\partial t} - f(u)\frac{\partial^n u}{\partial x^n} \right] = 0.$$ \hspace{1cm} (2.7)

**Proof.** Multiplying Eq. (2.6) by $\mu(u)$ and taking the formal Lagrangian

$$\mathcal{L} = v\mu(u)[u_t - f(u)u_x],$$

where $u_n = D^n_x(u)$, we have:

$$\frac{\delta \mathcal{L}}{\delta u} = -D_t[\mu(u)v] - D^n_x[\mu(u)f(u)v] + v\mu'(u)u_t - v[\mu(u)f(u)]'u_n.$$  

Noting that $-D_t[\mu(u)v] + v\mu'(u)u_t = -\mu(u)v_t$ and letting $v = u$ we obtain

$$\left. \frac{\delta \mathcal{L}}{\delta u} \right|_{v=u} = -\mu(u)u_t - D^n_x[\mu(u)f(u)u] - [\mu(u)f(u)]'u u_n.$$
If we take $\mu(u) = [uf(u)]^{-1}$, then $\mu(u)f(u)u = 1$, $\mu(u)f(u) = u^{-1}$, and hence
\[
\left. \frac{\delta L}{\delta u} \right|_{v=u} = -\frac{1}{uf(u)} [ut - f(u)u_n].
\]
Thus, Eq. (2.7) satisfies the strict self-adjointness condition (1.28) with $\lambda = -1$.

### 2.2 Linear heat equation

Taking in (2.5) $f(u) = 0$, we rewrite the classical linear heat equation $u_t = u_{xx}$ in the following strictly self-adjoint form:
\[
\frac{1}{u} [ut - u_{xx}] = 0.
\]
(2.8)

This result can be extended to the heat equation
\[
u_t - \Delta u = 0,
\]
(2.9)

where $\Delta u$ is the Laplacian with $n$ variables $x = (x^1, \ldots, x^n)$. Namely, the strictly self-adjoint form of Eq. (2.9) is
\[
\frac{1}{u} [ut - \Delta u] = 0.
\]
(2.10)

Indeed, the formal Lagrangian (1.9) for Eq. (2.10) has the form
\[
L = \frac{v}{u} [ut - \Delta u].
\]

Substituting it in (1.19) we obtain
\[
F^* = -Dt \left( \frac{v}{u} \right) - \Delta \left( \frac{v}{u} \right) - \frac{v}{u^2} [ut - \Delta u].
\]

Upon letting $v = u$ it becomes
\[
F^* = -\frac{1}{u} [ut - \Delta u].
\]

Hence, Eq. (2.10) satisfies the condition (1.28) with $\lambda = -1$.  

2.3 Nonlinear heat equation

Consider the nonlinear heat equation $u_t - D_x (k(u)u_x) = 0$, or

$$u_t - k(u)u_{xx} - k'(u)u_x^2 = 0. \quad (2.11)$$

Its adjoint equation has the form

$$v_t + k(u)v_{xx} = 0.$$

Therefore it is obvious that (2.11) does not satisfy Definition 1.2. But it becomes strictly self-adjoint if we rewrite it in the form

$$\frac{1}{u} \left[ u_t - k(u)u_{xx} - k'(u)u_x^2 \right] = 0. \quad (2.12)$$

Indeed, the formal Lagrangian (1.9) for Eq. (2.12) is written

$$\mathcal{L} = \frac{v}{u} \left[ u_t - k(u)u_{xx} - k'(u)u_x^2 \right].$$

Substituting it in (1.19) we obtain

$$F^* = -D_t \left( \frac{v}{u} \right) - D_x^2 \left( \frac{v}{u} k(u) \right) + 2D_x \left( \frac{v}{u} k'(u)u_x \right) - \frac{v}{u} k'(u)u_{xx} - \frac{v}{u} k''(u)u_x^2 - \frac{v}{u^2} \left[ u_t - k(u)u_{xx} - k'(u)u_x^2 \right].$$

Letting here $v = u$ we have:

$$F^* = -\frac{1}{u} \left[ u_t - k(u)u_{xx} - k'(u)u_x^2 \right].$$

Hence, Eq. (2.10) satisfies the strict self-adjointness condition (1.28) with $\lambda = -1$.

2.4 The Burgers equation

Taking in (2.5) $f(u) = u$ we obtain the strictly self-adjoint form

$$\frac{1}{u} \left[ u_t - u_{xx} \right] - u_x = 0 \quad (2.13)$$

of the Burgers equation $u_t = u_{xx} + uu_x$. 
2.5 Heat conduction in solid hydrogen

According to [67], the heat conduction in solid crystalline molecular hydrogen at low pressures is governed by the nonlinear equation (up-to positive constant coefficient)

\[ u_t = u^2 \Delta u. \]  

(2.14)

It is derived from the Fourier equation

\[ \rho c_\ast \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) \]

using the empirical information that the density \( \rho \) at low pressures has a constant value, whereas the specific heat \( c_\ast \) and the thermal conductivity \( k \) have the estimations

\[ c_\ast \approx T^3, \quad k \approx T^3 (1 + T^4)^{-2}. \]

It is also shown in [67] that the one-dimensional equation (2.14),

\[ u_t = u^2 u_{xx}, \]  

(2.15)

is related to the linear heat equation by a non-point transformation (Eq. (5) in [67]). A similar relation was found in [8] for another representation of Eq. (2.15). The non-point transformation of Eq. (2.15) to the linear heat equation

\[ w_s = w_{\xi\xi} \]  

(2.16)

is written in [21] as the differential substitution

\[ t = s, \quad x = w, \quad u = w_\xi. \]  

(2.17)

It is also demonstrated in [21], Section 20, that Eq. (2.15) is the unique equation with nontrivial Lie-Bäcklund symmetries among the equations of the form

\[ u_t = f(u) + h(u, u_\xi), \quad f'(u) \neq 0. \]

The connection between Eq. (2.15) and the heat equation is treated in [65] as a reciprocal transformation [65]. It is shown in [66] that this connection, together with its extensions, allows the analytic solution of certain moving boundary problems in nonlinear heat conduction.

Our Example 2.2 from Section 2.1 reveals one more remarkable property of Eq. (2.15). Namely, taking \( n = 2 \) and \( f(u) = u^2 \) in Eq. (2.7) we see that Eq. (2.15) becomes strictly self-adjoint if we rewrite it in the form

\[ \frac{u_t}{u^3} = u_{xx} \cdot \frac{1}{u}. \]  

(2.18)
2.6 Harry Dym equation

Taking in Example 2.2 from Section 2.1 \( n = 3 \) and \( f(u) = u^3 \) we see that the Harry Dym equation

\[ u_t - u^3 u_{xxx} = 0 \]  

(2.19)

becomes strictly self-adjoint upon rewriting it in the form

\[ \frac{u_t}{u^3} - \frac{u_{xxx}}{u} = 0. \]

2.7 Kompaneets equation

The equations considered in Sections 2.1 - 2.6 are quasi self-adjoint. For example, for Eq. (2.6) we have

\[ F^* = -v_t - D_x^n(f(u)v) - vf'(u)u_n, \]

whence making the substitution

\[ v = \frac{1}{f(u)} \]

we obtain

\[ F^* = \frac{f'}{f^2} u_t - \frac{f'}{f} u_n = \frac{f'}{f^2} [u_t - f(u) u_n]. \]

Hence, Eq. (2.6) is quasi self-adjoint.

Example 2.3. The Kompaneets equation

\[ u_t = \frac{1}{x^2} D_x \left[ x^4(u_x + u + u^2) \right] \]  

(2.20)

provides an example of an equation that is not quasi self-adjoint. Indeed, Eq. (2.20) has the formal Lagrangian

\[ \mathcal{L} = v[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)]. \]

The calculation yields the following adjoint equation to (2.20):

\[ \frac{\delta \mathcal{L}}{\delta u} \equiv v_t + x^2 v_{xx} - x^2 (1 + 2u)v_x + 2(x + 2xu - 1)v = 0. \]  

(2.21)

Letting \( v = \varphi(u) \) one obtains:

\[ \frac{\delta \mathcal{L}}{\delta u} \bigg|_{v=\varphi(u)} = \varphi'(u)[u_t + x^2 u_{xx} - x^2 (1 + 2u)u_x] \]

\[ + \varphi''(u)x^2 u_x^2 + 2(x + 2xu - 1)\varphi(u). \]
Writing the quasi self-adjointness condition (1.36) in the form
\[ \frac{\delta \mathcal{L}}{\delta u} \bigg|_{v=\phi(u)} = \lambda [-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u) u_x + 4x(u + u^2)] \]
and comparing the coefficients for \( u_t \) in both sides one obtains \( \lambda = -\phi'(u) \), so that the quasi self-adjointness condition takes the form
\[ \phi'(u)[u_t + x^2 u_{xx} - x^2(1 + 2u) u_x] + \phi''(u)x^2 u_x^2 + 2(x + 2xu - 1)\phi(u) = \phi'(u)[u_t - x^2 u_{xx} - (x^2 + 4x + 2x^2 u) u_x - 4x(u + u^2)]. \]
Comparing the coefficients for \( u_{xx} \) in both sides we obtain \( \phi'(u) = 0 \). Then the above equation becomes
\[ (x + 2xu - 1)\phi(u) = 0 \]
and yields \( \phi(u) = 0 \). Hence the Kompaneets equation is not quasi self-adjoint because the condition (1.43) is not satisfied.

But we can rewrite Eq. (2.20) in the strictly self-adjoint form by using a more general multiplier than above, namely, the multiplier
\[ \mu = \frac{x^2}{u}. \tag{2.22} \]
Indeed, upon multiplying by this \( \mu \) Eq. (2.20) is written
\[ \frac{x^2}{u} u_t = \frac{1}{u} D_x \left[ x^4(u_x + u + u^2) \right]. \]
Its formal Lagrangian
\[ \mathcal{L} = \frac{v}{u} \left\{ -x^2 u_t + D_x \left[ x^4(u_x + u + u^2) \right] \right\} \]
satisfies the strict self-adjointness condition (1.28) with \( \lambda = -1 \):
\[ \frac{\delta \mathcal{L}}{\delta u} \bigg|_{v=u} = -\frac{1}{u} \left\{ -x^2 u_t + D_x \left[ x^4(u_x + u + u^2) \right] \right\}. \]

**Remark 2.1.** Note that \( v = x^2 \) solves Eq. (2.21) for any \( u \). The connection of this solution with the multiplier (2.22) is discussed in Section 3.7. See also Section 4.
3 Nonlinear self-adjointness

Motivated by the examples discussed in Sections 1 and 2 as well as other similar examples, I introduce here the general concept of nonlinear self-adjointness of systems consisting of any number of equations with \( m \) dependent variables. This concept encapsulates Definition 1.2 of strict self-adjointness and Definition 1.3 of quasi self-adjointness. The new concept has two different features. They are expressed below by two different but equivalent definitions.

3.1 Two definitions and their equivalence

Definition 3.1. The system of \( m \) differential equations (see Eqs. (1.6))

\[
F_{\bar{\alpha}}(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \bar{\alpha} = 1, \ldots, m, \quad (3.1)
\]

with \( m \) dependent variables \( u = (u^1, \ldots, u^m) \) is said to be nonlinearly self-adjoint if the adjoint equations

\[
F_{\bar{\alpha}}^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta (v^\beta F_{\bar{\beta}})}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \quad (3.2)
\]

are satisfied for all solutions \( u \) of the original system (3.1) upon a substitution

\[
v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u), \quad \bar{\alpha} = 1, \ldots, m, \quad (3.3)
\]

such that

\[
\varphi(x, u) \neq 0. \quad (3.4)
\]

In other words, the equations

\[
F_{\bar{\alpha}}^*(x, u, \varphi(x, u), \ldots, u_{(s)}, \varphi_{(s)}) = \lambda^\beta \frac{\partial}{\partial u_{\bar{\beta}}} F_{\bar{\beta}}(x, u, \ldots, u_{(s)}) \quad (3.5)
\]

hold for \( \alpha = 1, \ldots, m \), where \( \lambda^\beta \) are undetermined coefficients, and \( \varphi_{(\sigma)} \) are the derivatives of (3.3):

\[
\varphi_{(\sigma)} = \{D_{u_1} \cdots D_{u_\sigma} (\varphi^{\bar{\alpha}}(x, u))\}, \quad \sigma = 1, \ldots, s.
\]

Here \( v \) and \( \varphi \) are the \( m \)-dimensional vectors

\[
v = (v^1, \ldots, v^m), \quad \varphi = (\varphi^1, \ldots, \varphi^m),
\]

and Eq. (3.4) means that not all components of \( \varphi \) vanish simultaneously.
Remark 3.1. If the system (3.1) is over-determined, i.e. \( \overline{m} > m \), then the adjoint system (3.2) is sub-definite since it contains \( m < \overline{m} \) equations for \( \overline{m} \) new dependent variables \( v \). Vise versa, if \( \overline{m} < m \), then the system (3.1) is sub-definite and the adjoint system (3.2) is over-determined.

Remark 3.2. The adjoint system (3.2), upon substituting there any solution \( u(x) \) of Eqs. (3.1), becomes a linear homogeneous system for the new dependent variables \( v^\alpha \). The essence of Eqs. (3.5) is that for the self-adjoint system (3.1) there exist functions (3.3) that provide a non-trivial (not identically zero) solution to the adjoint system (3.2) for all solutions of the original system (3.1). This property can be taken as the following alternative definition of the nonlinear self-adjointness.

Definition 3.2. The system (3.1) is nonlinearly self-adjoint if there exist functions \( v^\alpha \) given by (3.3) that solve the adjoint system (3.2) for all solutions \( u(x) \) of Eqs. (3.1) and satisfy the condition (3.4).

Proposition 3.1. The above two definitions are equivalent.

Proof. Let the system (3.1) be nonlinearly self-adjoint by Definition 3.1. Then, according to Remark 3.2, the system (3.1) satisfies the condition of Definition 3.2.

Conversely, let the system (3.1) be nonlinearly self-adjoint by Definition 3.2. Namely, let the functions \( v^\alpha \) given by (3.3) and satisfying the condition (3.4) solve the adjoint system (3.2) for all solutions \( u(x) \) of Eqs. (3.1). This is possible if and only if Eqs. (3.5) hold. Then the system (3.1) is nonlinearly self-adjoint by Definition 3.1.

Example 3.1. It has been mentioned in Example 1.2 that the KdV equation

\[ u_t = u_{xxx} + uu_x \]  

(3.6)

is strictly self-adjoint. In terms of Definition 3.2 it means that \( v = u \) solves the adjoint equation

\[ v_t = v_{xxx} + uv_x \]

(3.7)

for all solutions of the KdV equation (3.6). One can verify that the general substitution of the form (3.3), \( v = \varphi(t, x, u) \), satisfying Eq. (3.5) is given by

\[ v = A_1 + A_2u + A_3(x + tu), \]

(3.8)

where \( A_1, A_2, A_3 \) are arbitrary constants. One can also check that \( v \) given by Eq. (3.8) solves the adjoint equation (3.7) for all solutions \( u \) of the KdV equation. The solution \( v = x + tu \) is an invariant of the Galilean
transformation of the KdV equation and appears in different approaches (see [21], Section 22.5, and [7]). Thus, the KdV equation is nonlinearly self-adjoint with the substitution (3.8).

Proposition 3.2. Any linear equation is nonlinearly self-adjoint.

Proof. This property is the direct consequence of Definition 3.2 because the adjoint equation \( F^*[v] = 0 \) to a linear equation \( F[u] = 0 \) does not involve the variable \( u \).

3.2 Remark on differential substitutions

One can further extend the concept of self-adjointness by replacing the point-wise substitution (3.3) with differential substitutions of the form

\[
v^\alpha = \varphi^\alpha(x, u, u_{(1)}, \ldots, u_{(r)}), \quad \bar{\alpha} = 1, \ldots, \bar{m}.
\]

Then Eqs. (3.5) will be written, e.g. in the case \( r = 1 \), as follows:

\[
F^*_\alpha(x, u, \varphi, \ldots, u_{(s)}, \varphi_{(s)}) = \lambda_{\bar{\alpha}}^\beta F_\beta + \lambda_{\bar{\alpha}}^\beta D_j(F_\beta).
\]

Example 3.2. The reckoning shows that the equation

\[
u_{xy} = \sin u
\]

is not self-adjoint via a point-wise substitution \( v = \varphi(x, y, u) \), but it is self-adjoint in the sense of Definition 3.1 with the following differential substitution:

\[
v = \varphi(x, y, u_x, u_y) \equiv A_1[xu_x - yu_y] + A_2u_x + A_3u_y,
\]

where \( A_1, A_2, A_3 \) are arbitrary constants. The adjoint equation to Eq. (3.11) is

\[v_{xy} - v \cos u = 0,
\]

and the self-adjointness condition (3.10) with the function \( \varphi \) given by (3.12) is satisfied in the form

\[
\varphi_{xy} - \varphi \cos u = (A_1x + A_2)D_x(u_{xy} - \sin u) + (A_3 - A_1y)D_y(u_{xy} - \sin u).
\]
3.3 Nonlinear heat equation

3.3.1 One-dimensional case

Let us apply the new viewpoint to the nonlinear heat equation (2.11), $u_t = (k(u)u_x)_x$, discussed in Section 2.3. We will take it in the expanded form

$$ u_t - k(u)u_{xx} - k'(u)u_x^2 = 0, \quad k(u) \neq 0. \quad (3.14) $$

The adjoint equation (1.18) to Eq. (3.14) is

$$ v_t + k(u)v_{xx} = 0. \quad (3.15) $$

We take the substitution (3.3) written together with the necessary derivatives:

$$ v = \varphi(t, x, u), $$

$$ v_t = \varphi_u u_t + \varphi_t, \quad v_x = \varphi_u u_x + \varphi_x, \quad v_{xx} = \varphi_u u_{xx} + \varphi_u u_x^2 + 2\varphi_x u_x + \varphi_{xx}, \quad (3.16) $$

and arrive at the following self-adjointness condition (3.5):

$$ \varphi_u u_t + \varphi_t + k(u)[\varphi_u u_{xx} + \varphi_u u_x^2 + 2\varphi_x u_x + \varphi_{xx}] = \lambda[u_t - k(u)u_{xx} - k'(u)u_x^2]. \quad (3.17) $$

The comparison of the coefficients of $u_t$ in both sides of Eq. (3.17) yields $\lambda = \varphi_u$. Then, comparing the terms with $u_{xx}$ we see that $\varphi_u = 0$. Hence Eq. (3.17) reduces to

$$ \varphi_t + k(u)\varphi_{xx} = 0 \quad (3.18) $$

and yields $\varphi_t = 0$, $\varphi_{xx} = 0$, whence $\varphi = C_1 x + C_2$, where $C_1, C_2 = \text{const}$. We have demonstrated that Eq. (3.14) is nonlinearly self-adjoint by Definition 3.1 and that the substitution (3.3) has the form

$$ v = C_1 x + C_2. \quad (3.19) $$

The same result can be easily obtained by using Definition 3.2. We look for the solution of the adjoint equation (3.15) in the form $v = \varphi(t, x)$. Then Eq. (3.15) has the form (3.18). Since it should be satisfied for all solutions $u$ of Eq. (3.14), we obtain $\varphi_t = 0$, $\varphi_{xx} = 0$, and hence Eq. (3.19).
3.3.2 Multi-dimensional case

The similar analysis can be applied to the nonlinear heat equation with several variables \( x = (x^1, \ldots, x^n) \):

\[ u_t = \nabla \cdot (k(u) \nabla u), \quad (3.20) \]

or

\[ u_t - k(u) \Delta u - k'(u) |\nabla u|^2 = 0. \quad (3.21) \]

The reckoning shows that the adjoint equation (1.18) to Eq. (3.21) is written

\[ v_t + k(u) \Delta v = 0. \quad (3.22) \]

It is easy to verify the nonlinear self-adjointness by Definition 3.2. Namely, searching the solution of the adjoint equation (3.22) in the form

\[ v = \varphi(t, x^1, \ldots, x^n), \]

one obtains

\[ \varphi_t + k(u) \Delta \varphi = 0, \]

whence

\[ \varphi_t = 0, \quad \Delta \varphi = 0. \]

We conclude that Eq. (3.21) is self-adjoint and that the substitution (3.3) is given by

\[ v = \varphi(x^1, \ldots, x^n), \quad (3.23) \]

where \( \varphi(x^1, \ldots, x^n) \) is any solution of the \( n \)-dimensional Laplace equation \( \Delta \varphi = 0 \).

3.4 Anisotropic nonlinear heat equation

3.4.1 Two-dimensional case

Consider the heat diffusion equation

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y \quad (3.24) \]

in an anisotropic two-dimensional medium (see [23], Section 10.8) with arbitrary functions \( f(u) \) and \( g(u) \). The adjoint equation is

\[ v_t + f(u)v_{xx} + g(u)v_{yy} = 0. \quad (3.25) \]

Using Definition 3.2 we obtain the following equations for nonlinear self-adjointness of Eq. (3.24):

\[ \varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0. \quad (3.26) \]

Integrating Eqs. (3.26) we obtain the following substitution (3.3):

\[ v = C_1 xy + C_2 x + C_3 y + C_4. \quad (3.27) \]
3.4.2 Three-dimensional case

The three-dimensional anisotropic nonlinear heat diffusion equation has the following form (see [23], Section 10.9):

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z. \]

(3.28)

Its adjoint equation is

\[ v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} = 0. \]

(3.29)

Eq. (3.28) is nonlinearly self-adjoint. In this case the substitution (3.27) is replaced by

\[ v = C_1 x y z + C_2 x y + C_3 x z + C_4 y z + C_5 x + C_6 y + C_7 z + C_8. \]

(3.30)

3.5 Nonlinear wave equations

3.5.1 One-dimensional case

Consider the following one-dimensional nonlinear wave equation:

\[ u_{tt} = (k(u)u_x)_x, \quad k(u) \neq 0, \]

(3.31)

or in the expanded form

\[ u_{tt} - k(u)u_{xx} - k'(u)u_x^2 = 0. \]

(3.32)

The adjoint equation (1.18) to Eq. (3.31) is written

\[ v_{tt} - k(u)v_{xx} = 0. \]

(3.33)

Proceeding as in Section 3.3.1 or applying Definition 3.2 to Eqs. (3.32), (3.33) by letting \( v = \varphi(t,x) \), we obtain the following equations that guarantee the nonlinear self-adjointness of Eq. (3.31):

\[ \varphi_{tt} = 0, \quad \varphi_{xx} = 0. \]

(3.34)

Integrating Eqs. (3.34) we obtain the following substitution:

\[ v = C_1 t x + C_2 t + C_3 x + C_4. \]

(3.35)
3.5.2 Multi-dimensional case

The multi-dimensional version of Eq. (3.31) with \( x = (x^1, \ldots, x^\nu) \) is written

\[
\frac{\partial^2 u}{\partial t^2} = \nabla \cdot (k(u) \nabla u), \tag{3.36}
\]

or

\[
\frac{\partial^2 u}{\partial t^2} - k(u) \Delta u - k'(u) |\nabla u|^2 = 0. \tag{3.37}
\]

The adjoint equation is

\[
\frac{\partial^2 v}{\partial t^2} - k(u) \Delta v = 0. \tag{3.38}
\]

Using Definition 3.2 and searching the solution of the adjoint equation (3.38) in the form \( v = \varphi(t, x^1, \ldots, x^\nu) \), we obtain the equations

\[
\varphi_{tt} = 0, \quad \Delta \varphi = 0.
\]

Solving them we arrive at the following substitution (3.3):

\[
v = a(x) t + b(x), \tag{3.39}
\]

where \( a(x) \) and \( b(x) \) solve the \( \nu \)-dimensional Laplace equation,

\[
\Delta a(x^1, \ldots, x^\nu) = 0, \quad \Delta b(x^1, \ldots, x^\nu) = 0.
\]

Hence Eq. (3.36) is nonlinearly self-adjoint.

3.5.3 Nonlinear vibration of membranes

Vibrations of a uniform membrane whose tension varies during deformations are described by the following Lagrangian:

\[
L = \frac{1}{2} \left[ \frac{u^2_t}{2} - k(u) \left( u^2_x + u^2_y \right) \right], \quad k'(u) \neq 0. \tag{3.40}
\]

The corresponding Euler-Lagrange equation

\[
\frac{\partial L}{\partial u} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial u_t} \right) - D_x \left( \frac{\partial L}{\partial u_x} \right) - D_y \left( \frac{\partial L}{\partial u_y} \right) = 0
\]

provides the nonlinear wave equation

\[
\frac{\partial^2 u}{\partial t^2} = k(u) (u_{xx} + u_{yy}) + \frac{1}{2} k'(u) (u^2_x + u^2_y). \tag{3.41}
\]

Note that Eq. (3.41) differs from the two-dimensional nonlinear wave equation (3.37) by the coefficient \( 1/2 \). Let us find out if this difference affects self-adjointness.
By applying (3.2) to the formal Lagrangian of Eq. (3.41) we obtain:

\[ F^* = v_{tt} - k(u)(v_{xx} + v_{yy}) - k'(u)(u_x v_x + u_y v_y + v u_{xx} + v u_{yy}) - \frac{v}{2} k''(u)(u_x^2 + u_y^2). \]

We take the substitution (3.3) together with the necessary derivatives (see Eqs. (3.16)):

\[
\begin{align*}
v &= \varphi(t, x, y, u), \\
v_t &= \varphi_u u_t + \varphi_t, \\
v_x &= \varphi_u u_x + \varphi_x, \\
v_y &= \varphi_u u_y + \varphi_y, \\
v_{xx} &= \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2 \varphi_{ux} u_x + \varphi_{xx}, \\
v_{yy} &= \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2 \varphi_{uy} u_y + \varphi_{yy}, \\
v_{tt} &= \varphi_u u_{tt} + \varphi_{uu} u_t^2 + 2 \varphi_{ut} u_t + \varphi_{tt},
\end{align*}
\]

and substitute the expressions (3.42) in the self-adjointness condition (3.5):

\[ F^* \big|_{v=\varphi} = \lambda [u_{tt} - k(u)(u_{xx} + u_{yy}) - \frac{1}{2} k'(u)(u_x^2 + u_y^2)]. \]

Comparing the coefficients of \( u_{tt} \) we obtain \( \lambda = \varphi_u \). Then we compare the coefficients of \( u_{xx} \) and obtain \( \varphi k'(u) = 0 \). This equation yields \( \varphi = 0 \) because \( k'(u) \neq 0 \). Thus, the condition (3.4) is not satisfied for the pointwise substitution (3.3). Further investigation of Eq. (3.41) for the nonlinear self-adjointness requires differential substitutions.

### 3.6 Anisotropic nonlinear wave equation

#### 3.6.1 Two-dimensional case

The two-dimensional anisotropic nonlinear wave equation is (see [23], Section 12.6)

\[ u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y. \quad (3.43) \]

Its adjoint equation has the form

\[ v_{tt} - f(u)v_{xx} - g(u)v_{yy} = 0. \quad (3.44) \]

Proceeding as in Section 3.4 we obtain the following equations that guarantee the self-adjointness of Eq. (3.43):

\[ \varphi_{tt} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0. \quad (3.45) \]

Integrating Eqs. (3.45) we obtain the following substitution (3.3):

\[ v = C_1 t x y + C_2 t x + C_3 t y + C_4 x y + C_5 t + C_6 x + C_7 y + C_8. \quad (3.46) \]
Remark 3.3. I provide here detailed calculations in integrating Eqs. (3.45). The general solution to the linear second-order equation \( \varphi_{tt} = 0 \) is given by
\[
\varphi = A(x, y)t + B(x, y)
\] (3.47)
with arbitrary functions \( A(x, y) \) and \( B(x, y) \). Substituting this expression for \( \varphi \) in the second and third equations (3.45) and splitting with respect to \( t \) we obtain the following equations for \( A(x, y) \) and \( B(x, y) \):
\[
A_{xx} = 0, \quad A_{yy} = 0,
B_{xx} = 0, \quad B_{yy} = 0.
\]
Substituting the general solution
\[
A = a_1(y)x + a_2(y)
\]
of the equation \( A_{xx} = 0 \) in the equation \( A_{yy} = 0 \) and splitting with respect to \( x \), we obtain \( a_1'' = 0 \), \( a_2'' = 0 \), whence
\[
a_1 = c_{11}y + c_{12}, \quad a_2 = c_{21}y + c_{22},
\]
where \( c_{11}, \ldots, c_{22} \) are arbitrary constants. Substituting these in the above expression for \( A \) we obtain
\[
A = c_{11}xy + c_{12}x + c_{21}y + c_{22}.
\]
Proceeding likewise with the equations for \( B(x, y) \), we have
\[
B = d_{11}xy + d_{12}x + d_{21}y + d_{22}
\]
with arbitrary constant coefficients \( d_{11}, \ldots, d_{22} \). Finally, we substitute the resulting \( A \) and \( B \) in the expression (3.47) for \( \varphi \) and, changing the notation, arrive at (3.46).

3.6.2 Three-dimensional case

The three-dimensional anisotropic nonlinear wave equation
\[
u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z
\] (3.48)
has the following adjoint equation
\[
v_{tt} - f(u)v_{xx} - g(u)v_{yy} - h(u)v_{zz} = 0.
\] (3.49)
In this case Eqs. (3.45) are replaced by
\[
\varphi_{tt} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0
\]
and yield the following substitution (3.3):
\[
v = C_1 txyz + C_2 txy + C_3 txz + C_4 tyz + C_5 tx + C_6 ty + C_7 tz
\]
\[
+ C_8 xy + C_9 xz + C_{10} yz + C_{11} t + C_{12} x + C_{13} y + C_{14} z + C_{15}.\] (3.50)
3.7 Nonlinear self-adjointness and multipliers

The approach of this section is not used for constructing conservation laws. But it may be useful for other applications of the nonlinear self-adjointness.

**Theorem 3.1.** The differential equation (1.17),

\[
F(x, u, u_(1), \ldots, u_(s)) = 0,
\]

is nonlinearly self-adjoint (Definition 3.1) if and only if it becomes strictly self-adjoint (Definition 1.2) upon rewriting in the equivalent form

\[
\mu(x, u)F(x, u, u_(1), \ldots, u_(s)) = 0, \quad \mu(x, u) \neq 0,
\]

with an appropriate multiplier \(\mu(x, u)\).

**Proof.** We will write the condition (3.5) for nonlinear self-adjointness of Eq. (3.51) in the form

\[
\frac{\delta (vF)}{\delta u} \bigg|_{v=\phi(x,u)} = \lambda(x, u)F(x, u, u_(1), \ldots, u_(s)).
\]

Furthermore, invoking that the equations (3.52) and (3.51) are equivalent, we will write the condition (1.28) for strict self-adjointness of Eq. (3.52) in the form

\[
\frac{\delta (w\mu F)}{\delta u} \bigg|_{w=\mu} = \tilde{\lambda}(x, u)F(x, u, u_(1), \ldots, u_(s)).
\]

Since \(w\) is a dependent variable and \(\mu = \mu(x, u)\) is a certain function of \(x, u\), the variational derivative in the left-hand side of (3.54) can be written as follows:

\[
\frac{\delta (w\mu F)}{\delta u} = w \frac{\partial \mu}{\partial u} F + \mu w \frac{\partial F}{\partial u} - D_i \left( \mu w \frac{\partial F}{\partial u_i} \right) + D_i D_j \left( \mu w \frac{\partial F}{\partial u_{ij}} \right) - \cdots
\]

\[
= w \frac{\partial \mu}{\partial u} F + \frac{\delta (vF)}{\delta u},
\]

where \(v\) is the new dependent variable defined by

\[
v = \mu(x, u)w.
\]

is the new dependent variable instead of \(w\). Now the left side of Eq. (3.54) is written

\[
\frac{\delta (w\mu F)}{\delta u} \bigg|_{w=\mu} = u \frac{\partial \mu}{\partial u} F + \frac{\delta (vF)}{\delta u} \bigg|_{v=\mu(x,u)}.
\]
Let us assume that Eq. (3.51) is nonlinearly self-adjoint. Then Eq. (3.53) holds with a certain given function \(\varphi(x, u)\). Therefore, we take the multiplier
\[
\mu(x, u) = \frac{\varphi(x, u)}{u}
\] (3.57)
and reduce Eq. (3.56) to the following form:
\[
\left.\frac{\delta(w \mu F)}{\delta u}\right|_{w=u} = \left(\lambda + \frac{\partial \varphi}{\partial u} - \frac{\varphi}{u}\right) F.
\]
This proves that Eq. (3.54) holds with
\[
\tilde{\lambda} = \frac{\partial \varphi}{\partial u} - \frac{\varphi}{u} + \lambda.
\]
Hence, Eq. (3.52) with the multiplier \(\mu\) given by (3.57) is strictly self-adjoint.

Let us assume now that Eq. (3.52) with a certain multiplier \(\mu(x, u)\) is strictly self-adjoint. Then Eq. (3.54) holds. Therefore, if we take the function \(\varphi\) defined by (see (3.57))
\[
\varphi(x, u) = u \mu(x, u),
\] (3.58)
Eq. (3.56) yields:
\[
\left.\frac{\delta(v F)}{\delta u}\right|_{v=\varphi(x, u)} = \left(\tilde{\lambda} - u \frac{\partial \mu}{\partial u}\right) F.
\]
It follows that Eq. (3.53) holds with
\[
\lambda = \tilde{\lambda} - u \frac{\partial \mu}{\partial u}.
\]
We conclude that Eq. (3.51) is nonlinearly self-adjoint, thus completing the proof.

**Example 3.3.** The multiplier (2.22) used in Example 2.3 and the function \(\varphi = x^2\) that provides a solution of the adjoint equation (2.21) to the Kompaneets equation are related by Eq. (3.58).

**Example 3.4.** Let us consider the one-dimensional nonlinear wave equation (3.32),
\[
u_{tt} - k(u)u_{xx} - k'(u)u_x^2 = 0.
\]
If we substitute in (3.57) the function \(\varphi\) given by the right-hand side of (3.35) we will obtain the multiplier that maps Eq. (3.32) into the strictly
self-adjoint equivalent form. For example, taking (3.35) with $C_2 = 1$ and
$C_1 = C_3 = C_4 = 0$ we obtain the multiplier
\[
\mu = \frac{t}{u}.
\]
The corresponding equivalent equation to Eq. (3.32) has the formal Lagrangian
\[
L = \frac{tv}{u} \left[ u_{tt} - k(u)u_{xx} - k'(u)u_x^2 \right].
\]

We have
\[
\frac{\delta L}{\delta u} = D_t^2 \left( \frac{tv}{u} \right) - \frac{tv}{u^2} u_{tt} - D_x^2 \left( \frac{tv}{u} k(u) \right) - \frac{tv}{u} k'(u)u_{xx} + \frac{tv}{u^2} k(u)u_{xx}
\]
\[
+ 2D_x \left( \frac{tv}{u} k'(u)u_x \right) - \frac{tv}{u} k''(u)u_x^2 + \frac{tv}{u^2} k'(u)u_x^2.
\]

Letting here $v = u$ we see that the strict self-adjointness condition is satisfied in the following form:
\[
\frac{\delta L}{\delta u} \bigg|_{v=u} = -\frac{t}{u} \left[ u_{tt} - k(u)u_{xx} - k'(u)u_x^2 \right].
\]

4 Generalized Kompaneets equation

4.1 Introduction

The equation
\[
\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n}{\partial x} + n + n^2 \right) \right],
\]
known as the Kompaneets equation or the photon diffusion equation, was derived independently by A.S. Kompaneets\footnote{He mentions in his paper that the work has been done in 1950 and published in Report N. 336 of the Institute of Chemical Physics of the USSR Acad. Sci.} \[56\] and R. Weymann \[82\]. They take as a starting point the kinetic equations for the distribution function of a photon gas\footnote{Weymann uses Dreicer’s kinetic equation \[12\] for a photon gas interacting with a plasma which is slightly different from the equation used by Kompaneets.} and arrive, at certain idealized conditions, at Equation (4.1). This equation provides a mathematical model for describing the time development of the energy spectrum of a low energy homogeneous photon gas interacting with a rarefied electron gas via the Compton scattering. Here $n$
is the density of the photon gas (photon number density), \( t \) is time and \( x \) is connected with the photon frequency \( \nu \) by the formula
\[
x = \frac{h \nu}{k T_e},
\] (4.2)
where \( h \) is Planck’s constant and \( k T_e \) is the electron temperature with the standard notation \( k \) for Boltzmann’s constant. According to this notation, \( h \nu \) has the meaning of the photon energy. The nonrelativistic approximation is used, i.e. it is assumed that the electron temperatures satisfy the condition \( k T_e \ll mc^2 \), where \( m \) is the electron mass and \( c \) is the light velocity.

The term low energy photon gas means that \( h \nu \ll mc^2 \).

The question arises if the idealized conditions assumed in deriving Eq. (4.1) may be satisfied in the real world. For discussions of theoretical and observational evidences for such possibility in astrophysical environments, for example in intergalactic gas, see e.g. [83], [76] and the references therein. See also the recent publication [9].

\section*{4.2 Discussion of self-adjointness of the Kompaneets equation}

For unifying the notation, the dependent variable \( n \) in Eq. (4.1) will be denoted by \( u \) and Eq. (4.1) will be written further in the form
\[
\frac{u_t}{x^2} = D_x \left[ x^4 (u_x + u + u^2) \right].
\] (4.3)

Writing it in the expanded form
\[
u_t = x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2),
\] (4.4)
we have the following formal Lagrangian for Eq. (4.3):
\[
\mathcal{L} = v [-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2 u)u_x + 4x(u + u^2)].
\]

Working out the variational derivative of this formal Lagrangian,
\[
\frac{\delta \mathcal{L}}{\delta u} = D_t(v) + D_x^2 (x^2 v) - D_x [(x^2 + 4x + 2x^2 u)v] + 2x^2 vu_x + 4x(1 + 2u)v,
\]
we obtain the adjoint equation to Eq. (4.3):
\[
\frac{\delta \mathcal{L}}{\delta u} = v_t + x^2 v_{xx} - x^2 (1 + 2u)v_x + 2(x + 2xu - 1)v = 0.
\] (4.5)
If \( v = \varphi(u) \), then
\[
\begin{align*}
    v_t &= \varphi'(u)u_t, \\
    v_x &= \varphi'(u)u_x, \\
    v_{xx} &= \varphi'(u)u_{xx} + \varphi''(u)u_x^2.
\end{align*}
\]

It follows that the quasi self-adjointness condition (1.36),
\[
\left. \frac{\delta L}{\delta u} \right|_{v=\varphi(u)} = \lambda[-u_t + x^2u_{xx} + (x^2 + 4x + 2x^2u)u_x + 4x(u + u^2)],
\]
is not satisfied.

Let us check if this condition is satisfied in the more general form (3.5):
\[
\left. \frac{\delta L}{\delta u} \right|_{v=\varphi(t,x,u)} = \lambda[-u_t + x^2u_{xx} + (x^2 + 4x + 2x^2u)u_x + 4x(u + u^2)]. \quad (4.6)
\]

In this case
\[
\begin{align*}
    v_t &= D_t[\varphi(t, x, u)] = \varphi_u u_t + \varphi_t, \\
    v_x &= D_x[\varphi(t, x, u)] = \varphi_u u_x + \varphi_x, \\
    v_{xx} &= D_x(v_x) = \varphi_u u_{xx} + \varphi_u u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx}.
\end{align*}
\]

Inserting (4.7) in the expression for the variational derivative given by (4.5) and singling out in Eq. (4.6) the terms containing \( u_t \) and \( u_{xx} \), we obtain
\[
\varphi_u [u_t + x^2u_{xx}] = \lambda[-u_t + x^2u_{xx}].
\]

Since this equation should be satisfied identically in \( u_t \) and \( u_{xx} \), it yields \( \lambda = \varphi_u = 0 \). Hence \( \varphi = \varphi(t, x) \) and Eq. (4.6) becomes:
\[
\varphi_t + x^2\varphi_{xx} - x^2(1 + 2u)\varphi_x + 2(x + 2xu - 1)\varphi = 0. \quad (4.8)
\]

This equation should be satisfied identically in \( t, x \) and \( u \). Therefore we nullify the coefficient for \( u \) and obtain
\[
x\varphi_x - 2\varphi = 0,
\]
whence
\[
\varphi(t, x) = c(t)x^2.
\]

Substitution in Eq. (4.8) yields \( c'(t) = 0 \). Hence, \( v = \varphi(t, x) = Cx^2 \) with arbitrary constant \( C \). Since \( \lambda = 0 \) in (4.6) and the adjoint equation (4.5) is linear and homogeneous in \( v \), one can let \( C = 1 \). Thus, we have demonstrated the following statement.
Proposition 4.1. The adjoint equation (4.5) has the solution
\[ v = x^2 \] (4.9)
for any solution \( u \) of Equation (4.3). In another words, the Kompaneets equation (4.3) is nonlinearly self-adjoint with the substitution (3.3) given by (4.9).

Remark 4.1. The substitution (4.9) does not depend on \( u \). The question arises on existence of a substitution \( v = \varphi(t, x, u) \) involving \( u \) if we rewrite Eq. (4.3) in an equivalent form
\[ \alpha(t, x, u)[-u_t + x^2 u_{xx} + (x^2 + 4x + 2x^2u)u_x + 4x(u + u^2)] = 0 \] (4.4')
with an appropriate multiplier \( \alpha \neq 0 \). This question is investigated in next section for a more general model.

4.3 The generalized model

In the original derivation of Eq. (4.1) the following more general equation appears accidentally (see [56], Eqs. (9), (10) and their discussion):
\[ \frac{\partial n}{\partial t} = \frac{1}{g(x)} \frac{\partial}{\partial x} \left[ g^2(x) \left( \frac{\partial n}{\partial x} + f(n) \right) \right] \] (4.10)
with undetermined functions \( f(u) \) and \( g(x) \). Then, using a physical reasoning, Kompaneets takes \( f(u) = n(1 + n) \) and \( g(x) = x^2 \). This choice restricts the symmetry properties of the model significantly. Namely, Equation (4.1) has only the time-translational symmetry with the generator
\[ X = \frac{\partial}{\partial t} . \] (4.11)
The symmetry (4.11) provides only one invariant solution, namely the stationary solution \( n = n(x) \) defined by the Riccati equation
\[ \frac{dn}{dx} + n^2 + n = \frac{C}{x^4} . \]
The generalized model (4.10) can be used for extensions of symmetry properties via the methods of preliminary group classification [1, 52]. In this way, exact solutions known for particular approximations to the Kompaneets equation can be obtained. This may also lead to new approximations of the solutions by taking into account various inevitable perturbations of the idealized situation assumed in the Kompaneets model (4.1).
So, we will take with minor changes in notation the generalized model (4.10):

\[ u_t = \frac{1}{h(x)} D_x \left\{ h^2(x)[u_x + f(u)] \right\}, \quad h'(x) \neq 0. \quad (4.12) \]

It is written in the expanded form as follows:

\[ u_t = h(x) (u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u)). \quad (4.13) \]

We will write Eq. (4.13) in the equivalent form similar to (4.4'):

\[ \alpha(t, x, u)[ - u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u))] = 0, \quad (4.14) \]

where \( \alpha \neq 0 \). This provides the following formal Lagrangian:

\[ L = v \alpha(t, x, u)[ - u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u))], \quad (4.15) \]

where \( v \) is a new dependent variable. For this Lagrangian, we have

\[ \frac{\delta L}{\delta u} = D_t(v\alpha) + hD_x^2(v\alpha) - D_x[h(x)f'(u)v\alpha + 2h'(x)v\alpha] + h(x)f''(u)v\alpha \]

\[ + v\alpha [ - u_t + h(x)(u_{xx} + f'(u)u_x) + 2h'(x)(u_x + f(u))]. \quad (4.16) \]

The reckoning shows that

\[ \frac{\delta L}{\delta u} = D_t(v\alpha) + hD_x^2(v\alpha) - h f' D_x(v\alpha) + (h' f' - h'') v\alpha \]

\[ + v\alpha [ - u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h']. \]

Now we write the condition for the self-adjointness of Eq. (4.13) in the form

\[ \frac{\delta L}{\delta u} \bigg|_{v=\varphi(t, x, u)} = \lambda \left[ - u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right] \quad (4.17) \]

with an undetermined coefficient \( \lambda \). Substituting (4.16) in (4.17) we have:

\[ D_t(\varphi \alpha) + hD_x^2(\varphi \alpha) - h f' D_x(\varphi \alpha) + (h' f' - h'') \varphi \alpha \]

\[ + \varphi \alpha [ - u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h'] \quad (4.18) \]

\[ = \lambda \left[ - u_t + (u_{xx} + f'u_x)h + 2(u_x + f)h' \right]. \]
Here $\varphi = \varphi(t, x, u)$, $\alpha = \alpha(t, x, u)$ and consequently (see (4.7))

\[
\begin{align*}
D_t(\varphi\alpha) &= (\varphi\alpha)_u u_t + (\varphi\alpha)_t, \\
D_x(\varphi\alpha) &= (\varphi\alpha)_u u_x + (\varphi\alpha)_x, \\
D^2_x(\varphi\alpha) &= (\varphi\alpha)_u u_{xx} + (\varphi\alpha)_{uu} u_x^2 + 2(\varphi\alpha)_{ux} u_x + (\varphi\alpha)_{xx}.
\end{align*}
\]

We substitute (4.16) in Eq. (4.18), equate the coefficients for $u_t$ in both sides of the resulting equation and obtain $(\varphi\alpha)_u - \varphi \alpha_u = -\lambda$. Hence,

\[\lambda = -\alpha \varphi_u.\]

Using this expression for $\lambda$ and equating the coefficients for $hu_{xx}$ in both sides of Eq. (4.18) we get $(\varphi\alpha)_u + \varphi \alpha_u = -\alpha \varphi_u$. It follows that $(\varphi\alpha)_u = 0$ and hence

\[\alpha \varphi = k(t, x).\]

Now Eq. (4.18) becomes:

\[k_t + h(x)k_{xx} - h''(x)k + f'(u)[h'(x)k - h(x)k_x] = 0.\]

If $f''(u) \neq 0$, the above equation splits into two equations:

\[h'(x)k - h(x)k_x = 0, \quad k_t + h(x)k_{xx} - h''(x)k.\]

The first of these equations yields $k(t, x) = c(t)h(x)$, and then the second equation shows that $c'(t) = 0$. Hence, $k = C h(x)$ with $C = \text{const}$. Letting $C = 1$, we have:

\[\alpha \varphi = h(x).\] (4.20)

Eq. (4.20) can be satisfied by taking, e.g.

\[\alpha = \frac{h(x)}{u}, \quad \varphi = u.\] (4.21)

Thus, we have proved the following statement.

**Proposition 4.2.** Eq. (4.12) written in the equivalent form

\[\frac{h(x)}{u} u_t = \frac{1}{u} D_x\{h^2(x)[u_x + f(u)]\}\] (4.22)

is strictly self-adjoint. In another words, the adjoint equation to Eq. (4.22) coincides with (4.22) upon the substitution

\[v = u.\] (4.23)
In particular, let us verify by direct calculations that the original equation (4.3) becomes strictly self-adjoint if we rewrite it in the equivalent form
\[ \frac{x^2}{u} u_t = \frac{1}{u} D_x \left[ x^4 (u_x + u + u^2) \right]. \] (4.24)

Eq. (4.24) reads
\[ -\frac{x^2}{u} u_t + \frac{x^4}{u} u_{xx} + \left[ (x^4 + 4x^3) \frac{1}{u} + 2x^4 \right] u_x + 4x^3 (1 + u) = 0 \] (4.25)
and has the formal Lagrangian
\[ \mathcal{L} = -\frac{x^2 v}{u} u_t + \frac{x^4 v}{u} u_{xx} + \left[ (x^4 + 4x^3) \frac{v}{u} + 2x^4 v \right] u_x + 4x^3 (v + uv). \]

Accordingly, the adjoint equation to Eq. (4.25) is written
\[ D_t \left( \frac{x^2 v}{u} \right) + D_x^2 \left( \frac{x^4 v}{u} \right) - D_x \left[ (x^4 + 4x^3) \frac{v}{u} + 2x^4 v \right] \]
\[ + \frac{x^2 v}{u^2} u_t - \frac{x^4 v}{u^2} u_{xx} - \left( x^4 + 4x^3 \right) \frac{v}{u^2} u_x + 4x^3 v = 0. \]

Letting here \( v = u \) one has \( v/u = 1 \) and after simple calculations arrives at Eq. (4.25).

5 Quasi self-adjoint reaction-diffusion models

Let us consider the one-dimensional reaction-diffusion model described by the following system (see, e.g. [80]):
\[ \frac{\partial u}{\partial t} = f(u, v) + A \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left( \phi(u, v) \frac{\partial v}{\partial x} \right), \]
\[ \frac{\partial v}{\partial t} = g(u, v) + B \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x} \left( \psi(u, v) \frac{\partial u}{\partial x} \right). \] (5.1)

It is convenient to write Eqs. (5.1) in the form
\[ D_t(u) = AD_x^2(u) + D_x [\phi(u, v) D_x(v)] + f(u, v), \]
\[ D_t(v) = BD_x^2(v) + D_x [\psi(u, v) D_x(u)] + g(u, v). \] (5.2)
The total differentiations have the form
\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \cdots, \]
\[ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_t} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x} + \cdots \]
and Eqs. (5.2) are written
\[ u_t = Au_{xx} + \phi v_{xx} + [\phi u_x + \phi v_x] v_x + f, \tag{5.3} \]
\[ v_t = Bv_{xx} + \psi u_{xx} + [\psi u_x + \psi v_x] u_x + g. \]

The formal Lagrangian for the system (5.3) is
\[ \mathcal{L} = z(Au_{xx} - u_t + \phi v_{xx} + \phi u_x v_x + \phi v_x^2 + f) + w(Bv_{xx} - v_t + \psi u_{xx} + \psi u_x^2 + \psi u_x v_x + g), \tag{5.4} \]
where \( z \) and \( w \) are new dependent variables. Eqs. (1.8) are written:
\[ F_1^* = \frac{\delta \mathcal{L}}{\delta u} = D_x^2 \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) - D_t \left( \frac{\partial \mathcal{L}}{\partial u_t} \right) - D_x \left( \frac{\partial \mathcal{L}}{\partial u_x} \right) + \frac{\partial \mathcal{L}}{\partial u}, \]
\[ F_2^* = \frac{\delta \mathcal{L}}{\delta v} = D_x^2 \left( \frac{\partial \mathcal{L}}{\partial v_{xx}} \right) - D_t \left( \frac{\partial \mathcal{L}}{\partial v_t} \right) - D_x \left( \frac{\partial \mathcal{L}}{\partial v_x} \right) + \frac{\partial \mathcal{L}}{\partial v}. \]
Substituting here the expression (5.4) for \( \mathcal{L} \) we obtain after simple calculations the following adjoint equations (3.2) to the system (5.3):
\[ A z_{xx} + z_t + \psi v_x w_x - \phi u_x z_x + \psi w_{xx} + zf_u + wg_u = 0, \tag{5.5} \]
\[ B w_{xx} + w_t + \phi u_x z_x - \psi u_x w_x + \phi z_{xx} + zf_v + wg_v = 0. \tag{5.6} \]

Let us investigate the system (5.3) for quasi self-adjointness (Definition 1.3). We write the left-hand sides of Eqs. (5.5) and (5.6) as linear combinations of the left-hand sides of Eqs. (5.3):
\[ A z_{xx} + z_t + \psi v_x w_x - \phi u_x z_x + \psi w_{xx} + zf_u + wg_u \tag{5.7} \]
\[ = (Au_{xx} - u_t + \phi v_{xx} + \phi u_x v_x + \phi v_x^2 + f)P \]
\[ + (Bv_{xx} - v_t + \psi u_{xx} + \psi u_x^2 + \psi u_x v_x + g)Q, \]
\[ B w_{xx} + w_t + \phi u_x z_x - \psi u_x w_x + \phi z_{xx} + zf_v + wg_v \tag{5.8} \]
\[ = (Au_{xx} - u_t + \phi v_{xx} + \phi u_x v_x + \phi v_x^2 + f)M \]
\[ + (Bv_{xx} - v_t + \psi u_{xx} + \psi u_x^2 + \psi u_x v_x + g)N, \]
where $P, Q, M$ and $N$ are undetermined coefficients. We write the substitution (1.42) in the form

$$z = Z(u, v), \quad w = W(u, v)$$  \hspace{1cm} (5.9)

and insert in the left-hand sides of Eqs. (5.7)-(5.8) these expressions for $z, w$ together with their derivatives

$$z_t = Z_u u_t + Z_v v_t, \quad z_x = Z_u u_x + Z_v v_x,$$
$$z_{xx} = Z_u u_{xx} + Z_v v_{xx} + Z_{uu} u_x^2 + 2Z_{uv} u_x v_x + Z_{vv} v_x^2,$$
$$w_t = W_u u_t + W_v v_t, \quad w_x = W_u u_x + W_v v_x,$$
$$w_{xx} = W_u u_{xx} + W_v v_{xx} + W_{uu} u_x^2 + 2W_{uv} u_x v_x + W_{vv} v_x^2.$$

Equating the coefficients for $u_t$ and $v_t$ in both sides of Eqs. (5.7)-(5.8) we obtain

$$P = -Z_u, \quad Q = -Z_v,$$
$$N = -W_v, \quad M = -W_u. \hspace{1cm} (5.10)$$

Now we calculate the coefficients for $u_{xx}$ and $v_{xx}$, take into account Eqs. (5.10) and arrive at the following equations:

$$2AZ_u + \psi Z_v + \psi W_u = 0,$$
$$(A + B)Z_v + \phi Z_u + \psi W_v = 0,$$
$$2BW_v + \phi Z_v + \psi W_u = 0,$$
$$(A + B)W_u + \phi Z_u + \psi W_v = 0. \hspace{1cm} (5.11)$$

Eqs. (5.11) provide a linear homogeneous algebraic equations for the quantities

$$Z_u, \quad Z_v, \quad W_u, \quad W_v$$

with the matrix

$$\begin{pmatrix}
2A & \psi & \psi & 0 \\
\phi & A + B & 0 & \psi \\
0 & \phi & \phi & 2B \\
\phi & 0 & A + B & \psi
\end{pmatrix}.$$ 

This matrix has the inverse because its determinant is equal to

$$4(A + B)^2(\phi \psi - AB).$$
and does not vanish in the case of arbitrary $A, B, \phi$ and $\psi$. Hence, Eqs. (5.11) yield:

$$Z_u = Z_v = W_u = W_v = 0.$$  (5.12)

It follows that $Z(u, v) = C_1$, $W(u, v) = C_2$. Thus, the substitution (1.42) has the form

$$z = C_1, \quad w = C_2$$  (5.13)

with arbitrary constants $C_1, C_2$. Then Eqs. (5.7)-(5.8) become

$$(C_1f + C_2g)_u = 0, \quad (C_1f + C_2g)_v = 0$$

and yield

$$\tilde{f} + \tilde{g} = C,$$

where $\tilde{f} = C_1f$, $\tilde{g} = C_2g$, and $C = \text{const}$. Since $\tilde{f}$ and $\tilde{g}$, along with $f$ and $g$, are arbitrary functions, we can omit the “tilde” and write

$$f + g = C.$$  (5.14)

Eq. (5.14) provides the necessary and sufficient condition for the quasi self-adjointness of the system (5.1). Thus, we have proved the following statement.

**Theorem 5.1.** The system (5.1) is quasi self-adjoint if and only if it has the form

$$\frac{\partial u}{\partial t} = f(u, v) + A \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left( \phi(u, v) \frac{\partial v}{\partial x} \right),$$

$$\frac{\partial v}{\partial t} = C - f(u, v) + B \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x} \left( \psi(u, v) \frac{\partial u}{\partial x} \right),$$  (5.15)

where $\phi(u, v)$, $\psi(u, v)$, $f(u, v)$ are arbitrary functions and $A, B, C$ are arbitrary constants. The substitution (1.42) is given by (5.13).

**Remark 5.1.** If we replace (5.9) by the general substitution (3.3), i.e. take

$$z = Z(t, x, u, v), \quad w = W(t, x, u, v),$$  (5.16)

then Eqs. (5.13) will be replaced by

$$z = Z(t, x), \quad w = W(t, x),$$  (5.17)

with functions $Z(t, x)$, $W(t, x)$ satisfying the following equations:

$$(\psi W - \phi u Z)_x = 0,$$  (5.18)

$$AZ_{xx} + Z_t + \psi W_{xx} + (fZ + gW)_u = 0,$$

$$BW_{xx} + W_t + \phi Z_{xx} + (fZ + gW)_v = 0.$$  (5.19)
6 A model of an irrigation system

Let us consider the second-order nonlinear partial differential equation

\[ C(\psi) \psi_t = [K(\psi) \psi_x]_x + [K(\psi) (\psi_z - 1)]_z - S(\psi). \]  

(6.1)

It serves as a mathematical model for investigating certain irrigation systems (see [24], Section 9.8 and the references therein). The dependent variable \( \psi \) denotes the soil moisture pressure head, \( C(\psi) \) is the specific water capacity, \( K(\psi) \) is the unsaturated hydraulic conductivity, \( S(\psi) \) is a source term. The independent variables are the time \( t \), the horizontal axis \( x \) and the vertical axis \( z \) which is taken to be positive downward.

The adjoint equation (3.2) to Eq. (6.1) has the form

\[ C(\psi) v_t + K(\psi) [v_{xx} + v_{zz}] + K'(\psi) v_z - S'(\psi) v = 0. \]  

(6.2)

It follows from (6.2) that Eq. (6.1) is not nonlinearly self-adjoint if \( C(\psi), K(\psi) \) and \( S(\psi) \) are arbitrary functions. Indeed, using Definition 3.2 of the nonlinear self-adjointness and nullifying in (6.2) the term with \( S'(\psi) \) we obtain \( v = 0 \). Hence, the condition (3.4) of the nonlinear self-adjointness is not satisfied.

However, Eq. (6.1) can be nonlinearly self-adjoint if there are certain relations between the functions \( C(\psi), K(\psi) \) and \( S(\psi) \). For example, let us suppose that the specific water capacity \( C(\psi) \) and the hydraulic conductivity \( K(\psi) \) are arbitrary, but the source term \( S(\psi) \) is related with \( C(\psi) \) by the equation

\[ S'(\psi) = aC(\psi), \quad a = \text{const}. \]  

(6.3)

Then Eq. (6.2) becomes

\[ C(\psi)[v_t - av] + K(\psi) [v_{xx} + v_{zz}] + K'(\psi) v_z = 0 \]

and yields:

\[ v_z = 0, \quad v_{xx} = 0, \quad v_t - av = 0. \]  

(6.4)

We solve the first two equations (6.4) and obtain

\[ v = p(t)x + q(t). \]

We substitute this in the third equation (6.4),

\[ [p'(t) - ap(t)]x + q'(t) - aq(t) = 0, \]

split it with respect to \( x \) and obtain:

\[ p'(t) - ap(t) = 0, \quad q'(t) - aq(t) = 0, \]
whence
\[ p(t) = be^{at}, \quad q(t) = le^{at}, \quad b, l = \text{const}. \]
Thus, Eq. (6.1) satisfying the condition Eq. (6.3) is nonlinearly self-adjoint, and the substitution (3.3) has the form
\[ v = (bx + l)e^{at}. \quad (6.5) \]

One can obtain various nonlinearly self-adjoint Equations (6.1) by considering other relations between \( C(\psi), K(\psi) \) and \( S(\psi) \) different from (6.3).
PART 2

Construction of conservation laws using symmetries

7 Discussion of the operator identity

7.1 Proof of Noether’s theorem based on the operator identity

Let us discuss some consequences of the operator identity*

\[ X + D_i (\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i \, . \]  

(7.1)

Here

\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^{\alpha i} \frac{\partial}{\partial u^{\alpha i}} + \cdots , \]  

(7.2)

\[ W^\alpha = \eta^\alpha - \xi^i u^\alpha_i, \quad \alpha = 1, \ldots, m, \]  

(7.3)

\[ \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m, \]  

(7.4)

and

\[ N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha} + \sum_{s=1}^{\infty} D_{i_1} \cdots D_{i_s} (W^\alpha) \frac{\delta}{\delta u^{i_1 \cdots i_s}}, \quad i = 1, \ldots, n, \]  

(7.5)

where the Euler-Lagrange operators with respect to derivatives of \( u^\alpha \) are obtained from (7.4) by replacing \( u^\alpha \) by the corresponding derivatives, e.g.

\[ \frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s=1}^{\infty} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^{j_1 \cdots j_s}}. \]  

(7.6)

The coefficients \( \xi^i, \eta^\alpha \) in (7.2) are arbitrary differential functions (see Section 1.1) and the other coefficients are determined by the prolongation formulae

\[ \zeta^\alpha_i = D_i (W^\alpha) + \xi^i u^\alpha_{ij}, \quad \zeta^{\alpha i} = D_i D_{i_2} (W^\alpha) + \xi^i u^{\alpha i_2}, \ldots \]  

(7.7)

*A detailed proof of the identity (7.1) can be found in [68] where Eq. (19) is the same as Eq. (7.1) except for notation. The operator identity (7.1) was rediscovered in [20] and used for simplifying the proof of Noether’s theorem. Accordingly, Eq. (7.1) was called in [20] the Noether identity. See also [26], Section 8.4.
The derivation of Eq. (7.1) is essentially based on Eqs. (7.7).

Recall that Noether’s theorem, associating conservation laws with symmetries of differential equations obtained from variational principles, was originally proved by calculus of variations. The alternative proof of this theorem given in [20] (see also [21, 26]) is based on the identity (7.1) and is simple. Namely, let us consider the Euler-Lagrange equations

\[
\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{7.8}
\]

If we assume that the operator (7.2) is admitted by Eqs. (7.8) and that the variational integral

\[
\int \mathcal{L}(x, u, u^{(1)}, \ldots) dx
\]

is invariant under the transformations of the group with the generator \(X\) then the following equation holds:

\[
X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = 0. \tag{7.9}
\]

Therefore, if we act on \(\mathcal{L}\) by both sides of the identity (7.1),

\[
X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = W^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i[N^i(\mathcal{L})],
\]

and take into account Eqs. (7.8), (7.9), we see that the vector with the components

\[
C^i = N^i(\mathcal{L}), \quad i = 1, \ldots, n, \tag{7.10}
\]

satisfies the conservation equation

\[
D_i(C^i)\big|_{(7.8)} = 0. \tag{7.11}
\]

For practical applications, when we deal with law order Lagrangians \(\mathcal{L}\), it is convenient to restrict the operator (7.5) on the derivatives involved in \(\mathcal{L}\) and write the expressions (7.10) in the expanded form

\[
C^i = \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \ldots \right]
\]

\[
+ D_j(W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \ldots \right] + D_j D_k (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} - \ldots \right]. \tag{7.12}
\]

Thus, Noether’s theorem can be formulated as follows.
**Theorem 7.1.** If the operator (7.2) is admitted by Eqs. (7.8) and satisfies the condition (7.9) of the invariance of the variational integral, then the vector (7.12) constructed by Eqs. (7.12) satisfies the conservation law (7.11).

**Remark 7.1.** The identity (7.1) is valid also in the case when the coefficients $\xi^i$, $\eta^\alpha$ of the operator $X$ involve not only the local variables $x, u, u^{(1)}, u^{(2)}, \ldots$ but also nonlocal variables (see Section 11.5). Accordingly, the formula (7.12) associates conserved vectors with nonlocal symmetries as well.

**Remark 7.2.** If the invariance condition (7.9) is replaced by the divergence condition

$$X(\mathcal{L}) + D_i(\xi^i)\mathcal{L} = D_i(B^i),$$

then the identity (7.1) leads to the conservation law (7.11) where the conserved vector (7.10) is replaced with

$$C^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \ldots, n.$$ (7.13)

**Remark 7.3.** If we write the operator (7.2) in the equivalent form

$$X = W^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha_{i} \frac{\partial}{\partial u_{i}^\alpha} + \zeta^\alpha_{i_1i_2} \frac{\partial}{\partial u_{i_1i_2}^\alpha} + \cdots,$$ (7.14)

then the prolongation formulae (7.7) become simpler:

$$\zeta^\alpha_i = D_i(W^\alpha), \quad \zeta^\alpha_{i_1i_2} = D_{i_1}D_{i_2}(W^\alpha), \ldots.$$ (7.15)

**7.2 Test for total derivative and for for divergence**

I recall here the well-known necessary and sufficient condition for a differential function to be divergence, or total derivative in the case of one independent variable.

One can easily derive from the definition (1.1) of the total differentiation $D_i$ the following lemmas (see also [26], Section 8.4.1).

**Lemma 7.1.** The following infinite series of equations hold:

$$\frac{\partial}{\partial u^\alpha} D_i = D_i \frac{\partial}{\partial u^\alpha},$$

$$D_j \frac{\partial}{\partial u^\alpha_j} D_i = D_i \frac{\partial}{\partial u^\alpha_j} + D_i D_j \frac{\partial}{\partial u^\alpha_j},$$

$$D_j D_k \frac{\partial}{\partial u^\alpha_{jk}} D_i = D_i D_k \frac{\partial}{\partial u^\alpha_{jk}} + D_i D_j D_k \frac{\partial}{\partial u^\alpha_{jk}},$$

$$\cdots$$
Lemma 7.2. The following operator identity holds for every $i$ and $\alpha$:

$$\frac{\delta}{\delta u^\alpha} D_i = 0.$$ 

Proof. Using Lemma 7.1 and manipulating with summation indices we obtain:

$$\frac{\delta}{\delta u^\alpha} D_i = \left( \frac{\partial}{\partial u^\alpha} - D_j \frac{\partial}{\partial u_j^\alpha} + D_j D_k \frac{\partial}{\partial u_j^\alpha_k} - D_j D_k D_l \frac{\partial}{\partial u_j^\alpha_k_l} + \cdots \right) D_i$$

$$= \frac{\partial}{\partial u^\alpha} D_i - D_i \frac{\partial}{\partial u^\alpha} - D_i D_j \frac{\partial}{\partial u_j^\alpha} + D_i D_k \frac{\partial}{\partial u_k^\alpha} + D_i D_j D_k \frac{\partial}{\partial u_j^\alpha_k}$$

$$- D_i D_k D_l \frac{\partial}{\partial u_k^\alpha_l} - \cdots = 0.$$

Proposition 7.1. A differential function $f(x, u, u^{(1)}, \ldots, u^{(s)}) \in A$ is divergence,

$$f = D_i (h^i), \quad h^i(x, u, \ldots, u^{(s-1)}) \in A, \quad (7.16)$$

if and only if the following equations hold identically in $x, u, u^{(1)}, \ldots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \quad (7.17)$$

The statement that (7.16) implies (7.17) follows immediately from Lemma 7.2. For the proof of the inverse statement that (7.17) implies (7.16), see [10], Chapter 4, § 3.5, and [68]. See also [26], Section 8.4.1.

We will use Proposition 7.1 also in the particular case of one independent variable $x$ and one dependent variable $u = y$. Then it is formulated as follows.

Proposition 7.2. A differential function $f(x, y, y', \ldots, y^{(s)}) \in A$ is the total derivative,

$$f = D_x (g), \quad g(x, y, y', \ldots, y^{(s-1)}) \in A, \quad (7.18)$$

if and only if the following equation holds identically in $x, y, y', \ldots$:

$$\frac{\delta f}{\delta y} = 0. \quad (7.19)$$

Here $\delta f/\delta y$ is the Euler-Lagrange operator (7.6):

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''} - D_x^3 \frac{\partial}{\partial y'''}, \quad (7.20)$$
7.3 Adjoint equation to linear ODE

Let us consider an arbitrary $s$th-order linear ordinary differential operator

$$L[y] = a_0 y^{(s)} + a_1 y^{(s-1)} + \cdots + a_{s-2} y'' + a_{s-1} y' + a_s y,$$  \hfill (7.21)

where $a_i = a_i(x)$. We know from Section 1.5 that the adjoint operator to (7.21) can be calculated by using Eq. (1.8). I give here the independent proof based on the operator identity (7.1).

**Proposition 7.3.** The adjoint operator to (7.21) can be calculated by the formula

$$L^* [z] = \frac{\delta (z L[y])}{\delta y}. \hfill (7.22)$$

**Proof.** Let

$$X = w \frac{\partial}{\partial y} + w' \frac{\partial}{\partial y'} + w'' \frac{\partial}{\partial y''} + \cdots \hfill (7.23)$$

be the operator (7.14) with one independent variable $x$ and one dependent variable $u = y$, where the prolongation formulae (7.15) are written using the notation

$$w' = D_x(w), \quad w'' = D_x^2(w), \ldots. \hfill (7.24)$$

In this notation the operator (7.5) is written

$$N = w \frac{\delta}{\delta y'} + w' \frac{\delta}{\delta y''} + \cdots + w^{(s)} \frac{\delta}{\delta y^{(s)}}. \hfill (7.25)$$

Having in mind its application to the differential function $L[y]$ given by (7.21) we consider the following restricted form of $N$ :

$$N = w \frac{\delta}{\delta y'} + w' \frac{\delta}{\delta y''} + \cdots + w^{(s-1)} \frac{\delta}{\delta y^{(s-1)}}. \hfill (7.25)$$

The identity (7.1) has the form

$$X = w \frac{\delta}{\delta y} + D_x N. \hfill (7.26)$$

We act by both sides of this identity on $z L[y]$, where $z$ is a new dependent variable:

$$X(z L[y]) = w \frac{\delta (z L[y])}{\delta y} + D_x N(z L[y]). \hfill (7.27)$$

Since the operator (7.23) does not act on the variables $x$ and $z$, we have

$$X(z L[y]) = z X(L[y]). \hfill (7.28)$$
Furthermore we note that

\[ X(L[y]) = L[w]. \]  

(7.29)

Inserting (7.28) and (7.29) in Eq. (7.27) we obtain

\[ zL[w] - w \frac{\delta(zL[y])}{\delta y} = D_x(\Psi), \]  

(7.30)

where \( \Psi \) is a quadratic form \( \Psi = \Psi[w, z] \) defined by

\[ \Psi = N(zL[y]). \]  

(7.31)

After replacing \( w \) with \( y \) Eq. (7.30) coincides with Eq. (1.3) for the adjoint operator,

\[ zL[y] - yL^*[z] = D_x(\psi), \]  

(7.32)

where \( L^*[z] \) is given by the formula (7.22) and \( \psi = \psi[y, z] \) is defined by

\[ \psi[y, z] = \Psi[w, z] \big|_{w=y} \equiv N(zL[y]) \big|_{w=y}. \]  

(7.33)

Remark 7.4. Let us find the explicit formula for \( \psi \) in Eq. (7.32) We write the operator \( N \) given by Eq. (7.25) in the expanded form

\[
N = w \left[ \frac{\partial}{\partial y'} - D_x \frac{\partial}{\partial y''} + \cdots + (-D_x)^{s-1} \frac{\partial}{\partial y^{(s)}} \right] \\
+ w' \left[ \frac{\partial}{\partial y''} - D_x \frac{\partial}{\partial y^{(s)}} + \cdots + (-D_x)^{s-2} \frac{\partial}{\partial y^{(s)}} \right] + \cdots \\
+ w^{(s-2)} \left[ \frac{\partial}{\partial y^{(s-1)}} - D_x \frac{\partial}{\partial y^{(s)}} \right] + w^{(s-1)} \frac{\delta}{\delta y^{(s)}},
\]

act on \( zL[y] \) written in the form

\[ zL[y] = a_s y z + a_{s-1} y' z + a_{s-2} y'' z + \cdots + a_1 y^{(s-1)} z + a_0 y^{(s)} z, \]

and obtain \( \Psi \). We replace \( w \) with \( y \) in \( \Psi = \Psi[w, z] \) and \( \psi = \psi[y, z] \):

\[
\psi[y, z] = y \left[ a_{s-1} z - (a_{s-2} z)' + \cdots + (-1)^{s-1} (a_0 z)^{(s-1)} \right] \\
+ y' \left[ a_{s-2} z - (a_{s-3} z)' + \cdots + (-1)^{s-2} (a_0 z)^{(s-2)} \right] + \cdots \\
+ y^{(s-2)} \left[ a_1 z - (a_0 z)' \right] + y^{(s-1)} a_0 z.
\]

(7.34)

The expression (7.34) is obtained in the classical literature using integration by parts (see, e.g. [75], Chapter 5, §4, Eq. (31')).
7.4 Conservation laws and integrating factors for linear ODEs

Consider an \( s \)th-order homogeneous linear ordinary differential equation

\[
L[y] = 0, \tag{7.35}
\]

where \( L[y] \) is the operator defined by Eq. (7.21). If \( L[y] \) is a total derivative,

\[
L[y] = D_x \left( \psi(x, y, y', \ldots, y^{(s-1)}) \right), \tag{7.36}
\]

Eq. (7.35) is written as a conservation law

\[
D_x \left( \psi(x, y, y', \ldots, y^{(s-1)}) \right) = 0,
\]

whence upon integration one obtains a linear equation of order \( s - 1 \):

\[
\psi(x, y, y', \ldots, y^{(s-1)}) = C_1. \tag{7.37}
\]

We can also reduce the order of the non-homogeneous equation

\[
L[y] = f(x) \tag{7.38}
\]

by rewriting it in the conservation form

\[
D_x \left[ \psi(x, y, y', \ldots, y^{(s-1)}) - \int f(x) dx \right] = 0. \tag{7.39}
\]

Integrating it once we obtain the non-homogeneous linear equation of order \( s - 1 \):

\[
\psi(x, y, y', \ldots, y^{(s-1)}) = C_1 + \int f(x) dx.
\]

**Example 7.1.** Consider the second-order equation

\[
y'' + y' \sin x + y \cos x = 0.
\]

We have

\[
y'' + y' \sin x + y \cos x = D_x(y' + y \sin x).
\]

Therefore the second-order equation in question reduces to the first-order equation

\[
y' + y \sin x = C_1.
\]

Integrating the latter equation we obtain the general solution

\[
y = \left[ C_2 + C_1 \int e^{-\cos x} dx \right] e^{\cos x}
\]
to our second-order equation. Dealing likewise with the non-homogeneous equation
\[ y'' + y' \sin x + y \cos x = 2x \]
we obtain its general solution
\[ y = \left[ C_2 + \int (C_1 + x^2) e^{-\cos x} \, dx \right] e^{\cos x}. \]

If \( L[y] \) in Eq. (7.35) is not a total derivative, one can find an appropriate factor \( \phi(x) \neq 0 \), called an *integrating factor*, such that \( \phi(x)L[y] \) becomes a total derivative:
\[ \phi(x)L[y] = D_x \left( \psi(x, y, y', \ldots, y^{(s-1)}) \right). \]  \hfill (7.40)

A connection between integrating factors and the adjoint equations for linear equations is well known in the classical literature (see, e.g. [75], Chapter 5, §4). Proposition 7.2 gives a simple way to establish this connection and prove the following statement.

**Proposition 7.4.** A function \( \phi(x) \) is an integrating factor for Eq. (7.35) if and only if
\[ z = \phi(x), \quad \phi(x) \neq 0, \] \hfill (7.41)
is a solution of the adjoint equation * to Eq. (7.35):
\[ L^*[z] = 0. \] \hfill (7.42)

Knowledge of a solution (7.41) to the adjoint equation (7.42) allows to reduce the order of Eq. (7.35) by integrating Eq. (7.40):
\[ \psi(x, y, y', \ldots, y^{(s-1)}) = C_1. \] \hfill (7.43)
Here \( C_1 \) is an arbitrary constants and \( \psi \) defined according to Eqs. (7.31)-(7.32), i.e.
\[ \psi = N(zL[y]) \big|_{w=y}. \] \hfill (7.44)

**Proof.** If (7.41) is a solution of the adjoint equation (7.42), we substitute it in Eq. (7.32) and arrive at Eq. (7.40). Hence \( \phi(x) \) is an integrating factor for Eq. (7.35). Conversely, if \( \phi(x) \) is an integrating factor for Eq. (7.35), then Eq. (7.40) is satisfied. Now Proposition 7.2 yields
\[ \frac{\delta (\phi(x)L[y])}{\delta y} = 0. \]
Hence (7.41) is a solution of the adjoint equation (7.42). Finally, Eq. (7.44) follows from (7.32).

*This statement is applicable to nonlinear ODEs as well, see [31].
Example 7.2. Let us apply the above approach to the first-order equation
\[ y' + P(x)y = Q(x). \]  
(7.45)

Here \( L[y] = y' + P(x)y \). The adjoint equation (7.42) is written
\[ z' - P(x)z = 0. \]
Solving it we obtain the integrating factor
\[ z = e^{\int P(x)dx}. \]

Therefore we rewrite Eq. (7.45) in the equivalent form
\[ [y' + P(x)y] e^{\int P(x)dx} = Q(x) e^{\int P(x)dx}, \]
(7.46)
and compute the function \( \Psi \) given by Eq. (7.31):
\[ \Psi = N(zL[y]) = w \frac{\partial}{\partial y} [z(y' + P(x)y)] = wz = w e^{\int P(x)dx}. \]

Eq. (7.44) yields
\[ \psi = y e^{\int P(x)dx}. \]  
(7.47)

Now we can take (7.46) instead of Eq. (7.38) and write it in the form (7.39) with \( \psi \) given by (7.47). Then we obtain
\[ D_x \left[ ye^{\int P(x)dx} - \int Q(x) e^{\int P(x)dx} dx \right] = 0, \]
whence
\[ ye^{\int P(x)dx} = C_1 + \int Q(x) e^{\int P(x)dx} dx. \]
Solving the latter equation for \( y \) we obtain the general solution of Eq. (7.45):
\[ y = \left[ C_1 + \int Q(x) e^{\int P(x)dx} dx \right] e^{-\int P(x)dx}. \]  
(7.48)

Example 7.3. Let us consider the second-order homogeneous equation
\[ y'' + \frac{\sin x}{x^2} y' + \left( \frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) y = 0. \]  
(7.49)

Its left-hand side does not satisfy the total derivative condition (7.19) because
\[ \frac{\delta}{\delta y} \left[ y'' + \frac{\sin x}{x^2} y' + \left( \frac{\cos x}{x^2} - \frac{\sin x}{x^3} \right) y \right] = \frac{\sin x}{x^2}. \]
Therefore we will apply Proposition 7.4. The adjoint equation to Eq. (7.49) is written
\[ z'' - \frac{\sin x}{x^2} z' + \frac{\sin x}{x^3} z = 0. \]
We take its obvious solution \( z = x \), substitute it in Eq. (7.31) and using (7.33) find
\[ \Psi = N \left[ xy'' + \frac{\sin x}{x} y' + \left( \frac{\cos x}{x} - \frac{\sin x}{x^2} \right) y \right] = \frac{\sin x}{x} w - w + xw'. \]
Therefore Eq. (7.43) is written
\[ xy' + \left( \frac{\sin x}{x} - 1 \right) y = C_1. \]
Integrating this first-order linear equation we obtain the general solution of Eq. (7.49):
\[ y = \left( C_2 + C_1 \int \frac{1}{x^2} e^{\frac{\sin x}{x} dx} \right) x e^{-\frac{\sin x}{x} dx}. \] (7.50)

### 7.5 Application of the operator identity to linear PDEs

Using the operator identity (7.1) one can easily extend the equations (7.32)-(7.33) for linear ODEs to linear partial differential equations and systems. Let us consider the second-order linear operator
\[ L[u] = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u \] (7.51)
considered in Section 1.5, Remark 1.1. The adjoint operator is
\[ L^*[v] \equiv \frac{\delta (vF[u])}{\delta u} = D_i D_j (a^{ij}v) - D_i (b^i v) + cv. \] (7.52)
Let us take the operator identity (7.1),
\[ X = W \frac{\delta}{\delta u} + D_i N^i, \] (7.53)
where \( X \) is the operator (7.14) with one dependent variable \( u \),
\[ X = W \frac{\partial}{\partial u} + W_i \frac{\partial}{\partial u_i} + W_{ij} \frac{\partial}{\partial u_{ij}}, \]
and \( N^i \) are the operators (7.5),
\[ N^i = W \frac{\delta}{\delta u_i} + W_j \frac{\delta}{\delta u_{ij}} = W \left[ \frac{\partial}{\partial u_i} - D_j \frac{\partial}{\partial u_{ij}} \right] + W_j \frac{\partial}{\partial u_{ij}}. \]
We use above the notation $W_i = D_i(W)$, $W_{ij} = D_iD_j(W)$. Now we proceed as in Section 7.3. Namely, we act on $vL[u]$ by both sides of the identity (7.53),

$$X(vL[u]) = W \frac{\delta (vL[u])}{\delta u} + D_i N^i(vL[u]),$$

take into account that $X$ does not act on the variables $x^i$, $v$, and that $X(L[u]) = L[W]$, use Eq. (7.52) and obtain:

$$vL[W] - WL^*[v] = D_i N^i(vL[u]).$$

Letting here $W = u$ we arrive at the following generalization of the equation (7.32):

$$vL[u] - uL^*[v] = D_i(\psi^i),$$

(7.54)

where $\psi^i$ are defined as in (7.33)-(7.34):

$$\psi^i = N^i(vL[u])|_{W=u} \equiv a^{ij}(x)[vu_i - uv_i] + [b^i(x) - D_i(a^{ij}(x))]uv.$$  (7.55)

### 7.6 Application of the operator identity to nonlinear equations

Let us apply the constructions of Section 7.5 to nonlinear equations (1.6),

$$F_\alpha(x, u, u^{(1)}, \ldots, u^{(s)}) = 0, \quad \alpha = 1, \ldots, m.$$    (7.56)

We write the operator (7.14) in the form

$$X = W^\alpha \frac{\partial}{\partial u^\alpha} + W^\alpha_i \frac{\partial}{\partial u^\alpha_i} + W^\alpha_{ij} \frac{\partial}{\partial u^\alpha_{ij}} + \cdots,$$

where $W^\alpha_i = D_i(W^\alpha)$, $W^\alpha_{ij} = D_iD_j(W^\alpha)$, $\ldots$. Then the operator (7.5) is written

$$N^i = W^\alpha_j \frac{\delta}{\delta u^\alpha_i} + W^\alpha_i \frac{\delta}{\delta u^\alpha_{ij}} + \cdots.$$  

We act on $v^\beta F_\beta$ by both sides of the operator identity (7.1)

$$X = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i,$$

denote by $F^*_\alpha[v]$ the adjoint operator defined by Eq. (1.8) and obtain

$$v^\beta \tilde{F}_\beta[W] - W^\alpha F^*_\alpha[v] = D_i(\Psi^i),$$

(7.57)

where

$$\Psi^i = N^i(v^\beta F^*_\beta)$$
and \( \hat{F}_\beta[W] \) is the linear approximation to \( F_\beta \) defined by (see also Section 1.3)

\[
\hat{F}_\beta[W] = X(F_\beta) \equiv W^\alpha \frac{\partial F_\beta}{\partial u^\alpha} + W_i^\alpha \frac{\partial F_\beta}{\partial u_i^\alpha} + W_{ij}^\alpha \frac{\partial F_\beta}{\partial u_{ij}^\alpha} + \cdots .
\]

**Remark 7.5.** Eq. (7.57) shows that \( F^*_\alpha[v] = \hat{F}^*_\beta[W] \), i.e. the adjoint operator \( F^*_\alpha \) to nonlinear Eqs. (7.56) is the usual adjoint operator \( \hat{F}^*_\beta \) to the linear operator \( \hat{F}_\beta[W] \) (see also [2]). But the linear self-adjointness of \( \hat{F}_\beta[W] \) is not identical with the nonlinear self-adjointness of Eqs. (7.56). For example, the KdV equation

\[
F \equiv u_t - u_{xxx} - uu_x = 0
\]

is nonlinearly self-adjoint (see Example 1.2 in Section 1.6). But its linear approximation \( \hat{F}[W] = W_t - W_{xxx} - uW_x - Wu_x \) is not a self-adjoint linear operator. Moreover, all linear equations are nonlinearly self-adjoint.

# 8 Conservation laws: Generalities and explicit formula

## 8.1 Preliminaries

Let us consider a system of \( m \) differential equations

\[
F_\bar{\alpha}(x, u, u^{(1)}, \ldots, u^{(s)}) = 0, \quad \bar{\alpha} = 1, \ldots, m,
\]

where \( u = (u^1, \ldots, u^m) \) and \( x = (x^1, \ldots, x^n) \).

A conservation law for Eqs. (8.1) is written

\[
[D_i(C^i)]_{(8.1)} = 0.
\]

The subscript \( |_{(8.1)} \) means that the left-hand side of (8.2) is restricted on the solutions of Eqs. (8.1). In practical calculations this restriction can be achieved by solving Eqs. (8.1) with respect to certain derivatives of \( u \) and eliminating these derivatives from the left-hand side of (8.2). For example, if (8.1) is an evolution equation

\[
u_t = \Phi(t, x, u, u_x, u_{xx}),
\]

the restriction \( |_{(8.1)} \) can be understood as the elimination of \( u_t \). The \( n \)-dimensional vector

\[
C = (C^1, \ldots, C^m)
\]

satisfying Eq. (8.2) is called a conserved vector for the system (8.1). If its components are functions \( C^i = C^i(x, u, u^{(1)}, \ldots) \) of \( x, u \) and derivatives
of a finite order, the conserved vector (8.3) is called a *local conserved vector*.

Since the conservation equation (8.2) is linear with respect to $C^i$, any linear combination with constant coefficients of a finite number of conserved vectors is again a conserved vector. It is obvious that if the divergence of a vector (8.3) vanishes identically, it is a conserved vector for any system of differential equations. This is a *trivial* conserved vectors for all differential equations. Another type of *trivial conserved vectors* for Eqs. (8.1) are provided by those vectors whose components $C^n$ vanish on the solutions of the system (8.1). One ignores both types of trivial conserved vectors. In other words, conserved vectors (8.3) are simplified by considering them up to addition of these trivial conserved vectors.

The following less trivial operation with conserved vectors is particularly useful in practice. Let

$$C^1|_{(8.1)} = \tilde{C}^1 + D_2(H^2) + \cdots + D_n(H^n). \quad (8.4)$$

Then the conserved vector (8.3) can be replaced with the equivalent conserved vector

$$\tilde{C} = (\tilde{C}^1, \tilde{C}^2, \ldots, \tilde{C}^n) = 0 \quad (8.5)$$

with the components

$$\tilde{C}^1, \quad \tilde{C}^2 = C^2 + D_1(H^2), \quad \ldots, \quad \tilde{C}^n = C^n + D_1(H^n). \quad (8.6)$$

The passage from (8.3) to the vector (8.5) is based on the commutativity of the total differentiations. Namely, we have

$$D_1D_2(H^2) = D_2D_1(H^2), \quad D_1D_n(H^n) = D_nD_1(H^n),$$

and therefore the conservation equation (8.2) for the vector (8.3) is equivalent to the conservation equation

$$\left[D_1(\tilde{C}^n)\right]_{(8.1)} = 0$$

for the vector (8.5). If $n \geq 3$, the simplification (8.6) of the conserved vector can be iterated: if $\tilde{C}^2$ contains the terms

$$D_3(\tilde{H}^3) + \cdots + D_n(\tilde{H}^n)$$

one can subtract them from $\tilde{C}^2$ and add to $\tilde{C}^3, \ldots, \tilde{C}^n$ the corresponding terms

$$D_2(\tilde{H}^3), \ldots, D_2(\tilde{H}^n).$$
Note that the conservation law (8.2) for Eqs. (8.1) can be written in the form
\[ D_i(C^i) = \mu^{\alpha} F_{\alpha}(x, u, u^{(1)}, \ldots, u^{(s)}) \]  
(8.7)
with undetermined coefficients \( \mu^{\alpha} = \mu^{\alpha}(x, u, u^{(1)}, \ldots) \) depending on a finite number of variables \( x, u, u^{(1)}, \ldots \). If \( C^i \) depend on higher-order derivatives, Eq. (8.7) is replaced with
\[ D_i(C^i) = \mu^{\alpha} F_{\alpha} + \mu^{\alpha i} D_i(F_{\alpha}) + \mu^{\alpha ij} D_i D_j(F_{\alpha}) + \cdots. \]  
(8.8)
It is manifest from Eq. (8.7) or Eq. (8.8) that the total differentiations of a conserved vector (8.3) provide again conserved vectors. Therefore, e.g. the vector
\[ D_1(C) = (D_1(C^1), \ldots, D_1(C^n)) \]  
(8.9)
obtained from a known vector (8.3) is not considered as a new conserved vector.

If one of the independent variables is time, e.g. \( x^1 = t \), then the conservation equation (8.2) is often written, using the divergence theorem, in the integral form
\[ \frac{d}{dt} \int_{\mathbb{R}^{n-1}} C^1 dx^2 \cdots dx^n = 0. \]  
(8.10)
But the differential form (8.2) of conservation laws carries, in general, more information than the integral form (8.10). Using the integral form (8.10) one may even lose some nontrivial conservation laws. As an example, consider the two-dimensional Boussinesq equations
\[ \Delta \psi_t - g \rho_x - f v_z = \psi_x \Delta \psi_z - \psi_z \Delta \psi_x, \]
\[ v_t + f \psi_z = \psi_x v_z - \psi_z v_x, \]  
\[ \rho_t + \frac{N^2}{g} \psi_x = \psi_x \rho_z - \psi_z \rho_x, \]  
(8.11)
used in geophysical fluid dynamics for investigating uniformly stratified incompressible fluid flows in the ocean. Here \( \Delta \) is the two-dimensional Laplacian,
\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \]
and \( \psi \) is the stream function so that the \( x, z \)-components \( u, w \) of the velocity \( (u, v, w) \) of the fluid are given by
\[ u = \psi_z, \quad w = -\psi_x. \]  
(8.12)
Eqs. (8.11) involve the physical constants: $g$ is the gravitational acceleration, $f$ is the Coriolis parameter, and $N$ is responsible for the density stratification of the fluid. Each equation of the system (8.11) has the conservation form (8.2), namely

$$
\begin{align*}
D_t(\Delta \psi) + D_x(-g \rho + \psi_z \Delta \psi) + D_z(-fv - \psi_x \Delta \psi) &= 0, \\
D_t(v) + D_x(v\psi_z) + D_z(f \psi - v\psi_z) &= 0, \\
D_t(\rho) + D_x \left( \frac{N^2}{g} \psi + \rho \psi_z \right) + D_z(-\rho \psi_x) &= 0.
\end{align*}
$$

(8.13)

In the integral form (8.10) these conservation laws are written

$$
\begin{align*}
\frac{d}{dt} \int \int \Delta \psi \, dxdz &= 0, \\
\frac{d}{dt} \int \int v \, dxdz &= 0, \\
\frac{d}{dt} \int \int \rho \, dxdz &= 0.
\end{align*}
$$

(8.14)

We can rewrite the differential conservation equations (8.13) in an equivalent form by using the operations (8.4)-(8.6) of the conserved vectors. Namely, let us apply these operations to the first equation (8.13), i.e. to the conserved vector

$$
C^1 = \Delta \psi, \quad C^2 = -g \rho + \psi_z \Delta \psi, \quad C^3 = -fv - \psi_x \Delta \psi.
$$

(8.15)

Noting that

$$
C^1 = D_x(\psi_x) + D_z(\psi_z),
$$

and using the operations (8.4)-(8.6) we transform the vector (8.15) to the form

$$
\tilde{C}^1 = 0, \quad \tilde{C}^2 = -g \rho + \psi_{tx} + \psi_z \Delta \psi, \quad \tilde{C}^3 = -fv + \psi_{tz} - \psi_x \Delta \psi.
$$

(8.16)

The integral conservation equation (8.10) for the vector for (8.16) is trivial, $0 = 0$. Thus, after the transformation of the conserved vector (8.15) to the equivalent form (8.16) we have lost the first integral conservation law in (8.14). But it does not mean that the conserved vector (8.16) has no physical significance. Indeed, if write the differential conservation equation with the vector (8.16), we again obtain the first equation of the system (8.11):

$$
D_x(\tilde{C}^2) + D_z(\tilde{C}^3) = \Delta \psi_t - g \rho_x - f v_z - \psi_x \Delta \psi_z + \psi_z \Delta \psi_x.
$$
Let us assume that Eqs. (8.1) have a nontrivial local conserved vector satisfying Eq. (8.7). Then not all $\mu^\beta$ vanish simultaneously due to nontriviality of the conserved vector. Furthermore, since $\mu^\beta F_\beta$ depends on $x, u$ and a finite number of derivatives $u^{(1)}, u^{(2)}, \ldots$ (i.e. it is a differential function) and has a divergence form, the following equations hold (for a detailed discussion see [26], Section 8.4.1):

$$\frac{\delta}{\delta \alpha} \left[ \mu^\beta F_\beta(x, u, u^{(1)}, \ldots, u^{(s)}) \right] = 0, \quad \alpha = 1, \ldots, m. \quad (8.17)$$

Note that Eqs. (8.17) are identical with Eqs. (3.2) where the differential substitution (3.9) is made with $\varphi^\alpha = \mu^\alpha$. Hence, the system (8.1) is nonlinearly self-adjoint. I formulate this simple observation as a theorem since it is useful in applications (see Section 11).

**Theorem 8.1.** Any system of differential equations (8.1) having a nontrivial local conserved vector satisfying Eq. (8.7) is nonlinearly self-adjoint.

### 8.2 Explicit formula for conserved vectors

Using Definition 3.1 of nonlinear self-adjointness and the theorem on conservation laws proved in [29] by using the operator identity (7.1), we obtain the explicit formula for constructing conservation laws associated with symmetries of any nonlinearly self-adjoint system of equations. The method is applicable independently on the number of equations in the system and the number of dependent variables. The result is as follows.

**Theorem 8.2.** Let the system of differential equations (8.1) be nonlinearly self-adjoint. Specifically, let the adjoint system (3.2) to (8.1) be satisfied for all solutions of Eqs. (8.1) upon a substitution (3.3),

$$\bar{v}^\alpha = \varphi^\alpha(x, u), \quad \bar{\alpha} = 1, \ldots, \bar{m}. \quad (8.18)$$

Then any Lie point, contact or Lie-Bäcklund symmetry

$$X = \xi^i(x, u, u^{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u^{(1)}, \ldots) \frac{\partial}{\partial u^\alpha}, \quad (8.19)$$

as well as a nonlocal symmetry of Eqs. (8.1) leads to a conservation law (8.2) constructed by the following formula:

$$C^i = \xi^i L + W^\alpha \left[ \frac{\partial L}{\partial u^\alpha_i} - D_j \left( \frac{\partial L}{\partial u^\alpha_{ij}} \right) + D_j D_k \left( \frac{\partial L}{\partial u^\alpha_{ijk}} \right) - \ldots \right]$$

$$+ D_j (W^\alpha) \left[ \frac{\partial L}{\partial u^\alpha_{ij}} - D_k \left( \frac{\partial L}{\partial u^\alpha_{ijk}} \right) + \ldots \right] + D_j D_k (W^\alpha) \left[ \frac{\partial L}{\partial u^\alpha_{ijk}} - \ldots \right], \quad (8.20)$$
where
\[ W^\alpha = \eta^\alpha - \xi^i u^\alpha_j \]  
(8.21)
and \( \mathcal{L} \) is the *formal Lagrangian* for the system (8.1),
\[ \mathcal{L} = v^\beta F_{\beta j}. \]  
(8.22)
In (8.20) the formal Lagrangian \( \mathcal{L} \) should be written in the symmetric form with respect to all mixed derivatives \( u^\alpha_{ij}, u^\alpha_{ijk}, \ldots \) and the "non-physical variables" \( v^\alpha \) should be eliminated via Eqs. (8.18).

One can omit in (8.20) the term \( \xi^i \mathcal{L} \) when it is convenient. This term provides a trivial conserved vector mentioned in Section 8.1 because \( \mathcal{L} \) vanishes on the solutions of Eqs. (8.1). Thus, the conserved vector (8.20) can be taken in the following form:
\[ C^i = W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \ldots \right] \]  
(8.23)
\[ + D_j \left( W^\alpha \right) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \ldots \right] + D_j D_k \left( W^\alpha \right) \left[ \frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} - \ldots \right]. \]

**Remark 8.1.** One can use Eqs. (8.23) for constructing conserved vectors even if the system (8.1) is not self-adjoint, in particular, if one cannot find explicit formulae (8.18) or (3.9) for point or differential substitutions, respectively. The resulting conserved vectors will be *nonlocal* in the sense that they involve the variables \( v \) connected with the physical variables \( u \) via differential equations, namely, adjoint equations to (8.1).

**Remark 8.2.** Theorem 8.2, unlike Noether’s theorem 7.1, does not require additional restrictions such as the invariance condition (7.9) or the divergence condition mentioned in Remark 7.2.

**9 A nonlinearly self-adjoint irrigation system**

Let us apply Theorem 8.2 to Eq. (6.1) satisfying the condition (6.3):
\[ C(\psi)\dot{\psi}_t = [K(\psi)\dot{\psi}_x]_x + [K(\psi) (\psi_x - 1)]_x - S(\psi), \]  
(9.1)
\[ S'(\psi) = aC(\psi), \quad a = \text{const}. \]  
(9.2)
The formal Lagrangian (8.22) for Eq. (9.1) has the form
\[ L = -C(\psi)\psi_t + K(\psi)(\psi_{xx} + \psi_{zz}) + K'(\psi)(\psi_x^2 + \psi_z^2 - \psi_z) - S(\psi) \] \tag{9.3}

We will use the substitution (6.5) of the particular form
\[ \nu = e^{at}. \] \tag{9.4}

Denoting \( t = x^1, x = x^2, z = x^3 \) we write the conservation equation (8.2) in the form
\[ D_t(C^1) + D_x(C^2) + D_z(C^3) = 0. \] \tag{9.5}

This equation should be satisfied on the solutions of Eq. (9.1).

The formal Lagrangian (9.3) does not contain derivatives of order higher than two. Therefore in our case Eqs. (8.23) take the simple form
\[ C^i = W \left[ \frac{\partial L}{\partial \psi_i} - D_j \left( \frac{\partial L}{\partial \psi_{ij}} \right) \right] + D_j(W) \frac{\partial L}{\partial \psi_{ij}} \] \tag{9.6}
and yield:
\[ C^1 = -WC(\psi)e^{at}, \]
\[ C^2 = W[2K'(\psi)\psi_x - D_x(K(\psi)e^{at})] + D_x(W)K(\psi)e^{at}, \]
\[ C^3 = W[K'(\psi)(2\psi_z - 1) - D_z(K(\psi)e^{at})] + D_z(W)K(\psi)e^{at}. \]

Substituting here the expression (9.3) for \( L \) we obtain
\[ C^1 = -WC(\psi)v, \]
\[ C^2 = W[2K'(\psi)v\psi_x - D_x(K(\psi)v)] + D_x(W)K(\psi)v, \]
\[ C^3 = W[K'(\psi)v(2\psi_z - 1) - D_z(K(\psi)v)] + D_z(W)K(\psi)v, \]
where \( v \) should be eliminated by means of the substitution (9.4). So, we have:
\[ C^1 = -WC(\psi)e^{at}, \]
\[ C^2 = [WK'(\psi)\psi_x + D_x(W)K(\psi)]e^{at}, \] \tag{9.7}
\[ C^3 = [WK'(\psi)(\psi_z - 1) + D_z(W)K(\psi)]e^{at}. \]

Since Eq. (9.1) does not explicitly involve the independent variables \( t, x, z \), it is invariant under the translations of these variables. Let us construct the conserved vector (9.7) corresponding to the time translation group with the generator
\[ X = \frac{\partial}{\partial t}. \] \tag{9.8}
For this operator Eq. (8.21) yields

\[ W = -\psi_t. \]  \hfill (9.9)

Substituting (9.9) in Eqs. (9.7) we obtain

\[ C^1 = C(\psi)\psi_t e^{at}, \]
\[ C^2 = -[K'(\psi)\psi_t \psi_x + K(\psi)\psi_{tx}]e^{at}, \]  \hfill (9.10)
\[ C^3 = -[K'(\psi)\psi_t (\psi_z - 1) + K(\psi)\psi_{tz}]e^{at}. \]

Now we replace in \( C^1 \) the term \( C(\psi)\psi_t \) by the right-hand side of Eq. (9.1) to obtain:

\[ C^1 = -S(\psi)e^{at} + D_x(K(\psi)\psi_x e^{at}) + D_z(K(\psi)(\psi_z - 1)e^{at}). \]

When we substitute this expression in the conservation equation (9.5), we can write

\[ D_t(D_x(K(\psi)\psi_x e^{at})) = D_x(D_t(K(\psi)\psi_x e^{at})). \]

Therefore we can transfer the terms \( D_x(...) \) and \( D_t(...) \) from \( C^1 \) to \( C^2 \) and \( C^3 \), respectively (see (8.6)). Thus, we rewrite the vector (9.10), changing its sign, as follows:

\[ C^1 = S(\psi)e^{at}, \]
\[ C^2 = aK(\psi)\psi_x e^{at} - D_t(K(\psi)\psi_x e^{at}), \]
\[ C^3 = aK'(\psi)\psi_t (\psi_z - 1)e^{at} - D_t(K(\psi)(\psi_z - 1)e^{at}). \]

Working out the differentiation \( D_t \) in the last terms of \( C^2 \) and \( C^3 \) we finally arrive at the following vector:

\[ C^1 = S(\psi)e^{at}, \]
\[ C^2 = aK(\psi)\psi_x e^{at}, \]
\[ C^3 = aK(\psi)(\psi_z - 1)e^{at}. \]  \hfill (9.11)

The reckoning shows that the vector (9.11) satisfies the conservation equation (9.5) due to the condition (9.2). Note that \( C^1 \) is the density of the conserved vector (9.11).

The use of the general substitution (6.5) instead of its particular case (9.4) leads to the conserved vector with the density

\[ C^1 = S(\psi)(bx + l)e^{at}. \]

This approach opens a new possibility to find a variety of conservation laws for the irrigation model (6.1) by considering other self-adjoint cases of the model and using the extensions of symmetry Lie algebras (see [24], Section 9.8).
10 Utilization of differential substitutions

10.1 Equation $u_{xy} = \sin u$

We return to Section 3.2 and calculate the conservation laws for Eq. (3.11),
\[ u_{xy} = \sin u, \tag{10.1} \]
using the differential substitution (3.12),
\[ v = A_1[xu_x - yu_y] + A_2u_x + A_3u_y, \tag{10.2} \]
and the admitted three-dimensional Lie algebra with the basis
\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \tag{10.3} \]

The conservation equation for Eq. (10.1) will be written in the form
\[ D_x(C^1) + D_y(C^2) = 0. \]

We write the formal Lagrangian for Eq. (10.1) in the symmetric form
\[ \mathcal{L} = \left( \frac{1}{2} u_{xy} + \frac{1}{2} u_{yx} - \sin u \right) v. \tag{10.4} \]

Eqs. (8.23) yield:
\[ C^1 = \frac{1}{2} D_y(W)v - \frac{1}{2} Wv_y, \quad C^2 = \frac{1}{2} D_x(W)v - \frac{1}{2} Wv_x. \tag{10.5} \]

where we have to eliminate the variable $v$ via the differential substitution (10.2).

Substituting in (10.5) $W = -u_x$ corresponding to the operator $X_1$ from (10.3), replacing $v$ with (10.2) and $u_{xy}$ with $\sin u$, then transferring the terms of the form $D_y(\ldots)$ from $C^1$ to $C^2$ (see the simplification (8.6)) we obtain:
\[ C^1 = A_1 \cos u, \quad C^2 = \frac{1}{2} A_1 u_x^2. \]

We let $A_1 = 1$ and conclude that the application of Theorem 8.2 to the symmetry $X_1$ yields the conserved vector
\[ C^1 = \cos u, \quad C^2 = \frac{1}{2} u_x^2. \tag{10.6} \]
The similar calculations with the operator $X_2$ from (10.3) lead to the conserved vector
\[ C^1 = \frac{1}{2} u_y^2, \quad C^2 = \cos u. \] (10.7)

The third symmetry, $X_3$ from (10.3), does not lead to a new conserved vector. Indeed, in this case $W = yu_y - xu_x$. Substituting it in the first formula (10.5) we obtain after simple calculations
\[ C^1 = \frac{1}{2} A_3 u_y^2 - A_2 \cos u + D_y \left[ (A_2 y + A_3 x) \left( \frac{1}{2} u_x u_y + \cos u \right) \right]. \]

Hence, upon transferring the term $D_y(\ldots)$ from $C^1$ to $C^2$ the resulting $C^1$ will be a linear combination with constant coefficients of the components $C^1$ of the conserved vectors (10.6) and (10.7). The same will be true for $C^2$. Therefore the conserved vector provided by the symmetry $X_3$ will be a linear combination with constant coefficients of the conserved vectors (10.6) and (10.7).

One can also use the Noether theorem because Eq. (10.1) has the classical Lagrangian, namely
\[ L = -\frac{1}{2} u_x u_y + \cos u. \] (10.8)

Then the symmetries $X_1$ and $X_2$ provide again the conserved vectors (10.6) and (10.7), respectively. But now we obtain one more conserved vector using $X_3$, namely
\[ C^1 = x \cos u - \frac{y}{2} u_y^2, \quad C^2 = \frac{x}{2} u_x^2 - y \cos u. \] (10.9)

### 10.2 Short pulse equation

The differential equation (up to notation and appropriate scaling the physical variables)
\[ D_t D_x (u) = u + \frac{1}{6} D_x^2(u^3) \] (10.10)

was suggested in [71] (see there Eq. (11), also [70]) as a mathematical model for the propagation of ultra-short light pulses in media with nonlinearities, e.g. in silica fibers. The mathematical model is derived in [71] by considering the propagation of linearly polarized light in a one-dimensional medium and assuming that the light propagates in the infrared range. The final step in construction of the model is based on the method of multiple scales.

Eq. (10.10) is connected with Eq. (10.1) by a non-point transformation which is constructed in [69] as a chain of differential substitutions (given
also in [70] by Eqs. (2)). Using this connection, an exact solitary wave solution (a pulse solution) to Eq. (10.10) is constructed in [70]. One can also find in [69] a Lax pair and a recursion operator for Eq. (10.10).

Note that Eq. (10.10) does not have a conservation form. I will find a conservation law of Eq. (10.10) thus showing that it can be rewritten in a conservation form. A significance of this possibility is commonly known and is not discussed here.

We write the short pulse equation (10.10) in the expanded form

\[ u_{xt} = u + \frac{1}{2} u^2 u_{xx} + uu_x^2 \]  

(10.11)

so that the formal Lagrangian is written

\[ \mathcal{L} = v \left[ u_{xt} - u - \frac{1}{2} u^2 u_{xx} - uu_x^2 \right]. \]  

(10.12)

Substituting (10.12) in (3.2) we obtain the following adjoint equation to Eq. (10.11):

\[ v_{xt} = v + \frac{1}{2} u^2 v_{xx}. \]  

(10.13)

We first demonstrate the following statement.

**Proposition 10.1.** Eq. (10.10) is not nonlinearly self-adjoint with a substitution

\[ v = \phi(t, x, u) \]  

(10.14)

but it is nonlinearly self-adjoint with the differential substitution

\[ v = u_t - \frac{1}{2} u^2 u_x. \]  

(10.15)

**Proof.** We write the nonlinear self-adjointness condition (3.5),

\[ \left[ v_{xt} - v - \frac{1}{2} u^2 v_{xx} \right] = \lambda \left[ u_{xt} - u - \frac{1}{2} u^2 u_{xx} - uu_x^2 \right], \]  

substitute here the expression (10.14) for \( v \) and its derivatives

\[ v_{xx} = \phi_u u_{xx} + \phi_{uu} u_x^2 + 2\phi_{xu} u_x + \phi_{xx}, \]

\[ v_{xt} = \phi_u u_{xt} + \phi_{uu} u_x u_t + \phi_{xu} u_t + \phi_{tu} u_x + \phi_{xt}, \]  

(10.16)

and, comparing the terms with the second-order derivatives of \( u \), obtain \( \lambda = \phi_u \). Then the nonlinear self-adjointness condition becomes

\[ \phi_{uu} u_x u_t + \phi_{xu} u_t + \phi_{tu} u_x + \phi_{xt} - \phi - \frac{1}{2} u^2 (\phi_{uu} u_x^2 + 2\phi_{xu} u_x + \phi_{xx}) \]

\[ = -\phi_u [u + uu_x^2], \]  

(10.17)
The terms with $u_t$ in Eq. (10.17) yield $\varphi_{uu} = \varphi_{xu} = 0$. Then we take the term with $u_x^2$ and obtain $\varphi_u = 0$. Hence

$$\varphi = a(t,x).$$

Now Eq. (10.17) gives $a_{xx} = 0$, $a_{xt} - a = 0$, whence $a = 0$. Thus

$$\varphi = 0,$$

i.e. the substitution (10.14) is trivial. This proves the first part of Proposition 10.1. Its second part is proved by similar calculations with the substitution

$$v = \varphi(t,x,u,u_x,u_t).$$

I will not reproduce these rather lengthy calculations, but instead we will verify that the substitution (10.15) maps any solution of Eq. (10.11) into a solution of the adjoint equation (10.13). First we calculate

$$v_x = u_{xt} - \frac{1}{2} u^2 u_{xx} - uu_x^2$$

and see that on the solutions of Eq. (10.11) we have $v_x = u$. Now we calculate other derivatives and verify that on the solutions of Eq. (10.11) the following equations hold:

$$v_x = u, \quad v_t = u_{tt} - \frac{1}{2} u^2 u_{xt} - uu_x u_t, \quad v_{xt} = u_t, \quad v_{xx} = u_x. \quad (10.18)$$

It is easily seen that Eq. (10.13) is satisfied. Namely, using (10.15) and (10.18) we have:

$$v_{xt} - v - \frac{1}{2} u^2 v_{xx} = u_t - \left( u_t - \frac{1}{2} u^2 u_x \right) - \frac{1}{2} u^2 u_x = 0.$$

The maximal Lie algebra of point symmetries of Eq. (10.10) is the three-dimensional algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t}. \quad (10.19)$$

Let us construct the conservation laws

$$D_t(C^1) + D_x(C^2) = 0 \quad (10.20)$$

for the basis operators (10.19).
Since the formal Lagrangian (10.12) does not contain derivatives of order higher than two, Eqs. (8.23) are written (see (9.6))

\[ C_i = W \left[ \frac{\partial L}{\partial u_i} - D_j \left( \frac{\partial L}{\partial u_{ij}} \right) \right] + D_j(W) \frac{\partial L}{\partial u_{ij}}. \]

In our case we have:

\[ C_1 = -WD_x \left( \frac{\partial L}{\partial u_{tx}} \right) + D_x(W) \frac{\partial L}{\partial u_{tx}}, \quad (10.21) \]

\[ C_2 = W \left[ \frac{\partial L}{\partial u_x} - D_t \left( \frac{\partial L}{\partial u_{xt}} \right) \right] + D_t(W) \frac{\partial L}{\partial u_{xt}} + D_x(W) \frac{\partial L}{\partial u_{xx}}. \]

Substituting in (10.21) the expression (10.12) for \( L \) written in the symmetric form

\[ L = v \left[ \frac{1}{2} u_{tx} + \frac{1}{2} u_{xt} - u - \frac{1}{2} u^2 u_{xx} - uu_x \right] \]

we obtain

\[ C_1 = -\frac{1}{2} W v_x + \frac{1}{2} v D_x(W), \quad (10.23) \]

\[ C_2 = -W \left[ uv u_x + \frac{1}{2} v_t - \frac{1}{2} u^2 v_x \right] + \frac{1}{2} v D_t(W) - \frac{1}{2} u^2 v D_x(W). \]

Since \( v \) should be eliminated via the differential substitution (10.15), we further simplify this vector by replacing \( v_x \) with \( u \) according to the first equation (10.18) and obtain:

\[ C_1 = -\frac{1}{2} W u + \frac{1}{2} v D_x(W), \quad (10.24) \]

\[ C_2 = -W \left[ uv u_x + \frac{1}{2} v_t - \frac{1}{2} u^3 \right] + \frac{1}{2} v D_t(W) - \frac{1}{2} u^2 v D_x(W), \]

where \( v \) and \( v_t \) should be replaced with their values given in Eqs. (10.15), (10.18).

Let us construct the conserved vectors using the symmetries (10.19). Their commutators are

\[ [X_1, X_3] = -X_1, \quad [X_2, X_3] = X_2. \]

Hence, according to [21], Section 22.4, the operator \( X_3 \) plays a distinguished role. Namely, the conserved vectors associated with \( X_1 \) and \( X_2 \) can be obtained from the conserved vector provided by \( X_3 \) using the adjoint actions.
of the operators $X_1$ and $X_2$, respectively. Therefore we start with $X_3$.

Substituting in (10.24) the expression

$$W = u + tu_t - xu_x$$

corresponding to the symmetry $X_3$, eliminating the terms of the form $D_x(A)$ from $C^1$ and adding them to $C^2$ in the form $D_t(A)$ according to the simplification (8.6), we obtain after routine calculations the following conserved vector:

$$C^1 = u^2, \quad (10.25)$$
$$C^2 = u^2u_uu_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4u_x^2.$$  

The conservation equation (10.20) for the vector (10.25) holds in the form

$$D_t(C^1) + D_x(C^2) = 2\left(u_t - \frac{1}{2}u^2u_x\right)\left(u + \frac{1}{2}u^2u_xx + uu_x^2 - u_xt\right). \quad (10.26)$$

Let us turn now to the operators $X_1$ and $X_2$ from (10.19). To simplify the calculations it is useful to modify Eqs. (10.24) as follows. Noting that $vD_x(W) = D_x(vW) - Wv_x$ we rewrite the vector (10.23) in the form

$$C^1 = -Wv_x,$$
$$C^2 = -W\left[uvu_x - \frac{1}{2}u^2v_x\right] + vD_t(W) - \frac{1}{2}u^2vD_x(W).$$

Then (10.24) is replaced with

$$C^1 = -uW, \quad (10.27)$$
$$C^2 = -W\left[uvu_x - \frac{1}{2}u^3\right] + vD_t(W) - \frac{1}{2}u^2vD_x(W).$$

Substituting in the first formula (10.27) to expression $W = -u_t$ corresponding the operator $X_1$ we obtain $C^1 = uu_t$. This is the time derivative of $C^1$ from (10.25). Hence the symmetry $X_1$ leads to a trivial conserved vector obtained from the vector (10.25) by the differentiation $D_t$, in accordance with [21]. Likewise, it is manifest from (10.27) that the operator $X_2$ leads to a trivial conserved vector obtained from the conserved vector (10.25) by the differentiation $D_x$. Thus we have demonstrated the following statement.

**Proposition 10.2.** The Lie point symmetries (10.19) of Eq. (10.11) yield one non-trivial conserved vector (10.25). Accordingly, the short pulse equation (10.11) can be written in the following conservation form:

$$D_t\left(u^2\right) + D_x\left(u^2u_uu_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4u_x^2\right) = 0. \quad (10.28)$$
11 Gas dynamics

11.1 Classical symmetries and conservation laws

Let us consider the polytropic gasdynamic equations

\[\begin{align*}
v_t + (v \cdot \nabla)v + \frac{1}{\rho} \nabla p &= 0, \\
\rho_t + v \cdot \nabla \rho + \rho \nabla \cdot v &= 0, \\
p_t + v \cdot \nabla p + \gamma p \nabla \cdot v &= 0,
\end{align*}\]  \hspace{1cm} (11.1)

where \(\gamma\) is a constant known as the polytropic (or adiabatic) exponent. The independent variables are the time and the space coordinates:

\[t, \quad x = (x^1, \ldots, x^n), \quad n \leq 3.\]  \hspace{1cm} (11.2)

The dependent variables are the velocity, the density and the pressure:

\[v = (v^1, \ldots, v^n), \quad \rho, \quad p.\]  \hspace{1cm} (11.3)

Eqs. (11.1) with arbitrary \(\gamma\) have the Lie algebra of point symmetries spanned by

\[\begin{align*}
X_0 &= \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad Y_0 = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i}, \quad Y_i = t \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i}, \\
X_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j}, \quad (i < j), \\
Z_0 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}, \quad Z_i = t \frac{\partial}{\partial t} - v^i \frac{\partial}{\partial v^i} + 2\rho \frac{\partial}{\partial \rho}, \quad i, j = 1, \ldots, n,
\end{align*}\]  \hspace{1cm} (11.4)

and the following classical conservation laws:

\[\begin{align*}
\frac{d}{dt} \int_{\Omega(t)} \rho d\omega &= 0 \quad \text{– Conservation of mass} \\
\frac{d}{dt} \int_{\Omega(t)} \left( \frac{1}{2} \rho |v|^2 + \frac{p}{\gamma - 1} \right) d\omega &= - \int_{S(t)} p \nu \cdot \nu dS \quad \text{– Energy} \\
\frac{d}{dt} \int_{\Omega(t)} \rho \nu d\omega &= - \int_{S(t)} p \nu dS \quad \text{– Momentum} \\
\frac{d}{dt} \int_{\Omega(t)} \rho (x \times \nu) d\omega &= - \int_{S(t)} p (x \times \nu) dS \quad \text{– Angular momentum} \\
\frac{d}{dt} \int_{\Omega(t)} \rho (tv - x) d\omega &= - \int_{S(t)} tp \nu dS \quad \text{– Center-of-mass theorem.}
\end{align*}\]
The conservation laws are written in the integral form by using the standard symbols:
- $\Omega(t)$ - arbitrary $n$-dimensional volume, moving with fluid,
- $S(t)$ - boundary of the volume $\Omega(t)$,
- $\nu$ - unit (outer) normal vector to the surface $S(t)$.

If we write the above conservation laws in the general form

$$
\frac{d}{dt} \int_{\Omega(t)} T d\omega = - \int_{S(t)} (\chi \cdot \nu) dS, \quad (11.5)
$$

then the differential form of these conservation laws will be

$$
D_t(T) + \nabla \cdot (\chi + Tv) = 0. \quad (11.6)
$$

### 11.2 One-dimensional case

Theorem 8.1 from Section 8.1 shows that the system of gasdynamic equations (11.1) is nonlinearly self-adjoint. Let us illustrate this statement in the one-dimensional case:

$$
\begin{align*}
v_t + vv_x + \frac{1}{\rho} p_x &= 0, \\
\rho_t + v\rho_x + \rho v_x &= 0, \\
p_t + \rho v_x + \gamma p v_x &= 0.
\end{align*} \quad (11.7)
$$

We write the formal Lagrangian in the form

$$
\mathcal{L} = U \left( v_t + vv_x + \frac{1}{\rho} p_x \right) + R(\rho_t + v\rho_x + \rho v_x) + P(p_t + \rho v_x + \gamma p v_x) \quad (11.8)
$$

and obtain the following adjoint system for the new dependent variables $U, R, P$:

$$
\begin{align*}
\frac{\delta \mathcal{L}}{\delta v} &\equiv -U_t - vU_x - \rho R_x + (1 - \gamma) P p_x - \gamma p P_x = 0, \\
\frac{\delta \mathcal{L}}{\delta \rho} &\equiv -R_t - vR_x - \frac{1}{\rho^2} U p_x = 0, \\
\frac{\delta \mathcal{L}}{\delta P} &\equiv -P_t - \frac{1}{\rho} U_x + \frac{1}{\rho^2} U \rho_x + (\gamma - 1) P v_x - v P_x = 0.
\end{align*} \quad (11.9)
$$

Let us take, e.g. the conservation of energy from Section 8.1. Then we have

$$
T = \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1}, \quad \chi = pv,
$$
and using the differential form (11.6) of the energy conservation we obtain the following equation (8.7):

\[ D_t \left( \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} \right) + D_x \left( \frac{1}{2} \rho v^3 + \frac{\gamma}{\gamma - 1} pv \right) = \rho v \left( v_t + vv_x + \frac{1}{\rho} p_x \right) + \frac{v^2}{2} (\rho_t + v \rho_x + \rho v_x) + \frac{1}{\gamma - 1} (p_t + v p_x + \gamma pv_x). \]  

(11.10)

Hence, the adjoint equations (11.9) are satisfied for all solutions of the gasdynamic equations (11.1) upon the substitution

\[ U = \rho v, \quad R = \frac{v^2}{2}, \quad P = \frac{1}{\gamma - 1}. \]  

(11.11)

This conclusion can be easily verified by the direct substitution of (11.11) in the adjoint system (11.9). Namely, we have:

\[
\begin{align*}
\left. \frac{\delta L}{\delta v} \right|_{(11.11)} &= -\rho \left( v_t + vv_x + \frac{1}{\rho} p_x \right) - v (\rho_t + v \rho_x + \rho v_x), \\
\left. \frac{\delta L}{\delta \rho} \right|_{(11.11)} &= -v \left( v_t + vv_x + \frac{1}{\rho} p_x \right), \\
\left. \frac{\delta L}{\delta p} \right|_{(11.11)} &= 0.
\end{align*}
\]  

(11.12)

### 11.3 Adjoint system to Equations (11.1) with \( n \geq 2 \)

For gasdynamic equations (11.1) with two and three space variables \( x^i \) the formal Lagrangian (11.8) is replaced by

\[
\mathcal{L} = U \cdot \left( v_t + (v \cdot \nabla) v + \frac{1}{\rho} \nabla \rho \right) + R (\rho_t + v \cdot \nabla \rho + \rho \nabla \cdot v) \\
+ P (p_t + v \cdot \nabla p + \gamma p \nabla \cdot v),
\]  

(11.13)

where the vector \( U = (U^1, \ldots, U^n) \) and the scalars \( R, P \) are new dependent variables. Using this formal Lagrangian, we obtain the following adjoint
system instead of (11.9):

\[
\frac{\delta \mathcal{L}}{\delta v} \equiv - U_t - (v \cdot \nabla)U + (U \cdot \nabla)v - (\nabla \cdot v)U
- \rho \nabla R + (1 - \gamma)P \nabla p - \gamma p \nabla P = 0,
\]

\[
\frac{\delta \mathcal{L}}{\delta \rho} \equiv - R_t - v \cdot \nabla R - \frac{1}{\rho^2} U \cdot \nabla R = 0,
\]

(11.14)

\[
\frac{\delta \mathcal{L}}{\delta p} \equiv - P_t - \frac{1}{\rho} (\nabla \cdot U) + \frac{1}{\rho^2} U \cdot \nabla \rho - (\gamma - 1)P(\nabla \cdot v) - v \cdot \nabla P = 0.
\]

The nonlinear self-adjointness of the system (11.1) can be demonstrated as in the one-dimensional case discussed in Section 11.2.

11.4 Application to nonlocal symmetries of the Chaplygin gas

The Chaplygin gas is described by the one-dimensional gasdynamic equations (11.7) with \( \gamma = -1 \):

\[
\begin{align*}
v_t + v v_x + \frac{1}{\rho} p_x &= 0, \\
\rho_t + v \rho_x + \rho v_x &= 0, \\
p_t + v p_x - \rho v_x &= 0.
\end{align*}
\]

(11.15)

Eqs. (11.15) have the same maximal Lie algebra of Lie point symmetries as Eqs. (11.7) with arbitrary \( \gamma \). This algebra is spanned by the symmetries (11.4) in the one-dimensional case, namely

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \\
X_4 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
X_5 &= \rho \frac{\partial}{\partial \rho} + \frac{\partial}{\partial p}, \\
X_6 &= \frac{t}{\rho} \frac{\partial}{\partial t} - \frac{v}{\rho} \frac{\partial}{\partial v} + \frac{2}{\rho} \frac{\partial}{\partial \rho}.
\end{align*}
\]

(11.16)

The Chaplygin gas has more symmetries than an arbitrary one-dimensional polytropic gas upon rewriting it in Lagrange’s variables obtained by replacing \( x \) and \( \rho \) with \( \tau \) and \( q \), respectively, obtained by the following nonlocal transformation:

\[
\tau = \int \rho dx, \quad q = \frac{1}{\rho}.
\]

(11.17)
Then the system (11.15) becomes

\begin{align*}
q_t - v_r &= 0, \\
v_t + p_r &= 0, \\
p_t - \frac{p}{q} v_r &= 0
\end{align*}
(11.18)

and admits the 8-dimensional Lie algebra with the basis

\begin{align*}
Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial \tau}, \quad Y_3 = \frac{\partial}{\partial v}, \quad Y_4 = t \frac{\partial}{\partial t} + \tau \frac{\partial}{\partial \tau}, \\
Y_5 &= \tau \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p} - q \frac{\partial}{\partial q}, \quad Y_6 = v \frac{\partial}{\partial v} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}, \\
Y_7 &= \frac{\partial}{\partial p} + \frac{q}{p} \frac{\partial}{\partial q}, \quad Y_8 = t \frac{\partial}{\partial v} - s \frac{\partial}{\partial p} - \frac{yq}{p} \frac{\partial}{\partial q}.
\end{align*}
(11.19)

It is shown in [1] that the operators $Y_7, Y_8$ from (11.19) lead to the following nonlocal symmetries for Eqs. (11.15):

\begin{align*}
X_7 &= \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial \rho} + \frac{\rho}{p} \frac{\partial}{\partial \rho}, \\
X_8 &= \left( \frac{t^2}{2} + s \right) \frac{\partial}{\partial x} + t \frac{\partial}{\partial v} - \tau \frac{\partial}{\partial p} + \frac{\rho \tau}{p} \frac{\partial}{\partial \rho},
\end{align*}
(11.20)

where $\tau, s, \sigma$ are the following nonlocal variables:

\begin{align*}
\tau &= \int \rho dx, \quad s = - \int \frac{\tau}{p} dx, \quad \sigma = - \int \frac{dx}{p}.
\end{align*}
(11.21)

They can be equivalently defined by the compatible over-determined systems

\begin{align*}
\tau_x &= \rho, & \tau_t + v \tau_x &= 0, \\
s_x &= - \frac{\tau}{p}, & s_t + v s_x &= 0, \\
\sigma_x &= - \frac{1}{p}, & \sigma_t + v \sigma_x &= 0,
\end{align*}
(11.22)

or

\begin{align*}
\tau_x &= \rho, & \tau_t &= - v \rho, \\
s_x &= - \frac{\tau}{p}, & s_t &= \frac{v \tau}{p}, \\
\sigma_x &= - \frac{1}{p}, & \sigma_t &= \frac{v}{p}.
\end{align*}
(11.23)
Let us verify that the operator \( X_7 \) is admitted by Eqs. (11.15). Its first prolongation is obtained by applying the usual prolongation procedure and eliminating the partial derivatives \( \sigma_x \) and \( \sigma_t \) via Eqs. (11.23). It has the form

\[
X_7 = \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p} \frac{\partial}{\partial p} - \frac{\rho v_x}{p} \frac{\partial}{\partial v_t} + \frac{v_x}{p} \frac{\partial}{\partial v_x} - \frac{v p_x}{p} \frac{\partial}{\partial p_t} + \frac{p_x}{p} \frac{\partial}{\partial p_x} + \left( \frac{\rho_t}{p} - \frac{\rho p_t}{p^2} - \frac{v p_x}{p} \right) \frac{\partial}{\partial p_t} + \left( \frac{2 \rho_x}{p} - \frac{\rho p_x}{p^2} \right) \frac{\partial}{\partial \rho_x}.
\]

(11.24)

The calculation shows that the invariance condition is satisfied in the following form:

\[
X_7 \left( v_t + v v_x + \frac{1}{\rho} p_x \right) = 0,
\]

\[
X_7 (\rho_t + v \rho_x + \rho v_x) = \frac{1}{p} (\rho_t + v \rho_x + \rho v_x) - \frac{\rho}{p^2} (p_t + v p_x - pv_x),
\]

\[
X_7 (p_t + v p_x - pv_x) = 0.
\]

One can verify likewise that the invariance test for the operator \( X_8 \) is satisfied in the following form:

\[
X_8 \left( v_t + v v_x + \frac{1}{\rho} p_x \right) = 0,
\]

\[
X_8 (\rho_t + v \rho_x + \rho v_x) = \frac{\tau}{p} (\rho_t + v \rho_x + \rho v_x) - \frac{\rho \tau}{p^2} (p_t + v p_x - pv_x),
\]

\[
X_8 (p_t + v p_x - pv_x) = 0.
\]

The operators \( Y_1, \ldots, Y_6 \) from (11.19) do not add to the operators (11.16) new symmetries of the system (11.15).

Thus, the Chaplygin gas described by Eqs. (11.15) admits the eight-dimensional vector space spanned by the operators (11.16) and (11.20). However this vector space is not a Lie algebra. Namely, the commutators of the dilation generators \( X_4, X_5, X_6 \) from (11.16) with the operators (11.20) are not linear combinations of the operators (11.16), (11.20) with constants coefficients. The reason is that the operators \( X_4, X_5, X_6 \) are not admitted by the differential equations (11.22) for the nonlocal variables \( \tau, s, \sigma \). Therefore I will extend the action of the dilation generators to \( \tau, s, \sigma \) so that the extended operators will be admitted by Eqs. (11.22).

Let us take the operator \( X_4 \). We write it in the extended form

\[
X'_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \sigma}.
\]
where $\alpha, \beta, \mu$ are unknown functions of $t, x, v, \rho, p, \tau, s, \sigma$. Then we make the prolongation of $X'_4$ to the first-order partial derivatives of the nonlocal variables with respect to $t$ and $x$ by treating $\tau, s, \sigma$ as new dependent variables and obtain

$$X'_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial \tau} + \beta \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial \sigma}$$

$$+ [D_t(\alpha) - \tau_t] \frac{\partial}{\partial \tau} + [D_x(\alpha) - \tau_x] \frac{\partial}{\partial \tau_x}$$

$$+ [D_t(\beta) - s_t] \frac{\partial}{\partial s} + [D_x(\beta) - s_x] \frac{\partial}{\partial s_x}$$

$$+ [D_t(\mu) - \sigma_t] \frac{\partial}{\partial \sigma} + [D_x(\mu) - \sigma_x] \frac{\partial}{\partial \sigma_x}.$$

Now we require the invariance of Eqs. (11.22):

$$X'_4(\tau_x - \rho) = 0, \quad X'_4(\tau_t + v\tau_x) = 0,$$

$$X'_4\left(s_x + \frac{\tau}{p}\right) = 0, \quad X'_4(s_t + vs) = 0,$$

$$X'_4\left(\sigma_x + \frac{1}{p}\right) = 0, \quad X'_4(\sigma_t + v\sigma_x) = 0.$$ (11.25)

As usual, Eqs. (11.25) should be satisfied on the solutions of Eqs. (11.22). Let us solve the equations $X'_4(\tau_x - \rho) = 0$, $X'_4(\tau_t + v\tau_x) = 0$. They are written

$$[D_x(\alpha) - \tau_x]_{(11.22)} = 0, \quad [D_t(\alpha) - \tau_t + v(D_x(\alpha) - \tau_x)]_{(11.22)} = 0.$$ (11.26)

Since $\tau_x = D_x(\alpha)$, the first equation in (11.26) is satisfied if we take

$$\alpha = \tau$$

With this $\alpha$ the second equation in (11.26) is also satisfied because $\tau_t + v\tau_x = 0$. Now the first equation in the second line of Eqs. (11.25) becomes

$$\left[ D_x(\beta) - s_x + \frac{\tau}{p} \right]_{(11.22)} = D_x(\beta) - 2s_x = 0$$

and yields

$$\beta = 2s.$$
The second equation in the second line of Eqs. (11.25) is also satisfied with this $\beta$. Applying the same approach to the third line of Eqs. (11.25) we obtain

$$\mu = \sigma.$$  

After similar calculations with $X_5$ and $X_6$ we obtain the following extensions of the dilation generators:

$$X'_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s} + \sigma \frac{\partial}{\partial \sigma},$$
$$X'_5 = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p} + \tau \frac{\partial}{\partial \tau} - \sigma \frac{\partial}{\partial \sigma},$$
$$X'_6 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} + 2\rho \frac{\partial}{\partial \rho} + 2\tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s}. \quad (11.27)$$

The operators (11.20), (11.27) together with the operators $X_1, X_2, X_3$ from (11.16) span the eight-dimensional Lie algebra $L_8$ admitted by Eqs. (11.15) and Eqs. (11.22). The algebra $L_8$ has the following commutator table:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X'_4$</th>
<th>$X'_5$</th>
<th>$X'_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0</td>
<td>0</td>
<td>$X_2$</td>
<td>$X_1$</td>
<td>0</td>
<td>0</td>
<td>$X_3$</td>
<td></td>
</tr>
<tr>
<td>$X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$-X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-X_3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X'_4$</td>
<td>$-X_1$</td>
<td>$-X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$X_8$</td>
</tr>
<tr>
<td>$X'_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-X_7$</td>
<td>0</td>
</tr>
<tr>
<td>$X'_6$</td>
<td>$-X_1$</td>
<td>0</td>
<td>$X_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2$X_8$</td>
</tr>
<tr>
<td>$X_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$X_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_8$</td>
<td>$-X_3$</td>
<td>0</td>
<td>0</td>
<td>$-X_8$</td>
<td>0</td>
<td>$-2X_8$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let us apply Theorem 8.2 to the nonlocal symmetries (11.20) of the Chaplygin gas. The formal Lagrangian (11.8) for Eqs. (11.15) has the form

$$\mathcal{L} = U \left( v_t + v v_x + \frac{1}{\rho} p_x \right) + R \left( \rho_t + v \rho_x + \rho v_x \right) + P \left( p_t + v p_x - p v_x \right). \quad (11.28)$$
Accordingly, the adjoint system (11.9) for the Chaplygin gas is written

\[ \frac{\delta L}{\delta v} \equiv -U_t - vU_x - \rho R_x + 2P\rho_x + pP_x = 0, \]
\[ \frac{\delta L}{\delta \rho} \equiv -R_t - vR_x - \frac{1}{\rho^2} Up_x = 0, \]
\[ \frac{\delta L}{\delta p} \equiv -P_t - \frac{1}{\rho} U_x + \frac{1}{\rho^2} U\rho_x - 2Pv_x - vP_x = 0. \]

(11.29)

Let us proceed as in Section 11.2. Namely, let us first construct solutions to the adjoint system (11.29) by using the known conservation laws given in Section 11.1. Since the one-dimensional gasdynamic system does not have the conservation of angular momentum, we use the conservation of mass, energy, momentum and center-of-mass and obtain the respective differential conservation equations (see the derivation of Eq. (11.10)):

\[ D_t(\rho) + D_x(\rho v) = \rho_t + v\rho_x + \rho v_x, \]
\[ D_t(\rho v^2 - p) + D_x(p\rho + \rho v^3) = 2\rho v(v_t + vv_x + \frac{1}{\rho} p_x) + v^2(\rho_t + v\rho_x + \rho v_x) - (p_t + v\rho_x - \rho v_x), \]
\[ D_t(\rho v) + D_x(p + \rho v^2) = \rho(v_t + vv_x + \frac{1}{\rho} p_x) + v(\rho_t + v\rho_x + \rho v_x), \]
\[ D_t(tpv - x\rho) + D_x(tp + tpv^2 - xv) = t\rho(v_t + vv_x + \frac{1}{\rho} p_x) + (tv - x)(\rho_t + v\rho_x + \rho v_x). \]

(11.30)-(11.33)

Eqs. (11.30)- (11.33) give the following solutions to the adjoint equations (11.29):

\[ U = 0, \quad R = 1, \quad P = 0, \]
\[ U = 2\rho v, \quad R = v^2, \quad P = -1, \]
\[ U = \rho, \quad R = v, \quad P = 0, \]
\[ U = t\rho, \quad R = tv - x, \quad P = 0. \]

(11.34)-(11.37)

The formal Lagrangian (11.28) contains the derivatives only of the first order. Therefore Eqs. (8.23) for calculating the conserved vectors take the simple form

\[ C^i = W^\alpha \frac{\partial L}{\partial u^\alpha_i}, \quad i = 1, 2. \]

(11.38)

We denote

\[ t = x^1, \quad x = x^2, \quad v = u^1, \quad \rho = u^2, \quad p = u^3. \]
In this notation conservation equation (8.2) will be written in the form
\[
[D_t(C^1) + D_x(C^2)]_{(11.15)} = 0. \tag{11.39}
\]
Writing (11.38) in the form
\[
C^1 = W^1 \frac{\partial L}{\partial v_t} + W^2 \frac{\partial L}{\partial \rho_t} + W^3 \frac{\partial L}{\partial p_t},
\]
\[
C^2 = W^1 \frac{\partial L}{\partial v_x} + W^2 \frac{\partial L}{\partial \rho_x} + W^3 \frac{\partial L}{\partial p_x}
\]
and substituting the expression (11.28) for \(L\) we obtain the following final expressions for computing the components of conserved vectors:

\[
C^1 = UW^1 + RW^2 + PW^3, \tag{11.40}
\]
\[
C^2 = (vU + \rho R - p P)W^1 + vRW^2 + \left(\frac{1}{\rho} U + vP\right) W^3, \tag{11.41}
\]
where

\[
W^\alpha = \eta^\alpha - \xi^i u^\alpha_i, \quad \alpha = 1, 2, 3. \tag{11.42}
\]

We will apply Eqs. (11.40)-(11.41) to the nonlocal symmetries (11.20).

First we write the expressions (11.42) for the operator \(X_7\) from (11.20):
\[
W^1 = -\sigma v_x, \quad W^2 = \frac{\rho}{p} - \sigma \rho_x, \quad W^3 = -(1 + \sigma p_x). \tag{11.43}
\]
Then we substitute (11.43) in (11.40)-(11.41) and obtain four conserved vectors by replacing \(U, R, P\) with each of four different solutions (11.34)-(11.37) of the adjoint system (11.29). Some of these conserved vectors may be trivial. We select only the nontrivial ones.

Let us calculate the conserved vector obtained by eliminating \(U, R, P\) by using the solution (11.34), \(U = 0, \quad R = 1, \quad P = 0\). In this case (11.40)-(11.41) and (11.43) yield
\[
C^1 = W^2 = \frac{\rho}{p} - \sigma \rho_x, \tag{11.44}
\]
\[
C^2 = \rho W^1 + v W^2 = -\sigma \rho v_x + \frac{\rho}{p} v - \sigma v \rho_x.
\]
We write
\[-\sigma \rho_x = -D_x(\sigma \rho) + \rho \sigma_x,\]
replace \(\sigma_x\) with \(-1/p\) according to Eqs. (11.22) and obtain
\[C^1 = -D_x(\sigma \rho).\]
Therefore application of the operations (8.4)-(8.6) yields \( \tilde{C}_1 = 0 \) and
\[
\tilde{C}_2 = -\sigma p v_x + \frac{\rho}{p} v - \sigma v p_x - D_t(\sigma p)
= -\sigma p v_x + \frac{\rho}{p} v - \sigma v p_x - \sigma \rho_t - \sigma \rho_x
= -\sigma (\rho_t + v p_x + \rho v_x).
\]
We have replaced \( \sigma_t \) with \( v/p \) according to Eqs. (11.23). The above expression for \( \tilde{C}_2 \) vanishes on Eqs. (11.15). Hence, the conserved vector (11.44) is trivial.

Utilization of the solutions (11.35) and (11.36) also leads to trivial conserved vectors only. Finally, using the solution (11.37),
\[
U = t \rho, \quad R = tv - x, \quad P = 0,
\]
we obtain, upon simplifying by using the operations (8.4)-(8.6), the following nontrivial conserved vector:
\[
C_1 = \sigma \rho, \quad C_2 = \sigma \rho v + t. \tag{11.45}
\]
The conservation equation (11.39) is satisfied in the following form:
\[
D_t(C_1) + D_x(C_2) = \sigma (\rho_t + v p_x + \rho v_x). \tag{11.46}
\]

Note that we can write \( C_2 \) in (11.45) without \( t \) since it adds only the trivial conserved vector with the components \( C_1 = 0, \ C_2 = t \). Thus, removing \( t \) in (11.45) and using the definition of \( \sigma \) given in (11.21) we formulate the result.

**Proposition 11.1.** The nonlocal symmetry \( X_7 \) of the Chaplygin gas gives the following nonlocal conserved vector:
\[
C_1 = -\rho \int \frac{dx}{p}, \quad C_2 = -\rho v \int \frac{dx}{p}. \tag{11.47}
\]

Now we use the operator \( X_8 \) from (11.20). In this case
\[
W^1 = t - \left( \frac{t^2}{2} + s \right) v_x,
W^2 = \frac{\rho \tau}{p} - \left( \frac{t^2}{2} + s \right) \rho_x,
W^3 = -\tau - \left( \frac{t^2}{2} + s \right) p_x. \tag{11.48}
\]
Substituting in (11.40)-(11.41) the expressions (11.48) and the solution (11.34) of the adjoint system, i.e. letting $U = 0$, $R = 1$, $P = 0$, we obtain

\[ C^1 = W^2 = \frac{\rho \tau}{p} - \left( \frac{t^2}{2} + s \right) \rho_x, \]
\[ C^2 = \rho W^1 + v W^2 = t p + \frac{\rho v \tau}{p} - \left( \frac{t^2}{2} + s \right) (\rho v_x + v \rho_x). \]

Noting that

\[- \left( \frac{t^2}{2} + s \right) \rho_x = - \frac{\rho \tau}{p} - D_x \left( \frac{t^2}{2} \rho + \rho s \right)\]

we reduce the above vector to the trivial conserved vector $\tilde{C}^1 = 0$, $\tilde{C}^2 = 0$.

Taking the solution (11.35) of the adjoint system, i.e. letting

\[ U = 2 \rho v, \quad R = v^2, \quad P = -1, \]

we obtain

\[ C^1 = 2 \rho v W^1 + v^2 W^2 - W^3 \]
\[ = 2 t p v + \frac{\rho \tau v^2}{p} + \tau - \left( \frac{t^2}{2} + s \right) D_x (\rho v^2 - p), \]
\[ C^2 = (3 \rho v^2 + p) W^1 + v^3 W^2 + v W^3 \]
\[ = t (3 \rho v^2 + p) + \frac{\rho \tau v^3}{p} - v \tau \]
\[ - \left( \frac{t^2}{2} + s \right) (3 \rho v^2 v_x + v^3 \rho_x + p v_x + v p_x). \]

Then, upon rewriting $C^1$ in the form

\[ C^1 = 2 t \rho v + 2 \tau - D_x \left[ \left( \frac{t^2}{2} + s \right) (\rho v^2 - p) \right]\]

and applying the operations (8.4)-(8.6) we arrive at the following conserved vector:

\[ C^1 = t \rho v + \tau, \quad C^2 = t (\rho v^2 + p). \quad (11.49) \]

The conservation equation (11.39) is satisfied for (11.49) in the following form:

\[ D_t(C^1) + D_x(C^2) = t p \left( v_t + v v_x + \frac{1}{\rho} p_x \right) + t v (\rho_t + v \rho_x + \rho v_x). \quad (11.50) \]
Taking the solution (11.36) of the adjoint system, i.e. letting
\[ U = \rho, \quad R = v, \quad P = 0, \]
we obtain
\[ C^1 = \rho W^1 + v W^2, \quad C^2 = 2 \rho v W^1 + v^2 W^2 + W^3. \]
Substituting the expressions (11.48) for \( W^1, W^2, W^3 \) and simplifying as in the previous case we obtain the conserved vector
\[ C^1 = t \rho, \quad C^2 = t \rho v - \tau. \quad (11.51) \]
The conservation equation (11.39) is satisfied for (11.49) in the following form:
\[ D_t(C^1) + D_x(C^2) = t(\rho_t + v \rho_x + \rho v_x). \quad (11.52) \]
Finally, we take the solution (11.37), \( U = t \rho, \quad R = tv - x, \quad P = 0 \), and obtain
\[ C^1 = t \rho W^1 + (tv - x)W^2, \quad C^2(2t \rho v - x \rho)W^1 + (tv^2 - xv)W^2 + tW^3. \]
Simplifying as above, we arrive at the conserved vector
\[ C^1 = \left( \frac{t^2}{2} - s \right) \rho, \quad C^2 = \left( \frac{t^2}{2} - s \right) \rho v - t \tau. \quad (11.53) \]
The conservation equation (11.39) is satisfied for (11.49) in the following form:
\[ D_t(C^1) + D_x(C^2) = \left( \frac{t^2}{2} - s \right) (\rho_t + v \rho_x + \rho v_x). \quad (11.54) \]
Substituting in the conserved vectors (11.49), (11.51) and (11.53) the definition (11.21) of the nonlocal variables we formulate the result.

**Proposition 11.2.** The nonlocal symmetry \( X_8 \) of the Chaplygin gas gives the following nonlocal conserved vectors:
\[ C^1 = t \rho v + \int \rho dx, \quad C^2 = t(\rho v^2 + p); \quad (11.55) \]
\[ C^1 = t \rho, \quad C^2 = t \rho v - \int \rho dx; \quad (11.56) \]
\[ C^1 = \left[ \frac{t^2}{2} + \int \frac{1}{p} \left( \int \rho dx \right) \right] \rho, \quad (11.57) \]
\[ C^2 = \left[ \frac{t^2}{2} + \int \frac{1}{p} \left( \int \rho dx \right) \right] \rho v - t \int \rho dx. \]

**Theorem 11.1.** Application of Theorem 8.2 to two nonlocal symmetries (11.20) gives four nonlocal conservation laws (11.47), (11.55)-(11.57) for the Chaplygin gas (11.15).
11.5 The operator identity for nonlocal symmetries

Example 11.1. Let us verify that the operator identity (7.1) is satisfied for the nonlocal symmetry $X_7$ of the Chaplygin gas. Specifically, let us check that the coefficients of

$$\frac{\partial}{\partial v}, \frac{\partial}{\partial p}, \frac{\partial}{\partial p}, \frac{\partial}{\partial v_t}, \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho_x}, \frac{\partial}{\partial p_t}, \frac{\partial}{\partial p_t}$$

in both sides of (7.1) are equal. Using the first prolongation (11.24) of $X_7$ and the definition of the nonlocal variable $\sigma$ given in Eqs. (11.23) we see that the left-hand side of the identity (7.1) is written

$$X_7 + D_t(\xi^i) = \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial p} + \frac{\rho}{p} \frac{\partial}{\partial \rho} - \frac{v_x}{p} \frac{\partial}{\partial v_t} + \frac{v_x}{p} \frac{\partial}{\partial v_x} - \frac{v_p}{p} \frac{\partial}{\partial p_t}$$

$$+ \frac{p_x}{p} \frac{\partial}{\partial p_x} + \left( \frac{\rho_t}{p} - \frac{\rho p_t}{p^2} - \frac{v_p}{p} \right) \frac{\partial}{\partial \rho_t} + \left( 2 \frac{\rho_x}{p} - \frac{\rho p_x}{p^2} \right) \frac{\partial}{\partial \rho_x} - \frac{1}{p} \cdot$$

Then we use the expressions (11.43) of $W^\alpha$ for the operator $X_7$, substitute them in the definition (7.5) of $N_i$ and obtain in our approximation:

$$N^1 = -\sigma v_x \frac{\partial}{\partial v_t} + \left( \frac{\rho}{p} - \sigma p_x \right) \frac{\partial}{\partial p_t} - (1 + \sigma p_x) \frac{\partial}{\partial p_x}$$

$$N^2 = \sigma - \sigma v_x \frac{\partial}{\partial v_x} + \left( \frac{\rho}{p} - \sigma p_x \right) \frac{\partial}{\partial \rho_x} - (1 + \sigma p_x) \frac{\partial}{\partial \rho_x} \cdot$$

Now the right-hand side of (7.1) is written:

$$W^1 \frac{\delta}{\delta v} + W^2 \frac{\delta}{\delta p} + W^3 \frac{\delta}{\delta p} + D_t N^1 + D_x N^2$$

$$= -\sigma v_x \left[ \frac{\partial}{\partial v} = -D_t \frac{\partial}{\partial v_t} - D_{xx} \frac{\partial}{\partial v_x} \right]$$

$$+ \left( \frac{\rho}{p} - \sigma p_x \right) \left[ \frac{\partial}{\partial p} = -D_t \frac{\partial}{\partial p_t} - D_{xx} \frac{\partial}{\partial p_x} \right] \frac{\partial}{\partial p_t}$$

$$- (1 + \sigma p_x) \left[ \frac{\partial}{\partial p} = -D_t \frac{\partial}{\partial p_t} - D_{xx} \frac{\partial}{\partial p_x} \right] \frac{\partial}{\partial p_x}$$

$$+ D_t \left[ -\sigma v_x \frac{\partial}{\partial v_t} + \left( \frac{\rho}{p} - \sigma p_x \right) \frac{\partial}{\partial p_t} - (1 + \sigma p_x) \frac{\partial}{\partial p_t} \right]$$

$$+ D_x \left[ \sigma - \sigma v_x \frac{\partial}{\partial v_x} + \left( \frac{\rho}{p} - \sigma p_x \right) \frac{\partial}{\partial \rho_x} - (1 + \sigma p_x) \frac{\partial}{\partial \rho_x} \right] \cdot$$
Making the changes in two last lines of Eq. (11.60) such as

\[ D_t \left[ -\sigma v_x \frac{\partial}{\partial v_t} \right] = -\sigma v_x D_t \frac{\partial}{\partial v_t} - D_t(\sigma v_x) \frac{\partial}{\partial v_t} = -\sigma v_x D_t \frac{\partial}{\partial v_t} - \left( \frac{v}{p} v_x + \sigma v_{tx} \right) \frac{\partial}{\partial v_t} \]

one can see that the coefficients of the differentiations (11.58) in (11.59) and (11.60) coincide. Inspection of the coefficients of the differentiations in higher derivatives \( v_{tt}, v_{tx}, v_{xx}, \ldots \) requires the higher-order prolongations of the operator \( X_7 \).

**Exercise 11.1.** Verify that the operator identity (7.1) is satisfied in the same approximation as in Example 11.1 for the nonlocal symmetry operator \( X_8 \) from (11.20).

## 12 Comparison with the “direct method”

### 12.1 General discussion

Theorem 8.2 allows to construct conservation laws for equations with known symmetries simply by substituting in Eqs. (8.23) the expressions \( W^\alpha \) and \( L \) given by Eqs. (8.21) and (8.21), respectively.

The “direct method” means the determination of the conserved vectors (8.3) by solving Eq. (8.2) for \( C_i \). Upon restricting the highest order of derivatives of \( u \) involved in \( C_i \), Eq. (8.2) splits into several equations. If one can solve the resulting system, one obtains the desired conserved vectors. Existence of symmetries is not required.

To the best of my knowledge, the direct method was used for the first time in 1798 by Laplace [57]. He applied the method to Kepler’s problem in celestial mechanics and found a new vector-valued conserved quantity (see [57], Book II, Chap. III, Eqs. (P)) known as Laplace’s vector.

The application of the direct method to the gasdynamic equations (11.1) allowed to demonstrate in [79] that all conservation laws involving only the independent and dependent variables (11.2), (11.3) were provided by the classical conservation laws (mass, energy, momentum, angular momentum and center-of-mass) given in Section 11.1 and the following two special conservation laws

\[
\frac{d}{dt} \int_{\Omega(t)} \left\{ t(\rho|v|^2 + np) - \rho x \cdot v \right\} d\omega = - \int_{S(t)} \left( 2t v - x \right) \cdot v dS,
\]

\[
\frac{d}{dt} \int_{\Omega(t)} \left\{ t^2(\rho|v|^2 + np) - \rho x \cdot \left( 2tv - x \right) \right\} d\omega = - \int_{S(t)} \left( 2tp \left( tv - x \right) \right) \cdot v dS
\]
that were found in [19] in the case \( \gamma = (n + 2)/n \) by using the symmetry ideas.

All local conservation laws for the heat equation \( u_t - u_{xx} = 0 \) have been found by the direct method in [11] (see in [23], Section 10.1; see also [74]). Namely it has been shown by considering the conservation equations of the form
\[
D_t[\tau(t, x, u, u_x, u_{xx}, \ldots)] + D_x[\psi(t, x, u, u_x, u_{xx}, \ldots)] = 0
\]
that all such conservation laws are given by
\[
D_t[\varphi(t, x)u] + D_x[u\varphi_x(t, x) - u_x\varphi(t, x)] = 0,
\]
where \( v = \varphi(t, x) \) is an arbitrary solution of the adjoint equation \( v_t + v_{xx} = 0 \) to the heat equation. Similar result can be obtained by applying Theorem 8.2 for any linear equation, e.g. for the heat equation \( u_t - \Delta u = 0 \) with any number of spatial variables \( x = (x^1, \ldots, x^n) \). Namely, applying formula (8.23) to the scaling symmetry \( X = u\partial/\partial u \) we obtain the conservation law
\[
D_t[\varphi(t, x)u] + \nabla \cdot [u\nabla \varphi(t, x) - \varphi(t, x)\nabla u] = 0,
\]
where \( v = \varphi(t, x) \) is an arbitrary solution of the adjoint equation \( v_t + \Delta v = 0 \) to the heat equation. This conservation law embraces the conservation laws associated with all other symmetries of the heat equation.

Various mathematical models for describing the geological process of segregation and migration of large volumes of molten rock were proposed in the geophysical literature (see the papers [72], [6], [73], [18], [63] and the references therein). One of them is known as the generalized magma equation and has the form
\[
\frac{\partial u}{\partial t} + D_z \left[ u^n - u^nD_z \left( u^{-m}u_t \right) \right] = 0, \quad n, m = \text{const.} \quad (12.1)
\]
It is accepted as a reasonable mathematical model for describing melt migration through the Earth’s mantle. Several conservation laws for this model have been calculated by the direct method in [6], [18] and interpreted from symmetry point of view in [63]. It is shown in [54] that Eq. (12.1) is quasi self-adjoint with the substitution (1.34) given by \( v = u^{1-n-m} \) if \( m + n \neq 1 \) and \( v = \ln |u| \) if \( m + n = 1 \). These substitutions show that Eq. (12.1) is strictly self-adjoint (Definition 1.2) if \( m + n = 0 \). Using the quasi self-adjointness, the conservation laws are easily computed in [54].

A simplification of the direct method was suggested in [2]. Namely, one writes the conservation equation in the form (8.7),
\[
D_t(C^i) = \mu^\alpha F_\alpha(x, u, u(1), \ldots, u(s)), \quad (8.7)
\]
and first finds the undetermined coefficients $\mu^\alpha$ by satisfying the integrability condition of Eqs. (8.7), i.e. by solving the equations (see Proposition 7.1 in Section 7.2)

$$\frac{\delta}{\delta u^\alpha}\left[\mu^\beta(x, u, u(1), \ldots) F_\beta(x, u, u(1), \ldots, u(s))\right] = 0, \quad \alpha = 1, \ldots, m. \quad (12.2)$$

Then, for each solution $\mu^\alpha$ of Eqs. (12.2), the components $C_i$ of the corresponding conserved vector are computed from Eq. (8.7). In simple situations $C_i$ can be detected merely by looking at the right-hand side of Eq. (8.7), see further Example 12.1. In more complicated situations, one needs to use computer manipulations.

**Remark 12.1.** Note that Eq. (12.2) should be satisfied on the solutions of Eqs. (8.1). Then the left-hand side of (12.2) can be written as

$$F_{\alpha}^*(x, u, v, \ldots, u(s), v(s)) \bigg|_{v=\mu(x, u, u(1), \ldots)}$$

with $F_{\alpha}^*$ defined by Eq. (3.2).

The reader can find a detailed discussion of the direct method in the recent book [7]. I will compare two methods by considering few examples and exercises.

### 12.2 Examples and exercises

**Example 12.1.** (See [7], Sec. 1.3). Let us consider the KdV equation (3.6),

$$u_t = u_{xxx} + uu_x, \quad (3.6)$$

and write the condition (12.2) for $\mu = \mu(t, x, u)$. We have:

$$\frac{\delta}{\delta u}[\mu(t, x, u)(u_t - u_{xxx} - uu_x)] = -D_t(\mu) + D_x^3(\mu) + D_x(\mu) - \mu u_x + (u_t - u_{xxx} - uu_x) \frac{\partial \mu}{\partial u}$$

$$= -D_t(\mu) + D_x^3(\mu) + uD_x(\mu) + (u_t - u_{xxx} - uu_x) \frac{\partial \mu}{\partial u}.$$ 

In accordance with Remark 12.1, we consider this expression on the solutions of the KdV equation and see that Eq. (12.2) coincides with the adjoint equation (3.7) to (3.6):

$$D_t(\mu) = D_x^3(\mu) + uD_x(\mu). \quad (12.3)$$
Its solution is given in Example 3.1 and has the form (3.8),

$$\mu = A_1 + A_2 u + A_3 (x + tu), \quad A_1, A_2, A_3 = \text{const.}$$

Thus, we have the following three linearly independent solutions of Eq. (12.3):

$$\mu_1 = 1, \quad \mu_2 = u, \quad \mu_3 = (x + tu).$$

and the corresponding three equations (8.7):

\begin{align*}
D_t(C^1) + D_x(C^2) &= u_t - u_{xxx} - uu_x, \quad (12.4) \\
D_t(C^1) + D_x(C^2) &= u(u_t - u_{xxx} - uu_x), \quad (12.5) \\
D_t(C^1) + D_x(C^2) &= (x + tu)(u_t - u_{xxx} - uu_x). \quad (12.6)
\end{align*}

In this simple example the components $C^1, C^2$ of the conserved vector can be easily seen from the right-hand sides of Eqs. (12.4)-(12.6). In the case of (12.4), (12.5) it is obvious. Therefore let us consider the right-hand side of Eq. (12.6). We see that

\begin{align*}
(x + tu)u_t &= D_t \left( xu + \frac{1}{2} tu^2 \right) - \frac{1}{2} u^2, \\
-(x + tu)uu_x &= -D_x \left( \frac{1}{2} xu^2 + \frac{1}{3} tu^3 \right) + \frac{1}{2} u^2, \\
-(x + tu)u_{xxx} &= -D_x (xu_{xx} + tu_{xx}) + u_{xx} + tu_x u_{xx}, \\
&= D_x \left( u_x + \frac{1}{2} tu_x^2 - xu_{xx} - tuu_{xx} \right).
\end{align*}

Hence, the right-hand side of Eq. (12.6) can be written in the divergence form:

\begin{align*}
(x + tu)(u_t - u_{xxx} - uu_x) \\
&= D_t \left( t \frac{u^2}{2} + xu \right) + D_x \left[ u_x + t \left( \frac{u^2}{2} - uu_{xx} - \frac{u^3}{3} \right) - x \left( \frac{u^2}{2} + u_{xx} \right) \right].
\end{align*}

The expressions under $D_t(\cdots)$ and $D_x(\cdots)$ give $C^1$ and $C^2$, respectively, in (12.6). Note that the corresponding conservation law

$$D_t \left( t \frac{u^2}{2} + xu \right) + D_x \left[ u_x + t \left( \frac{u^2}{2} - uu_{xx} - \frac{u^3}{3} \right) - x \left( \frac{u^2}{2} + u_{xx} \right) \right] = 0$$

was derived from the Galilean invariance of the KdV equation (see [21], Section 22.5) and by the direct method (see [7], Section 1.3.5).
The similar treatment of the right-hand sides of the equations (12.4) and (12.5) leads to Eq. (3.6) and to the conservation law
\[ D_t(u^2) + D_x \left( u_x^2 - 2uu_{xx} - \frac{2}{3} u^3 \right) = 0, \]  
(12.7)
respectively. Theorem 8.2 associates the conservation law (12.7) with the scaling symmetry of the KdV equation.

**Exercise 12.1.** Apply the direct method to the short pulse equation (10.11) using the differential substitution (10.15). In this case Eq. (8.7) is written
\[ D_t(C^1) + D_x(C^2) = u_t u_{xt} - \frac{1}{2} u^2 u_x u_{xt} \]
\[ - \left( u + \frac{1}{2} u^2 u_{xx} + uu_x^2 \right) u_t + \frac{1}{2} u^3 u_x + \frac{1}{4} u^4 u_{xx} + \frac{1}{2} u^3 u_x^3. \]
(12.8)

**Exercise 12.2.** Consider the Boussinesq equations (8.11). Taking its formal Lagrangian
\[ L = \omega \left[ \Delta \psi_t - g \rho_x - f v_z - \psi_x \Delta \psi_z + \psi_z \Delta \psi_x \right] \]
\[ + \mu \left[ v_t + f \psi_z - \psi_x v_z + \psi_z v_x \right] + \rho \left[ (N^2/g) \psi_x - \psi_x \rho_z + \psi_z \rho_x \right], \]
where \( \omega, \mu, r \) are new dependent variables, we obtain the adjoint system to Eqs. (8.11):
\[ \frac{\delta L}{\delta \psi} = 0, \quad \frac{\delta L}{\delta v} = 0, \quad \frac{\delta L}{\delta \rho} = 0. \]
(12.9)
It is shown in [43] that the system (8.11) is self-adjoint. Namely, the substitution
\[ \omega = \psi, \quad \mu = -v, \quad r = -(g^2/N^2) \rho \]
(12.10)
maps the adjoint system (12.9) into the system (8.11). Using the self-adjointness, nontrivial conservation laws were constructed via Theorem 8.2. Apply the direct method to the system (8.11). Note that knowledge of the substitution (12.10) gives the following equation (8.7):
\[ D_t(C^1) + D_x(C^2) + D_z(C^3) \]
\[ = \psi \left[ \psi_{xxx} + \psi_{zzz} - g \rho_x - f v_z - \psi_x (\psi_{xxx} + \psi_{zzz}) + \psi_z (\psi_{xxx} + \psi_{zzz}) \right] \]
\[ - \left[ v_t + f \psi_z - \psi_x v_z + \psi_z v_x \right] - \frac{g^2}{N^2} \rho \left[ \rho + \frac{N^2}{g} \psi_x - \psi_x \rho_z + \psi_z \rho_x \right]. \]
Example 12.2. Let us consider the conservation equation (11.46),

\[ D_t(C^1) + D_x(C^2) = \sigma(\rho_t + v\rho_x + \rho v_x), \]

where \( \sigma \) is connected with the velocity \( v \) and the pressure \( p \) of the Chaplygin gas by Eqs. (11.22),

\[ \sigma_x = -\frac{1}{\rho}, \quad \sigma_t + v\sigma_x = 0. \]

In this example Eqs. (12.2) are not satisfied. Indeed, we have

\[ \frac{\delta}{\delta v} [\sigma(\rho_t + v\rho_x + \rho v_x)] = \sigma \rho_x - D_x(\sigma \rho) = -\rho \sigma = \rho \int \frac{dx}{p} \neq 0, \]

\[ \frac{\delta}{\delta \rho} [\sigma(\rho_t + v\rho_x + \rho v_x)] = \sigma_t - D_x(\sigma v) + \sigma v_x = -(\sigma_t + v\sigma_x) = 0, \]

\[ \frac{\delta}{\delta p} [\sigma(\rho_t + v\rho_x + \rho v_x)] = 0. \]

Example 12.3. Let us consider the conservation equation (11.50),

\[ D_t(C^1) + D_x(C^2) = t \rho \left( v_t + vv_x + \frac{1}{\rho} p_x \right) + tv(\rho_t + v\rho_x + \rho v_x). \]

Here Eqs. (12.2) are not satisfied. Namely, writing

\[ t \rho \left( v_t + vv_x + \frac{1}{\rho} p_x \right) + tv(\rho_t + v\rho_x + \rho v_x) = t \rho v_t + 2t \rho vv_x + tp_x + tv\rho_t + tv^2 \rho_x \]

we obtain:

\[ \frac{\delta}{\delta v} [t \rho v_t + 2t \rho vv_x + tp_x + tv\rho_t + tv^2 \rho_x] = -\rho, \]

\[ \frac{\delta}{\delta \rho} [t \rho v_t + 2t \rho vv_x + tp_x + tv\rho_t + tv^2 \rho_x] = -v, \]

\[ \frac{\delta}{\delta p} [t \rho v_t + 2t \rho vv_x + tp_x + tv\rho_t + tv^2 \rho_x] = 0. \]

Exercise 12.3. Check if Eqs. (12.2) are satisfied for the conservation equations (11.52) and (11.54).
PART 3
Utilization of conservation laws
for constructing solutions of PDEs

13 General discussion of the method

As mentioned in Section 7.4, one can integrate or reduce the order of linear
ordinary differential equations by rewriting them in a conservation form
(7.39). Likewise one can integrate or reduce the order of a nonlinear ordinary
differential equation as well as a system of ordinary differential equations
using their conservation laws. Namely, a conservation law

$$D_x \left( \psi(x, y, y', \ldots, y^{(s-1)}) \right) = 0$$  \hspace{1cm} (13.1)

for a nonlinear ordinary differential equation

$$F(x, y, y', \ldots, y^{(s)}) = 0$$  \hspace{1cm} (13.2)

yields the first integral

$$\psi(x, y, y', \ldots, y^{(s-1)}) = C_1.$$  \hspace{1cm} (13.3)

We will discuss now an extension of this idea to partial differential equa-
tions. Namely, we will apply conservation laws for constructing particular
exact solutions of systems of partial differential equations. Detailed calcu-
lations are given in examples considered in the next sections.

Let us assume that the system (8.1),

$$F_{\alpha} \left( x, u, u_{(1)}, \ldots, u_{(s)} \right) = 0, \quad \alpha = 1, \ldots, m,$$  \hspace{1cm} (13.4)

has a conservation law (8.2),

$$\left[ D_i(C^i) \right]_{(13.4)} = 0,$$  \hspace{1cm} (13.5)

with a known conserved vector

$$C = \left( C^1, \ldots, C^m \right),$$  \hspace{1cm} (13.6)

where

$$C^i = C^i \left( x, u, u_{(1)}, \ldots \right), \quad i = 1, \ldots, n.$$  

We write the conservation equation (13.5) in the form (8.7),

$$D_i(C^i) = \mu^\alpha F_{\alpha} \left( x, u, u_{(1)}, \ldots, u_{(s)} \right).$$  \hspace{1cm} (13.7)
For a given conserved vector (13.6) the coefficients $\mu^{\bar{a}}$ in Eq. (13.7) are known functions $\mu^{\bar{a}} = \mu^{\bar{a}}(x, u, u(1), \ldots)$.

We will construct particular solutions of the system (13.4) by requiring that on these solutions the vector (13.6) reduces to the following trivial conserved vector:

$$C = (C^1(x^2, \ldots, x^n), \ldots, C^n(x^1, \ldots, x^{n-1})) .$$

(13.8)

In other words, we look for particular solutions of the system (13.4) by adding to Eqs. (13.4) the differential constraints

$$C^1(x, u, u(1), \ldots) = h^1(x^2, x^3, \ldots, x^n),$$

$$C^2(x, u, u(1), \ldots) = h^2(x^1, x^3, \ldots, x^n),$$

$$\ldots$$

$$C^n(x, u, u(1), \ldots) = h^n(x^1, \ldots, x^{n-1}),$$

(13.9)

where $C^i(x, u, u(1), \ldots)$ are the components of the known conserved vector (13.6). Due to the constraints (13.9), the left-hand side of Eq. (13.7) vanishes identically. Hence the number of equations in the system (13.4) will be reduced by one.

The differential constraints (13.9) can be equivalently written as follows:

$$D_1 [C^1(x, u, u(1), \ldots)] = 0,$$

$$D_2 [C^2(x, u, u(1), \ldots)] = 0,$$

$$\ldots$$

$$D_n [C^n(x, u, u(1), \ldots)] = 0.$$  

(13.10)

Remark 13.1. The overdetermined system of $m+n$ equations (13.4), (13.10) reduces to $m+n-1$ equations due to the conservation law (13.5).

14 Application to the Chaplygin gas

14.1 Detailed discussion of one case

Let us apply the method to the Chaplygin gas equations (11.15),

$$v_t + vv_x + \frac{1}{\rho} p_x = 0,$$

$$\rho_t + \rho v_x = 0,$$

$$p_t + \rho p_x - pv_x = 0.$$ 

(14.1)
We will construct a particular solution of the system (14.1) using the simplest conservation law (11.30),

\[ D_t(\rho) + D_x(\rho v) = \rho_t + v\rho_x + \rho v_x. \] (14.2)

The conservation equation (14.2) is written in the form (13.7) with the following conserved vector (13.6):

\[ C^1 = \rho, \quad C^2 = \rho v. \] (14.3)

The differential constraints (13.9) are written as follows:

\[ \rho = g(x), \quad \rho v = h(t). \] (14.4)

Thus we look for solutions of the form

\[ \rho = g(x), \quad v = \frac{h(t) \cdot g(x)}{g(x)}. \] (14.5)

The functions (14.5) solve the second equation in (14.1) because the conservation law (14.2) coincides with the second equation (14.1) (see Remark 13.1). Therefore it remains to substitute (14.5) in the first and third equations of the system (14.1). The result of this substitution can be solved for the derivatives of \( p \):

\[ p_x = -h' + \frac{h^2 g'}{g^2}, \]
\[ p_t = -\frac{hg'}{g^2} p + \frac{hh'}{g} - \frac{h^3 g'}{g^3}. \] (14.6)

The compatibility condition \( p_{xt} = p_{tx} \) of the system (14.6) gives the equation

\[ \left( g'' - 2 \frac{g'^2}{g} \right) p = g^2 \frac{h''}{h} - 2g'h' - h^2 \frac{g''}{g} + 2h^2 \frac{g'^2}{g^2}. \] (14.7)

For illustration purposes I will simplify further calculations by considering the particular case when the coefficient for \( p \) in Eq. (14.6) vanishes:

\[ g'' - 2 \frac{g'^2}{g} = 0. \] (14.8)

The solution of Eq. (14.8) is

\[ g(x) = \frac{1}{ax + b}, \quad a, b = \text{const}. \] (14.9)
Substituting (14.9) in Eq. (14.7) we obtain
\[ h'' + 2ahh' = 0, \quad (14.10) \]
whence
\[ h(t) = k \tan(c - akt) \quad (14.11) \]
if \( a \neq 0 \), and
\[ h(t) = At + B \quad (14.12) \]
if \( a = 0 \).

If the constant \( a \) in (14.9) does not vanish, we substitute (14.9) and (14.11) in Eqs. (14.6), integrate them and obtain
\[ p = k^2(ax + b) + Q \cos(c - akt), \quad Q = \text{const.} \quad (14.13) \]
In the case \( a = 0 \) the similar calculations yield
\[ p = -Ax + b \left( \frac{A^2 t^2}{2} + ABt + Q \right), \quad Q = \text{const.} \quad (14.14) \]

Thus, using the conservation law (14.2) we have arrived at the solutions
\[ \rho = \frac{1}{ax + b}, \quad v = k(ax + b) \tan(c - akt), \quad (14.15) \]
and
\[ \rho = \frac{1}{b}, \quad v = b(At + B), \quad p = -Ax + \frac{b}{2} A^2 t^2 + ABbt + Q. \quad (14.16) \]

**14.2 Differential constraints provided by other conserved vectors**

The conservation laws (11.31)-(11.33) give the following differential constraints (13.9):
\[ \rho v^2 - p = g(x), \quad pv + \rho v^3 = h(t), \quad (14.17) \]
\[ \rho v = g(x), \quad p + \rho v^2 = h(t), \quad (14.18) \]
\[ t\rho v - x\rho = g(x), \quad tp + t\rho v^2 - xpv = h(t). \quad (14.19) \]
The nonlocal conserved vectors (11.49, (11.51) and (11.53) lead to the following differential constraints (13.9):

\[
t\rho v + \tau = g(x), \quad p + \rho v^2 = h(t), \tag{14.20}
\]

\[
t\rho = g(x), \quad t\rho v - \tau = h(t), \tag{14.21}
\]

\[
\left(\frac{t^2}{2} - s\right) \rho = g(x), \quad \left(\frac{t^2}{2} - s\right) \rho v - t\tau = h(t). \tag{14.22}
\]

The constraints (14.20) are not essentially different from the constraints (14.18). It is manifest if we write them in the form (13.10).

15 Application to nonlinear equation describing an irrigation system

The method of Section 13 can be used for constructing particular solutions not only of a system, but of a single partial differential equations as well.

Let us consider the nonlinear equation (6.1),

\[
C(\psi)\psi_t = [K(\psi)\psi_x]_x + [K(\psi)(\psi_z - 1)]_z - S(\psi), \tag{15.1}
\]

satisfying the nonlinear self-adjointness condition (6.3),

\[
S'(\psi) = aC(\psi), \quad a = \text{const.} \tag{15.2}
\]

and apply the method of Section 13 to the conserved vector (9.11),

\[
C^1 = S(\psi)e^{at}, \quad C^2 = aK(\psi)\psi_x e^{at}, \quad C^3 = aK(\psi)(\psi_z - 1)e^{at}. \tag{15.3}
\]

The conditions (13.9) are written:

\[
S(\psi)e^{at} = f(x, z), \quad aK(\psi)\psi_x e^{at} = g(t, z), \quad aK(\psi)(\psi_z - 1)e^{at} = h(t, x).
\]

These conditions mean that the left-hand sides of the first, second and third equation do not depend on \(t, x\) and \(z\), respectively. Therefore they can be equivalently written as the following differential constraints (see Eqs. (13.10)):

\[
aS(\psi) + S'(\psi)\psi_t = 0, \tag{15.4}
\]

\[
[K(\psi)\psi_x]_x = 0,
\]

\[
[K(\psi)(\psi_z - 1)]_z = 0.
\]
The constraints (15.4) reduce Eq. (15.1) to Eq. (15.2). Hence, the particular solutions of Eq. (15.1) provided by the conserved vector (15.3) are described by the system

\[
\begin{align*}
   aC(\psi) - S'(\psi) &= 0, \\
   aS(\psi) + S'(\psi)\psi_t &= 0, \\
   [K(\psi)\psi_x]_x &= 0, \\
   [K(\psi)(\psi_z - 1)]_z &= 0.
\end{align*}
\] (15.5)
Approximate self-adjointness and approximate conservation laws

The methods developed in this paper can be extended to differential equations with a small parameter in order to construct approximate conservation laws using approximate symmetries. I will illustrate this possibility by examples. The reader interested in approximate symmetries can find enough material in [25], Chapters 2 and 9. A brief introduction to the subject can be found also in [50].

16 The van der Pol equation

The van der Pol equation has the form

\[ F \equiv y'' + y + \varepsilon (y'^3 - y') = 0, \quad \varepsilon = \text{const.} \neq 0. \]  \hspace{1cm} (16.1)

16.1 Approximately adjoint equation

We have:

\[ \frac{\delta}{\delta y} \left\{ z \left[ y'' + y + \varepsilon (y'^3 - y') \right] \right\} = z'' + z + \varepsilon D_x \left( z - 3zy'^2 \right). \]

Thus, the adjoint equation to the van der Pol equation is

\[ F^* \equiv z'' + z + \varepsilon \left( z' - 3z'y'^2 - 6zy'y'' \right) = 0. \]

We eliminate here \( y'' \) by using Eq. (16.1), consider \( \varepsilon \) as a small parameter and write \( F^* \) in the first order of precision with respect to \( \varepsilon \). In other words, we write

\[ y'' \approx -y. \]  \hspace{1cm} (16.2)

Then we obtain the following approximately adjoint equation to Eq. (16.1):

\[ F^* \equiv z'' + z + \varepsilon \left( z' - 3z'y'^2 + 6zyy' \right) = 0. \]  \hspace{1cm} (16.3)

16.2 Approximate self-adjointness

Let us investigate Eq. (16.1) for approximate self-adjointness. Specifically, I will call Eq. (16.1) approximately self-adjoint if there exists a non-trivial (not vanishing identically) approximate substitution

\[ z \approx f(x, y, y') + \varepsilon g(x, y, y') \]  \hspace{1cm} (16.4)
such that $F$ given by Eq. (16.1) and $F^*$ defined by Eq. (16.3) approximately satisfy the condition (3.5) of nonlinear self-adjointness. In other words, the following equation is satisfied in the first-order of precision in $\varepsilon$:

$$
F^*\big|_{z=f+\varepsilon g} = \lambda F. \quad (16.5)
$$

Note, that the unperturbed equation $y'' + y = 0$ is nonlinearly self-adjoint. Namely it coincides with the adjoint equation $z'' + z = 0$ upon the substitution

$$
z = \alpha y + \beta \cos x + \gamma \sin x, \quad \alpha, \beta, \gamma = \text{const}. \quad (16.6)
$$

Therefore we will consider the substitution (16.4) of the following restricted form:

$$
z \approx f(x, y) + \varepsilon g(x, y, y'). \quad (16.7)
$$

In differentiating $g(x, y, y')$ we will use Eq. (16.2) because we make out calculations in the first order of precision with respect to $\varepsilon$. Then we obtain:

$$
z' = D_x(f) + \varepsilon D_x(g)\big|_{y'=y} = f_x + y'f_y + \varepsilon(g_x + y'g_y - yg'),
$$

$$
z'' = D_x^2(f) + \varepsilon D_x^2(g)\big|_{y'=y} = f_{xx} + 2y'f_{xy} + y'^2f_{yy} + y''f_y + \varepsilon(g_{xx} + 2y'g_{xy} - 2yg_{xy'} + y'^2g_{yy} - 2y'g_{yy'} + y^2g_{yy'} - yg_y - y'g_y).
$$

Substituting (16.8) in (16.3) and solving Eq. (16.5) with $\varepsilon = 0$ we see that $f$ is given by Eq. (16.6). Then $\lambda = C$ and the terms with $\varepsilon$ in Eq. (16.5) give the following second-order linear partial differential equation for $g(x, y, y')$:

$$
g + D_x^2(g)\big|_{y'=y} = \alpha \left(4y'^3 - 6y'^2y' - 2y'\right)
$$

$$
+ \beta \left(\sin x - 3y'^2 \sin x - 6yy' \cos x\right) + \gamma \left(3y'^2 \cos x - \cos x - 6yy' \sin x\right).
$$

The standard existence theorem guarantees that Eq. (16.9) has a solution. It is manifest that the solution does not vanish because $g = 0$ does not satisfy Eq. (16.9). We conclude that the van der Pol equation (16.1) with a small parameter $\varepsilon$ is approximately self-adjoint. The substitution (16.7) satisfying the approximate self-adjointness condition (16.5) has the form

$$
z \approx \alpha y + \beta \cos x + \gamma \sin x + \varepsilon g(x, y, y'), \quad (16.10)
$$

where $\alpha, \beta, \gamma$ are arbitrary constants and $g(x, y, y')$ solves Eq. (16.9).
16.3  Exact and approximate symmetries

If $\varepsilon$ is treated as an arbitrary constant, Eq. (16.1) has only one point symmetry, namely the one-parameter group of translations of the independent variable $x$. We will write the generator $X_1 = \partial / \partial x$ of this group in the form (7.14):

$$X_1 = y' \frac{\partial}{\partial y}. \quad (16.11)$$

If $\varepsilon$ is a small parameter, then Eq. (16.1) has, along with the exact symmetry (16.11), the following 7 approximate symmetries ([25], Section 9.1.3.3):

$$X_2 = \{4y - \varepsilon [y^2 y' + 3xy (y^2 + y'^2)] \} \frac{\partial}{\partial y},$$

$$X_3 = \{8 \cos x + \varepsilon [(4 - 3y'^2 - 9y^2) x \cos x + 3(xy^2)' \sin x] \} \frac{\partial}{\partial y},$$

$$X_4 = \{8 \sin x + \varepsilon [(4 - 3y'^2 - 9y^2) x \sin x - 3(xy^2)' \cos x] \} \frac{\partial}{\partial y},$$

$$X_5 = \{24y^2 \cos x - 24yy' \sin x + \varepsilon [(12yy' + 9y^3y' + 9y^3) x \sin x + (12y^2 - 9y^2 y'^2 - 6y^4) \sin x - (12y^2 - 9y^2 y'^2 - 9y^4)x \cos x - 3y^3 y' \cos x] \} \frac{\partial}{\partial y}, \quad (16.12)$$

$$X_6 = \{24y^2 \sin x + 24yy' \cos x - \varepsilon [(12yy' + 9y^3y' + 9y^3) x \cos x + (12y^2 - 9y^2 y'^2 - 6y^4) \cos x + (12y^2 + 9y^2 y'^2 + 9y^4)x \sin x + 3y^3 y' \sin x] \} \frac{\partial}{\partial y},$$

$$X_7 = \{4y \cos 2x - 4y' \sin 2x + \varepsilon [3(yy'^2 - y^3) x \cos 2x$$

$$- 3y^2 y' \cos 2x + 6y^2 y' \sin 2x + 2(y - y^3) \sin 2x] \} \frac{\partial}{\partial y},$$

$$X_8 = \{4y \sin 2x + 4y' \cos 2x - \varepsilon [3(y^3 - yy'^2) x \sin 2x$$

$$+ 3y^2 y' \sin 2x + 6y^2 y' \cos 2x + 2(y - y^3) \cos 2x] \} \frac{\partial}{\partial y}.$$

16.4  Approximate conservation laws

We can construct now approximate conserved quantities for the van der Pol equation using the formula (8.23) and the approximate substitution (16.10).
Inserting in (8.23) the formal Lagrangian
\[ L = z \left[ y'' + y + \varepsilon (y'^3 - y') \right] \]
we obtain
\[ C = W \left[ -z' + \varepsilon (3y'^2 z - z) \right] + W' z. \quad (16.13) \]

Let us calculate the conserved quantity (16.13) for the operator \( X_1 \) given by Eq. (16.11). In this case \( W = y' \), \( W' = y'' \), and therefore (16.13) has the form
\[ C = -y' z' + \varepsilon \left( 3y'^2 z - y \right) z + y'' z. \]
We eliminate here \( y'' \) via Eq. (16.1), use the approximate substitution (16.10) and and obtain (in the first order of precision with respect to \( \varepsilon \)) the following approximate conserved quantity:
\[ C = -\alpha \left( y^2 + y'^2 \right) + \beta \left( y' \sin x - y \cos x \right) - \gamma \left( y' \cos x + y \sin x \right) \]
\[ + \varepsilon \left( 2\alpha y^3 + 2\beta y'^3 \cos x + 2\gamma y'^3 \sin x - yg - y'D_x(g) \right) \bigg|_{y''=-y}. \quad (16.14) \]
Differentiating it and using the equations (16.1) and (16.2) we obtain
\[ D_x(C) = \varepsilon \left[ \alpha \left( 4y'^3 - 6y^2 y' - 2y' \right) + \beta \left( \sin x - 3y'^2 \sin x - 6yy' \cos x \right) \right. \]
\[ + \gamma \left( 3y'^2 \cos x - \cos x - 6yy' \sin x \right) - g - D^2_x(g) \bigg|_{y''=-y} \bigg] + o(\varepsilon), \quad (16.15) \]
where \( o(\varepsilon) \) denotes the higher-order terms in \( \varepsilon \). The equations (16.9) and (16.15) show that the quantity (16.14) satisfies the approximate conservation law
\[ D_x(C) \bigg|_{(16.1) \approx 0}. \quad (16.16) \]

Let us consider the operator \( X_2 \) from (16.12). In this case we have
\[ W = 4y - \varepsilon \left[ y'^2 + 3xy \left( y'^2 + y^2 \right) \right], \]
\[ W' \approx 4y' - \varepsilon \left[ 2y^3 + 5yy'^2 + 3x \left( y'^2 + y^2 \right) \right]. \quad (16.17) \]
Proceeding as above we obtain the following approximate conserved quantity:
\[ C = 4y' \left( \beta \cos x + \gamma \sin x \right) - 4y \left( \gamma \cos x - \beta \sin x \right) \]
\[ + \varepsilon \left\{ 2\alpha y^2 \left( 4y'^2 - y^2 - 2 \right) + 4y'g - 4yD_x(g) \bigg|_{y''=-y} \right. \]
\[ + \left[ 7yy'^2 - 3xy' \left( y^2 + y'^2 \right) - 2y'^3 - 4y \right] \left( \beta \cos x + \gamma \sin x \right) \]
\[ + \left[ y^2 y' + 3xy \left( y^2 + y'^2 \right) \right] \left( \gamma \cos x - \beta \sin x \right) \bigg\}. \quad (16.18) \]
The calculation shows that the quantity (16.18) satisfies the approximate conservation law (16.16) in the following form:

\[
D_x(C) = 4(\beta \cos x + \gamma \sin x) \left[ y'' + y + \varepsilon \left( y'^3 - y' \right) \right] + 4\varepsilon y \left[ \alpha \left( 4y'^3 - 6y^2y' - 2y' \right) + \beta \left( \sin x - 3y'^2 \sin x - 6yy' \cos x \right) + \gamma \left( 3y'^2 \cos x - \cos x - 6yy' \sin x \right) - g - D_x^2(g) \right] + o(\varepsilon).
\]

Continuing this procedure, one can construct approximate conservation laws for the remaining approximate symmetries (16.12).

17 Perturbed KdV equation

Let us consider again the KdV equation (3.6),

\[
u_t = u_{xxx} + uu_x,
\]

and the following perturbed equation:

\[F \equiv u_t - u_{xxx} - uu_x - \varepsilon u = 0.\] (17.1)

We will follow the procedure described in Section 16.

17.1 Approximately adjoint equation

Let us write the formal Lagrangian for Eq. (17.1) in the form

\[\mathcal{L} = v [-u_t + u_{xxx} + uu_x + \varepsilon u].\]

(17.2)

Then

\[
\frac{\delta \mathcal{L}}{\delta u} = v_t - v_{xxx} - D_x(uv) + vu_x + \varepsilon v = v_t - v_{xxx} - uv_x + \varepsilon v.
\]

Hence, the approximately adjoint equation to Eq. (17.1) has the form

\[F^* \equiv v_t - v_{xxx} - uv_x + \varepsilon v = 0.\] (17.3)
17.2 Approximate self-adjointness

As mentioned in Section 3.1, Example 3.1, the KdV equation (3.6) is nonlinearly self-adjoint with the substitution (3.8),

\[ v = A_1 + A_2 u + A_3 (x + tu). \]  

(3.8)

Therefore in the case of the perturbed equation (17.1) we look for the substitution

\[ v = \phi(t, x, u) + \varepsilon \psi(t, x, u), \]

satisfying the nonlinear self-adjointness condition

\[ F^* \big|_{v=\phi+\varepsilon \psi} = \lambda F \]  

(17.4)

in the first-order of precision in \( \varepsilon \), in the following form:

\[ v = A_1 + A_2 u + A_3 (x + tu) + \varepsilon \psi(t, x, u). \]  

(17.5)

When we substitute the expression (17.5) in the definition (17.3) of \( F^* \), the terms without \( \varepsilon \) in Eq. (17.4) disappear by construction of the substitution (3.8) and give \( \lambda = A_2 + A_3 t \). Then we write Eq. (17.4), rearranging the terms, in the form

\[ \varepsilon \psi_u [u_t - u_{xxx} - u u_x] - 3 \varepsilon u_{xx} [u_x \psi_{uu} + \psi_{xu}] - \varepsilon u_x [u_x^2 \psi_{uuu} + 3 u_x \psi_{xuu} + 3 \psi_{xxx}] + \varepsilon [\psi_t - \psi_{xxx} - u \psi_x + A_1 + A_2 u + A_3 (x + tu)] = - \varepsilon (A_2 + A_3 t) u. \]  

(17.6)

In view Eq. (17.1), the first term in the first line of Eq. (17.6) is written \( \varepsilon^2 u \psi_u \). Hence, this term vanishes in our approximation. The terms with \( u_{xx} \) in the first line of Eq. (17.6) yield

\[ \psi_{uu} = 0, \quad \psi_{xu} = 0, \]

whence

\[ \psi = f(t) u + g(t, x). \]

The third bracket in the first line of Eq. (17.6) vanishes, and Eq. (17.6) becomes

\[ [f'(t) - g_x(t, x)] u + g_t(t, x) - g_{xxx}(t, x) + 2[A_2 + A_3 t] u + A_1 + A_3 x = 0. \]

After rather simple calculations we solve this equation and obtain

\[ g(t, x) = A_4 - A_1 t + (A_5 + 2 A_2 - A_3 t) x, \quad f(t) = A_6 + A_5 t - \frac{3}{2} A_3 t^2. \]

We conclude that the perturbed KdV equation (17.1) is approximately self-adjoint. The approximate substitution (17.5) has the following form:

\[ v \approx A_1 + A_2 u + A_3 (x + tu) \]

+ \( \varepsilon \left[ \left( A_6 + A_5 t - \frac{3}{2} A_3 t^2 \right) u + A_4 - A_1 t + (A_5 + 2 A_2 - A_3 t) x \right] \).
17.3 Approximate symmetries

Recall that the Lie algebra of point symmetries of the KdV equation (3.6) is spanned by the following operators:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t \frac{\partial}{\partial x} - \frac{\partial}{\partial u},
\]

\[
X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}.
\]  

Following the method for calculating approximate symmetries and using the terminology presented in [25], Chapter 2, we can prove that all symmetries (17.8) are stable. Namely the perturbed equation (17.1) inherits the symmetries (17.8) of the KdV equation in the form of the following approximate symmetries:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t \frac{\partial}{\partial x} - \frac{\partial}{\partial u} + \varepsilon \left( \frac{1}{2} t^2 \frac{\partial}{\partial x} - t \frac{\partial}{\partial u} \right),
\]

\[
X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} + \varepsilon \left[ \frac{9}{2} t^2 \frac{\partial}{\partial t} + 3tx \frac{\partial}{\partial x} - (6tu + 3x) \frac{\partial}{\partial u} \right].
\]

17.4 Approximate conservation laws

We can construct now the approximate conservation laws

\[
[D_t(C^1) + D_x(C^2)]_{(17.1)} \approx 0
\]

for the perturbed KdV equation (17.1) using its approximate symmetries (17.9), the general formula (8.23) and the approximate substitution (17.7). Inserting in (8.23) the formal Lagrangian (17.2) we obtain

\[
C^1 = -Wv, \\
C^2 = W \left[ uv + v_{xx} \right] - v_xD_x(W) + vD_x^2(W).
\]

I will calculate here the conserved vector (17.11) for the operator \(X_4\) from (17.9). In this case we have

\[
W = -2u - 3tu_t - xu_x + \varepsilon \left( 6tu + 3x + \frac{9}{2} t^2 u_t + 3txu_x \right).
\]

We further simplify the calculations by taking the particular substitution (17.7) with \(A_2 = 1, A_1 = A_3 = \cdots = A_6 = 0\). Then

\[
v = u + 2\varepsilon x.
\]
Substituting (17.12), (17.13) in the first component of the vector (17.11) and then eliminating $u_t$ via Eq. (17.1) we obtain:

\[
C^1 \approx (2u + 3tu_t + xu_x)(u + 2\varepsilon x) - \varepsilon \left(6tu + 3x + \frac{9}{2} t^2 u_t + 3txu_x \right) u
\]
\[
= 2u^2 + 3tu_xxxx + 3tu^2 u_x + xu_x + \varepsilon \left( xu + 6txu_{xxx} + 3txu_x 
+ 2x^2 u_x - 3tu^2 - \frac{9}{2} t^2 uu_{xxx} - \frac{9}{2} t^2 u_x^2 \right).
\]

Upon singling out the total derivatives in $x$, it is written:

\[
C^1 \approx \frac{3}{2} u^2 - 3\varepsilon \left[ xu + \frac{3t}{2} u^2 \right] + D_x \left[ \frac{x}{2} u^2 + tu^3 - \frac{3t}{2} u_x^2 + 3tuu_{xx} \right] + \varepsilon \left( 2x^2 u + \frac{3tx}{2} u^2 - \frac{3t^2}{2} u^3 - 6tu_x + 6txu_{xx} + \frac{9t^2}{4} u_x^2 - \frac{9t^2}{2} uu_{xx} \right).
\]

Then we substitute (17.12), (17.13) in the second component of the vector (16.14), transfer the term $D_x(\ldots)$ from $C^1$ to $C^2$, multiply the resulting vector $(C^1, C^2)$ by $2/3$ and arrive at the following vector:

\[
C^1 = u^2 - 2\varepsilon \left[ xu + \frac{3t}{2} u^2 \right], \tag{17.15}
\]
\[
C^2 = u_x^2 - 2\varepsilon u^2 + 2uu_{xx} + \varepsilon \left[ xu^2 - 2u_x + 2xu_{xx} + 2tu^3 - 3tu_x^2 + 6tuu_{xx} \right].
\]

The approximate conservation law (17.10) for the vector (17.15) is satisfied in the following form:

\[
D_t(C^1) + D_x(C^2) = 2u(u_t - u_{xxx} - uu_x - \varepsilon u) - 2\varepsilon (x + 3tu)(u_t - u_{xxx} - uu_x) + o(\varepsilon).
\]

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Abstract. The paper is devoted to the group analysis of equations of motion of two-dimensional uniformly stratified rotating fluids used as a basic model in geophysical fluid dynamics. It is shown that the system of the nonlinear equations in question is self-adjoint. This property is used for constructing conservation laws associated with the symmetries of the system. The group invariant solutions are investigated and used for defining internal wave beams.

Keywords: Self-adjointness, Formal Lagrangian, Energy, Invariant solutions, Internal wave beams.

1 Introduction

One of basic models in geophysical fluid dynamics is provided by the Euler equations in the rotating reference frame (see, e.g. [16], Section 7.2):

\[ \begin{align*}
    \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{\rho} \nabla p - 2\Omega \times \mathbf{u} - g \mathbf{k}, \\
    \rho_t + \mathbf{u} \cdot \nabla \rho &= 0, \\
    \text{div} \mathbf{u} &= 0,
\end{align*} \]

(1.1)

where \( \mathbf{u} = (u, v, w) \) is the velocity, \( \Omega \) is rate of the rotation of the earth, the constant \( g \) is the gravitational acceleration, and \( \mathbf{k} \) is the unit vector in the vertical direction.
We will apply Lie group analysis to the system of nonlinear equations

\[ \Delta \psi_t - g \rho_x - f v_z = \psi_x \Delta \psi_z - \psi_z \Delta \psi_x, \]

\[ v_t + f \psi_z = \psi_x v_z - \psi_z v_x, \]

\[ \rho_t + N^2 \frac{\psi_x}{g} \psi = \psi_x \rho_z - \psi_z \rho_x, \]

obtained from Eqs. (1.1) in the Boussinesq approximation, rooted on the assumption that the density does not depart much from its mean value (for details, see e.g. [45], Part I, and the references therein). Here \( \psi \) is the stream function related to the components \( u \) and \( w \) of the velocity \( \mathbf{u} \) by

\[ u = \psi_z, \quad w = -\psi_x, \]

the latter equations being compatible due to the incompressibility condition

\[ u_x + w_z = 0. \]

In Eqs. (1.2) \( \Delta \) is the two-dimensional Laplacian:

\[ \Delta = D_x^2 + D_z^2, \quad \text{e.g.} \quad \Delta \psi_t = \frac{\partial^2 \psi_t}{\partial x^2} + \frac{\partial^2 \psi_t}{\partial z^2} \equiv D_t(\Delta \psi), \]

and \( f, N \) are constants. Namely, \( f \) is the Coriolis frequency defined as being twice the vertical component of the Coriolis force:

\[ f = 2\Omega \sin \theta, \]

where \( \theta \) is the latitude. The quantity \( N \) is the buoyancy frequency that appears due to the density stratification of a fluid and is constant under the linear stratification hypothesis (see [45], Chapter 2).

The system (1.2) is used in geophysical fluid dynamics for investigating plane internal waves in uniformly stratified incompressible fluids (oceans). For instance, Eqs. (1.2) were used in [55], in the particular case \( f = 0 \), to study two non-unidirectional wave beams propagating and interacting in stratified fluid. An exact solution of the same system, again in the case when \( f = 0 \), was employed in [61] for investigating stability of a single internal plane wave. Weakly nonlinear effects in colliding of internal wave beams were investigated in [77], [78] by using Eqs. (1.2) with \( f = 0 \). The system (1.2) with \( f \neq 0 \) was used in [53] to model weakly nonlinear wave interactions governing the time behavior of the oceanic energy spectrum.

We will show in what follows that the system of equations (1.2) is self-adjoint (in the terminology of [28, 29], see also [38]) and use this remarkable
property of the system for calculating conservation laws associated with symmetry properties of the system (1.2).

In some calculations, e.g. in Sections 5.6, 5.4, 5.7 it is convenient to write Eqs. (1.2) by using the Jacobians $J(\psi, v) = \psi_x v_z - \psi_z v_x$, etc., in the following form:

$$
\begin{align*}
\Delta \psi_t - g \rho_x - f v_z &= J(\psi, \Delta \psi), \\
v_t + f \psi_z &= J(\psi, v), \\
\rho_t + \frac{N^2}{g} \psi_x &= J(\psi, \rho).
\end{align*}
$$

(1.3)

2 Symmetries

2.1 Obvious symmetries

The equations (1.2) do not contain the dependent and independent variables explicitly and therefore they are invariant with respect to addition of arbitrary constants to all these variables. It means that Eqs. (1.2) admit the one-parameter groups of translations in all variables,

$$
\bar{\psi} = \psi + a_3, \quad \bar{\rho} = \rho + a_2, \quad \bar{\psi} = \psi + a_3, \quad \bar{t} = t + a_4, \quad \bar{x} = x + a_5, \quad \bar{z} = z + a_6,
$$

with the generators

$$
\begin{align*}
X_1 &= \frac{\partial}{\partial \psi}, & X_2 &= \frac{\partial}{\partial \rho}, & X_3 &= \frac{\partial}{\partial \psi}, \\
X_4 &= \frac{\partial}{\partial t}, & X_5 &= \frac{\partial}{\partial x}, & X_6 &= \frac{\partial}{\partial z}.
\end{align*}
$$

(2.1)

One can also find by simple calculations the dilations (scaling)

$$
\bar{\psi} = a \psi, \quad \bar{\rho} = b \rho, \quad \bar{\psi} = c \psi, \quad t = \alpha \bar{t}, \quad x = \beta \bar{x}, \quad z = \beta \bar{z}
$$

(2.2)

admitted by Eqs. (1.2). These transformations are defined near the identity transformation if the parameters $a, \ldots, \beta$ are positive. The dilations of $x$ and $z$ are taken by the same parameter $\beta$ in order to keep invariant the Laplacian $\Delta$. Let us find the parameters $a, \ldots, \beta$ from the invariance condition of Eqs. (1.2). The transformations (2.2) change the derivatives
involved in Eqs. (1.2) as follows:
\[
\begin{align*}
\ddot{v}_t &= a\alpha v_t, \quad \ddot{v}_x = a\beta v_x, \quad \ddot{v}_z = a\beta v_z, \\
\ddot{\rho}_t &= b\alpha \rho_t, \quad \ddot{\rho}_x = b\beta \rho_x, \quad \ddot{\rho}_z = b\beta \rho_z, \\
\ddot{\psi}_t &= c\alpha \psi_t, \quad \ddot{\psi}_x = c\beta \psi_x, \quad \ddot{\psi}_z = c\beta \psi_z, \\
\ddot{\Delta}_\psi &= c\alpha^2 \Delta \psi_t, \quad \ddot{\Delta}_\psi_x = c\beta^2 \Delta \psi_x, \quad \ddot{\Delta}_\psi_z = c\beta^2 \Delta \psi_z,
\end{align*}
\]

where \( \ddot{\Delta} \) is the Laplacian written in the variables \( \bar{x}, \bar{z} \). The invariance of Eqs. (1.2) under the transformations (2.2) means that the following equations are satisfied:
\[
\begin{align*}
\ddot{\Delta}_\psi_t - g\ddot{\rho}_x - f\ddot{\psi}_x - \psi_x \ddot{\Delta}_\psi_x + \psi_x \ddot{\Delta}_\psi_x &= 0, \\
\dddot{v} + f\ddot{\psi}_x - \ddot{\psi}_x \dddot{v} + \dddot{v} \ddot{\psi}_x &= 0, \\
\dddot{\rho} + \frac{N^2}{g} \ddot{\psi}_x - \ddot{\psi}_x \dddot{\rho} + \dddot{\psi} \ddot{\rho} &= 0,
\end{align*}
\]

whenever Eqs. (1.2) hold. Substituting here the expressions (2.3) we have:
\[
\begin{align*}
\ddot{\Delta}_\psi_t - g\ddot{\rho}_x - f\ddot{\psi}_x - \psi_x \ddot{\Delta}_\psi_x + \psi_x \ddot{\Delta}_\psi_x &= c\alpha^2 \ddot{\psi}_t - b\beta \ddot{\psi}_x - c^2 \beta^4 (\psi_x \ddot{\psi}_x - \psi_x \ddot{\psi}_x), \\
\dddot{v} + f\ddot{\psi}_x - \ddot{\psi}_x \dddot{v} + \dddot{v} \ddot{\psi}_x &= a\alpha v_t + c\beta f \psi_x - a\beta^2 (\psi_x v_x - \psi_x v_x), \\
\dddot{\rho} + \frac{N^2}{g} \ddot{\psi}_x - \ddot{\psi}_x \dddot{\rho} + \dddot{\psi} \ddot{\rho} &= b\alpha \rho_t + c\beta^2 N^2 \psi_x - b\beta^2 (\psi_x \rho_x - \psi_x \rho_x).
\end{align*}
\]

These equations show that the invariance of Eqs. (1.2) is guaranteed by the following six equations for five undetermined parameters \( a, b, c, \alpha, \beta \):
\[
\begin{align*}
c\alpha \beta^2 &= b\beta = c^2 \beta^4, & a\alpha &= c\beta = ac\beta^2, & b\alpha &= c\beta = bc\beta^2.
\end{align*}
\]

It can be verified by simple calculations that Eqs. (2.4) yield
\[
\begin{align*}
\alpha &= 1, & b &= a, & c &= a^2, & \beta &= \frac{1}{a},
\end{align*}
\]

where \( a \) is an arbitrary parameter. We substitute these values of the parameters in (2.2), denote the positive parameter \( a \) by \( e^a \), drop the tilde and conclude that Eqs. (1.2) admit the non-uniform dilation group
\[
\begin{align*}
\tilde{t} &= t, \quad \tilde{x} = xe^a, \quad \tilde{z} = ze^a, \quad \tilde{v} = ve^a, \quad \tilde{\rho} = \rho e^a, \quad \tilde{\psi} = \psi e^{2a},
\end{align*}
\]

with the generator
\[
X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}.
\]

The operators (2.1)-(2.5) are called obvious symmetries of Eqs. (1.2).
2.2 Maximal algebra of symmetries

The determining equations for Lie point symmetries of Eqs. (1.2) have been solved using the DIMSYM 2.3 package. The computation shows that Eqs. (1.2) with arbitrary constants $f$ and $N$, under the assumption that $f \neq 0$, admit the infinite-dimensional Lie algebra spanned by the operators

\[
X_1 = \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_{a(t)} = a(t) \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t},
\]

\[
X_{b(t)} = b(t) \left[ \frac{\partial}{\partial x} - f \frac{\partial}{\partial v} \right] + b'(t) z \frac{\partial}{\partial \psi},
\]

\[
X_{c(t)} = c(t) \left[ \frac{\partial}{\partial z} + \frac{N^2}{g} \frac{\partial}{\partial \rho} \right] - c'(t) x \frac{\partial}{\partial \psi},
\]

\[
(2.6)
\]

\[
X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2 \psi \frac{\partial}{\partial \psi},
\]

\[
X_8 = t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3 \psi \frac{\partial}{\partial \psi} - 2f x \frac{\partial}{\partial v} + 2 \frac{N^2}{g} z \frac{\partial}{\partial \rho},
\]

\[
X_9 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} - \frac{1}{f} \left[ g \rho + (f^2 - N^2) z \right] \frac{\partial}{\partial v} + \frac{1}{g} \left[ f v + (f^2 - N^2) x \right] \frac{\partial}{\partial \rho},
\]

where $a(t)$, $b(t)$ and $c(t)$ are arbitrary functions.

**Remark 2.1.** In the case $f = 0$ the maximal Lie algebra of symmetries is spanned by the operators

\[
X_1 = h(v, g \rho - N^2 z) \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = a(t) \frac{\partial}{\partial \psi},
\]

\[
X_4 = \frac{\partial}{\partial t}, \quad X_5 = b(t) \frac{\partial}{\partial x} + b'(t) z \frac{\partial}{\partial \psi},
\]

\[
X_6 = c(t) \left[ \frac{\partial}{\partial z} + \frac{N^2}{g} \frac{\partial}{\partial \rho} \right] - c'(t) x \frac{\partial}{\partial \psi},
\]

\[
(2.7)
\]

\[
X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2 \psi \frac{\partial}{\partial \psi},
\]

\[
X_8 = t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3 \psi \frac{\partial}{\partial \psi} + 2 \frac{N^2}{g} z \frac{\partial}{\partial \rho},
\]

where $h(v, g \rho - N^2 z)$ is an arbitrary function of two variables.
3 Self-adjointness

3.1 Preliminaries

We will use the terminology and the following definitions from [28, 29] (see also [27]).

Let \( x = (x^1, \ldots, x^n) \) be \( n \) independent variables, and \( u = (u^1, \ldots, u^m) \) be \( m \) dependent variables. The partial derivatives of \( u^\alpha \) with respect to \( x^i \) are denoted by \( u^\alpha_i = \{u^\alpha_i\}, u^\alpha_{ij} = \{u^\alpha_{ij}\}, \ldots \) with

\[
D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, \ldots, n.
\]  

(3.1)

Even though the operators \( D_i \) are given by formal infinite sums, their action \( D_i(f) \) is well defined for functions \( f(x, u, u^{(1)}, \ldots) \) depending on a finite number of the variables \( x, u, u^{(1)}, u^{(2)}, \ldots \). The usual summation convention on repeated indices \( \alpha \) and \( i \) is assumed in expressions like Eq. (3.1).

The variational derivatives (the Euler-Lagrange operator) are defined by

\[
\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_i \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m,
\]  

(3.2)

where the summation over the repeated indices \( i_1 \ldots i_s \) runs from 1 to \( n \).

**Definition 3.1.** The adjoint equations to nonlinear partial differential equations

\[
F^{\alpha}(x, u, \ldots, u^{(s)}) = 0, \quad \alpha = 1, \ldots, m,
\]  

(3.3)

are given by (see also [5])

\[
F^{\alpha*}(x, u, \mu, \ldots, u^{(s)}, \mu^{(s)}) = 0, \quad \alpha = 1, \ldots, m,
\]  

(3.4)

where \( \mu = (\mu^1, \ldots, \mu^m) \) are new dependent variables, and \( F^{\alpha*} \) are defined by

\[
F^{\alpha*}(x, u, \mu, \ldots, u^{(s)}, \mu^{(s)}) = \frac{\delta (\mu^\beta F_\beta)}{\delta u^\alpha}.
\]  

(3.5)

In the case of linear equations, Definition 3.1 is equivalent to the classical definition of the adjoint equation.

Consider the function

\[
\mathcal{L} = \mu^\beta F_\beta(x, u, \ldots, u^{(s)})
\]  

(3.6)
Eq. (3.3) and their adjoint equations (3.4) can be obtained from (3.5) by taking the variational derivatives (3.2) with respect to the dependent variables $u$ and the similar variational derivatives with respect to the new dependent variables $\mu$,

$$\frac{\delta}{\delta \mu^\alpha} = \frac{\partial}{\partial \mu^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \mu^{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \quad (3.7)$$

Namely:

$$\frac{\delta L}{\delta \mu^\alpha} = F^\alpha(x, u, \ldots, u^{(s)}), \quad (3.8)$$

$$\frac{\delta L}{\delta u^\alpha} = F^*\alpha(x, u, \mu, \ldots, u^{(s)}, \mu^{(s)}). \quad (3.9)$$

This circumstance justifies the following definition.

**Definition 3.2.** The differential function (3.6) is called a *formal Lagrangian* for the differential equations (3.3). For the sake of brevity, formal Lagrangians are also referred to as Lagrangians.

If the variables $u$ are known, the new variables $\mu$ are obtained by solving Eqs. (3.4) which are, according to (3.5), linear partial differential equations (3.4) with respect to $\mu^\alpha$. Using the existing terminology (see, e.g. [1]), we will call $\mu^\alpha$ *nonlocal variables*.

Nonlocal variables can be excluded from physical quantities such as conservation laws if Eqs. (3.3) are *self-adjoint* ([28]) or, in general, *quasi-self-adjoint* ([30]) in the following sense.

**Definition 3.3.** Eqs. (3.3) are said to be *self-adjoint* if the system obtained from the adjoint equations (3.4) by the substitution $\mu = u$:

$$F^\alpha(x, u, \ldots, u^{(s)}, u^{(s)}) = 0, \quad \alpha = 1, \ldots, m, \quad (3.10)$$

is equivalent to the original system (3.3), i.e.

$$F^\alpha(x, u, \ldots, u^{(s)}, u^{(s)}) = \Phi^\beta_\alpha F_\beta(x, u, \ldots, u^{(s)}), \quad \alpha = 1, \ldots, m,$$

with regular (in general, variable) coefficients $\Phi^\beta_\alpha$.

**Definition 3.4.** Eqs. (3.3) are said to be quasi-self-adjoint if the system of adjoint equations (3.4) becomes equivalent to the original system (3.3) upon the substitution

$$\mu = h(u) \quad (3.11)$$

with a certain function $h(u)$ such that $h'(u) \neq 0$. 
3.2 Adjoint system to Eqs. (1.2)

Let us apply the methods from Section 3.1 to Eqs. (1.2). In this case the formal Lagrangian (3.6) for Eqs. (1.2) is written

\[ L = \varphi [\Delta \psi_t - gp_x - fv_z - \psi_x \Delta \psi_z + \psi_z \Delta \psi_x] \]

\[ + \mu [v_t + f \psi_z - \psi_x v_z + \psi_z v_x] + r \left[ \rho_t + \frac{N^2}{g} \psi_x - \psi_x \rho_z + \psi_z \rho_x \right], \quad (3.12) \]

where \( \varphi, \mu \) and \( r \) are new dependent variables. The adjoint equations to Eqs. (1.2) are obtained by taking the variational derivatives of \( L \), namely:

\[ \frac{\delta L}{\delta \psi} = 0, \quad \frac{\delta L}{\delta v} = 0, \quad \frac{\delta L}{\delta \rho} = 0, \quad (3.13) \]

where (see (3.2); see also Eqs. (4.6))

\[ \frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_z \frac{\partial}{\partial v_z}, \]

\[ \frac{\delta}{\delta \rho} = \frac{\partial}{\partial \rho} - D_x \frac{\partial}{\partial \rho_x} - D_z \frac{\partial}{\partial \rho_z}, \]

\[ \frac{\delta}{\delta \psi} = \frac{\partial}{\partial \psi} - D_x \frac{\partial}{\partial \psi_x} - D_z \frac{\partial}{\partial \psi_z} + D_x D_t \frac{\partial}{\partial \psi_t} + D_z D_t \frac{\partial}{\partial \psi_z} + \cdots. \]

Taking into account the special form (3.12) of \( L \), we have:

\[ \frac{\delta L}{\delta \psi} = -D_x \frac{\partial L}{\partial \psi_x} - D_z \frac{\partial L}{\partial \psi_z} - (D_x^2 + D_z^2) \left[ D_t \frac{\partial L}{\partial \Delta \psi_t} + D_x \frac{\partial L}{\partial \Delta \psi_x} + D_z \frac{\partial L}{\partial \Delta \psi_z} \right] \]

\[ = D_x (\varphi \Delta \psi_z + \mu v_z - \frac{N^2}{g} r + r \rho_z) - D_z (\varphi \Delta \psi_x + f \mu + \mu v_x + r \rho_x) \]

\[ - (D_x^2 + D_z^2) \left[ D_t (\varphi) + D_x (\varphi \psi_z) - D_z (\varphi \psi_x) \right] \]

\[ = \varphi_x \Delta \psi_z - \varphi_z \Delta \psi_x + \mu_x v_z - \frac{N^2}{g} r_x + r_x \rho_z - f \mu_x - \mu_z v_x - r_z \rho_x \]

\[ - \Delta \varphi_t + 2 \left[ \varphi_{xx} \psi_{xx} + \varphi_{zz} \psi_{zz} - \varphi_{xx} \psi_{xz} - \varphi_{zz} \psi_{zx} \right], \]

\[ \frac{\delta L}{\delta v} = -D_t \frac{\partial L}{\partial v_t} - D_x \frac{\partial L}{\partial v_x} - D_z \frac{\partial L}{\partial v_z} = -\mu_t - \mu_x \psi_z + f \varphi_z + \mu_z \psi_x, \]

\[ \frac{\delta L}{\delta \rho} = -D_t \frac{\partial L}{\partial \rho_t} - D_x \frac{\partial L}{\partial \rho_x} - D_z \frac{\partial L}{\partial \rho_z} = -r_t + g \varphi_x - r_x \psi_z + r_z \psi_x. \]
Hence, the adjoint equations (3.13) can be written as follows:

\begin{align}
\Delta \varphi_t + \frac{N^2}{g} r_x + f \mu_z - \varphi_x \Delta \psi_z + \varphi_z \Delta \psi_x - \Theta &= 0, \quad (3.14) \\
- \mu_t - \mu_x \varphi_z + f \varphi_z + \mu_z \psi_x &= 0, \quad (3.15) \\
- r_t + g \varphi_x - r_x \psi_z + r_z \psi_x &= 0, \quad (3.16)
\end{align}

where

\[ \Theta = J(\mu, v) + J(r, \rho) + 2 \left[ \varphi_{zz} \psi_{xx} + \varphi_{xx} \psi_{zz} - \varphi_{xx} \psi_{zz} - \varphi_{xx} \psi_{xx} \right]. \quad (3.17) \]

### 3.3 Self-adjointness of Eqs. (1.2)

**Theorem 3.1.** The system (1.2) satisfies the quasi-self-adjointness condition with the substitution (3.11) of the following form:

\[ \varphi = \psi, \quad \mu = -v, \quad r = -\frac{g^2}{N^2} \rho. \quad (3.18) \]

**Proof.** One can verify that after the substitution (3.18) the quantity \( \Theta \) given by Eq. (3.17) vanishes. Therefore the adjoint equations (3.14)-(3.16) become identical with Eqs. (1.2) after the substitution (3.18). Hence, according to Definition 3.4, Eqs. (1.2) are quasi-self-adjoint. Since Eqs. (3.18) are obtained just by simple scaling of the equations \( \varphi = \psi, \mu = v, r = \rho \) required for the (strict) self-adjointness, we will say that Eqs. (1.2) are self-adjoint.

### 4 General discussion of conservation laws

#### 4.1 Preliminaries

Along with the individual notation \( t, x, z \) for the the independent variables, and \( v, \rho, \psi \) for the dependent variables, we will also use the index notation \( x^1 = t, \ x^2 = x, \ x^3 = z \) and \( u^1 = v, \ u^2 = \rho, \ u^3 = \psi \), respectively. We will write the conservation laws both in the differential form

\[ D_t(C^1) + D_x(C^2) + D_z(C^3) = 0 \quad (4.1) \]

and the integral form

\[ \frac{d}{dt} \int \int C^1 dx dz = 0, \quad (4.2) \]
where the double integral in taken over the the \((x, z)\) plane \(\mathbb{R}^2\). The equations (4.1) and (4.2) provide a conservation law for Eqs. (1.2) if they hold for the solutions of Eqs. (1.2). The vector \(\mathbf{C} = (C^1, C^2, C^3)\) satisfying the conservation equation (4.1) is termed a conserved vector. Its component \(C^1\) is called the density of the conservation law due to Eq. (4.2). The two-dimensional vector \((C^2, C^3)\) defines the flux of the conservation law.

The integral form (4.2) of a conservation law follows from the differential form (4.1) provided that the solutions of Eqs. (1.2) vanish or rapidly decrease at the infinity on \(\mathbb{R}^2\). Indeed, integrating Eq. (4.1) over an arbitrary region \(\Omega \subset \mathbb{R}^2\) we have:

\[
\frac{d}{dt} \int \int_{\Omega} C^1 \, dx \, dz = - \int \int_{\Omega} \left[ D_x(C^2) + D_z(C^3) \right] \, dx \, dz.
\]

According to Green’s theorem, the integral on the right-hand side reduces to the integral along the boundary \(\partial \Omega\) of \(\Omega\):

\[
- \int \int_{\Omega} \left[ D_x(C^2) + D_z(C^3) \right] \, dx \, dz = \int_{\partial \Omega} C^3 \, dx - C^2 \, dz,
\]

and hence vanishes as \(\Omega\) expands and becomes the plane \(\mathbb{R}^2\).

**Remark 4.1.** It is manifest from this discussion that one can ignore in \(C^1\) “divergent type” terms because they do not change the integral in the conservation equation (4.2). Specifically if \(C^1\) evaluated on the solutions of Eqs. (1.2) has the form

\[
C^1 = \widetilde{C}^1 + D_x(h^2) + D_z(h^3) \tag{4.3}
\]

with some functions \(h^2, h^3\), then the conservation equation (4.1) can be equivalently rewritten in the form (see [34], Paper 1, Section 20.1)

\[
D_t(\widetilde{C}^1) + D_x(\widetilde{C}^2) + D_z(\widetilde{C}^3) = 0,
\]

where

\[
\widetilde{C}^2 = C^2 + D_t(h^2), \quad \widetilde{C}^3 = C^3 + D_t(h^3).
\]

Accordingly, we have

\[
\int \int C^1 \, dx \, dz = \int \int \widetilde{C}^1 \, dx \, dz,
\]

and hence the integral conservation equation (4.2) provided by the conservation density \(C^1\) of the form (4.3) coincides with that provided by the density \(\widetilde{C}^1\).
In particular, if $\tilde{C}_1 = 0$ the integral in Eq. (4.2) vanishes. This kind of conservation laws are *trivial* from physical point of view. Therefore we single out physically useless conservation laws by the following definition.

**Definition 4.1.** The conservation law is said to be *trivial* if its density $C^1$ evaluated on the solutions of Eqs. (1.2) is the divergence,

$$C^1 = D_x(h^2) + D_z(h^3).$$

The following statement ([26], Section 8.4.1; see also [29]) simplifies calculations while dealing with conservation equations.

**Lemma 4.1.** A function $F(v, \rho, \psi, v_x, v_z, \rho_x, \rho_z, \psi_x, \psi_z, \psi_{xt}, \psi_{zt}, \ldots)$ is the divergence,

$$F = D_x(C^1) + D_z(C^2),$$

if and only if satisfies the following equations:

$$\frac{\delta F}{\delta v} = 0, \quad \frac{\delta F}{\delta \rho} = 0, \quad \frac{\delta F}{\delta \psi} = 0. \quad (4.5)$$

Here the variational derivatives act on $F$ as usual (see also Section 3.2):

$$\begin{align*}
\frac{\delta F}{\delta v} &= \frac{\partial F}{\partial v} - D_x \left( \frac{\partial F}{\partial v_x} \right) - D_z \left( \frac{\partial F}{\partial v_z} \right), \\
\frac{\delta F}{\delta \rho} &= \frac{\partial F}{\partial \rho} - D_x \left( \frac{\partial F}{\partial \rho_x} \right) - D_z \left( \frac{\partial F}{\partial \rho_z} \right), \\
\frac{\delta F}{\delta \psi} &= \frac{\partial F}{\partial \psi} - D_x \left( \frac{\partial F}{\partial \psi_x} \right) - D_z \left( \frac{\partial F}{\partial \psi_z} \right) \\
&\quad + D_x D_t \left( \frac{\partial F}{\partial \psi_{xt}} \right) + D_z D_t \left( \frac{\partial F}{\partial \psi_{zt}} \right) + \cdots.
\end{align*} \quad (4.6)$$

**Corollary 4.1.** A function $C^1$ is the density of a conservation law (4.1) if and only if the function

$$F = D_t(C^1) \bigg|_{(1.2)}$$

satisfies Eqs. (4.5). Here $\bigg|_{(1.2)}$ means that the quantity $D_t(C^1)$ is evaluated on the solutions of Eqs. (1.2).

In particular, Lemma 4.1 allows one to single out trivial conservation laws as follows.
Corollary 4.2. The conservation law (4.1) is trivial if and only if its density $C^1$ evaluated on the solutions of Eqs. (1.2), i.e. the quantity

$$C^1_* = C^1|_{(1.2)}$$

satisfies Eqs. (4.5),

$$\frac{\delta C^1_*}{\delta v} = 0, \quad \frac{\delta C^1_*}{\delta \rho} = 0, \quad \frac{\delta C^1_*}{\delta \psi} = 0,$$

(4.9)
on the solutions of Eqs. (1.2).

4.2 Variational derivatives of expressions with Jacobians

We will use in our calculations the following statement on the behaviour of certain expressions with Jacobians under the action of the variational derivatives (4.6).

Proposition 4.1. The following equations hold:

$$\frac{\delta J(\psi, v)}{\delta v} = 0, \quad \frac{\delta J(\psi, v)}{\delta \psi} = 0,$$

(4.10)

$$\frac{\delta [vJ(\psi, v)]}{\delta v} = 0, \quad \frac{\delta [vJ(\psi, v)]}{\delta \psi} = 0,$$

(4.11)

$$\frac{\delta [\rho J(\psi, \rho)]}{\delta \rho} = 0, \quad \frac{\delta [\rho J(\psi, \rho)]}{\delta \psi} = 0,$$

(4.12)

$$\frac{\delta J(\psi, \Delta \psi)}{\delta \psi} = 0, \quad \frac{\delta [\psi J(\psi, \Delta \psi)]}{\delta \psi} = 0.$$

(4.13)

Proof. Let us verify that the first equation (4.10) holds. We have (see Eqs. (4.6)):

$$\frac{\delta J(\psi, v)}{\delta v} = \frac{\delta (\psi_x v_z - \psi_z v_x)}{\delta v} = -D_z(\psi_x) + D_x(\psi_z) = -\psi_{xz} + \psi_{zx} = 0.$$  

Replacing $v$ by $\psi$ one obtains the second equation (4.10). Let us verify now that Eqs. (4.11) are satisfied. We have:

$$\frac{\delta [vJ(\psi, v)]}{\delta v} = \frac{\delta [v(\psi_x v_z - \psi_z v_x)]}{\delta v} = \frac{\partial [v(\psi_x v_z - \psi_z v_x)]}{\partial v} - D_z(v\psi_x) + D_x(v\psi_z)$$

$$= J(\psi, v) - D_z(v\psi_x) + D_x(v\psi_z) = J(\psi, v) - J(\psi, v) - v\psi_{xz} + v\psi_{zx} = 0.$$
and
\[
\frac{\delta[v J(\psi, v)]}{\delta \psi} = -D_x(v v_z) + D_z(v v_x) = -v_x v_z - v v_{zx} + v_z v_x + v v_{xx} = 0.
\]
Replacing \( v \) by \( \rho \) one obtains Eqs. (4.12). Eqs. (4.13) are derived likewise even though they involve higher-order derivatives. We have:
\[
\delta J(\psi, \Delta \psi) = \delta \left[ \psi_x (\psi_{xxx} + \psi_{xzz}) - \psi_z (\psi_{xxx} + \psi_{xzz}) \right] = 0.
\]

Derivation of the second equation (4.13) requires only a simple modification of the previous calculations. Namely:
\[
\delta \left[ \psi J(\psi, \Delta \psi) \right] = \delta \left[ \psi_x (\Delta \psi_z) + \psi_z (\Delta \psi_x) - \Delta D_x (\psi_x) + \Delta D_z (\psi_z) \right] = 0.
\]

4.3 Conserved vectors associated with symmetries

It has been demonstrated in [27, 29] that for any operator
\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}
\]

admitted by the system (1.2), the quantities
\[
C^i = \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_j^\alpha} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) \right] + D_j (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] + D_j D_k (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right], \quad i = 1, 2, 3,
\]

define the components of a conserved vector for Eqs. (1.2) considered together with the adjoint equations (3.14)-(3.16). Here
\[
W^\alpha = \eta^\alpha - \xi^i u_i^\alpha, \quad \alpha = 1, 2, 3.
\]
The formula (4.15) is written by taking into account that the Lagrangian (3.12) involves the derivatives up to third order. Moreover, noting that the Lagrangian (3.12) vanishes on the solutions of Eqs. (1.2), we can drop the first term in (4.15) and use the conserved vector in the abbreviated form

\[ C^i = \tilde{W}^i [\partial \tilde{L} / \partial u^\alpha_{ij} - D_j \left( \partial \tilde{L} / \partial u^\alpha_{ij} \right) + D_j D_k \left( \partial \tilde{L} / \partial u^\alpha_{ijk} \right) ] \]

(4.17)

For computing the conserved vectors (4.17), the Lagrangian (3.12) containing the mixed derivatives should be written in the symmetric form

\[ \mathcal{L} = \frac{1}{3} \varphi \left[ \psi_{txx} + \psi_{xxt} + \psi_{xtx} + \psi_{txz} + \psi_{ztx} - 3g \varphi_x - 3f \varphi_z \right] \]

(4.18)

\[ - \psi_x \left( \psi_{xxx} + \psi_{xzx} + 3 \psi_{xzz} \right) + \psi_z \left( 3 \psi_{xxx} + \psi_{xzx} + \psi_{xxz} + \psi_{zzx} \right) \]

\[ + \mu \left[ v + f \varphi_z - \varphi_x v_x + \varphi_z v_x \right] + r \left[ \rho_t + \frac{N^2}{g} \varphi_x - \varphi_z \rho_z + \varphi_z \rho_x \right]. \]

Since the Lagrangian \( \mathcal{L} \), and hence the components (4.17) of a conserved vector contain the nonlocal variables \( \varphi, \mu, r \), we obtain in this way nonlocal conserved vectors.

### 4.4 Computation of conserved vectors

The substitution of (4.18) in (4.17) yields:

\[ C^1 = W^1 \frac{\partial \tilde{L}}{\partial v_t} + W^2 \frac{\partial \tilde{L}}{\partial \rho_t} + W^3 \left[ D_x \left( \frac{\partial \tilde{L}}{\partial \psi_{txx}} \right) + D_z \left( \frac{\partial \tilde{L}}{\partial \psi_{txz}} \right) \right] \]

\[ - \left[ D_x (W^3) D_x \left( \frac{\partial \tilde{L}}{\partial \psi_{txx}} \right) + D_z (W^3) D_z \left( \frac{\partial \tilde{L}}{\partial \psi_{txz}} \right) \right] \]

\[ + D_x^2 (W^3) \frac{\partial \tilde{L}}{\partial \psi_{txx}} + D_z^2 (W^3) \frac{\partial \tilde{L}}{\partial \psi_{txz}}, \]

or

\[ C^1 = W^1 \mu + W^2 r + \frac{1}{3} W^3 \left[ D_x^2 (\varphi) + D_z^2 (\varphi) \right] \]

(4.19)

\[ - \frac{1}{3} \left[ \varphi_x D_x (W^3) + \varphi_z D_z (W^3) \right] + \frac{1}{3} \varphi \left[ D_x^2 (W^3) + D_z^2 (W^3) \right]. \]
Furthermore, using the same procedure, we obtain:

\[
C^2 = W^1 \frac{\partial \mathcal{L}}{\partial v_x} + W^2 \frac{\partial \mathcal{L}}{\partial \rho_x} + W^3 \left[ \frac{\partial \mathcal{L}}{\partial \psi_x} + D_x^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} \right) + D_z^2 \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} \right) \right] + D_t D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{txx}} + \frac{\partial \mathcal{L}}{\partial \psi_{xxt}} \right) \\
- D_x (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} \right) - D_t (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{txx}} \right) - D_x (W^3) D_t \left( \frac{\partial \mathcal{L}}{\partial \psi_{xtx}} \right) \\
- D_z (W^3) D_z \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} \right) - D_z (W^3) D_x \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxt}} \right) - D_z (W^3) D_t \left( \frac{\partial \mathcal{L}}{\partial \psi_{xzt}} \right) \\
+ D_x^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} + D_z^2 (W^3) \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} + D_t D_x (W^3) \left( \frac{\partial \mathcal{L}}{\partial \psi_{txx}} + \frac{\partial \mathcal{L}}{\partial \psi_{xxt}} \right) \\
+ D_x D_z (W^3) \left( \frac{\partial \mathcal{L}}{\partial \psi_{xxx}} + \frac{\partial \mathcal{L}}{\partial \psi_{xzz}} \right),
\]

or

\[
C^2 = W^1 \mu \psi_x + W^2 (r \psi_x - g \varphi) + W^3 \left[ -\Delta \psi_x - \mu v_x + \frac{N^2}{g} r - r \rho_x \right] + D_x^2 (\varphi \psi_x) + \frac{1}{3} D_z^2 (\varphi \psi_x) + \frac{2}{3} \varphi_{xt} - \frac{2}{3} D_x D_z (\varphi \psi_x) \right) \\
- D_x (W^3) \left[ D_x (\varphi \psi_x) + \frac{1}{3} \varphi_t - \frac{1}{3} D_z (\varphi \psi_x) \right] - \frac{1}{3} D_t (W^3) \varphi_x \\
- \frac{1}{3} D_z (W^3) \left[ D_z (\varphi \psi_x) - D_x (\varphi \psi_x) \right] + \left[ D_x^2 (W^3) + \frac{1}{3} D_z^2 (W^3) \right] \varphi \psi_x \\
+ \frac{2}{3} \varphi D_t D_x (W^3) - \frac{2}{3} \varphi \psi_x D_z D_x (W^3). \tag{4.20}
\]

Likewise we get

\[
C^3 = -W^1 (\mu \psi_x + f \varphi) - W^2 r \psi_x + W^3 \left[ -\Delta \psi_x + (f + v_x) \mu + r \rho_x \right] - \frac{1}{3} D_x^2 (\varphi \psi_x) - \frac{2}{3} D_z^2 (\varphi \psi_x) + \frac{2}{3} \varphi_{xt} + \frac{2}{3} D_x D_z (\varphi \psi_x) \right) \\
+ \frac{1}{3} D_x (W^3) \left[ D_x (\varphi \psi_x) - D_z (\varphi \psi_x) \right] - \frac{1}{3} D_t (W^3) \varphi_x \right) \\
- D_z (W^3) \left[ \frac{1}{3} \varphi_t - D_z (\varphi \psi_x) + \frac{1}{3} D_x (\varphi \psi_x) \right] \\
- \left[ \frac{1}{3} D_x^2 (W^3) + D_z^2 (W^3) \right] \varphi \psi_x + \frac{2}{3} \varphi D_t D_z (W^3) + \frac{2}{3} \varphi \psi_x D_z D_x (W^3). \tag{4.21}
\]
4.5 Elimination of new dependent variables

The quantities (4.19)-(4.21) define a nonlocal conserved vector because they contain the nonlocal variables \( \varphi, \mu, r \). In consequence, the conservation equation (4.1) requires not only the basic equations (1.2), but also the adjoint equations (3.14)-(3.16).

However, we can eliminate the nonlocal variables using the self-adjointness of Eqs. (1.2) thus transforming the nonlocal conserved vector into a local one. Namely, we substitute in Eqs. (4.19)-(4.21) the expressions (3.18) for \( \varphi, \mu, r \):

\[
\varphi = \psi, \quad \mu = -v, \quad r = \frac{g^2}{N^2} \rho.
\]

Then the adjoint equations (3.14)-(3.16) are satisfied for any solutions of the basic equations (1.2), and hence the quantities (4.19)-(4.21) satisfy the conservation equation (4.1) on all solutions of Eqs. (1.2).

Let us apply the procedure to \( C^1 \). We eliminate the nonlocal variables in (4.19) by substituting there the expressions (3.18) and write \( C^1 \) in the following form:

\[
C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 + \frac{1}{3} [W^3 \Delta \psi - \psi_x D_x (W^3) - \psi_z D_z (W^3)] + \frac{1}{3} \Delta \left( \psi W^3 \right),
\]

where

\[
\Delta \psi = D_x^2(\psi) + D_z^2(\psi), \quad \Delta W^3 = D_x^2(W^3) + D_z^2(W^3).
\]

We further simplify the expression for \( C^1 \) by using the identities

\[
W^3 D_x^2(\psi) = D_x [W^3 D_x(\psi)] - \psi_x D_x (W^3),
\]

\[
W^3 D_z^2(\psi) = D_z [W^3 D_z(\psi)] - \psi_z D_z (W^3)
\]

and

\[
\psi D_x^2(W^3) = D_x \left[ \psi D_x (W^3) \right] - \psi_x D_x (W^3),
\]

\[
\psi D_z^2(W^3) = D_z \left[ \psi D_z (W^3) \right] - \psi_z D_z (W^3).
\]

Then we have:

\[
C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 - \psi_x D_x (W^3) - \psi_z D_z (W^3) + \frac{1}{3} \Delta (\psi W^3). \quad (4.22)
\]

Dropping in (4.22) the divergent type term

\[
\frac{1}{3} \Delta (\psi W^3) = D_x \left[ \frac{1}{3} D_x (\psi W^3) \right] + D_z \left[ \frac{1}{3} D_z (\psi W^3) \right]
\]
in accordance with Remark 4.1, we finally obtain:

\[ C^1 = -v W^1 - \frac{g^2}{N^2} \rho W^2 - \psi_z D_x (W^3) - \psi_z D_z (W^3). \]  \hspace{1cm} (4.23)

We will not dwell on a similar modification of the expressions (4.20), (4.21) for the components \( C^2 \) and \( C^3 \) of conserved vectors. We will see further in Section 5.4 that they can be found by simpler calculations when a density \( C^1 \) is known.

5 Conserved vectors due to obvious symmetries

We will use here the obvious symmetries (2.1), (2.5) for computing the conserved vectors.

5.1 Translation of \( v \)

For the operator \( X_1 \) from (2.1) Eqs. (4.16) yield

\[ W^1 = 1, \quad W^2 = 0, \quad W^3 = 0. \]

Substituting these expressions in Eq. (4.23) we obtain

\[ C^1 = -v. \]

In this case the equations (4.20) and (4.21) are also simple. They are written

\[ C^2 = u \psi_z, \quad C^3 = -(u \psi_x + f \phi) \]

and upon using Eqs. (3.18) yield:

\[ C^2 = -v \psi_z, \quad C^3 = v \psi_x - f \psi. \]

Since any conserved vector is defined up to multiplication by an arbitrary constant, we change the sign of \( C^1, C^2, C^3 \) and obtain the following conserved vector:

\[ C^1 = v, \quad C^2 = v \psi_z, \quad C^3 = f \psi - v \psi_x. \]  \hspace{1cm} (5.1)

One can verify that the conservation equation (4.1) for the vector (5.1) coincides with second equation in (1.2):

\[ D_t (C^1) + D_x (C^2) + D_z (C^3) = v_t + v_x \psi_z + f \psi_z - v_z \psi_x. \]
5.2 Translation of $\rho$

For the operator $X_2$ from (2.1) Eqs. (4.16) yield

\[ W^1 = 0, \quad W^2 = 1, \quad W^3 = 0. \]

Substituting these expressions in Eq. (4.23) we obtain

\[ C^1 = -\frac{g^2}{N^2} \rho. \]

Furthermore, Eqs. (4.20), (4.21) and Eqs. (3.18) yield:

\[ C^2 = -g\psi - \frac{g^2}{N^2} \rho \psi_z, \quad C^3 = \frac{g^2}{N^2} \rho \psi_x. \]

Multiplying $C^1$, $C^2$, $C^3$ by $-N^2/g^2$ we arrive at the following conserved vector:

\[ C^1 = \rho, \quad C^2 = \frac{N^2}{g} \psi + \rho \psi_z, \quad C^3 = -\rho \psi_x. \quad (5.2) \]

One can readily verify that the conservation equation (4.1) for the vector (5.2) is also satisfied. Namely, it coincides with the third equation in (1.2).

5.3 Translation of $\psi$

For the operator $X_3$ from (2.1) Eqs. (4.16) yield

\[ W^1 = 0, \quad W^2 = 0, \quad W^3 = 1. \]

Substituting these expressions in Eq. (4.23) we obtain

\[ C^1 = 0. \]

Hence, the invariance of Eqs. (1.2) under the translation of $\psi$ furnishes only a trivial conservation law (see Definition 4.1).

5.4 Derivation of the flux of conserved vectors with known densities

We will show here how to find the components $C^2$ and $C^3$ of the conserved vector (5.1) without using Eqs. (4.20), (4.21), provided that we know the conserved density $C^1 = v$.

Let us first verify that $C^1 = v$ satisfies Corollary 4.1. In this case $D_t(C^1) = v_t$, and hence Eq. (4.7) yields

\[ F = D_t(C^1) \big|_{(1.2)} = J(\psi, v) - f\psi_z. \quad (5.3) \]
Using Proposition 4.1 we see that Eqs. (4.5) are satisfied:

$$\frac{\delta F}{\delta v} = \frac{\delta F}{\delta \rho} = 0, \quad \frac{\delta F}{\delta \psi} = D_z(f) = 0.$$  \hfill (5.4)

Therefore Corollary 4.1 guarantees that $F$ defined by Eq. (5.3) satisfies Eq. (4.4):

$$\psi_x v_z - \psi_z v_x - f \psi_z = D_z(H^1) + D_z(H^2)$$ \hfill (5.5)

with certain functions $H^1$, $H^2$.

To find $H^1$, $H^2$, we write

$$\psi_x v_z - f \psi_z = D_z(v \psi_x - f \psi) - v \psi_{xx}, \quad -\psi_z v_x = D_x(-v \psi_z) + v \psi_{xx}$$

and obtain:

$$\psi_x v_z - \psi_z v_x - f \psi_z = D_x(-v \psi_z) + D_z(v \psi_x - f \psi).$$

Thus, $H^1 = -v \psi_z$, $H^2 = v \psi_x - f \psi$. Denoting $C^2 = -H^1$, $C^3 = -H^2$, i.e.

$$C^1 = v \psi_z, \quad C^2 = f \psi - v \psi_x,$$

and invoking Eq. (5.3), we write Eq. (5.5) in the form

$$D_t(C^1)\big|_{(1,2)} + D_x(C^1) + D_z(C^2) = 0.$$  

This is precisely the conservation equation (4.1) for the vector (5.1). Thus, we have obtained the components $C^2$, $C^3$ of the conserved vector (5.1) without using Eqs. (4.20), (4.21).

The components $C^2$, $C^3$ of the conserved vector (5.2) can be derived likewise.

### 5.5 Translation of $x$

For the operator $X_5$ from (2.1) Eqs. (4.16) yield

$$W^1 = -v_x, \quad W^2 = -\rho_x, \quad W^3 = -\psi_x.$$  

Substituting these expressions in Eq. (4.23) we obtain

$$C^1 = vv_x + \frac{g^2}{N^2} \rho \rho_x + \psi_x \psi_{xx} + \psi_x \psi_{xz} = D_x \left( \frac{1}{2} v^2 + \frac{1}{2} g^2 \rho^2 + \frac{1}{2} \psi_x^2 + \frac{1}{2} \psi_z^2 \right).$$

Hence, the invariance of Eqs. (1.2) under the translation of $x$ furnishes only a trivial conservation law (see Definition 4.1). Similar calculations show that the invariance under the translation of $z$ provides also a trivial conservation law.
5.6 Time translation

For the operator $X_4$ from (2.1) Eqs. (4.16) yield

$$W^1 = -v_t, \quad W^2 = -\rho_t, \quad W^3 = -\psi_t.$$  

Substituting these expressions in Eq. (4.23) we obtain

$$C^1 = vv_t + \frac{g^2}{N^2} \rho \rho_t + \psi_x \psi_{xt} + \psi_z \psi_{zt}. \quad (5.6)$$

Changing the last two terms of $C^1$ by using the identity

$$\psi_x \psi_{xt} + \psi_z \psi_{zt} = D_x (\psi \psi_{xt}) - \psi \phi_x \phi_{xt} + D_z (\psi \psi_{zt}) - \psi \phi_x \phi_{zt}$$

$$= D_x (\psi \psi_{xt}) + D_z (\psi \psi_{zt}) - \psi \phi_t \phi$$ \quad (5.7)

and dropping the divergent type terms, we rewrite $C^1$ given by Eq. (5.6) in the form

$$C^1 = vv_t + \frac{g^2}{N^2} \rho \rho_t - \psi \phi_t \phi. \quad (5.8)$$

Let us clarify if the conservation law with the density (5.8) is trivial or non-trivial. According to Definition 4.1, we have to evaluate the density (5.8) on the solutions of Eqs. (1.2). In this case it is convenient to use Eqs. (1.2) in the form (1.3) and replace Eq. (4.8) by

$$C^1_* = \left. C^1 \right|_{(1.3)}.$$

Then we have

$$C^1_* = \left\{ vJ(\psi, v) + \frac{g^2}{N^2} \rho J(\psi, \rho) - \psi J(\psi, \phi) \right\} - fD_z (v\psi) - gD_x (\rho \psi)$$

and Corollary 4.2 shows that the conservation law is trivial. Indeed, the last two terms of $C^1_*$ have the divergent form. The expression in braces for $C^1_*$ satisfies Eqs. (4.9) according to Proposition 4.1, and hence it also has the divergent form.

Thus, the invariance of Eqs. (1.2) under the time translation furnishes only a trivial conservation law.

5.7 Use of the dilation. Conservation of energy

Consider the generator (2.5) of the dilation group,

$$X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2 \psi \frac{\partial}{\partial \psi}.$$
In this case the quantities (4.16) have the form
\[ W^1 = v - xv_x - zv_z, \quad W^2 = \rho - x\rho_x - z\rho_z, \quad W^3 = 2\psi - x\psi_x - z\psi_z. \] (5.9)

The substitution of (5.9) in (4.23) yields:
\[ C^1 = -v^2 + xvv_x + zvv_z + \frac{g^2}{N^2} (\rho^2 + x\rho_x + z\rho_z) \]
\[ - \psi_x^2 + x\psi_x\psi_{xx} + z\psi_x\psi_{xz} - \psi_z^2 + x\psi_z\psi_{xz} + z\psi_z\psi_{zz}. \] (5.10)

We modify (5.10) by using the identities
\[ xv_x + zv_z = \frac{1}{2} D_x (xv^2) + \frac{1}{2} D_z (zv^2) - v^2, \]
\[ x\rho_x + z\rho_z = \frac{1}{2} D_x (x\rho^2) + \frac{1}{2} D_z (z\rho^2) - \rho^2, \]
\[ x\psi_x\psi_{xx} + x\psi_z\psi_{xz} = \frac{1}{2} D_x \left[ x (\psi_x^2 + \psi_z^2) \right] - \frac{1}{2} (\psi_x^2 + \psi_z^2), \]
\[ z\psi_x\psi_{xz} + z\psi_z\psi_{zz} = \frac{1}{2} D_z \left[ z (\psi_x^2 + \psi_z^2) \right] - \frac{1}{2} (\psi_x^2 + \psi_z^2). \]

Substituting these in (5.10) and dropping the divergent type terms we have:
\[ C^1 = -2 \left( v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right), \]
where
\[ |\nabla \psi|^2 = \psi_x^2 + \psi_z^2. \]

Dividing \( C^1 \) by the inessential coefficient \((-2)\) we finally obtain the following conservation law in the integral form (4.2):
\[ \frac{d}{dt} \int \int \left[ v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right] dx dz = 0. \] (5.11)

Eq. (5.11) represents the conservation of the energy with the density
\[ E = v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2. \] (5.12)

Let us find the components \( C^2 \) and \( C^3 \) of this conservation law written in the differential form (4.1). We will use the procedure suggested in Section
5.4. Let us first verify that $E$ defined by Eq. (5.12) satisfies Corollary 4.1 for densities of conservation laws. We have

$$D_t(E) = 2 \left( vv_t + \frac{g^2}{N^2} \rho \rho_t + \psi_x \psi_{xt} + \psi_z \psi_{zt} \right).$$  \hspace{1cm} (5.13)

Since the expression in the brackets in Eq. (5.13) is identical with (5.6) it can be rewritten in the form (5.8), and hence satisfies Eqs. (4.5). Corollary 4.1 guarantees that $E$ is the density of a conservation law. It is manifest from Eq. (5.12) that this conservation law is non-trivial.

According to Corollary 4.1, $D_t(E)$ defined by Eq. (5.13) and evaluated on the solutions of Eqs. (1.2) satisfies Eq. (4.4),

$$D_t(E) \bigg|_{(1.2)} = D_x(H^1) + D_z(H^2),$$ \hspace{1cm} (5.14)

with certain functions $H^1$, $H^2$. In order to find $H^1$, $H^2$, we use Eq. (5.7),

$$2(\psi_x \psi_{xt} + \psi_z \psi_{zt}) = D_x(2\psi \psi_{xt}) + D_z(2\psi \psi_{zt}) - 2\psi \Delta \psi_t,$$ \hspace{1cm} (5.15)

and write:

$$2vv_t = 2\psi_x vv_z - \psi_z vv_x - 2fv_z \Rightarrow$$

$$D_t(E) = 2 \left( v^2 vv_z - \psi_z vv_x - 2fv_z \right)$$

$$= D_x(v^2 \psi_z) - D_x(v^2 \psi_z) - 2fD_x(v\psi) + 2f\psi v_z,$$ \hspace{1cm} (5.16)

$$2 \frac{g^2}{N^2} \rho \rho_t = 2 \frac{g^2}{N^2} \left( \psi_x \rho \rho_z - \psi_z \rho \rho_x \right) - 2g\rho \psi_x$$

$$= \frac{g^2}{N^2} \left[ D_z(\rho^2 \psi_z) - D_x(\rho^2 \psi_x) \right] - 2gD_x(\rho \psi) + 2g\rho \psi_x, \hspace{1cm} (5.17)$$

$$-2\psi \Delta \psi_t = -2g\rho \rho_x - 2f \psi v_z - 2\psi \psi_z \Delta \psi_z + 2\psi \psi_z \Delta \psi_x$$

$$= -2g\psi \rho_x - 2f \psi v_z - D_x(\psi^2 \Delta \psi_z) + D_x(\psi^2 \Delta \psi_x). \hspace{1cm} (5.18)$$

Substituting the expressions (5.16), (5.17) and (5.15), (5.18) in the right-hand side of Eq. (5.13), we arrive at Eq. (5.14) with

$$H^1 = -v^2 \psi_z - \frac{g^2}{N^2} \rho^2 \psi_z - 2g\rho \psi + 2\psi \psi_{xt} - \psi^2 \Delta \psi_z,$$

$$H^2 = v^2 \psi_z + \frac{g^2}{N^2} \rho^2 \psi_x - 2fv\psi + 2\psi \psi_{zt} + \psi^2 \Delta \psi_x.$$

Thus, denoting $C^2 = -H^1$, $C^3 = -H^2$ we arrive at the following differential form (4.1) of the conservation of energy for Eqs. (1.2):

$$D_t(E) + D_x(C^2) + D_z(C^3) = 0 \hspace{1cm} (5.19)$$
with the density $E$ given by Eq. (5.12) and the flux given by the equations

$$C^2 = 2g\rho\psi + v^2\psi_z + \frac{g^2}{N^2} \rho^2 \psi_z - 2\psi\psi_{zt} + \psi^2 \Delta \psi_z,$$

$$C^3 = 2fv\psi - v^2\psi_x - \frac{g^2}{N^2} \rho^2 \psi_x - 2\psi\psi_{zt} - \psi^2 \Delta \psi_x.$$  \hfill (5.20)

6 Conservation law provided by the semi-dilation

Let us consider the operator $X_8$ from (2.6),

$$X_8 = t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3\psi \frac{\partial}{\partial \psi} - 2fx \frac{\partial}{\partial v} + 2\frac{N^2}{g} z \frac{\partial}{\partial \rho}$$ \hfill (6.1)

It generates the following one-parameter group with the parameter $\varepsilon$:

$$\bar{t} = te^{\varepsilon}, \quad \bar{x} = xe^{\varepsilon}, \quad \bar{z} = ze^{\varepsilon}, \quad \bar{\psi} = \psi e^{3\varepsilon},$$

$$\bar{v} = v + fx (1 - e^{2\varepsilon}), \quad \bar{\rho} = \rho - \frac{N^2}{g} (1 - e^{2\varepsilon}).$$ \hfill (6.2)

Since some of variables, namely $t, x, z$ and $\psi$ are subjected to dilations while two other variables transform otherwise, we call (6.2) the semi-dilation group. Let us construct the conserved vector provided by this group.

6.1 Computation of the conservation density

We will compute the density of conservation laws using Eq. (4.23):

$$C^1 = -vW^1 - \frac{g^2}{N^2} \rho W^2 - \psi_x D_x (W^3) - \psi_z D_z (W^3).$$ \hfill (4.23)

where $W^1, W^2, W^3$ are given by Eqs. (16):

$$W^\alpha = \eta^\alpha - \xi^\alpha u_j^\alpha, \quad \alpha = 1, 2, 3.$$ \hfill (4.16)

In the case of the operator (6.1) the quantities (16) are written:

$$W^1 = -2fx - tv_t - 2xv_x - 2zv_z,$$

$$W^2 = 2\frac{N^2}{g} z - t\rho_t - 2x\rho_x - 2z\rho_z,$$

$$W^3 = 3\psi - tv_t - 2xv_x - 2zv_z.$$ \hfill (6.3)
Substituting (6.3) in (4.23) we obtain after simple calculations:
\[ C_1 = 2(f xv - gz \rho) - |\nabla \psi|^2 + (xD_x + zD_z) \left( v^2 + \frac{g^2}{N^2} \rho^2 \right) \]
\[ + (xD_x + zD_z) (|\nabla \psi|^2) + t \left[ uv_t + \frac{g^2}{N^2} \rho v_t + \psi_{t} \psi_{xt} + \psi_{z} \psi_{zt} \right]. \]

We can drop the last term in (6.4) because it can be written in the divergent form upon eliminating \(v_t, \rho_t\) and \(\psi_t\) using Eqs. (1.2). Indeed, it is shown in Section 5.6 that the expression in the square brackets (cf. Eq. (5.6)) evaluated on the solutions of Eqs. (1.2) has the divergent form. Multiplication by \(t\) does not violate this property.

Now we use the identities
\[ (xD_x + zD_z) \left( v^2 + \frac{g^2}{N^2} \rho^2 \right) = -2 \left( v^2 + \frac{g^2}{N^2} \rho^2 \right) \]
\[ + D_x \left[ x \left( v^2 + \frac{g^2}{N^2} \rho^2 \right) \right] + D_z \left[ z \left( v^2 + \frac{g^2}{N^2} \rho^2 \right) \right], \]
\[ (xD_x + zD_z) (|\nabla \psi|^2) = -2|\psi|^2 + D_x \left[ x(|\nabla \psi|^2) \right] + D_z \left[ z(|\nabla \psi|^2) \right], \]
drop the divergent type terms and obtain the following conserved density:
\[ C_1 = 2 \left( f xv - gz \rho - \frac{1}{2} |\nabla \psi|^2 \right) - 2 \left( v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right). \]

Finally we note that the last term in (6.5) is the energy density (5.12). Therefore we eliminate it and conclude that the invariance under the semidilation with the generator (6.1) provides the conservation law with the density
\[ P = f xv - gz \rho - \frac{1}{2} |\nabla \psi|^2. \]

### 6.2 The conserved vector

Let us find the components \(C^2, C^3\) of the conserved vector with the density (6.6). We will apply the procedure described in Section (5.4). We have:
\[ D_t(P) = f xv_t - gz \rho_t - (\psi_{x} \psi_{xt} + \psi_{z} \psi_{zt}). \]

Using Eqs. (1.2), we obtain:
\[ D_t(P) \bigg|_{(1.2)} = -f^2 x \psi_z + f x \psi_z v_z - f x \psi_z v_x + N^2 z \psi_x \]
\[ - gz \psi_x \rho_z + gz \psi_z \rho_x - D_x (\psi \psi_{xt}) - D_z (\psi \psi_{zt}) + \psi \Delta \psi, \]
we rewrite this equation, invoking Eq. (5.18), in the form
\[ D_t(P) = D_x \left( N^2 z \psi + f x \psi v_z - g z \psi \rho_z - \psi \psi_{zt} + \frac{1}{2} \psi^2 \Delta \psi_z \right) \]
\[ - D_z \left( f^2 x \psi + f x \psi v_x - g z \psi \rho_x + \psi \psi_{xt} + \frac{1}{2} \psi^2 \Delta \psi_x \right). \]

Thus, the generator (6.1) provides the conservation law
\[ D_t(P) + D_x(C^2) + D_z(C^3) = 0 \]
with the density \( P \) given by (6.6) and the flux given by the equations
\[ C^2 = -N^2 z \psi - f x \psi v_z + g z \psi \rho_z + \psi \psi_{zt} - \frac{1}{2} \psi^2 \Delta \psi_z, \]
\[ C^3 = f^2 x \psi + f x \psi v_x - g z \psi \rho_x + \psi \psi_{xt} + \frac{1}{2} \psi^2 \Delta \psi_x. \]

7 Conservation law provided by the rotation

Taking the rotation generator \( X_9 \) from (2.6) and proceedings as in Section 6 we obtain the following conserved density:
\[ Q = v \rho + f x \rho - \frac{N^2}{g} z v. \] (7.1)

Taking Eqs. (1.2) in the form (1.3), we have:
\[ D_t(Q)\big|_{(1.3)} = v \left[ J(\psi, \rho) - \frac{N^2}{g} \psi_x \right] + \rho \left[ J(\psi, v) - f \psi_z \right] \]
\[ + f x \left[ J(\psi, \rho) - \frac{N^2}{g} \psi_x \right] - \frac{N^2}{g} z \left[ J(\psi, v) - f \psi_z \right]. \] (7.2)

The reckoning shows that
\[ v J(\psi, \rho) + \rho J(\psi, v) = D_z (v \rho \psi_z) - D_x (v \rho \psi_x), \]
\[ x J(\psi, \rho) - \rho \psi_z = D_z (x \rho \psi_x) - D_x (x \rho \psi_z), \]
\[ z J(\psi, v) + v \psi_x = D_x (z v \psi_x) - D_z (z v \psi_x), \]
\[ z \psi_z - x \psi_x = D_z (z \psi) - D_x (x \psi). \]

Substituting these expressions in Eq. (7.2) we conclude that the rotation generator \( X_9 \) provides the conservation law
\[ D_t(Q) + D_x(C^2) + D_z(C^3) = 0 \]
with the density \( P \) given by (7.1) and the flux given by the equations

\[
C^2 = \left[ v\rho + f x\rho - \frac{N^2}{g} zv \right] \psi_z + \frac{N^2}{g} f z\psi,
\]

\[
C^3 = \left[ \frac{N^2}{g} zv - v\rho - f x\rho \right] \psi_x - \frac{N^2}{g} f x\psi.
\] (7.3)

8 Summary of conservation laws

The calculations of Sections 5, 6 and 7 provide five nontrivial conservation laws for the system of nonlinear equations (1.2). These conservation laws are summarized below. For the convenience of the reader, we formulate them in the integral and the differential forms.

8.1 Conservation laws in integral form

\[
\frac{d}{dt} \int \int v \, dx \, dz = 0.
\] (8.1)

\[
\frac{d}{dt} \int \int \rho \, dx \, dz = 0.
\] (8.2)

\[
\frac{d}{dt} \int \int \left[ v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right] \, dx \, dz = 0.
\] (8.3)

\[
\frac{d}{dt} \int \int \left[ f x v - g z \rho - \frac{1}{2} |\nabla \psi|^2 \right] \, dx \, dz = 0.
\] (8.4)

\[
\frac{d}{dt} \int \int \left[ v\rho + f x\rho - \frac{N^2}{g} zv \right] \, dx \, dz = 0.
\] (8.5)

8.2 Conservation laws in differential form

\[
D_t(v) + D_x(v\psi_z) + D_z(f \psi - v\psi_x) = 0.
\] (8.1’)

\[
D_t(\rho) + D_x \left( \frac{N^2}{g} \psi + \rho \psi_z \right) + D_z(-\rho \psi_x) = 0.
\] (8.2’)
\[ D_t \left( v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right) \]
\[ + D_x \left( 2g\rho \psi + v^2 \psi_z + \frac{g^2}{N^2} \rho^2 \psi_z - 2\psi \psi_{zt} + \psi^2 \Delta \psi_z \right) \] (8.3')
\[ + D_z \left( 2fv\psi - v^2 \psi_x - \frac{g^2}{N^2} \rho^2 \psi_x - 2\psi \psi_{zt} - \psi^2 \Delta \psi_x \right) = 0. \]

\[ D_t \left( fxv - g\rho - \frac{1}{2} |\nabla \psi|^2 \right) \]
\[ + D_x \left( -N^2 z\psi - fxv \psi_z + g\rho \psi_{zt} - \frac{1}{2} \psi^2 \Delta \psi_z \right) \] (8.4')
\[ + D_z \left( f^2 x\psi + fxv \psi_x - g\rho \psi_{xt} + \psi_{zt} + \frac{1}{2} \psi^2 \Delta \psi_x \right) = 0. \]

\[ D_t \left( v\rho + fx\rho - \frac{N^2}{g} zv \right) \]
\[ + D_x \left( \left[ v\rho + fx\rho - \frac{N^2}{g} zv \right] \psi_z + \frac{N^2}{g} f z\psi \right) \] (8.5')
\[ + D_z \left( \left[ \frac{N^2}{g} zv - v\rho - fx\rho \right] \psi_x - \frac{N^2}{g} f x\psi \right) = 0. \]

The conservation laws (8.2) and (8.3) express the mass and the energy conservations, respectively. It seems that the other conservation laws do not have direct analogies in mechanics and should be investigated from point of view of their physical significance.

9 Solutions based on translation and dilation

9.1 Computation of the invariant solution

Let us find the invariant solution based on the following two operators:
\[ mX_5 - kX_6 = m \frac{\partial}{\partial x} - k \frac{\partial}{\partial z} \quad (m, k = \text{const.}), \]
\[ X_7 = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}. \] (9.1)
We first find their invariants \( J(t, x, z, v, \rho, \psi) \) by solving the equations
\[
(mx_5 - kx_6)J = 0, \quad X_7J = 0. \tag{9.2}
\]
The characteristic equation \( kd\lambda + m\epsilon = 0 \) for the first equation (9.2) yields that the operator \( mx_2 - kx_3 \) has, along with \( t, v, \rho, \psi \), the following invariant:
\[
\lambda = k\lambda + mz. \tag{9.3}
\]
Therefore we have to find the invariants \( J(t, \lambda, v, \rho, \psi) \) for the operator \( X_7 \). To this end, we write the action of \( X_7 \) on the variables \( t, \lambda, v, \rho, \psi \) by the standard formula
\[
X_7 = X_7(\lambda) \frac{\partial}{\partial \lambda} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}
\]
and obtain
\[
X_7 = \lambda \frac{\partial}{\partial \lambda} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}. \tag{9.4}
\]
To solve the equation \( X_7J(t, \lambda, v, \rho, \psi) = 0 \) for the invariants, we calculate the first integrals for the characteristic system
\[
\frac{d\lambda}{\lambda} = \frac{dv}{v} = \frac{d\rho}{\rho} = \frac{d\psi}{2\psi}
\]
and see that a basis of invariants for the operators (9.1) is given by
\[
t, \quad V = \frac{v}{\lambda}, \quad R = \frac{\rho}{\lambda}, \quad \phi = \frac{\psi}{\lambda^2}.
\]
Accordingly, we assign the invariants \( V, R, \phi \) to be functions of the invariant \( t \) and arrive at the following general form of the candidates for the invariant solutions:
\[
v = \lambda V(t), \quad \rho = \lambda R(t), \quad \psi = \lambda^2 \phi(t), \quad \lambda = k\lambda + mz. \tag{9.5}
\]
In order to find the functions \( V(t), R(t), \phi(t) \), we have to substitute the expressions (9.5) in Eqs. (1.2).

We have:
\[
\psi_t = \lambda^2 \phi'(t), \quad \psi_x = 2k\lambda \phi(t), \quad \psi_z = 2m\lambda \phi(t),
\]
\[
\nabla^2 \psi_t = 2(k^2 + m^2) \phi'(t), \quad \nabla^2 \psi_x = 0, \quad \nabla^2 \psi_z = 0,
\]
\[
\psi_x v_z = 2km \lambda \phi(t) V(t), \quad \psi_z v_x = 2km \lambda \phi(t) V(t),
\]
\[
\psi_x \rho_z = 2km \lambda \phi(t) R(t), \quad \psi_z \rho_x = 2km \lambda \phi(t) R(t).
\]
Therefore Eqs. (1.2) yield the following system of first-order linear ordinary differential equations:

\[
2(k^2 + m^2)\phi' - gkR - fmV = 0,
\]
\[
\lambda V' + 2fm\lambda \phi = 0,
\]
\[
\lambda R' + 2\frac{k\lambda}{g} N^2\phi = 0,
\]
or

\[
\phi' = \frac{1}{2(k^2 + m^2)}(gkR + fmV), \quad \text{(9.6)}
\]
\[
V' = -2fm\phi, \quad \text{(9.7)}
\]
\[
R' = -2\frac{k}{g} N^2\phi. \quad \text{(9.8)}
\]

Let us integrate Eqs. (9.6)-(9.8). Differentiating Eq. (9.6) and using Eqs. (9.7)-(9.8), we obtain

\[
\phi'' + \omega^2 \phi = 0, \quad \text{(9.9)}
\]

where

\[
\omega^2 = \frac{k^2 N^2 + m^2 f^2}{k^2 + m^2}. \quad \text{(9.10)}
\]

The general solution of Eq. (9.9) is given by

\[
\phi(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \quad C_1, C_2 = \text{const.} \quad \text{(9.11)}
\]

Substituting (9.11) in Eqs. (9.7)-(9.8) and integrating, we obtain

\[
V = C_3 - \frac{2fm}{\omega} \left[ C_1 \sin(\omega t) - C_2 \cos(\omega t) \right],
\]
\[
R = C_4 - \frac{2k}{g\omega} N^2 \left[ C_1 \sin(\omega t) - C_2 \cos(\omega t) \right].
\]

To determine the constants \(C_3\) and \(C_4\), we substitute in Eq. (9.6) the above expressions for \(V, R\) and the expression (9.11) for \(\phi\) and obtain

\[
fmC_3 + gkC_4 = 0.
\]
Thus, the solution to Eqs. (9.6)-(9.8) has the following form:

\[ \phi(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t), \tag{9.12} \]
\[ V(t) = \frac{2fm}{\omega} \left[ C_2 \cos(\omega t) - C_1 \sin(\omega t) \right] + C_3, \tag{9.13} \]
\[ R(t) = \frac{2k}{g \omega} N^2 \left[ C_2 \cos(\omega t) - C_1 \sin(\omega t) \right] - \frac{fm}{gk} C_3. \tag{9.14} \]

Finally, substituting (9.12)-(9.14) in (9.5), we arrive at the following solution to the system (1.2):

\[ \rho = \frac{2k}{g \omega} N^2 \left[ C_2 \cos(\omega t) - C_1 \sin(\omega t) \right] \lambda - \frac{fm}{gk} C_3 \lambda, \tag{9.15} \]
\[ v = \frac{2fm}{\omega} \left[ C_2 \cos(\omega t) - C_1 \sin(\omega t) \right] \lambda + C_3 \lambda, \tag{9.16} \]
\[ \psi = \left[ C_1 \cos(\omega t) + C_2 \sin(\omega t) \right] \lambda^2, \tag{9.17} \]

where \( \lambda \) is given by (9.3), \( \omega \) is defined by Eq. (9.10) and \( C_1, C_2, C_3 \) are arbitrary constants.

### 9.2 Generalized invariant solution

It is natural to generalize the candidates (9.5) for the invariant solutions and look for particular solutions of the system (1.2) in the following form of separated variables:

\[ v = F(\lambda) V(t), \quad \rho = \alpha(\lambda) R(t), \quad \psi = \beta(\lambda) \phi(t), \quad \lambda = kx + mz. \tag{9.18} \]

The reckoning shows that then the right-hand sides of Eqs. (1.2) vanish and Eqs. (1.2) become:

\[ (k^2 + m^2) \beta''(\lambda) \phi'(t) - gk \alpha'(\lambda) R(t) - fm F'(\lambda) V(t) = 0, \tag{9.19} \]
\[ F(\lambda) V'(t) + fm \beta'(\lambda) \phi(t) = 0, \tag{9.20} \]
\[ \alpha(\lambda) R'(t) + \frac{kN^2}{g} \beta'(\lambda) \phi(t) = 0. \tag{9.21} \]

Differentiating Eq. (9.19) with respect to \( t \), using Eqs. (9.20)-(9.21) and dividing by \( \beta' \), we obtain

\[ (k^2 + m^2) \frac{\beta''}{\beta'} \phi'' + \left( N^2 k^2 \frac{\alpha'}{\alpha} + fm^2 \frac{F'}{F} \right) \phi = 0. \]
Assuming that the ratios $\frac{\beta''}{\beta'}, \frac{\alpha'}{\alpha}, \frac{F'}{F}$ are proportional with constant coefficients and have one and the same sign, we arrive at an equation of the form (9.9). For example, letting

$$\frac{\beta''}{\beta'} = \frac{\alpha'}{\alpha} = \frac{F''}{F},$$  

we obtain Eq. (9.9). Then, according to (9.11), we can set in (9.18) $\phi(t) = \cos(\omega t)$ and $\phi(t) = \sin(\omega t)$, i.e.

$$\psi = A(\lambda) \cos(\omega t) \quad \text{and} \quad \psi = B(\lambda) \sin(\omega t).$$  

For each function $\psi$ given by (9.23) we determine the functions $V(t)$, $R(t)$ using Eqs. (9.20), (9.21), (9.21), then take the linear combinations of the resulting functions and arrive at the following form of the “generalized invariant solution” (9.18):

$$\psi = A(\lambda) \cos(\omega t) + B(\lambda) \sin(\omega t),$$  

$$v = \frac{fm}{\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)] + F(\lambda),$$  

$$\rho = \frac{kN^2}{g\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)] + H(\lambda),$$

where $\omega$ is given by Eq. (9.10).

The reckoning shows that the functions (9.24)-(9.26) with arbitrary $A(\lambda), B(\lambda)$ solve Eqs. (1.2) provided that $F(\lambda), H(\lambda)$ satisfy the following equation:

$$gkH'(\lambda) + fmF'(\lambda) = 0.$$  

One can readily verify that the invariant solution (9.15)-(9.17), which is a particular case of (9.24)-(9.26), obeys the condition (9.27).

### 9.3 Wave beams

Let us find the energy of the generalized invariant solution. We set $F(\lambda) = H(\lambda) = 0$ in Eqs. (9.24)-(9.26) and substitute in Eq. (5.12) the expressions

$$\psi = A(\lambda) \cos(\omega t) + B(\lambda) \sin(\omega t),$$  

$$v = \frac{fm}{\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)],$$  

$$\rho = \frac{kN^2}{g\omega} [B'(\lambda) \cos(\omega t) - A'(\lambda) \sin(\omega t)],$$
where
\[ \lambda = kx + mz, \quad \omega^2 = \frac{k^2 N^2 + m^2 f^2}{k^2 + m^2}. \]

After simple calculations we obtain:
\[ E = (k^2 + m^2) \left[ A'(\lambda)^2 + B'(\lambda)^2 \right]. \]

Invoking that any conserved vector is defined up to multiplication by an arbitrary constant, we divide the above expression for \( E \) by \((k^2 + m^2)\) and obtain the following energy:
\[ E = A'(\lambda)^2 + B'(\lambda)^2. \] (9.31)

Since the energy density (9.31) depends only on \( \lambda = kx + mz \), it is constant along the straight line
\[ kx + mz = \text{const.} \] (9.32)

Accordingly, the “local energy” (9.31) has one and the same value at points \((x_0, z_0)\) and \((x_1, z_1)\) provided that
\[ kx_0 + mz_0 = kx_1 + mz_1. \] (9.33)

The energy density (9.31) describes the local behavior of the solutions. Therefore it is significant to understand its distribution on the \((x, z)\) plane. Suppose that the functions \( A(\lambda) \), \( B(\lambda) \) and their derivatives rapidly decrease as \( \eta \to \infty \). If we take, as an example, the functions
\[ A(\lambda) = \frac{a}{1 + \lambda^2}, \quad B(\lambda) = \frac{a\lambda}{1 + \lambda^2}, \] (9.34)

where \( a \) is a positive constant, then the energy density (9.31) of the wave beams has the form
\[ E = \frac{a^2}{(1 + \lambda^2)^2}. \]

Hence, the energy is localized along the straight line (9.33). Therefore we can define a wave beam through a point \((x_0, z_0)\) as the totality of the points \((x_1, z_1)\) satisfying Eq. (9.33). In other words, we identify a wave beam with the straight line (9.33)
Abstract. Group analysis is used for investigating the propagation of nonlinear waves in a rotating column of uniformly stratified fluid under the Boussinesq approximation. One class of the obtained invariant solutions can be visualized as rotating whirlpools along which the pressure deviation from the mean state is zero, is positive inside and negative outside of the whirlpools. The radii of the whirlpools increase near the earth’s equator and decrease near the poles.

Keywords: Rotating stratified fluid, Boussinesq approximation, Nonlinear waves, Invariant solutions, Rotating whirlpools.

1 Model and its symmetries

Our starting point is the system (1.1) from Paper 5 in this volume, i.e. the Euler equations in the rotating reference frame. Then we employ the Boussinesq approximation, use the spherical coordinates and the assumption due to Kelvin which, in the present context, says that the radial component of the velocity vector is zero everywhere. As a result, we have the following nonlinear over-determined system for describing a rotating column of
uniformly stratified fluid:

\[
\begin{align*}
\frac{u_\theta^2}{r} + f u_\theta &= \frac{\partial p}{\partial r}, \\
\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + w \frac{\partial u_\theta}{\partial z} &= -\frac{1}{r} \frac{\partial p}{\partial \theta}, \\
\frac{\partial w}{\partial t} + \frac{u_\theta}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} - \rho g, \\
\frac{\partial p}{\partial t} + \frac{u_\theta}{r} \frac{\partial p}{\partial \theta} &= \frac{N^2}{g} w, \\
\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z} &= 0,
\end{align*}
\]  

(1.1)

where \( f, g \) and \( N^2 \) are the physical constants defined in Paper 5 in this volume.

The symmetries of Eqs. contain two arbitrary functions, \( \varphi(z) \), \( \psi(t) \), and are spanned by the following operators:

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \theta}, \quad X_3 = \frac{\partial}{\partial z}, \\
X_4 &= r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z} + u_\theta \frac{\partial}{\partial u_\theta} + w \frac{\partial}{\partial w} + 2p \frac{\partial}{\partial p} + \rho \frac{\partial}{\partial \rho}, \\
X_5 &= 2(f t + 2 \theta) \frac{\partial}{\partial \theta} - 4r \frac{\partial}{\partial r} + 2f r \frac{\partial}{\partial u_\theta} + f^2 r^2 \frac{\partial}{\partial p}, \\
X_\varphi &= g \varphi(z) \frac{\partial}{\partial p} - \varphi'(z) \frac{\partial}{\partial \rho}, \quad X_\psi = \psi(t) \frac{\partial}{\partial p}.
\end{align*}
\]

2 Self-adjointness

Our aim is to construct conservation laws for the system (1.1) using the method developed in [38] (Paper 4 in this volume). According to this method, one can associate conservation laws with symmetries on any system of differential equations provided that the system under consideration is nonlinearly self-adjoint in the terminology of [38]. The system in question can be determined (the number of the equations in the system is equal to the number of dependent variables), over-determined (the number of the equations in the system is more than the number of dependent variables) or sub-definite (the number of the equations in the system is less than the number of dependent variables).

We will show in this section that the over-determined system of equations (1.1) is nonlinearly self-adjoint.
2.1 Adjoint system

Let us rewrite Eqs. (1.1), denoting \( u_\theta = u \), in the following form:

\[
F_1 \equiv \frac{\partial u}{\partial t} + \frac{u}{r} \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial p}{\partial \theta} = 0,
\]

\[
F_2 \equiv \frac{\partial w}{\partial t} + \frac{u}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + \rho g = 0,
\]

\[
F_3 \equiv \frac{\partial p}{\partial r} - \frac{u^2}{r} - f u = 0,
\]

\[
F_4 \equiv \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} = 0.
\]

The formal Lagrangian for the system (2.1) is

\[
\mathcal{L} = UF_1 + VF_2 + PF_3 + RF_4 + QF_5,
\]

where \( U, V, P, R \) and \( Q \) are new dependent variables.

The adjoint system to Eqs. (2.1) is written

\[
F_1^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \quad F_2^* \equiv \frac{\delta \mathcal{L}}{\delta w} = 0, \quad F_3^* \equiv \frac{\delta \mathcal{L}}{\delta p} = 0, \quad F_4^* \equiv \frac{\delta \mathcal{L}}{\delta \rho} = 0. \quad (2.3)
\]

Substituting in (2.3) the the expressions (2.2) and (2.1) we obtain:

\[
F_1^* = -U_t - \frac{u}{r} U_\theta - wU_z - U \frac{\partial w}{\partial z} + \frac{V}{r} \frac{\partial w}{\partial \theta} + \frac{R}{r} \frac{\partial \rho}{\partial \theta} - \frac{2u}{r} P - \frac{1}{r} Q_\theta - f P,
\]

\[
F_2^* = -V_t - \frac{u}{r} V_\theta - \frac{V}{r} \frac{\partial u}{\partial \theta} - wV_z - Q_z + U \frac{\partial u}{\partial z} - \frac{N^2}{g} R,
\]

\[
F_3^* = -P_r - \frac{1}{r} U_\theta - V_z, \quad F_4^* = -R_t - \frac{u}{r} R_\theta - \frac{R}{r} \frac{\partial u}{\partial \theta} + gV,
\]

where the subscripts in \( U_t, U_\theta, U_z, \ldots \) denote the differentiations.

2.2 Proof of nonlinear self-adjointness

According to [38] the system (2.1) is nonlinearly self-adjoint if there exists a substitution

\[
U = \varphi^1, \quad V = \varphi^2, \quad P = \varphi^3, \quad R = \varphi^4, \quad Q = \varphi^5, \quad (2.5)
\]
where the functions $\varphi^1, \ldots, \varphi^5$ depend on $t, \theta, r, z, u, w, p, \rho$, do not vanish simultaneously and satisfy the equations

$$F^*_\alpha \big|_{(2.5)} = \lambda^3_\alpha F^*_\beta, \quad \alpha = 1, \ldots, 4.$$ (2.6)

Here $F^*_\beta$ ($\beta = 1, \ldots, 5$) and $F^*_\alpha$ ($\alpha = 1, \ldots, 4$) are given by the equations (2.1) and (2.4), respectively, $\lambda^3_\alpha$ are undetermined coefficients, the symbol $\big|_{(2.5)}$ indicates that the non-physical variables $U, \ldots, Q$ and their derivatives are eliminated by means of the substitution (2.5). The derivatives are computed in a usual way, e.g.

$$U_t = D_t(\varphi^1) \equiv \varphi^1_t + \varphi^1_u \frac{\partial u}{\partial t} + \varphi^1_w \frac{\partial w}{\partial t} + \varphi^1_p \frac{\partial p}{\partial t} + \varphi^1_\rho \frac{\partial \rho}{\partial t},$$

$$U_\theta = D_\theta(\varphi^1) \equiv \varphi^1_\theta + \varphi^1_u \frac{\partial u}{\partial \theta} + \varphi^1_w \frac{\partial w}{\partial \theta} + \varphi^1_p \frac{\partial p}{\partial \theta} + \varphi^1_\rho \frac{\partial \rho}{\partial \theta},$$

$$U_z = D_z(\varphi^1) \equiv \varphi^1_z + \varphi^1_u \frac{\partial u}{\partial z} + \varphi^1_w \frac{\partial w}{\partial z} + \varphi^1_p \frac{\partial p}{\partial z} + \varphi^1_\rho \frac{\partial \rho}{\partial z},$$

$$Q_\theta = D_\theta(\varphi^5) \equiv \varphi^5_\theta + \varphi^5_u \frac{\partial u}{\partial \theta} + \varphi^5_w \frac{\partial w}{\partial \theta} + \varphi^5_p \frac{\partial p}{\partial \theta} + \varphi^5_\rho \frac{\partial \rho}{\partial \theta},$$

(2.7)

where

$$\varphi^1_t, \ldots \varphi^1_\rho \quad \text{and} \quad \varphi^5_\theta, \ldots \varphi^5_\rho$$

are the respective partial derivatives of the function

$$\varphi^1(t, \theta, r, z, u, w, p, \rho) \quad \text{and} \quad \varphi^5(t, \theta, r, z, u, w, p, \rho).$$

The coefficients $\lambda^3_\alpha$ and the functions $\varphi^1, \ldots, \varphi^5$ are found by solving four equations (2.6) corresponding to $\alpha = 1, \ldots, 4$. The first equation corresponding to $\alpha = 1$ is written

$$F^*_1 = \lambda^1_1 F_1 + \lambda^1_2 F_2 + \lambda^1_3 F_3 + \lambda^1_4 F_4 + \lambda^1_5 F_5.$$ (2.8)

We substitute in $F^*_1$ given by the first equation (2.4) the expressions (2.7) for $U_t, \ldots, Q_\theta$, equate the coefficients for

$$\frac{\partial u}{\partial t}, \frac{\partial w}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial \rho}{\partial t}$$

in both sides of Eq. (2.8) and obtain:

$$\lambda^1_1 = -\varphi^1_u, \quad \lambda^1_2 = -\varphi^1_w, \quad \varphi^1_p = 0, \quad \lambda^1_5 = -\varphi^1_\rho.$$ (2.9)

Likewise, considering the coefficients for

$$\frac{\partial p}{\partial r}, \frac{\partial p}{\partial z}, \frac{\partial \rho}{\partial r}, \frac{\partial \rho}{\partial z}$$

and others, we obtain the remaining coefficients.
we obtain:

\[ \lambda_3^1 = 0, \quad \lambda_2^1 = 0, \quad \varphi_1^1 = 0. \]  

(2.10)

Summarizing the equations (2.9) and (2.10) we see that

\[ \varphi^1 = \varphi^1(t, \theta, r, z, u) \]  

(2.11)

and that

\[ \lambda_1^1 = -\varphi_1^1 u, \quad \lambda_2^1 = \lambda_3^1 = \lambda_4^1 = 0. \]  

(2.12)

In view of (2.12), Eq. (2.8) becomes

\[ F_1^* = -\varphi_1^1 F_1 + \lambda_1^5 F_5. \]  

(2.13)

We continue the similar calculations with Eq. (2.13) and with the remaining three equations (2.6) corresponding to \( \alpha = 2, 3, 4 \). After lengthy but regular calculations we conclude that the substitution (2.5) satisfying Eqs. (2.6) has the following form:

\[ U = a(t, \theta) r + b(t, r), \quad V = 0, \]
\[ P = -\frac{\partial a(t, \theta)}{\partial \theta} r, \quad R = 0, \]  

(2.14)

\[ Q = [a(t, \theta) r + b(t, r)] u + c(t, \theta, r), \]

where \( a(t, \theta), b(t, r) \) are arbitrary functions, and \( c(t, \theta, r) \) is determined by the differential equation

\[ \frac{1}{r} \frac{\partial c}{\partial \theta} = f r \frac{\partial a(t, \theta)}{\partial \theta} - \left[ r \frac{\partial a(t, \theta)}{\partial t} + \frac{\partial b(t, r)}{\partial t} \right]. \]  

(2.15)

Substituting (2.14) in (2.4) one can verify that the nonlinear self-adjointness conditions (2.6) of Eqs. (2.1) are satisfied in the following form:

\[ F_1^* = -[a(t, \theta) r + b(t, r)] F_5, \quad F_2^* = 0, \quad F_3^* = 0, \quad F_4^* = 0. \]

Thus the system (2.1) is nonlinearly self-adjoint.

3 Conservation laws

According to [38]), Eq. (8.23), symmetries

\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \]  

\[ \xi^i = \frac{\partial \xi}{\partial x^i}, \quad \eta^\alpha = \frac{\partial \eta}{\partial u^\alpha} \]  

(2.16)
of any nonlinearly self-adjoint system of first-order differential equations

\[ F_{\bar{\alpha}}(x, u, u_{(1)}) = 0, \quad \bar{\alpha} = 1, \ldots, m, \]

is generate the conserved vectors with following components:

\[ C^\alpha = W^\alpha \frac{\partial L}{\partial u_{i}^\alpha}. \]  

(3.1)

Here \( L = v^\beta F_{\beta} \) is the formal Lagrangian for the system in question and

\[ W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \]

The “non-physical variables” \( v^{\bar{\alpha}} \) should be eliminated from the vector (3.1) by using the substitution

\[ v^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x, u), \quad \bar{\alpha} = 1, \ldots, \bar{m}, \]

connecting the adjoint system with the system under consideration.

We will apply the formula (3.1) to our system (2.1) by using the notation

\[ x_1 = t, \quad x_2 = \theta, \quad x_3 = r, \quad x_4 = z, \]
\[ u_1 = u, \quad u_2 = w, \quad u_3 = p, \quad u_4 = \rho. \]  

(3.2)

According to the notation (3.2), the conservation laws will be written in the form

\[ [D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4)]_{(2.1)} = 0, \]  

(3.3)

where \( D_t, \ldots, D_z \) denote the total differentiations in \( t, \ldots, z. \)

Substituting in (3.1) the expression (2.2) of the formal Lagrangian by taking into account the notation (3.2) and the equation \( V = R = 0 \) due to Eqs. (2.14) we obtain:

\[ C^1 \equiv W^\alpha \frac{\partial L}{\partial u_{i}^\alpha} = UW^1, \]
\[ C^2 \equiv W^\alpha \frac{\partial L}{\partial u_{i}^\alpha} = \frac{1}{r}(uU + Q)W^1 + \frac{1}{r} UW^3, \]
\[ C^3 \equiv W^\alpha \frac{\partial L}{\partial u_{i}^\alpha} = PW^3, \]
\[ C^4 \equiv W^\alpha \frac{\partial L}{\partial u_{i}^\alpha} = wUW^1 + QW^2. \]
We replace here $U$, $P$ and $Q$ with their values given in (2.14) and arrive at the following final formula for calculating the conserved vectors:

$$
C^1 = (ar + b)W^1,
$$

$$
C^2 = \frac{1}{r} [2(ar + b)u + c]W^1 + \frac{1}{r} (ar + b)W^3,
$$

$$
C^3 = -ra_\theta W^3,
$$

$$
C^4 = (ar + b)wW^1 + [(ar + b)u + c]W^2,
$$

where $a_\theta = \partial a(t, \theta)/\partial \theta$.

### 3.1 Time translation: Energy

Let us compute the conserved vector provided by the time translation symmetry $X_1$ from (1.2). In this case we have

$$
W^1 = -\frac{\partial u}{\partial t}, \quad W^2 = -\frac{\partial w}{\partial t}, \quad W^3 = -\frac{\partial p}{\partial t}.
$$

Inserting these expressions in (3.4), eliminating in $C^1$ the derivative $\partial u/\partial t$ via the first equation (2.1) and using Eqs. (2.14) we write the conserved vector in the following form:

$$
C^1 = -a_\theta (u^2 + p) + D_\theta \left[ \left( a + \frac{b}{r} \right) (u^2 + p) \right] + D_z [(ar + b)uw],
$$

$$
C^2 = -\frac{1}{r} (ar + b) \left( \frac{\partial p}{\partial t} + 2u \frac{\partial u}{\partial t} \right) - \frac{c}{r} \frac{\partial u}{\partial t},
$$

$$
C^3 = ra_\theta \frac{\partial p}{\partial t},
$$

$$
C^4 = -(ar + b) \left( w \frac{\partial u}{\partial t} + u \frac{\partial w}{\partial t} \right) - c \frac{\partial w}{\partial t}.
$$
Finally, removing the terms $D_\theta(\cdots)$ and $D_z(\cdots)$ from $C^1$ to $C^2$ and $C^4$ we obtain:

$$C^1 = -(u^2 + p) a_\theta,$$

$$C^2 = \frac{1}{r} \left[ (ra_t + bt)(u^2 + p) - c \frac{\partial u}{\partial t} \right],$$

$$C^3 = ra_\theta \frac{\partial p}{\partial t},$$

$$C^4 = (ra_t + bt)uw - c \frac{\partial w}{\partial t},$$

where $a_\theta, a_t, b_t$ denote the partial derivatives.

The calculation shows that the conservation law (3.3) is satisfied for the vector (3.5) in the following form:

$$D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4)$$

$$= (ra_t + bt) \left( \frac{\partial u}{\partial t} + \frac{u \partial u}{r \partial \theta} + \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial p}{\partial \theta} \right)$$

$$+ ra_\theta \left( \frac{\partial p}{\partial r} - \frac{u^2}{r} - f u \right) + \left[ (ra_t + bt)u - c \frac{\partial}{\partial t} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} \right) \right].$$

Invoking that $u = u_\theta$, we obtain the conserved vector for Eqs. (1.1):

$$C^1 = -\frac{\partial a(t, \theta)}{\partial \theta} (u_\theta^2 + p),$$

$$C^2 = \frac{1}{r} \left[ \left( r \frac{\partial a(t, \theta)}{\partial t} + \frac{\partial b(t, r)}{\partial t} \right) (u_\theta^2 + p) - c(t, \theta, r) \frac{\partial u_\theta}{\partial t} \right],$$

$$C^3 = r \frac{\partial a(t, \theta)}{\partial \theta} \frac{\partial p}{\partial t},$$

$$C^4 = \left( r \frac{\partial a(t, \theta)}{\partial t} + \frac{\partial b(t, r)}{\partial t} \right) wu_\theta - c(t, \theta, r) \frac{\partial w}{\partial t}. $$

The integral form of the conservation law with the vector (3.7) gives the following energy conservation for the system (1.1):

$$\frac{d}{dt} \int_{R^3} \frac{\partial a(t, \theta)}{\partial \theta} (u_\theta^2 + p) d\theta dr dz = 0.$$
3.2 Rotation: Angular momentum

For the horizontal rotation, i.e. to the \( \theta \)-translation symmetry \( X_2 \) from (1.2) we have

\[
W^1 = -\frac{\partial u}{\partial \theta}, \quad W^2 = -\frac{\partial w}{\partial \theta}, \quad W^3 = -\frac{\partial p}{\partial \theta}.
\]

Substituting these expressions in the formula (3.4) and making simplifications as above we arrive at the following conserved vector:

\[
\begin{align*}
C^1 &= ra_\theta u, \\
C^2 &= -\frac{1}{r} [(ar + b)u + c] \frac{\partial u}{\partial \theta} + (ar + b)w \frac{\partial u}{\partial z} - (a_ir + b_i)u, \\
C^3 &= ra_\theta \frac{\partial p}{\partial \theta}, \\
C^4 &= -(ar + b)w \frac{\partial u}{\partial \theta} - [(ar + b)u + c] \frac{\partial w}{\partial \theta}.
\end{align*}
\]

The vector (3.9) satisfies the conservation law (3.3) in the following form:

\[
\begin{align*}
D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) \\
= ra_\theta \left( \frac{\partial u}{\partial t} + \frac{u}{r} \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial p}{\partial \theta} \right) + ra_\theta \frac{\partial p}{\partial \theta} \left( \frac{\partial p}{\partial r} - \frac{u^2}{r} - f u \right) \\
- (ar + b) \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} \right) \frac{\partial u}{\partial \theta} - [(ar + b)u + c] \frac{\partial w}{\partial \theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} \right).
\end{align*}
\]

Remark 3.1. One can simplify the vector (3.9) by removing the term of the form \( D_z(\cdots) \) from \( C^2 \) to \( C^3 \). Then the vector (3.9) becomes

\[
\begin{align*}
C^1 &= ra_\theta u, \\
C^2 &= -\frac{c}{r} \frac{\partial u}{\partial \theta} - (a_ir + b_i)u, \\
C^3 &= ra_\theta \frac{\partial p}{\partial \theta}, \\
C^4 &= ra_\theta uw - c \frac{\partial w}{\partial \theta}.
\end{align*}
\]
and the conservation equation (3.10) is written

\[
D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) \\
= r a_\theta \left( \frac{\partial u}{\partial t} + \frac{u \partial u}{r \partial \theta} + w \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial p}{\partial \theta} \right) \\
+ r a_\theta \frac{\partial}{\partial \theta} \left( \frac{\partial p}{\partial r} - \frac{u^2}{r} - f u \right) + \left( r a_\theta u - c \frac{\partial}{\partial \theta} \right) \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} \right).
\]

Writing the integral form of the conservation law with the vector (3.10) and returning to the physical notation \( u = u_\theta \) we obtain the following law of conservation of the angular momentum for the system (1.1):

\[
\frac{d}{dt} \int_{R^3} \frac{\partial a(t, \theta)}{\partial \theta} r u_\theta d\theta dr dz = 0.
\]

### 3.3 Conserved vector associated with scaling transformation

For the scaling transformation generator \( X_4 \) from (1.2) we have

\[
W^1 = u - r \frac{\partial u}{\partial r} - z \frac{\partial u}{\partial r}, \\
W^2 = w - r \frac{\partial w}{\partial r} - z \frac{\partial w}{\partial r}, \\
W^3 = 2p - r \frac{\partial p}{\partial r} - z \frac{\partial p}{\partial r}.
\]

Substituting these expressions in the formula (3.4) and making simplifications as above we arrive at the following conserved vector:

\[
C^1 = (4ar + 3b + rb_r) u, \\
C^2 = \frac{1}{r} (4ar + 3b + rb_r)(u^2 + p) + \left( \frac{2c}{r} + c_r \right) u, \\
C^3 = -4ra_\theta p - c \frac{\partial u}{\partial \theta}, \\
C^4 = (4ar + 3b + rb_r) uw + c \left( w - r \frac{\partial w}{\partial r} \right).
\]
The vector (3.12) satisfies the conservation law (3.3) in the following form:

\[
D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) = c \left( \frac{2}{r} - \frac{\partial}{\partial r} \right) \left( \frac{\partial u}{\partial \theta} + r \frac{\partial w}{\partial z} \right) \\
+ (4a r + 3b + rb_r) \left[ \frac{\partial u}{\partial t} + \frac{u \partial u}{r \partial \theta} + w \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial p}{\partial \theta} + u \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} \right) \right].
\]

(3.13)

The corresponding integral conservation law for the system (1.1) is written:

\[
\frac{d}{dt} \int_{R^3} \left[ 4a(t, \theta)r + 3b(t, r) + r \frac{\partial b(t, r)}{\partial r} \right] u_\theta \, d\theta dr dz = 0.
\]

(3.14)

### 3.4 Other symmetries from (1.2) provide trivial conserved vectors

The conserved vector (3.4) associated with the \(z\)-translation symmetry \(X_3\) vanishes on the solution manifold of the system (2.1). Hence, it is a trivial conserved vector. Moreover, the symmetries \(X_5, X_\phi\) and \(X_\psi\) also provide trivial conserved vectors.

### 3.5 Conclusion

Thus, the symmetries (1.2) provide three infinite sets of nontrivial integral conservation laws, (3.8), (3.11) and (3.14), depending on three arbitrary functions, namely \(a(t, \theta), b(t, r)\) and an arbitrary function involved in \(c(t, \theta, r)\).

### 4 Invariant solutions

The symmetries (1.2) can be used for obtaining exact solutions of the system (2.1) by computing the invariant and partially invariant solutions (see, e.g. \[64\]). In order to obtain all possible invariant and partially invariant solutions one has to construct optimal systems of subalgebras of the Lie algebra with the basis (1.2). Note that invariant solutions based on three-dimensional subalgebras are of a particular interest because they are described by systems of ordinary differential equations. We will consider two solutions of this type.
4.1 Non-stationary solution

Let us construct the invariant solutions based on the three-dimensional sub-algebra spanned by the operators $X_3, X_4, X_5$ from (1.2). According to the theorem on representation of non-singular invariant manifolds ([64], Section 14.3), the invariant solutions can be represented via invariants $J(t, \theta, r, z, u, w, p, \rho)$ of the operators $X_3, X_4, X_5$. The invariance under $X_3$ requires that $J$ does not depend on $z$. Thus, we have to solve the system

$$
X_4(J) \equiv r \frac{\partial J}{\partial r} + u \frac{\partial J}{\partial u} + w \frac{\partial J}{\partial w} + 2p \frac{\partial J}{\partial p} + \rho \frac{\partial J}{\partial \rho} = 0,
$$

$$
X_5(J) \equiv 2(f t + 2\theta) \frac{\partial J}{\partial \theta} - 4r \frac{\partial J}{\partial r} + 2fr \frac{\partial J}{\partial u} + f^2 r^2 \frac{\partial J}{\partial p} = 0
$$

for the function $J = J(t, \theta, r, u, w, p, \rho)$. Integration of the system (4.1) gives the following basis of invariants:

$$
J_1 = t, \quad J_2 = \frac{2u + fr}{(ft + 2\theta)r}, \quad J_3 = \frac{w}{(ft + 2\theta)r},
$$

$$
J_4 = \frac{8p + f^2 r^2}{(ft + 2\theta)^2 r^2}, \quad J_5 = \frac{\rho}{(ft + 2\theta)r}.
$$

The representation of the invariant solutions via the basic invariants (4.2) is obtained by assuming that $J_2, J_3, J_4, J_5$ are unknown functions of $J_1$. This yields the following candidates for the invariant solutions:

$$
u = -\frac{fr}{2} + (ft + 2\theta)r F(t), \quad w = (ft + 2\theta)r G(t),
$$

$$
p = -\frac{f^2 r^2}{8} + (ft + 2\theta)^2 r^2 H(t), \quad \rho = (ft + 2\theta)r K(t).
$$

Substituting (4.3) in Eqs. (2.1) and solving the resulting equations for the unknown function $F(t), G(t), H(t), K(t)$ we obtain

$$
F(t) = 0, \quad G(t) = A \cos(Nt) + B \sin(Nt),
$$

$$
H(t) = 0, \quad K(t) = \frac{N}{g} [A \sin(Nt) - B \cos(Nt)],
$$

where $A$ and $B$ are arbitrary constants. Substituting (4.4) in (2.3) and returning to the notation $u = u_\theta$ we obtain the following transient-state solution of the system (1.1):

$$
u_\theta = -\frac{fr}{2}, \quad w = (ft + 2\theta) r [A \cos(Nt) + B \sin(Nt)],
$$

$$
p = -\frac{f^2 r^2}{8}, \quad \rho = \frac{N}{g} (ft + 2\theta) r [A \sin(Nt) - B \cos(Nt)].
$$
4.2 Stationary solution

Let us construct the invariant solutions based on the three-dimensional subalgebra spanned by the operators $X_1, X_3, X_4$ from (1.2). Proceeding as in Section 4.1 we obtain the following candidates for the invariant solutions:

$$u = r F(\theta), \quad w = r G(\theta), \quad p = r^2 H(\theta), \quad \rho = r K(\theta).$$

(4.6)

Substituting (4.6) in Eqs. (2.1) we obtain

$$F(\theta) = F_0, \quad G(t) = C_1 \cos(k\theta) + C_2 \sin(k\theta),$$

$$H(t) = \frac{1}{2} (F_0^2 + f F_0), \quad K(t) = \frac{N}{g} \left[ C_1 \sin(k\theta) - C_2 \cos(k\theta) \right],$$

(4.7)

where $C_1, C_2$ are arbitrary constants, $F_0$ is an arbitrary constant different from zero, and

$$k = \frac{N}{F_0}.$$

If $F_0 = 0$ the solution collapses to the trivial solution $u = w = p = \rho = 0$.

Substituting (4.7) in (2.15) and returning to the notation $u = u_\theta$ we obtain the following steady-state solution of the system (1.1):

$$u_\theta = F_0 r, \quad w = r \left[ C_1 \cos(k\theta) + C_2 \sin(k\theta) \right],$$

$$p = \frac{1}{2} (F_0^2 + f F_0) r^2, \quad \rho = \frac{N}{g} r \left[ C_1 \sin(k\theta) - C_2 \cos(k\theta) \right].$$

(4.8)

4.3 Nonlinear whirlpools

Constructing as above the invariant solutions based on the three-dimensional subalgebra spanned by the operators $X_2, X_4, X_5$ from (1.2) we obtain the following candidates for the invariant solutions:

$$u = -\frac{f r}{2} + z F(t), \quad w = z G(t), \quad p = -\frac{f^2 r^2}{8} + z^2 H(t), \quad \rho = z K(t).$$

Substituting these expressions in Eqs. (2.1) we obtain

$$F(t) = 0, \quad G(t) = 0, \quad K(t) = k = \text{const.}, \quad H(t) = -\frac{k}{2} g.$$

Invoking that $u = u_\theta$, we have the following solution of the system (1.1):

$$u_\theta = -\frac{f r}{2}, \quad w = 0, \quad p = -\frac{f^2 r^2}{8} - \frac{k}{2} g z^2, \quad \rho = k z,$$

(4.9)
where \( k \) is an arbitrary constant. If we take the coordinate system so that \( z \) is directed upward then \( k \) should be positive. In this coordinate system \( g < 0 \), and hence the pressure deviation \( p \) vanishes on the surface of the funnel defined by the equation

\[
r = \frac{2}{f} \sqrt{-g k z}
\]  

and is positive inside the funnel, i.e. under the condition

\[
r < \frac{2}{f} \sqrt{-g k z}.
\]

This situation is illustrated in Figure 1 showing the solution (4.9) in the form of a stationary whirlpool narrowing (expanding) with the depths. For this reason, we call the invariant solution (4.9) the whirlpool associated with nonlinear waves in the stratified fluid, or briefly the nonlinear whirlpool. The intensity of rotation of fluid particles inside the whirlpool is monotonically increasing from zero at the apex \((z = 0)\) of the whirlpool to its maximum value on the whirlpool’s shell. It is manifest from Eqs. (4.9) and the definition \( f = 2\Omega \sin \theta \) (see Introduction to Paper 5 in this volume), that the domain of the positive pressure is narrowing to zero at higher latitudes as shown in Figure 2. Note that we have considered in Figure 2 the whirlpools lying within the latitude range \( \theta \in [1^\circ, 90^\circ] \) North.

Let us discuss the energy of the nonlinear whirlpool. Substituting the solution (4.9) in the expressions for the components of the conserved vector
(3.7) we obtain

\[ C^1 = \left( \frac{k}{2} g z^2 - \frac{f^2 r^2}{8} \right) \frac{\partial a(t, \theta)}{\partial \theta}, \]
\[ C^2 = \left( \frac{f^2 r^2}{8} - \frac{k}{2} g z^2 \right) \frac{\partial a(t, \theta)}{\partial t}, \]
\[ C^3 = 0, \quad C^4 = 0. \]

(4.12)

It is manifest that the vector (4.12) satisfies the conservation equation

\[ D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) = 0. \]

The energy density \( C^1 \) given in (4.12) changes from zero at the apex of the whirlpool to the value

\[ C^1 = k g \frac{\partial a(t, \theta)}{\partial \theta} z^2 \]

on the whirlpool’s shell. The total energy of mass of water inside the whirlpool is equal to

\[ E = \int \left( \frac{k}{2} g z^2 - \frac{f^2 r^2}{8} \right) \frac{\partial a(t, \theta)}{\partial \theta} d\theta dr dz, \]

(4.13)

where the integral is taken over the volume of the cone

\[ r \leq \frac{2}{f} \sqrt{-gk z}. \]

4.4 Invariant solution based on \( X_2, X_3 \)

Invariance with respect to the translations in \( \theta \) and \( z \) generated by \( X_2 \) and \( X_3 \), respectively requires that the dependent variables are functions of \( t \) and \( r \) only. Accordingly, the system (1.1) reduces to the form

\[ \frac{\partial u_\theta}{\partial t} = 0, \]
\[ \frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} + fu_\theta, \]
\[ \frac{\partial w}{\partial t} = -\rho g, \]
\[ \frac{\partial \rho}{\partial t} = \frac{N^2}{g} w. \]

(4.14)
Integration of the equations (4.12) gives the invariant solution

\[ u_\theta = U(r), \]
\[ p = \int \left[ \frac{1}{r} U^2(r) + fU(r) \right] dr + V(t), \]
\[ w = W_1(r) \cos(Nt) + W_2(r) \sin(Nt), \]
\[ \rho = \frac{N^2}{g} [W_1(r) \sin(Nt) - W_2(r) \cos(Nt)] \]

with arbitrary functions \( U(r), W_1(r), W_2(r) \) and \( V(t) \).
Paper 7

Group analysis of a model of atmospheric flows

Revised preprints [46], [3]

Abstract. We investigate group invariant solutions and conservation laws of a third-order nonlinear partial differential equation with three independent variables. The equation under consideration describes incompressible fluid flows on a three-dimensional spherical rotating surface. The equation can be used to model zonal west-to-east flows in the upper atmosphere.

Keywords: Atmospheric zonal flows, Nonlinear model, Invariant solutions, Self-adjointness, Conservation laws.

1 Introduction

The considered model is described by the following third-order nonlinear equation (for a physical discussion of the model, see [4])

$$\frac{\partial \Delta_s \psi}{\partial t} + \frac{\psi_\varphi}{R_0} + \frac{F}{\sin \theta} \frac{\partial \Delta_s \psi}{\partial \varphi} - \frac{\psi L_1 F}{\sin \theta} + \frac{\varepsilon}{\sin \theta} J(\psi_\theta, \Delta_s \psi) = 0. \quad (1.1)$$

Here

$$\Delta_s = \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \quad (1.2)$$

is the Laplace-Beltrami operator on the unit sphere written in the spherical angles, whereas

$$L_1 = \frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d}{d \theta} \right) - \frac{1}{\sin^2 \theta} \quad (1.3)$$

is the Stourn-Liouville operator for the associated Legendre functions.
Substituting (1.2) and (1.3) in (1.1) we write Equation (1.1) in the expanded form

\[
\Omega \equiv \psi_t \cos \theta + \psi_{\theta \theta} \sin \theta + \frac{1}{\sin \theta} \psi_{\varphi \varphi} + \frac{\sin \theta}{R_0} \psi_{\varphi} \\
+ \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta \varphi} + \psi_{\theta \theta \varphi} + \frac{1}{\sin^2 \theta} \psi_{\varphi \varphi \varphi} \right) \left[ F(\theta) + \varepsilon \psi_{\theta} \right] \\
- \left( F''(\theta) + \frac{\cos \theta}{\sin \theta} F'(\theta) - \frac{1}{\sin^2 \theta} F(\theta) \right) \psi_{\varphi} \\
- \varepsilon \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta \theta} + \psi_{\theta \theta \theta} + \frac{1}{\sin^2 \theta} (\psi_{\theta \varphi \varphi} - \psi_{\theta}) - \frac{2 \cos \theta}{\sin^3 \theta} \psi_{\varphi \varphi} \right) \psi_{\varphi} = 0.
\]

(1.4)

Here \( \theta, \varphi \) and \( t \) are three independent variables, \( \psi \) is the dependent variable with the partial derivatives \( \psi_{\theta}, \psi_{\varphi}, \) etc. Equation (1.4) contains an arbitrary function \( F(\theta) \) and two parameters \( \varepsilon, R_0 \) which are supposed in what follows different from zero.

Note that the spherical components \( u_{\varphi}, u_{\theta} \) of the velocity vector are written

\[ u_{\varphi} = \psi_{\theta}, \quad u_{\theta} = -\frac{\psi_{\varphi}}{\sin \theta}. \]

(1.5)

2 Symmetries

Solving the determining equation one can demonstrate that Equation (1.1) with an arbitrary function \( F(\theta) \) admits the Lie algebra with the following basis:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = \lambda(t) \frac{\partial}{\partial \psi}, \\
X_4 = 2R_0 \varepsilon t \frac{\partial}{\partial t} - \varepsilon t \frac{\partial}{\partial \varphi} + (\cos \theta - 2R_0 H(\theta) - 2\varepsilon R_0 \psi) \frac{\partial}{\partial \psi}, \\
X_5 = \varepsilon \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \theta} + \varepsilon \cos \theta \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \varphi} \\
- \left( F(\theta) + \frac{\sin \theta}{2R_0} \right) \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \psi}, \\
X_6 = \varepsilon \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \theta} - \varepsilon \cos \frac{\theta}{\sin \theta} \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \varphi} \\
- \left( F(\theta) + \frac{\sin \theta}{2R_0} \right) \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \psi}.
\]

(2.1)
Here $\lambda(t)$ is an arbitrary function and

$$H(\theta) = \int F(\theta) d\theta.$$  (2.2)

Computing the commutators of the operators (2.1) one obtains the following commutator table:

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0</td>
<td>0</td>
<td>$X_3'$</td>
<td>$2\varepsilon R_0 X_1 - \varepsilon X_2$</td>
<td>$\frac{1}{2\varepsilon R_0} X_6$</td>
<td>$-\frac{1}{2\varepsilon R_0} X_5$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$X_6$</td>
<td>$-X_5$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$-X_3'$</td>
<td>0</td>
<td>0</td>
<td>$X_3''$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_4$</td>
<td>$\varepsilon X_2 - 2\varepsilon R_0 X_1$</td>
<td>0</td>
<td>$-X_3''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_5$</td>
<td>$-\frac{1}{2\varepsilon R_0} X_6$</td>
<td>$-X_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\varepsilon^2 X_2$</td>
</tr>
<tr>
<td>$X_6$</td>
<td>$\frac{1}{2\varepsilon R_0} X_5$</td>
<td>$X_5$</td>
<td>0</td>
<td>0</td>
<td>$-\varepsilon^2 X_2$</td>
<td>0</td>
</tr>
</tbody>
</table>

Here

$$X_3' = \lambda'(t) \frac{\partial}{\partial \psi}, \quad X_3'' = -2\varepsilon R_0 [\lambda(t) + t \lambda'(t)] \frac{\partial}{\partial \psi}.$$  

To verify that the operators $X_4$ and $X_5$ is admitted by Equation (1.4), we prolong the operators $X_4$ and $X_5$ and obtain after calculations:

$$X_4(\Omega) = -4\varepsilon R_0 \Omega, \quad X_5(\Omega) = \varepsilon \frac{\cos \theta}{\sin \theta} \sin \left( \varphi + \frac{t}{2R_0} \right) \Omega.$$  

This proves that the operators $X_4$ and $X_5$ are admitted by Equation (1.4). This fact for the operator $X_6$ follows from the commutator table because the commutator of any two admitted operators is also admitted.

### 3 Invariant solutions

#### 3.1 Invariant solution based on $X_2$ and $X_4$

The commutator table from Section 2 shows that $X_2$ and $X_4$ span an Abelian two-dimensional subalgebra. Let us find the invariant solution with respect to this subalgebra. The equations

$$X_2 J(t, \varphi, \theta, \psi) = 0, \quad X_4 J(t, \varphi, \theta, \psi) = 0$$
provide two functionally independent invariants
\[ \lambda = \theta, \quad \mu = t \left( \psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos \theta}{2\varepsilon R_0} \right). \]

Letting \( \mu = \Phi(\lambda) \), we obtain the following general form for the invariant solutions:
\[ \psi = \frac{\cos \theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{1}{t} \Phi(\theta). \quad (3.1) \]

Substituting (3.1) in Equation (1.4) we obtain the second-order ordinary differential equation
\[ \Phi'' + \frac{\cos \theta}{\sin \theta} \Phi' = 0, \]
whence
\[ \Phi = C_1 + C_2 \ln |\tan(\theta/2)|. \]
Substituting this in (3.1) we obtain the following invariant solution:
\[ \psi = \frac{\cos \theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{C_1}{t} + \frac{C_2}{t} \ln |\tan(\theta/2)|, \quad (3.2) \]
where \( C_1, C_2 \) are arbitrary constants.

**Remark.** The operator \( X_2 \) generates the obvious translation \( \overline{\varphi} = \varphi + \beta_2 \) of the angle \( \varphi \), where \( \beta_2 \) is the group parameter (the subscript in the group parameter coincides with that of the group generator). The operator \( X_4 \) generates the one-parameter group of a more complex form, namely:
\[ \bar{t} = t e^{2\varepsilon R_0 \beta_4}, \quad \bar{\theta} = \theta, \quad \overline{\varphi} = \varphi + \frac{t}{2R_0} \left( 1 - e^{2\varepsilon R_0 \beta_4} \right), \]
\[ \overline{\psi} = \psi e^{-2\varepsilon R_0 \beta_4} + \frac{\cos \theta - 2R_0 H(\theta)}{2\varepsilon R_0} \left( 1 - e^{-2\varepsilon R_0 \beta_4} \right). \quad (3.3) \]
Thus, the solution (3.2) is invariant under the translation of the angle \( \varphi \) (due to the independence on this angle) and the transformation (3.3).

### 3.2 Invariant solution based on \( X_4 \) and \( X_5 \)

These operators also span an Abelian subalgebra. Let us find a basis of invariants of this subalgebra. Three functionally independent invariants for \( X_4 \) are
\[ \theta, \quad \nu = t + 2R_0 \varphi, \quad \mu = t \left( \psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos \theta}{2\varepsilon R_0} \right). \]
Acting on the invariants by the operator $X_5$ we obtain:

$$X_5(\theta) = \varepsilon \sin \left( \phi + \frac{t}{2R_0} \right) = \varepsilon \sin \frac{\nu}{2R_0},$$

$$X_5(\nu) = 2\varepsilon R_0 \frac{\cos \theta \cos \left( \phi + \frac{t}{2R_0} \right)}{\sin \theta} = 2\varepsilon R_0 \frac{\cos \theta \cos \frac{\nu}{2R_0}}{\sin \theta}.$$

The reckoning also shows that $X_5(\mu) = 0$. Hence, $X_5$ is written in terms of the invariants as follows:

$$X_5 = \varepsilon \sin \frac{\nu}{2R_0} \frac{\partial}{\partial \theta} + 2\varepsilon R_0 \frac{\cos \theta}{\sin \theta} \cos \frac{\nu}{2R_0} \frac{\partial}{\partial \nu}. \quad (3.4)$$

Solving the equation $X_5 J(\theta, \nu, \mu) = 0$ with the operator $X_5$ given by (3.4) one readily obtains the invariants $\mu$ and

$$\lambda = \sin \theta \cos \frac{\nu}{2R_0}.$$

Thus, the subalgebra spanned by $X_4, X_5$ has the following two functionally independent invariants:

$$\lambda = \sin \theta \cos \left( \phi + \frac{t}{2R_0} \right), \quad \mu = t \left( \psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos \theta}{2\varepsilon R_0} \right). \quad (3.5)$$

Letting $\mu = \Phi(\lambda)$, we obtain the following general form for the invariant solutions:

$$\psi = \frac{\cos \theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{1}{t} \Phi(\lambda). \quad (3.6)$$

Here $\lambda$ is given in (3.5) and therefore

$$\lambda_\varepsilon = -\frac{1}{2R_0} \sin \theta \sin \left( \phi + \frac{t}{2R_0} \right),$$

$$\lambda_\theta = \cos \theta \cos \left( \phi + \frac{t}{2R_0} \right), \quad (3.7)$$

$$\lambda_\varphi = -\sin \theta \sin \left( \phi + \frac{t}{2R_0} \right).$$

For the sake of brevity, we will use the notation

$$a = \cos \theta, \quad b = \sin \theta, \quad A = \cos \left( \phi + \frac{t}{2R_0} \right), \quad B = \sin \left( \phi + \frac{t}{2R_0} \right). \quad (3.8)$$
In this notation Equations (3.7) are written:

\[ \lambda = bA, \quad \lambda_t = \frac{-bB}{2R_0}, \quad \lambda_\theta = aA, \quad \lambda_\varphi = -bB. \]  

(3.9)

In view of the equations (3.9) and (3.8) the derivatives of the function (3.6) are written as follows:

\[ \psi_\theta = -\frac{b}{2\varepsilon R_0} - \frac{F'}{\varepsilon} + \frac{aA}{t} \Phi', \quad \psi_\varphi = -\frac{bB}{t} \Phi', \]

\[ \psi_\theta\theta = -\frac{a}{2\varepsilon R_0} - \frac{F'}{\varepsilon} - \frac{b^2AB}{t} \Phi' + \frac{a^2A^2}{t^2} \Phi'', \quad \psi_\varphi\varphi = -\frac{bA}{t^2} \Phi' + \frac{b^2B^2}{t^2} \Phi'', \]

\[ \psi_{\theta\varphi} = -\frac{aB}{t} \Phi' - \frac{abAB}{t^2} \Phi'', \quad \psi_{\theta\theta} = -\frac{aA}{t^2} \Phi' - \frac{1}{2tR_0} (aB\Phi' + abAB\Phi''), \]

\[ \psi_{\theta\varphi\varphi} = \frac{bB}{2tR_0} \Phi' + \frac{bA}{t^2} \Phi' - \frac{b^2B^2}{t^2} \Phi'' + \frac{3b^2AB}{2tR_0} \Phi'' - \frac{b^3B^3}{2tR_0} \Phi''', \]

\[ \psi_{\theta\theta\theta} = \frac{b}{2\varepsilon R_0} - \frac{F''}{\varepsilon} - \frac{3abA^2}{t^3} \Phi'+ \frac{a^3A^3}{t^3} \Phi''', \]

\[ \psi_{\theta\theta\varphi} = \frac{bB}{t} \Phi' + \frac{b^2AB}{t} \Phi'' - \frac{2a^2AB}{t^2} \Phi'' - \frac{a^2bA^2B}{t} \Phi''' , \]

\[ \psi_{\theta\varphi\varphi\varphi} = -\frac{aA}{t} \Phi' - \frac{abA}{t} \Phi'' + \frac{2abB^2}{t^2} \Phi'' + \frac{ab^2AB^2}{t^2} \Phi''' , \]

\[ \psi_{\varphi\varphi\varphi\varphi} = -\frac{bB}{t} \Phi' + \frac{3b^2AB}{t} \Phi'' - \frac{b^3B^3}{t} \Phi''' . \]

One can verify after some calculations that due to these expression for the derivatives of \( \psi \) Equation (1.4) reduces to the following form:

\[ [B^2 + a^2A^2] \Phi'' - bA\Phi' = 0. \]  

(3.10)

According to the notation (3.8), we have

\[ B^2 = 1 - A^2, \quad a^2 = 1 - b^2. \]

Substituting these expressions in (3.10) and invoking that \( bA = \lambda \) (the first equation (3.9)) we obtain

\[ (1 - \lambda^2)\Phi'' - 2\lambda\Phi' = 0. \]
We integrate this second-order linear ordinary differential equation, note that $|\lambda| \leq 1$ by definition of $\lambda$, and write the solution in the form
\[ \Phi(\lambda) = C_1 + C_2 \ln \frac{1 - \lambda}{1 + \lambda} \]
when $|\lambda| \neq 1$. Here $C_1, C_2$ are arbitrary constants. Inserting the above expression for $\Phi(\lambda)$ in (3.6) we obtain the following invariant solution:
\[ \psi = C_1 + \frac{\cos \theta}{2 \varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{C_2}{t} \ln \frac{1 - \lambda}{1 + \lambda}, \quad \lambda = \sin \theta \cos \left( \varphi + \frac{t}{2R_0} \right). \quad (3.11) \]

4 Self-adjointness

For investigating the self-adjointness, it is convenient to write Equation (1.4) in the form
\[ \Omega_1 + \Omega_2 + \varepsilon \Omega_3 = 0 \quad (4.1) \]
where
\[ \Omega_1 = \psi_{t\theta} \cos \theta + \psi_{t\theta\theta} \sin \theta + \frac{1}{\sin \theta} \psi_{t\varphi\varphi} + \frac{\sin \theta}{R_0} \psi_{\varphi}, \quad (4.2) \]
\[ \Omega_2 = \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2 \theta} \psi_{\varphi\varphi} \right) F(\theta) \]
\[ - \left( F''(\theta) + \frac{\cos \theta}{\sin \theta} F'(\theta) - \frac{1}{\sin^2 \theta} F(\theta) \right) \psi_{\varphi}, \quad (4.3) \]
\[ \Omega_3 = \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2 \theta} \psi_{\varphi\varphi\varphi} \right) \psi_{\theta} \]
\[ - \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2 \theta} \left( \psi_{\varphi\varphi} - \psi_{\theta} \right) - \frac{2 \cos \theta}{\sin^3 \theta} \psi_{\varphi\varphi} \right) \psi_{\varphi}. \quad (4.4) \]
Then the adjoint equation to Equation (4.1) will be written
\[ \Omega_1^* + \Omega_2^* + \varepsilon \Omega_3^* = 0 \quad (4.5) \]
with
\[ \Omega_1^* = \frac{\delta(\nu \Omega_1)}{\delta \psi}, \quad \Omega_2^* = \frac{\delta(\nu \Omega_2)}{\delta \psi}, \quad \Omega_3^* = \frac{\delta(\nu \Omega_3)}{\delta \psi}, \quad (4.6) \]
where $\nu$ is a new dependent variable.

According to the definition of the (strict) self-adjointness given in Section 1.6 of the work [38] (Paper 4 in this volume), Eq. (4.1) is self-adjoint if
\[ [\Omega_1^* + \Omega_2^* + \varepsilon \Omega_3^*]_{\psi=\psi} = \lambda(t, \varphi, \theta, \psi)[\Omega_1 + \Omega_2 + \varepsilon \Omega_3]. \quad (4.7) \]
By definition of the variational derivative $\delta/\delta \psi$, we have, e.g.

$$\Omega_1^* = D_tD_\theta(v \cos \theta) - D_tD_\theta^2(v \sin \theta) - D_tD_\varphi^2\left(\frac{v}{\sin \theta}\right) - D_\varphi\left(\frac{\sin \theta}{R_0} v\right).$$

Working out the differentiations, we obtain

$$\Omega_1^* = -\left[v_{t\theta} \cos \theta + v_{t\theta\theta} \sin \theta + \frac{1}{\sin \theta} v_{t\varphi\varphi} + \frac{\sin \theta}{R_0} v_\varphi\right].$$

It is manifest that letting here

$$v = \psi$$

we obtain

$$\Omega_1^* = -\Omega_1.$$  \tag{4.9}

Hence, we take $\lambda = -1$ in Eq. (4.7).

The similar calculation for $\Omega_2^*$ show that the equation

$$\Omega_2^* = -\Omega_2$$  \tag{4.10}

holds when $v = \psi$ if and only if the function $F(\theta)$ solves the equation

$$\frac{dF}{d\theta} = \frac{\cos \theta}{\sin \theta} F.$$  \tag{4.11}

The general solution to Equation (4.11) is

$$F(\theta) = k \sin \theta, \quad k = \text{const.}$$  \tag{4.12}

Finally, the reckoning shows that the equation

$$\Omega_3^* = -\Omega_3$$  \tag{4.13}

holds when $v = \psi$.

Equations (4.9), (4.10) and (4.13) yield to the following result.

**Theorem 4.1.** The self-adjointness condition (4.7) for Equation (4.1) is satisfied if $F(\theta)$ has the form (4.12).

Upon inserting in (1.4) the functions

$$F(\theta) = k \sin \theta, \quad F'(\theta) = k \cos \theta, \quad F''(\theta) = -k \sin \theta,$$

(4.14)
the self-adjoint equation (1.4) is written as follows:

\[ \Lambda \equiv \left[ \psi_t \cos \theta + \psi_{t \theta} \sin \theta + \frac{1}{\sin \theta} \psi_{t \varphi \varphi} \right] + \left( \frac{1}{R_0} + 2k \right) \psi_{\varphi} \sin \theta + k \left( \psi_{\theta \varphi} \cos \theta + \psi_{\theta \theta \varphi} \sin \theta + \frac{1}{\sin \theta} \psi_{\varphi \varphi \varphi} \right) + \varepsilon \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta \varphi} + \psi_{\theta \theta \varphi} + \frac{1}{\sin^2 \theta} \psi_{\varphi \varphi \varphi} \right) \psi_{\theta} - \varepsilon \left( \frac{\cos \theta}{\sin \theta} \psi_{\theta \varphi} + \psi_{\theta \theta \varphi} + \frac{1}{\sin^2 \theta} (\psi_{\theta \varphi \varphi} - \psi_{\varphi}) - \frac{2 \cos \theta}{\sin^3 \theta} \psi_{\varphi \varphi} \right) \psi_{\varphi} = 0. \]

## 5 Preliminary discussion of conservation laws

The conserved vectors associated with every symmetry

\[ X = \xi^i(x, \psi) \frac{\partial}{\partial x^i} + \eta(x, \psi) \frac{\partial}{\partial \psi} \]

of the self-adjoint equation (4.15) will be computed by the general formula

\[ C^i = W \left[ \frac{\partial L}{\partial \psi_i} - D_j \left( \frac{\partial L}{\partial \psi_{ij}} \right) + D_j D_k \left( \frac{\partial L}{\partial \psi_{ijk}} \right) \right] + D_j(W) \left[ \frac{\partial L}{\partial \psi_{ij}} - D_k \left( \frac{\partial L}{\partial \psi_{ijk}} \right) \right] + D_j D_k(W) \frac{\partial L}{\partial \psi_{ijk}}, \]

where

\[ W = \eta - \xi^j \psi_j \]

and the following notation is used:

\[ x^1 = t, \ x^2 = \theta, \ x^3 = \varphi. \]

In the final formulae (5.2) we let \( v = \psi \) according to Eq. (4.8). In accordance with the notation (5.4), the conservation laws will be written in the form

\[ \left[ D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) \right]_{(1.4)} = 0. \]

Due to this notation, the component \( C^1 \) of the conserved vector is the conserved density.
In (5.2) the formal Lagrangian \( L = v \Lambda \) of Eq. (4.15) should be written in following symmetric form with respect to the mixed derivatives of \( \psi \):

\[
L = v \left[ \frac{\cos \theta}{2} (\psi_{\theta \theta} + \psi_{\theta \theta}) + \frac{\sin \theta}{3} (\psi_{\theta \theta} + \psi_{\theta \theta} + \psi_{\theta \theta}) \right. \\
+ \left( \frac{1}{R_0} + 2k \right) \psi_\phi \sin \theta + \frac{k \cos \theta}{2} (\psi_{\theta \phi} + \psi_{\phi \theta}) + \frac{\psi_{\theta \phi} + \psi_{\phi \theta} + \psi_{\phi \phi}}{3 \sin \theta} \\
+ \frac{k \sin \theta}{3} (\psi_{\theta \phi} + \psi_{\phi \theta} + \psi_{\theta \phi}) + \frac{k \sin \theta}{3} \psi_{\phi \phi} (5.6) \\
+ \varepsilon \psi_\theta \left( \frac{\cos \theta}{2 \sin \theta} (\psi_{\theta \phi} + \psi_{\phi \theta}) + \frac{1}{3} (\psi_{\theta \phi} + \psi_{\phi \theta} + \psi_{\theta \phi}) + \frac{1}{\sin^2 \theta} \psi_{\phi \phi} \right) \\
- \varepsilon \psi_\phi \left( \frac{\cos \theta}{\sin \theta} (\psi_{\phi \theta} + \psi_{\theta \phi} + \psi_{\theta \phi} + \psi_{\phi \phi} - 3 \psi_\theta) - \frac{2 \cos \theta}{\sin^3 \theta} \psi_{\phi \phi} \right].
\]

Substituting (5.6) in (5.2) and the letting \( v = \psi \), we obtain the following expression for the conserved density:

\[
C^1 = D_\theta (\psi \cos \theta) + \frac{1}{\sin \theta} \psi_{\phi \phi} W \\
+ \left[ \frac{1}{2} \psi \cos \theta - \frac{1}{3} D_\theta (\psi \sin \theta) \right] D_\theta (W) - \frac{\psi_\phi}{3 \sin \theta} D_\phi (W) (5.7) \\
+ \frac{\psi}{3 \sin \theta} D_\phi (W) + \frac{\psi}{3 \sin \theta} D_\phi (W).
\]

It can be simplified significantly by transferring the terms of the forms \( D_\theta (A) \) and \( D_\phi (B) \) from \( C^1 \) to \( C^2 \) and \( C^3 \), respectively. Namely, we write (5.7) in the form

\[
C^1 = D_\theta (\psi \sin \theta) + \frac{1}{\sin \theta} \psi_{\phi \phi} W \\
+ D_\theta \left[ \frac{1}{2} W \psi \cos \theta + \frac{1}{3} D_\theta (W) \psi \sin \theta - \frac{2}{3} W D_\theta (\psi \sin \theta) \right] \\
+ D_\phi \left[ \frac{1}{3 \sin \theta} \{ \psi D_\phi (W) - 2W \psi_\phi \} \right]
\]

and after the above simplification obtain

\[
C^1 = D_\theta (\psi \sin \theta) + \frac{1}{\sin \theta} \psi_{\phi \phi} W. (5.8)
\]
We will see that in each particular case the expression (5.8) can be simplified further.

Let us find the conserved density (5.7) associated with the first symmetry from (2.1), i.e. with the time translation generator $X_1$. Here $W = -\psi_t$ and Equation (5.8) yields

$$C^1 = -\psi_t \left( D_\theta (\psi_t \sin \theta) + \frac{1}{\sin \theta} \psi_{\varphi \varphi} \right)$$

$$= \psi_\theta \psi_{t\theta} \sin \theta + \frac{1}{\sin \theta} \psi_\varphi \psi_{t\varphi} - D_\theta (\psi_t \psi_\theta \sin \theta) - D_\varphi \left( \frac{1}{\sin \theta} \psi_t \psi_\varphi \right).$$

Hence the symmetry $X_1$ leads to the conserved density

$$C^1 = \psi_\theta \psi_{t\theta} \sin \theta + \frac{1}{\sin \theta} \psi_\varphi \psi_{t\varphi}.$$  \hspace{1cm} (5.9)

We can nullify this conserved density. Indeed, let us write the expression (5.9) for $C^1$ in the form

$$C^1 = D_\theta \left( \psi \psi_{t\theta} \sin \theta \right) + D_\varphi \left( \frac{1}{\sin \theta} \psi_t \psi_\varphi \right)$$

$$- \left[ \psi_{t\theta} \cos \theta + \psi_{t\theta} \sin \theta + \frac{1}{\sin \theta} \psi_{t\varphi} \right] \psi.$$ 

We eliminate the expression in the square brackets using Eq. (4.15) and after some calculations obtain:

$$C^1 = D_\theta (\Theta) + D_\varphi (\Phi) - \psi \Lambda,$$  \hspace{1cm} (5.10)

where $\Lambda$ is defined in Equation (4.15) and

$$\Theta = \psi \psi_{t\theta} \sin \theta + k \psi \psi_{\theta \varphi} \sin \theta - \varepsilon \left( \psi \psi_\varphi \psi_{\theta \theta} + \frac{\cos \theta}{\sin \theta} \psi_\theta \psi_\varphi \psi_{\varphi} \psi - \frac{1}{2 \sin^2 \theta} \psi_\varphi^3 \right),$$

$$\Phi = \frac{1}{\sin \theta} \psi \psi_{t\varphi} + \left( \frac{1}{2 R_0} + k \right) \psi^2 \sin \theta + \frac{k}{2 \sin \theta} \left( 2 \psi \psi_{\varphi \varphi} - \psi_\varphi^2 \right) - \frac{k}{2} \psi_\theta^3 \sin \theta$$

$$+ \varepsilon \left[ \psi \psi_{\theta} \psi_{t\theta} + \frac{1}{2 \sin^2 \theta} \left( 2 \psi \psi_\theta \psi_{t\varphi} - 2 \psi \psi_\varphi \psi_{t\theta} - \psi_\theta^2 \psi_\varphi \right) \right] + \frac{\cos \theta}{\sin \theta} \psi_\theta^2 + \frac{\cos \theta}{\sin^3 \theta} \psi_\varphi^2.$$

Therefore $C^1$ vanishes on Eq. (4.15) after transferring the terms $D_\theta (\Theta)$ and $D_\varphi (\Phi)$ into $C^2$ and $C^3$, respectively. This proves our statement.
6 Computation of conserved vectors

The construction of all components of the conserved vectors requires lengthy calculations. Therefore, they were computed [3] using the package MAPLE.

6.1 Symmetries

We let $F(\theta) = k \sin \theta$ and $H(\theta) = k \cos \theta$ in (2.1) and obtain the following symmetries of Eq. (4.15):

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = \lambda(t) \frac{\partial}{\partial \psi},
\]

\[
X_4 = 2R_0 \varepsilon t \frac{\partial}{\partial t} - \varepsilon t \frac{\partial}{\partial \varphi} + [(1 + 2kR_0) \cos \theta - 2\varepsilon R_0 \psi] \frac{\partial}{\partial \psi},
\]

\[
X_5 = \varepsilon \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \theta} + \varepsilon \cos \theta \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \varphi} - \left( k + \frac{1}{2R_0} \right) \sin \theta \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \psi},
\]

\[
X_6 = \varepsilon \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \theta} - \varepsilon \cos \theta \sin \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \varphi} - \left( k + \frac{1}{2R_0} \right) \sin \theta \cos \left( \varphi + \frac{t}{2R_0} \right) \frac{\partial}{\partial \psi}.
\]

6.2 Conserved vector associated with $X_1$

Let us find the conserved vector (5.2) associated with the symmetry $X_1$. Here $W = -\psi_t$ and the computation gives the following components of the conserved vector:

\[
C^1 = \psi_\varphi \psi_\theta \sin \theta + \frac{1}{\sin \theta} \psi_\varphi \psi_\varphi,
\]

\[
C^2 = - (\psi_\varphi \psi_t \theta + \psi_\varphi \psi_{tt} \theta + k \psi_\varphi \psi_{\theta \varphi} + k \psi_\varphi \psi_{t \theta \varphi}) \sin \theta + \varepsilon \left[ \psi_\varphi \psi_\varphi \psi_\theta \theta - \psi_\varphi \psi_\varphi \psi_{\theta \varphi} + \frac{\cos \theta}{\sin \theta} (\psi \varphi \psi_\varphi \psi_\theta - \psi \psi_\varphi \psi_{\theta \varphi}) \right],
\]

\[
+ \frac{1}{\sin^2 \theta} \left( \psi \varphi \psi_\varphi \psi_{\varphi \varphi} - \psi_\psi \varphi \psi_{\varphi \varphi \varphi} \right),
\]
\[ C^3 = -\frac{\sin \theta}{R_0} \psi_{\psi_t} - \frac{1}{\sin \theta} (\psi_{\psi_t\psi_\varphi} + \psi_{\psi_\varphi}) \]
\[-k \left[ (2\psi_{\psi_t} - \psi_{\psi_\varphi}) \sin \theta + \frac{1}{\sin \theta} (\psi_{\psi_t\psi_\varphi} - \psi_{\psi_\varphi}) \right] \]
\[+ \varepsilon \left[ \psi_{\psi_t\psi_\theta} - \psi_{\psi_\theta} \psi_{\psi_t} + \frac{1}{\sin \theta} (\psi_{\psi_t\psi_\theta} - \psi_{\psi_\theta}) \right] \]
\[+ \frac{1}{\sin^2 \theta} (\psi_{\psi_t\psi_{\varphi\varphi} - \psi_{\psi_\varphi}} - \psi_{\psi_t} - \psi_{\psi_{\varphi\varphi}} + \psi_{\psi_\varphi}^2) \].

We rewrite the conserved density (6.2) in the form (5.10),
\[ C^1 = -\psi \Lambda + D_\theta (\Theta) + D_\varphi (\Phi), \]
and transfer the terms \( D_\theta (\Theta) \) and \( D_\varphi (\Phi) \) into \( C^2 \) and \( C^3 \), respectively. As a result, we obtain
\[ \tilde{C}^2 \equiv C^2 + D_t (\Theta) = D_\varphi (\varepsilon \tilde{\Phi}), \]
\[ \tilde{C}^3 \equiv C^3 + D_t (\Phi) = D_\theta (-\varepsilon \tilde{\Phi}), \]
where
\[ \tilde{\Phi} = -\psi_{\psi_t\psi_\theta} - \frac{\cos \theta}{\sin \theta} \psi_{\psi_t} + \frac{1}{2 \sin^2 \theta} (2\psi_{\psi_\varphi} - 2\psi_{\psi_t\psi_\varphi} + \psi_{\psi_\varphi}^2). \]

It follows that the vector (6.2) - (6.4) is equivalent to the trivial conserved vector
\[ \tilde{C} = (-\psi \Lambda, 0, 0). \]

### 6.3 Conserved vector associated with \( X_2 \)

For the symmetry \( X_2 \) from (6.1) we obtain
\[ C^1 = -\psi_{\varphi} \left( D_\theta (\psi \sin \theta) + \frac{1}{\sin \theta} \psi_{\varphi\varphi} \right) \]
\[= D_\varphi \left[ \frac{1}{2} \psi_{\theta}^2 \sin \theta - \frac{1}{2 \sin \theta} \psi_{\psi_\varphi}^2 \right] - D_\theta (\psi_{\psi_\varphi} \sin \theta). \]
Calculating further we see that the conserved vector has the form
\[ C^1 = 0, \]
\[ C^2 = D_\varphi (\Phi), \]
\[ C^3 = -D_\theta (\Phi) - \psi \Lambda, \]
where
\[
\Phi = \varepsilon \left( \frac{1}{3} \sin^2 \theta \left( \psi_\varphi^3 - \psi_\varphi \psi_\varphi \psi_\varphi \right) - \frac{\cos \theta}{2 \sin \theta} \psi_\varphi \psi_\theta \psi_\varphi + \frac{1}{3} \psi_\varphi \psi_\theta \psi_\varphi + \frac{1}{3} \psi_\theta^2 \psi_\varphi \\
- \frac{2}{3} \psi_\varphi \psi_\theta \psi_\varphi - \frac{k}{6} \psi_\varphi \cos \theta + \frac{1}{3} (k \psi_\theta \psi_\varphi - 2 k \psi_\psi_\varphi - 3 \psi_\theta \psi_\theta) \sin \theta \right).
\]

Hence the symmetry $X_2$ provides the trivial conserved vector

\[(0, 0, -\psi \Lambda).\]

### 6.4 Conserved vector associated with $X_3$

For the symmetry $X_3$ we obtain the following nontrivial conserved vector:

\[
C^1 = 0,
\]

\[
C^2 = \lambda(t) \left[ \psi_\varphi \sin \theta + k \psi_\varphi \sin \theta - \varepsilon \left( \frac{\psi_\varphi \psi_\varphi \psi_\varphi}{\sin^2 \theta} + \psi_\varphi \psi_\theta \psi_\varphi + \frac{\cos \theta}{\sin \theta} \psi_\theta \psi_\varphi \right) \right],
\]

\[
C^3 = \lambda(t) \left[ \left( \frac{1}{R_0} + 2k \right) \psi \sin \theta + \frac{1}{\sin \theta} (\psi_\varphi + k \psi_\varphi) \right. \\
+ \varepsilon \left( \frac{\psi_\varphi \psi_\varphi \psi_\varphi}{\sin^2 \theta} + \psi_\theta \psi_\theta \psi_\varphi + \frac{\cos \theta}{\sin \theta} \psi_\theta \psi_\varphi \right) \right].
\]

(6.7)

For this vector the conservation equation (5.5) is satisfied in the form

\[D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = \lambda(t) \Lambda.\]

### 6.5 Conserved vector associated with $X_4$

The symmetry $X_4$ provides the conserved density (see the notation (1.5))

\[
C^1 = \left[ (1 + 2kR_0)u_\varphi \sin \theta + 2 \varepsilon R_0 \left( u_\theta^2 + u_\varphi^2 \right) + \varepsilon t R_0 \left( u_\theta^2 + u_\varphi^2 \right) \right] \sin \theta.
\]

It can be reduced to the form

\[C^1(1 + 2kR_0)u_\varphi \sin^2 \theta + 2 \varepsilon R_0 \left( u_\theta^2 + u_\varphi^2 \right) \sin \theta \]

because $\varepsilon t R_0 \left( u_\theta^2 + u_\varphi^2 \right) \sin \theta = 2 \varepsilon R_0 \left[ D_\theta (t \Theta) + D_\varphi (t \Phi) - t \psi \Lambda \right]$, where $\Theta$ and $\Phi$ are the expressions used in (5.10). Thus, invoking Eqs. (1.5), we get

\[C^1 = (1 + 2kR_0) \psi_\theta \sin^2 \theta + 2 \varepsilon R_0 \left( \psi_\theta \sin \theta + \frac{1}{\sin \theta} \psi_\varphi \right).\]
For $C^2$, $C^3$ we obtain

$$C^2 = 4\varepsilon^2 R_0 \psi \left( \psi_\varphi \psi_{\theta\theta} + \frac{\cos \theta}{\sin \theta} \psi_\theta \psi_\varphi + \frac{1}{\sin^2 \theta} \psi_\varphi \psi_{\varphi\varphi} \right)$$

$$- \varepsilon \left[ 4R_0 \psi (k \psi_{\theta\varphi} + \psi_{\varphi\theta}) \sin \theta + (2R_0 k + 1) \left( \frac{\psi_\theta \psi_\varphi}{\sin \theta} + \psi_\varphi \psi_{\theta\theta} \cos \theta \right) \right]$$

$$+ (2R_0 k + 1) \psi_{\theta\theta} \sin \theta \cos \theta,$$

$$C^3 = -4\varepsilon^2 R_0 \psi \left( \psi_\theta \psi_{\theta\theta} + \frac{\cos \theta}{\sin \theta} \psi_\theta^2 + \frac{\psi_\varphi \psi_{\varphi\varphi}}{\sin^2 \theta} \right) - \varepsilon \left[ 4R_0 \sin \theta \left( \frac{\psi_{\varphi\varphi} + \psi_{\varphi\theta}}{\sin \theta} \right) \right]$$

$$- (2R_0 k + 1) \left( \frac{\psi_\varphi^2}{\sin^2 \theta} + \psi_\theta \psi_{\theta\theta} \cos \theta + \frac{\cos \theta}{\sin^2 \theta} (\psi_\theta \psi_{\varphi\varphi} - \psi_\varphi \psi_{\theta\theta}) + \frac{\psi_\theta^2}{\sin \theta} \right)$$

$$- 2\psi_\varphi^2 \sin \theta + \frac{\psi_\varphi^2}{\sin \theta} + \left( \frac{1}{2} - R_0 k \right) \psi_\theta^2 \sin \theta ] + (2R_0 k + 1) \left( k \psi_\theta \cos^2 \theta \right)$$

$$+ \frac{\cos \theta}{\sin \theta} (k \psi_{\varphi\varphi} + \psi_{\varphi\theta}) + \left( \frac{2k + 1}{R_0} \right) \psi + k \psi_{\theta\theta} \right) \sin \theta \cos \theta \right].$$

In this case the conservation equation (5.5) has the form

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = ((2kR_0 + 1) \cos \theta - 4\varepsilon R_0 \psi) \Lambda.$$

### 6.6 Conserved vector associated with $X_5$

Here

$$C^1 = \left( k + \frac{1}{2R_0} \right) (\psi_\theta \sin \theta \cos \theta - \psi_{\varphi\varphi}) \sin h,$$

$$C^2 = \varepsilon^2 (\psi_\varphi \cos h + \psi \sin h) \left[ \psi_\varphi \psi_{\theta\theta} - \psi_\theta \psi_{\theta\varphi} + \frac{\cos \theta}{\sin \theta} (\psi_\varphi \psi_{\theta\theta} - \psi_\theta \psi_{\varphi\varphi}) \right]$$

$$+ \frac{1}{\sin^2 \theta} (\psi_\varphi \psi_{\theta\varphi} - \psi_\theta \psi_{\varphi\varphi}) - \frac{2 \cos \theta}{\sin^3 \theta} \psi_\varphi \psi_{\varphi\varphi} \right]$$

$$+ \varepsilon \left( k + \frac{1}{2R_0} \right) \left( \frac{2}{\sin \theta} \psi_\varphi \psi_{\varphi\varphi} + (\psi_\varphi \psi_{\theta\theta} - \psi_\theta \psi_{\varphi\varphi}) \sin \theta \right) \sin h$$

$$- \varepsilon (\psi_\varphi \cos h + \psi \sin h) \left[ \frac{1}{\sin \theta} (k \psi_{\varphi\varphi} + \psi_{\varphi\theta}) + (k \psi_{\theta\varphi} + \psi_{\varphi\theta}) \cos \theta \right]$$

$$+ \left( \left( k + \frac{1}{2R_0} \right) 2\psi_\varphi + k \psi_{\theta\varphi} + \psi_{\varphi\theta} \right) \sin \theta \right] + \left[ k \psi_\varphi \cos \theta \sin h \right]$$

$$- (k \psi_{\theta\varphi} + \psi_{\varphi\theta}) \sin \theta \sin \theta \frac{1}{2R_0} \psi \cos \theta \cos h \right] \left( k + \frac{1}{2R_0} \right) \sin \theta ,$$
\[ C^3 = \frac{\varepsilon}{\sin \theta} \left( k + \frac{1}{2R_0} \right) [\psi_\theta \psi_{\varphi} \cos h - (\psi_\varphi \psi_{\theta\varphi} + \psi_\theta \psi_{\varphi\varphi}) \sin h] \]
\[ + \left( k + \frac{1}{2R_0} \right)^2 \psi_\varphi \cos h + \left( k + \frac{1}{2R_0} \right) \left( \frac{1}{2R_0} \psi(1 - 2\sin^2 \theta) - k \psi_{\varphi\varphi} \right) \sin h, \]

where
\[ h = \varphi + \frac{t}{2R_0}. \]

The conservation equation (5.5) is satisfied in the form
\[ D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = -\varepsilon (\psi \sin h + \psi_\varphi \cos h) D_\theta(\Lambda) \]
\[ - \left( \varepsilon (\psi_\theta \sin h + \psi_{\theta\varphi} \cos h) + \left( k + \frac{1}{2R_0} \right) \sin \theta \sin h \right) \Lambda. \]

### 6.7 Conserved vector associated with \( X_6 \)

The conserved vector associated with symmetry \( X_6 \) and the corresponding conservation equation are obtained from the conserved vector provided by \( X_5 \) and the corresponding conservation equation by replacing \( \sin h \) with \( \cos h \) and \( \cos h \) with \( -\sin h \). For example,
\[ C^1 = \left( k + \frac{1}{2R_0} \right) (\psi_\theta \sin \theta \cos \theta - \psi_{\varphi\varphi}) \cos h, \]

\[ C^3 = -\frac{\varepsilon}{\sin \theta} \left( k + \frac{1}{2R_0} \right) [\psi_\theta \psi_\varphi \sin h + (\psi_\varphi \psi_{\theta\varphi} + \psi_\theta \psi_{\varphi\varphi}) \cos h] \]
\[ - \left( k + \frac{1}{2R_0} \right)^2 \psi_\varphi \sin h + \left( k + \frac{1}{2R_0} \right) \left( \frac{1}{2R_0} \psi(1 - 2\sin^2 \theta) - k \psi_{\varphi\varphi} \right) \cos h, \]

where \( h \) is the same as in the case of the symmetry \( X_5 \). The conservation equation has the form
\[ D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = \varepsilon (\psi_\varphi \sin h - \psi \cos h) D_\theta(\Lambda) \]
\[ + \left( \varepsilon (\psi_{\theta\varphi} \sin h - \psi_\theta \cos h) - \left( k + \frac{1}{2R_0} \right) \sin \theta \cos h \right) \Lambda. \]

**Exercise.** Write down the component \( C^2 \) of the conserved vector associated with the symmetry \( X_6 \).
Abstract. A new method is proposed for constructing exact solutions for systems of nonlinear partial differential equations. It is called the method of conservation laws. Application of the method to the Chaplygin gas allowed to construct new solutions containing several arbitrary parameters. It is shown that these solutions cannot be obtained, in general, as group invariant solutions.

Keywords: Method of conservation laws, Nonlinear PDEs, Chaplygin gas, Symmetries, Nonlocal conservation laws, Exact solutions.

1 Introduction

Despite the common appreciation of a significance of conservation laws as a useful tool in formulating and investigating mathematical models, numerical solution of differential equations, etc., I often here the question if conservation laws may help in solving systems of partial differential equations analytically.

I have considered some examples from this point of view and give an affirmative answer to the above question in the present paper.

In this introduction we discuss a difference between conservation laws for ordinary and partial differential equations from the point of view of their use in solving differential equations.
1.1 Ordinary differential equations

In the case of ordinary differential equations (ODEs)

\[ F(x, y, y', \ldots, y^{(s)}) = 0 \]  

(1.1)

any nontrivial conservation law

\[ D_x \left( \psi(x, y, y', \ldots, y^{(s-1)}) \right) \bigg|_{(1.1)} = 0 \]  

(1.2)

provides a first integral

\[ \psi(x, y, y', \ldots, y^{(s-1)}) = C, \quad C = \text{const.} \]  

(1.3)

Equation (1.2) should be satisfied on all solutions of Equation (1.1).

Thus, knowledge of conservation laws for a system of ordinary differential equations allows to reduce the order of the system, or find its general solution provided that we know enough conservation laws.

Often, significant physical properties of a problem described by ODEs can be expressed directly in terms of conservation laws without solving the ODEs in question. Consider, e.g. the two body problem (a planet and the sun) described by Newton’s equation

\[ m\ddot{x} = \frac{\alpha x}{r^3} \]  

(1.4)

for the position vector \(x\) of the planet. Here \(r = |x|\) is the distance of the planet from the sun, \(v = x'\) is the velocity of the planet, \(\ddot{x}\) is the second-order time derivative of the vector \(x\), and \(\alpha\) is the constant proportional to the mass \(m\) of the planet. Equation (1.4) has, in particular, the following conserved vectors:

\[ M = m(x \times v) \quad \text{(the angular momentum)} \]  

(1.5)

and

\[ A = v \times M + \frac{\alpha x}{r} \quad \text{(the Laplace vector)} \]  

(1.6)

It was shown already by Laplace [57] in 1798 that the second and the first Kepler’s laws can be derived from the conservation of the vectors (1.5) and (1.6), respectively, without solving the system of second-order ODEs (1.4).

Remark 1.1. It is widely known that the conservation of the angular momentum (1.5) follows from the invariance of Newton’s equation (1.4) with respect to the rotation group in the \(x\) space. The conservation of the Laplace vector (1.6) follows from a more complicated, infinite-order tangent (Lie-Bäcklund) symmetry group (see [21], Section 25).
1.2 Systems of partial differential equations

The situation is quite different in the case of partial differential equations (PDEs). Consider, e.g. the system of \( m \) PDEs

\[
F_\alpha(x, u, u(1), \ldots, u(s)) = 0, \quad \alpha = 1, \ldots, m, \tag{1.7}
\]

with \( n \) independent variables \( x = (x^1, \ldots, x^n) \) and \( m \) dependent variables \( u = (u^1, \ldots, u^m) \). The symbols \( u(1), \ldots, u(s) \) denote the sets of the partial derivatives of the orders 1, \ldots, \( s \). A conservation law for the system (1.7) is written

\[
D_i(C^i) = 0, \tag{1.8}
\]

where \( D_i \) is the total derivative in \( x^i \) and the summation in the repeated index \( i \) is assumed. Equation (1.8) should be satisfied on all solutions of the system (1.7). The vector

\[
C = (C^1, \ldots, C^n), \tag{1.9}
\]

with the components

\[
C^i = C^i(x, u, u(1), \ldots), \quad i = 1, \ldots, n, \tag{1.10}
\]

satisfying Equation (1.8) is called a conserved vector. For some PDEs the components (1.10) of a conserved vector may depend on additional, e.g. so-called nonlocal variables. Then one has nonlocal conservation laws. The Chaplygin gas considered in the present paper has nonlocal conservation laws.

Knowledge of a conserved vector (1.9) does not lead, in general, to a solution of the system (1.7). I propose in this paper (see also Paper 4 in this volume) a method for constructing solutions to systems of PDEs using conservation laws.

2 Formulation of the method

Let the vector (1.9) with the components (1.10) be a conserved vector for the system (1.7). We look for particular solutions of the system (1.7) by adding to Eqs. (1.7) the differential constraints

\[
D_1 \left[ C^1(x, u, u(1), \ldots) \right] = 0, \\
D_2 \left[ C^2(x, u, u(1), \ldots) \right] = 0, \\
\vdots \\
D_n \left[ C^n(x, u, u(1), \ldots) \right] = 0. \tag{2.1}
\]
Note that the over-determined system of \( m + n \) equations (1.7), (2.1) contains only \( m + n - 1 \) independent equations due to the conservation law (1.8).

The differential constraints (2.1) can be equivalently written in the integrated form

\[
\begin{align*}
C^1 (x, u, u(1), \ldots) &= h^1(x^2, x^3, \ldots, x^n), \\
C^2 (x, u, u(1), \ldots) &= h^2(x^1, x^3, \ldots, x^n), \\
&\vdots \\
C^n (x, u, u(1), \ldots) &= h^n(x^1, \ldots, x^{n-1}).
\end{align*}
\tag{2.2}
\]

3 Application to the Chaplygin gas

The Chaplygin gas is described by the polytropic one-dimensional gasdynamic equations with the polytropic exponent \( \gamma = -1 \):

\[
\begin{align*}
v_t + vv_x + \frac{1}{\rho} p_x &= 0, \\
\rho_t + v\rho_x + \rho v_x &= 0, \\
p_t + v p_x - p v_x &= 0.
\end{align*}
\tag{3.1}
\]

3.1 Local and nonlocal symmetries

Equations (3.1) have the six-dimensional Lie algebra of local, namely Lie point symmetries spanned by the operators

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= t \frac{\partial}{\partial x} + x \frac{\partial}{\partial x}, & X_4 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
X_5 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}, & X_6 &= t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} + 2 \rho \frac{\partial}{\partial \rho}.
\end{align*}
\tag{3.2}
\]

Moreover, it has been shown in [1] that the Chaplygin gas has two additional, nonlocal symmetries

\[
\begin{align*}
X_7 &= \sigma \frac{\partial}{\partial x} - \frac{\partial}{\partial \rho} + \frac{\rho}{p} \frac{\partial}{\partial \rho}, \\
X_8 &= \left(\frac{t^2}{2} + s\right) \frac{\partial}{\partial x} + t \frac{\partial}{\partial v} - \tau \frac{\partial}{\partial \rho} + \frac{\rho \tau}{p} \frac{\partial}{\partial \rho}.
\end{align*}
\tag{3.3}
\]

Here \( \tau, s, \sigma \) are the following nonlocal variables:

\[
\begin{align*}
\tau &= \int \rho \, dx, & s &= - \int \frac{\tau}{p} \, dx, & \sigma &= - \int \frac{dx}{p}.
\end{align*}
\tag{3.4}
\]
They can be equivalently defined by the compatible over-determined systems

\[
\begin{align*}
\tau_x &= \rho, \\
\tau_t &= -v\rho, \\
s_x &= -\frac{\tau}{p}, \\
s_t &= \frac{v\tau}{p}, \\
\sigma_x &= -\frac{1}{p}, \\
\sigma_t &= \frac{v}{p}.
\end{align*}
\]  

The dilation generators \(X_4, X_5, X_6\) are not admitted by the differential equations (3.5) for the nonlocal variables \(\tau, s, \sigma\). It follows that the eight-dimensional vector space spanned by the operators (3.2) and (3.3) is not a Lie algebra. However, we can extend the action of the operators \(X_4, X_5, X_6\) to the variables \(\tau, s, \sigma\) so that the extended operators will be admitted by Equations (3.5). The calculation yields the following extensions ([38], Sec. 11.4):

\[
\begin{align*}
X'_4 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s} + \sigma \frac{\partial}{\partial \sigma}, \\
X'_5 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p} + \tau \frac{\partial}{\partial \tau} - \sigma \frac{\partial}{\partial \sigma}, \\
X'_6 &= t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v} + 2\rho \frac{\partial}{\partial \rho} + 2\tau \frac{\partial}{\partial \tau} + 2s \frac{\partial}{\partial s}.
\end{align*}
\]  

The operators (3.3) and (3.6) together with the operators \(X_1, X_2, X_3\) from (3.2) span an eight-dimensional Lie algebra \(L_8\) with the commutator table

<table>
<thead>
<tr>
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<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
<th>(X'_4)</th>
<th>(X'_5)</th>
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<tr>
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<td>0</td>
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<td>(-2X_8)</td>
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</table>
3.2 Local conservation laws

The local conservation laws for the Chaplygin gas, i.e. those involving only the physical variables $t, x, v, \rho, p$, express the classical statements on conservation of mass, energy, momentum and center-of-mass theorem. These conservation laws are written in the differential form (1.8) as follows:

$$D_t(\rho) + D_x(\rho v) = \rho_t + \rho v_x, \quad (3.7)$$

$$D_t(\rho v^2 - p) + D_x(pv + \rho v^3) = 2\rho v(v_t + vv_x + \frac{1}{\rho} p_x) + v^2(\rho_t + \rho v_x + \rho v_x) - (p_t + \rho p_x - \rho v_x), \quad (3.8)$$

$$D_t(\rho v) + D_x(p + \rho v^2) = \rho(v_t + vv_x + \frac{1}{\rho} p_x) + v(\rho_t + \rho v_x + \rho v_x), \quad (3.9)$$

$$D_t(tpv - x\rho) + D_x(tp + t\rho v^2 - xpv) = t\rho(v_t + vv_x + \frac{1}{\rho} p_x) + (tv - x)(\rho_t + \rho v_x + \rho v_x). \quad (3.10)$$

3.3 Nonlocal conservation laws

The Chaplygin gas, unlike the general polytropic gas, has nonlocal conservation laws. They can be obtained by applying to nonlocal symmetries (3.3) the general procedure [38] for constructing conserved vectors of nonlinearly self-adjoint equations associated with their symmetries. The construction is based on the following statement demonstrated in [38].

**Proposition 3.1.** All systems of differential equations having local conservation laws are nonlinearly self-adjoint.

Therefore we use the local conservation laws (3.7)-(3.10) and obtain the following result [38].

**Proposition 3.2.** The nonlocal symmetries (3.3) of the Chaplygin gas provide four nonlocal conservation laws

$$[D_t(C^1) + D_x(C^2)]_{(3.1)} = 0 \quad (3.11)$$

with conserved vectors $C = (C^1, C^2)$ having the following coordinates:

$$C^1 = \sigma \rho, \quad C^2 = \sigma \rho v, \quad (3.12)$$

$$C^1 = tpv + \tau, \quad C^2 = t(\rho v^2 + p), \quad (3.13)$$

$$C^1 = tp, \quad C^2 = tv - \tau. \quad (3.14)$$
The calculation shows that the conservation equation (3.11) for the vectors (3.12) - (3.15) is satisfied in the following forms:

\[ D_t(\sigma \rho) + D_x(\sigma \rho v) = \sigma(\rho_t + v\rho_x + \rho v_x), \]  
\[ D_t(t\rho v + \tau) + D_x(t(\rho v^2 + p)) = t\rho \left( v_t + vv_x + \frac{1}{\rho} p_x \right) + tv(\rho_t + v\rho_x + \rho v_x), \]  
\[ D_t(t\rho) + D_x(t\rho v - \tau) = t(\rho_t + v\rho_x + \rho v_x), \]  
\[ D_t \left( (t^2 - 2s)\rho \right) + D_x \left( (t^2 - 2s)\rho v - 2t\tau \right) = (t^2 - 2s)(\rho_t + v\rho_x + \rho v_x). \]  

3.4 Solutions obtained from mass conservation

We use the conservation law (3.7) with the conserved vector \( C^1 = \rho, \ C^2 = \rho v \). The differential constraints (2.1) are written

\[ D_t(\rho) = 0, \quad D_x(\rho v) = 0, \]

or in the integrated form (2.2):

\[ \rho = g(x), \quad \rho v = h(t). \]

Thus we look for the solutions of the form

\[ \rho = g(x), \quad v = \frac{h(t)}{g(x)}. \]  

The functions (3.20) solve the second equation in (3.1) due to the conservation law (3.7). Therefore it remains to substitute (3.20) in the first and third equations of the system (3.1). The result of this substitution can be solved for the derivatives of \( p \):

\[ p_x = -h' + \frac{h^2 g'}{g^2}, \]  
\[ p_t = -\frac{hg'}{g^2} p + \frac{hh'}{g} - \frac{h^3 g'}{g^3}. \]  

The compatibility condition \( p_{xt} = p_{tx} \) of the system (3.21) gives the equation

\[ \left( g'' - 2 \frac{g'^2}{g} \right) p = g^2 \frac{h''}{h} - 2gh' - h^2 \frac{g''}{g} + 2h^2 \frac{g'^2}{g^2}. \]
One can continue the procedure by investigating the compatibility of
the over-determined system (3.21)-(3.22) for \( p \) in order to obtain a system
of differential equations for determining the functions \( g(x) \) and \( h(t) \). But
instead I will simplify the calculations by considering the particular case
when the coefficient for \( p \) in Eq. (3.21) vanishes:

\[
g'' - 2 \frac{g'^2}{g} = 0. \tag{3.23}
\]

The solution of Eq. (3.23) is

\[
g(x) = \frac{1}{ax + b}, \quad a, b = \text{const.} \tag{3.24}
\]

Substituting (3.24) in Eq. (3.22) we obtain

\[
h'' + 2ahh' = 0, \tag{3.25}
\]

whence

\[
h(t) = k \tan(c - akt) \tag{3.26}
\]

if \( a \neq 0 \), and

\[
h(t) = At + B \tag{3.27}
\]

if \( a = 0 \). We will assume that \( A \neq 0 \) in order to avoid the trivial solution
with constant \( v, \rho, p \).

If the constant \( a \) in (3.24) does not vanish, we substitute (3.24) and
(3.26) in Eqs. (3.21), integrate them and obtain

\[
p = k^2(ax + b) + Q \cos(c - akt), \quad Q = \text{const.} \tag{3.28}
\]

In the case \( a = 0 \) the similar calculations yield

\[
p = -Ax + \frac{b}{2} A^2t^2 + ABt + Q, \quad Q = \text{const.} \tag{3.29}
\]

Thus, using the conservation of mass we have arrived at the exact solu-
tions

\[
\rho = \frac{1}{ax + b}, \quad a \neq 0,
\]

\[
v = k(ax + b) \tan(c - akt),
\]

\[
p = k^2(ax + b) + Q \cos(c - akt)
\]
and

\[
\begin{align*}
\rho &= \frac{1}{b}, \quad b \neq 0, \\
v &= b(A t + B), \\
p &= -Ax + \frac{b}{2} A^2 t^2 + ABbt + Q.
\end{align*}
\tag{3.31}
\]

The question arises if the solutions (3.30) and (3.31) can be obtained as invariant solutions with respect to certain subalgebras of the Lie algebra of Lie point symmetries (3.2). This is possible if there exist constants \(l_1, \ldots, l_6\), not all zero, such that the operator

\[
X = l_1 X_1 + \cdots + l_6 X_6
\]

\[
= [l_1 + (l_4 + l_6)t] \frac{\partial}{\partial t} + [l_2 + l_3 t + l_4 x] \frac{\partial}{\partial x}
\]

\[
+ [l_3 - l_6 v] \frac{\partial}{\partial v} + [l_5 + 2l_6] \rho \frac{\partial}{\partial \rho} + l_5 p \frac{\partial}{\partial p}
\]

satisfies the invariant tests

(i) \[X \left( \rho - \frac{1}{ax + b} \right) \] \tag{3.30} 

(ii) \[X \left( v - k(ax + b) \tan(c - akt) \right) \] \tag{3.30} 

(iii) \[X \left( p - k^2(ax + b) - Q \cos(c - akt) \right) \] \tag{3.30} 

and

(i) \[X \left( \rho - \frac{1}{b} \right) \] \tag{3.31} 

(ii) \[X \left( v - b(A t + B) \right) \] \tag{3.31} 

(iii) \[X \left( p + Ax - \frac{b}{2} A^2 t^2 - ABbt - Q \right) \] \tag{3.31}

for the solutions (3.30) and (3.31), respectively.

**Proposition 3.3.** The solution (3.30) is not an invariant solution.

**Proof.** Substituting the expression (3.32) of \(X\) in the equation (i) of the system (3.33) and multiplying by \((ax + b)^2\) we obtain

\[(l_5 + 2l_6)(ax + b) + a(l_2 + l_3 t + l_4 x) = 0.\]
Whence, “splitting with respect to $t, x$”, i.e. equating to zero the terms with $t, x$ and the remaining term, we get the following three equations:

\[ l_3 = 0, \quad (3.35) \]
\[ l_5 + 2l_6 + l_4 = 0, \quad (3.36) \]
\[ a l_2 + b (l_5 + 2l_6) = 0. \quad (3.37) \]

Proceeding likewise with the equation (ii) of the system (3.33), taking into account Equation (3.35) and multiplying by $\cos(c - akt)$ we obtain

\[ \frac{ak(ax + b)[l_1 + (l_4 + l_6)t]}{\cos(c - akt)} - [l_6(ax + b) + a(l_2 + l_4 x)] \sin(c - akt) = 0. \]

Whence, equating to zero the terms with the different trigonometric functions and splitting with respect to $t, x$ as above, we arrive at the following equations:

\[ l_1 = 0, \quad (3.38) \]
\[ l_4 + l_6 = 0, \quad (3.39) \]
\[ a l_2 + b l_6 = 0. \quad (3.40) \]

Note that the equations (3.36) and (3.39) yield

\[ l_5 + l_6 = 0 \]

and that due to the latter equation two equations, (3.37) and (3.40), coincide. Hence, Equations (3.35)-(3.40) become

\[ l_1 = l_3 = 0, \quad l_4 + l_6 = 0, \quad l_5 + l_6 = 0, \quad a l_2 + b l_6 = 0. \quad (3.41) \]

Finally, we write the equation (iii) of the system (3.33) by taking into account Equation (3.41) and obtain

\[ l_5[k^2(ax + b) + Q \cos(c - akt)] - ak^2(l_2 + l_4 x) = 0, \]

whence

\[ l_2 = l_4 = l_5 = 0. \quad (3.42) \]

Now the second equation in (3.41) yields $l_6 = 0$ and Equations (3.41)-(3.42) become

\[ l_1 = l_2 = l_3 = l_4 = l_5 = l_6 = 0. \]

Thus the operator (3.32) vanishes. This proves our statement.

**Proposition 3.4.** The solution (3.31) is not an invariant solution.
Proof. Equation (3.34) (i) is written

\[ l_5 + 2l_6 = 0. \]  

Equation (3.34) (ii) is written

\[ l_3 - b(At + B)l_6 - bA[l_1 + (l_4 + l_6)t] = 0 \]

and, upon “splitting with respect to \( t' \), yields

\[ l_4 + 2l_6 = 0, \]  
\[ l_3 - b(Al_1 + Bl_6) = 0. \]

Equation (3.34) (iii) is written

\[
\left( -Ax + \frac{b}{2} A^2 t^2 + ABbt + Q \right)l_5 + A(l_2 + l_3 t + l_4 x) \\
+ Ab(At - B)[l_1 + (l_4 + l_6)t] = 0.
\]

Equation to zero the coefficient for \( t^2 \) in (3.46) we obtain

\[ l_5 + 2l_6 + 2l_4 = 0, \]

whence due to Equation (3.43) we get \( l_4 = 0 \). This equation together with Equations (3.43)-(3.44) yield

\[ l_4 = l_5 = l_6 = 0. \]  

(3.47)

We also have Equation (3.45) which is written now

\[ l_3 - bAl_1 = 0. \]  

(3.48)

Due to the equations (3.47), Equation (3.46) becomes

\[ A(l_2 + l_3 t) + Ab(At - B)l_1 = 0, \]

whence

\[ l_3 + bAl_1 = 0, \quad l_2 - bBl_1 = 0. \]  

(3.49)

The first equation in (3.49) together with Equation (3.48) show that

\[ l_1 = l_3 = 0 \]

and the second equation in (3.49) gives \( l_2 = 0 \). Hence we have arrived at the equations

\[ l_1 = l_2 = l_3 = l_4 = l_5 = l_6 = 0, \]

thus proving our statement.
3.5 Solutions provided by conservation of momentum

The conservation laws (3.8)-(3.10) of energy, momentum and the center-of-mass give Equations (2.2) of the following forms:

\[ \rho v^2 - p = g(x), \quad \rho v + \rho v^3 = h(t), \]  
(3.8')

\[ \rho v = g(x), \quad p + \rho v^2 = h(t), \]  
(3.9')

\[ t\rho v - x\rho = g(x), \quad tp + t\rho v^2 - x\rho v = h(t). \]  
(3.10')

The nonlocal conserved vectors (3.12)-(3.15) lead to Equations (2.2) of the following forms:

\[ \sigma \rho = g(x), \quad \sigma \rho v = h(t), \]  
(3.12')

\[ t\rho v + \tau = g(x), \quad p + \rho v^2 = h(t), \]  
(3.13')

\[ t\rho = g(x), \quad t\rho v - \tau = h(t), \]  
(3.14')

\[ (t^2 - 2s)\rho = g(x), \quad (t^2 - 2s)\rho v - 2t\tau = h(t). \]  
(3.15')

Here I will apply the method of conservation laws to the momentum conservation, i.e. use Equation (3.9'). Thus we seek the solutions of the form

\[ \rho = \frac{g(x)}{v}, \quad p = h(t) - g(x)v. \]  
(3.50)

We substitute (3.50) in the first equation of the system (3.1) and obtain:

\[ v_t - \frac{g'(x)}{g(x)} v = 0. \]  
(3.51)

Due to Equations (3.50), (3.51) the third equation of the system (3.1) becomes

\[ h'(t) - 2g'(x)v^2 - h(t)v_x = 0, \]  
(3.52)

whereas the second equation of the system (3.1) is satisfied identically.

The general solution of Equation (3.51) is readily obtained by integration with respect to \( t \) and has the form

\[ v = \frac{g(x)}{F(x) - tg'(x)} \]  
(3.53)

with an arbitrary function \( F(x) \). One can substitute the expression (3.53) for \( v \) in (3.52) and investigate the resulting equation for two unknown functions \( g(x) \) and \( h(t) \). But I will simplify the calculations by letting \( F(x) = 0 \). Then

\[ v = -\frac{g(x)}{tg'(x)} \]  
(3.54)
and Equation (3.52) is written

\[ h'(t) - 2 \frac{g'(x)}{g''(x)} \frac{1}{t^2} + \frac{h(t)}{t} - \frac{g(x)g''(x)}{g''(x)} = 0. \]  

(3.55)

In order to separate the variables, I take

\[ h(t) = \frac{\alpha}{t} \]  

(3.56)

with an arbitrary constant \( \alpha \neq 0 \) and reduce Equation (3.55) to the form

\[ \alpha g'' + 2gg' = 0 \]

or

\[ (\alpha g')' + (g^2)' = 0, \]

whence

\[ \alpha g' + g^2 = K, \quad K = \text{const.} \]  

(3.57)

The solution of Equation (3.57) depends on the sign of the constant \( K \).

Let

\[ K = \beta^2 > 0. \]

Then we write Equation (3.57) in the form

\[ \frac{dg}{g^2 - \beta^2} = -\frac{1}{\alpha} \, dx, \]

integrate it and obtain

\[ g = \frac{\alpha \kappa}{2} \left( \frac{1 + Ce^{-\kappa x}}{1 - Ce^{-\kappa x}} \right), \]  

(3.58)

where \( C \) is an arbitrary constant and

\[ \kappa = \frac{2\beta}{\alpha}. \]

If

\[ K = -\mu^2 < 0 \]

the solution of Equation (3.57) is given by

\[ g = \alpha \omega \tan(C - \omega x), \]  

(3.59)

where \( C \) is an arbitrary constant and

\[ \omega = \frac{\mu}{\alpha}. \]
Equations (3.58), (3.54), (3.50) and (3.56) give the following solution of the system (3.1):

\[ v = \frac{1}{2C\kappa t} \left( 1 - C^2 e^{-2\kappa x} \right) e^{\kappa x}, \]
\[ \rho = C \frac{\alpha \kappa^2 t e^{-\kappa x}}{(1 - Ce^{-\kappa x})^2}, \]
\[ p = \frac{\alpha}{t} \left[ 1 - \frac{1}{4C} (1 + Ce^{-\kappa x})^2 e^{\kappa x} \right]. \] (3.60)

Equations (3.59), (3.54), (3.50) and (3.56) give the following solution of the system (3.1):

\[ v = \frac{1}{\omega t} \sin(C - \omega x) \cos(C - \omega x), \]
\[ \rho = \frac{\alpha \omega^2 t}{\cos^2(C - \omega x)}, \]
\[ p = \frac{\alpha}{t} \cos^2(C - \omega x). \] (3.61)

**Exercise.** Answer the question on whether or not the solutions (3.60) and (3.61) can be obtained as invariant solutions with respect to certain subalgebras of the Lie algebra of Lie point symmetries (3.2).

### 3.6 Use of nonlocal conservation law

Let us use the conservation law (3.16) with the conserved vector (3.12):

\[ C^1 = \sigma \rho, \quad C^2 = \sigma \rho v, \]

where \( \sigma \) is the nonlocal variable determined by the equations (see (3.5)):

\[ \sigma_x = -\frac{1}{p}, \quad \sigma_t = \frac{v}{p}. \] (3.62)

The differential constraints (2.1) are written \( D_t(\sigma \rho) = 0, \ D_x(\sigma \rho v) = 0 \) and give

\[ \sigma \rho = f(x), \quad \sigma \rho v = g(t). \]

Thus we look for the solutions with the following specific forms of \( \rho \) and \( v \):

\[ \rho = \frac{f(x)}{\sigma}, \quad v = \frac{g(t)}{f(x)}. \] (3.63)

The functions (3.63) solve the second equation in (3.1) due to the conservation equation (3.16). Therefore it remains to substitute (3.63) in the first and third equations of the system (3.1).
Paper 9

Anisotropic diffusion equations

See [40], [41], [42]

Abstract. The nonlinear self-adjointness of anisotropic diffusion equations with an external source is investigated. Conservation laws associated with symmetries of self-adjoint equations are obtained. The conservation laws are used for constructing exact solutions different from group-invariant solutions.

Keywords: Diffusion equation, Nonlinear self-adjointness, Conservation laws, Exact solutions.

1 Preliminaries

1.1 Anisotropic diffusion equation with a source

Diffusion processes in anisotropic materials whose physical characteristics such as the thermal conductivity are affected by the temperature, are described by the nonlinear second-order evolution equation

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z \]  

(1.1)

known as the anisotropic diffusion equation. The modification

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z + q(u) \]  

(1.2)

of the anisotropic diffusion equation (1.1) describes the diffusion phenomena in the case of existence of an external source \( q(u) \). It is called the anisotropic diffusion equation with a source.

Physical applications and interesting mathematical properties of Eqs. (1.1), (1.2) are discussed in [15] (see also [23], Section 10.9).
The functions $f(u), g(u), h(u)$ are positive according to their physical meaning. So, we consider Eqs. (1.1) and (1.2) with arbitrary positive coefficients $f(u), g(u), h(u)$. Furthermore, we will assume that these coefficients are linearly independent and that none of them is constant, i.e.

$$f'(u) \neq 0, \quad g'(u) \neq 0, \quad h'(u) \neq 0. \quad (1.3)$$

Note, that Eq. (1.1) has a conservation form whereas Eq. (1.2) with $q(u) \neq 0$ does not have such form. A conservation form is useful in many respects, e.g. in qualitative and numerical analysis. Moreover, possibility of different conservation forms can be helpful. Therefore we will construct various conservation laws for Eq. (1.1) using the method of nonlinear self-adjointness [38] and investigate the question on existence of conservation laws for Eq. (1.2) with specific values of the source term $q(u) \neq 0$.

### 1.2 Definition of nonlinear self-adjointness

Recall the definition of nonlinear self-adjointness (see Section 3 of Paper 4 in this volume). Let us consider a second-order partial differential equation

$$F(x, u, u_{(1)}, u_{(2)}) = 0, \quad (1.4)$$

where $u$ is the dependent variable, $u_{(1)}$ and $u_{(2)}$ are the sets of the first-order partial derivatives $u_i$ and the second-order derivatives $u_{ij}$ of $u$ with respect to the independent variables $x = (x^1, \ldots, x^n)$. The adjoint equation to Eq. (1.4) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}) = 0, \quad (1.5)$$

where $F^*$ is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}) = \frac{\delta(vF)}{\delta u}. \quad (1.6)$$

Here $v$ is a new dependent variable and $v_{(1)}, v_{(2)}$ are the sets of its partial derivatives. Furthermore, $\delta(vF)/\delta u$ denotes the variational derivative of $vF$:

$$\frac{\delta(vF)}{\delta u} = \frac{\partial(vF)}{\partial u} - D_i \left( \frac{\partial(vF)}{\partial u_i} \right) + D_i D_k \left( \frac{\partial(vF)}{\partial u_{ik}} \right) - \cdots,$$

where the total differentiations are extended to the new dependent variable $v$:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + u_{ij} \frac{\partial}{\partial u_j} + v_{ij} \frac{\partial}{\partial v_j} + \cdots. \quad (1.7)$$
Eq. (1.4) is said to be nonlinearly self-adjoint [38] if the adjoint equation (1.5) is satisfied for all solutions $u$ of the original equation (1.4) upon a substitution

$$v = \varphi(x,u,u(1),u(2)), \quad \varphi \neq 0. \quad (1.8)$$

In what follows, we will simplify the calculations by considering the point-wise substitutions

$$v = \varphi(x,u), \quad \varphi(x,u) \neq 0, \quad (1.9)$$

instead of the differential substitutions (1.8). The condition that the function $\varphi$ that does not vanish is significant. The condition for the nonlinear self-adjointness can be written in the form

$$F^*(x,u,\varphi,u(1),\varphi(1),u(2),\varphi(2)) = \lambda F(x,u,u(1),u(2)), \quad (1.10)$$

where $\lambda = \lambda(x,u,u(1),\ldots)$ is an undetermined variable coefficient, $\varphi(1), \varphi(2)$ denote the derivatives of the function $\varphi(x,u)$. For instance, $\varphi(1)$ is the set of the first-order total derivatives

$$D_i(\varphi) = \frac{\partial \varphi(x,u)}{\partial x^i} + u_i \frac{\partial \varphi(x,u)}{\partial u}, \quad i = 1, \ldots, n.$$ 

Eq. (1.10) should be satisfied identically in all variables $x, u, u(1), u(2)$.

### 1.3 Conserved vectors associated with symmetries of nonlinearly self-adjoint equations

The general result on construction of conserved vectors associated with symmetries of nonlinearly self-adjoint equations demonstrated in [38] leads to the following statement for the second-order equation (1.4).

Let (1.4) be nonlinearly self-adjoint and admit a one-parameter point transformation group with the generator

$$X = \xi^i(x,u) \frac{\partial}{\partial x^i} + \eta(x,u) \frac{\partial}{\partial u}. \quad (1.11)$$

Then the vector

$$C^i = W \left[ \frac{\partial L}{\partial u_i} - D_j \left( \frac{\partial L}{\partial u_{ij}} \right) \right] + D_j(W) \frac{\partial L}{\partial u_{ij}}, \quad i = 1, \ldots, n, \quad (1.12)$$

is a conserved vector for Eq. (1.4), i.e. satisfies the conservation equation

$$[D_i(C^i)]_{(1.4)} = 0. \quad (1.13)$$
Here
\[ W = \eta - \xi^j u_j \]  \hspace{1cm} (1.14)
and \( \mathcal{L} \) is the formal Lagrangian for Eq. (1.4) given by
\[ \mathcal{L} = vF. \]  \hspace{1cm} (1.15)
It is assumed that the variable \( v \) and its derivatives are eliminated from the right-hand side of Eq. (1.12) by using the substitution (1.9), where the function \( \varphi(x, u) \) is found by solving the nonlinear self-adjointness condition (1.10).

2 Investigation for nonlinear self-adjointness

2.1 Equation (1.1)

Since Eq. (1.1) has the conservation form (1.13), it is nonlinearly self-adjoint by Theorem 8.1 from [38]. Let us find the corresponding substitution (1.9).

We write Eq. (1.1) in the form (1.4) with
\[ F = f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} - u_t + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2, \]  \hspace{1cm} (2.1)
insert the expression for \( F \) in (1.6) and after simple calculations obtain the following adjoint equation (1.5) with
\[ F^* = v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz}. \]  \hspace{1cm} (2.2)

In our case the substitution (1.9) has the form
\[ v = \varphi(t, x, y, z, u). \]  \hspace{1cm} (2.3)

Its derivatives are written
\[ v_t \equiv D_t(\varphi) = \varphi_u u_t + \varphi_t, \quad v_x \equiv D_x(\varphi) = \varphi_u u_x + \varphi_x, \]
\[ v_y \equiv D_y(\varphi) = \varphi_u u_y + \varphi_y, \quad v_z \equiv D_z(\varphi) = \varphi_u u_z + \varphi_z, \]
\[ v_{xx} \equiv D_x^2(\varphi) = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}, \]
\[ v_{yy} \equiv D_y^2(\varphi) = \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2\varphi_{yu} u_y + \varphi_{yy}, \]
\[ v_{zz} \equiv D_z^2(\varphi) = \varphi_u u_{zz} + \varphi_{uu} u_z^2 + 2\varphi_{zu} u_z + \varphi_{zz}. \]  \hspace{1cm} (2.4)

Now we take the nonlinear self-adjointness condition (1.10), where \( F \) and \( F^* \) are given by (2.1) and (2.2), respectively:
\[ v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} \]
\[ = \lambda \left[ -u_t + f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2 \right] \]
where the corresponding derivatives of \( v \) in the left-hand side should be replaced with their expressions (2.4). First we compare the coefficients for \( u_t \) in both sides of Eq. (2.5) and obtain
\[
\lambda = -\varphi_u.
\]
Then the coefficients for \( u_{xx}, u_{yy}, u_{zz} \) yield:
\[
\begin{align*}
 f(u)\varphi_u &= -f(u)\varphi_u, \\
 g(u)\varphi_u &= -g(u)\varphi_u, \\
 h(u)\varphi_u &= -h(u)\varphi_u.
\end{align*}
\]
By our assumption, the functions \( f(u), g(u), h(u) \) do not vanish. Therefore the above equations yield that \( \varphi_u = 0 \). Hence, \( \varphi = \varphi(t, x, y, z) \) and therefore
\[
\lambda = 0, \quad v_t = \varphi_t, \quad v_{xx} = \varphi_{xx}, \quad v_{yy} = \varphi_{yy}, \quad v_{zz} = \varphi_{zz}.
\]
Then Eq. (2.5) becomes
\[
\varphi_t + f(u)\varphi_{xx} + g(u)\varphi_{yy} + h(u)\varphi_{zz} = 0. \quad (2.6)
\]
Since \( f(u), g(u), h(u) \) are linearly independent and obey the conditions (1.3), whereas \( \varphi \) does not depend on \( u \), Eq. (2.6) yields
\[
\varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi_{zz} = 0. \quad (2.7)
\]
The general solution of Eqs. (2.7) is given by
\[
\varphi = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8
\]
with arbitrary constant coefficients \( a_1, \ldots a_8 \). This proves the following.

**Proposition 2.1.** Eq. (1.1) satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[
v = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8. \quad (2.8)
\]

### 2.2 Two-dimensional equation with a source

Let us consider Eq. (1.2), for the sake of simplicity, in the case of two spatial variables \( x, y \):
\[
u_t = (f(u)u_x)_x + (g(u)u_y)_y + q(u). \quad (2.9)
\]
The adjoint equation has the form
\[
F^* \equiv v_t + f(u)v_{xx} + g(u)v_{yy} + q'(u)v = 0. \quad (2.10)
\]
Repeating the calculations of Section 2.1 we obtain the following equation for the nonlinear self-adjointness of Eq. (2.9) (compare with Eq. (2.6)):

\[ \varphi_t + f(u)\varphi_{xx} + g(u)\varphi_{yy} + q'(u)\varphi = 0. \]  
(2.11)

If \( f(u), g(u) \) and \( q(u) \) are arbitrary functions, Eq. (2.11) yields (compare with Eqs. (2.7)):

\[ \varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi = 0. \]  
(2.12)

These equations show that a substitution of the form (1.9) does not exist. Indeed, the last equation in (2.12) contradicts the condition \( \varphi \neq 0 \). Hence, Eq. (2.9) with the arbitrary source \( q(u) \) is not nonlinearly self-adjointness with the substitution of the form (1.9).

However, Eq. (2.9) with sources of particular forms can be nonlinearly self-adjoint. For example, let

\[ q'(u) = rf(u), \quad r = \text{const}. \]  
(2.13)

Then Eq. (2.11) becomes

\[ \varphi_t + f(u)[\varphi_{xx} + r\varphi] + g(u)\varphi_{yy} = 0 \]

and yields (compare with Eqs. (2.12)):

\[ \varphi_t = 0, \quad \varphi_{yy} = 0, \quad \varphi_{xx} + r\varphi = 0. \]  
(2.14)

The solution to Eqs. (2.14) has the form

\[ \varphi = a(x)y + b(x), \]  
(2.15)

where \( a(x) \) and \( b(x) \) arbitrary solutions of the linear second-order ODE

\[ w'' + rw = 0. \]  
(2.16)

Eq. (2.13) shows that the source strength increases together with the temperature, i.e. \( q'(u) > 0 \), if \( r = \omega^2 > 0 \), and decreases, \( q'(u) < 0 \), if \( r = -\delta^2 < 0 \). Having this in mind and denoting

\[ \mathcal{F}(u) = \int f(u)du \]  
(2.17)

we consider two particular forms of Eq. (2.9):

\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y + \omega^2\mathcal{F}(u), \quad \omega = \text{const.}, \]  
(2.18)
and
\[ u_t = (f(u)u_x)_x + (g(u)u_y)_y - \delta^2 F(u), \quad \delta = \text{const} \]  
\hspace{2cm} (2.19)

In the case (2.18) Eq. (2.16) is written
\[ w'' + \omega^2 w = 0 \]
and yields
\[ w = C_1 \cos(\omega x) + C_2 \sin(\omega x). \]

Hence
\[ a(x) = A_1 \cos(\omega x) + A_2 \sin(\omega x), \quad b(x) = B_1 \cos(\omega x) + B_2 \sin(\omega x) \]
with arbitrary constants \( A_1, A_2, B_1, B_2 \). We we substitute these expressions in Eq. (2.15) and arrive at the following statement.

**Proposition 2.2.** Eq. (2.18) satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[ v = (A_1y + B_1) \cos(\omega x) + (A_2y + B_2) \sin(\omega x). \]  
\hspace{2cm} (2.20)

In the case (2.19) Eq. (2.16) is written
\[ w'' - \delta^2 w = 0 \]
and yields
\[ w = C_1 e^{\delta x} + C_2 e^{-\delta x}. \]

Proceeding as above we arrive at the following statement.

**Proposition 2.3.** Eq. (2.19) satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[ v = (A_1y + B_1) e^{\delta x} + (A_2y + B_2) e^{-\delta x}. \]  
\hspace{2cm} (2.21)

2.3 **Remark on materials with specific anisotropy**

The situation is different if the conditions (1.3) are not satisfied. Let, e.g. \( g(u) \) be a positive constant, \( g = k \). Then Eq. (1.1) has the form
\[ u_t = (f(u)u_x)_x + ku_{yy} + (h(u)u_z)_z. \]  
\hspace{2cm} (2.22)

In this case Eqs. (2.7) are replaced by the following equations:
\[ \varphi_t + k\varphi_{yy} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{zz} = 0. \]  
\hspace{2cm} (2.23)
The second and third equations of the system (2.23) yield
\[ \varphi = \alpha(t, y)xz + \beta(t, y)x + \gamma(t, y)z + \sigma(t, y). \]

The first equation (2.23) shows that \( \alpha(t, y), \beta(t, y), \gamma(t, y) \) and \( \sigma(t, y) \) solve the adjoint equation
\[ v_t + kv_{yy} = 0 \quad (2.24) \]
to the linear heat equation
\[ u_t - ku_{yy} = 0. \quad (2.25) \]

Thus, we have demonstrated the following statement.

**Proposition 2.4.** Eq. (2.22) satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[ v = \alpha(t, y)xz + \beta(t, y)x + \gamma(t, y)z + \sigma(t, y), \quad (2.26) \]
where \( \alpha(t, y), \beta(t, y), \gamma(t, y) \) and \( \sigma(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).

Combining Propositions 2.2 and 2.3 with Proposition 2.4, we obtain the following statements.

**Proposition 2.5.** The equation
\[ u_t = (f(u)u_x)_x + ku_{yy} + \omega^2 F(u), \quad f(u) = F'(u), \quad (2.27) \]
satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[ v = \alpha(t, y) \cos(\omega x) + \beta(t, y) \sin(\omega x), \quad (2.28) \]
where \( \alpha(t, y) \) and \( \beta(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).

**Proposition 2.6.** The equation
\[ u_t = (f(u)u_x)_x + ku_{yy} - \delta^2 F(u), \quad f(u) = F'(u), \quad (2.29) \]
satisfies the nonlinear self-adjointness condition (1.10) with the substitution (2.3) of the form
\[ v = \alpha(t, y)e^{\delta x} + \beta(t, y)e^{-\delta x}, \quad (2.30) \]
where \( \alpha(t, y) \) and \( \beta(t, y) \) are any solutions of the adjoint equation (2.24) to the linear heat equation (2.25).
3 Conservation laws

3.1 Computation of conserved vectors for Eq. (1.1)

Here we construct the conserved vector (1.12) for Eq. (1.1),

\[ u_t = f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2, \]

(3.1)

associated with its translational symmetries. We specify the notation by writing the symmetry generator (1.11) in the form

\[ X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}. \]

(3.2)

Then the expression (1.14) becomes

\[ W = \eta - \xi^1 u_t - \xi^2 u_x - \xi^3 u_y - \xi^4 u_z \]

(3.3)

and the conservation equation (1.13) means that the following equation holds on the solutions of Eq. (3.1):

\[ D_t(C_1) + D_x(C_2) + D_y(C_3) + D_z(C_4) = 0. \]

(3.4)

The formal Lagrangian (1.15) for Eq. (3.1) is

\[ \mathcal{L} = v[f(u)u_{xx} + g(u)u_{yy} + h(u)u_{zz} + f'(u)u_x^2 + g'(u)u_y^2 + h'(u)u_z^2 - u_t]. \]

(3.5)

Due to the specific dependence of the formal Lagrangian (3.5) on the derivatives \( u_i, u_{ij} \), the components of the vector (1.12) are written

\[ C_1 = W \frac{\partial \mathcal{L}}{\partial u_t}, \]

\[ C_2 = W \left[ \frac{\partial \mathcal{L}}{\partial u_x} - D_x \left( \frac{\partial \mathcal{L}}{\partial u_{xx}} \right) \right] + D_x(W) \frac{\partial \mathcal{L}}{\partial u_{xx}}, \]

\[ C_3 = W \left[ \frac{\partial \mathcal{L}}{\partial u_y} - D_y \left( \frac{\partial \mathcal{L}}{\partial u_{yy}} \right) \right] + D_y(W) \frac{\partial \mathcal{L}}{\partial u_{yy}}, \]

\[ C_4 = W \left[ \frac{\partial \mathcal{L}}{\partial u_z} - D_z \left( \frac{\partial \mathcal{L}}{\partial u_{zz}} \right) \right] + D_z(W) \frac{\partial \mathcal{L}}{\partial u_{zz}}. \]

Substituting here the explicit expression (3.5) of \( \mathcal{L} \) we obtain:

\[ C_1 = -Wv, \]

\[ C_2 = W[f'(u)u_xv - f(u)v_x] + f(u)vD_x(W), \]

\[ C_3 = W[g'(u)u_yv - g(u)v_y] + g(u)vD_y(W), \]

\[ C_4 = W[h'(u)u_zv - h(u)v_z] + h(u)vD_z(W). \]

(3.6)
Eqs. (1.1) and (1.2) with arbitrary coefficients \( f(u), g(u), h(u) \) are invariant under the groups of translations of \( t, x, y, z \) with the generators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}.
\] (3.7)

Eq. (1.1) has also a dilation symmetry. Moreover, both equations (1.1) and (1.2) may have more symmetries in certain particular cases [23], but we don’t consider them here.

Let us apply the formula (3.6) to the symmetry \( X_2 \). The corresponding quantity (3.3) equals \( W = -u_x \). We substitute it in (3.6) and obtain

\[
\begin{align*}
C_1 &= vu_x, \\
C_2 &= -f'(u)vu_x^2 + f(u)u_xv_x - f(u)vu_{xx}, \\
C_3 &= -g'(u)vu_xu_y + g(u)u_xv_y - g(u)vu_{xy}, \\
C_4 &= -h'(u)vu_xu_z + h(u)u_xv_z - h(u)vu_{xz}.
\end{align*}
\] (3.8)

We have to substitute here the expression (2.8) for \( v \),

\[
v = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8.
\] (2.8)

Since \( v \) is a given function whereas \( u \) is any solution of Eq. (3.1), we want to simplify the conserved vector (3.8) by transforming it in an equivalent conserved vector which conserved density \( C_1 \) contains \( u \) instead of \( u_x \). To this end, we use the identity \( vu_x = D_x(uv) - uv_x \) and write \( C_1 \) in (3.8) in the form

\[
C_1 = \tilde{C}_1 + D_x(uv),
\]

where

\[
\tilde{C}_1 = -uv_x.
\] (3.9)

Then we transfer the term \( D_x(uv) \) from \( C_1 \) to \( C_2 \) using the usual procedure (see, e.g. [38], Section 8.1). Namely, since the total differentiations commute with each other, we have

\[
D_l(\tilde{C}_1 + D_x(uv)) + D_x(C_2) = D_l(\tilde{C}_1) + D_x(C_2 + D_l(uv)).
\]

Therefore the conservation equation (3.4) for the vector (3.8) can be equivalently rewritten in the form

\[
D_l(\tilde{C}_1) + D_x(\tilde{C}_2) + D_y(C_3) + D_z(C_4) = 0,
\] (3.10)

where \( \tilde{C}_1 \) has the form (3.9) and \( \tilde{C}_2 \) is given by

\[
\tilde{C}_2 = C_2 + D_l(uv).
\]
Let us work out the above expression for $\tilde{C}^2$. Invoking that $D_t(v) = 0$ due to Eq. (2.8), and using Eqs. (1.1), (3.1) we have

$$\tilde{C}^2 = C^2 + vu_t = C^2 + v\left[f(u)u_{xx} + f'(u)u_x^2 + D_y(g(u)u_y) + D_z(h(u)u_z)\right].$$

We substitute here the expression of $C^2$ from Eqs. (3.8) and obtain:

$$\tilde{C}^2 = f(u)u_xv_x + vD_y(g(u)u_y) + vD_z(h(u)u_z).$$

We simplify the latter expression for $\tilde{C}^2$ by noting that

$$vD_y(g(u)u_y) = D_y(g(u)vu_y) - g(u)u_yv_y,$$
$$vD_z(h(u)u_z) = D_z(h(u)vu_z) - h(u)u_zv_z.$$

Therefore we transfer the terms $D_y(g(u)vu_y)$ and $D_z(h(u)vu_z)$ to $C^3$ and $C^4$, respectively, and obtain:

$$\tilde{C}^2 = f(u)u_xv_x - g(u)u_yv_y - h(u)u_z.$$  \hspace{1cm} (3.11)

Now the components $C^3$ and $C^4$ of the vector (3.8) become:

$$\tilde{C}^3 = C^3 + D_x(g(u)vu_y), \quad \tilde{C}^4 = C^4 + D_z(h(u)vu_z).$$

After substituting here the expressions of $C^3$ and $C^4$ from (3.8) we have

$$\tilde{C}^3 = g(u)(u_xv_y + u_yv_x), \quad \tilde{C}^4 = h(u)(u_xv_z + u_zv_x).$$

Combining these expressions with (3.9), (3.11) and ignoring the tilde, we arrive at the following conserved vector which is equivalent to (3.8):

$$C^1 = -uv_x,$$
$$C^2 = f(u)u_xv_x - g(u)u_yv_y - h(u)u_zv_z,$$
$$C^3 = g(u)(u_xv_y + u_yv_x),$$
$$C^4 = h(u)(u_xv_z + u_zv_x).$$  \hspace{1cm} (3.12)

The vector (3.12) involves the first-order derivatives of the variable $v$ given by Eq. (2.8). Therefore the vector (3.12) contains seven parameters $a_1, \ldots, a_7$. In fact, it is a linear combination of seven linearly independent conserved vectors obtained from (3.12) setting by turns one of the parameters $a_i$ equal to 1 and the others equal to 0. But some of these seven vectors are trivial.
in the sense that their divergence is identically zero, i.e. the conservation equation (3.4) is satisfied identically. For example, setting in (3.12) $a_6 = 1, a_1 = \cdots = a_7 = 0$, i.e. $v = y$, we obtain the vector
\[ C^1 = 0, \quad C^2 = -g(u)u_y, \quad C^3 = g(u)u_x, \quad C^4 = 0. \]

For this vector Eq. (3.4) is satisfied identically,
\[ D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = -D_x(g(u)u_y) + D_y(g(u)u_x) \equiv 0. \]

Let us single out the nontrivial conserved vectors. Since $v$ given by Eq. (2.8), the conservation equation (3.4) for the vector (3.12) is written as
\[ D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = v_x F, \quad (3.13) \]
where $F$ is given by Eq. (2.1),
\[ F = -u_t + (f(u)u_x)_x + (g(u)u_y) y + (h(u)u_z) z. \]

Then, specifying the expression of $v_x$ from (2.8), we write (3.13) in the form
\[ D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = (a_1 y z + a_2 y + a_3 z + a_5) F. \quad (3.14) \]
Eq. (3.14) shows that we have only four nontrivial conserved vectors. They correspond to $a_1, a_2, a_3$ and $a_5$, i.e. they are obtained from (3.12) setting by turns one of these four parameters to be equal to 1, the others equal to 0. For example, the nontrivial conserved vector (3.12) corresponding to $a_5$ is
\[ C^1 = -u, \quad C^2 = f(u)u_x, \quad C^3 = g(u)u_y, \quad C^4 = h(u)u_z. \quad (3.15) \]
The conservation equation (3.4) for the vector (3.15) coincides with Eq. (1.1).

Thus, the nontrivial conserved vectors are obtained by substituting in (3.12) the expression (2.8) for $v$ with $a_4 = a_6 = a_7 = 0$. The resulting vector
\[ C^1 = -(a_1 y z + a_2 y + a_3 z + a_5) u, \]
\[ C^2 = (a_1 y z + a_2 y + a_3 z + a_5) f(u) u_x \]
\[ - (a_1 z + a_2) xo(u) u_y - (a_1 y + a_3) xh(u) u_z, \]
\[ C^3 = (a_1 z + a_2) g(u)(x u_x + y u_y) + (a_3 z + a_5) g(u) u_y, \]
\[ C^4 = (a_1 y + a_3) h(u)(x u_x + z u_z) + (a_2 y + a_5) h(u) u_z \]
is the linear combination with the coefficients $a_5, a_1, a_2, a_3$ of four linearly independent vectors, namely the vector (3.15) and the following three vectors:

\[ C^1 = -yzu, \quad C^2 = yz f(u) u_x - xz g(u) u_y - xy h(u) u_z, \]
\[ C^3 = zg(u)(xu_x + yu_y), \quad C^4 = yh(u)(xu_x + zu_z); \quad (3.17) \]

\[ C^1 = -yu, \quad C^2 = yf(u) u_x - xg(u) u_y, \]
\[ C^3 = g(u)(xu_x + yu_y), \quad C^4 = yh(u) u_z; \quad (3.18) \]

\[ C^1 = -zu, \quad C^2 = zf(u) u_x - xh(u) u_z, \]
\[ C^3 = zg(u) u_y, \quad C^4 = h(u)(xu_x + zu_z). \quad (3.19) \]

The conserved vectors associated with the symmetry $X_3$ from (3.7) can be obtained from the above results merely by the permutation $x \leftrightarrow y$, followed by the permutations $f \leftrightarrow g$ and $C^2 \leftrightarrow C^3$. This procedure maps the vector (3.12) to the following conserved vector:

\[ C^1 = -uv_y, \]
\[ C^2 = f(u)(yu_x + xu_y), \]
\[ C^3 = g(u)yv_y - f(u)ux_x - h(u)ux_z, \]
\[ C^4 = h(u)(yu_x + u_z). \quad (3.20) \]

Accordingly, Eq. (3.14) becomes the following conservation equation for the vector (3.20):

\[ D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = (a_1 xz + a_2 x + a_4 z + a_6)F. \quad (3.21) \]

It shows that the nontrivial conserved vectors are obtained by substituting in (3.20) the expression (2.8) for $v$ with $a_3 = a_5 = a_7 = 0$. The resulting vector

\[ C^1 = -(a_1 xz + a_2 x + a_4 z + a_6)u, \]
\[ C^2 = (a_1 z + a_2) f(u)(xu_x + yu_y) + (a_4 z + a_6) f(u)ux_x, \]
\[ C^3 = (a_1 xz + a_2 x + a_4 z + a_6) g(u)u_y \]
\[ - (a_1 z + a_2) yf(u)ux_x - (a_1 x + a_4) yh(u)ux_z, \]
\[ C^4 = (a_4 z + a_1) h(u)(yu_y + zu_z) + (a_2 x + a_6) h(u)ux_x \]
is the linear combination with the coefficients \(a_6, a_1, a_2, a_4\) of four linearly independent vectors, namely the vector (3.15) and the following three vectors:

\[
\begin{align*}
C^1 &= -xz u, \quad C^2 = zf(u)(xu_x + yu_y), \\
C^3 &= xzg(u)y - yz f(u)u_x - xyh(u)u_z, \\
C^4 &= xh(u)(yu_y + zu_z);
\end{align*}
\] (3.23)

\[
\begin{align*}
C^1 &= -x u, \quad C^2 = f(u)(xu_x + yu_y), \\
C^3 &= xg(u)y - yf(u)u_x, \quad C^4 = xh(u)u_z;
\end{align*}
\] (3.24)

\[
\begin{align*}
C^1 &= -zu, \quad C^2 = zf(u)u_x, \\
C^3 &= zg(u)y - yh(u)u_z, \quad C^4 = h(u)(yu_y + zu_z).
\end{align*}
\] (3.25)

Proceeding as above with the symmetry \(X_4\) from (3.7) we obtain, in addition to (3.15), (3.17)-(3.19) and (3.23)-(3.25), the following conserved vectors:

\[
\begin{align*}
C^1 &= -xy u, \quad C^2 = yf(u)(xu_x + zu_z), \\
C^3 &= xg(u)(yu_y + zu_z), \\
C^4 &= xyh(u)u_z - yz f(u)u_x - xzg(u)u_y;
\end{align*}
\] (3.26)

\[
\begin{align*}
C^1 &= -x u, \quad C^2 = f(u)(xu_x + zu_z), \\
C^3 &= xg(u)u_y, \quad C^4 = xh(u)u_z - zf(u)u_x;
\end{align*}
\] (3.27)

\[
\begin{align*}
C^1 &= -yu, \quad C^2 = yf(u)u_x, \\
C^3 &= g(u)(yu_y + zu_z), \quad C^4 = yh(u)u_z - zg(u)u_y.
\end{align*}
\] (3.28)

Finally, we turn to the time-translational symmetry \(X_1\) from (3.7). In this case \(W = -u_t\). Replacing \(u_t\) by the right-hand side of Eq. (1.1) we obtain from the first equation (3.6):

\[
C^1 = v \left[ D_x(f(u)u_x) + D_y(g(u)u_y) + D_z(h(u)u_z) \right]. \quad \text{(3.29)}
\]

Now we observe that

\[
v D_x(f(u)u_x) = D_x \left[ v f(u)u_x - v_x F(u) \right],
\]
where we denote
\[ F(u) = \int f(u) du \]
and use the equation \( v_{xx} = 0 \) resulting from the representation (2.8) of \( v \). Transforming likewise two other terms in (3.29) we write \( C_1 \) in the divergent form:
\[ C_1 = D_x [vf(u)u_x - v_x F(u)] + D_y [vg(u)u_y - v_y G(u)] + D_z [vh(u)u_z - v_z H(u)] , \]
where \( G(u) = \int g(u) du, \quad H(u) = \int h(u) du \). Now we can transfer all terms of \( C_1 \) to the components \( C_2, C_3, C_4 \) and obtain \( C_1 = 0 \). The calculation shows that after this transfer we will have \( C_1 = C_2 = C_3 = C_4 = 0 \). Hence, \( X_1 \) does not lead to a non-trivial conservation law.

Thus, we have proved the following statement.

**Theorem 3.1.** The translational symmetries (3.7) of Eq. (3.1) with arbitrary coefficients \( f(u), g(u), h(u) \) provide ten linearly independent conserved vectors (3.15), (3.17)-(3.19) and (3.23)-(3.28).

**Remark 3.1.** Eq. (2.22) is nonlinearly self-adjoint with the substitution (2.30) containing arbitrary solutions \( \alpha(t, y), \beta(t, y), \gamma(t, y), \sigma(t, y) \) of the adjoint equation (2.24) to the one-dimensional linear heat equation. Therefore the conserved vector constructed by the above procedure for Eq. (2.22) will contain arbitrary solutions of Eq. (2.24).

### 3.2 Conserved vectors for Equation (2.18)

As mentioned in Section 2.2, the anisotropic heat equation (1.2) with an external source \( q(u) \neq 0 \) does not have a conservation form. However, Eq. (1.2) can be rewritten in a conservation form
\[ D_t(C^1) + D_x(C^2) + D_y(C^3) = 0 \]
if it is nonlinearly self-adjoint, for example, in the special cases (2.18) and (2.19). We will find here the conservation form for Eq. (2.18). The calculations are similar for Eq. (2.19).

We write Eq. (2.18) in the form
\[ u_t = f(u)u_{xx} + g(u)u_{yy} + f'(u)u_x^2 + g'(u)u_y^2 + \omega^2 F(u), \quad \omega = \text{const.}, \quad (3.30) \]
and have the formal Lagrangian
\[ L = v[f(u)u_{xx} + g(u)u_{yy} + f'(u)u_x^2 + g'(u)u_y^2 + \omega^2 F(u) - u_t]. \quad (3.31) \]
For this formal Lagrangian Eqs. (1.12) yield (cf. Eqs. (3.6))
\[ C^1 = -Wv, \]
\[ C^2 = W [f'(u)u_xv - f(u)v_x] + f(u)vD_x(W), \] (3.32)
\[ C^3 = W [g'(u)u_yv - g(u)v_y] + g(u)vD_y(W). \]

Eq. (3.30) admits the three-dimensional Lie algebra spanned by the operators \(X_1, X_2, X_3\) from (3.7). Let us apply the formula (3.32) to the symmetry \(X_2\). In this case \(W = -u_x\) and (3.32) is written (see Eqs. (3.8))
\[ C^1 = vu_x, \]
\[ C^2 = -f'(u)vu_x^2 + f(u)u_xv_x - f(u)vuv_x, \] (3.33)
\[ C^3 = -g'(u)vu_xu_y + g(u)u_xv_y - g(u)vuv_y, \]
where \(v\) should be replaced by its expression (2.20),
\[ v = (A_1y + B_1)\cos(\omega x) + (A_2y + B_2)\sin(\omega x). \] (2.20)

Let us simplify the vector (3.33) in the same way as in Section 3.2. We write
\[ C^1 = \tilde{C}^1 + D_x(uv), \]
where
\[ \tilde{C}^1 = -uv_x, \] (3.34)
and replace \(C^2\) by
\[ \tilde{C}^2 = C^2 + D_t(uv). \]

Hence
\[ \tilde{C}^2 = C^2 + vu_t \]
\[ = C^2 + v\left[ f(u)u_{xx} + f'(u)u_x^2 + D_y(g(u)v_y) + \omega^2 F(u) \right]. \]
The substitution of the expression (3.32) for \(C^2\) yields:
\[ \tilde{C}^2 = f(u)u_xv_x + \omega^2 F(u) + vD_y(g(u)v_y). \]

One can verify that the following equation holds:
\[ vD_y(g(u)v_y) = D_y [vg(u)v_y - G(u)v_y]. \]

It is obtained by introducing the function \(G(u) = \int g(u)du\) and noting that the equation \(v_{yy} = 0\) is valid for the representation (2.20) of \(v\). Hence \(\tilde{C}^2\) can be reduced to
\[ \tilde{C}^2 = f(u)u_xv_x + \omega^2 F(u), \] (3.35)
whereas $C^3$ becomes
\[
\tilde{C}^3 = C^3 + D_x [vg(u)u_y - G(u)v_y].
\]
The latter equation upon inserting the expression (3.32) for $C^3$ yields:
\[
\tilde{C}^3 = g(u)u_yv_x - G(u)v_{xy}. \tag{3.36}
\]
Collecting Eqs. (3.34)-(3.36) and ignoring the tilde we arrive at the vector
\[
C^1 = -uv_x, \\
C^2 = f(u)u_xv_x + \omega^2 F(u)v, \tag{3.37} \\
C^3 = g(u)u_yv_x - G(u)v_{xy},
\]
where $v$ should be replaced by its expression (2.20). The vector (3.37) satisfies the conservation equation in the following form:
\[
D_t(C^1) + D_x(C^2) + D_y(C^3) = v_x [ (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 F(u) - u_t].
\]
The expression (2.20) for $v$ contains four arbitrary constants $A_1, A_2, B_1, B_2$. Accordingly, the vector (3.37) is a linear combination of four linearly independent vectors. Hence, we have demonstrated the following statement.

**Theorem 3.2.** The invariance of Eq. (3.18) with respect to the one-parameter group of translations of $x$ with the generator $X_2$ provides the following four linearly independent conserved vectors:

\[
\begin{align*}
C^1 &= \sin(\omega x)u, & C^2 &= -\sin(\omega x) f(u)u_x + \omega \cos(\omega x) F(u), \\
C^3 &= -\sin(\omega x) g(u)u_y; \tag{3.38}
\end{align*}
\]
\[
\begin{align*}
C^1 &= \cos(\omega x)u, & C^2 &= -\cos(\omega x)f(u)u_x - \omega \sin(\omega x) F(u), \\
C^3 &= -\cos(\omega x) g(u)u_y; \tag{3.39}
\end{align*}
\]
\[
\begin{align*}
C^1 &= y \sin(\omega x)u, & C^2 &= -y \sin(\omega x) f(u)u_x + \omega y \cos(\omega x) F(u), \\
C^3 &= -y \sin(\omega x) g(u)u_y + \sin(\omega x)G(u); \tag{3.40}
\end{align*}
\]
\[
\begin{align*}
C^1 &= y \cos(\omega x)u, & C^2 &= -y \cos(\omega x)f(u)u_x - \omega y \sin(\omega x) F(u), \\
C^3 &= -y \cos(\omega x) g(u)u_y + \cos(\omega x)G(u). \tag{3.41}
\end{align*}
\]
Remark 3.2. The conserved vectors provided by the generator $X_3$ of the
group of translations of $y$ can be computed likewise. The generator $X_1$ of
the time-translations provides only the trivial conserved vector.

Remark 3.3. Using the conserved vectors (3.38)-(3.41) one can write Eq.
(2.18) in four different conservation forms. For example, the vector (3.38)
satisfies the conservation equation

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = \sin(\omega x) \left[ u_t - (f(u)u_x)_x - (g(u)u_y)_y - \omega^2 F(u) \right].$$

Accordingly, Eq. (2.18) can be replaced by the following conservation equa-
tion:

$$D_t [\sin(\omega x) u] - D_x [\sin(\omega x) f(u)u_x + \omega \cos(\omega x) F(u)]$$
$$- D_y [\sin(\omega x) g(u)u_y] = 0. \quad (3.42)$$
Paper 10

Anisotropic wave equations

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Abstract. Anisotropic nonlinear wave equations with a source are discussed from point of view of their conservation laws and exact solutions associated with the conservation laws. Nonlinearly self-adjoint equations are singled out. Conservation laws associated with symmetries are found and used for constructing exact solutions for nonlinearly self-adjoint equations.

Keywords: Method of conservation laws, Anisotropic wave equation, Symmetries, Conservation laws, Exact solutions.

1 Investigation of nonlinear self-adjointness

We will consider the anisotropic nonlinear wave equations with a source $q(u)$:

$$u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z + q(u).$$

(1.1)

The nonlinearity means that at least one of the functions $f(u), g(u), h(u)$, and $q'(u)$ is not constant.

Equation (1.1) with arbitrary functions $f(u), g(u), h(u), q(u)$ admits three translation groups with the generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial z}.$$
For the sake of brevity, we will dwell on the two-dimensional case and write our equation in the form
\[ F \equiv u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y - q(u) = 0. \] (1.2)

The adjoint equation to Equation (1.2) has the form
\[ F^* \equiv v_{tt} - f(u)v_{xx} - g(u)v_{yy} - q'(u)v = 0. \] (1.3)

The nonlinear self-adjointness condition of Equation (1.2) is written
\[ F^*\bigg|_{v=\varphi} = \lambda F, \] (1.4)
where \(\lambda\) is an undetermined variable coefficient and the symbol \(\bigg|_{v=\varphi}\) in the left-hand side of Equation (1.4) indicates that \(v\) and its derivatives are eliminated by using a substitution
\[ v = \varphi(t, x, y, u), \quad \varphi \neq 0. \] (1.5)

The derivatives of \(v\) are
\[ v_t \equiv D_t(\varphi) = \varphi_u u_t + \varphi_t, \quad v_x \equiv D_x(\varphi) = \varphi_u u_x + \varphi_x, \]
\[ v_y \equiv D_y(\varphi) = \varphi_u u_y + \varphi_y, \]
\[ v_{tt} \equiv D_t^2(\varphi) = \varphi_u u_{tt} + \varphi_{uu} u_t^2 + 2\varphi_{tu} u_t + \varphi_{tt}, \] (1.6)
\[ v_{xx} \equiv D_x^2(\varphi) = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{ux} u_x + \varphi_{xx}, \]
\[ v_{yy} \equiv D_y^2(\varphi) = \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2\varphi_{uy} u_y + \varphi_{yy}. \]

The nonlinear self-adjointness condition (1.4) is written
\[ v_{tt} - f(u)v_{xx} - g(u)v_{yy} - q'(u)v = \lambda \left[ u_{tt} - f(u)u_{xx} - g(u)u_{yy} - f'(u)u_x^2 - g'(u)u_y^2 - q(u) \right], \] (1.7)
where \(v\) and its derivatives in the left-hand side should be replaced with their expressions (1.5) and (1.6). The comparison of the coefficients for \(u_{tt}\) in both sides of Equation (1.7) gives
\[ \lambda = \varphi_u. \]

The similar comparison of the terms with \(u_t^2\) gives
\[ \varphi_{uu} = 0, \] (1.8)
and Equation (1.7) becomes
\[ 2\varphi_{tu}u_t + \varphi_u - f(u) [2\varphi_{xu}u_x + \varphi_{xx}] - g(u) [2\varphi_{yu}u_x + \varphi_{yy}] \\
- q'(u)\varphi = -\varphi_u \left[ f'(u)u_x^2 + g'(u)u_y^2 + q(u) \right]. \tag{1.9} \]

The terms with \( u_x^2; u_y^2 \) in (1.9) give
\[ f'(u)\varphi_u = 0, \quad g'(u)\varphi_u = 0, \]
whence
\[ \varphi_u = 0, \tag{1.10} \]
unless
\[ f'(u) = g'(u) = 0. \tag{1.11} \]

In the case (1.11) we have \( f = A = \text{const.}, \ g = B = \text{const.}, \) and Equation (1.9) yields
\[ \varphi_{tu} = \varphi_{xu} = \varphi_{yu} = 0. \]

These equations together with (1.8) yield
\[ \varphi = \alpha u + \phi(t, x, y), \quad \alpha = \text{const.} \]

Substituting this expression of \( \varphi \) in Equation (1.9) and taking into account Equations (1.11), we obtain \( q''(u) = 0, \) whence
\[ q(u) = Cu + \beta, \quad C, \beta = \text{const.} \]

After further simple analysis of Equation (1.9), we conclude that in the case (1.11) we have the linear equation (1.2),
\[ u_{tt} = Au_{xx} + Bu_{yy} + Cu + \beta \tag{1.12} \]
satisfying the nonlinear self-adjointness condition with the substitution
\[ v = \alpha u + \phi(t, x, y), \quad \alpha = \text{const.}, \tag{1.13} \]
where \( \phi(t, x, y) \) is any solution of the equation
\[ \phi_{tt} = A\phi_{xx} + B\phi_{yy} + C\phi - \alpha \beta \]

Since we are interested in nonlinear equations (1.1), we dwell on the case (1.10), \( \varphi_u = 0. \) Then the substitution (1.5) has the form
\[ v = \varphi(t, x, y), \]
and Equation (1.9) is written
\[ \varphi_{tt} - f(u)\varphi_{xx} - g(u)\varphi_{yy} - q'(u)\varphi = 0. \] (1.14)

If \( f(u), g(u) \) and \( q(u) \) are arbitrary functions, then Equation (1.14) yields
\[ \varphi_{tt} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi = 0. \] (1.15)

These equations show that a substitution of the form (1.5) does not exist because the last equation in (1.15) contradicts the condition \( \varphi \neq 0 \).

Hence, Equation (1.2) with the arbitrary source \( q(u) \) is not nonlinearly self-adjointness with the substitution of the form (1.5).

However, Equation (1.2) with sources of particular forms can be nonlinearly self-adjoint as it happens for the heat diffusion in anisotropic media with a source (see [42], see also Paper 9 in the present volume). This happens if
\[ q'(u) = C, \quad C = \text{const.}, \] (1.16)
or if \( q'(u) \) is proportional to \( f(u) \):
\[ q'(u) = rf(u), \quad r = \text{const.} \] (1.17)

If \( q'(u) \) is proportional to \( g(u) \), one obtains the case (1.17) by interchanging the variables \( x \) and \( y \).

Let us consider the case (1.17) with a positive constant \( r = \omega^2, \omega \neq 0 \).

Then Equation (1.2) has the form
\[ u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 F(u), \] (1.18)
where \( F'(u) = f(u) \). Now the equation (1.14) is replaced by the equation
\[ \varphi_{tt} + f(u)[\varphi_{xx} + \omega^2 \varphi] + g(u)\varphi_{yy} = 0. \]

Consequently, we have the equations
\[ \varphi_{tt} = 0, \quad \varphi_{xx} + \omega^2 \varphi = 0, \quad \varphi_{yy} = 0 \] (1.19)
instead of Equations (1.15). Dealing with Equations (1.19) as in the case of diffusion equations [42], we can verify that the nonlinear self-adjointness condition (1.4) is satisfied for Equation (1.18) with the following substitution (1.5):
\[ v = [(A_1y + B_1)t + (a_1y + b_1)]\cos(\omega x) \]
\[ + [(A_2y + B_2)t + (a_2y + b_2)]\sin(\omega x), \] (1.20)
where \( A_1, \ldots, b_2 \) are arbitrary constants.
2 Conservation law

Let us construct the conservation law for Equation (1.18),

\[ u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 F(u), \tag{2.1} \]

using the space translation symmetry

\[ X = \frac{\partial}{\partial x}. \tag{2.2} \]

and taking the substitution (1.20) of the particular form

\[ v = t \sin(\omega x) \tag{2.3} \]

corresponding to \( B_2 = 1, A_1 = B_1 = A_2 = a_1 = b_1 = a_2 = b_2 = 0. \)

The formal Lagrangian for Equation (2.1) has the form

\[ \mathcal{L} = v \left[ u_{tt} - (f(u)u_x - f'(u)u_x^2 - (g(u)u_y - g'(u)u_y^2 - \omega^2 F(u)) \right], \tag{2.4} \]

and the general formula [38] for constructing a conserved vector associated with a symmetry

\[ X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u} \]

of a self-adjoint equation is written as follows (see also Paper 4, Section 8.2 and Paper 9, Section 1.3 in the present volume):

\[ C^i = W \left[ \frac{\partial \mathcal{L}}{\partial u_t} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{tj}} \right) \right] + D_j(W) \frac{\partial \mathcal{L}}{\partial u_{ij}}, \tag{2.5} \]

where

\[ W = \eta - \xi^j u_j, \]

the variable \( v \) should be eliminated by using the substitution (2.3).

In our case we write the conservation equation in the form

\[ \left[ D_t(C^1) + D_x(C^2) + D_y(C^3) \right]_{(2.1)} = 0. \tag{2.6} \]

Inserting in Equations (2.4) the formal Lagrangian (2.4) and the expression \( W = -u_x \) corresponding to the operator (2.2) we obtain the conserved vector with the following components:

\[ C^1 = \omega t \cos(\omega x) u_t - \omega \cos(\omega x) u, \]
\[ C^2 = -\omega t \sin(\omega x) F(u) \sin(\omega x), \tag{2.7} \]
\[ C^3 = -\omega t \cos(\omega x) g(u) u_y. \]
We have:

\[ D_t(C^1) = \omega t \cos(\omega x)u_{tt}, \]
\[ D_x(C^2) = -\omega t \cos(\omega x)((f(u)u_x)_x - \omega^2 F(u)), \] \hspace{1cm} (2.8)
\[ D_y(C^3) = -\omega t \cos(\omega x)((g(u)u_y)_y). \]

It is transparent from Equations (2.8) that the conservation equation (2.1) is satisfied.

3 Solutions provided by the conservation law

According to Equations (2.8), the method of conservation laws (see Paper 8 or Paper 4 Part 3, in the present volume) for constructing particular solutions of Equation (2.1) gives the following differential constraints:

\[ \omega t \cos(\omega x)u_{tt} = 0, \]
\[ \omega t \cos(\omega x)((f(u)u_x)_x - \omega^2 F(u)) = 0, \] \hspace{1cm} (3.1)
\[ \omega t \cos(\omega x)((g(u)u_y)_y) = 0. \]

Hence, we have to solve the following over-determined system:

\[ u_{tt} = 0, \]
\[ (f(u)u_x)_x - \omega^2 F(u) = 0, \] \hspace{1cm} (3.2)
\[ (g(u)u_y)_y = 0. \]

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Bibliography


[8] G. W. Bluman and S. Kumei. On the remarkable nonlinear diffusion equation $\frac{\partial}{\partial x} \left[ a(u + b)^{-2} \frac{\partial u}{\partial x} \right] - \frac{\partial u}{\partial t} = 0$. *J. Math. Phys.*, 21, No. 5:1019–1023, 1980.

291


[80] M. A. Tsyganov, V. N. Biktashev, J. Brindley, A. V. Holden, and 
G.R. Ivanitsky. Waves in systems with cross-diffusion as a new class 


with a plasma via the Compton effect. *Physics of Fluids*, 8:2112–2114, 
1965.

[83] R. Weymann. The energy spectrum of radiation in the expanding 
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Volume V contains preprints written during 2008-2014 as advanced topics to be added to the textbook (b). They include, e.g. a discussion of a wide class of linear ordinary differential equations whose integration is reducible to solution of algebraic equations. This class contains the constant coefficient equations and Euler’s equations as particular cases. The recent method of nonlinear self-adjointness for constructing conservations laws associated with symmetries of partial differential equations is also presented in this volume.